Density of stochastic differential equations driven by gamma processes

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Let a, b and T be positive constants. Let $\{J^i(t)\,;\, t \in [0,T]\}$ $(i = 1,\ldots,d)$ be independent gamma processes with the parameter (a,b), that is, each process J^i is a one-sided pure-jump Lévy process without any Gaussian components with the Lévy measure

$$\nu(\mathrm{d}z) = g(z)\,\mathbb{I}_{(0,+\infty)}(z)\,\mathrm{d}z, \quad g(z) = a\,\exp(-bz)/z.$$

The marginal $J^i(t)$ at time $t \in [0,T]$ has the density in closed form:

$$p_{J^i(t)}(y) = b^{at}\,y^{at-1}\exp(-by)/\Gamma(at), \quad y \in [0,+\infty).$$

Let $A_0, A_1, \ldots, A_d : \mathbb{R}^d \to \mathbb{R}^d$ be smooth and bounded. Suppose that the functions A_i $(i = 1,\ldots,d)$ satisfy the invertible condition:

$$\inf_{y\in\mathbb{R}^d}\ \inf_{z\in(0,+\infty)} \left|\det\left(I_d + \partial A_i(y)\,z\right)\right| > 0 \tag{1}$$

for any $i = 1,\ldots,d$. For a non-random point $x \in \mathbb{R}^d$, we shall consider the $\mathbb{R}^d$-valued process $\{X(t)\,;\, t \in [0,T]\}$ determined by the stochastic differential equation of the form:

$$dX(t) = A_0\big(X(t)\big)\,\mathrm{d}t + A_i\big(X(t-)\big)\,\mathrm{d}J(t), \quad X(0) = x, \tag{2}$$

where $A = \big(A_1,\ldots,A_d\big)$ and $J(t) = \big(J^1(t),\ldots,J^d(t)\big)$. Then, there exists a unique solution $\{X(t)\,;\, t \in [0,T]\}$ to the equation (2) such that, for each $t \in [0,T]$, the function $\mathbb{R}^d \ni x \longmapsto X(t) \in \mathbb{R}^d$ has a C^∞-modification, and its Jacobi matrix is invertible a.s. In this talk, we shall focus on the sensitivity, and the error estimate on the densities between the solution and the driving gamma process, which can be applied to the strict positivity of the density. This is based upon joint work with Vlad Bally (Université Paris-Est Marne-la-Vallée, France).

Let C_1 be a positive constant, and $\Xi \in C_b^\infty\big(\mathbb{R}^d \otimes \mathbb{R}^d\,;\,[0,1]\big)$ such that

$$\Xi(B) = 0 \ \ \big(0 \le |\det B| \le C_1/2\big), \quad \Xi(B) = 1 \ \ \big(|\det B| \ge C_1\big).$$

The Girsanov transform on the driving process leads to get the integration by parts formula.

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Theorem 1 *For $\Phi \in C_b^1(\mathbb{R}^d; \mathbb{R})$, the following equality holds:*

$$\mathbb{E}\Big[\partial_k \Phi\big(X(T)\big)\,\Xi\big(V_X(T)\big)\Big] = \mathbb{E}\Big[\Phi\big(X(T)\big)\,\Theta_k\big(X(T), \Xi\big(V_X(T)\big)\big)\Big] \tag{3}$$

for $k = 1, \ldots, d$, where $V_X(T)$ is the Malliavin covariance matrix for $X(T)$.

Moreover, suppose that the functions A_i $(1 \le i \le d)$ satisfy the uniformly elliptic condition:

$$\inf_{\zeta \in \mathbb{S}^{d-1}} \inf_{y \in \mathbb{R}^d} \zeta \cdot A(y)\, A(y)^* \zeta \ge C_2, \tag{4}$$

under which there exists a C^∞-Lebesgue density for $X(T)$. Theorem 1 enables us to see that

Theorem 2 *It holds that*

$$\mathbb{E}\big[\mathbb{I}_{(X(T)\in D)}\,\Xi\big(V_X(T)\big)\big] = \int_D \sum_{k=1}^d \mathbb{E}\Big[\partial_k Q_d\big(X(T) - y\big)\,\Theta_k\big(X(T), \Xi\big(V_X(T)\big)\big)\Big]\,\mathrm{d}y \tag{5}$$

for $D \in \mathscr{B}\big(\mathbb{R}^d\big)$, where Q_d is the fundamental solution to the equation $\Delta Q_d = \delta_0$.

Let C_3 be a positive constant, and $\psi_{1,i} \in C_b^\infty\big([0, +\infty); [0,1]\big)$ $(i = 1, \ldots, d)$ with

$$\psi_{1,i}(u_i) = 1 \ \ \big(u_i \ge C_3\big), \quad \psi_{1,i}(u_i) = 0 \ \ \big(u_i \le C_3/2\big).$$

Define $\psi_1(u) = \prod_{i=1}^d \psi_{1,i}(u_i)$ for $u = \big(u_1, \ldots, u_d\big) \in [0, +\infty)^d$. Rewrite (2) as follows:

$$\begin{aligned} X(T) &= \Big(x + A(x)J(T)\Big) + \Big(\int_0^T A_0\big(X(s)\big)\mathrm{d}s + \int_0^T \big\{A\big(X(s-)\big) - A(x)\big\}\mathrm{d}J(s)\Big) \\ &=: G(T) + R(T). \end{aligned}$$

Theorem 3 *It holds that*

$$p_{X(T)}(y) \ge \tilde{p}_{G(T)}(y) - \mathscr{E}_T, \tag{6}$$

where $p_{J(T)}(y)$ is the density for $J(T)$, $\mathscr{E}_T = C_4\big(|R(T)|_p + \|V_{\tilde{R}}(T)\|_p + \|H_{\tilde{R}}(T)\|_p\big)$, and

$$\tilde{p}_{G(T)}(y) = \psi_1\big(A(x)^{-1}(y - x)\big)\,p_{J(T)}\big(A(x)^{-1}(y - x)\big).$$

References

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