An estimate of the gap of spectrum of Schrödinger operators which generate hyperbounded semigroup [1] [2]

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§1 Setting

Notations

$(X, \mathfrak{B}, m)$: Probability space

$dm_\varphi = \varphi^2 dm$: Weighted measure ($\varphi \in L^2(m)$).

$V \in L^1(X, m)$.

$\langle u \rangle_m = \int_X udm$

In this section, we are working in more general situations than hyperbounded setting. We assume (A1) (A2) on $(\mathcal{E}, \mathrm{D}(\mathcal{E}))$.

(A1) (conservativeness and derivation property)

P_t : a conservative L^2-Markovian symmetric semigroup on $L^2(m)$.

$(\mathcal{E}, \mathrm{D})$: the corresponding Dirichlet form.

Then $\mathcal{E}(1,1) = 0$ holds.

For any φ which is a C^1-function on $\mathbb{R}^n$ with bounded derivative and $\{u_i\}_{i=1}^n \in \mathrm{D}(\mathcal{E})$, it holds that

$$\begin{aligned}&\Gamma(\varphi(u_1, \dots, u_n), \varphi(u_1, \dots, u_n)) \\ &\quad = \sum_{i,j=1}^n \Gamma(u_i, u_j) \partial_i \varphi(u_1, \dots, u_n) \partial_j \varphi(u_1, \dots, u_n).\end{aligned} \tag{1}$$

(A2) Let $V_\pm(x) = \max(\pm V(x), 0)$. Then $V_+ \in L^1(m)$ and there exist $a \in (0,1)$ and $b \in [0, \infty)$ such that for all $u \in \mathrm{D}(\mathcal{E}) \cap L^2(V_+ \cdot m)$,

$$\int_X V_- u^2 dm \le a \left\{ \mathcal{E}(u,u) + \int_X V_+ u^2 dm \right\} + b\|u\|_{L^2(m)}^2. \tag{2}$$

Definition 1

$$\mathcal{D} \;:=\; \mathrm{D}(\mathcal{E}) \cap L^\infty(X, m) \tag{3}$$

$$\mathcal{E}_V(u,v) \;:=\; \mathcal{E}(u,v) + \int_X Vuvdm \qquad (u, v \in \mathcal{D}). \tag{4}$$

[1] to appear in J. Funct. Anal.

[2] The latest version is in “ http://www.sigmath.es.osaka-u.ac.jp/~aida/paper/paper.html”

Theorem 2 (1) *Under* (A2), *for any* $u \in \mathcal{D}$,

$$\mathcal{E}_V(u,u) \geq -b\|u\|_{L^2}^2.$$

(2) *The closure of* $\mathcal{D}$ *with respect to* $\mathcal{E}_V + b\| \ \|_{L^2}$ *is* $\mathrm{D}(\mathcal{E}) \cap L^2(X, |V|m)$.

Definition 3 (Definition of Schrödinger operators) *Let* $-L_V$ *be the semibounded self-adjoint operator corresponding to* $\mathcal{E}_V$ *with the domain* $\mathrm{D}(\mathcal{E}) \cap L^2(|V|m)$. *We denote the corresponding* L^2*-semigroup by* T_t. *Let* $\sigma(-L_V)$ *denote the spectral set and*

$$\lambda_0(V) = \inf \sigma(-L_V), \tag{5}$$

$$\lambda_1(V) = \inf \left(\sigma(-L_V) \setminus \{\lambda_0(V)\}\right). \tag{6}$$

We are concerned with an estimate on $\lambda_1(V) - \lambda_0(V)$.

Further we assume

(A3) $\lambda_0(V)$ is an simple eigenvalue and the corresponding eigenfunction is almost everywhere positive or negative.

We denote the eigenfunction(=ground state) by Ω such that $\|\Omega\|_{L^2(m)} = 1$ and $\Omega > 0$ almost everywhere.

We will define an unitarily equivalent semigroup on $L^2(X, m_\Omega)$ by

$$\hat{T}_t f = \Omega^{-1} e^{t\lambda_0(V)} T_t(f\Omega).$$

Then $\hat{T}_t$ is the symmetric contraction semigroup on $L^2(m_\Omega)$. Let $(\hat{\mathcal{E}}, \hat{\mathrm{D}})$ be the corresponding closed form.

It follows from the definition that

$$\mathrm{D}(\hat{\mathcal{E}}) = \left\{ v\Omega^{-1} \mid v \in \mathrm{D}(\mathcal{E}_V) \right\}, \tag{7}$$

$$\hat{\mathcal{E}}(u,u) = \mathcal{E}_V(u\Omega, u\Omega) - \lambda_0(V)\|u\Omega\|_{L^2(m)}^2 \quad (u \in \mathrm{D}(\hat{\mathcal{E}})). \tag{8}$$

Let

$$\mathcal{D}_\Omega \overset{\mathrm{def}}{=} \left\{ u \in \mathcal{D} \;\middle|\; \int_X \Gamma(u,u) dm_\Omega < \infty \right\}.$$

$$\mathcal{E}_\Omega(u,u) \overset{\mathrm{def}}{=} \int_X \Gamma(u,u) dm_\Omega \ (u \in \mathcal{D}_\Omega). \tag{9}$$

Formally

$$\hat{\mathcal{E}}(u,u) = \mathcal{E}_\Omega(u,u).$$

Concerning this formal identity, we have

Lemma 4 (1) $\Omega \in \mathrm{D}(\mathcal{E})$.

(2) **(I. Shigekawa, 1992)**

$$\int_X \frac{\Gamma(\Omega)}{\Omega^2} dm = \int_X (V - \lambda_0(V)) dm. \tag{10}$$

(3) $(\mathcal{E}_\Omega, \mathcal{D}_\Omega)$ *is a densely defined Markovian symmetric form and the smallest closed extension is* $(\hat{\mathcal{E}}, \mathrm{D}(\hat{\mathcal{E}}))$.

Let

$$\lambda_V = \inf \left\{ \hat{\mathcal{E}}(u,u) \;\middle|\; u \in \mathrm{D}(\hat{\mathcal{E}}) \text{ and } \int_X u dm_\Omega = 0, \|u\|_{L^2(m_\omega)} = 1 \right\}. \tag{11}$$

Then by the definition

$$\lambda_V = \lambda_0(V) - \lambda_1(V). \tag{12}$$

Also by Lemma 4 (3),

$$\lambda_V = \inf \left\{ \mathcal{E}_\Omega(u,u) \;\middle|\; u \in \mathcal{D}_\Omega \text{ and } \int_X u dm_\Omega = 0, \|u\|_{L^2(m_\omega)} = 1 \right\}. \tag{13}$$

§2 Weak Poincare inequality and some estimates on ground state

(WPI) [27] For any $\delta > 0$, there exists a constant $\xi(\cdot) > 0$ such that for any $u \in \mathrm{D}(\mathcal{E})$,

$$\|u - \langle u \rangle_m\|^2_{L^2(m)} \le \xi(\delta)\mathcal{E}(u,u) + \delta \|u\|^2_\infty. \tag{14}$$

Lemma 5 *Assume* (14) *holds. Let* $\varphi \in \mathrm{D}(\mathcal{E})$ *and assume that* $\varphi > 0$ *a.e. and* $\Gamma(\varphi)/\varphi^2 \in L^1(m)$. *For* $u \in \mathrm{D}(\mathcal{E}) \cap L^\infty(m)$, *let*

$$\mathcal{E}_\varphi(u,u) = \int_X \Gamma(u) dm_\varphi. \tag{15}$$

Then for any $r > 0$, $\varepsilon > 0$, $K > 0$, $\delta > 0$

$$\|u - \langle u \rangle_{m_\varphi}\|^2_{L^2(m_\varphi)} \le \frac{\xi(\delta)K^4}{\varepsilon^2}(1+r)^2 \mathcal{E}_\varphi(u,u) + \zeta_\varphi(r,\varepsilon,K,\delta)\|u\|^2_\infty, \tag{16}$$

where

$$\begin{aligned} &\zeta_\varphi(r,\varepsilon,K,\delta) \\ &\quad = K^4 \Big\{ (1+r)(1+\frac{1}{r})\xi(\delta) \int_{\{\varphi \le \varepsilon\}} \frac{\Gamma(\varphi)}{\varphi^2} dm + (1+r)\delta + (1+\frac{1}{r}) m(\varphi \le \varepsilon) \Big\} \\ &\quad + 4 \int_{\{\varphi \ge K\}} \varphi^2 dm \end{aligned} \tag{17}$$

and

$$\inf \{ \zeta_\varphi(r,\varepsilon,K,\delta) \mid \varepsilon > 0, K > 0, \delta > 0 \} = 0. \tag{18}$$

Lemma 6 (1) *Let f be a C^1-function on $\mathbb{R}$ with compact support. Then it holds that*

$$\int_X f(\frac{1}{\Omega})^2 \frac{\Gamma(\Omega)}{\Omega^2} dm + 2\int_X f(\frac{1}{\Omega}) f'(\frac{1}{\Omega}) \frac{\Gamma(\Omega)}{\Omega^3} dm = \int_X (V - \lambda_0(V)) f(\frac{1}{\Omega})^2 dm. \tag{19}$$

(2) *For $R \geq 0$,*

$$\int_{\{\Omega^{-1} \geq R\}} \frac{\Gamma(\Omega)}{\Omega^2} dm \leq \int_{\{\Omega^{-1} \geq R\}} (V - \lambda_0(V)) dm \tag{20}$$

(3) *Assume (WPI) holds. Let*

$$p_\Omega = m(\Omega \leq e^{-1}), \tag{21}$$

$$n_\Omega = \left[\frac{4p_\Omega}{1-p_\Omega}\right] + 1, \tag{22}$$

$$\gamma_\Omega = e^{-n_\Omega}, \tag{23}$$

where $[x]$ denotes the greatest integer less than or equal to x. Then for $S \geq \exp(\exp(n_\Omega))$ and $\delta > 0$,

$$\begin{aligned} m(\Omega^{-1} \geq S) &\leq 2(1-p_\Omega)^{-1} \left(\xi(\delta) \gamma_\Omega^{-2} (\log S)^{-2} \int_{\{\Omega^{-1} \geq S^{\gamma_\Omega}\}} \frac{\Gamma(\Omega)}{\Omega^2} dm + \delta \right) \\ &\leq 2(1-p_\Omega)^{-1} \left(\xi(\delta) \gamma_\Omega^{-2} (\log S)^{-2} \int_{\{\Omega^{-1} \geq S^{\gamma_\Omega}\}} |V - \lambda_0(V)| dm + \delta \right). \end{aligned} \tag{24}$$

(4) *Assume that $\mathcal{E}$ satisfies Poincaré's inequality and $V \in L^p(m)$ $(p > 1)$. Then $\log \Omega \in L^q(X, m)$ for $1 \leq q < 2p$ and $\log \Omega \in \mathrm{D}(\mathcal{E})$.*

Note When $X = L_x(M)$, $m =$ pinned measure and M is a hyperbolic space, we can prove that for sufficietnly small δ,

$$\xi(\delta) = C_1 \log(\frac{C_2}{\delta}) + C_3.$$

§3 Main estimate

We already proved that if WPI holds for $\mathcal{E}$, then it holds for $\mathcal{E}_\Omega$ too. So it suffices to prove some Sobolev type (or weaker certain inequality) for $\mathcal{E}_\Omega$ to prove $\lambda_V > 0$. To this end, we assume that

(A4) $V \in L^2(m)$ and for any $p \geq 1$, $\|e^{V_-}\|_{L^p(m)} < \infty$, where $V_-(x) = \max(-V(x), 0)$

and

(A5) There exists $\alpha > 0$ such that for any $u \in \mathcal{D}$,

$$\int_X u^2 \log(u^2/\|u\|^2_{L^2(m)})dm \le \alpha \mathcal{E}_V(u,u). \tag{25}$$

Then note that (A4) and (A5) implies (A2). So we can define $\mathcal{E}_V, L_V, T_t$.
Further we assume that

(A6) There exists a symmetric diffusion process X^x_t which corresponds to the diffusion semigroup P_t. x denotes the starting point.

Lemma 7 *Assume that $(\mathcal{E}, \mathrm{D}(\mathcal{E}))$ is irreducible and $(A4), (A5), (A6)$ hold. Then*
(1) *There exists a unique ground state Ω satisfying (A3).*
(2) *Ω has the following estimate.*

$$\|\Omega\|_{L^4} \le \|e^{t_\alpha (V_- + \lambda_0(V))}\|_{L^8}, \tag{26}$$

where $t_\alpha = \frac{\alpha \log 13}{4}$.

We will apply the following to the case where $\varphi = \Omega$.

Lemma 8 *Let $\varphi \in L^2(X,m)$ be a positive measurable function. Let u be a measurable function on X such that*

$$\int_X (u(x)\varphi(x))^2 \log (u(x)\varphi(x))^2 \, dm(x) \le C, \tag{27}$$

where C is a positive number. Then it holds that for any $\eta > 0, S > 1, R > \eta^{-1}$

$$\begin{aligned} &\int_{\{|u| \ge R\}} u^2(x) dm_\varphi(x) dm \\ &\quad \le \frac{1}{2}\left(\frac{1}{\log(R\eta)} + \frac{1}{\log S}\right)(C + e^{-1}) + S^2 m(\varphi \le \eta) \end{aligned} \tag{28}$$

We will apply the following by replacing m by m_Ω.

Lemma 9 *Let $u \in L^2(X,m)$ and assume that $\langle u \rangle_m = 0$. Let $R > 0$ and ψ_R be the function such that $\psi_R(t) = t$ for $-R \le t \le R$, $\psi_R(t) = R$ for $t \ge R$ and $\psi_R(t) = -R$ for $t \le -R$. Then it holds that*

$$\|u\|^2_{L^2(m)} \le \|\psi_R(u) - \langle \psi_R(u) \rangle_m\|^2_{L^2(m)} + \left(1 + \frac{1}{R^2}\right) \int_{\{|u| > R\}} (u^2 - R^2) dm. \tag{29}$$

For any $0 < r < 1$, let

$$f_r(\varepsilon, \delta, K) \stackrel{\text{def}}{=} K^4 \Big\{ r^{-1}(1+r)(1+r^{-1})\xi(\delta)\|V - \lambda_0(V)\|_{L^2} \cdot m\,(\Omega \leq \varepsilon)^{1/2} + (1+r)\delta + (1+r^{-1})m(\Omega \leq \varepsilon) \Big\} + 4C_V m(\Omega \geq K)^{1/2}. \tag{30}$$

Note that $\zeta_\Omega(r, \varepsilon, K, \delta) \leq f_r(\varepsilon, \delta, K)$ and

$$\inf \{ f_r(\varepsilon, \delta, K) \mid \varepsilon > 0, \delta > 0, K > 0 \} = 0. \tag{31}$$

Let

$$h_r(S, \eta, R, \varepsilon, \delta, K) \stackrel{\text{def}}{=} 1 - f_r(\varepsilon, \delta, K)R^2 - (1 + \frac{1}{R^2}) \left\{ \frac{1}{2} \left(\frac{1}{\log(R\eta)} + \frac{1}{\log S} \right) (\alpha\lambda_0(V) + e^{-1}) + S^2 m(\Omega \leq \eta) \right\}. \tag{32}$$

Then

$$\sup \Big\{ h_r(S, \eta, R, \varepsilon, \delta, K) \mid S > 1, \eta > 0, R > \eta^{-1}, \varepsilon > 0, \delta > 0, K > 0 \Big\} > 0. \tag{33}$$

Theorem 10 (Main estimate) *Recall the standing assumptions* (A1)–(A5). *Then let*

$$\tilde{\lambda}_V = \sup \left\{ \frac{h_r(S, \eta, R, \varepsilon, \delta, K)}{k_r(S, \eta, R, \varepsilon, \delta, K)} \;\middle|\; 0 < r < 1, S > 1, \eta > 0, R > \eta^{-1}, \varepsilon > 0, \delta > 0, K > 0 \right\}, \tag{34}$$

where

$$k_r(S, \eta, R, \varepsilon, \delta, K) = \frac{(1+r)^2 \xi(\delta)}{\varepsilon^2} K^4 + \frac{\alpha}{2} \left(1 + \frac{1}{R^2} \right) \left(\frac{1}{\log(R\eta)} + \frac{1}{\log S} \right). \tag{35}$$

Then it holds that

$$\lambda_V \geq \tilde{\lambda}_V > 0. \tag{36}$$

Proof of Main theorem:

Let $u \in \mathcal{D}_\Omega$ be a function such that $\|u\|_{L^2(m_\Omega)} = 1$, $\langle u \rangle_{m_\Omega} = 0$. By Lemma 4 , $u\Omega \in \mathrm{D}(\mathcal{E}_V)$ and

$$\mathcal{E}_V(u\Omega, u\Omega) = \int_X \Gamma(u) dm_\Omega + \lambda_0(V)\|u\Omega\|^2_{L^2(m)}. \tag{37}$$

We write $\lambda = \mathcal{E}_\Omega(u,u)$ for simplicity. By the LSI (25), we have

$$\int_X (u\Omega)^2 \log\left((u\Omega)^2\right) dm \leq \alpha(\lambda + \lambda_0(V)). \tag{38}$$

Hence by Lemma 8 for $\eta > 0$,$S > 1$ and $R > \eta^{-1}$,

$$\int_{\{|u|\geq R\}} u^2(x) dm_\Omega \leq \frac{1}{2}\left(\frac{1}{\log(R\eta)} + \frac{1}{\log S}\right)\left(\alpha(\lambda + \lambda_0(V)) + e^{-1}\right) + S^2 m(\Omega \leq \eta). \tag{39}$$

Let ψ_R be the function which was defined in Lemma 9. We have

$$\begin{aligned}
1 &\leq \|\psi_R(u) - \langle \psi_R(u)\rangle_{m_\Omega}\|^2_{L^2(m_\Omega)} + \left(1 + \frac{1}{R^2}\right)\int_{\{|u|>R\}} u^2 dm_\Omega \text{ (by Lemma 9)} \\
&\leq \frac{\xi(\delta)K^4}{\varepsilon^2}(1+r)^2\mathcal{E}_\Omega(\psi_R)(u), \psi_R(u)) + f_r(\varepsilon,\delta,K)R^2 \\
&\quad +(1+\frac{1}{R^2})\left\{\frac{1}{2}\left(\frac{1}{\log(R\eta)} + \frac{1}{\log S}\right)\cdot\left(\alpha(\lambda+\lambda_0(V)) + e^{-1}\right) + S^2 m(\Omega \leq \eta)\right\} \\
&\quad \text{(by Lemma 5, (39))} \\
&\leq k_r(S,\eta,R,\varepsilon,\delta,K)\lambda + 1 - h_r(S,\eta,R,\varepsilon,\delta,K).
\end{aligned} \tag{40}$$

■

Note: To obtain $\lambda_V > 0$, it suffices to show that for some $0 \leq b < 1$ and $a > 0$ $R > 0$, it holds that for any $u \in \mathcal{D}_\Omega$ with $\langle u \rangle_{m_\Omega} = 0$ and $\|u\|_{L^2(m_\Omega)} = 1$,

$$\int_{|u|\geq R} \left(u^2 - R^2\right) dm_\Omega \leq a\mathcal{E}_\Omega(u,u) + b. \tag{41}$$

Note that this inequality is necessary condition for $\lambda_V > 0$, that is the validity of the Poincare inequality for $\mathcal{E}_\Omega$. This might be a infinitesimal version of Hino's condition (I): There exists $t > 0$ and $K > 0$ such that

$$\sup\left\{\|(T_t^\Omega u - K)_+\|_{L^2} \mid \|u\|_{L^2(m_\Omega)} = 1\right\} < 1.$$

§4 Schrödinger operator on Wiener space

(X, H, m): an abstract Wiener space.

$$\mathcal{E}(u,u) \stackrel{\text{def}}{=} \int_X |Du(x)|_H^2 d\mu(x), \tag{42}$$

$$\mathbf{D}(\mathcal{E}) \stackrel{\text{def}}{=} \mathbb{D}_2^1(X) \tag{43}$$

L : generator of $\mathcal{E}$ (=Ornstein-Uhlenbeck operator)
The following inequalities hold.
(1) Gross' LSI

$$\int_X u^2 \log(u^2/\|u\|_{L^2(m)}^2) dm \leq 2\mathcal{E}(u,u)$$

(2) Poincaré inequality

$$\int_X (u - \langle u\rangle_m)^2 dm \leq \mathcal{E}(u,u)$$

Let $U \in L^2(m)$. Assume
(A7) It holds that $U \in L^2(m)$ and for any $p > 1$,

$$E[e^{pU_-}] < \infty. \tag{44}$$

Then we can prove that

Proposition 11 *For $\rho > 1$ and any $u \in \mathcal{D}$, it holds that*

$$\int_X u^2 \log(u^2/\|u\|_{L^2(m)}^2) dm \leq \frac{2\rho}{\rho-1}\mathcal{E}_V(u,u), \tag{45}$$

where

$$V(x) = U(x) + \log\|e^{-U}\|_{L^{2\rho}}. \tag{46}$$

We can apply main theorem to the operator L_V.
Note that $\lambda_V = \lambda_1(U) - \lambda_0(U)$.

Lemma 12 *Let $0 < r < 1$ and $p > 0$. Suppose that for some $q > \max(p, \frac{1}{2})\log(16p+1)$,*

$$E[e^{qV}] < \infty. \tag{47}$$

Then it holds that

$$\|\Omega^{-1}\|_{L^p}^p \leq \gamma_{p,q,r}, \tag{48}$$

where $t_{p,q} = \min(q, \frac{q}{2p})$ *and*

$$\gamma_{p,q,r} = \left\{(1-r)^{1/2} D_{r,V}^4\right\}^{-p} \left\{\frac{e^{2t_{p,q}}-1}{e^{2t_{p,q}}-1-16p}\right\}^{1/4} \cdot \exp\left(\frac{4pF^{-1}(D_{r,V})^2}{e^{2t_{p,q}}-1}\right) \|e^{V-\lambda_0(V)}\|_{L^q}^{pt_{p,q}} \tag{49}$$

$$F(x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du \tag{50}$$

$$D_{r,V} = 1 - \sqrt{C_{r,V}-1} = 1 - \sqrt{\frac{\|e^{t_\alpha(V_-+\lambda_0(V))}\|_{L^8}^4 - 1}{\|e^{t_\alpha(V_-+\lambda_0(V))}\|_{L^8}^4 - 1 + r^2}} \tag{51}$$

$$C_{r,V} = \left(1 + r^{-2}(C_V^2 - 1)\right)^{-1}. \tag{52}$$

Corollary 13 *Let* $p \geq 1/2$ *and* $q > p\log(16p+1)$. *Assume that* $E[e^{qU}] < \infty$. *Then*

$$\lambda_U \geq \frac{5}{8}\Bigg[\frac{e}{2} + C_{p,q}\exp\left(512\lambda_0(V)\left(1+\frac{2}{p}\right)\left(1+\frac{5}{p}\right)\right)\left(\frac{3}{2}\|V-\lambda_0(V)\|_{L^2}+1\right)^{4/p} \cdot \|e^{t_4(V_-+\lambda_0(V))_-}\|_{L^8}^{2\left(1+\frac{4}{p}\right)\left(84+\frac{1}{e^{q/p}-1}\right)} \|e^{V-\lambda_0(V)}\|_{L^q}^{\frac{5q}{p}(1+\frac{4}{p})}\Bigg]^{-1}, \tag{53}$$

where

$$C_{p,q} = \frac{9}{4} \cdot 2^{5(1+\frac{4}{p})(25+\frac{64}{e^{q/p}-1})} \cdot 2^{\frac{32}{p}(\frac{5}{p}+1)} (48)^{4/p} (64)^{4+\frac{16}{p}} \exp\left(128(1+\frac{2}{p})(1+\frac{5}{p})e^{-1}\right) \cdot \left(\frac{e^{q/p}-1}{e^{q/p}-1-16p}\right)^{\frac{5}{2p}(1+\frac{4}{p})} \tag{54}$$

and $V(x) = U(x) + \log\|e^{-U}\|_{L^4}$ *and* $\lambda_0(V)$ *has the upper bound*

$$\lambda_0(V) \leq \|U + \log\|e^{-U}\|_{L^4}\|_{L^1}. \tag{55}$$

§5 Remarks on $-\Delta + V$ on $L^2(\mathbb{R}^n, dx)$

Let

$$A_V = -\Delta + V, \tag{56}$$

$$V \stackrel{\text{def}}{=} \frac{|x|^2}{4} - \frac{n}{2} + U(x). \tag{57}$$

Let $\varphi_0 = \frac{1}{(2\pi)^{n/4}} e^{-\frac{|x|^2}{4}}$ and set $dm = \varphi_0^2 dm$. Assume that

(A8) $U \in L^p(m)$ for some $p > 2$ and $e^{U_-} \in L^{\infty-}(m)$.

Then A_V is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

Let us consider the finite dimensional case, $X = \mathbb{R}^n$, $dm = \varphi_0^2 dx$. Then $L = \Delta - x \cdot \nabla$.

Theorem 14 *$L_U = L - U$ and A_V are unitarily equivalent by the transformation*

$$T_{\varphi_0} \; : \; L^2(\mathbb{R}^n, dm) \to L^2(\mathbb{R}^n, dx)$$

by $T_{\varphi_0} u = \varphi_0 \cdot u$.

So we can apply our results to estimate the gap of spectrum of A_V.
(e.g. $-\Delta + |x|^a$, $a \geq 2$).
Let Ω, φ be the ground states of L_U and A_V respectively. Then

$$\varphi = \varphi_0 \Omega.$$

NOTE (On relations integrability of Ω^{-1} and the growth order of V)
(1) Lemma 6 (4) and WKB approximation
Consider A_V on $\mathbb{R}^1$. Assume $V(x) = \frac{|x|^2}{4} + U(|x|)$ and $U(x) \geq |x|^a$, where $a > 2$. Formally by the WKB approximation,

$$\begin{aligned} \varphi(x) &\sim C \cdot (V(x) - \lambda_0(V))^{-1/4} \exp\left(-W(x)\right) \qquad (|x| \to \infty) && (58) \\ W(x) &= \int_0^{|x|} \left(\sqrt{V(t)} - \frac{\lambda_0(V)}{2\sqrt{V(t)}} \right) dt && (59) \end{aligned}$$

Then

$$\begin{aligned} \Omega^{-1} &= \varphi_0 \varphi^{-1} \\ &\sim C \cdot \exp\left(-\frac{x^2}{4}\right) \exp\left(\int_0^{|x|} \frac{t}{2} \sqrt{1 + \frac{4}{t^2} U(t) dt} \right) && (60) \end{aligned}$$

Note that there exists $0 < C < 1$ and for any $t > 0$,

$$\frac{t}{2} + C_1 \sqrt{U(t)} \leq \frac{t}{2} \sqrt{1 + \frac{4}{t^2} U(t)} \leq \frac{t}{2} + \sqrt{U(t)}. \tag{61}$$

So

$$\log \Omega^{-1}(x) \sim \int_0^{|x|} \sqrt{U(t)} dt. \tag{62}$$

Thus if $U \in L^p(m)$, then $\log \Omega^{-1} \in L^{2p-}(m)$.

(2) When $V(x) = |x|^a$ $(a > 2)$,

$$\varphi(x) \leq C_1 \exp\left(-C_2 |x|^{1+\frac{\alpha}{2}}\right). \tag{63}$$

So

$$\Omega(x)^{-1} \geq C_1^{-1}\varphi_0(x)\exp\left(-C_2|x|^{1+\frac{\alpha}{2}}\right). \tag{64}$$

This shows that $\Omega^{-1} \notin L^p(m)$ for any $p > 0$.

§6 Stability Property of WPI under Connected Sum of State Spaces

- $(X, \mathfrak{B}, m)$ a probability space
- $(\mathcal{E}, \mathrm{D}(\mathcal{E}))$ Dirichlet space on $L^2(X, m)$.
- Γ carré du champ

Proposition 15 *Let $X_1, X_2 \subset X$ and assume that $m(Y_3) > 0$ where $Y_3 = X_1 \cap X_2$. Set $X_3 = X_1 \cup X_2$. Also assume that there exist functions $\xi_i(\cdot)$ $(i = 1, 2)$ on $\mathbb{R}^+$ such that for any $u \in \mathrm{D}(\mathcal{E})$ it holds that*

$$\left\| u - \frac{1}{m(X_i)}\int_{X_i} udm \right\|^2_{L^2(X_i,m)} \leq \xi_i(\delta)\int_{X_i}\Gamma(u,u)dm + \delta\|u\|^2_{L^\infty(X_i)}. \tag{65}$$

Then it holds that

$$\left\| u - \frac{1}{m(X_3)}\int_{X_3} udm \right\|^2_{L^2(X_3)} \leq C_1(\delta)\int_{X_3}\Gamma(u,u)dm + C_2(\delta)\|u\|^2_{L^\infty(X_3)}, \tag{66}$$

where

$$\begin{aligned} C_1(\delta) &= m(X_3)^{-1}\Big\{2m(X_1)\xi_1(\delta)(4m(Y_2)m(Y_3)^{-1}+1) \\ &\quad +2m(X_2)\xi_2(\delta)(4m(Y_1)m(Y_3)^{-1}+1)\Big\} \qquad (67) \\ C_2(\delta) &= 2\delta m(X_3)^{-1}\Big\{m(X_1)\left(4m(Y_2)m(Y_3)^{-1}+1\right) \\ &\quad +m(X_2)\left(4m(Y_1)m(Y_3)^{-1}+1\right)\Big\}. \qquad (68) \end{aligned}$$

Let us apply Proposition 15 to a diffusion process on Wiener space. Let U be an connected open set in abstract Wiener space (B, H, μ). Let us consider the following bilinear form.

$$\mathcal{E}_U(u,u) = \int_U |Du(x)|^2 d\mu, \tag{69}$$

where $u \in \mathfrak{F}C_b^\infty$ and Du denotes the H-derivative of u. Kusuoka proved that this is a closable Markovian form. Let us consider the smallest closed extension Dirichlet form $\mathcal{E}_U$ on $L^2(U, \mu)$. Then we have

Corollary 16 *WPI holds for $\mathcal{E}_U$.*

Remark 17 Let us consider a ball.

$$B_\varepsilon(x) := \{z \in B \mid \|z - x\| < \varepsilon\}.$$

On $B_\varepsilon(x)$, LSI and Poincare inequality holds for $\mathcal{E}_{B_\varepsilon(x)}$.

Remark 18 Kusuoka proved that WPI holds for H-connected domain U with a certain property. Also the domain of the Kusuoka's Dirichlet form is larger than the above. Our assumption is quite stronger than his and cannot be applied to loop space case!

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