The distributions of sliding block patterns in finite samples and the inclusion-exclusion principles for partially ordered sets

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Let $X \in A^n$ with finite alphabet A and $w \in A^*$. Let $|w|$ be the length of the word w. We consider the following random variable,

$$N_w := \sum_{i=1}^{n} I_{X_i^{i+|w|-1}=w} \text{ where } I_{X_i^{i+|w|-1}=w} = 1 \text{ if } X_i^{i+|w|-1} = w \text{ else } 0.$$

We also call this statistics sliding block patterns. In particular if we count the occurrence of multiple words, it is called suffix tree.

The distributions of sliding block patterns have been shown via generating functions based on induction of sample size, see [1, 2, 3, 5, 4].

In this paper we show the distributions of sliding block patterns for Bernoulli processes with finite alphabet, which is not based on the induction on sample size. We show a new inclusion-exclusion formula in multivariate generating function form on partially ordered sets, and show a simpler expression of generating functions of the number of pattern occurrences in finite samples.

We say that a word w is overlapping if there is a word x with $|w| < |x| < 2|w|$ and w appears in x at least 2 times, and w is called non-overlapping if there is no such x. We write $x \sqsubset y$ if x is a prefix of y.

Theorem 1 *Let P be an i.i.d. process of fixed sample size n of finite alphabet. Let $s_1 \sqsubset s_2 \sqsubset \cdots \sqsubset s_l$ be an increasing non–overlapping words of finite alphabet, i.e., s_i is a prefix of s_j and $m_i < m_j$, where m_i is the length of s_i, for all $i < j$. Let $P(s_i)$ be the probability of s_i for $i = 1, \ldots, l$. Let*

$$\begin{aligned}
A(k_1, \ldots, k_l) &= \binom{n - \sum_i m_i k_i + \sum_i k_i}{k_1, \ldots, k_l} \prod_{i=1}^{l} P^{k_i}(s_i), \\
B(k_1, \ldots, k_l) &= P(\sum_{i=1}^{n} I_{X_i^{i+m_i-1}=s_j} = k_j,\ j = 1, \ldots, l), \qquad (1)\\
F_A(z_1, \ldots, z_l) &= \sum_{k_1,\ldots,k_l} A(k_1, \ldots, k_l) z^{k_1} \cdots z^{k_l}, \text{ and} \\
F_B(z_1, \ldots, z_l) &= \sum_{k_1,\ldots,k_l} B(k_1, \ldots, k_l) z^{k_1} \cdots z^{k_l}.
\end{aligned}$$

Then

$$F_A(z_1, z_2, \ldots, z_l) = F_B(z_1 + 1, z_1 + z_2 + 1, \ldots, z_1 + \cdots + z_l + 1).$$

With slight modification of Theorem 1, we can compute the number of the occurrence of the overlapping increasing words. For example, let us consider increasing self-overlapping words 11, 111, 1111 and the number of their occurrences. Let 011, 0111, 01111 then these words are increasing non-self-overlapping words. The number of occurrences 11, 111, 1111 in sample of length n is equivalent to the number of occurrences 011, 0111, 01111 in sample of length $n+1$ that starts with 0. We can apply Theorem 1 to derive the distribution of increasing overlapping words with this manner.

In [5], expectation, variance, and CLTs for the sliding block pattern are shown. We show the general higher moments for non-overlapping words.

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Theorem 2 *Let w be a non-overlapping pattern.*

$$\forall t \ E(N_w^t) = \sum_{s=1}^{\min\{T,t\}} A_{t,s} \binom{n - s|w| + s}{s} P^s(w).$$

$$A_{t,s} = \sum_r \binom{s}{r} r^t (-1)^{s-r}, \ T = \max\{t \in \mathbb{N} \mid n - t|w| \geq 0\}.$$

In the above theorem, $A_{t,s}$ is the number of surjective functions from $\{1,2,\ldots,t\} \to \{1,2,\ldots,s\}$ for $t, s \in \mathbb{N}$, see [6].

In [5], it is shown that central limit theorem holds for sliding block patterns,

$$P(\frac{N_w - E(N_w)}{\sqrt{V_w}} < x) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx,$$

where w is non-overlapping pattern, $E(N_w) = (n - |w| + 1)P(w)$ and $V(N_w) = E(N_w) + (n - 2|w| + 2)(n - 2|w| + 1)p^2(w) - E^2(N_w)$.

Let

$$N'_w := \sum_{i=1}^{\lfloor n/|w| \rfloor} I_{X_{i*|w|}^{(i+1)*|w|-1} = w}.$$

N'_w obeys binomial law if the process is i.i.d. We call N'_w *block-wise sampling*. As an application of CLT approximation, we compare power functions of sliding block sampling N_w and block-wise sampling N'_w. We consider the following test for sliding block patterns: We write $E_\theta = E(N_w)$ and $V_\theta = V(N_w)$ if $P(w) = \theta$. Null hypothesis: $P(w) = \theta^*$ vs alternative hypothesis $P(w) < \theta^*$. Reject null hypothesis if and only if $N_w < E_{\theta^*} - 5\sqrt{V_{\theta^*}}$. The likelihood of the critical region is called power function, i.e., $Pow(\theta) := P_\theta(N_w < E_{\theta^*} - 5\sqrt{V_{\theta^*}})$ for $\theta \leq \theta^*$.

We construct a test for block-wise sampling: Null hypothesis: $P(w) = \theta^*$ vs alternative hypothesis $P(w) < \theta^*$. Reject null hypothesis if and only if $N'_w < E'_{\theta^*} - 5\sqrt{V'_{\theta^*}}$, where $E'_\theta = \lfloor n/|w| \rfloor \theta$ and $V'_\theta = \lfloor n/|w| \rfloor \theta(1-\theta)$. The following table shows powers of tests for sliding block patterns and block wise sampling at $\theta = 0.2, 0.18, 0.16$ under the condition that $\theta^* = 0.25, |w| = 2,$ and $n = 500$.

θ	0.2	0.18	0.16
Power of Sliding block	0.316007	0.860057	0.995681
Power of Block wise	0.000295	0.002939	0.021481

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References

[1] F. Bassino, J. Clément, and P. Micodème. Counting occurrences for a finite set of words: combinatorial methods. *ACM Trans. Algor.*, 9(4):Article No. 31, 2010.

[2] I. Goulden and D. Jackson. *Combinatorial Enumeration.* John Wiley, 1983.

[3] L. Guibas and A. Odlyzko. String overlaps, pattern matching, and nontransitive games. *J. Combin. Theory Ser. A*, 30:183–208, 1981.

[4] P. Jacquet and W. Szpankowski. *Analytic Pattern Matching.* Cambridge University Press, 2015.

[5] M. Régnier and W. Szpankowski. On pattern frequency occurrences in a markovian sequence. *Algorithmica*, 22(4):631–649, 1998.

[6] J. Riordan. *Introduction to combinatorial analysis.* John Wiley, 1958.

Central limit theorem for random walks on nilpotent covering graphs with weak asymmetry

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(Jointwork with Satoshi ISHIWATA (Yamagata) and Hiroshi KAWABI (Keio))

Long time behaviors of random walks (RWs) on an infinite graph is a well-studied topic in geometry, harmonic analysis and graph theory, to say nothing of probability theory. It is known that geometric features such as the *periodicity* and the *volume growth* of the underlying graph affect long time behaviors of RWs. By putting an emphasis on them, Ishiwata, Kawabi and Kotani [1] considered a non-symmetric random walk $\{w_n\}_{n=0}^{\infty}$ on a Γ-crystal lattice X, a covering graph of a finite graph whose covering transformation group Γ is abelian. Through a discrete analogue of the harmonic map from X into a Euclidean space $\Gamma \otimes \mathbb{R}$, they established two kinds of functional central limit theorems (CLTs) for $\{w_n\}_{n=0}^{\infty}$. In fact, since a diverging drift term arising from the non-symmetry prevents us from taking the CLT-scaling limit directly, it is difficult to prove such CLTs. To overcome the difficulty, two schemes were introduced in [1]. One is to replace the usual transition operator by the transition-shift operator to "delete" the diverging drift term. The other is to introduce a family of non-symmetric RWs on X to "weaken" the diverging drift term. (The latter scheme is also applied in the study of the hydrodynamic limit of *weakly asymmetric* simple exclusion processes.)

Let Γ be a finitely generated nilpotent group. In [2], we considered a non-symmetric RW $\{w_n\}_{n=0}^{\infty}$ on a Γ-*nilpotent covering graph*, a generalization of both crystal lattices and Cayley graphs of a finitely generated group of polynomial volume growth. By extending the former scheme to the nilpotent case, we established a functional CLT for $\{w_n\}_{n=0}^{\infty}$ in [2]. The main purpose of this talk is to extend the latter scheme to the nilpotent case and to establish another functional CLT for $\{w_n\}_{n=0}^{\infty}$. This talk is based on our recent preprint [3].

Let $X = (V, E)$ be a Γ-nilpotent covering graph. Here V is the set of all vertices and E the set of all oriented edges in X. For $e \in E$, we denote the origin, terminus and inverse edge of e by $o(e), t(e)$ and $\overline{e}$, respectively. We set $E_x := \{e \in E \,|\, o(e) = x\}$ for $x \in V$. Let $p : E \longrightarrow (0, 1]$ be a Γ-invariant transition probability and $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})$ a RW on X starting from $x \in V$ associated with p. Through the covering map $\pi : X \longrightarrow X_0$, we also consider the RW $(\Omega_{\pi(x)}(X_0), \mathbb{P}_{\pi(x)}, \{\pi(w_n)\}_{n=0}^{\infty})$ and the corresponding transition probability is also denoted by $p : E_0 \longrightarrow (0, 1]$. We denote by $m : V_0 \longrightarrow (0, 1]$ the normalized invariant measure on X_0 and also write $m : V \longrightarrow (0, 1]$ for the Γ-invariant lift of m to X. Let $\mathrm{H}_1(X_0, \mathbb{R})$ be the first homology group of X_0. We define the *homological direction* of the RW on X_0 by $\gamma_p := \sum_{e \in E_0} p(e)m\big(o(e)\big)e \in \mathrm{H}_1(X_0, \mathbb{R})$. We call the RW on X_0 (m-)*symmetric* if $p(e)m\big(o(e)\big) = p(\overline{e})m\big(t(e)\big)$ for $e \in E_0$. Otherwise, it is called (m-)*non-symmetric*. Note that the RW on X_0 is (m-)symmetric if and only if $\gamma_p = 0$.

Thanks to the celebrated theorem of Malćev, we find a connected and simply connected nilpotent Lie group G such that Γ is isomorphic to a cocompact lattice in G. The nilpotent Lie group G is equipped with the canonical dilations $(\tau_\varepsilon)_{\varepsilon \geq 0}$, which gives a scalar multiplication on

G. By realizing X into G, CLTs for RWs on X can be discussed. Let $\mathfrak{g}$ be the corresponding Lie algebra of G and $\mathfrak{g}^{(1)} \cong G/[G,G]$ the generating part of $\mathfrak{g}$. We take a canonical surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \mathfrak{g}^{(1)}$ by using the general theory of covering spaces. Thanks to the map $\rho_{\mathbb{R}}$ and the discrete Hodge–Kodaira theorem, a flat metric g_0 associated with the transition probability p, called the *Albanese metric*, is induced on $\mathfrak{g}^{(1)}$. A periodic realization $\Phi_0 : X \longrightarrow G$ is said to be *modified harmonic* if

$$\sum_{e \in E_x} p(e) \log\Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big)\Big)\Big|_{\mathfrak{g}^{(1)}} = \rho_{\mathbb{R}}(\gamma_p) \qquad (x \in V).$$

The quantity $\rho_{\mathbb{R}}(\gamma_p) \in \mathfrak{g}^{(1)}$ is called the *asymptotic direction*, which also appears in the law of large numbers for $\mathfrak{g}^{(1)}$-valued RW $\{\log\big(\Phi_0(w_n)\big)\big|_{\mathfrak{g}^{(1)}}\}_{n=0}^{\infty}$. It should be noted that $\gamma_p = 0$ implies $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$, however, the converse does not hold in general.

For the given transition probability p, we introduce a family of Γ-invariant transition probabilities $(p_\varepsilon)_{0\le\varepsilon\le1}$ on X by $p_\varepsilon(e) := p_0(e) + \varepsilon q(e)$ for $e \in E$, where

$$p_0(e) := \frac{1}{2}\Big(p(e) + \frac{m\big(t(e)\big)}{m\big(o(e)\big)}p(\overline{e})\Big), \quad q(e) := \frac{1}{2}\Big(p(e) - \frac{m\big(t(e)\big)}{m\big(o(e)\big)}p(\overline{e})\Big).$$

Namely, the family $(p_\varepsilon)_{0\le\varepsilon\le1}$ is given by the linear interpolation between the given transition probability $p = p_1$ and the (m-)symmetric probability p_0. Moreover, the homological direction γ_{p_ε} equals $\varepsilon\gamma_p$ for every $0 \le \varepsilon \le 1$, which plays a key role in the proof of main theorems.

We now fix a reference point $x_* \in V$ such that $\Phi_0^{(0)}(x_*) = \mathbf{1}_G$, where $\mathbf{1}_G$ is the unit element of G. We write $g_0^{(\varepsilon)}$ for the Albanese metric on $\mathfrak{g}^{(1)}$ associated with p_ε and $\Phi_0^{(\varepsilon)} : X \longrightarrow G$ be the (p_ε-)modified harmonic realization for every $0 \le \varepsilon \le 1$. We set $\mathcal{Y}_{k/n}^{(\varepsilon,n)}(c) := \tau_{n^{-1/2}}\big(\Phi_0^{(\varepsilon)}(w_k(c))\big)$ for $n \in \mathbb{N}$, $k = 0, 1, \ldots, n$, $c \in \Omega_{x_*}(X)$ and $0 \le \varepsilon \le 1$. We then define a G-valued continuous stochastic process $\mathcal{Y}^{(\varepsilon,n)} = (\mathcal{Y}_t^{(\varepsilon,n)})_{0\le t\le1}$ by the geodesic interpolation of $\{\mathcal{Y}_{k/n}^{(\varepsilon,n)}\}_{k=0}^{n}$ with respect to the Carnot–Carathéodory metric on G. We take an orthonormal basis $\{V_1, V_2, \ldots, V_{d_1}\}$ of $(\mathfrak{g}^{(1)}, g_0^{(0)})$ and consider a stochastic differential equation (SDE)

$$dY_t = \sum_{i=1}^{d_1} V_i(Y_t) \circ dB_t^i + \rho_{\mathbb{R}}(\gamma_p)(Y_t)\, dt, \qquad Y_0 = \mathbf{1}_G,$$

where $(B_t)_{0\le t\le1} = (B_t^1, B_t^2, \ldots, B_t^{d_1})_{0\le t\le1}$ is an $\mathbb{R}^{d_1}$-standard Brownian motion starting from $B_0 = \mathbf{0}$. Let $(Y_t)_{0\le t\le1}$ be the G-valued diffusion process which solves the SDE above.

We now state our main result as follows:

Theorem. *Under several natural assumptions on* $\{\Phi_0^{(\varepsilon)}\}_{0\le\varepsilon\le1}$*, the sequence* $\{\mathcal{Y}^{(n^{-1/2},n)}\}_{n=1}^{\infty}$ *converges in law to a* G*-valued diffusion process* Y *in* $C^{0,\alpha\text{-Höl}}([0,1];G)$ *as* $n \to \infty$ *for all* $\alpha < 1/2$.

References

[1] S. Ishiwata, H. Kawabi and M. Kotani: J. Funct. Anal. **272** (2017), pp.1553–1624.
[2] S. Ishiwata, H. Kawabi and R. Namba: preprint (2018), `arXiv:1806.03804`.
[3] S. Ishiwata, H. Kawabi and R. Namba: preprint (2018), `arXiv:1808.08856`.

A limit theorem for inverse local times of jumping-in diffusion processes

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(joint work with Kouji Yano(Kyoto University)

Let X be a strong Markov process X on the half line $[0,\infty)$ which is a natunal scale diffusion up to the first hitting time of 0 and, as soon as X hits 0, X jumps into the interior $(0,\infty)$ and starts afresh. This kind of processes are studied by Feller[1] and Itô[2] and shown that such processes are characterized by the speed measure dm which characterizes the diffusion on the interior $(0,\infty)$ and the jumping-in measure j which characterizes the law of jumps from the boundary 0 to the interior $(0,\infty)$. We denote this process by $X_{m,j}$ and call it a *jumping-in diffusion.*

Let us consider the inverse local time $\eta_{m,j}$ at 0 of a jumping-in diffusion $X_{m,j}$. We study the fluctuation scaling limit of the inverse local time $\eta_{m,j}$ of the form:

$$\frac{1}{\lambda^{1/\alpha}}(\eta_{m,j}(\lambda t)-b\lambda t)\xrightarrow[\lambda\to\infty]{d}T(t)\text{ in }\mathbb{D}\tag{1}$$

for some constants $b\geq 0$ and $\alpha\in(0,2]$. Here $\mathbb{D}$ denotes the space of càdlàg paths from $[0,\infty)$ to $\mathbb{R}$ equipped with Skorokhod's J_1-topology. In order to obtain the limit, we establish the continuity theorem for jumping-in diffusion processes which roughly asserts the following: for jumping-in diffusions $\{X_{m_n,j_n}\}_n$, if their speed measures $\{dm_n\}_n$ converge to a speed measure dm in a certain sense and the measures $\{j_n(dx)\}_n$ degenerate to the point mass at the origin in a certain sense, then for the appropriate constants $\{b_n\}_n$, it holds that

$$\eta_{m_n,j_n}(t)-b_n t\xrightarrow[n\to\infty]{d}\sigma B(t)+T(m;\kappa t)\text{ in }\mathbb{D}\tag{2}$$

for some constants σ and κ. Here B denotes a standard Brownian motion and $T(m;t)$ the spectrally positive Lévy process without Gaussian part associated to m which is independent of B. In order to prove the continuity theorem, we introduce a class of λ-eigenfunctions of the generalized second order differential operator $\frac{d}{dm}\frac{d}{dx}$ and apply Krein-Kotani correspondence and its continuity established in Kotani[4].

As an application of the above result, we study the occupation time of two-sided jumping-in diffusions which are constructed by connecting two jumping-in diffusion processes with respect to 0. Let X be such a process and define $A(t) = \int_0^t 1_{(0,\infty)}(X_s)ds$. We give conditions for the existence of the limit distribution $\frac{1}{t}A(t)$ as $t \to \infty$. Moreover, in the case where the limit degenerate, that is,

$$\frac{1}{t}A(t) \xrightarrow[t\to\infty]{P} p \in [0,1] \tag{3}$$

holds, we show the scaling limit of the fluctuation around the limit constant along the exponential clock, that is, the following limit:

$$\frac{1}{\mathbf{e}_{f(q)}}(A(\mathbf{e}_q t) - p\mathbf{e}_q t) \xrightarrow[q\to+0]{d} Z(t) \tag{4}$$

for a process $Z(t)$ and a positive function $f(q)$ which converges to 0 as $q \to +0$. Here $\mathbf{e}_q$ denotes an exponentially distributed random variable with parameter $q > 0$ and is independent of X. This result is a jumping-in version of the result proved for diffusions in Kasahara and Watanabe[3].

References

[1] W. Feller. The parabolic differential equations and the associated semi-groups of transformations. Ann. of Math. (2), 55:468-519, 1952.

[2] K. Itô. Poisson point processes and their application to Markov processes. Springer-Briefs in Probability and Mathematical Statistics. Springer, Singapore, 2015. With a foreword by Shinzo Watanabe and Ichiro Shigekawa.

[3] Y. Kasahara and S. Watanabe. Brownian representation of a class of Lévy processes and its application to occupation times of diffusion processes. Illinois J. Math., 50(1-4):515-539, 2006.

[4] S.Kotani. Krein's strings with singular left boundary. Rep. Math. Phys., 59(3):305-316, 2007.

On optimal periodic dividend strategies for Lévy risk processes

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1 Introduction

This talk is based on [4] and [5]. In this talk, we revisit the optimal periodic dividend problem, in which dividend payments can only be made at the jump times of an independent Poisson process. In the dual (spectrally positive Lévy) model, recent results have shown the optimality of a periodic barrier strategy, which pays dividends at Poissonian dividend-decision times, if and only if the surplus is above some level. In this talk, we show the optimality of this strategy for a spectrally negative Lévy process whose dual has a completely monotone Lévy density. We also consider the version with bail-outs where the surplus must be non-negative uniformly in time. There are many previous studies of spectrally negative cases. Loeffen([3]) and Kyprianou et al.([2]) showed the optimality of a barrier strategy in the classical case and that of a threshold strategy under the absolutely continuous assumption on the dividend strategy, respectively. Avram et al.([1]) and Pérez et al.([7]) showed the optimality of those of the version with bail-outs. In this draft, we give the main results for the models without bail-outs.

2 Preliminary facts and main results

Let X be a spectrally negative Lévy process. Suppose that the Lévy measure of $-X$ has a completely monotone with respect to the Lebesgue measure. Let N^r be the Poisson process with rate $r > 0$ which are independent from X. Let $\mathbb{F}$ be the filtration generated from X and N^r. In this setting, a strategy $\pi = \{L^\pi_t : t \geq 0\}$ is a non-decreasing, right-continuous, and $\mathbb{F}$-adapted process such that the cumulative amount of dividends L^π admits the form

$$L^\pi(t) = \int_{[0,t]} \nu^\pi(s) dN^r(s), \tag{2.1}$$

for some $\mathbb{F}$-adapted càglàd process ν^π. The surplus process U^π after dividends are deducted is such that

$$U^\pi(t) = X(t) - L^\pi(t) \tag{2.2}$$

where $\sigma^\pi_0 = \inf\{t > 0 : U^\pi(t) < 0\}$ is the corresponding ruin time. We assume that the payment cannot exceed the available surplus and hence

$$0 \leq \nu^\pi(s) \leq U^\pi(s-), \qquad s \geq 0. \tag{2.3}$$

We fix $q > 0$ which is the discount rate. We define the expected net present value of dividends paid until ruin as the following:

$$v_\pi(x) = \mathbb{E}_x\left[\int_{[0,\sigma^\pi_0]} e^{-qt} dL^\pi(t)\right]. \tag{2.4}$$

Let $\mathcal{A}_r$ be the set of all admissible strategies. The problem is to compute the value function

$$v_{\pi^*}(x) = v(x) := \sup_{\pi \in \mathcal{A}_r} v_\pi(x). \tag{2.5}$$

For $b \geq 0$, the periodic barrier strategy π^b is the strategy which satisfies the

$$\nu^{\pi^b}(t) = (U_r^{\pi^b}(t-) - b) \vee 0, \qquad t > 0. \tag{2.6}$$

The strategy π^b was constructed by [6]. The expected NPV v^{π^b} was computed by [6, Corollary 4.4] using the scale functions.

Let Φ be the inverse Laplace exponent of X and $W^{(q)}$ be the q-scale function of X. We denote

$$Z^{(q)}(x, \Phi(q+r)) = r \int_0^\infty e^{-\Phi(q+r)z} W^{(q)}(z+x) dz, \qquad x \in \mathbb{R}, \tag{2.7}$$

$$h(b) = e^{-\Phi(q+r)b} \left(r W^{(q)\prime}(b) - \Phi(q+r) Z^{(q)\prime}(b, \Phi(q+r)) \right), \quad b > 0. \tag{2.8}$$

We define

$$b^* = \inf\{b > 0 : h(b) \leq 0\}. \tag{2.9}$$

Then we have the following theorem:

Theorem 2.1. *For $x > 0$, we have $v_{\pi^{b^*}}(x) = v(x)$.*

References

[1] F. Avram, Z. Palmowski, and M. R. Pistorius. On the optimal dividend problem for a spectrally negative Lévy process. *Ann. Appl. Probab.*, Vol. 17, No. 1, pp. 156–180, 2007.

[2] A. E. Kyprianou, R. Loeffen, and J. L. Pérez. Optimal control with absolutely continuous strategies for spectrally negative Lévy processes. *J. Appl. Probab.*, Vol. 49, No. 1, pp. 150–166, 2012.

[3] R. L. Loeffen. On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes. *Ann. Appl. Probab.*, Vol. 18, No. 5, pp. 1669–1680, 2008.

[4] K. Noba, J. L. Pérez, K. Yamazaki, and K. Yano. On optimal periodic dividend strategies for Lévy risk processes. *Insurance Math. Econom.*, Vol. 80, pp. 29–44, 2018.

[5] K. Noba, J. L. Pérez, K. Yamazaki, and K. Yano. On optimal periodic dividend and capital injection strategies for spectrally negative Lévy models. *J. Appl. Probab., arXiv:1801.00088*, 2018, to appear.

[6] J. L. Pérez and K. Yamazaki. Mixed periodic-classical barrier strategies for Lévy risk processes. *Risks*, Vol. 6, No. 2, p. 33, 2018.

[7] J. L. Pérez, K. Yamazaki, and X Yu. On the bail-out optimal dividend problem. *J. Optim. Theory Appl., to appear.*

Sierpinski gasket 格子上の長距離浸透モデルにおけるランダムウォークの混合時間と等周定数

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基本的なフラクタル格子の 1 つである Sierpinski gasket 格子上での長距離浸透モデルに対して、ランダムグラフ上のランダムウォークの混合時間の評価 ([3]) と、ランダムグラフの等周定数について考える。

平面上の点 $\mathrm{O}=(0,0)$, $u_0=(\frac{1}{2},\frac{\sqrt{3}}{2})$, $v_0=(1,0)$ に対して、三角形 $\mathrm{O}u_0v_0$ の 3 個の頂点と 3 本の辺からなるグラフを G_0 とする。さらに、$u_n=2^nu_0$, $v_n=2^nv_0$ とし、有限グラフの列 $\{G_n\}_{n=0}^{\infty}$ を

$$G_{n+1}=G_n\cup(G_n+u_n)\cup(G_n+v_n)\quad(n=0,1,2,\cdots)$$

によって定義する。$G=\cup_{n=0}^{\infty}G_n$ を Sierpinski gasket 格子 (pre-Sierpinski gasket) と呼ぶ。以下では G_n の頂点集合を V_n で表す。

n を固定し、有限グラフ G_n 上で長距離浸透モデルの問題を考える。すなわち、$p(1)=1$,

$$p(k)=1-\exp(-\beta k^{-s})\quad(k=2,3,4,\cdots)$$

（β,s は正の実数）として、各 $x,y\in V_n$ $(x\neq y)$ に対して独立に、確率 $p(|x-y|)$ で、2 点 x,y がランダムな辺で結ばれるとする。（$|x-y|$ は、G_n 上における 2 点間の最短ステップ数。）V_n に属する頂点と、上記の規則によって作られるランダムな辺からなるランダムグラフを G'_n とおく。なお、ここでは向き付けられた辺集合を考え、x,y が辺で結ばれているときには辺 (x,y) と 辺 (y,x) が存在しているとみなす。また、G_n 上の長距離浸透モデルに対する確率測度を $\mathbb{P}$ で表す。

ランダムグラフ G'_n の形状を固定するごとに、その上で、推移確率 $P(x,y)=P(X_{t+1}=y|X_t=x)$ が

$$P(x,y)=\begin{cases}\frac{1}{2\deg(x)} & (x\neq y \text{ かつ } x,y \text{ が } G'_n \text{上の隣接点のとき}) \\ \frac{1}{2} & (x=y \text{ のとき}) \\ 0 & (\text{その他})\end{cases}\qquad(x,y\in V_n)$$

（$\deg(x)$ は x から出ている辺の本数）で与えられる離散時間 lazy random walk $\{X_t\}_{t=0}^{\infty}$ を考え、$\{X_t\}_{t=0}^{\infty}$ の混合時間を $\tau(G'_n)$ とおく。なお、以下では $d=\log 3/\log 2$ とする。

定理 1 ([3]) $d < s < 2d$ のとき、正の定数 c_1, c_2 が存在して次が成り立つ。

$$\lim_{n\to\infty} \mathbb{P}\left(c_1 2^{(s-d)n} \le \tau(G'_n) \le n^{c_2} 2^{(s-d)n}\right) = 1.$$

$\mathbb{Z}/n\mathbb{Z}$ 上の長距離浸透モデルに対しては、$\tau(G'_n)$ に相当する量が $1 < s < 2$ のとき n^{s-1} のオーダー、$s > 2$ のとき n^2 のオーダーとなり、$s = 2$ の前後で不連続に変化することが [1] で示されている。([2] で証明の一部が修正されている。) 一方、Sierpinski gasket 格子上の長距離浸透モデルの場合には、$s > 2d$ のときの $\tau(G'_n)$ の評価はまだ得られていない。

以下では、$\tau(G'_n)$ と深い関係を持つ量であるランダムグラフの等周定数について考える。$\pi = (\pi(x))_{x\in V_n}$ を G'_n 上の lazy random walk $\{X_t\}_{t=0}^{\infty}$ の定常分布とし、$Q(x,y) = \pi(x)P(x,y)$ $(x, y \in V_n)$ とおく。また、$A, B \subset V_n$ に対して $\pi(A) = \sum_{x\in A} \pi(x)$, $Q(A,B) = \sum_{x\in A}\sum_{y\in B} Q(x,y)$ と書く。

$$\phi^* = \min_{\substack{D\subset V_n \\ 0<\pi(D)\le\frac{1}{2}}} \frac{Q(D,D^c)}{\pi(D)}$$

を等周定数と呼ぶ。

命題 2 (1) $d < s < 2d$ のとき、正の定数 c_3, c_4 が存在して次が成り立つ。

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-c_3} 2^{(d-s)n} \le \phi^* \le c_4 2^{(d-s)n}\right) = 1.$$

(2) $s \ge 2d$ のとき、正の定数 c_5, c_6 が存在して次が成り立つ。

$$\lim_{n\to\infty} \mathbb{P}\left(c_5 n^{-1} 3^{-n} \le \phi^* \le c_6 n 3^{-n}\right) = 1.$$

参考文献

[1] Benjamini, I., Berger, N., Yadin, A. Long-range percolation mixing time. *Combin. Probab. Comput.* **17**, 487–494 (2008)

[2] Benjamini, I., Berger, N., Yadin, A. Long-range percolation mixing time. arXiv:math/0703872v2. (2009)

[3] Misumi, J. The mixing time of a random walk on a long-range percolation cluster in pre-Sierpinski gasket. *J. Stat. Phys.* **165**, 153-163 (2016)

Functional limit theorem for intermittent interval maps

Toru Sera (Kyoto University)

Interval maps with indifferent fixed points have been studied as models of intermittent phenomena, such as intermittent transitions to turbulent flow in convective fluid. In this context, the occupations near indifferent fixed points correspond to long regular or *laminar phases*, while the occupations away from them correspond to short irregular or *turbulent* bursts. There have been many studies of scaling limits of the occupations near and away from them, e.g., [1, 7, 8, 10, 4, 6]. In this talk, we present a functional and joint-distributional refinement of them, based on [5]. It is motivated particularly by [2, 3, 9].

We impose the following assumption from now on:

Assumption. An interval map $T : [0, 1] \to [0, 1]$ satisfies the following conditions:

(1) (for simplicity) T is point-symmetric, i.e., $Tx = 1 - T(1-x)$, $x \in (1/2, 1]$.

(2) the restriction $T|_{[0,1/2]} : [0, 1/2] \to [0, 1]$ is a C^2-bijective map.

(3) $T0 = 0$, $T'0 = 1$ and $T''x > 0$, $x \in (0, 1/2)$.

Note that 0 and 1 are indifferent fixed points of T. We know that T has a unique (up to scalar multiplication) σ-finite invariant measure $\mu(\mathrm{d}x)$ equivalent to the Lebesgue measure $\mathrm{d}x$. From now on, let us fix $\delta \in (0, 1/2)$. Then, it holds that $\mu([0, \delta)) = \mu((1-\delta, 1]) = \infty$ and $\mu([\delta, 1-\delta]) < \infty$. Hence Birkhoff's pointwise ergodic theorem implies

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{\{T^k x \in [\delta, 1-\delta]\}} \underset{n\to\infty}{\to} 0 \quad \left(\text{equivalently,} \quad \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{\{T^k x \notin [\delta, 1-\delta]\}} \underset{n\to\infty}{\to} 1\right), \quad \text{a.e.} x.$$

Roughly speaking, the orbit $(x, Tx, T^2x, \dots)$ of almost every starting point x is concentrated close to 0 and 1. We are interested in non-trivial scaling limits of occupation times for $[0, \delta), [\delta, 1-\delta]$ or $(1-\delta, 1]$. Let us denote by $\varphi(N) = \varphi(N, x)$ the Nth hitting time of $(x, Tx, T^2x, \dots)$ for $[\delta, 1-\delta]$:

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(N+1) = \min\{k > \varphi(N) : T^k x \in [\delta, 1-\delta]\}, \quad N \geq 0.$$

We will denote by $\overline{\mu} = \mu([\delta, 1-\delta] \cap \cdot)/\mu([\delta, 1-\delta])$ the normalized restriction of μ over $[\delta, 1-\delta]$. We now present our main result.

Theorem (S. [5]). *Let $\alpha \in (0, 1)$, and let ξ be a $[0, 1]$-valued random variable with $\mathbb{P}[\xi \in \mathrm{d}x] \ll \mathrm{d}x$. Then the following conditions are equivalent:*

(i) *$Tx - x = (1-x) - T(1-x)$ is regularly varying of index $(1+1/\alpha)$ at 0.*

(ii) *it holds that*

$$\left(\frac{1}{n}\sum_{k=0}^{\varphi([b_n t])} \mathbb{1}_{\{T^k\xi<\delta\}},\ \frac{1}{n}\sum_{k=0}^{\varphi([b_n t])} \mathbb{1}_{\{T^k\xi>1-\delta\}} : t\geq 0\right) \xrightarrow[n\to\infty]{\mathrm{d}} \left(S_-^{(\alpha)}(t),\ S_+^{(\alpha)}(t) : t \geq 0\right), \quad \text{in } D,$$

where $b_n := 1/\overline{\mu}[\varphi(1)>n]$, and $S_-^{(\alpha)}(t)$ and $S_+^{(\alpha)}(t)$ are i.i.d. α-stable subordinators with Lévy measure $\frac{\alpha}{2}r^{-1-\alpha}\mathrm{d}r$, $r>0$.

(iii) *it holds that*

$$\left(\frac{1}{n}\sum_{k=0}^{[nt]} \mathbb{1}_{\{T^k\xi<\delta\}},\ \frac{\Gamma(1-\alpha)}{b_n}\sum_{k=0}^{[nt]} \mathbb{1}_{\{T^k\xi\in[\delta,1-\delta]\}},\ \frac{1}{n}\sum_{k=0}^{[nt]} \mathbb{1}_{\{T^k\xi>1-\delta\}} : t\geq 0\right)$$
$$\xrightarrow[n\to\infty]{\mathrm{d}} \left(\int_0^t \mathbb{1}_{\{Z^{(\alpha)}(s)<0\}}\mathrm{d}s,\ L^{(\alpha)}(t),\ \int_0^t \mathbb{1}_{\{Z^{(\alpha)}(s)>0\}}\mathrm{d}s : t\geq 0\right), \quad \text{in } D,$$

where $Z^{(\alpha)}(t)$ denotes a $(2-2\alpha)$-dimensional symmetric Bessel diffusion process starting from the origin, and $L^{(\alpha)}(t)$ denotes the local time of $Z^{(\alpha)}(t)$ at the origin in the Blumenthal–Getoor normalization.

References

[1] J. Aaronson. The asymptotic distributional behaviour of transformations preserving infinite measures. *J. Analyse Math.*, 39:203–234, 1981.

[2] M. Barlow, J. Pitman and M. Yor. Une extension multidimensionnelle de la loi de l'arc sinus. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 294–314. Springer, Berlin, 1989.

[3] E. Fujihara, Y. Kawamura and Y. Yano. Functional limit theorems for occupation times of Lamperti's stochastic processes in discrete time. *J. Math. Kyoto Univ.*, 47(2):429–440, 2007.

[4] T. Owada and G. Samorodnitsky. Functional central limit theorem for heavy tailed stationary infinitely divisible processes generated by conservative flows. *Ann. Probab.*, 43(1):240–285, 2015.

[5] T. Sera. Functional limit theorem for occupation time processes of infinite ergodic transformations. preprint available at arXiv:1810.04571.

[6] T. Sera and K. Yano. Multiray generalization of the arcsine laws for occupation times of infinite ergodic transformations. *Trans. Amer. Math. Soc.*, to appear.

[7] M. Thaler. A limit theorem for sojourns near indifferent fixed points of one-dimensional maps. *Ergodic Theory Dynam. Systems*, 22(4):1289–1312, 2002.

[8] M. Thaler and R. Zweimüller. Distributional limit theorems in infinite ergodic theory. *Probab. Theory Related Fields*, 135(1):15–52, 2006.

[9] M. Tyran-Kamińska. Convergence to Lévy stable processes under some weak dependence conditions. *Stochastic Process. Appl.*, 120(9):1629–1650, 2010.

[10] R. Zweimüller. Infinite measure preserving transformations with compact first regeneration. *J. Anal. Math.*, 103:93–131, 2007.

The Laplacian on some round Sierpiński carpets and Weyl's asymptotics for its eigenvalues

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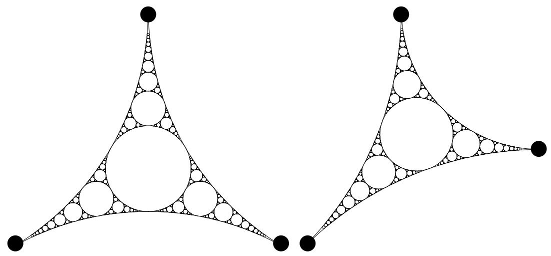

Fig. 1. Apollonian gaskets $K_{\alpha,\beta,\gamma}$

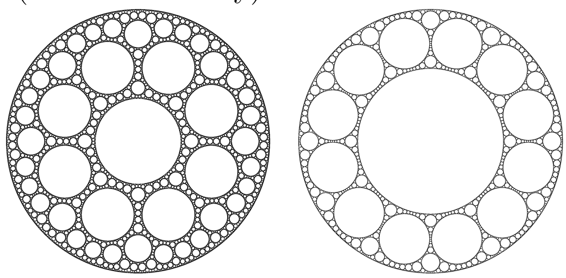

Fig. 2. Round Sierpiński carpets $\partial_\infty G_8$, $\partial_\infty G_{12}$

The purpose of this talk is to present the speaker's recent results on the construction of a "canonical" Laplacian on round Sierpiński carpets invariant with respect to certain Kleinian groups (i.e., discrete subgroups of the group $\mathrm{Möb}(\widehat{\mathbb{C}})$ of (orientation preserving or reversing) Möbius transformations on $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$) and on Weyl's asymptotics for its eigenvalues. Here a *round Sierpiński carpet* refers to a subset of $\widehat{\mathbb{C}}$ homeomorphic to the standard Sierpiński carpet whose complement consists of disjoint open disks in $\widehat{\mathbb{C}}$.

1. Preceding results for the Apollonian gasket

The construction of the Laplacian is based on the speaker's preceding study of the simplest case of the *Apollonian gasket* $K_{\alpha,\beta,\gamma}$. This is a compact fractal subset of $\mathbb{C}$ obtained from an ideal triangle, i.e., the closed subset of $\mathbb{C}$ enclosed by mutually *externally* tangent three circles, with the radii of the circles $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ and with the set $V_0^{\alpha,\beta,\gamma}$ of its three vertices (see Fig. 1). Set $\mathcal{C}(K_{\alpha,\beta,\gamma}) := \{f \mid f : K_{\alpha,\beta,\gamma} \to \mathbb{R},\ f \text{ is continuous}\}$.

Theorem 1.1 (K., cf. [5]). *There exist a finite Borel measure μ on $K_{\alpha,\beta,\gamma}$ with full support and an irreducible, strongly local, regular symmetric Dirichlet form $(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$ on $L^2(K_{\alpha,\beta,\gamma}, \mu)$ such that for any affine function $h : \mathbb{C} \to \mathbb{R}$, $h|_{K_{\alpha,\beta,\gamma}} \in \mathcal{F}_{\alpha,\beta,\gamma}$ and*

$$\mathcal{E}^{\alpha,\beta,\gamma}(h|_{K_{\alpha,\beta,\gamma}}, v) = 0 \qquad \text{for any } v \in \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma}) \text{ with } v|_{V_0^{\alpha,\beta,\gamma}} = 0 \tag{1.1}$$

(i.e., $h|_{K_{\alpha,\beta,\gamma}}$ is $\mathcal{E}^{\alpha,\beta,\gamma}$*-harmonic* on $K_{\alpha,\beta,\gamma} \setminus V_0^{\alpha,\beta,\gamma}$). *Moreover, $\mathcal{C}_{\alpha,\beta,\gamma} := \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma} \times \mathcal{C}_{\alpha,\beta,\gamma}}$ are unique (up to positive constant multiples of $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma} \times \mathcal{C}_{\alpha,\beta,\gamma}}$).*

Theorem 1.2 (K.). *$\mathcal{C}^{\mathrm{LIP}}_{\alpha,\beta,\gamma} := \{u|_{K_{\alpha,\beta,\gamma}} \mid u : \mathbb{C} \to \mathbb{R},\ u \text{ is Lipschitz}\} \subset \mathcal{C}_{\alpha,\beta,\gamma}$ and*

$$\mathcal{E}^{\alpha,\beta,\gamma}(u, v) = \sum\nolimits_{C \in \mathcal{A}_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \int_C \langle \nabla_C u, \nabla_C v \rangle \, d\mathrm{vol}_C \quad \text{for any } u, v \in \mathcal{C}^{\mathrm{LIP}}_{\alpha,\beta,\gamma}, \tag{1.2}$$

where $\mathcal{A}_{\alpha,\beta,\gamma}$ denotes the set of all the arcs appearing in the construction of $K_{\alpha,\beta,\gamma}$, $\mathrm{rad}(C)$ *the radius of C, ∇_C the gradient on C and* vol_C *the length measure on C.*

Theorem 1.3 (K.). *As the measure μ in Theorem* 1.1, *$\mu^{\alpha,\beta,\gamma} := \sum_{C \in \mathcal{A}_{\alpha,\beta,\gamma}} \mathrm{rad}(C)\mathrm{vol}_C$ can be taken. Moreover, the Laplacian associated with $(K_{\alpha,\beta,\gamma}, \mu^{\alpha,\beta,\gamma}, \mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$ has discrete spectrum and its eigenvalues $\{\lambda_n^{\alpha,\beta,\gamma}\}_{n \in \mathbb{N}}$ (with each repeated according to multiplicity) satisfy, with*[1] *$d_{\mathsf{AG}} := \dim_{\mathrm{H}} K_{\alpha,\beta,\gamma}$ and some $c_0 \in (0, \infty)$ independent of α, β, γ,*

$$\lim\nolimits_{\lambda \to \infty} \lambda^{-d_{\mathsf{AG}}/2} \#\{n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \le \lambda\} = c_0 \mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}) \in (0, \infty). \tag{1.3}$$

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Keywords: Kleinian groups, round Sierpiński carpets, Laplacian, Weyl's eigenvalue asymptotics.

[1] $\dim_{\mathrm{H}}$ and $\mathcal{H}^d$ denote Hausdorff dimension and the d-dimensional Hausdorff measure, respectively.

2. Kleinian groups G_m with limit set a round Sierpiński carpet

Let $m \in \mathbb{N}$, $m > 6$. Since $\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} < \pi$ there exists a geodesic triangle with inner angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$, which is unique up to hyperbolic isometry, in the Poincaré disk model $\mathbb{B}^2 := \{z \in \mathbb{C} \mid |z| < 1\}$ of the hyperbolic plane. More specifically, set $\ell_1 := \mathbb{R}$, $\ell_3 := \{te^{\pi i/m} \mid t \in \mathbb{R}\}$ and choose $t, r \in (0, \infty)$ so that $\ell_2 := \{z \in \mathbb{C} \mid |z - te^{\pi i/m}| = r\}$ is orthogonal to $\partial\mathbb{B}^2 := \{z \in \mathbb{C} \mid |z| = 1\}$ and intersects ℓ_1 with angle $\frac{\pi}{3}$; there is a unique such choice of t, r by virtue of $\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} < \pi$. Let $\triangle_0$ denote the closed geodesic triangle formed by ℓ_1, ℓ_2, ℓ_3 and define a subgroup Γ_m of $\mathrm{M\ddot{o}b}(\widehat{\mathbb{C}})$ by $\Gamma_m := \langle \{\mathrm{Inv}_{\ell_k}\}_{k=1}^3 \rangle$, where Inv_ℓ denotes the inversion (reflection) in a circle or a straight line ℓ. Then *Poincaré's polygon theorem* (see, e.g., [2, Section 8]) tells us that $\mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_0)$, where $\tau(\triangle_0)$ and $\sigma(\triangle_0)$ intersect only on their boundaries for any $\tau, \sigma \in \Gamma_m$ with $\tau \neq \sigma$.

Next, choose $r_m \in (0, 1)$ so that $S := \{z \in \mathbb{C} \mid |z| = r_m\}$ intersects ℓ_2 with angle $\frac{\pi}{3}$; it is elementary to see that there is a unique such choice of r_m. Then it turns out (see, e.g., [1]) that the subgroup G_m of $\mathrm{M\ddot{o}b}(\widehat{\mathbb{C}})$ defined by $G_m := \langle \Gamma_m, \mathrm{Inv}_S \rangle$ is a Kleinian group and that $\partial_\infty G_m := \overline{\bigcup_{g \in G_m} g(\partial\mathbb{B}^2)}$ is the *limit set* of G_m (i.e., the minimum non-empty closed G_m-invariant subset of $\widehat{\mathbb{C}}$) and is in fact a round Sierpiński carpet (being homeomorphic to the standard Sierpiński carpet follows from [6]).

Set $K_0 := (\partial_\infty G_m) \cap \mathbb{B}^2$, $\mathcal{G} := \bigl\{g \in \mathrm{M\ddot{o}b}(\widehat{\mathbb{C}}) \bigm| g^{-1}(\infty) \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2}\bigr\}$ and $K_g := g(K_0)$ for $g \in \mathcal{G}$. Also set $\mathcal{D}_g := \bigl\{gh(\widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2}) \bigm| h \in G_m\bigr\} \setminus \bigl\{g(\widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2})\bigr\}$, so that $\mathcal{D}_g$ is a family of disjoint open disks in $\mathbb{C}$ with $K_g = g(\mathbb{B}^2) \setminus \bigcup_{D \in \mathcal{D}_g} D$. Now we adopt (1.2) as the *definition* of the Dirichlet form on K_g and similarly for the volume measure on K_g.

Definition 2.1 (K.). Let $g \in \mathcal{G}$ and set $\mathcal{C}_g := \{u|_{K_g} \mid u : \mathbb{C} \to \mathbb{R},\ u \text{ is Lipschitz}\}$. We define a Borel measure ν^g on K_g and a symmetric bilinear form $\mathcal{E}^g : \mathcal{C}_g \times \mathcal{C}_g \to \mathbb{R}$ by

$$\nu^g := \sum\nolimits_{D \in \mathcal{D}_g} \mathrm{rad}(\partial D) \mathrm{vol}_{\partial D}, \quad \mathcal{E}^g(u, v) := \sum\nolimits_{D \in \mathcal{D}_g} \mathrm{rad}(\partial D) \int_{\partial D} \langle \nabla_{\partial D} u, \nabla_{\partial D} v \rangle \, d\mathrm{vol}_{\partial D}.$$

Proposition 2.2 (K.). *On $L^2(K_g, \nu^g)$, $(\mathcal{E}^g, \mathcal{C}_g)$ is closable and its closure $(\mathcal{E}^g, \mathcal{F}_g)$ is a strongly local regular Dirichlet form whose associated Laplacian has discrete spectrum.*

Since G_m is convex cocompact (hence Gromov hyperbolic), $d_m := \dim_{\mathrm{H}} K_g \in (1, 2)$ and $\mathcal{H}^{d_m}(K_g) \in (0, \infty)$ by [4, Theorem 7]. The following is the main result of this talk.

Theorem 2.3 (K.). *There exists $c_m \in (0, \infty)$ such that for any $g \in \mathcal{G}$, the eigenvalues $\{\lambda_n^g\}_{n \in \mathbb{N}}$ (with each eigenvalue repeated according to its multiplicity) of the (non-negative definite) Laplacian associated with $(K_g, \nu^g, \mathcal{E}^g, \mathcal{F}_g)$ satisfy*

$$\lim\nolimits_{\lambda \to \infty} \lambda^{-d_m/2} \#\{n \in \mathbb{N} \mid \lambda_n^g \leq \lambda\} = c_m \mathcal{H}^{d_m}(K_g). \tag{2.1}$$

Theorem 2.3 is proved by applying *Kesten's renewal theorem* [3, Theorem 2] to a certain Markov chain on the *space of "all possible Euclidean shapes of K_g"* defined by $H \backslash \mathcal{G} := \{Hg \mid g \in \mathcal{G}\}$, where H denotes the group of Euclidean similarities of $\mathbb{C}$.

References

[1] S. Bullett and G. Mantica, *Nonlinearity* **5** (1992), 1085–1109.

[2] D. B. A. Epstein and C. Petronio, *Enseign. Math.* **40** (1994), 113–170.

[3] H. Kesten, *Ann. Probab.* **2** (1974), 355–386.

[4] D. Sullivan, *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 171–202.

[5] A. Teplyaev, in: M. L. Lapidus and M. van Frankenhuijsen (eds.), Proc. Sympos. Pure Math., vol. 72, Part 1, Amer. Math. Soc., 2004, pp. 131–154.

[6] G. T. Whyburn, *Fund. Math.* **45** (1958), 320–324.

Ground state of the renormalized Nelson model: final version

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この研究は Oliver Matte (Aalborg 大学) との共同研究 [9] である．シュレディンガー作用素と結合した量子場の模型でスペクトルがよく研究されている典型的なものに Nelson 模型がある．過去 2013 年の「確率論シンポ」, 2014 年と 2017 年の「確率解析とその周辺」で Nelson 模型のくりこみ理論について発表した．今回大きく研究が進展しほぼ最終的な形を得ることができた．簡単に物理的な説明をする．Nelson 模型は非相対論的なスピンのない核子とスカラー中間子の線形相互作用を表している．相互作用は Yukawa 型相互作用と呼ばれる．スピンのある場合は [10, 8] を参照．Nelson 模型を自己共役作用素として定義するためにはまず紫外切断が必要で, そのとき

$$H = H_p \otimes 1\!\!1 + 1\!\!1 \otimes H_f + g\phi$$

と定義される．ここで, $g \in \mathbb{R}$ は結合定数を表す．この紫外切断がくりこめることを Nelson 自身が約 50 年前に証明している [12]: "$L^2(\mathbb{R}^3) \otimes \mathscr{F}$ 上の自己共役作用素 H_∞ で $s-\lim_{\varepsilon\downarrow 0} e^{-T(H-E)} = e^{-TH_\infty}$ となるものが存在する．ここで E はくりこみ項".

注意:
(1) Path 測度を用いた別証明が [4] で与えられている．
(2) 多様体上の Nelaon 模型のくりこみについては [2] を参照.
(3) Nelson は強位相での収束を示したが, 一様位相で収束を示すことができる [11].
(4) H_∞ の明示的な形はわかっていないが, Nelson 自身は H_∞ を二次形式で与えた.
(5) [4, 11] では e^{-tH_∞} の Feynman-Kac 公式が与えられた.
(6) H_∞ の基底状態に関しては [6] で結合定数が十分小さいときに存在が示されている．
(7) 紫外切断があるとき基底状態の性質はギブス測度を用いて [1, 7, 5] で調べられている.

H_∞ の基底状態 $\varphi_{\rm g}$ に関して次の結果を得た．V は Kato-分解可能クラスで binding 条件 [3] を満たすと仮定する．$\kappa \geq 0$ を赤外切断パラメター, N は個数作用素とする．

存在: $\kappa > 0$ のとき H_∞ の基底状態 $\varphi_{\rm g}$ が存在し, 一意的である．

局所性 1: $\kappa > 0$ とする．このとき $\|e^{\beta N}\varphi_{\rm g}\| < \infty$ が任意の $\beta > 0$ で成り立つ.

局所性 2: $\kappa > 0$ とする．このとき
(1) $\|e^{\beta\phi(h)^2}\varphi_{\rm g}\| < \infty$ が任意の $\beta < 1/\|h/\sqrt{|k|}\|^2$ で成り立つ.
(2) $\lim_{\beta\uparrow 1/\|h/\sqrt{|k|}\|^2} \|e^{\beta\phi(h)^2}\varphi_{\rm g}\| = \infty$

非存在: $\kappa = 0$ のとき基底状態は存在しない.

非 Fock 表現: H_∞^G を H_∞ の Gross 変換とする．H_∞^G は $\kappa > 0$ で H_∞ とユニタリー同値．さらに, 任意の $\kappa \geq 0$ で基底状態が存在する．

基底状態の存在証明のアイデアを簡単に述べる．仮想的な質量 $\nu > 0$ を導入し，さらに [11] の Feynman-Kac 公式を使って有界な開集合 $\mathcal{G} \subset \mathbb{R}^3$ 上に stopping time を使って H を定義し直す．それを $H(\mathcal{G}, \nu)$ とおく．$e^{-tH(\mathcal{G},\nu)}$ の hypercontractivity を示して $H(\mathcal{G}, \nu)$ が基底状態をもつことを示す．それを $\varphi_{\rm g}(\mathcal{G}, \nu)$ とする．massless 極限 $\nu_n \to 0$, 無限体積極限 $\mathcal{G}_n \to \mathbb{R}^3$, 紫外切断除去の極限を順に取り $\varphi_{\rm g}(\mathcal{G}_n, \nu_n)$ の弱収束極限を $\varphi_{\rm g}$ とおく．Riesz-Kolmogorove 定理型のコンパクト性の議論を使って部分列が強収束することを示して $\varphi_{\rm g} \neq 0$ を示す．これが H_∞ の基底状態を与える．局所性 1,2 は基底状態から「くりこまれたギブス測度」を構成して証明する．

参考文献

[1] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos, and H. Spohn. Ground state properties of the Nelson Hamiltonian - A Gibbs measure-based approach. *Rev. Math. Phys.*, 14:173–198, 2002.

[2] C. Gérard, F. Hiroshima, A. Panati, and A. Suzuki. Removal of UV cutoff for the Nelson model with variable coefficients. *Lett. Math. Phys.*, 101:305–322, 2012.

[3] M. Griesemer, E. Lieb, and M. Loss. Ground states in non-relativistic quantum electrodynamics. *Invent. Math.*, 145:557–595, 2001.

[4] M. Gubinelli, F. Hiroshima, and J. Lőrinczi. Ultraviolet renormalization of the Nelson Hamiltonian through functional integration. *J. Funct. Anal.*, 267:3125–3153, 2014.

[5] M. Hirokawa, F. Hiroshima, and J. Lőrinczi. Spin-boson model through a Poisson driven stochastic process. *Math. Zeitschrift*, 277:1165–1198, 2014.

[6] M. Hirokawa, F. Hiroshima, and H. Spohn. Ground state for point particles interacting through a massless scalar bose field. *Adv. Math.*, 191:339–392, 2005.

[7] F. Hiroshima. Functional integral approach to semi-relativistic Pauli-Fierz models. *Adv. Math.*, 259:784–840, 2014.

[8] F. Hiroshima and J. Lőrinczi. Functional integral representations of the Pauli-Fierz model with spin 1/2. *J. Funct. Anal.*, 254:2127–2185, 2008.

[9] F. Hiroshima and O. Matte. Ground states and their associated Gibbs measures in the renormalized Nelson model. preprint, 2018.

[10] F. Hiroshima and H. Spohn. Ground state degeneracy of the Pauli-Fierz model with spin. *Adv. Theor. Math. Phys.*, 5:1091–1104, 2001.

[11] O. Matte and J. Møller. Feynman-Kac formulas for the ultra-violet renormalized Nelson model. *arXiv:1701.02600*, preprint, 2017.

[12] E. Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Math. Phys.*, 5:1990–1997, 1964.