Slow movement of a random walk on the range of a random walk in the presence of an external field

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In studying random walks in random environments, there is a particular focus at the moment on understanding the effect of an external field. In this talk, I will consider this issue in the case when the random environment is the graph generated by a simple random walk path in high dimensions ($d \geq 5$), and the external field has the effect of introducing a unidirectional bias on the random walk that moves within this medium. The localization result that I describe for this model demonstrates that, unlike in the supercritical percolation setting, a slowdown effect occurs as soon as a non-trivial bias is introduced. The proof applies a decomposition of the underlying simple random walk path at its cut-times to relate the associated biased random walk to a one-dimensional random walk in a random environment in Sinai's regime.

To state the main result more precisely, we first need to formally define a biased random walk on the range of a random walk. Let $(S_n)_{n\in\mathbb{Z}}$ be a two-sided random walk on $\mathbb{Z}^d$, i.e. suppose that $(S_n)_{n\geq 0}$ and $(S_{-n})_{n\geq 0}$ are independent random walks on $\mathbb{Z}^d$ starting from 0 built on a probability space with probability measure $\mathbf{P}$. The range of this process is defined to be the random graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ with vertex set

$$V(\mathcal{G}) := \{S_n : n \in \mathbb{Z}\}, \tag{1}$$

and edge set

$$E(\mathcal{G}) := \{\{S_n, S_{n+1}\} : n \in \mathbb{Z}\}. \tag{2}$$

Now, fix a bias parameter $\beta \geq 1$, and to each edge $e = \{e_-, e_+\} \in E(\mathcal{G})$, assign a conductance

$$\mu_e := \beta^{\max\{e_-^{(1)}, e_+^{(1)}\}}, \tag{3}$$

where $e_\pm^{(1)}$ is the first coordinate of $e_\pm$. The biased random walk on $\mathcal{G}$ is then the time-homogenous Markov chain $X = ((X_n)_{n\geq 0}, \mathbf{P}_x^{\mathcal{G}}, x \in V(\mathcal{G}))$ on

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$V(\mathcal{G})$ with transition probabilities

$$P_{\mathcal{G}}(x,y) := \frac{\mu_{\{x,y\}}}{\mu(\{x\})},$$

where μ is a measure on $V(\mathcal{G})$ defined by $\mu(\{x\}) := \sum_{e \in E(\mathcal{G}): x \in e} \mu_e$. Note that, if β is strictly greater than 1, then the biased random walk X prefers to move in the first coordinate direction. If, on the other hand, there is no bias, i.e. $\beta = 1$, then the preceding definition leads to the usual simple random walk on $\mathcal{G}$. Finally, as is the usual terminology for random walks in random environments, for $x \in V(\mathcal{G})$, we say that $\mathbf{P}_x^{\mathcal{G}}$ is the quenched law of X started from x. Since 0 is always an element of $V(\mathcal{G})$, we can also define an annealed law $\mathbb{P}$ for the biased random walk on $\mathcal{G}$ started from 0 by setting

$$\mathbb{P} := \int \mathbf{P}_0^{\mathcal{G}}(\cdot) \mathrm{d}\mathbf{P}. \tag{4}$$

When there is no bias, under this law, the essentially one-dimensional nature of the environment leads to the random walk X behaving diffusively along the path. However, whenever $\beta > 1$, the following result regarding the slow movement of the biased random walk holds, and it will be the main aim of the talk to explain the ideas behind its proof.

Theorem 0.1. *Fix a bias parameter $\beta > 1$ and $d \geq 5$. If $X = (X_n)_{n \geq 0}$ is the biased random walk on the range $\mathcal{G}$ of the two-sided simple random walk S in $\mathbb{Z}^d$, then there exists an S-measurable random variable L_n taking values in $\mathbb{R}^d$ such that*

$$\mathbb{P}\left(\left|\frac{X_n}{\log n} - L_n\right| > \varepsilon\right) \to 0,$$

for any $\varepsilon > 0$. Moreover, $(L_n)_{n \geq 1}$ converges in distribution under $\mathbf{P}$ to a random variable L_β whose distribution can be characterized explicitly.