Phase-type fitting approximation of overshoot/undershoot distributions for spectrally negative Lévy processes

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a *spectrally negative Lévy process* $X = \{X_t; t \geq 0\}$ and $\mathbb{P}^x$ be the conditional probability under which $X_0 = x$. The process X is uniquely characterized by its *Laplace exponent*

$$\psi(s) := \log \mathbb{E}^0 \left[e^{sX_1} \right] = \hat{\mu} s + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{0} (e^{sz} - 1 - sz1_{\{z>-1\}})\Pi(\mathrm{d}z), \quad s \in \mathbb{C}$$

where Π is a Lévy measure with the support $(-\infty, 0)$ and satisfies the integrability condition $\int_{(-\infty,0)} (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty$. Our aim is to compute the joint distribution of overshoot and undershoot (with discounting):

$$h_q(x; A, B) := \mathbb{E}^x \left[e^{-q\tau_0^-} 1_{\{X_{\tau_0^- -} \in B, X_{\tau_0^-} \in A, \tau_0^- < \infty\}} \right], \quad B \in \mathcal{B}(0, \infty),\ A \in \mathcal{B}(-\infty, 0)$$

where $q > 0$ and $\tau_0^- := \inf \{t \geq 0 : X_t < 0\}$.

We focus on the case Π has a *completely monotone* density and approximate $h_q(x; A, B)$ by using the spectrally negative Lévy process of the form

$$X_t - X_0 = \mu t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty,$$

for some $\mu \in \mathbb{R}$ and $\sigma \geq 0$. Here $B = \{B_t; t \geq 0\}$ is a standard Brownian motion, $N = \{N_t; t \geq 0\}$ is a Poisson process with arrival rate λ, and $Z = \{Z_n; n = 1, 2, \ldots\}$ is an i.i.d. sequence of hyperexponential random variables with density function

$$f(z) = \sum_{j=1}^{m} p_j \eta_j e^{-\eta_j z}, \quad z \geq 0,$$

for some $0 < \eta_1 < \cdots < \eta_m < \infty$ and $p_1 + \cdots + p_m = 1$. Its Laplace exponent is

$$\psi(s) = \mu s + \frac{1}{2}\sigma^2 s^2 - \lambda \sum_{j=1}^{m} p_j \frac{s}{\eta_j + s}.$$

In this case, the *Cramér-Lundberg* equation, $\psi(s) = q$, has a unique positive root

$$\zeta_q := \sup\{s \geq 0 : \psi(s) = q\},$$

while, regarding the negative roots,

1. when $\sigma > 0$, there are $m+1$ roots $-\xi_{1,q}, \ldots, -\xi_{m+1,q}$ such that

$$0 < \xi_{1,q} < \eta_1 < \xi_{2,q} < \cdots < \eta_m < \xi_{m+1,q} < \infty;$$

2. when $\sigma = 0$ and $\mu > 0$, there are m roots $-\xi_{1,q}, \ldots, -\xi_{m,q}$ such that

$$0 < \xi_{1,q} < \eta_1 < \xi_{2,q} < \cdots < \xi_{m,q} < \eta_m < \infty.$$

We let $\mathcal{I}_q$ be $\{1, \ldots, m+1\}$ and $\{1, \ldots, m\}$ when $\sigma > 0$ and $\sigma = 0$, respectively.

Proposition 1. *Suppose $B = (\underline{b}, \overline{b})$ and $A = (-\overline{a}, -\underline{a})$ for some $0 \leq \underline{a} \leq \overline{a}$ and $0 \leq \underline{b} \leq \overline{b}$. Then*

$$h_q(x; A, B) = \lambda \sum_{j=1}^{m} p_j (e^{-\eta_j \underline{a}} - e^{-\eta_j \overline{a}}) \kappa_{j,q}(x; B)$$

where, for each $1 \leq j \leq m$,

$$\begin{aligned}\kappa_{j,q}(x; B) := & \frac{e^{\zeta_q x}}{\psi'(\zeta_q)(\eta_j + \zeta_q)} \left(e^{-(\eta_j + \zeta_q)(\underline{b} \vee x)} - e^{-(\eta_j + \zeta_q)(\overline{b} \vee x)} \right) \\ & + \sum_{i \in \mathcal{I}_q} C_{i,q} e^{-\xi_{i,q} x} \left[\frac{1}{(\eta_j - \xi_{i,q})} \left(e^{-(\eta_j - \xi_{i,q})(\underline{b} \wedge x)} - e^{-(\eta_j - \xi_{i,q})(\overline{b} \wedge x)} \right) \right. \\ & \left. - \frac{1}{(\eta_j + \zeta_q)} \left(e^{-(\eta_j + \zeta_q)\underline{b}} - e^{-(\eta_j + \zeta_q)\overline{b}} \right) \right],\end{aligned}$$

for some constants $C_{i,q}$, $i \in \mathcal{I}_q$.

Using this proposition, we approximate $h_q(x; A, B)$ for arbitrary spectrally negative Lévy process with a completely monotone Lévy density. We verify the effectiveness of this approximation procedure through a series of numerical examples.

References

[1] Egami, M. and Yamazaki, K., *On scale functions of spectrally negative Levy processes with phase-type jumps*, arXiv:1005.0064, 2010.