Moduli of Prismatic $(\mathcal{G}, \mu)$-Apertures and Applications

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Setup

Fix a rational prime p.

Throughout this talk, we will encounter several discrete parameters.

- $h \geq 1$ encodes *height*
- $0 \leq d \leq h$ encodes *dimension*
- $n \geq 1$ encodes *level*
- $c \geq 1$ encodes *degree of p-nilpotence*

Setup

We adopt a resolutely ∞-categorical approach: all objects and operations should be understood in an appropriately derived, animated, or homotopy-coherent sense unless otherwise stated.

Fix for now an animated commutative ring R.

- CAlg_R is the ∞-category of *animated* commutative R-algebras.
- Superscript $\heartsuit$ signals that we are working with *discrete* or *classical* objects or operators.
- Mod_R is the stable ∞-category of R-modules. We equip this with the usual t-structure and tensor product $\otimes_R$. Note that, even when R is discrete, the latter may not agree with the classical tensor product $\otimes_R^{\heartsuit}$ on discrete R-modules.
- Given $a \in \pi_0(R)$ and $M \in \mathrm{Mod}_R$, $M/^{\mathbb{L}}a$ denotes the *derived* quotient. If R and M are discrete then $M/^{\mathbb{L}}a$ agrees with the usual quotient M/aM if and only if M is a-torsion-free.

Setup

From the perspective of algebraic geometry, we want to work with both derived prestacks and their p-adic formal analogues.

- *Derived prestacks* over R are functors from CAlg_R to the ∞-category Spc of *spaces* (also incarnated as *anima* or ∞*-groupoids*).
- $\mathrm{CAlg}_R^{p\text{-nilp}}$ is spanned by *p-nilpotent* objects (condition on π_0).
- $\mathrm{CAlg}_R^{p\text{-comp}}$ and $\mathrm{Mod}_R^{p\text{-comp}}$ are spanned by *(derived) p-complete* objects. We refer to discrete objects satisfying the classical condition as *p-adically complete* – both conditions agree in the p-adically separated case. Other completeness conditions should similarly be understood in a *derived* sense unless otherwise stated.
- Derived *p-adic formal* prestacks over R should be understood as functors from $\mathrm{CAlg}_R^{p\text{-nilp}}$ to Spc. When R is discrete, this is compatible with the usual notion assuming the full p-power-torsion $R[p^\infty]$ is bounded.

Setup

We will also need the following *Shimurian* setup.

- $\mathcal{G}$ is a smooth affine group scheme over $\mathbb{Z}_p$, with generic fiber G.
- $\mathcal{O}$ is the ring of integers of a finite *unramified* extension of $\mathbb{Q}_p$, with residue field k (more on this later...).
- μ is a cocharacter of $\mathcal{G}$ defined over $\mathcal{O}$.
- There is an adjoint action of $\mathbb{G}_{m,\mathcal{O}}$ on $\mathcal{G}_{\mathcal{O}}$ induced by μ, with *attractor* $\mathcal{P}_\mu^-$, *repeller* $\mathcal{P}_\mu^+$, and *fixed point locus* $\mathcal{Z}_\mu$. We have $\mathcal{P}_\mu^\pm = \mathcal{Z}_\mu \ltimes \mathcal{U}_\mu^\pm$.
- $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}_{\mathcal{O}}$, which via a similar action has an induced grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i^{(\mu)}$. We will always impose the condition that μ is 1-*bounded* – i.e., $\mathfrak{g}_i^{(\mu)} = 0$ for $i > 1$ (more on this later...).

An important example is given by taking $\mathcal{O} = \mathbb{Z}_p$, $\mathcal{G} = \mathrm{GL}_{h,\mathbb{Z}_p}$, and $\mu = \mu_{h,d} : z \mapsto \mathrm{diag}(z^{(d)}, 1^{(h-d)})$.

Basics of Prismatic $(\mathcal{G}, \mu)$-Apertures

With the basic setup out of the way, our goal is to study the moduli stack $\mathrm{BT}_n^{\mathcal{G},\mu}$ of *level-n prismatic* $(\mathcal{G}, \mu)$*-apertures*.

- $\mathrm{BT}_n^{\mathcal{G},\mu}$ is the derived p-adic formal stack over $\mathrm{Spf}\,\mathcal{O}$ whose R-points are the data of $\mathcal{G}$-torsors over $\mathrm{Syn}_n(R)$ of *type* μ.
- $\mathrm{Syn}_n(R)$ is the level-n reduction (obtained as the derived p^n-quotient or pullback to $\mathrm{Spec}\,\mathbb{Z}/p^n$) of the *syntomification* $\mathrm{Syn}(R)$ of $\mathrm{Spf}\,R$.
- $\mathrm{Syn}_n(R)$ comes equipped with a line bundle $\mathscr{O}_{\mathrm{syn},n} : \mathrm{Syn}_n(R) \to B\mathbb{G}_{m,\mathcal{O}}$. The type μ condition on a $\mathcal{G}$-torsor $\mathscr{Q} : \mathrm{Syn}_n(R) \to B\mathcal{G}_{\mathcal{O}}$ can be viewed as saying that, étale-locally on R, $\mathscr{Q}$ factors through $B\mu : B\mathbb{G}_{m,\mathcal{O}} \to B\mathcal{G}_{\mathcal{O}}$ via $\mathscr{O}_{\mathrm{syn},n}$.
- Let $\mathrm{BT}_n^{h,d} := \mathrm{BT}_n^{\mathrm{GL}_h,\mu_{h,d}}$. Given $\mathscr{E} \in \mathrm{Vect}(\mathrm{Syn}_n(R))$, the type (h, d) condition says that there is an associated *finitely filtered* vector bundle whose associated graded is nonzero only in degree 0 (with rank $h - d$) and degree -1 (with rank d).

The Main Theorem

Theorem (G.-Madapusi-Mathew)

1. $\mathrm{BT}_n^{\mathcal{G},\mu}$ *is a zero-dimensional quasi-compact smooth derived p-adic formal Artin stack over* $\mathrm{Spf}\,\mathcal{O}$ *with affine diagonal.*
2. *The level truncation map* $\mathrm{BT}_{n+1}^{\mathcal{G},\mu} \to \mathrm{BT}_n^{\mathcal{G},\mu}$ *is smooth and surjective.*
3. *Let* $(C' \twoheadrightarrow C, \gamma)$ *be a PD-thickening in* $\mathrm{CAlg}_{\mathcal{O}}^{p\text{-dfin}}$. *Then, there is a canonical commutative* Grothendieck-Messing *square*

$$\begin{tikzcd} \mathrm{BT}_n^{\mathcal{G},\mu}(C') \arrow[r] \arrow[d] & B\mathcal{P}_\mu^{-,(n)}(C') \arrow[d] \\ \mathrm{BT}_n^{\mathcal{G},\mu}(C) \arrow[r] & B\mathcal{P}_\mu^{-,(n)}(C) \times_{B\mathcal{G}^{(n)}(C)} B\mathcal{G}^{(n)}(C') \end{tikzcd}$$

Moreover, this square is Cartesian if γ *is nilpotent.*

The precise details of this Grothendieck-Messing (GM) theory will be explained later.

Towards a Comparison: (Prismatic) Dieudonné Theory

Let's look at comparisons for the general linear case. Let $p\text{-div}_n^{h,d}$ denote the p-adic formal moduli stack of *level-n Barsotti-Tate groups* of height h and dimension d. Work of Grothendieck and others shows that this satisfies an analogue of the previous theorem.

Relatedly, we have an equivalence between $\mathrm{BT}_n^{h,d}(R)$ and $p\text{-div}_n^{h,d}(R)$ in the following cases, each generalizing the one before it.

1. R perfect: Zink and Lau-Zink
2. R perfectoid: Scholze-Weinstein and Lau
3. R quasiregular semiperfectoid (qrspd): Anschütz-Le Bras
4. R quasisyntomic: Anschütz-Le Bras and Mondal

It is therefore reasonable that BT_n and $p\text{-div}_n$ (both defined by sampling over all choices of h, d) should be closely linked.

The Comparison Theorem

Let $R \in \mathrm{CAlg}^{p\text{-nilp},\heartsuit}$ be *p-differentially finite* (see the appendices).

Theorem (G.-Madapusi-Mathew)

1. *There is a canonical functor* $\mathbf{G}_n : \mathrm{BT}_n(R) \to p\text{-}\mathrm{div}_n(R)$, *which preserves height and dimension and is compatible with the natural notions of* Cartier duality *on both sides.*
2. *Suppose in addition that* R *is qrspd (for simplicity). Then, there is a canonical equivalence of categories*

$$\mathbf{D}_n : p\text{-}\mathrm{div}_n(R) \rightleftarrows \mathrm{BT}_n(R) : \mathbf{G}_n,$$

   where $\mathbf{D}_n$ *is the* (level-n truncated) covariant prismatic Dieudonné functor *(which preserves height and dimension).*

It follows that $\mathbf{G}_n$ extends to a canonical isomorphism from the classical truncation $\mathrm{BT}_{n,\mathrm{cl}}$ to $p\text{-}\mathrm{div}_n$, and thus that $\mathrm{BT}_{n,\mathrm{cl}}$ is discretely-valued.

Basics of 1-Bounded Derived Stacks

It turns out that the proofs of both theorems are linked via the formalism of 1-*bounded derived stacks*, which are certain pairs $\mathcal{X} = (\mathcal{X}^{\square}, X^0)$ to which we can associate a *syntomic cohomology stack* $\Gamma_{\mathrm{syn}}(\mathcal{X})$.

Example

Consider the morphism $\mathscr{O}_{\mathrm{syn},n}\{1\} : \mathrm{Syn}_n(R) \to B\mathbb{G}_{m,\mathcal{O}}$ induced by the *Breuil-Kisin (BK) twist*. $\mathrm{BT}_n^{\mathcal{G},\mu}$ is obtained by applying Γ_{syn} to the base change along $\mathscr{O}_{\mathrm{syn},n}\{1\}$ of the pair $\mathfrak{B}(\mathcal{G},\mu) := (B\mathbb{G}_{m,\mathcal{O}} \times B\mathcal{G}_{\mathcal{O}}, B\mathcal{Z}_{\mu})$.

Following Simpson, we say that a derived prestack is *graded* (resp. *filtered*) if it is equipped with a morphism to $B\mathbb{G}_m$ (resp. $\mathbb{A}^1/\mathbb{G}_m$). Let $y^0 : B\mathbb{G}_{m,C} \to \mathcal{Y}$ be a map of graded derived prestacks. We take $\mathcal{X}$ as above to live over $(\mathcal{Y}, y^0)$, with $\mathcal{X}^{\square} \to \mathcal{Y}$ a relative locally finitely presented derived Artin stack and X^0 an *open* derived derived substack of the 1-*bounded* fixed point locus of the base change $\mathcal{X}^{\square}_{y^0}$. The 1-boundedness here can be expressed as a weight constraint on an appropriate graded cotangent complex.

Basics of Frames

The Artin-Lurie representability theorem provides us with checkable criteria to establish the respresentability (and other nice properties) of $\Gamma_{\mathrm{syn}}(\mathcal{X})$. To check these criteria as well as establish GM theory, it's helpful to have "linear algebra" models for $\Gamma_{\mathrm{syn}}(\mathcal{X})$ and other similar objects. These are provided by the formalism of *(animated higher) frames*.

Definition

A **frame** is the data of a tuple $\underline{A} := (I \xrightarrow{s} A, \mathrm{Fil}^\bullet A, \Phi, A\{1\}, \iota_{\mathrm{BK}})$ with

- $I \xrightarrow{s} A$ a generalized Cartier divisor on A (with $\overline{A} := A/I$);
- $\mathrm{Fil}^\bullet A$ a non-negatively (decreasingly) filtered (p, I)-complete animated commutative ring;
- $\Phi : \mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_I A$ a *filtered Frobenius* map, with underlying Frobenius lift $\phi : A \to A$;
- $A\{1\}$ an invertible A-module (the *(abstract) BK twist*) equipped with an A-linear isomorphism $\iota_{\mathrm{BK}} : \phi^* A\{1\} \otimes_A I \xrightarrow{\sim} A\{1\}$.

Basics of Frames

We say that $\underline{A}$ is **prismatic** if (A, I) defines an animated prism with Frobenius ϕ and $A\{1\}$ is the *BK twist* of A (as an animated prism, satisfying some compatibilities with Φ). This is a large and important class of frames, with the following a crucial example.

Example

Suppose R is qrspd with prismatic cohomology (Δ_R, I) and Frobenius ϕ. Consider the *Nygaard filtration* with terms $\mathrm{Fil}^i_{\mathcal{N}} \Delta_R := \phi^{-1}(I^i)$ and transition maps given by inclusion. There is an induced filtered Frobenius $\Phi_{\mathcal{N}}$, and we take $\Delta_R\{1\}$ to be the BK twist. This defines the **Nygaard frame** $\underline{\Delta}_R$ over R.

We can make sense of this construction for any semiperfectoid R, though (Δ_R, I) may be non-discrete and so some care is needed (its structure is closely related to derived crystalline cohomology). We will make heavy use of the full range of semiperfect inputs.

Basics of Frames

Other key examples of frames come from working with Witt vectors.

Example

1. Let $R \in \mathrm{CAlg}^{p\text{-comp},\heartsuit}$ with associated prism $(W(R),(p))$ and Frobenius F. Let $\mathrm{Fil}^{\bullet}_{\mathrm{Lau}} W(R)$ be the filtration given by
$$\cdots \xrightarrow{p} VW(R) \xrightarrow{p} VW(R) \hookrightarrow W(R) = \cdots$$
noting that $W(R)$ may have p-torsion. We equip $\mathrm{Fil}^{\bullet}_{\mathrm{Lau}} W(R)$ with the multiplication given in positive degrees by $Vx \cdot Vy := V(xy)$. There is an induced filtered Frobenius Φ_{Lau}, and we take $W(R)\{1\}$ to be trivial. This defines the **Witt frame** $\underline{W}(R)$ over R, which is prismatic.
2. Given $R \in \mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}$, we define $\underline{W}_n(R)$ so that the natural map $W(R) \to W_n(R)$ induces a natural frame map $\underline{W}(R) \to \underline{W}_n(R)$. This is the **level-$n$ truncated Witt frame** over R, which is not prismatic. If $n = 1$ then this is also called the **zip frame** over R.

Basics of Frames

- All of the previous frames were viewed as living over R. In general, we think of a frame $\underline{A}$ as living over $R_A := \mathrm{gr}^0 A$.
- If $\underline{A}$ is prismatic then there is a canonical frame map $\underline{A} \to \underline{W}(R_A)$, and if in addition $R_A \in \mathrm{CAlg}_{\mathbb{F}_p}$ then there is a compatible canonical frame map $\underline{A} \otimes \mathbb{F}_p \to \underline{W}_1(R_A)$.
- Let $\mathcal{R}(\underline{A}) := \mathrm{Spf}(\mathrm{Rees}(\mathrm{Fil}^\bullet A))/\mathbb{G}_m$. There are natural disjoint open immersions $\tau, \sigma : \mathrm{Spf}\, A \to \mathcal{R}(\underline{A})$, with τ arising from the trivial filtration on A and σ induced by Φ. Taking the coequalizer of τ and σ produces the **(abstract) syntomification** $\mathrm{Syn}(\underline{A})$ of $\underline{A}$. Both of these notions admit *level-n truncated* variants $\mathcal{R}_n(\underline{A}) := \mathcal{R}(\underline{A}) \otimes \mathbb{Z}/p^n$ and $\mathrm{Syn}_n(\underline{A}) := \mathrm{Syn}(\underline{A}) \otimes \mathbb{Z}/p^n$.
- Frames are *rigid*: given a p-completely étale map $R_A \to R_{A'}$, there is a unique lift $\underline{A} \to \underline{A}'$ with underlying $A \to A'$ a (p, I)-completely étale frame map. Hence, $\underline{A}$ induces a *sheaf* of frames on $(\mathrm{Spf}\, R_A)_{\text{ét}}$.

Basics of Frames

- A **level-n truncated** $(\mathcal{G},\mu)$**-window** over $\underline{A}$ is a $\mathcal{G}$-torsor over $\mathrm{Syn}_n(\underline{A})$ satisfying a suitable local triviality condition that makes it of *type* μ. In this way, we obtain the sheaf $\mathrm{Wind}^{\mathcal{G},\mu}_{\underline{A},n}$ on $(\mathrm{Spf}\, R_A)_{\text{ét}}$. Working instead with $\mathrm{Syn}(\underline{A})$ gives the *untruncated* variant $\mathrm{Wind}^{\mathcal{G},\mu}_{\underline{A}}$.
- Over $\mathrm{Spec}\, k$ we have the derived moduli stack $\mathrm{Disp}^{\mathcal{G},\mu}_n$ of **level-n truncated Witt** $(\mathcal{G},\mu)$**-displays**, given by $R \mapsto \mathrm{Wind}^{\mathcal{G},\mu}_{\underline{W}_n(R)}(R)$. If $n = 1$ then we also use the term $(\mathcal{G},\mu)$**-zip**. This agrees with the construction of Pink-Wedhorn-Ziegler of *F-zips with $\mathcal{G}$-structure of type* μ, up to some re-normalization.
- A **syntomic structure** on $\underline{A}$ is the data of a factorization of $A \twoheadrightarrow \overline{A}$ through the base change of $A \twoheadrightarrow R_A$ along $\phi : A \to A$. For many frames of interest (e.g., $\underline{W}(R)$) there is a natural choice of syntomic structure. A choice of syntomic structure induces a comparison morphism $\mathrm{Syn}(\underline{A}) \to \mathrm{Syn}(R_A)$.

Basics of Frames

Proposition (Bhatt-Lurie, G.-Madapusi-Mathew)

Let R be a semiperfectoid ring.

1. *There are compatible natural isomorphisms* $\mathrm{Syn}_n(\mathbb{\Delta}_R) \xrightarrow{\sim} \mathrm{Syn}_n(R)$.
2. *If R is semiperfect, there is a natural comparison map* $\mathbb{\Delta}_R \to \underline{W}(R)$ *sending the mod-p^n reduction of $\mathbb{\Delta}_R$ to $\underline{W}_n(R)$. If R is perfect then this comparison map is an isomorphism.*

Corollary

Let R be a semiperfectoid $\mathcal{O}$-algebra.

1. *There are compatible natural isomorphisms* $\mathrm{BT}_n^{\mathcal{G},\mu}(R) \xrightarrow{\sim} \mathrm{Wind}_{\mathbb{\Delta}_R,n}^{\mathcal{G},\mu}(R)$.
2. *If R is semiperfect, there are natural comparison maps* $\mathrm{Wind}_{\mathbb{\Delta}_R,n}^{\mathcal{G},\mu}(R) \to \mathrm{Disp}_n^{\mathcal{G},\mu}(R)$ *which are isomorphisms if R is perfect.*
3. $\mathrm{Wind}_{\mathbb{\Delta}_R,n}^{\mathcal{G},\mu}$ *and the restriction of* $\mathrm{BT}_n^{\mathcal{G},\mu}$ *agree as sheaves on* $(\mathrm{Spf}\, R)_{\text{ét}}$.

Proof of the Main Theorem

With these frames in hand, recall that we want to prove the following.

Claim

1. $\mathrm{BT}_n^{\mathcal{G},\mu}$ *is a zero-dimensional quasi-compact smooth derived p-adic formal Artin stack over* $\mathrm{Spf}\,\mathcal{O}$ *with affine diagonal.*
2. *The level truncation map* $\mathrm{BT}_{n+1}^{\mathcal{G},\mu} \to \mathrm{BT}_n^{\mathcal{G},\mu}$ *is smooth and surjective.*

The broad strategy is to induct on the level parameter n and the p-nilpotence parameter c, working with frame models whenever possible.

- The base case is given by taking $n = 1 = c$, which corresponds to studying the special fiber $\mathrm{BT}_1^{\mathcal{G},\mu} \otimes \mathbb{F}_p$ (i.e., we input $R \in \mathrm{CAlg}_k$).
- We need a well-behaved toy model for comparison. By prior work, we have natural maps $\mathrm{BT}_1^{\mathcal{G},\mu}(R) \to \mathrm{Disp}_1^{\mathcal{G},\mu}(R)$ for semiperfect k-algebras. The key is that this extends to the whole special fiber.

Proof of the Main Theorem

Theorem (G.-Madapusi-Mathew)

There is a natural map $\mathrm{BT}_1^{\mathcal{G},\mu} \otimes \mathbb{F}_p \to \mathrm{Disp}_1^{\mathcal{G},\mu}$ *that is relatively representable by a smooth* 0*-dimensional derived Artin stack over* $\operatorname{Spec} k$ *with relatively affine diagonal – in fact, it is a gerbe banded by a certain height* 1 *finite flat group scheme* $\mathrm{Lau}_1^{\mathcal{G},\mu}$ *over* $\operatorname{Spec} k$*. In particular,* $\mathrm{BT}_1^{\mathcal{G},\mu} \otimes \mathbb{F}_p$ *is a smooth zero-dimensional derived Artin stack over* $\operatorname{Spec} k$ *with affine diagonal.*

Taking $\mathcal{O} = \mathbb{Z}_p$ and restricting to the (discrete) smooth locus, the above result is originally due to Drinfeld and uses a totally different method of proof. The key to our approach is that, for R semiperfect, the underlying map $\mathrm{Syn}_1(\mathbb{\Delta}_R) \to \mathrm{Syn}(\underline{W}_1(R))$ factors through the *square-zero thickening* $\mathrm{Syn}_1(\underline{W}(R)) \to \mathrm{Syn}(\underline{W}_1(R))$. The *Lau group scheme* $\mathrm{Lau}_1^{\mathcal{G},\mu}$ appears thanks to a result of Bragg-Olsson on representability of fppf cohomology.

Proof of the Main Theorem

The next step is to induct on n, holding $c = 1$ fixed. Given $\mathscr{Q} \in \mathrm{BT}_n^{\mathcal{G},\mu}(R)$ with $R \in \mathrm{CAlg}_k$, the key is that the fiber of $\mathrm{BT}_{n+1}^{\mathcal{G},\mu} \otimes \mathbb{F}_p \to \mathrm{BT}_n^{\mathcal{G},\mu} \otimes \mathbb{F}_p$ above $\mathscr{Q}$ is a torsor for the syntomic cohomology stack associated to the 1-shifted perfect complex $\mathfrak{g}(\mathscr{Q})[1]$ over $\mathrm{Syn}_1(R)$, where $\mathfrak{g}(\mathscr{Q})$ is obtained by twisting the adjoint representation of $\mathcal{G}_{\mathcal{O}}$ on $\mathfrak{g}$ by $\mathscr{Q}$.

To flesh this out a little, let $R \in \mathrm{CAlg}^{p\text{-dfin}}$ and $\mathscr{M} \in \mathrm{Perf}(\mathrm{Syn}_n(R))$ with *Hodge-Tate weights* contained in $[0, 1]$. Using the associated *derived vector stack* $\mathbb{V}(\mathscr{M})$, we can associate a 1-bounded derived stack to $\mathscr{M}$ and thus obtain the syntomic cohomology stack $\Gamma_{\mathrm{syn}}(\mathscr{M})$.

Proposition

Suppose that $\mathscr{M}$ has Tor-*amplitude* $\geq -r$. *Then,* $\Gamma_{\mathrm{syn}}(\mathscr{M})$ *is a locally finitely presented derived p-adic formal Artin r-stack over* $\mathrm{Spf}\, R$.

This works because many nice properties of 1-bounded derived stacks $\mathcal{X}$ transfer over to $\Gamma_{\mathrm{syn}}(\mathcal{X})$.

Proof of the Main Theorem

What about induction on c? The trick is to combine so-called Tot-*descent* with GM theory. A good example of the former is the following result.

Proposition (G.-Madapusi-Mathew)

Let $R \in \mathrm{CAlg}_{\mathcal{O}}^{p\text{-dfin}}$. *Then, the natural map*

$$\mathrm{BT}_n^{\mathcal{G},\mu}(R) \to \mathrm{Tot}(\mathrm{BT}_n^{\mathcal{G},\mu}(R \otimes_{\mathbb{Z}} \mathbb{F}_p^{\otimes_{\mathbb{Z}}(\bullet+1)}))$$

to the totalization is an isomorphism. Working with pro-objects,

$$\mathrm{BT}_n^{\mathcal{G},\mu}(R) \xrightarrow{\sim} \varprojlim_{i \geq 1} \mathrm{BT}_n^{\mathcal{G},\mu}(R/^{\mathbb{L}} p^i).$$

It follows that Tot-descent can be viewed as a kind of derived algebraization result. Note that, even if R is discrete, the terms $R \otimes_{\mathbb{Z}} \mathbb{F}_p^{\otimes_{\mathbb{Z}}(i+1)}$ are necessarily non-discrete for $i \geq 1$ since $\mathbb{F}_p$ is p-torsion.

Proof of the Main Theorem

Tot-descent is closely related to the following result.

Proposition (Halpern-Leistner-Preygel)

Let $A \in \mathrm{CAlg}$ and $J \subseteq \pi_0(A)$ a finitely generated ideal such that A is J-complete. Given $\mathcal{Y}$ a derived J-adic formal Artin stack and $R \in \mathrm{CAlg}_A$, the natural map

$$\mathcal{Y}(R) \to \mathrm{Tot}(\mathcal{Y}(R \otimes_A (A/J)^{\otimes_A(\bullet+1)}))$$

is an isomorphism.

We want to apply this result to $\mathrm{BT}_n^{\mathcal{G},\mu}$, but don't *a priori* know that $\mathrm{BT}_n^{\mathcal{G},\mu}$ is Artin. We can get around this by building $\mathrm{BT}_n^{\mathcal{G},\mu}$ up from appropriately chosen Artin stacks, to which the result does apply. That we can do this is maybe not so surprising: a result of Bhatt-Lurie shows that (relative) prismatic cohomology satisfies Tot-descent, so intuitively we just need to compatibly accommodate the Nygaard filtration.

Proof of the Main Theorem

Let's now recall the statement of GM theory.

Claim

Let $(C' \twoheadrightarrow C, \gamma)$ be a PD-thickening in $\mathrm{CAlg}_{\mathcal{O}}^{p\text{-dfin}}$. Then, there is a canonical commutative Grothendieck-Messing *square*

$$\begin{tikzcd} \mathrm{BT}_n^{\mathcal{G},\mu}(C') \arrow[r] \arrow[d] & B\mathcal{P}_\mu^{-,(n)}(C') \arrow[d] \\ \mathrm{BT}_n^{\mathcal{G},\mu}(C) \arrow[r] & B\mathcal{P}_\mu^{-,(n)}(C) \times_{B\mathcal{G}^{(n)}(C)} B\mathcal{G}^{(n)}(C') \end{tikzcd}$$

Moreover, this square is Cartesian if γ is nilpotent.

This square is built from the natural map $\mathrm{BT}_n^{\mathcal{G},\mu} \to B\mathcal{P}_\mu^{-,(n)}$, which encodes some generalization of the *Hodge filtration* for a prismatic $(\mathcal{G}, \mu)$-aperture. GM theory then roughly says that lifting a prismatic $(\mathcal{G}, \mu)$-aperture along a nilpotent PD-thickening is the same as lifting just its Hodge filtration.

Proof of the Main Theorem

Note that this construction requires a map $\mathrm{BT}_n^{\mathcal{G},\mu}(C) \to B\mathcal{G}^{(n)}(C')$, which arises from the choice of γ. This is natural since γ itself can be viewed as the data of a certain factorization involving PD envelopes.

Remark

Given a derived p-adic formal prestack $\mathcal{Y}$ *and* $i \geq 1$, *the associated* (ith) Weil twist *of* $\mathcal{Y}$ *is the derived p-adic formal prestack* $\mathcal{Y}^{(i)}$ *given by* $\mathcal{Y}^{(i)}(S) := \mathcal{Y}(S/^{\mathbb{L}}p^i)$. *If* $\mathcal{Y}$ *is a stack (resp. Artin stack) then* $\mathcal{Y}^{(i)}$ *is as well.*

Given $R \in \mathrm{CAlg}^{p\text{-nilp}}$, if $p > 2$ then $R \twoheadrightarrow R/^{\mathbb{L}}p$ admits a natural nilpotent PD-structure $\gamma_{p,\mathrm{can}}$. If $p = 2$ then $R \twoheadrightarrow R/^{\mathbb{L}}2$ also has a natural PD-structure but it isn't nilpotent, so we instead define $\gamma_{2,\mathrm{can}}$ using $R \twoheadrightarrow R/^{\mathbb{L}}4$. In either case, applying GM theory to $\gamma_{p,\mathrm{can}}$ allows us to write $\mathrm{BT}_n^{\mathcal{G},\mu}$ as the pullback of nice derived p-adic formal stacks and thereby control it as desired. The same strategy gives the desired control on the level truncation maps $\mathrm{BT}_{n+1}^{\mathcal{G},\mu} \to \mathrm{BT}_n^{\mathcal{G},\mu}$.

Proof of the Main Theorem

To establish GM theory, we again need to bootstrap.

1. Use Tot-descent to reduce to the case $c = 1$.
2. Use left Kan extension and quasisyntomic descent to reduce to the semiperfect case, where we can work with frames.
3. Factor $(C' \twoheadrightarrow C, \gamma)$, so that we may assume it's a square-zero extension with trivial PD-structure (this uses the nilpotence of γ).
4. Reduce to the case that $C' \twoheadrightarrow C$ is a split square-zero extension and thus 1-connective.

There are then two key facts: $\mathbb{\Delta}_{C'} \to \mathbb{\Delta}_C$ is surjective, and the homotopy kernel $\mathrm{hker}(\mathbb{\Delta}_{C'} \to \mathbb{\Delta}_C)$ admits a *locally nilpotent* divided Frobenius. The rest of the proof is then a "by hand" computation.

Proof of the Comparison Theorem

Let's now discuss the proof of the comparison theorem, starting with a recollection of the objects involved.

Definition

Let $R \in \mathrm{CAlg}^{p\text{-nilp},\heartsuit}$. An n-**truncated Barsotti-Tate group** over R is a finite flat commutative group scheme $\mathfrak{G}$ over R such that

- $\mathfrak{G}$ is p^n-torsion;
- $\mathfrak{G} \xrightarrow{p^{n-m}} \mathfrak{G} \xrightarrow{p^m} \mathfrak{G}$ is an exact sequence of abelian fppf sheaves for all $m < n$;
- if $n = 1$, $\ker F = \mathrm{im}\, V$ inside $\mathfrak{G}_{R/p}$.

The **height** h and **dimension** d are locally given by the conditions that $\mathfrak{G}[p]$ has degree p^h over R and $\ker F \subseteq \mathfrak{G}[p]_{R/p}$ has degree p^d over R/p.

The stack $p\text{-div}_n$ comes equipped with the *Cartier duality* functor $\mathfrak{G} \mapsto \mathfrak{G}^* := \underline{\mathrm{Hom}}(\mathfrak{G}, \mu_{p^n})$. This defines an anti-involution which sends type (h, d) to type $(h, h-d)$.

Proof of the Comparison Theorem

Suppose now that R is in addition p-differentially finite, and let $\mathscr{E} \in \mathrm{BT}_n(R)$. Given $m \leq n$, consider the group scheme $\mathfrak{G}_m(\mathscr{E})$ over R defined by

$$\mathfrak{G}_m(\mathscr{E})(C) := \tau^{\leq 0} R\Gamma(\mathrm{Syn}_m(C), \mathscr{E}|_{\mathrm{Syn}_m(C)}) \simeq \Gamma_{\mathrm{syn}}(\mathscr{E}|_{\mathrm{Syn}_m(C)}),$$

where the group structure essentially comes from addition of cosections of $\mathscr{E}^\vee$. For $m < n$, the exact sequence

$$0 \to \mathbb{Z}/p^m \xrightarrow{p^{n-m}} \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-m} \to 0$$

induces an exact sequence $\mathfrak{G}_m(\mathscr{E}) \to \mathfrak{G}_n(\mathscr{E}) \to \mathfrak{G}_{n-m}(\mathscr{E})$, which we can use to deduce that $\mathfrak{G}_n(\mathscr{E}) \xrightarrow{p^{n-m}} \mathfrak{G}_n(\mathscr{E}) \xrightarrow{p^m} \mathfrak{G}_n(\mathscr{E})$ is exact. Realizing the rest of the expected structure of $\mathfrak{G}_n(\mathscr{E})$ is slightly technical (the key is to bootstrap from the case $n = 1$), but this at least gives a sketch of the construction of $\mathbf{G}_n(\mathscr{E})$.

Proof of the Comparison Theorem

The stack BT_n comes equipped with the *Cartier duality* functor

$$\mathscr{E} \mapsto \mathscr{E}^* := \mathscr{E}^\vee\{1\} = \mathscr{E}^\vee \otimes \mathscr{O}_{\mathrm{syn},n}\{1\}.$$

This is an anti-involution sending type (h, d) to type $(h, h-d)$, with

$$\mathbf{G}_n(\mathscr{O}_{\mathrm{syn},n}) \simeq \underline{\mathbb{Z}/p^n}, \qquad \mathbf{G}_n(\mathscr{O}_{\mathrm{syn},n}\{1\}) \simeq \mu_{p^n}.$$

This goes most of the way toward establishing that $\mathbf{G}_n$ is compatible with Cartier duality, as $\underline{\mathbb{Z}/p^n}$ and μ_{p^n} are Cartier dual to each other and serve as fundamental building blocks for $p\text{-div}_n$. This compatibility is helpful for showing that $\mathbf{G}_n$ interacts as desired with $\mathbf{D}_n$, which is easily shown to satisfy the mirror result

$$\mathbf{D}_n(\underline{\mathbb{Z}/p^n}) \simeq \mathscr{O}_{\mathrm{syn},n}, \qquad \mathbf{D}_n(\mu_{p^n}) \simeq \mathscr{O}_{\mathrm{syn},n}\{1\}.$$

Proof of the Comparison Theorem

To better discuss $\mathbf{D}_n$, let's recall one definition of a p-divisible group.

Definition

A p-**divisible group** over R is an abelian fppf sheaf $\mathfrak{G}$ over R such that

- $\mathfrak{G} \xrightarrow{p} \mathfrak{G}$ is an epimorphism of abelian fppf sheaves;
- $\mathfrak{G}(n) := \ker(\mathfrak{G} \xrightarrow{p^n} \mathfrak{G})$ is representable by a finite flat commutative group scheme over R for all $n \geq 1$;
- the natural map $\varinjlim_{n \geq 1} \mathfrak{G}(n) \to \mathfrak{G}$ is an isomorphism.

To $\mathfrak{G}$ we can associate the *classifying stack* $B\mathfrak{G} := \varinjlim_{n\geq 1} B\mathfrak{G}(n)$ and the *prismatic cohomology* $R\Gamma_{\Delta}(B\mathfrak{G})$ equipped with the *Nygaard filtration* $\mathrm{Fil}^{\bullet}_{\mathcal{N}} R\Gamma_{\Delta}(B\mathfrak{G})$. For R quasisyntomic, Mondal's prismatic Dieudonné functor is given by $\mathbf{D}(\mathfrak{G}) := H^2(\mathrm{Fil}^{\bullet}_{\mathcal{N}} R\Gamma_{\Delta}(B\mathfrak{G}))$. Note that $\mathbf{D}$ is naturally *contravariant*.

Proof of the Comparison Theorem

A similar recipe works to define $\mathbf{D}_n$ in the qrspd case, though there are some technicalities and we actually want a construction which is naturally *covariant*. To avoid these matters, we will simply mention the untruncated version of some of Mondal's results. Let $\mathbf{p\text{-}div}(R)$ denote the full category of p-divisible groups over R, and $\mathbf{Vect}^{[0,1]}(\mathrm{Syn}(R))$ the full ∞-category of vector bundles over $\mathrm{Syn}(R)$ with Hodge-Tate weights contained in $[0, 1]$ (both notions allow for non-invertible morphisms).

Theorem (Mondal)

$\mathbf{D} : \mathbf{p\text{-}div}(R)^{\mathrm{op}} \to \mathbf{Vect}^{[0,1]}(\mathrm{Syn}(R))$ *is an equivalence of (∞-)categories, compatible with Cartier duality and preserving height and dimension.*

It should be noted that Mondal also provides a similar *linearization* of the category $\mathrm{FFGS}(R)$ of all finite flat commutative group schemes over R.

FAQ

There are some natural questions that arise from these results.

1. Why do we need $\mathcal{O}$ to be unramified?
2. To what extent can we relax the condition that μ is 1-bounded?
3. The isomorphism $\mathrm{BT}_{n,\mathrm{cl}} \xrightarrow{\sim} p\text{-div}_n$ tells us that BT_n has essentially no interesting derived structure. Does the same hold true for $\mathrm{BT}_n^{\mathcal{G},\mu}$?
4. Witt-theoretic methods require us to parametrize the Hodge filtration as data, so why don't we need to do that in our approach?

For reference later, we let $\mathrm{BT}_\infty^{\mathcal{G},\mu} := \varprojlim_{n\geq 1} \mathrm{BT}_n^{\mathcal{G},\mu}$ denote the moduli stack of *prismatic* $(\mathcal{G},\mu)$*-apertures*. This is pro-Artin and pro-smooth by the main theorem.

FAQ

Question: Why do we need $\mathcal{O}$ to be unramified?

The fact that $\mathcal{O}$ is unramified supplies a natural isomorphism $\mathcal{O} \simeq W(k)$ and thus a natural map $\operatorname{Spf} \mathcal{O} \simeq \operatorname{Spf} W(k) \simeq \mathbb{\Delta}(k) \to \mathbb{\Delta}(\mathcal{O})$ (see the appendix for more on prismatization). This is used implicitly when working with $\mathrm{BT}_n^{\mathcal{G},\mu}$ using 1-bounded derived stacks.

Note that a similar trick is used in the Witt display literature, where we have $W(\mathcal{O}) \to W(k) \simeq \mathcal{O}$. Our approach riffs on this by heuristically replacing $W(\mathcal{O})$ with $\mathbb{\Delta}_{\mathcal{O}}$ (though for several reasons we work with $\mathbb{\Delta}(\mathcal{O})$ instead).

FAQ

Question: To what extent can we relax the condition that μ is 1-bounded?

Our proof of GM theory intimately depends on the 1-boundedness assumption, and GM theory isn't expected to hold if this assumption is dropped. It would be interesting to study deformation phenomena outside of the 1-bounded case and see if they are controlled by the Hodge filtration plus some additional information.

The map $\mathrm{BT}_1^{\mathcal{G},\mu} \otimes \mathbb{F}_p \to \mathrm{Disp}_1^{\mathcal{G},\mu}$ makes sense for general μ, and it seems reasonable from the perspective of $\mathcal{G}$-zip theory that this map should be well-behaved at some broad level of generality. Our study of this map makes essential use of the 1-boundedness of μ, but it's conceivable that we can get a less refined picture in a more general setting (i.e., we might not get something as nice as a gerbe description, or if we do it might have more complicated structure than a simple banding).

FAQ

Question: The isomorphism $\mathrm{BT}_{n,\mathrm{cl}} \xrightarrow{\sim} p\text{-div}_n$ tells us that BT_n has essentially no interesting derived structure. Does the same hold true for $\mathrm{BT}_n^{\mathcal{G},\mu}$?

At least for Hodge type, existing theory seems to suggest that $\mathrm{BT}_n^{\mathcal{G},\mu}$ shouldn't have any interesting derived structure. Outside of Hodge type it could go either way, but my guess (not implicating Keerthi or Akhil on this) is that $\mathrm{BT}_n^{\mathcal{G},\mu}$ can't have much nontrivial derived structure at least for abelian type (I make no claims beyond that!). Whatever nontrivial derived structure there is would help explain why we have struggled to find moduli descriptions in general. I somehow expect $\mathrm{BT}_{n,\mathrm{cl}}^{\mathcal{G},\mu}$ to be close to describable in concrete terms, so even if it doesn't take values in Set it would at worst take values in m-groupoids for small m (which we would hopefully be able to take independent of $(\mathcal{G},\mu)$).

FAQ

Question: Witt-theoretic methods require us to parametrize the Hodge filtration as data, so why don't we need to do that in our approach?

Basically, the Hodge filtration is already encoded by the Nygaard filtration. Let R be a discrete p-differentially finite ring and $\mathscr{E} \in \mathrm{BT}_{\infty}^{h,d}(R)$. The pullback $\rho_{\mathrm{Hdg}}^{*}\mathscr{E}$ (see the appendices) is a vector bundle over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spf} R$ and so can be identified with a finitely filtered vector bundle $\mathrm{Fil}_{\mathrm{Hdg}}^{\bullet} E$ over R. Indeed, by assumption we have $\operatorname{rank} \mathrm{gr}_{\mathrm{Hdg}}^{-1} E = d$ and $\operatorname{rank} \mathrm{gr}_{\mathrm{Hdg}}^{0} E = h - d$. Letting $\mathfrak{G}$ denote the p-divisible group associated with $\mathscr{E}$, there is a natural equivalence of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}_{\mathrm{Hdg}}^{0} E \arrow[r] \arrow[d, "\simeq"] & E \arrow[r] \arrow[d, "\simeq"] & \mathrm{gr}_{\mathrm{Hdg}}^{-1} E \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \omega_{\mathfrak{G}^*} \arrow[r] & \mathbb{D}(\mathfrak{G}^*)(R) \arrow[r] & \mathrm{Lie}(\mathfrak{G}) \arrow[r] & 0
\end{tikzcd}$$

Complements to the Results

Let X be the mod-p reduction of the integral canonical model of a Hodge type Shimura variety with *hyperspecial* level at p, with $(\mathcal{G}, \mu)$ specified accordingly. There is a smooth morphism $\zeta_1^{\mathcal{G},\mu} : X \to \mathrm{Disp}_1^{\mathcal{G},\mu}$, which induces the Ekedahl-Oort stratification on X and is the starting point for constructing (generalized) Hasse invariants.

Theorem (Shen)

There is a smooth morphism $\xi_1^{\mathcal{G},\mu} : X \to \mathrm{BT}_1^{\mathcal{G},\mu} \otimes \mathbb{F}_p$ lifting $\zeta_1^{\mathcal{G},\mu}$ – i.e., the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\xi_1^{\mathcal{G},\mu}"] \arrow[dr, "\zeta_1^{\mathcal{G},\mu}"'] & \mathrm{BT}_1^{\mathcal{G},\mu} \otimes \mathbb{F}_p \arrow[d] \\ & \mathrm{Disp}_1^{\mathcal{G},\mu} \end{tikzcd}$$

In fact, this result lifts to $\mathrm{BT}_1^{\mathcal{G},\mu}$ by work of Imai-Kato-Youcis.

Complements to the Results

Lau essentially defines the group scheme $\mathrm{Lau}_n^{h,d}$ by the property that the natural map $p\text{-div}_n^{h,d} \otimes \mathbb{F}_p \to \mathrm{Disp}_n^{h,d}$ is a gerbe banded by $\mathrm{Lau}_n^{h,d}$. It's natural to wonder whether $\mathrm{BT}_n^{h,d} \otimes \mathbb{F}_p \to \mathrm{Disp}_n^{h,d}$ is a gerbe banded by $\mathrm{Lau}_n^{h,d}$, and indeed this is a result of Drinfeld. Drinfeld goes on to define a group scheme $\mathrm{Lau}_n^{\mathcal{G},\mu}$ using so-called *Zink complexes*, and even gives a more explicit description of what he calls the *Cartier dual* $(\mathrm{Lau}_n^{\mathcal{G},\mu})^*$. Given his and our results, the following conjecture is then natural.

Conjecture (Drinfeld)

The natural map $\mathrm{BT}_n^{\mathcal{G},\mu} \otimes \mathbb{F}_p \to \mathrm{Disp}_n^{\mathcal{G},\mu}$ *is a gerbe banded by* $\mathrm{Lau}_n^{\mathcal{G},\mu}$.

If this conjecture were true then the underlying topological spaces of $\mathrm{BT}_n^{\mathcal{G},\mu}$ and $\mathrm{Disp}_n^{\mathcal{G},\mu}$ would be homeomorphic. The difficulty here lies in studying the behavior of the map $\mathrm{Syn}_n(\underline{W(R)}) \to \mathrm{Syn}(\underline{W_n(R)})$.

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Appendix A: Overview of Prismatization

Let's start with a list of the key players.

- $\Delta(\mathbb{Z}_p)$ the *prismatization*
- $F : \Delta(\mathbb{Z}_p) \to \Delta(\mathbb{Z}_p)$ the *Frobenius*, which is faithfully flat
- $\mathrm{HT}(\mathbb{Z}_p) \hookrightarrow \Delta(\mathbb{Z}_p)$ the *Hodge-Tate locus*, an effective Cartier divisor cut out by $\mathcal{J}_{\mathrm{HT}} \in \mathrm{Pic}(\Delta(\mathbb{Z}_p))$ the *Hodge-Tate ideal sheaf*
- $\mathrm{Nyg}(\mathbb{Z}_p)$ the *(Nygaard-)filtered prismatization*
- $\rho_{\mathrm{Hdg}} : \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, \mathbb{Z}_p \to \mathrm{Nyg}(\mathbb{Z}_p)$ the *Hodge map*
- $j_{\mathrm{dR}}, j_{\mathrm{HT}} : \Delta(\mathbb{Z}_p) \to \mathrm{Nyg}(\mathbb{Z}_p)$ the *de Rham* and *Hodge-Tate embeddings*, which are disjoint open immersions
- $\mathrm{Syn}(\mathbb{Z}_p)$ the *syntomification*, which is the coequalizer of j_{dR} and j_{HT}
- $\mathcal{O}_{\Delta}\{1\} \in \mathrm{Pic}(\Delta(\mathbb{Z}_p))$, $\mathcal{O}_{\mathcal{N}}\{1\} \in \mathrm{Pic}(\mathrm{Nyg}(\mathbb{Z}_p))$, and $\mathcal{O}_{\mathrm{syn}}\{1\} \in \mathrm{Pic}(\mathrm{Syn}(\mathbb{Z}_p))$ the *Breuil-Kisin (BK) twists*

Appendix A: Overview of Prismatization

Here are some important supplementary comments.

- All of these constructions admit analogues for arbitrary R, through a combination of pullback and a process called *transmutation*. Formally, the construction of $\mathbb{\Delta}(R)$ from $\mathbb{\Delta}(\mathbb{Z}_p)$ via transmutation mirrors how we go from the absolute prismatic site $(\mathbb{Z}_p)_{\mathbb{\Delta}}$ to $(R)_{\mathbb{\Delta}}$.
- $\mathbb{\Delta}(\mathbb{Z}_p)$, $\mathrm{Nyg}(\mathbb{Z}_p)$, and $\mathrm{Syn}(\mathbb{Z}_p)$ are all p-adic formal stacks, with level-n truncated variants $\mathbb{\Delta}_n(\mathbb{Z}_p)$, $\mathrm{Nyg}_n(\mathbb{Z}_p)$, and $\mathrm{Syn}_n(\mathbb{Z}_p)$.
- Important points include the *de Rham point* $\rho_{\mathrm{dR}} : \mathrm{Spf}\,\mathbb{Z}_p \to \mathbb{\Delta}(\mathbb{Z}_p)$ and *Hodge-Tate point* $\rho_{\mathrm{HT}} : \mathrm{Spf}\,\mathbb{Z}_p \to \mathrm{HT}(\mathbb{Z}_p) \hookrightarrow \mathbb{\Delta}(\mathbb{Z}_p)$.
- $\mathrm{Nyg}(\mathbb{Z}_p)$ comes equipped with a structure map $\pi_{\mathbb{\Delta}} : \mathrm{Nyg}(\mathbb{Z}_p) \to \mathbb{\Delta}(\mathbb{Z}_p)$ and the *Rees map* $r_{\mathcal{N}} : \mathrm{Nyg}(\mathbb{Z}_p) \to \mathbb{A}^1/\mathbb{G}_m$.
- $\mathbb{A}^1/\mathbb{G}_m$ comes equipped with the *Tate twist* $\mathscr{O}_{\mathrm{Tate}}(1)$, which acts on generalized Cartier divisors (over S) by $(L \xrightarrow{s} S) \mapsto (S \xrightarrow{s^\vee} L^\vee)$.

Appendix A: Overview of Prismatization

Let's now look at how these things fit together.

- $\pi_{\Delta} \circ j_{\mathrm{dR}} = \mathrm{id}$ while $\pi_{\Delta} \circ j_{\mathrm{HT}} = F$, hence j_{dR} and j_{HT} "differ by Frobenius."
- The Hodge map ρ_{Hdg} is filtered (i.e., it commutes with $r_{\mathcal{N}}$).
- There is an identification of the image of j_{dR} with the non-vanishing locus of ρ_{Hdg} – i.e., j_{dR} is obtained from $\mathrm{Spf}\,\mathbb{Z}_p \to \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\,\mathbb{Z}_p$ via pullback along $r_{\mathcal{N}}$.
- ρ_{Hdg} and j_{dR} fit into the commutative square

$$\begin{tikzcd} \mathrm{Spf}\,\mathbb{Z}_p \arrow[r, "\rho_{\mathrm{dR}}"] \arrow[d] & \Delta(\mathbb{Z}_p) \arrow[d, "j_{\mathrm{dR}}"] \\ \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\,\mathbb{Z}_p \arrow[r, "\rho_{\mathrm{Hdg}}"'] & \mathrm{Nyg}(\mathbb{Z}_p) \end{tikzcd}$$

Appendix A: Overview of Breuil-Kisin Twists

We can also look more specifically at the BK twists.

- There is a canonical isomorphism $F^*\mathscr{O}_{\Delta}\{1\} \otimes \mathcal{J}_{\mathrm{HT}} \xrightarrow{\sim} \mathscr{O}_{\Delta}\{1\}$.
- By prior work we have $F^*\mathscr{O}_{\Delta}\{1\} \simeq j_{\mathrm{HT}}^*(\pi_{\Delta}^*\mathscr{O}_{\Delta}\{1\})$ and $\mathscr{O}_{\Delta}\{1\} \simeq j_{\mathrm{dR}}^*(\pi_{\Delta}^*\mathscr{O}_{\Delta}\{1\})$.
- We have $\mathcal{J}_{\mathrm{HT}} \simeq j_{\mathrm{HT}}^*(r_{\mathcal{N}}^*\mathscr{O}_{\mathrm{Tate}}(1))$. Prior work tells us that $j_{\mathrm{dR}}^*(r_{\mathcal{N}}^*\mathscr{O}_{\Delta}\{1\}) \simeq \mathscr{O}_{\Delta(\mathbb{Z}_p)}$ (i.e., we get the *trivial* line bundle).
- We then *define* $\mathscr{O}_{\mathcal{N}}\{1\} := \pi_{\Delta}^*\mathscr{O}_{\Delta}\{1\} \otimes r_{\mathcal{N}}^*\mathscr{O}_{\mathrm{Tate}}(1)$, which comes equipped with a canonical isomorphism $j_{\mathrm{HT}}^*\mathscr{O}_{\mathcal{N}}\{1\} \xrightarrow{\sim} j_{\mathrm{dR}}^*\mathscr{O}_{\mathcal{N}}\{1\}$ and so we obtain $\mathscr{O}_{\mathrm{syn}}\{1\}$ over $\mathrm{Syn}(\mathbb{Z}_p)$ via descent.
- $\mathscr{O}_{\mathcal{N}}\{1\}$ and $r_{\mathcal{N}}$ induce the same grading on $\mathrm{Nyg}(\mathbb{Z}_p)$, which is an important compatibility for reconciling the different approaches to working with prismatic $(\mathcal{G}, \mu)$-apertures.

Appendix A: Witt and PD Envelope Constructions

Now for some actual constructions. We'll need the following notions.

- $\mathbb{W}$ is the *(p-typical) Witt ring scheme* given by $S \mapsto W(S)$. There is a canonical isomorphism $\mathbb{W} \simeq \operatorname{Spec} \mathbb{Z}\{x\}$ of ring schemes, where $\mathbb{Z}\{x\}$ is the free δ-algebra on one generator, which matches the natural notions of Frobenius on both sides. Passing to units gives the group scheme $\mathbb{W}^\times$.
- $\mathbb{W}_{\text{dist}}$ parametrizes *distinguished* Witt vectors, which are those $(a_0, a_1, \ldots)$ (written in Witt coordinates) such that a_0 is nilpotent and a_1 is a unit. Thus, $\mathbb{W}_{\text{dist}} \simeq \varinjlim_{i \geq 1} \operatorname{Spec} \mathbb{Z}[x_0, x_1^{\pm 1}, x_2, \ldots]/(x_0^i)$, with an action of $\mathbb{W}^\times$ given by multiplication.
- We have the PD-envelopes $\mathbb{G}_a^\# \simeq \operatorname{Spec} \mathbb{Z}[\{x^i/i!\}_{i \geq 1}]$ and $\mathbb{G}_m^\# \simeq \operatorname{Spec} \mathbb{Z}[x^{-1}, \{(x-1)^i/i!\}_{i \geq 1}]$. There are maps $\mathbb{W}[F] \to \mathbb{G}_a^\#$ and $\mathbb{W}^\times[F] \to \mathbb{G}_m^\#$ which are isomorphisms after base change to $\mathbb{Z}_{(p)}$.

Appendix A: Anatomy of Prismatization

View all of the previous constructions as living over $\operatorname{Spf}\mathbb{Z}_p$, for notational convenience. Given $S \in \mathrm{CAlg}^{p\text{-nilp},\heartsuit}$, a generalized Cartier divisor $J \to W(S)$ belongs to $\mathbb{\Delta}(\mathbb{Z}_p)(S)$ (and is called a *Cartier-Witt divisor* over S) if any of the following equivalent conditions are satisfied.

(i) $J \to W(S)$ defines an animated prism.

(ii) The image of $J \to W(S) \twoheadrightarrow S$ is nilpotent and the image of $J \to W(S) \xrightarrow{\delta} W(S)$ generates the unit ideal.

(iii) There is a Zariski cover $\{S_i\}_i$ of S such that each $W(S_i) \otimes_{W(S)} J$ is principal and some (hence any) generator is mapped via $W(S_i) \otimes_{W(S)} J \to W(S_i)$ to a distinguished element.

The last condition tells us that $\mathbb{\Delta}(\mathbb{Z}_p) \simeq \mathbb{W}_{\mathrm{dist}}/\mathbb{W}^{\times}$. Via transmutation, the data of an S-point of $\mathbb{\Delta}(R)$ is equivalent to the data of $J \to W(S)$ as above together with a map $R \to \overline{W(S)} := W(S)/J$ of animated rings (note that $J \to W(S)$ is a quasi-ideal).

Let's use this description to write some other things down.

- F is given on S-points (using the Witt vector Frobenius) by the pullback $(J \to W(S)) \mapsto (W(S) \otimes_{F,W(S)} J \to W(S))$.
- $\mathrm{HT}(\mathbb{Z}_p)$ is given on S-points by requiring $J \to W(S) \twoheadrightarrow S$ to vanish. Hence, $\mathrm{HT}(\mathbb{Z}_p) \simeq \mathbb{W}^0_{\mathrm{dist}}/\mathbb{W}^\times$ for $\mathbb{W}^0_{\mathrm{dist}} \simeq \operatorname{Spf} \mathbb{Z}_p[x_1^{\pm 1}, x_2, \ldots]$. Relatedly, $\mathcal{J}_{\mathrm{HT}} \in \operatorname{Pic}(\Delta(\mathbb{Z}_p))$ is given by $(J \to W(S)) \mapsto S \otimes_{W(S)} J$.
- Any (animated) prism (A, I) induces $\rho_{A,I} : \operatorname{Spf} A \to \Delta(\mathbb{Z}_p)$ (where $\operatorname{Spf} A$ is completed along (p, I)), given on S-points by $(A \to S) \mapsto (I \to A \to W(S))$ using the induced δ-algebra map $A \to W(S)$. More generally, $(A, I) \in (R)_\Delta$ induces $\rho_{A,I} : \operatorname{Spf} A \to \Delta(R)$. These morphisms provide a complete picture of quasicoherent sheaves over $\Delta(R)$, and give meaning to the heuristic that $\Delta(R)$ behaves like the *universal* prism over R (which doesn't exist in general).

Appendix A: Anatomy of Prismatization

- On S-points, ρ_{dR} picks out $W(S) \xrightarrow{p} W(S)$, while ρ_{HT} picks out $W(S) \xrightarrow{V(1)} W(S)$. These agree when S has characteristic p but differ in general. Note that $\rho_{\mathrm{dR}} = \rho_{\mathbb{Z}_p,(p)}$.
- $\mathrm{HT}(R) \hookrightarrow \Delta(R)$ is given by the pullback of $\mathrm{HT}(\mathbb{Z}_p) \hookrightarrow \Delta(\mathbb{Z}_p)$ along $\Delta(R) \to \Delta(\mathbb{Z}_p)$. We have $\rho_{\mathrm{dR}} : \mathrm{Spf}\, \mathbb{Z}_p \xrightarrow{\sim} \Delta(\mathbb{F}_p)$, and in fact $\mathrm{HT}(\mathbb{F}_p) \hookrightarrow \Delta(\mathbb{F}_p)$ identifies with $\mathrm{Spec}\, \mathbb{F}_p \hookrightarrow \mathrm{Spf}\, \mathbb{Z}_p$.
- We have $\mathrm{Aut}(\rho_{\mathrm{HT}}) \simeq \mathbb{W}^\times[F] \simeq \mathbb{G}_m^\#$, which induces $B\mathbb{G}_m^\# \xrightarrow{\sim} \mathrm{HT}(\mathbb{Z}_p)$ since ρ_{HT} is locally surjective with respect to the fpqc topology.

There are many other interesting things we could say about prismatization and the Hodge-Tate locus, but let's leave it at this for now.

Appendix B: Bootstrapping Under the Rug

Let's discuss an important technical notion we've so far neglected, which produces a broad class of rings which are suitably well-behaved.

Definition

$R \in \mathrm{CAlg}$ is *p*-**differentially finite** if $\Omega^1_{(\pi_0(R)/p)/\mathbb{F}_p}$ is a finitely generated $\pi_0(R)/p$-module and R is p-complete (for convenience). These form a full ∞-subcategory $\mathrm{CAlg}^{p\text{-dfin}}$.

Examples of such rings include R such that $\pi_0(R)$ is a

- semiperfectoid ring;
- finitely generated $\mathbb{Z}/p^c$-algebra;
- complete local Noetherian $\mathbb{Z}_p$-algebra with residue field κ such that $[\kappa : \kappa^p] < \infty$, where κ^p is the subfield of pth powers.

Appendix B: Bootstrapping Under the Rug

Here is the key technical result.

Proposition

Let R be a discrete p-differentially finite ring. Then, there exists a quasisyntomic cover $R \to R_\infty$ such that each term of $R_\infty^{\otimes_R \bullet}$ is semiperfectoid.

Given a derived p-adic formal stack, this allows us to bootstrap from semiperfect(-oid) inputs by appealing quasisyntomic descent.