Asymptotic behavior for traces of Hecke operators for holomorphic and quaternionic cusp forms

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Contents

1. Holomorphic Siegel cusp forms

Let $n \in \mathbb{N}$ and suppose $n \geq 2$.

- $J_{2n} \coloneqq \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$.
- $\underline{G} \coloneqq \mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid g J_{2n} {}^t g = J_{2n}\}$.
- $\mathfrak{H}_n \coloneqq \{Z \in \mathrm{M}_n(\mathbb{C}) \mid {}^t Z = Z,\, \mathrm{Im}(Z) > 0\}$.
- $G \coloneqq \underline{G}(\mathbb{R})$. K is a max. cpt. subgroup pf G ($K \simeq \mathrm{U}_n$).
- $Z \in \mathfrak{H}_n$, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$,
  G acts on $\mathfrak{H}_n$ as $g \cdot Z = (AZ + B)(CZ + D)^{-1}$. $(G/K \simeq \mathfrak{H}_n)$
  Set $J_k(g, Z) \coloneqq \det(CZ + D)^{-k}$.
- Γ is an arithmetic congruence subgroup in $\underline{G}(\mathbb{Q})$.
- Fix a G-invariant measure $\mathrm{d}Z$ on $\mathfrak{H}_n$.

A holom. fun. $f\colon \mathfrak{H}_n \to \mathbb{C}$ is called a Siegel cusp form of wt. k w.r.t. Γ if f satisfies

- $\forall \gamma \in \Gamma$, $\forall Z \in \mathfrak{H}_n$, $f(Z) = J_k(\gamma, Z)\, f(\gamma \cdot Z)$.
- $\sup\limits_{Z \in \mathfrak{H}_n} |\det(\mathrm{Im}(Z))^{k/2} f(Z)| < \infty$.

Let $S_k(\Gamma)$ denote the space of Siegel cusp forms of wt k w.r.t. Γ.

Suppose $-I_{2n} \in \Gamma$ for simplicity.

When n is odd, we also suppose k is even. ($S_k(\Gamma) = 0$ if k is odd.)

$\boxed{\text{Hecke operator}}$ $\alpha \in \underline{G}(\mathbb{Q})$, $T_{\Gamma\alpha\Gamma}\colon S_k(\Gamma) \to S_k(\Gamma)$,

$$(T_{\Gamma\alpha\Gamma} f)(Z) \coloneqq \sum_{\beta \in \Gamma\backslash\Gamma\alpha\Gamma} J_k(\beta, Z)\, f(\beta \cdot Z).$$

$d_n \coloneqq \dim_{\mathbb{R}} \mathfrak{H}_n = n^2 + n$.

$d(n,k) \coloneqq c \times \prod_{i=1}^{n} \prod_{j=i}^{n} (2k - i - j)$.

(formal degree, $c > 0$ is a constant depends only on $\mathrm{d}Z$)

$d(n,k) \asymp k^{\frac{d_n}{2}}$ $(k \to \infty)$.

$d(n,k) \asymp k^{\frac{d_n}{2}}$ $(k \to \infty)$.

$\delta_{\Gamma\alpha\Gamma} = 1$ when $I_{2n} \in \Gamma\alpha\Gamma$, and $\delta_{\Gamma\alpha\Gamma} = 0$ otherwise.

Theorem (Sugiyama-Tsuzuki-W.)

For any small $\varepsilon > 0$ we have (w.r.t. $k \to \infty$)

$$\mathrm{tr}(T_{\Gamma\alpha\Gamma}|_{S_k(\Gamma)}) = \mathrm{vol}(\Gamma\backslash\mathfrak{H}_n)\, d(n,k)\, \delta_{\Gamma\alpha\Gamma} + O_\varepsilon(k^{\frac{d_n - n}{2}+\varepsilon}\, \#(\Gamma\backslash\Gamma\alpha\Gamma)).$$

Remarks.

1. This theorem is proved by a result of Finis-Matz in the study of Weyl's law for the split reductive algebraic groups.
2. Dalal's asymptotic formula does not cover our result.
   Holomorphic discrete series of G with minimal K-type $\tau(k)$.
   His situation: $\dim\tau(k) \to \infty$ $(k \to \infty)$.
   Our situation: $\tau(k) = \det^k$, $\dim\tau(k) = 1$ $(k \to \infty)$.
   In addition, our remainder is better than his.
3. An asymptotic formula for $\mathrm{tr}(T_{\Gamma\alpha\Gamma}|_{S_k(\Gamma)})$ w.r.t. the level aspect of principal congruence subgroups was obtained by Kim-W.-Yamauchi.

Applications to families of automorphic representations

1. Plancherel density theorem (Delorme, Sauvageot, Shin).
2. Sato-Tate equidistribution theorem (Shin-Templier).
3. Low lying zeros (Katz-Sarnak philosophy, Shin-Templier).

   To obtain this, one must prove that the contribution of L-functions with poles is neglected. Kim-W.-Yamauchi proved this for Siegel cusp forms w.r.t. the level aspect (square free) using the endoscopic classification.

   A reminder like ours is needed to determine an explicit range of counting zeros.
4. Central limit theorem of Hecke eigenvalues

   Kim-W.-Yamauchi recently obtained this application in general by using the Sato-Tate equidistribution theorem. This is a generalization of Nagoshi's theorem (elliptic cusp forms).

Take a finite set S of primes and an open cpt. subgroup K_S of $\prod_{p\in S}\underline{G}(\mathbb{Q}_p)$ such that $\Gamma=\underline{G}(\mathbb{Q})\cap(K_S\prod_{p\notin S}\underline{G}(\mathbb{Z}_p))$.

Consider only prime numbers p satisfying $p\notin S$.

$\mathcal{F}_k$ is the set of Hecke eigenforms outside S, that forms a basis of $S_k(\Gamma)$.

$L_f(s,\mathrm{St})$ is the standard L-function of f (degree $2n+1$).

Set $L_f(s,\mathrm{St})=\sum_{n=1}^{\infty}a_f(n)\,n^{-s}$.

Theorem (Kim-W.-Yamauchi, the central limit formula)

Let h be a continuous bounded function on $\mathbb{R}$. If $\frac{\log k}{\log x}\to\infty$ as $x\to\infty$, then

$$\lim_{x\to\infty}\frac{1}{\dim S_k(\Gamma)}\sum_{f\in\mathcal{F}_k}h\left(\frac{\sum_{p\le x}a_f(p)}{\sqrt{\pi(x)}}\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h(t)\,e^{-\frac{t^2}{2}}\,\mathrm{d}t.$$

This means that Hecke eigenvalues behave like Random variables.

Proof

$$f_{n,k}(g^{-1}) \coloneqq d(n,k) \times 2^{nk} \det(A - iB + iC + D)^{-k}, \quad g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G.$$

Godement's formula When $k > 2n$, we have

$$\operatorname{tr}(T_{\Gamma\alpha\Gamma}|_{S_k(\Gamma)}) = \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma\alpha\Gamma} f_{n,k}(g^{-1}\gamma g)\, \mathrm{d}g$$

where $\mathrm{d}g$ (resp. $\mathrm{d}u$) is a Haar measure on G (resp. K) s.t. $\mathrm{d}g = \mathrm{d}Z\,\mathrm{d}u$ (resp. $\mathrm{vol}(K) = 1$).

$\boxed{\text{Some basic facts}}$ $k > n$.

- $f_{n,k}$ is a matrix coefficient of the holom. disc. ser. of G with the min. K-type $\det^k$. If $k > 2n$, then $f_{n,k}$ is integrable on G.
- $f_{n,k}$ is a cusp form in $L^2(G)$, that is, $\displaystyle\int_N f_{n,k}(x^{-1}ny)\,\mathrm{d}n = 0$ for any x, $y \in G$ and any unip. rad N.
- Uniform upper bound (Cogdell-Luo)
  $\exists c > 0$ s.t. $f_{n,k}(g) \le d(n,k) \times (1 + c(\|g\|^2 - 2n))^{-\frac{k}{2}}$,
  where $\|g\| = \mathrm{tr}(g\,{}^t g)^{\frac{1}{2}}$. If dist is the distance on $\mathfrak{H}_n = G/K$, for any cpt subset C of G we have
  $\mathrm{dist}(g \cdot iI_n, iI_n)^2 \ll_C \|g\|^2 - 2n \ll_C \mathrm{dist}(g \cdot iI_n, iI_n)^2$ for $\forall g \in C$.
- Semi-norm Suppose k is sufficiently large.
  Let $U(\mathfrak{g})$ denote the univ. envel. alg. of $\mathfrak{g} := \mathrm{Lie}(G)$.
  Fix a basis $\mathfrak{X}$ of the subspace $\{X \in U(\mathfrak{g}) \mid \deg X \le \dim G\}$.
  $\exists m \in \mathbb{N}$ s.t. $\displaystyle\sum_{X \in \mathfrak{X}} \|X * f_{n,k}\|_{L^1} \ll k^m$.

Recall $\underline{G} \coloneqq \mathrm{Sp}_{2n}$.

K_f is an open compact subgroup of $\underline{G}(\mathbb{A}_f)$ s.t. $\Gamma = \underline{G}(\mathbb{Q}) \cap K_f$.

Take a Haar measure on $\underline{G}(\mathbb{A}_f)$ as $\mathrm{vol}(K_f) = 1$.

Set

$$J_{\mathrm{nc}}(f_{n,k} \otimes h) \coloneqq \int_{\underline{G}(\mathbb{Q})\backslash \underline{G}(\mathbb{A})} \sum_{\gamma \in \underline{G}(\mathbb{Q}) \setminus \{\pm I_{2n}\}} (f_{n,k} \otimes h)(x^{-1}\gamma x)\, \mathrm{d}x$$

where $h \in C_c^\infty(K_f \backslash \underline{G}(\mathbb{A}_f)/K_f)$.

h_α is the characteristic function of $K_f \alpha^{-1} K_f$.

Then, Godement's formula is rewritten as

$$\mathrm{tr}(T_{\Gamma\alpha\Gamma}|_{S_k(\Gamma)}) = \mathrm{vol}(\Gamma\backslash\mathfrak{H}_n)\, d(n,k)\, \delta_{\Gamma\alpha\Gamma} + J_{\mathrm{nc}}(f_{n,k} \otimes h_\alpha).$$

Hence, it is sufficient to prove $J_{\mathrm{nc}}(f_{n,k} \otimes h) = O_\varepsilon(k^{\frac{d_n - n}{2} + \varepsilon}\, \|h\|_{L^1})$.

T is a truncation parameter. $F(\cdot, T)$ is the characteristic function of the truncation of $\underline{G}(\mathbb{Q})\backslash\underline{G}(\mathbb{A})$ at height T.

$$J_1^T(f) \coloneqq \int_{\underline{G}(\mathbb{Q})\backslash\underline{G}(\mathbb{A})} (1 - F(x,T)) \sum_{\gamma \in \underline{G}(\mathbb{Q}) \backslash \{\pm I_{2n}\}} f(x^{-1}\gamma x)\,\mathrm{d}x,$$

$$J_2^T(f) \coloneqq \int_{\underline{G}(\mathbb{Q})\backslash\underline{G}(\mathbb{A})} F(x,T) \sum_{\gamma \in \underline{G}(\mathbb{Q}) \backslash \{\pm I_{2n}\}} f(x^{-1}\gamma x)\,\mathrm{d}x.$$

Take a function $\phi \in C_c^\infty([0,\infty))$, $0 \le \phi \le 1$, and $\phi|_{[0,1]} = 1$.
A spherical function $F^{\phi,\delta}$ on G is defined by
$F^{\phi,\delta}(g) \coloneqq \phi(\delta^{-1}(\|g\|^2 - 2n))$ for $\delta > 0$.

$$\begin{aligned} J_{\mathrm{nc}}(f_{n,k} \otimes h) = {} & J_1^T(f_{n,k} \otimes h) && x \text{ is close to cusps} \\ & + J_2^T(f_{n,k}(1 - F^{\phi,\delta}) \otimes h) && \text{middle part} \\ & + J_2^T(f_{n,k} F^{\phi,\delta} \otimes h) && x^{-1}\gamma x \text{ is close to } iI_n. \end{aligned}$$

Convergence theorem of Finis-Lapid

Since $f_{n,k}$ is a cusp form, $J_1^T(f_{n,k} \otimes h)$ is rewritten as a modified kernel. Hence we can apply the conv. thm. of Finis-Lapis, and then $\exists r' \in \mathbb{N}$ s.t. for $d(T) \gg 1$,

$$J_1^T(f_{n,k} \otimes h) \ll (1 + d(T))^{r'} \, e^{-d(T)} \sum_{X \in \mathfrak{X}} \|X * (f_{n,k} \otimes h)\|_{L^1}.$$

We use the property of the semi-norm of $f_{n,k}$ and then

$$J_1^T(f_{n,k} \otimes h) \ll (1 + d(T))^{r'} \, e^{-d(T)} \, k^m \, \|h\|_{L^1}.$$

Fix a number $k_0 \gg 1$. Then

$$\begin{aligned} J_2^T(f_{n,k}(1 - F^{\phi,\delta}) \otimes h) &\le d(n,k) \, (1 + c\,\delta)^{\frac{-k+k_0}{2}} \, J_2^T(|f_{n,k_0}| \otimes |h|) \\ &\ll k^{\frac{d}{2}} \, (1 + c\,\delta)^{-\frac{k}{2}} \, (1 + d(T))^{r'} \, \|h\|_{L^1}. \end{aligned}$$

For the first inequality, we used the uniform bound of $f_{n,k}$.

For the second inequality, we applied the conv. thm. of Finis-Lapid.

Upper bound of Finis-Matz

Applying the uniform bound of $f_{n,k}$,

$$J_2^T(f_{n,k}F^{\phi,\delta} \otimes h) \ll k^{\frac{d}{2}} \, J_2^T(F^{\phi,\delta} \otimes |h|).$$

Then, by the upper bound of Finis-Matz

$$J_2^T(F^{\phi,\delta} \otimes |h|) \ll \delta^{\frac{n}{2}}(|\log \delta|^{n-1} + (1+d(T))^n) \, \|h\|_{L^1}.$$

Obtaining this bound of $J_2^T(F^{\phi,\delta} \otimes |h|)$ is technically the most difficult.

Conclusion

Set $d(T) = \left(m - \frac{d}{2} + \frac{n}{2}\right) \log k$ and $c\delta = k^{-1+2\varepsilon}$. Then,

$$J_1^T(f_{n,k} \otimes h) \ll_\varepsilon k^{\frac{d-n}{2}+\varepsilon} \|h\|_{L^1},$$
$$J_2^T(f_{n,k}F^{\phi,\delta} \otimes h) \ll_\varepsilon k^{\frac{d-n}{2}+(n+1)\varepsilon} \|h\|_{L^1}.$$

Since $(1+k^{-1+2\varepsilon})^{-\frac{k}{2}} \ll_\varepsilon k^{-\frac{n}{2}}$ (Cogdell-Luo),

$$J_2^T(f_{n,k}(1-F^{\phi,\delta}) \otimes h) \ll_\varepsilon k^{\frac{d}{2}+\varepsilon} (1+k^{-1+2\varepsilon})^{-\frac{k}{2}} \, \|h\|_{L^1} \ll_\varepsilon k^{\frac{d-n}{2}+\varepsilon} \|h\|_{L^1}.$$

Generality

$\underline{G}$ is a conn. s.s. alg. group over $\mathbb{Q}$.

$\{\sigma(k)\}_{k\gg 1}$ is a sequence of integrable disc. ser. of $G := \underline{G}(\mathbb{R})$.

$\tau(k)$ is the minimal K-type of $\sigma(k)$. (K is a max. cpt. subgp. of G.)

V_k is a representation space of $\tau(k)$.

$d(k)$ is the formal degree of $\sigma(k)$.

$$f_k(g) := \frac{d(k)}{\dim V_k} \overline{\mathrm{tr}(\mathrm{proj}_{\tau(k)} \circ \sigma(k)(g) \circ \mathrm{proj}_{\tau(k)})}$$

K_f is an open compact subgroup of $\underline{G}(\mathbb{A}_f)$.

$S_k(K_f)$ is a vector space of V_k-valued cusp forms associated to $\sigma(k)$.

For $h \in C_c^\infty(\underline{G}(\mathbb{A}_f))$, a Hecke op. $T_h \colon S_k(K_f) \to S_k(K_f)$ is defined.

Godement's formula

$$\mathrm{tr}(T_h|_{S_k(K_f)}) = \int_{\underline{G}(\mathbb{Q})\backslash\underline{G}(\mathbb{A})} \sum_{\gamma \in \underline{G}(\mathbb{Q})} (f_k \otimes h)(x^{-1}\gamma x)\, \mathrm{d}x.$$

(i) The convergence theorem of Finis-Lapid holds for any $\underline{G}$.

(ii) Upper bound of Finis-Matz

This is available for split reductive groups over $\mathbb{Q}$.

Eikemeier generalized their result to the $\mathbb{Q}$-quasi-split case.

(iii) Uniform upper bound of f_k and its derivative

A general upper bound of matrix coefficients was obtained by Miličić.

However, the implicit constant of his inequality is effected by K-types.

To apply this method to other groups, we must solve the problems related to (ii) and (iii).

List of simple Lie groups having holom. disc. ser.

G has a holom. disc. ser. if and only if K has a non-finite center.

- $G = \mathrm{Sp}_{2n}$, $K = \mathrm{U}_n$, tube domain, split.
- $G = \mathrm{SU}_{n,n}$, $K = S(\mathrm{U}_n \times \mathrm{U}_n)$, tube domain, quasi-split.
- $G = \mathrm{SO}^*_{4n}$, $K = \mathrm{U}_{2n}$, tube domain.
- $G = \mathrm{SO}_{2,n}$, $K = S(\mathrm{O}_2 \times \mathrm{O}_n)$, tube domain.
- $G = E_{7\,(-25)}$, $K = \mathrm{U}_1 \times E_6$, tube domain.
- $G = \mathrm{SU}_{p,q}$, $K = S(\mathrm{U}_p \times \mathrm{U}_q(q))$, $p < q$.
- $G = E_{6\,(-14)}$, $K = (\mathrm{U}_1 \times \mathrm{Spin}_{10})/\mu_4$.

Problem (iii) can be solved by using Jordan algebra for the tube domains.

3. Quaternionic cusp forms on G_2

$\underline{G}$ is the split adjoint algebraic group of type G_2 over $\mathbb{Q}$.

K_f is an open compact subgroup of $\underline{G}(\mathbb{A}_f)$.

$K \simeq \mathrm{SU}_2 \times \mathrm{SU}_2 /\{\pm 1\}$ is a maximal compact subgroup of $G = \underline{G}(\mathbb{R})$.

$\sigma(k)$ is the quaternionic discrete series of G with the minimal K-type $\tau(k) = \mathrm{Sym}^{2k}$.

$L^2_{\mathrm{cusp}}(\underline{G}(\mathbb{Q})\backslash\underline{G}(\mathbb{A})) \simeq \sum_{\pi \in \widehat{\underline{G}(\mathbb{A})}} m_\pi \cdot \pi$. $\pi \simeq \pi_\infty \otimes \pi_f$.

$$L_\pi(K_f) \coloneqq \langle \phi = \phi_\infty \otimes \phi_f \in m_\pi \cdot \pi \mid \phi_\infty \in m_\pi \cdot \tau(k),\ \ \pi_f(\pi, K_f)\phi_f = \phi_f \rangle,$$

$$\mathcal{L}_k(K_f) \coloneqq \oplus_{\pi_\infty \simeq \sigma(k)} L_\pi(K_f) \subset L^2_{\mathrm{cusp}}(\underline{G}(\mathbb{Q})\backslash\underline{G}(\mathbb{A})/K_f).$$

$V_k^\vee$ is the representation space of the contragredient rep. of $\tau(k)$.

$$S_k(K_f) \coloneqq (\mathcal{L}_k(K_f) \otimes V_k^\vee)^K.$$

Cusp forms in $S_k(K_f)$ are $V_k^\vee$-valued functions on $\underline{G}(\mathbb{Q})\backslash\underline{G}(\mathbb{A})/K_f$.

$h \in C_c^\infty(\underline{G}(\mathbb{A}_f))$, Hecke operator $T_h \colon S_k(K_f) \to S_k(K_f)$ is defined by

$$(T_h\phi)(g) := \int_{G(\mathbb{A}_f)} \phi(gx_f)\, h(x_f)\, \mathrm{d}x_f, \qquad (\mathrm{vol}(K_f) = 1).$$

$d(k)$ is the formula degree of $\sigma(k)$.

$$d(k) \asymp k^{\frac{d}{2}}, \quad d = \dim G/K + 1.$$

Theorem (Sugiyama-Tsuzuki-W.)

For any small $\varepsilon > 0$ and any $h \in C_c^\infty(K_f \backslash \underline{G}(\mathbb{A}_f)/K_f)$, we have (w.r.t. $k \to \infty$)

$$\mathrm{tr}(T_h|_{S_k(K_f)}) = \mathrm{vol}(\underline{G}(\mathbb{Q})\backslash\underline{G}(\mathbb{A}))\, d(k)\, h(1) + O_\varepsilon(k^{\frac{d}{2}-1+\varepsilon}\, \|h\|_{L^1}).$$

$\boxed{\text{Proof}}$

It is enough to prove a uniform bound of f_k and its derivative.

$P = LU$ is a maximal parabolic subgroup of G s.t. $L \simeq \mathrm{GL}_2(\mathbb{R})$ and $\dim U = 5$.

For the Iwasawa decomposition $g = luy \in G = LUK$ ($l \in L$, $u \in U$, $y \in K$), we set

$$H_P(luy) := \log|\det(l)| \in \mathbb{R}, \quad \kappa(g) = \overline{y} \in \mathrm{SO}_3 \simeq \mathrm{SU}_2/\{\pm 1\} = K/\mathrm{SU}_2\,.$$

p_k is the k-th Legendre polynomial.

$(x|y) := x^t y$ ($x, y \in \mathbb{R}^3$). $e_1 = (1,0,0)$.

(Wallach) $\sigma(k)$ is a subrepresentation of $\mathrm{Ind}_P^G \lambda_k$, where $\lambda_k = \mathrm{sgn}^k|\det|^{-k-1}$. Using the compact picture of $\mathrm{Ind}_P^G \lambda_k$, we obtain

$$f_k(g) = d(k) \int_{y \in K} e^{-k\, H_p(yg)}\, p_k((e_1\kappa(yg)|e_1\kappa(y)))\, \mathrm{d}y.$$

By this integral rep. of f_k and $|p_k((e_1 h_1|e_1 h_2))| \le 1$ ($h_1, h_2 \in \mathrm{SO}_3$), we can get desired bounds.

List of simple Lie groups having quater. disc. ser.

(Gross-Wallach)

G has a quater. disc. ser. if and only if K has a normal subgroup $\simeq \mathrm{SU}_2$.

- $G = \mathrm{Sp}_{1,n}$, $K = \mathrm{SU}_2 \times \mathrm{Sp}_n$.
- $G = \mathrm{SU}_{2,n}$, $K = S(\mathrm{U}_2 \times \mathrm{U}_n)$.
- $G = \mathrm{SO}_{4,n}$, $K = S(\mathrm{O}_4 \times \mathrm{O}_n)$, split $n = 3$, 4, 5.
- $G = G_{2\,(2)}$, $K = (\mathrm{SU}_2 \times \mathrm{SU}_2)/\{\pm 1\}$, split.
- $G = F_{4\,(4)}$, $K = (\mathrm{SU}_2 \times \mathrm{Sp}_3)/\{\pm 1\}$, split.
- $G = E_{6\,(2)}$, $K = (\mathrm{SU}_2 \times \mathrm{SU}_6)/\{\pm 1\}$, quasi-split.
- $G = E_{7\,(-5)}$, $K = (\mathrm{SU}_2 \times \mathrm{Spin}_{12})/\{\pm 1\}$.
- $G = E_{8\,(-24)}$, $K = (\mathrm{SU}_2 \times E_7)/\{\pm 1\}$.

(Wallach) In general (except $\mathrm{Sp}_{1,n}$), quater. dis. ser. is a subrep. of an induced rep. of λ_k. As for $\mathrm{Sp}_{1,n}$, f_k is explicitly constructed by Arakawa. Hence, Problem (iii) can be solved.

4. Optimal upper bound of the reminder

Let $\underline{G} = \mathrm{Sp}_{2n}$. For any small $\varepsilon > 0$ and any $h \in C_c^\infty(K_f \backslash \underline{G}(\mathbb{A}_f)/K_f)$ we have (w.r.t. $k \to \infty$)

$$\mathrm{tr}(T_h|_{S_k(K_f)}) = \mathrm{vol}(\Gamma \backslash \mathfrak{H}_n)\, d(n,k)\, h(1) + \, O_\varepsilon(k^{\frac{d_n - n}{2} + \varepsilon}\, \|h\|_{L^1}).$$

$\|h\|_{L^1}$ is optimal at the present, but $k^{\frac{d_n-n}{2}}$ is not.

$\Gamma(N)$ is the principal congruence subgroup of level $N > 2$.

Then, (W.)

$$\dim S_k(\Gamma(N)) = \mathrm{vol}(\Gamma \backslash \mathfrak{H}_n)\, d(n,k) + O(k^{\frac{d_n}{2} - c_n}),$$

$$c_n := \begin{cases} n & n \text{ is even,} \\ 2n-1 & n \text{ is odd.} \end{cases}$$

This means the exponent of k should be improved.

- The center of $\underline{G}$ is trivial. G is simple. G is not compact.
- disc. ser. $\sigma(k)$ is holom. or quater. with min. K-type $\det^k$ or Sym^{2k} respectively.
- $\underline{A}$ is a maximal $\mathbb{Q}$-split torus in $\underline{G}$ and $\underline{M}$ is the centralizer of $\underline{A}$ in $\underline{G}$. Suppose $\underline{M}(\mathbb{R})/\underline{A}(\mathbb{R})$ is compact.

G_γ the centralizer of γ in G.

U_G is the set of unipotent elements in G.

$$d_{\mathrm{ss}} := \min_{\gamma \in K \setminus \{1\}} \mathrm{dim} G_\gamma \backslash G/K, \;\; d_{\mathrm{unip}} := \min_{u \in U_G \setminus \{1\}} \mathrm{dim} G_u \backslash G$$

$$d(k) \asymp k^{\frac{d}{2}}, \;\; d := \begin{cases} \dim G/K & \sigma(k) \text{ is hol.,} \\ \dim G/K + 1 & \sigma(k) \text{ is quater.} \end{cases}$$

Theorem (Sugiyama-W.-Tsuzuki, Proof is being checked.)

Fix a function $h \in \mathbb{C}_c^\infty(K_f \backslash \underline{G}(\mathbb{A}_f)/K_f)$. Then we have (w.r.t. $k \to \infty$)

$$\mathrm{tr}(T_h|_{S_k(K_f)}) = \mathrm{vol}(\underline{G}(\mathbb{Q})\backslash \underline{G}(\mathbb{A}))\, d(k)\, h(1) + O\left(k^{\frac{1}{2}(d - \min(d_{\mathrm{ss}}, d_{\mathrm{unip}}))}\right).$$

Ex. $G = \mathrm{Sp}_{2n}(\mathbb{R})$, $d_{\mathrm{ss}} = 2n-2 < d_{\mathrm{unip}} = 2n$.

$G = G_2(\mathbb{R})$, $d_{\mathrm{ss}} = 8 > d_{\mathrm{unip}} = 6$.

Idea of Proof

Instead of the upper bound of Finis-Matz, we use the following bound.

Theorem (Brumely-Marshall)

For $\gamma \in K$, if $\delta \ll 4|D(\gamma)|^2$, then

$$\int_{G_\gamma \backslash G} F^{\phi,\delta}(g^{-1}\gamma g)\,\mathrm{d}g \ll \delta^{\frac{d_{\mathrm{ss}}(\gamma)}{2}} |D(\gamma)|^{-d_{\mathrm{ss}}(\gamma)}.$$

Here, $D(\gamma)$ is the Weyl denominator of γ in G and $d_{\mathrm{ss}}(\gamma) \coloneqq \dim G_\gamma \backslash G/K$.

This bound provides an optimal estimate of the contribution of semisimple elements.

Ramacher and I obtained a similar bound by Herb's Fourier inversion, but the exponent of δ does not reach theirs.