Crepant resolution of quotient singularities in positive characteristic

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Background: McKay correspondence

Crepant resolutions appear in the research of McKay correspondence, which is a bridge between representation theory and singularity theory. The classical McKay correspondence comes from McKay's observation [7] of two ways to construct the same ADE graph from a finite subgroup G of $\mathrm{SL}(2, \mathbb{C})$: one is to consider all non-trivial irreducible representations of G, and the other is to consider the dual graph of exceptional divisors for the minimal resolution of $\mathbb{C}^2/G$.

In higher dimensions, McKay correspondence is considered as a series of equalities between representation-theoretic invariant given by G and geometric invariant given by $\mathbb{C}^n/G$, where G is a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$.

One famous generalized version of McKay correspondence over $\mathbb{C}$ is Batyrev's theorem, in which we see crepant resolutions.

Definition

Let X be a normal algebraic variety and $f : Y \to X$ be a resolution of singularity. f is called crepant, if $K_Y = f^*K_X$.

Theorem (Batyrev[1],'99)

Let G be a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$. If there exists a crepant resolution $f : Y \to \mathbb{C}^n/G$, then the topological Euler number of Y is equal to the number of conjugacy classes of G. That is, $e(Y) = \#\mathrm{Conj}(G)$.

In Batyrev's theorem, the Euler number $e(Y)$ comes from the quotient singularity $\mathbb{C}^n/G$, while the number of conjugacy classes is equal to the number of irreducible representations of G over $\mathbb{C}$. This theorem therefore becomes a bridge connecting representation theory with geometry.

In dimension 2, crepant resolutions exist as the minimal resolutions for all quotient singularities associated to finite subgroups of $\mathrm{SL}(2, \mathbb{C})$; in dimension 3, G-Hilbert schemes give a standard construction of crepant resolutions[2]. For higher dimensions, there are examples with no crepant resolutions.

Question

How things change if we work over an algebraically closed field K of characteristic $p > 0$ instead?

Since McKay correspondence describes relationship between representation theory and geometry, we first see how representation theory changes by considering in positive characteristic.

Theorem (Maschke)

Let G be a finite group such that $\mathrm{Char} K \nmid \# G$. Then the group algebra $K[G]$ is semisimple.

This theorem inspires people to consider the following two cases of quotient singularities $\mathbb{A}_K^n/G$, where G is a finite subgroup of $\mathrm{SL}(n, K)$.

Two cases in positive characteristic

When the order of G cannot be divided by p, the case is called non-modular or tame.
When the order of G is divided by p, the case is called modular or wild.
In non-modular cases, both representation theory of G and the associated quotient singularity can be lifted to $\mathbb{C}$, and many similar results hold, such as the analogue statement of Batyrev's theorem. Compared with non-modular cases, the modular cases are usually with some worse properties.

Non-modular finite groups are similar to finite groups in characteristic 0 and therefore easier to be dealt with. In the right column of this page, we consider a generalized concept of non-modular finite groups and the associated quotient singularities, which is defined by focusing on semisimplicity that appears in Maschke's theorem.

Definition

An affine algebraic group scheme G over K is called linearly reductive, if any $K[G]$-module is semisimple.

- Non-modular finite groups are considered as constant linearly reductive finite group schemes.
- Over $\mathbb{C}$, linearly reductive finite group schemes are just all the (non-modular) finite groups. In positive characteristic, there exists non-constant linearly reductive finite group schemes.
- For a linearly reductive finite subgroup scheme of $\mathrm{SL}_{n,K}$, it has a linear action on $\mathbb{A}_K^n$ or $\widetilde{\mathbb{A}_K^n}$, which gives a linearly reductive quotient (lrq, for short) singularity.

Observation

The classification of linearly reductive finite subgroup schemes of $\mathrm{SL}_{2,K}$ is similar to the counterpart of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$[4]; similar result holds for classification of linearly reductive finite subgroup schemes of $\mathrm{SL}_{3,K}$[3].
In the thesis, we compute blow-ups for lrq singularities in dimension 2 and obtain their crepant resolutions with the same properties revealed by classical McKay correspondence.
This is an observation that lrq singularities are again similar to quotient singularities over $\mathbb{C}$, which leads to the following conjecture in dimension 3.

Conjecture (F)

Let K be an algebraically closed field of characteristic $p \geqslant 0$, and G be a linearly reductive finite subgroup scheme of $\mathrm{SL}_{3,K}$. Then G-$\mathrm{Hilb}(\mathbb{A}_K^3)$ is a crepant resolution of $\mathbb{A}_K^3/G$.

From this page, we discuss about modular cases, in which our main result lies. We start from two examples in Yasuda's p-cyclic McKay correspondence.

Theorem (Yasuda[8],'14)

1. *Let K be an algebraically closed field of characteristic 2, and C_2 be a 2-cyclic subgroup of $\mathrm{SL}(4, K)$ with no pesudo-reflections. Then $\mathbb{A}^4_K/C_2$ has a crepant resolution with Euler number 2.*
2. *Let K be an algebraically closed field of characteristic 3, and C_3 be a 3-cyclic subgroup of $\mathrm{SL}(3, K)$ with no pesudo-reflections. Then $\mathbb{A}^4_K/C_3$ has a crepant resolution with Euler number 3.*

It is furthermore known that for a cyclic quotient variety in positive characteristic, if its crepant resolution exists, then the Euler number of the resolution is equal to the order of the cyclic group - which coincides the analogue statement of Batyrev's theorem.

From these two examples, we can furthermore construct some crepant resolutions for quotient singularities associated to non-abelian modular groups.

Let $G = H \rtimes S$, where H is a non-modular abelian group consisting of diagonal matrices and S is a permutation group. In such cases, the quotient singularity given by H can be resolved by toric method and $\mathbb{A}^n/G$ can be seen as $\mathbb{A}^n/H/S$. Hence if $\mathbb{A}^n/H$ has a crepant S-equivariant resolution Y such that the action of S on Y can be locally seen as the permutation action on the affine space, then the crepant resolution can be given by the next diagram, according to Ito[5].

Construction of crepant resolution for $G = H \rtimes S$

$$
\begin{array}{ccccc}
 & & & & \widetilde{\mathbb{A}^n/G} \\
 & & & & \downarrow \\
 & & Y & \xrightarrow{/S} & Y/S \\
 & & \downarrow & & \downarrow \\
\mathbb{A}^n & \xrightarrow{/H} & \mathbb{A}^n/H & \xrightarrow{/S} & \mathbb{A}^n/G
\end{array}
$$

Here are crepant resolutions that follow from Yasuda's first example and Ito's construction.

Corollary (F)

Let K be an algebraically closed field of characteristic 2. For each positive integer n not dividing 2, let ζ_n be a primitive n-th root of unity in K. For integers a_1, a_2, a_3, a_4, denote the diagonal matrix $\mathrm{diag}(\zeta_n^{a_1}, \zeta_n^{a_2}, \zeta_n^{a_3}, \zeta_n^{a_4})$ by $\frac{1}{n}(a_1, a_2, a_3, a_4)$. Let S be a subgroup of $\mathrm{SL}(4, K)$ generated by the permutation element $(12)(34)$. Let $H \subseteq \mathrm{SL}(4, K)$ be one of the following:

1. $H = \langle \frac{1}{n}(1, 0, 0, -1), \frac{1}{n}(0, 1, 0, -1), \frac{1}{n}(0, 0, 1, -1) \rangle$, $2 \nmid n$;
2. $H = \langle \frac{1}{m}(1, -1, 0, 0), \frac{1}{n}(0, 0, 1, -1) \rangle$, $2 \nmid m, n$.

Then $G = \langle H, S \rangle$ is a finite subgroup of $\mathrm{SL}(4, K)$ with no pesudo-reflections, and $\mathbb{A}^4_K/G$ has a crepant resolution with Euler number equal to the number of conjugacy classes of G.

Similar construction can also be applied in characteristic 3 with Yasuda's second example. For the crepant resolutions above, they all agree with analogue statement of Batyrev's theorem. We will see a counterexample as our main result.

For main results of the thesis, we consider two quotient varieties in characteristic 2 such that the orders of the groups are divided by 2^2.
Using known results in modular invariant theory, we obtain the following defining equations by computation.

Proposition (F)

Let K be an algebraically closed field of characteristic 2, K_4 be the group consisting of permutations of order 2 in $\mathrm{SL}(4, K)$, and A_4 be the alternating group with permutation representation. Then

1. $\mathbb{A}^4_K/K_4 \cong V(AE + BC + CD + DB, BCD + E^2 + A^2F)$.
2. $\mathbb{A}^4_K/A_4 \cong V(E^2 + (A^2D + ABC + C^2)E + A^4D^2 + A^3C^3 + A^2B^3D + B^3C^2 + C^4)$.

Idea to deal with singularities

Our idea comes from Markushevich[6]. Assume that the singular locus is $C_1 \cup C_2$, where C_1 and C_2 are both smooth. We first compute the blow-up along C_1 and then repeatedly compute the blow-up along singular part in the exceptional divisor of the previous blow-up, until the singular locus is exactly the strict transform of C_2. Finally a series of blow-ups along singular locus can give resolution of singularity.

For $\mathbb{A}^4_K/K_4$, terminal singularity appears in computation, so it is impossible to obtain crepant resolution by this idea.
For $\mathbb{A}^4_K/A_4$, we obtain a crepant resolution which has Euler number 10. Since A_4 has 4 conjugacy classes, this is a new example of crepant resolution and a new counterexample to Batyrev's theorem in positive characteristic.

Theorem (F, main result)

Let K be an algebraically closed field of characteristic 2 and $X = \mathbb{A}^4_K/A_4$ be given by permutation action of alternating group A_4 on $\mathbb{A}^4_K$. Then X has a crepant resolution $\widetilde{X}$ with Euler number $e(\widetilde{X}) = 10$.

Further topics

- Further study on lrq singularities, such as the answer to the conjecture, and analogy of other methods in characteristic 0.
- Conceptual (likely to be representation-theoretic) explanation of the Euler number 10 that appears in the main result, and generalization on more quotient singularities by the same approach to the main result.

References

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