Complete classification of integrability and non-integrability for spin-1/2 chain with symmetric nearest-neighbor interaction

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Background and Claim

Integrable systems : well studied for a long time
Non-integrable systems: recently, a few models were shown to be non-integrable

Def: non-integrable systems
Systems without **local conserved quantities**
- **local**: sum of operators on spatially-local contiguous support
- **conserved**: commutative with the Hamiltonian

Expectation: almost all systems are non-integrable, based on thermalization, etc.

Main Theorem
Symmetric S=1/2 chains with n.n. interaction

$$H=\sum_{i=1}^{N}\left(\begin{array}{l} J_{XX}\,X_iX_{i+1}+J_{XY}\,X_iY_{i+1}+J_{XZ}\,X_iZ_{i+1}\\ +\,J_{YX}\,Y_iX_{i+1}+J_{YY}\,Y_iY_{i+1}+J_{YZ}\,Y_iZ_{i+1}\\ +\,J_{ZX}\,Z_iX_{i+1}+J_{ZY}\,Z_iY_{i+1}+J_{ZZ}\,Z_iZ_{i+1}\\ +\qquad h_X\,X_i+\qquad h_Y\,Y_i+\qquad h_Z\,Z_i \end{array}\right)$$

with $J_{XY}=J_{YX}\,,J_{YZ}=J_{ZY}\,,J_{ZX}=J_{XZ}$

are generally **non-integrable,**
except for already known integrable systems
(classical, transverse Ising, XY, XXZ, XYZ)

Proof Strategy

Expand arbitrary local quantity Q in the basis of local Pauli strings:

$$Q=\sum_i\left(\begin{array}{l} q_{X_i}X_i+q_{Y_i}Y_i+q_{Z_i}Z_i\\ +q_{X_iX_{i+1}}X_iX_{i+1}+\cdots+q_{Z_iZ_{i+1}}Z_iZ_{i+1}\\ +q_{X_iX_{i+1}X_{i+2}}X_iX_{i+1}X_{i+2}+\cdots\\ +\cdots \end{array}\right)$$

Note: commutator of two local Pauli strings is another local Pauli string (or 0)
ex.)

$$[X_iY_{i+1}Z_{i+2}\,,Y_{i-1}Y_i]=2i\,Y_{i-1}Z_i\,Y_i\,Z_{i+2}\quad\xRightarrow[\text{(drop }2i)]{}\quad \begin{array}{ccccc} & & X_i & Y_{i+1} & Z_{i+2}\\ & Y_{i-1} & Y_i & & \\ \hline + & Y_{i-1} & Z_i & Y_{i+1} & Z_{i+2} \end{array}$$

($[\,\bullet\,,H]$ is a linear map on local quantities space)

Non-integrability
= absence of local conserved quantities
= no solution $\{q\}$ satisfying $[Q,H]=0$

Goal: **derive that q of all local Pauli strings are 0** from $[Q,H]=0$

Proof Sketch

By local unitary transformation, any Hamiltonian can be reduced into

$$H=\sum_{i=1}^{N}\left(\begin{array}{l} J_X\,X_iX_{i+1}+J_Y\,Y_iY_{i+1}+J_Z\,Z_iZ_{i+1}\\ +\qquad h_X\,X_i+\qquad h_Y\,Y_i+\qquad h_Z\,Z_i \end{array}\right)$$

rank of $\mathrm{diag}(J_X,J_Y,J_Z)$ is crucial to the proof

rank 0, rank 1, part of rank 3: solved in prior studies

→ We focus on rank 2 case $(J_X,J_Y\neq 0\,,J_Z=0)$

$$\begin{cases}\text{integrable if } & (h_X,h_Y)=0 \quad \text{the XY model}\\ \text{non-integrable if } & (h_X,h_Y)\neq 0 \quad \leftarrow \text{prove it below}\end{cases}$$

Assume that Q is a k-support conserved quantity

Step 1 $(k+1)$-supp. coeff. of $[Q,H]$ are 0 ⇒ Most k-supp. coeff. of Q are 0

Coefficients are $\mathbf{0}$ if the leftmost is Z, XX, or XI

ex.)

$$\begin{array}{r|ccccc} & & \multicolumn{3}{c}{\xleftarrow{\quad k \quad}\!\!\xrightarrow{}} & \\ Q\text{'s term} & & Z_i & \cdots & X_{i+k-1} & \\ H\text{'s term} & & & & Y_{i+k-1} & Y_{i+k}\\ \hline [Q,H]\text{'s term} & + & Z_i & \cdots & Z_{i+k-1} & Y_{i+k}\\ & & \multicolumn{4}{c}{\xleftarrow{\quad k+1 \quad}\!\!\xrightarrow{}} \end{array}$$

(⋯ is arbitrary)

No other contributions to $Z_i\cdots Z_{i+k-1}Y_{i+k}$ exist

$$\text{❌}\ \begin{array}{cccc} ? & \cdots & ? & \\ & & ? & ?\\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}\qquad \text{❌}\ \begin{array}{ccccc} & ? & \cdots & & ?\\ ? & ? & & & \\ \hline Z_i & & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

$$\rightarrow\ \underbrace{\boxed{J_Y}}_{=0}\,q_{Z_i\cdots X_{i+k-1}}=0$$

Coefficients are $\mathbf{0}$ if the leftmost is XY

ex.)

$$\begin{array}{cccccc} & X & Y & \cdots & Y & \\ & & & & X & X\\ \hline - & X & Y & \cdots & Z & X \end{array}\ \xleftarrow{\ }\ \text{two}\ \xrightarrow{\ }\ \begin{array}{cccccc} & & Z & \cdots & Z & X\\ & X & X & & & \\ \hline + & X & Y & \cdots & Z & X \end{array}$$

two contributions

$$\rightarrow\ -\underbrace{\boxed{J_Y}}_{=0}\,q_{XY\cdots Y}+J_X\underbrace{\boxed{q_{Z\cdots ZX}}}_{=0}=0\qquad\rightarrow\ q_{XY\cdots Y}=0$$

Repeating similar arguments, we find that all coefficients other than $XZZ\cdots ZZX$, $XZZ\cdots ZZY$, $YZZ\cdots ZZX$, and $YZZ\cdots ZZY$ are 0

Step 2 k-supp. coeff. of $[Q,H]$ are 0 ⇒ Remaining k-supp. coeff. of Q are 0

(easy ex. of $k=3$)

$$\begin{array}{cccc} & Y & Z & Y\\ & & X & \\ \hline + & Y & Y & Y \end{array}\ \leftarrow \text{no other contributions}$$

$$\text{❌}\begin{array}{ccc} ? & ? & ?\\ ? & & \\ \hline Y & Y & Y \end{array}\quad \text{❌}\begin{array}{ccc} ? & ? & ?\\ & ? & \\ \hline Y & Y & Y \end{array}\quad \text{❌}\begin{array}{ccc} ? & ? & ?\\ & & ?\\ \hline Y & Y & Y \end{array}$$

$$\text{❌}\begin{array}{ccc} & ? & ?\\ ? & ? & \\ \hline Y & Y & Y \end{array}\quad \text{❌}\begin{array}{ccc} ? & ? & \\ & ? & ?\\ \hline Y & Y & Y \end{array}$$

$$\rightarrow\ h_X\,q_{YZY}=0$$