On convergence of elliptic operators on a Riemannian manifold

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Set up

- (M, g, μ): a weighted manifold with a density $\Psi > 0$
- $\Omega \subset M$ is an open set. $U \Subset M$ is a relatively compact open set.
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$$m_0(\Omega) = \inf \left\{ m \in \mathbb{R} \mid \inf_{1=\|u\|_{H_0^1(\Omega)}} \int_\Omega \left(mu^2 + g(\nabla u, \nabla u) \right) d\mu > 0 \right\}$$

where

$$H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1}, \quad (u, v)_{H^1} = \int_M uv \, d\mu + \int_M g(\nabla u, \nabla v) \, d\mu$$

Set up

- $\mathcal{M}(\Omega, \lambda, \Lambda)$ (with $\lambda, \Lambda > 0$) is a "coefficient fields"; that is, the set of measurable coefficient fields $\mathbb{L}$ such that

$$\begin{cases} g(\xi, \mathbb{L}\xi)(x) \geq \lambda g(\xi, \xi)(x), \quad \forall x \in M, \ \forall \xi \in T_x M \\ g(\xi, \mathbb{L}^{-1}\xi)(x) \geq \Lambda^{-1} g(\xi, \xi)(x), \quad \forall x \in M, \ \forall \xi \in T_x M \end{cases}$$

- Elliptic operators: $(\mathbb{L}_\epsilon)_{\epsilon > 0} \subset \mathcal{M}(\Omega, \lambda, \Lambda)$

$$\mathcal{L}_\epsilon = -\mathrm{div} \circ \mathbb{L}_\epsilon \circ \nabla : H_0^1(\Omega) \to H^{-1}(\Omega)$$

Definition 1.1

Let $(\mathbb{L}_\epsilon) \subset \mathcal{M}(\Omega, \lambda, \Lambda)$ *and* $\mathbb{L}_0 \in \mathcal{M}(\Omega, \lambda, \Lambda)$. *We say that the sequence* $(\mathbb{L}_\epsilon)$ *"H-convergence" to* $\mathbb{L}_0$ *iff for any* $U \Subset \Omega$ *and for any* $f \in H^{-1}(U)$, *the solutions* $u_\epsilon, u_0 \in H^1_0(U)$ *to*

$$\mathcal{L}_\epsilon u_\epsilon = \mathcal{L}_0 u_0 = f \quad \text{in } H^{-1}(U)$$

satisfy

$$\begin{cases} u_\epsilon \rightharpoonup u_0, & \text{weakly in } H^1(U) \\ \mathbb{L}_\epsilon \nabla u_\epsilon \rightharpoonup \mathbb{L}_0 \nabla u_0, & \text{weakly in } L^2(TU) \end{cases}$$

In that case, we denote

$$\mathbb{L}_\epsilon \xrightarrow{H} \mathbb{L}_0 \text{ in } (\Omega, g, \mu).$$

Main Result

Hereafter, $(\mathbb{L}_\epsilon) \subset \mathcal{M}(M, \lambda, \Lambda)$.

Theorem 2.1 (Hoppe - M - Neukamm)

There exists a subsequence (not relabeled) $(\mathbb{L}_\epsilon)$ *and* $\mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$ *such that*

$$\mathbb{L}_\epsilon \overset{H}{\to} \mathbb{L}_0 \text{ in } (M, g, \mu)$$

Main Result (Continuation)

Theorem 2.2 (Hoppe - M - Neukamm)

Let $(f_\epsilon) \subset L^2(\Omega)$ *and* $(F_\epsilon) \subset L^2(T\Omega)$ *be such that*

$$f_\epsilon \rightharpoonup f_0 \quad \text{weakly in } L^2(\Omega), \qquad F_\epsilon \to F_0 \quad \text{in } L^2(T\Omega).$$

Let $m > m_0(\Omega)/\lambda$ *and* $u_\epsilon, u_0 \in H_0^1(\Omega)$ *be the solutions to*

$$\begin{aligned} (\mathcal{L}_\epsilon + m)u_\epsilon &= f_\epsilon + \operatorname{div} F_\epsilon, \quad \text{in } H^{-1}(\Omega), \\ (\mathcal{L}_0 + m)u_0 &= f_0 + \operatorname{div} F_0, \quad \text{in } H^{-1}(\Omega). \end{aligned} \tag{1}$$

Then,

$$\mathbb{L}_\epsilon \stackrel{H}{\to} \mathbb{L}_0 \implies \begin{cases} u_\epsilon \rightharpoonup u_0 & \text{weakly in } H_0^1(\Omega), \\ \mathbb{L}_\epsilon \nabla u_\epsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 & \text{weakly in } L^2(T\Omega). \end{cases}$$

Additionally, if $m \neq 0$ *and* $f_\epsilon \to f$ *in* $L^2(M)$*, then* $u_\epsilon \to u$ *in* $L^2(M)$*.*

Definition 2.1 (Mosco convergence)

Let $\mathbb{L}_\epsilon \in \mathcal{L}(M, \lambda, \Lambda)$ *be symmetric with* $\epsilon \geq 0$. *Set*

$$Q_\epsilon(u, v) = \begin{cases} \int_M g(\mathbb{L}_\epsilon \nabla u, \nabla v)\, d\mu, & u, v \in H_0^1(M), \\ \infty, & \textit{else}. \end{cases}$$

We say $Q_\epsilon \to Q_0$ *in Mosco sense if*

- $\forall u \in L^2(M)$, $\exists u_\epsilon \in L^2(M)$ *such that* $\limsup Q_\epsilon(u_\epsilon, u_\epsilon) \leq Q_0(u, u)$.
- $v_\epsilon \rightharpoonup v$ *in* $L^2(M) \implies \liminf Q_\epsilon(v_\epsilon, v_\epsilon) \geq Q_0(v, v)$

Proposition 2.1 (H-convergence implies Mosco convergence)

Let $\mathbb{L}_\epsilon \in \mathcal{L}(M, \lambda, \Lambda)$ *be symmetric with* $\epsilon \geq 0$. *Then,*

$$\mathbb{L}_\epsilon \xrightarrow{H} \mathbb{L}_0 \implies Q_\epsilon \to Q_0 \textit{ in Mosco sense}$$

which implies

$$e^{t\mathcal{L}_\epsilon} \to e^{t\mathcal{L}_0} \textit{ in } L^2(M)$$

Main Result (Example)

Let $(\sigma_\epsilon)_{\epsilon \geq 0} \subset L^\infty(M)$ such that $\exists c > 0$ such that

$$c < \sigma_\epsilon < c^{-1}. \tag{2}$$

Consider

$$M_\epsilon = (M, g_\epsilon, \mu_\epsilon), \qquad g_\epsilon = \sigma_\epsilon g, \;\; \mu_\epsilon = (\sigma_\epsilon)^{n/2}.$$

Assume that $H_0^1(M)$ is compact in $L^2(M)$.

Proposition 2.2

There is $\Theta \in \mathcal{M}(M, \lambda, \Lambda)$ with ellipticity constants $0 < \lambda, \Lambda < \infty$ only depending on $d =$dim(M) and the constant in (2), and there exists a $\sigma_0 \in L^\infty(M)$ satisfying (2), such that the following holds for a subsequence (not relabeled):

(a) $\sigma_\epsilon^{d/2} \rightharpoonup \sigma_0^{d/2}$ *weakly-$*$ in* $L^\infty(M, g, \mu)$.

Main Result (Example)

Proposition 2.3 (Under the same situation in the previous proposition)

(b) *Let* $g_0 := \sigma_0 g$ *and* $\mu_0 := \sigma_0^{d/2}\mu$. *For all* $m > m'$, $m' \in \mathbb{R}$ *only depends on* $M = (M, g, \mu)$ *and the constant in* (2). *For* $f_\epsilon, f \in L^2(M, g, \mu)$, *let* $u_\epsilon \in H_0^1(M, g_\epsilon, \mu_\epsilon)$ *and* $u_0 \in H_0^1(M, g_0, \mu_0)$ *be the solutions to*

$$\begin{cases} mu_\epsilon + \Delta_\epsilon u_\epsilon = f_\epsilon & \text{in } H^{-1}(M, g_\epsilon, \mu_\epsilon), \\ mu_0 + div_0(\Theta \nabla u_0) = f_0 & \text{in } H^{-1}(M, g_0, \mu_0), \end{cases}$$

respectively. Then

$$f_\epsilon \to f \text{ in } L^2(M) \implies \begin{cases} u_\epsilon \rightharpoonup u_0 & \text{in } H^1(M), \\ \sigma_\epsilon^{\frac{d}{2}+1} \nabla u_\epsilon \rightharpoonup \sigma_0^{\frac{d}{2}+1} \Theta \nabla u_0 & \text{in } L^2(TM). \end{cases}$$

Main Result (Example)

Proposition 2.4 (Under the same situation in the previous proposition)

(c) *Let* $u_\epsilon \in H_0^1(M, g_\epsilon, \mu_\epsilon)$ *and* $u_0 \in H_0^1(M, g_0, \mu_0)$ *be the solutions to*

$$\begin{cases} mu_\epsilon + \Delta_\epsilon u_\epsilon = f_\epsilon, & f_\epsilon \in L^2(M, g_\epsilon, \mu_\epsilon), \\ mu_0 + div_0(\Theta \nabla u_0) = f_0, & f_0 \in L^2(M, g_0, \mu_0), \end{cases}$$

respectively. Then

$$f_\epsilon \to f_0 \text{ in } L^2 \implies u_\epsilon \to u_0 \text{ in } L^2.$$

(d) *Fix* $n \in \mathbb{N}$. *For every sequence* $(\lambda_{\varepsilon,n}, u_{\varepsilon,n})$ *of eigenpairs of* $m + \Delta_\varepsilon$ *in* $H^{-1}(M, g_\varepsilon, \mu_\varepsilon)$ *there are a (not relabeled) subsequence and an eigenpair* (λ_0, u_0) *of* $m + div(\Theta\nabla)$ *in* $H^{-1}(M, g_0, \mu_0)$ *such that* $\lambda_{n,\varepsilon} \to \lambda_0$ *and* $u_{n,\varepsilon} \to u_0$ *strongly in* L^2.

Key Lemmas

Lemma 1 (Div-Curl Lemma)

Let $(\xi_\epsilon) \subset L^2(T\Omega)$ *and* $(v_\epsilon) \subset H^1(\Omega)$ *be such that*

$$\begin{cases} \xi_\epsilon \rightharpoonup \xi & \textit{weakly in } L^2(T\Omega), \\ \mathrm{div}\xi_\epsilon \to \mathrm{div}\xi & \textit{in } H^{-1}(\Omega), \\ v_\epsilon \rightharpoonup v & \textit{weakly in } H^1(\Omega). \end{cases}$$

Then

$$\int_\Omega g(\xi_\epsilon, \nabla v_\epsilon)\rho \, d\mu \to \int_\Omega g(\xi, \nabla v)\rho \, d\mu \qquad \textit{for all } \rho \in C_c^\infty(\Omega).$$

Moreover, if $v_\epsilon, v \in H_0^1(\Omega)$, *then*

$$\int_\Omega g(\xi_\epsilon, \nabla v_\epsilon) \, d\mu \to \int_\Omega g(\xi, \nabla v) \, d\mu.$$

Lemma 2

Let V be a reflexive separable Banach space and (T_ϵ) be a sequence of linear operators $T_\epsilon \colon V \to V'$ that is uniformly bounded and coercive, i.e. there exists $C > 0$ (independent of ϵ) such that the operator norm of T_ϵ is bounded by C and

$$\langle T_\epsilon v, v \rangle_{V',V} \geq \frac{1}{C} \|v\|_V^2 \qquad \text{for all } v \in V. \tag{3}$$

Then there exists a a subsequence (not relabeled) (T_ϵ) and a linear bounded operator $T_0 : V \to V'$ satisfying (3) such that that is for all $f \in V'$ we have

$$T_\epsilon^{-1} f \rightharpoonup T_0^{-1} f \qquad \text{weakly in } V.$$

Proposition 3.1 (*H*-compactness on small balls)

Let $(\mathbb{L}_\epsilon) \subset \mathcal{M}(M, \lambda, \Lambda)$ *and let* $B_x(r)$ *with* $r < inj(x)$. *Then there exists* $\mathbb{L}_0 \in \mathcal{M}(B_x(r/2), \lambda, \Lambda)$ *and a (not relabeled) subsequence of* $(\mathbb{L}_\epsilon)$ *such that*

$$\mathbb{L}_\epsilon \xrightarrow{H} \mathbb{L}_0 \text{ in } B_x(r/2)$$

Lemma 3 (Uniqueness, locality, invariance w.r.t. transposition)

Let $\Omega \subset M$ *be open,* $U \Subset \Omega$ *and* $\mathbb{L}_\epsilon \xrightarrow{H} \mathbb{L}_0$, $\mathbb{L}'_\epsilon \xrightarrow{H} \mathbb{L}'_0$ *in* (Ω, g, μ).

1. $\mathbb{L}_\epsilon = \mathbb{L}'_\epsilon$ *on* $U \implies \mathbb{L}_0 = \mathbb{L}'_0$ *on* U μ*-a.e.*
2. $\mathbb{L}^*_\epsilon \xrightarrow{H} \mathbb{L}^*_0$ *in* (Ω, g, μ).

Lemma 4

Let $U \Subset \Omega \subset M$ *and* $\mathbb{L}_\epsilon, \mathbb{L}_0 \in \mathcal{M}(\Omega, \lambda, \Lambda)$. *Let* $f_\epsilon, f_0 \in L^2(U)$ *and* $G_\epsilon, F_\epsilon, G_0, F_0 \in L^2(TU)$ *be such that*

$$\begin{cases} f_\epsilon \rightharpoonup f_0 & \text{weakly in } L^2(U), \\ G_\epsilon \to G_0 & \text{in } L^2(TU), \\ F_\epsilon \to F_0 & \text{in } L^2(TU). \end{cases}$$

Let $u_\epsilon, u_0 \in H_0^1(\omega)$ *be the solutions to*

$$\begin{aligned} \mathcal{L}_\epsilon u_\epsilon &= f_\epsilon + \mathrm{div}(\mathbb{L}_\epsilon G_\epsilon) + \mathrm{div} F_\epsilon && \text{in } H^{-1}(U), \\ \mathcal{L}_0 u_0 &= f_0 + \mathrm{div}(\mathbb{L}_0 G_0) + \mathrm{div} F_0 && \text{in } H^{-1}(U). \end{aligned}$$

Then,

$$\mathbb{L}_\epsilon \overset{H}{\to} \mathbb{L}_0 \implies \begin{cases} u_\epsilon \rightharpoonup u_0 & \text{weakly in } H_0^1(U), \\ \mathbb{L}_\epsilon \nabla u_\epsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 & \text{weakly in } L^2(TU). \end{cases}$$

Approach to Proposition 3.1

Denote $2B = B_x(2r)$ and $B = B_x(r)$. Let $v^k \in C_c^\infty(B)$ be such that

$$\langle \nabla v^1(y), \ldots, \nabla v^n(y) \rangle = T_y(M), \quad \forall y \in B$$

Claim 1: There exist $\mathbb{L}_0$ on B and $v_\epsilon^k \in H_0^1(2B)$ such that

$$\begin{cases} v_\epsilon^k \rightharpoonup v^k, & H_0^1(2B) \\ v_\epsilon^k \to v^k, & L^2(2B) \\ \mathcal{L}_\epsilon^* v_\epsilon^k \to \mathcal{L}_0^* v^k, & H^{-1}(2B) \\ \mathbb{L}_\epsilon^* \nabla v_\epsilon^k \rightharpoonup \mathbb{L}_0^* \nabla v^k, & L^2(B) \end{cases}$$

Claim 2: $\mathcal{L}_0 = -\mathrm{div} \circ \mathbb{L}_0 \circ \nabla$ and $\mathbb{L}_\epsilon \xrightarrow{H} \mathbb{L}_0$

Proof of Claim 1

Since $(\mathcal{L}_\epsilon^* u, u)_{L^2} \geq C\|u\|^2_{H^1(2B)}$, there exists $\mathcal{L}_0^* : H_0^1(2B) \to H^{-1}(2B)$ such that

$$(\mathcal{L}_\epsilon^*)^{-1} f \rightharpoonup (\mathcal{L}_0^*)^{-1} f \quad H_0^1(2B)$$

Set

$$v_\epsilon^k := (\mathcal{L}_\epsilon^*)^{-1} \mathcal{L}_0^* v^k$$

Then

$$\begin{cases} v_\epsilon^k \rightharpoonup v^k, & H_0^1(2B) \\ v_\epsilon^k \to v^k, & L^2(2B) \\ \mathbb{L}_\epsilon^* \nabla v_\epsilon^k \rightharpoonup \exists l^k, & L^2(B) \end{cases}$$

Define $\mathbb{L}_0^*$ by

$$\mathbb{L}_0^* \nabla v^k = l^k, \quad 1 \leq k \leq n.$$

Proof of Claim 2

Let $U \Subset \frac{1}{2}B$. In a similar way, we find

$$(\mathcal{L}_\epsilon)^{-1} \rightharpoonup \exists(\mathcal{L}_0)^{-1} \quad \text{on } U$$

For $u \in H_0^1(U)$,

$$u_\epsilon := (\mathcal{L}_\epsilon)^{-1}\mathcal{L}_0 u \rightharpoonup u \text{ in } H_0^1(U), \quad J_\epsilon := \mathbb{L}_\epsilon \nabla u_\epsilon$$

Then

$$J_\epsilon \rightharpoonup \exists J_0 \quad L^2(TU)$$

and we find

$$\mathrm{div} J_0 = \mathcal{L}_0 u$$

Proof of Claim 2 (continuation)

Observe

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ in } H_0^1(U) \\ \mathbb{L}_\epsilon^* \nabla v_\epsilon^k \rightharpoonup \mathbb{L}_0^* \nabla v^k \\ \mathrm{div} \mathbb{L}_\epsilon^* \nabla v_\epsilon^k \to \mathrm{div} \mathbb{L}_0^* \nabla v^k \end{cases}$$

By Div-Curl Lemma,

$$(J_\epsilon, \rho \nabla v_\epsilon^k) = (\rho \nabla u_\epsilon, \mathbb{L}_\epsilon^* \nabla v_\epsilon^k) \to (\rho \nabla u, \mathbb{L}_0^* \nabla v^k) = (\mathbb{L}_0 \nabla u, \rho \nabla v^k)$$

On the other hand, we can prove

$$(J_\epsilon, \rho \nabla v_\epsilon^k) \to (J_0, \rho \nabla v^k)$$

Hence $J_0 = \mathbb{L}_0 \nabla u$, and (by $\mathrm{div} J_0 = \mathcal{L}_0 u$) we get

$$\mathrm{div} \mathbb{L}_0 \nabla = \mathcal{L}_0.$$

Finally, the fact $\mathbb{L}_0 \in \mathcal{M}(U, \lambda, \Lambda)$ can be proved by the uniformly ellipticity of $(\mathbb{L}_\epsilon)$ and Div-Curl Lemma.