ある無限グラフ上の因子に関するリーマン・ロッホの定理について

(On a Riemann-Roch theorem for divisors on an infinite graph)

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1. Riemann-Roch theorem on a weighted finite graph

Let $G = (V_G, E_G)$ be a connected graph consisting of finite set V_G of vertices and of finite set E_G of edges. We assume that weight $C_{x,y}$ is given at every edge $\{x, y\} \in E_G$.

For every vertex $x \in V_G$, define $N(x) = \{y \in V_G \mid \{x, y\} \in E_G\}$ and $i(x) = \min\{|\sum_{y\in N(x)} f(y)C_{x,y}| \in (0,\infty) \mid f : V_G \to \mathbb{Z}\}$.

notions	probabilistic materials
weight on edges	conductance $C_{x,y}$ between x and y
weight at vertices	$i(x)$
divisor	$D = \sum_{x\in V_G} \ell(x)i(x)1_{\{x\}}$
degree of divisor	$\deg(D) = \sum_{x\in V_G} \ell(x)i(x)$
canonical divisor	$K_G = \sum_{x\in V_G}\{\sum_{y\in N(x)} C_{x,y} - 2i(x)\}1_{\{x\}}$
Laplacian of f at $x \in V_G$	$\Delta f(x) = \sum_{y\in N(x)} C_{x,y}(f(x) - f(y))$
Euler-like characteristic	$\mathfrak{e}_{(V,C)} = \sum_{x\in V_G} i(x) - \sum_{\{x,y\}\in E_G} C_{x,y}$

A divisor $D = \sum_{x\in V_G} \ell(x)i(x)1_{\{x\}}$ is said to be effective, if $\ell(x) \geq 0$ for all $x \in V_G$. We need also the canonical divisor $K_G = \sum_{x\in V_G}\{\sum_{y\in N(x)} C_{x,y} - 2i(x)\}1_{\{x\}}$ and the family of total orders on V_G denoted by $\mathcal{O}$. For each $O \in \mathcal{O}$, its inverted total order $\overline{O}$ is defined by $x <_{\overline{O}} y$ for any $x, y \in V_G$ satisfying $y <_O x$. We introduce the divisor

$$\nu_O(x) = \sum_{y\in N(x), y<_O x} C_{x,y} - i(x), \quad x \in V_G$$

of degree $-\mathfrak{e}_{(V,C)} = \sum_{\{x,y\}\in E_G} C_{x,y} - \sum_{x\in V_G} i(x)$ admitting only non-effective equivalent divisors.

We introduce an equivalence between divisors D and D' and notation for the equivalence class given by

$$D \sim D' \Leftrightarrow D' = D + \Delta f \text{ for some } \mathbb{Z}\text{-valued function} f,$$
$$|D| = \{D' \mid D' \text{ is effective and equivalent with } D\}.$$

For any divisor D and non-negative integer k, we take

$$E_k(D) = \{ \text{ effective divisors } E \mid \deg\,(E) = i_{(V,C)}k \text{ satisfying } |D - E| \neq \emptyset\}$$

to define the dimension $r(D)$ of the divisor D by

$$r(D) = \begin{cases} -i_{(V,C)}, & \text{if } E_0(D) = \emptyset, \\ \max\{i_{(V,C)}k \mid E_k(D) \text{ consists of all effective divisors of degree } i_{(V,C)}k\}, & \text{otherwise.} \end{cases}$$

Theorem (Riemann-Roch theorem on weighted finite graph). For any divisor D,

$$r(D) - r(K_G - D) = \deg(D) + \mathfrak{e}_{(V,C)}.$$

Similarly to M. Baker and S. Norine's article [1], we can prove this assertion, the corner stones of which are the following assertions:

(RR) For each divisor D, there exists an $O \in \mathcal{O}$ such that either $|D|$ or $|\nu_O - D|$ is empty.

Proposition 1 (RR) implies $r(D) = \Big(\min_{D' \sim D, O \in \mathcal{O}} \deg^+(D' - \nu_O)\Big) - i_{(G,C)}$ for any divisor D, where $i_{(G,C)} = \min\{|\sum_{x \in V_G} \ell(x)i(x)| \in (0, \infty) \mid \ell : V_G \to \mathbb{Z}\}$ and $\deg^+(D) = \sum_{\ell(x)>0} \ell(x)i(x)$ for divisor $D = \sum_{x \in V_G} \ell(x)i(x)1_{\{x\}}$.

2. Riemann-Roch theorem in an infinite graph

Throughout this section we consider an infinite graph $G = (V_G, E_G)$ with locally finiteness and finite volume, namely, the function $\#N(x)$ given by $N(x) = \{y \mid \{x, y\} \in E_G\}$ is integer valued and the total volume $m(V) = \sum_{x \in V} m(x)$ given by $m(x) = \sum_{y \in N(x)} C_{x,y}$ is finite.

For any pair $\{x, y\}$ of distinct elements in V_G, we define the graph metric $d(x, y)$ between x, y by $d(x, y) = \min\{k \in \mathbb{N} \mid \{z_0, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-1}, z_k\} \in E_G$ for some $z_1, \ldots, z_{k-1} \in V_G$ with $z_0 = x, z_k = y\}$. We fix a reference vertex $v_0 \in V_G$ and take the sphere $S_k = \{y \in V_G \mid d(v_0, y) = k\}$ centered at the reference vertex v_0 with radius $k \in \mathbb{N}$ with respect to the graph metric d.

We consider a divisor $D = \sum_{x \in V_G} \ell(x)i(x)1_{\{x\}}$ on V_G satisfying $\sum_{x \in V_G} |\ell(x)|i(x)1_{\{x\}} < \infty$. We take an exhaustion sequence $G_1 \subset G_2 \subset \cdots$ of subgraphs of $G = (V_G, E_G)$ determined by $V_n = \{a \in V_G \mid d(v_o, a) \leq n\}$, $E_n = \{\{a, b\} \in E_G \mid a, b \in V_n\}$ and $G_n = (V_n, E_n)$ for each $n \in \mathbb{N}$.

We make an attempt to extend the Riemann-Roch theorem on finite graphs to one on an infinite graph by finding such sufficient conditions that sequence $\{r_n(D)\}$ consisting of so-called dimension of D on each G_n converges as n tends to ∞ for any divisor $D = \sum_{x \in V_G} \ell(x)i(x)1_{\{x\}}$ with finiteness of its support $\mathrm{supp}[D] = \{x \in V_G \mid \ell(x) \neq 0\}$. We will propose several conditions on the infinity of G for controlling the dimensions of the divisor by closely looking at the Laplacian naturally associated with $\{C_{x,y}\}$.

As a result, after redefinitions of the dimension $r(D)$, the canonical divisor K_G and Euler-like characteristic $\mathfrak{e}_{(V,C)}$, we can assert the same identity as in Theorem as a Riemann-Roch theorem for divisor $D = \sum_{x \in V_G} \ell(x)i(x)1_{\{x\}}$ on V_G with $\sum_{x \in V_G} |\ell(x)|i(x)1_{\{x\}} < \infty$ on an infinite graph satisfying specific conditions.

References

[1] M. Baker and S. Norine, Riemann-Roch and Abel-Jacobi theory on a finite graph, Advances in Mathematics, Volume 215, Issue 2, Pages 766-788.

定常過程に対する MA ブートストラップ

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$a \in \mathbb{R}^{q\times q}$ に対し，$\|a\| := \sup_{u\in\mathbb{R}^q, |u|=1} |au|$ をそのスペクトル・ノルムとする．$q \in \mathbb{N}$ とする．平均 $\mu_X \in \mathbb{R}^q$ を持つ $\mathbb{R}^q$-値の定常過程 $\{X_t\}_{t\in\mathbb{Z}}$ はMA (∞) 表現

$$X_k - \mu_X = \sum_{j=-\infty}^{k} \psi_{k-j}\epsilon_j, \qquad k \in \mathbb{Z} \tag{1}$$

により記述されるとする．ここで $\{\psi_k\}_{k=0}^{\infty}$ は $\mathbb{R}^{q\times q}$-値の列で，次を満たすとする：(A1) $\psi_0 = I_q$, (A2) $\sum_{j=0}^{\infty} j\|\psi_j\| < \infty$, (A3) $\Psi(z) := \sum_{j=0}^{\infty} z^j\psi_j$ は $\det\Psi(z) \neq 0$ $(z \in \mathbb{C},\ |z| \leq 1)$ を満たす．また，次も仮定する：

(A4) $\{\epsilon_k\}_{k\in\mathbb{Z}}$ は $\mathbb{R}^q$-値の IID 確率ベクトル列で，$E[\|\epsilon_0\|^4] < \infty$, $E[\epsilon_0] = 0$ および $E[\epsilon_0\epsilon_0^{\mathrm{T}}] > 0$ を満たす.

$X_1, \ldots, X_n$ を $\{X_t\}$ からの標本とする. 経験自己共分散関数 $\{\widehat{\gamma}(j)\}$ を次の様に定義する：

$$\widehat{\gamma}(j) := \begin{cases} \frac{1}{n}\sum_{k=1}^{n-j}(X_{k+j} - \bar{X}_n)(X_k - \bar{X}_n)^{\mathrm{T}}, & j = 0, 1, \ldots, n-1, \\ \frac{1}{n}\sum_{k=-j+1}^{n}(X_{k+j} - \bar{X}_n)(X_k - \bar{X}_n)^{\mathrm{T}}, & j = -n+1, \ldots, -1. \end{cases}$$

ここで $\bar{X}_n := \frac{1}{n}\sum_{k=1}^{n} X_k$ は標本平均を表す.

我々の MA ブートストラップは，次の 2 つのステップよりなる．

Step 1. このステップでは，標本のサイズ n が増加するにつれて増加する 次数 $p(n)$ の MA (移動平均) 過程を $\{X_n\}$ にフィットさせて，ノイズの経験分布を求める．

$1 \ll n$ に対し, $(\widehat{\phi}_{1,n}, \widehat{\phi}_{2,n}, \ldots, \widehat{\phi}_{n,n})$ を次の経験 Yule–Walker 方程式の解とする：

$$\sum_{j=1}^{n} \widehat{\phi}_{j,n}\widehat{\gamma}(i-j) = \widehat{\gamma}(i), \qquad i = 1, 2, \ldots, n.$$

我々は $p(n)$ を，$p(n) \to \infty$ $(n \to \infty)$ および $p(n) = o(n)$ $(n \to \infty)$ が成り立つようにとる. 以下，簡単のため，$p(n)$ を p と書く．次のようにおく：

$$\widehat{\psi}_{k,p} := \widehat{v}_{k,p}\widehat{v}_{0,p}^{-1}, \qquad k = 0, \ldots, p, \tag{2}$$

$$\widehat{v}_{k,p} := \widehat{\gamma}(k) - \sum_{j=1}^{p} \widehat{\gamma}(k+j)\widehat{\phi}_{j,p}^{\mathrm{T}}, \qquad k = 0, \ldots, p. \tag{3}$$

まず，$\epsilon_k = 0$ $(k = 0, \ldots, p-1)$ とし，

$$\epsilon_k = X_k - \bar{X}_n - \sum_{l=1}^{p} \widehat{\psi}_{l,p}\epsilon_{k-l}$$

から ϵ_k $(k = p+1, \ldots, n)$ を求め，

$$\widehat{\epsilon}_{k,p} = \epsilon_k, \qquad k = p+1, \ldots, n$$

を定める。次に

$$\tilde{\epsilon}_{k,p} := \widehat{\epsilon}_{k,p} - \frac{1}{n-p} \sum_{j=p+1}^{n} \widehat{\epsilon}_{j,p}, \qquad k = p+1, \ldots, n$$

と中心化し，$\{\tilde{\epsilon}_{k,p}\}_{k=p+1}^{n}$ の経験分布 $\frac{1}{n-p}\sum_{t=p+1}^{n} \delta_{\tilde{\epsilon}_{t,n}}$ の分布関数を $\widehat{F}_{\epsilon,p}$ と表す．

Step 2. このステップでは，$\{X_t\}$ の MA (移動平均) 近似を用いて標本 $X_1, \ldots, X_n$ のリサンプリング $\{X_t^*\}$ を構成する．

$\{\tilde{\epsilon}_{t,n}\}$ のリサンプリング$\{\epsilon_t^*\}_{t\in\mathbb{Z}}$ を

$$\{\epsilon_t^*\} \text{ は IID でかつ各 } \epsilon_t^* \text{ の分布は } \widehat{F}_{\epsilon,n} \text{ に従う}$$

ように取る．観測データ $X_1, \ldots, X_n$ のリサンプリング $\{X_t^*\}_{t\in\mathbb{Z}}$ を，次の**近似移動平均表現**に従い構成する：

$$X_t^* = \bar{X}_n + \sum_{j=0}^{p} \widehat{\psi}_{j,p} \epsilon_{t-j}^* \qquad (t \in \mathbb{Z}). \tag{4}$$

上の MA ブートストラップの構成は，標本 $X_1, \ldots, X_n$ による条件付確率 P^* を導く．我々は, P^* に関する量を * をつけて書く．

次のようにおく： $\widehat{\psi}_{k,p} := 0, \quad k \geq p+1$．次は [1, Lemma 5.1] の類似物である．

Lemma 1. (A1)–(A4) および $p(n) = O((n/\log n)^{1/4})$ $(n \to \infty)$ を仮定する. すると次を満たす確率変数n_1 が存在する：

$$\sup_{n \geq n_1} \sum_{j=0}^{\infty} j \|\widehat{\psi}_{j,p}\| < \infty \quad \text{almost surely.}$$

次の補題は [1, Lemma 5.2] の類似物である．

Lemma 2. (A1)–(A4) および $p(n) = O((n/\log n)^{1/2})$ $(n \to \infty)$ を仮定する. すると，次が成り立つ：

$$\sup_{0 \leq j < \infty} \|\widehat{\psi}_{j,p} - \psi_j\| = o(1) \quad (n \to \infty) \quad \text{almost surely.}$$

これらのことから，上の MA ブートストラップに対しては，[1] の AR ブートストラップの諸結果 (の多次元版) と同様のことが成り立つ．例えば，次は [1, Lemma 5.5] の類似物である．

Theorem 3. (A1)–(A4) および $p(n) = O((n/\log n)^{1/4})$ $(n \to \infty)$ を仮定する. すると次が成り立つ：

$$X_k^* \xrightarrow{d^*} X_k \quad \text{in prob.}$$

講演では，シミュレーション結果についても紹介する予定である．

参考文献

[1] Bühlmann, P. (1997). Sieve bootstrap for time series. *Bernoulli* **3** 123–148.

多変量 ARMA 過程の有限予測係数に対する閉形式表示

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Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ be the unit circle and the closed unit disk, in $\mathbb{C}$, respectively. Let $d \in \mathbb{N}$. In this talk, a d-variate *ARMA* (autoregressive moving-average) process $\{X_k : k \in \mathbb{Z}\}$ is a $\mathbb{C}^d$-valued, centered, weakly stationary process with spectral density w of the form

$$w(e^{i\theta}) = h(e^{i\theta})h(e^{i\theta})^*, \qquad \theta \in [-\pi, \pi) \tag{1}$$

with $h : \mathbb{T} \to \mathbb{C}^{d\times d}$ satisfying the following condition:

$$\begin{array}{l}\text{the entries of } h(z) \text{ are rational functions in } z \text{ that have}\\ \text{no poles in } \overline{\mathbb{D}}, \text{ and } \det h(z) \text{ has no zeros in } \overline{\mathbb{D}}.\end{array} \tag{C}$$

It is known that there exists $h_\sharp : \mathbb{T} \to \mathbb{C}^{d\times d}$ that satisfies (C) and

$$w(e^{i\theta}) = h(e^{i\theta})h(e^{i\theta})^* = h_\sharp(e^{i\theta})^* h_\sharp(e^{i\theta}), \qquad \theta \in [-\pi, \pi), \tag{2}$$

and $h_\sharp$ is unique up to a constant unitary factor. We may take $h_\sharp = h$ for the univariate case $d = 1$ but not so for $d \ge 2$, and this is one of the main difficulties when we deal with multivariate processes. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. We can write $h(z)^{-1}$ in the form

$$h(z)^{-1} = -\rho_0 - \sum_{\mu=1}^{K}\sum_{j=1}^{m_\mu} \frac{1}{(1-\overline{p}_\mu z)^j}\rho_{\mu,j} - \sum_{j=1}^{m_0} z^j \rho_{0,j}, \tag{3}$$

where

$$\begin{cases} K \in \mathbb{N} \cup \{0\}, \\ p_\mu \in \mathbb{D} \setminus \{0\} \quad (\mu = 1, \dots, K), \quad p_\mu \ne p_\nu \quad (\mu \ne \nu), \\ m_\mu \in \mathbb{N} \quad (\mu = 1, \dots, K), \quad m_0 \in \mathbb{N} \cup \{0\}, \\ \rho_{\mu,j} \in \mathbb{C}^{d\times d} \quad (\mu = 0, 1, \dots, K,\ j = 1, \dots, m_\mu), \quad \rho_0 \in \mathbb{C}^{d\times d}, \\ \rho_{\mu,m_\mu} \ne 0 \quad (\mu = 0, 1, \dots, K). \end{cases} \tag{4}$$

Here the convention $\sum_{k=1}^{0} = 0$ is adopted in the sums on the right-hand side of (3).

The next theorem shows that $h_\sharp^{-1}$ of a vector ARMA process has the same m_0 and the same poles with the same multiplicities as h^{-1}.

Theorem 1. *For m_0, K and $(p_1, m_1), \dots, (p_L, m_L)$ in (3) with (4), $h_\sharp^{-1}$ has the form*

$$h_\sharp(z)^{-1} = -\rho_0^\sharp - \sum_{\mu=1}^{K}\sum_{j=1}^{m_\mu} \frac{1}{(1-\overline{p}_\mu z)^j}\rho_{\mu,j}^\sharp - \sum_{j=1}^{m_0} z^j \rho_{0,j}^\sharp, \tag{5}$$

where

$$\begin{cases} \rho_{\mu,j}^\sharp \in \mathbb{C}^{d\times d} \quad (\mu = 0, 1, \dots, K,\ j = 1, \dots, m_\mu), \quad \rho_0^\sharp \in \mathbb{C}^{d\times d}, \\ \rho_{\mu,m_\mu}^\sharp \ne 0 \ (\mu = 0, 1, \dots, K). \end{cases} \tag{6}$$

We are concerned with the *finite predictor coefficients* $\phi_{n,j} \in \mathbb{C}^{d\times d}$ $(j = 1, \ldots, n)$ of a d-variate ARMA process $\{X_k\}$, defined by

$$P_{[-n,-1]}X_0 = \phi_{n,1}X_{-1} + \cdots + \phi_{n,n}X_{-n}, \tag{7}$$

where, for $n \in \mathbb{N}$, $P_{[-n,-1]}X_0$ stands for the best linear predictor of the future value X_0 based on the finite past $\{X_{-n}, \ldots, X_{-1}\}$.

The next theorem gives a closed-form expression for $\phi_{n,j}$.

Theorem 2. *Suppose that $m_\mu = 1$ $(\mu = 1, \ldots, K)$ and $m_0 = 0$. Then, for $n \geq 1$ and $j = 1, \ldots, n$, we have*

$$\phi_{n,j} = c_0 a_j + c_0 \mathbf{p}_0^{\mathrm{T}} (I_{dM} - \tilde{G}_n G_n)^{-1} (\Pi_n \Theta)^* \{\Lambda^{\mathrm{T}} \Pi_n \Theta \Xi_j \rho + \Xi_{n-j+1} \tilde{\rho}\}, \tag{8}$$

where $a_j = \sum_{\mu=1}^{K} \overline{p}_\mu^j \rho_{\mu,1}$ for $j \geq 1$, $\mathbf{p}_0^{\mathrm{T}} = (I_d, \ldots, I_d) \in \mathbb{C}^{d\times dK}$,

$$\Theta = \begin{pmatrix} p_1 h_\sharp(p_1)\rho_{1,1}^* & & & 0 \\ & p_2 h_\sharp(p_2)\rho_{2,1}^* & & \\ & & \ddots & \\ 0 & & & p_K h_\sharp(p_K)\rho_{K,1}^* \end{pmatrix} \in \mathbb{C}^{dK\times dK},$$

$$\Lambda = \begin{pmatrix} \frac{1}{1-p_1\overline{p}_1} I_d & \frac{1}{1-p_1\overline{p}_2} I_d & \cdots & \frac{1}{1-p_1\overline{p}_K} I_d \\ \frac{1}{1-p_2\overline{p}_1} I_d & \frac{1}{1-p_2\overline{p}_2} I_d & \cdots & \frac{1}{1-p_2\overline{p}_K} I_d \\ \vdots & \vdots & & \vdots \\ \frac{1}{1-p_K\overline{p}_1} I_d & \frac{1}{1-p_K\overline{p}_2} I_d & \cdots & \frac{1}{1-p_K\overline{p}_K} I_d \end{pmatrix} \in \mathbb{C}^{dK\times dK},$$

$$\Pi_n = \begin{pmatrix} p_1^n I_d & & & 0 \\ & p_2^n I_d & & \\ & & \ddots & \\ 0 & & & p_K^n I_d \end{pmatrix} \in \mathbb{C}^{dK\times dK}, \qquad n \geq 0,$$

$$\Xi_n = \begin{pmatrix} \frac{\overline{p}_1^n}{1-p_1\overline{p}_1} I_d & \frac{\overline{p}_2^n}{1-p_1\overline{p}_2} I_d & \cdots & \frac{\overline{p}_K^n}{1-p_1\overline{p}_K} I_d \\ \frac{\overline{p}_1^n}{1-p_2\overline{p}_1} I_d & \frac{\overline{p}_2^n}{1-p_2\overline{p}_2} I_d & \cdots & \frac{\overline{p}_K^n}{1-p_2\overline{p}_K} I_d \\ \vdots & \vdots & & \vdots \\ \frac{\overline{p}_1^n}{1-p_K\overline{p}_1} I_d & \frac{\overline{p}_2^n}{1-p_K\overline{p}_2} I_d & \cdots & \frac{\overline{p}_K^n}{1-p_K\overline{p}_K} I_d \end{pmatrix} \in \mathbb{C}^{dK\times dK}, \qquad n \geq 1,$$

$$\rho = (\rho_{1,1}^{\mathrm{T}}, \rho_{2,1}^{\mathrm{T}}, \ldots, \rho_{K,1}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{C}^{dK\times d},$$

$$\tilde{\rho} = \left(\overline{\rho_{1,1}^\sharp}, \overline{\rho_{2,1}^\sharp}, \ldots, \overline{\rho_{K,1}^\sharp}\right)^{\mathrm{T}} \in \mathbb{C}^{dK\times d}$$

and $G_n = \Pi_n \Theta \Lambda$, $\tilde{G}_n = (\Pi_n \Theta)^* \Lambda^{\mathrm{T}} \in \mathbb{C}^{dK\times dK}$.

The assumptions in Theorem 2 are just for simplicity of presentation. For the general result, see [1].

参考文献

[1] Inoue, A. (2018). Closed-form expression for finite predictor coefficients of vector ARMA processes, https://arxiv.org/pdf/1805.04820.pdf

Percolation と triangle condition

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1 Percolation

グラフ $G = (V, E)$ とパラメータ $p \in [0, 1]$ に対して, 各辺 $e \in E$ に確率 p で open, 確率 $1 - p$ で closed というラベルを付ける．このラベル付けのもと $(V, \{\text{open edge}\})$ という部分グラフを構成する．このようにして G の部分グラフからなる集合に確率測度を定めることを percolation という．本講演ではグラフとして infinite, connected, quasi-transitive を満たすもののみを考える, ここで quasi-transitive とはグラフの自己同型群 $\mathrm{Aut}(G)$ による G の商が有限集合である．つまりある有限個の頂点 $x_1, \ldots, x_n$ が存在して次の等式を満たす.

$$V = \bigcup_{i=1}^{n} \mathrm{Aut}(G) x_i.$$

このような仮定の任意のグラフ, 任意の p に対して infinite cluster (connected component) の個数は 確率 1 である定数になり, その定数は $0, \infty, 1$ の 3 通りしかない [5]. かつ p に関して単調性を持ちある種の臨界現象が起きる [3]. そこで figure 1 のようにそれぞれの閾値を p_c, p_u とおき, critical probability, uniqueness threshold と呼ぶ.

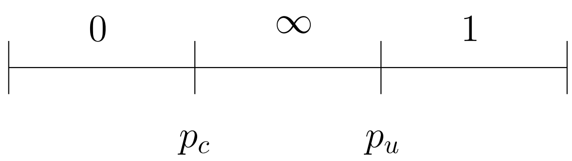

図 1: p_c and p_u

2 Triangle condition

各 2 頂点 $x, y \in V$ に対して, $\tau_p(x, y)$ を x と y が同じ cluster に含まれる確率とする．ある頂点 o を含む cluster の大きさの期待値を χ_p と置く．つまり次の等式で表される.

$$\chi_p = \sum_{x \in V} \tau_p(o, x).$$

これは p に関して単調増加な関数であり p_c で発散する [1]. この結果は上記の仮定を満たすすべてのグラフに対して成り立つことであるが, グラフが d-次元単位格子 $\mathbb{Z}^d$ の場合にはもっと早くから分かっていた. Aizenman と Newman [2] は $\mathbb{Z}^d$ において χ_p の漸近挙動を得る十分条件として triangle condition を導入した. グラフ G が p で triangle condition を満たすとは,

$$\nabla_p = \sum_{x,y \in V} \tau_p(o,x)\tau_p(x,y)\tau_p(y,o) < \infty$$

が成り立つことである. もし p_c で triangle condition を満たすならば $\lim_{p \uparrow p_c} \chi_p(p_c - p) = 1$ が成り立つことが示された. その後 $\mathbb{Z}^d$ で, より一般のグラフで, p_c で triangle condition を満たすなら ○○ が成り立つ, という趣旨の論文が多く出てきた.

3 主結果

グラフとして d-regular tree と $\mathbb{Z}$ の直積 $T_d \square \mathbb{Z}$ を考えたとき, 既に Hutchcroft [4] によって p_c で triangle condition を満たすことは知られている. かつ $p_c < p_u$ が任意の $d \geq 3$ で成り立つことも示されている. 主結果はどの範囲まで triangle condition を満たすかについてであり, 次の結果を得た.

$$\nabla_p \begin{cases} < \infty & (p < p_u) \\ = \infty & (p = p_u). \end{cases}$$

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可算マルコフシフトの大偏差原理とその連分数展開への応用

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Denote by X the set of all one-sided infinite sequences over the set $\mathbb{N}$ of positive integers, namely $X = \{x = (x_1, x_2, \ldots)\colon x_i \in \mathbb{N},\ i \in \mathbb{N}\}$, endowed with the product topology of the discrete topology on $\mathbb{N}$. Define the left shift $\sigma\colon X \to X$ by $(\sigma x)_i = x_{i+1}$ ($i \in \mathbb{N}$). For each $x \in X$ and $n \in \mathbb{N}$ define *an n-cylinder* by

$$[x_1, \ldots, x_n] = \{y = (y_i) \in X\colon x_i = y_i \text{ for every } i \in \{1, \ldots, n\}\}.$$

Let $\phi\colon X \to \mathbb{R}$ be a function. A Borel probability measure μ_ϕ on X is *Bowen's Gibbs measure for the potential ϕ* [1, 4, 5] if there exist constants $c_0 > 0$, $c_1 > 0$ and $P \in \mathbb{R}$ such that for every $x \in X$ and every $n \in \mathbb{N}$,

$$c_0 \leq \frac{\mu_\phi[x_1, \ldots, x_n]}{\exp\left(-Pn + \sum_{i=0}^{n-1} \phi(\sigma^i(x))\right)} \leq c_1.$$

Let $\mathcal{M}$ denote the space of Borel probability measures on X endowed with the weak*-topology. We are concerned with the following three sequences $\{\Delta_n\}$, $\{\Xi_n\}$, $\{\Upsilon_{y,n}\}$ of Borel probability measures on $\mathcal{M}$:

For each $x \in X$ and $n \in \mathbb{N}$ define $\delta_x^n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i x}$, with $\delta_{\sigma^i x}$ the unit point mass at $\sigma^i x$. Denote by Δ_n the distribution of the $\mathcal{M}$-valued random variable $x \mapsto \delta_x^n$ on the probability space (X, μ_ϕ);

For each integer $n \in \mathbb{N}$ define

$$\Xi_n = \left(\sum_{x \in \mathrm{Per}_n\sigma} \exp S_n\phi(x)\right)^{-1} \sum_{x \in \mathrm{Per}_n\sigma} \exp S_n\phi(x)\delta_{\delta_x^n},$$

$$\Upsilon_{y,n} = \left(\sum_{x \in \sigma^{-n}y} \exp S_n\phi(x)\right)^{-1} \sum_{x \in \sigma^{-n}y} \exp S_n\phi(x)\delta_{\delta_x^n},$$

where $\mathrm{Per}_n\sigma = \{x \in X\colon \sigma^n x = x\}$, $\sigma^{-n}y = \{x \in X\colon \sigma^n x = y\}$ and $y \in X$ is fixed.

Theorem A. ([6, Theorem A]). *Let $\phi\colon X \to \mathbb{R}$ be a measurable function and μ_ϕ a Bowen's Gibbs measure for the potential ϕ. Then $\{\Delta_n\}$, $\{\Xi_n\}$, $\{\Upsilon_{y,n}\}$ are exponentially tight and satisfy the Large Deviation Principle with the same convex good rate function I. All their weak*-limit points are supported on subsets of the set $I^{-1}(0)$.*

Under the hypotheses and notation of Theorem A, we call $\nu \in \mathcal{M}$ *a minimizer* if $I(\nu) = 0$. We give a sufficient condition for the uniqueness of minimizer. For a function $\phi\colon X \to \mathbb{R}$ put

$$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{x_1 \cdots x_n} \sup_{[x_1, \ldots, x_n]} \exp \sum_{i=0}^{n-1} \phi \circ \sigma^i,$$

where the sum runs over all n-cylinders. For $\gamma \in (0, 1]$ we introduce a metric d_γ on X by setting $d_\gamma(x, y) = \exp\left(-\gamma \inf\{i \in \mathbb{N}\colon x_i \neq y_i\}\right)$, with the convention $e^{-\infty} = 0$. A function $\phi\colon X \to \mathbb{R}$ is *Hölder continuous* if there exist $C > 0$ and $\gamma \in (0, 1]$ such that for every $k \in \mathbb{N}$ and all $x, y \in [k]$, $|\phi(x) - \phi(y)| \leq C d_\gamma(x, y)$.

Theorem B. *Let $\phi\colon X \to \mathbb{R}$ be a Hölder continuous function such that $\beta_\infty := \inf\{\beta \in \mathbb{R}\colon P(\beta\phi) < \infty\} < 1$. Then there exists a unique shift-invariant Bowen's Gibbs measure for the potential ϕ. It is the unique equilibrium state for ϕ, i.e., the unique measure which attains the supremum*

$$\sup\left\{h(\nu) + \int \phi d\nu \colon \nu \in \mathcal{M} \text{ is shift-invariant and } \int \phi d\nu > -\infty\right\}$$

($h(\nu)$ being the entropy of ν with respect to σ)), and it is the unique minimizer of the rate function I in Theorem A. The $\{\Delta_n\}$, $\{\Xi_n\}$, $\{\Upsilon_{y,n}\}$ converge in the weak-topology to the unit point mass at the minimizer.*

We apply Theorem B to the Gauss map $G\colon (0,1] \to [0,1)$ given by $G(x) = 1/x - \lfloor 1/x \rfloor$. For $x \in (0,1) \setminus \mathbb{Q}$, define $(a_i(x))_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ by $a_i(x) = \left\lfloor \frac{1}{G^{i-1}(x)} \right\rfloor$, and put

$$[a_1(x); a_2(x); \cdots; a_n(x)] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cdots + \cfrac{1}{a_n(x)}}}.$$

Then $x = \lim_{n\to\infty}[a_1(x); a_2(x); \cdots; a_n(x)]$. The map $\pi\colon x \in (0,1)\setminus\mathbb{Q} \to (a_i(x))_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ is a homeomorphism, and commutes with G and the left shift. Hence, the study of the behavior of $a_1(x), a_2(x), a_3(x), \ldots$ translates to that of the dynamics of G.

Define $\phi := -\log|DG| \circ \pi^{-1}$. Then $\beta_\infty = 1/2$ [3]. For each $\beta > 1/2$ the potential $\beta\phi$ satisfies the conditions in Theorem B. Denote by μ_β the G-invariant Borel probability measure which corresponds to the unique shift-invariant Bowen's Gibbs measure for the potential $\beta\phi$.

Corollary. (Equidistribution of weighted periodic points). *For every $\beta > 1/2$ the following convergence in the weak*-topology holds:*

$$\frac{1}{\sum_{x\in\mathrm{Per}_n(G)} |DG^n(x)|^{-\beta}} \sum_{x\in\mathrm{Per}_n(G)} |DG^n(x)|^{-\beta}\delta_x^n \longrightarrow \mu_\beta \quad (n\to\infty).$$

The convergence for $\beta = 1$ was first proved in [2] by directly showing the tightness of the sequence of measures. The μ_1 is the Gauss measure: $d\mu_1 = \frac{1}{\log 2}\frac{dx}{1+x}$.

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Resolution of sigma-fields for multiparticle finite-state evolution with infinite past

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Let us consider the stochastic recursive equation

$$X_k = N_k X_{k-1} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \tag{1}$$

with the *observation process* $X = \{X_k\}_{k\in\mathbb{Z}}$ taking values in a set V and with the *noise process* $N = \{N_k\}_{k\in\mathbb{Z}}$ doing in a composition semigroup Σ consisting of mappings from V to itself, where we write fv simply for the evaluation $f(v)$. For a given probability law on Σ, we call the pair $\{X, N\}$ a *μ-evolution* if the equation (1) holds and each N_k has law μ and is independent of $\mathcal{F}^{X,N}_{k-1} := \sigma(X_j, N_j : j \le k-1)$. Our problem here is to resolve the observation $\mathcal{F}^X_k = \sigma(X_j : j \le k)$ into three independent components as

$$\mathcal{F}^X_k = \mathcal{F}^Y_k \vee \mathcal{F}^X_{-\infty} \vee \sigma(U_k) \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z}, \tag{2}$$

where, for each k, the first component $\mathcal{F}^Y_k$ is a sub-σ-field of the noise $\mathcal{F}^N_k$, the second $\mathcal{F}^X_{-\infty} := \bigcap_{k\in\mathbb{Z}} \mathcal{F}^X_k$ is the remote past, and the third U_k is a random variable which is independent of $\mathcal{F}^Y_k \vee \mathcal{F}^X_{-\infty}$. For σ-fields $\mathcal{F}_1, \mathcal{F}_2, \dots$ we write $\mathcal{F}_1 \vee \mathcal{F}_2 \vee \cdots$ for $\sigma(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots)$.

If we assume that the product $N_j N_{j+1} \cdots N_k$ converges $\mathbb{P}$-a.s. as $j \to -\infty$ to some random mapping $\widetilde{N}_k$ and that X_j does to some random variable $X_{-\infty}$, then we obtain

$$\mathcal{F}^X_k \subset \mathcal{F}^{\widetilde{N}}_k \vee \mathcal{F}^X_{-\infty} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \tag{3}$$

with $\mathcal{F}^X_{-\infty} = \sigma(X_{-\infty})$, $\mathbb{P}$-a.s. We notice that, in typical cases, these a.s. convergences fail but the resolution (2) holds with the third random variable U_k being uniform in some sense.

Motivated by Tsirelson's example [2] of a stochastic differential equation without strong solutions, Yor [7] studied this problem in the case $V = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the one-dimensional torus, and $\Sigma = \mathbb{T}$ by identifying $z \in \mathbb{T}$ with the multiplication mapping $w \mapsto zw$. By means of the Fourier series and the martingale convergence theorems, he obtained a complete answer to the resolution problem. Akahori–Uenishi–Yano [1] and Hirayama–Yano [3] generalized Yor's results to compact groups; see also Yano–Yor [6] for a survey on this topic.

We now consider the resolution problem when the state space is a finite set $V = \{1, 2, \dots, \#V\}$ and $\Sigma = \mathrm{Map}(V)$ is the finite composition semigroup of all mappings from V to itself. In Yano [5] we gave a partial answer in the sense that the inclusion

$$\mathcal{F}^X_k \subset \mathcal{F}^N_k \vee \mathcal{F}^X_{-\infty} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \tag{4}$$

holds if and only if $\mathrm{Supp}(\mu)$ is *sync*, i.e., the image $g(V)$ is a singleton for some $g \in \langle \mathrm{Supp}(\mu) \rangle$, where $\langle \mathrm{Supp}(\mu) \rangle$ denotes the subsemigroup of Σ consisting of all finite compositions from $\mathrm{Supp}(\mu)$. Unfortunately, we have not so far obtained a general result nor a counterexample for the resolution of the form (2).

We thus focus on the resolution problem for multiparticle evolutions. For a probability law μ and for $m \in \mathbb{N}$, we mean by an *m-particle μ-evolution* the pair $\{\mathbb{X}, N\}$ of a V^m-valued process $\mathbb{X} = \{\mathbb{X}_k\}_{k\in\mathbb{Z}}$ with $\mathbb{X}_k = (X_k^1, \dots, X_k^m)$ and a Σ-valued process $N = \{N_k\}_{k\in\mathbb{Z}}$ such that the stochastic recursive equation

$$X_k^i = N_k X_{k-1}^i \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \text{ and } i = 1, \dots, m \tag{5}$$

holds and each N_k has law μ and is independent of $\mathcal{F}_{k-1}^{\mathbb{X},N}$. Choosing

$$m = \inf\{\#g(V) : g \in \langle \mathrm{Supp}(\mu) \rangle\}, \tag{6}$$

we shall give a complete answer to the resolution problem of the form

$$\mathcal{F}_k^{\mathbb{X}} = \mathcal{F}_k^Y \vee \mathcal{F}_{-\infty}^{\mathbb{X}} \vee \sigma(U_k) \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z}. \tag{7}$$

For this purpose, we utilize the *Rees decomposition* from the algebraic semigroup theory, which has played a fundamental role in the theory of infinite products of random variables taking values in topological semigroups; see, e.g., [4] for the details.

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Fréchet 空間上の quasi-regular non-local Dirichlet forms の 定式化と、その Φ_3^4 場の確率量子化への応用

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1 概要

Denote by S the Banach spaces of weighted real l^p, $1 \le p \le \infty$, spaces and the space of direct product $\mathbb{R}^{\mathbb{N}}$ (with $\mathbb{R}$ and resp. $\mathbb{N}$ the spaces of real numbers and resp. natural numbers), which are understood as Fréchet spaces. Let μ be a Borel probability measure on S. On the *real* $L^2(S;\mu)$ space, for each $0 < \alpha < 2$, we give an explicit formulation of α-stable type (cf., e.g., section 5 of [Fukushima,Uemura 2012] for corresponding formula on $L^2(\mathbb{R}^d)$, $d < \infty$) non-local quasi-regular (cf. section IV-3 of [M,R 92]) Dirichlet form $(\mathcal{E}_\alpha, \mathcal{D}(\mathcal{E}_\alpha))$ (with a domain $\mathcal{D}(\mathcal{E}_\alpha)$), and show an existence of S-valued Hunt processes properly associated to $(\mathcal{E}_\alpha, \mathcal{D}(\mathcal{E}_\alpha))$.

As an application of the above general results, we consider the problem of stochastic quantization of Euclidean free field, Φ_2^4 and Φ_3^4 fields, i.e., field with (self) interaction of 4-th power. By using the property that, for example, the support of the Euclidean Φ_3^4 field measure μ is in some *real* Hilbert space $\mathcal{H}_{-3}$, which is a sub space of the Schwartz space of *real* tempered distributions $\mathcal{S}'(\mathbb{R}^3 \to \mathbb{R})$, we define an isometric isomorphism τ_{-3} : $\mathcal{H}_{-3} \to$ "some weighted l^2 space". By making use of τ_{-3}, we then interpret the above general theorems formulated on the abstract $L^2(S;\mu)$ space to the Euclidean Φ_3^4 field, $L^2(\mathcal{H}_{-3};\mu)$, and for each $0 < \alpha \le 1$ we show the existence of an $\mathcal{H}_{-3}$-valued Hunt process $(Y_t)_{t\ge 0}$ the invariant measure of which is μ.

$(Y_t)_{t\ge 0}$ is understood as a stochastic quantization of Euclidean Φ_3^4 field realized by a Hunt process through the *non-local* Dirichlet form $(\mathcal{E}_\alpha, \mathcal{D}(\mathcal{E}_\alpha))$ for $0 < \alpha \le 1$.

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1) As far as we know, there has been no explicit proposal of general formulation of *non-local* quasi-regular Dirichlet form on infinite dimensional topological vector spaces (for the local case, i.e., the case where the associated Markov processes are (continuous) diffusions, much have been developed and known), which admits interpretations to Dirichlet forms on several concrete random fields on several Fréchet spaces.

2) Though there have been derived several results on the existence of (continuous) diffusions (i.e., roughly speaking, which associated to quadratic forms and generators of local type) corresponding with stochastic quantizations of Φ_2^4 or Φ_3^4 Euclidean fields (cf., the quotation given below), as far as we know, there exists no explicit corresponding consideration for *non-local* type Markov processes, which is performed through the Dirichlet form argument.

Hence, the present result is a first development that gives answers to the above mentioned open problems 1) and 2).

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