A generalization of the Center Theorem of the Thurston-Wolpert-Goldman Lie algebra

Aoi Wakuda
Graduate School of Mathematical Sciences, The University of Tokyo

Let S be an oriented surface (possibly with boundary and punctures). We assume the Euler characteristic of S is negative so that S admits a hyperbolic metric. We also assume that the interior of S is not homeomorphic to that of a pair of pants. Denote by $\hat{\pi}$ the set of free homotopy classes of directed closed curves on S. Unless otherwise specified, we assume K to be a commutative ring containing the ring $\mathbb{Z}\left[\frac{1}{2}\right]$. We denote by $K\hat{\pi}$ the free K-module generated by $\hat{\pi}$.

Definition

The *Goldman bracket* of α, $\beta \in \hat{\pi}$ is defined by the formula

$$[\alpha, \beta]_G := \sum_{P \in \alpha \cap \beta} \varepsilon_P(\alpha, \beta) |\alpha_P \beta_P|.$$

Here the representatives α and β are chosen so that they intersect transversely in a set of double points $\alpha \cap \beta$, $\varepsilon_P(\alpha, \beta)$ denotes the sign of the intersection between α and β at an intersection point P, and $\alpha_P \beta_P$ denotes the loop product of α and β at P.

Goldman proved the bracket defined above makes $K\hat{\pi}$ a Lie algebra. There is a Lie algebra involution $\iota : K\hat{\pi} \to K\hat{\pi}$ given by $\alpha \mapsto \alpha^{-1}$, which maps the curve α to the curve α with opposite orientation. The involution ι defines two submodules of the Goldman Lie algebra of S.

Definition

Let A_0 be a submodule of $K\hat{\pi}$ generated by the elements of the form $\alpha + \iota\alpha$ and A_1 a submodule of $K\hat{\pi}$ generated by the elements of the form $\alpha - \iota\alpha$.

The Goldman Lie algebra $K\hat{\pi}$ has the decomposition

$$K\hat{\pi} = A_0 \oplus A_1.$$

Since $\iota[\alpha, \beta]_G = [\iota\alpha, \iota\beta]_G$ for $\alpha, \beta \in K\hat{\pi}$, the submodule A_0 is a Lie subalgebra of $K\hat{\pi}$. The Lie algebra A_0 is called the *Thurston-Wolpert-Goldman Lie algebra* or, briefly, the *TWG Lie algebra*.

It is a fundamental problem to compute the center of a given Lie algebra. We also call a closed curve *non-essential* if it is homotopic to a point or a boundary curve or to a puncture.

Theorem(Chas-Kabiraj 2020)

The center of the TWG Lie algebra is generated by the class of non-essential curves as a K-module.

For any K-submodule A of $K\hat{\pi}$ and K-submodule M of a $K\hat{\pi}$-module, we denote

$$\mathrm{Ann}_M A := \{m \in M | [a, m] = 0 \text{ for all } a \in A\}.$$

The theorem above can be rephrased as determing the set $\mathrm{Ann}_{A_0} A_0$. We address the following natural question: how about the set of $\mathrm{Ann}_{A_i} A_j$ in the case $(i, j) = (0, 1), (1, 0), (1, 1)$? Our main result is to give an answer to this question.

Theorem(W.)

The annihilator $\mathrm{Ann}_{A_0} A_1$ of A_1 in A_0 is generated by the elements of the form $\alpha + \iota\alpha$ such that α is non-essential. The annihilator $\mathrm{Ann}_{A_1} A_i$ of A_i in A_1 $(i = 0, 1)$ is generated by the elements of the form $\alpha - \iota\alpha$ such that α is non-essential.

The following is a key lemma of the theorm above.

Lemma

Let α and β be elements of $\hat{\pi}$. Then the following conditions are equivalent.

(1) $\alpha - \iota\alpha = \pm(\beta - \iota\beta)$,
(2) $\alpha + \iota\alpha = \beta + \iota\beta$.

Sketch of the proof : (1) $\Rightarrow$ (2) : Denote by $\varphi : \pi_1(S) \to PSL(2, \mathbb{R})$ a natural injective group homomorphism induced by a hyperbolic metric $X \in Teich(S)$. By using this map, we can prove $\alpha \neq \iota\alpha$. Therefore, $\alpha = \beta, \iota\beta$. Thus, we obtain $\alpha + \iota\alpha = \beta + \iota\beta$.
(2) $\Rightarrow$ (1) : Clearly, $\alpha = \beta, \iota\beta$.