Gamma Integral Structure for the Blowup of $\mathbb{P}^n$ at a Point

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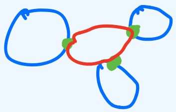
Gromov-Witten theory

In mathematics, enumerative geometry is the branch of algebraic geometry concerned with counting numbers of solutions to geometric questions.

Example:

(1) How many lines pass between two points in the plane?
This is a vary simple example in enumerative geometry. The answer is one.

(2) More generally, If we have a manifold X the Gromov-Witten invariants is going to count surfaces satisfying varies constraints. Such as pathing through a point of a giving sub-manifold or being tangent to a sub-manifold. For example the Gromov-Witten invariants could count the number of red spheres that have a common point with following three blue curves with three green points. (If the number is infinite, then the Gromov-Witten invariants is 0)



The Gromov-Witten invariants appears as correlator in topological string string theory. physicists have made predictions about the Gromov-Witten invariants and a large number of predictions are still open for mathematicians.

Reference:

https://arxiv.org/pdf/1407.1260.pdf
https://www.icts.res.in/program/JhGW2017

Research motivation and previous results

- Quantum Cohomology in Gromov-Witten theory gives us a Frobenius manifold structure. Under the semi-simplicity condition the Frobenius structure determines all higher-genus Gromov-Witten invariants. This was conjectured by Givental, who proved several special cases using fixed point localization. Teleman proved Givental's conjecture in general. The resulting theory is known as Givental-Teleman higher-genus reconstruction. The higher-genus reconstruction can be defined for any semi-simple Frobenius manifold.
- Semi-simple Frobenius manifold has a set of vectors known as reflection vectors.
  - Reflection vectors determine the monodromy group of the Frobenius manifold.
  - Reflection vectors have applications in integrable systems.
  - For Frobenius manifolds in singularity theory reflection vector is vanishing cycle.
- According to A. Bayer, if a smooth projective variety X has semi-simple quantum cohomology, then the blow-up of X at any number of points also has semi-simple quantum cohomology.

Question:

How the set of reflection vectors changes under the blow-up operation.

Every reflection vector decomposes into two parts: a cohomology class in $H^*(X)$ and a cohomology class in $H^*(E)$, where E is the exceptional divisor. The second part is essentially independent of X. In order to compute the reflection vectors, we work with the simplest possible choice $X = \mathbb{P}^n$.

Main result: Theorem 1.3

Suppose that $Q_1 = e_1^{\tau_1}$, $Q_2 = e^{\tau_2}$ where τ_1, $\tau_2 \in \mathbb{R}$. Put $\Psi_\tau(\mathcal{O}) = e^{-\tau_1 p_1 - \tau_2 p_2}\Psi(\mathcal{O})$. If $z \in \mathbb{R}_{<0}$, then

$$\int_{\mathbb{R}^n_{>0}} e^{f(x,\tau)z^{-1}}\omega = (2\pi)^{\frac{n-1}{2}}(-z)^{\frac{n}{2}}(S(0,Q,z)(-z)^{\theta}(-z)^{\rho})\Psi_\tau(\mathcal{O}), 1).$$

ρ is given by classical cup product multiplication by $c_1(T\,\mathrm{Bl}(\mathbb{P}^n))$. $S(t,Q,z)$ is the calibration. θ is the *Hodge grading operator*. $((\mathbb{C}^*)^n, f, \omega)$ is the *Givental mirror* for $\mathrm{Bl}(\mathbb{P}^n)$. Ψ is the *Iritani's map*. We will introduce these later.

About Theorem 1.3

Theorem 1.3 was proved also by Iritani[6]. We give a different proof which in some sense is simpler and we believe that our argument can be generalized for non-Fano toric manifolds. Using Theorem 1.3 we proved that $\Psi_\tau(\mathcal{O})$ is a reflection vector, where $\mathcal{O}$ is the structure sheaf of $\mathrm{Bl}(\mathbb{P}^n)$.

Future goal

We expect to find all of the reflection vectors. The key step is to find a cycle Γ_{n+2} such that the identity in Theorem 1.3 holds for $\mathbb{R}^n_{>0}$ replaced by Γ_{n+2} and $\mathcal{O}$ replaced by $\mathcal{O}_E$, where $\mathcal{O}_E$ is the structure sheaf of the exceptional divisor E.

Calibration

According to Givental, the calibration is an operator series $S = 1 + \sum_{k=1}^{\infty} S_k(t)z^{-k}$, $S_k \in \mathrm{End}(H)$, such that the Dubrovin's connection has a fundamental solution near $z = \infty$ of the form

$$S(t,z)z^{\theta}z^{-\rho},$$

where $\rho \in \mathrm{End}(H)$ is a nilpotent operator, $[\theta,\rho] = -\rho$, and the following symplectic condition holds

$$S(t,z)S(t,-z)^T = 1.$$

^{T} denotes transposition with respect to the Frobenius pairing.

Second structure connection, period vectors and reflection vectors

Our main interest is in the *2nd structure connection*

$$\nabla^{(n)}_{\partial/\partial t_i} = \frac{\partial}{\partial t_i} + (\lambda - E\bullet_t)^{-1}(\phi_i\bullet_t)(\theta - n - 1/2)$$
$$\nabla^{(n)}_{\partial/\partial \lambda} = \frac{\partial}{\partial \lambda} - (\lambda - E\bullet_t)^{-1}(\theta - n - 1/2),$$

where n in our case is an integer parameter. This is a connection on the trivial bundle

$$(M\times\mathbb{C})' \times H \to (M\times\mathbb{C})',$$

where $(M\times\mathbb{C})' = \{(t,\lambda) \mid \det(\lambda - E\bullet_t) \neq 0\}$.
Let us fix a reference point $(t^\circ,\lambda^\circ) \in (M\times\mathbb{C})'$ such that λ° is a sufficiently large real number. The following functions provide a fundamental solution to the 2nd structure connection

$$I^{(n)}(t,\lambda) = \sum_{k=0}^{\infty}(-1)^k S_k(t)\widetilde{I}^{(n+k)}(\lambda),$$

where

$$\widetilde{I}^{(m)}(\lambda) = e^{-\rho\partial_\lambda\partial_m}\Big(\frac{\lambda^{\theta-m-\frac{1}{2}}}{\Gamma(\theta-m+\frac{1}{2})}\Big).$$

Using the differential equations we extend $I^{(n)}$ to a multi-valued analytic function on $(M\times\mathbb{C})'$. We define *period vectors* as the following multi-valued functions taking values in H:

$$I^{(n)}_a(t,\lambda) := I^{(n)}(t,\lambda)\,a, \quad a\in H, \quad n\in\mathbb{Z}.$$

Using analytic continuation we get a representation

$$\pi_1((M\times\mathbb{C})', (t^\circ,\lambda^\circ)) \to \mathrm{GL}(H)$$

called the *monodromy representation* of the Frobenius manifold. The image W of the monodromy representation is called the *monodromy group*. The *intersection pairing*

$$(a|b) := (I^{(0)}_a(t,\lambda), (\lambda - E\bullet)I^{(0)}_b(t,\lambda))$$

is independent of t and λ.
Suppose now that γ is a simple loop. Up to a sign there exists a unique $a\in H$ such that $(a|a) = 2$ and the monodromy transformation of a along γ is $-a$. The monodromy transformation representing γ is the reflection defined by the following formula:

$$w_a(x) = x - (a|x)a.$$

Quantum Cohomology for $\mathrm{Bl}(\mathbb{P}^n)$

Let us consider the following diagram:

$$\begin{array}{ccc} X_{M,K} & \underset{j}{\overset{\pi_{n-1}}{\rightleftarrows}} & \mathbb{P}^{n-1}\,, \\ \downarrow{\scriptstyle \pi_n} & & \\ \mathbb{P}^n & & \end{array}$$

where $X_{M,K}$ is a toric manifold with a $2\times(n+2)$ matrix $M = \begin{pmatrix} 1 & \cdots & 1 & -1 & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$ and $K = \mathbb{R}^2_{>0}$. The maps π_{n-1}, π_n, j are defined by

$$\pi_{n-1}(z,\lambda_1,\lambda_2) := [z_1 : \cdots : z_n],$$
$$\pi_n(z,\lambda_1,\lambda_2) := [\lambda_1 z_1 : \cdots : \lambda_1 z_n : \lambda_2],$$
$$j([z_1 : \cdots : z_n]) := (z_1,\cdots,z_n,0,1).$$

Put $E := j(\mathbb{P}^{n-1}) \subset X_{M,K}$. We have

$$\pi_n(E) = [0 : \cdots : 0 : 1]$$

and

$$X_{M,K} = \mathrm{Bl}_{[0:\cdots:0:1]}(\mathbb{P}^n).$$

Let us denote

$$p_1 = \quad c_1(\pi^*_{n-1}\mathcal{O}_{\mathbb{P}^{n-1}}(1)),$$
$$p_2 = \quad c_1(\pi^*_n\mathcal{O}_{\mathbb{P}^n}(1)),$$

the generators of $H^\bullet(\mathrm{Bl}(\mathbb{P}^n);\mathbb{Z})$, with $p_1^n = 0$, $p_2(p_2 - p_1) = 0$. Let $e_1 :=$ a class of line in E, $e_2 :=$ a class of line in $\mathbb{P}^n$ avoiding $[0 : \cdots : 0 : 1]$. We have

$$H_2(\mathrm{Bl}(\mathbb{P}^n);\mathbb{Z}) = \mathbb{Z}e_1 + \mathbb{Z}e_2.$$

The novikov variables $Q = (Q_1, Q_2)$, $Q^d := Q_1^{D_1} Q_2^{D_2}$, for $d = d_1 e_1 + d_2 e_2 = D_1(e_1 - e_2) + D_2 e_2$. The quantum cup product $\bullet_Q$

$$(\Phi_a \bullet_Q \Phi_b, \Phi_c) = \sum_{d\in \mathrm{EFF}(X)} \langle \Phi_a, \Phi_b, \Phi_c\rangle_{0,3,d} Q^d,$$

where

$$\langle \Phi_a, \Phi_b, \Phi_c\rangle_{0,3} = \int_{[\overline{M}_{0,3}(X,D)]^{\mathrm{vir}}} \mathrm{ev}_1{}^* \Phi_a \cup \mathrm{ev}_2{}^* \Phi_b \cup \mathrm{ev}_3{}^* \Phi_c$$

is the number of spheres that have a common point with $P.D.(\Phi_a)$, $P.D.(\Phi_b)$ and $P.D.(\Phi_b)$. Note that if the number is infinite, then the value of the integral is zero.
For Fano toric manifold, by Givental[3], Iritani[6] and Brown[2] we have

$$S(0,Q,z)^T \cdot 1 = S(0,Q,-z)^{-1} \cdot 1 = \frac{1}{z} J_X(0,Q,-z) = I(Q,z).$$

Let $\Phi_{i,j} = p_1^{i-1} p_2^{j-1}$ be the basis of $H^\bullet(\mathrm{Bl}(\mathbb{P}^n))$. Using the formula for the J-function for Fano toric manifold we computed S^{-1}. Then we get that the quantum product of blow-up of $\mathbb{P}^n$ is given by

1) $p_1 \bullet \Phi_{i,j} = \Phi_{i+1,j}$, where $i \le n-1$, $j = 1,2$,
2) $p_2 \bullet \Phi_{i,1} = \Phi_{i,2}$, where $i \le n$,
3) $p_2 \bullet \Phi_{i,2} = \Phi_{i+1,2} + Q_2 \Phi_{i,1}$, where $i \le n-1$,
4) $p_1 \bullet \Phi_{n,2} = Q_1 Q_2 \Phi_{1,1}$,
5) $p_1 \bullet \Phi_{n,1} = Q_1 \Phi_{1,2} - Q_1 \Phi_{2,1}$,
6) $p_2 \bullet \Phi_{n,2} = Q_2 \Phi_{n,1} + Q_1 Q_2 \Phi_{1,1}$.

According to Givental, the mirror of $\mathrm{Bl}(\mathbb{P}^n)$ is given by the restriction of $f(x) := x_1 + \cdots + x_{n+2}$ to the complex torus $x \in (\mathbb{C}^*)^{n+2} : \prod_{j=1}^{n+2} x_j^{m_{ij}} = Q_i$ $(i = 1,2)$. Since $x_1 \cdots x_n x_{n+1}^{-1} = Q_1$, $x_{n+1}x_{n+2} = Q_2$. We have

$$f(x) = x_1 + \cdots + x_n + \frac{x_1 \cdots x_n}{Q_1} + \frac{Q_1 Q_2}{x_1 \cdots x_n}.$$

Put $\omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$. Then $\left((\mathbb{C}^*)^{n+2}, f, \omega\right)$ is a mirror model of $\mathrm{Bl}(\mathbb{P}^n)$ in the sense of Givental.
For projective manifold X, let us introduce *Iritani's map*.

$$\Psi : K^\circ(X)/\mathrm{torsion} \longrightarrow H^\bullet(X;\mathbb{C})$$
$$\Psi(E) = (2\pi)^{\frac{1-D}{2}} \hat{\Gamma}(X) \cup (2\pi i)^{\deg} \mathrm{ch}(E),$$

where $\hat{\Gamma}(X) = \Gamma(TX)$ and for a vector bundle E, Chern roots $x_1, \cdots .x_r$ we denote by

$$\Gamma(E) = \prod \Gamma(1 + x_i)$$

its $\Gamma-$*class*.
In our case the $\hat{\Gamma}(\mathrm{Bl}(\mathbb{P}^n)) = \Gamma(1+p_1)^n \Gamma(1+p_2)\Gamma(1+p_2-p_1)$ and $\Psi(\mathcal{O}) = (2\pi)^{\frac{1-n}{2}} \hat{\Gamma}(\mathrm{Bl}(\mathbb{P}^n))$.

Proof of main result

Theorem 1.3

$$\int_{\mathbb{R}^n_{>0}} e^{f(x,\tau)z^{-1}}\omega = (2\pi)^{\frac{n-1}{2}}(-z)^{\frac{n}{2}}(S(0,Q,z)(-z)^{\theta}(-z)^{\rho}\Psi_\tau(\mathcal{O}),1).$$

Proof: Using the formula for the J-function for Fano toric manifold [3][6][2] we have,

$$(-z)^{\theta}S(0,Q,z)^{-1}\cdot 1$$

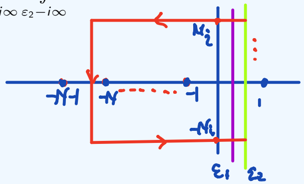
$$= \sum_{D_1,D_2\geq 0} \frac{Q_1^{D_1}Q_2^{D_2}}{\prod_{m=1}^{D_1}(-zp_1+mz)^n \prod_{m=1}^{D_2}(-zp_2+mz)\prod_{m=1}^{D_2-D_1}(-zp_2+zp_1+mz)}(-z)^{-\frac{n}{2}}.$$

Therefore,

$$\text{RHS} = \sum_{D_1=0}^{\infty}\sum_{D_2=0}^{\infty} \text{Res}_{p_1=-D_1}(\text{Res}_{p_2=p_1-D_2}+\text{Res}_{p_2=-D_2})\Gamma(p_1))^n\Gamma(p_2)\Gamma(p_2-p_1)e^{-p_1\tau_1-p_2\tau_2}dp_1dp_2.$$

For the LHS our key observation is that Fourier transformation of the oscillator integral w.r.t $\tau_i = \log Q_i$ is a product of Γ-functions. Therefore, using inverse Fourier transformation, we get a Melin-Barnes integral,

$$LHS = \left(\frac{1}{2\pi}\right)^2 \int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}\int_{\varepsilon_2-i\infty}^{\varepsilon_2+i\infty} e^{-p_2\tau_2-p_1\tau_1}\Gamma(p_2)\Gamma(p_2-p_1)\Gamma(p_1)^n dp_2dp_1.$$



We proved that the integral over red contour goes to 0 as N goes to $+\infty$.

$$LHS = \left(\frac{1}{2\pi}\right)\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}e^{-p_1t_1}\Gamma(p_1)^n\left(e^{(-j+p_1)t_2}\Gamma(-j+p_1)+e^{-jt_2}\Gamma(-j-p_1)\right)dp_1.$$

When $\text{Re}\, p_1 = \varepsilon_1$, for all j, we have

$$\left|e^{-p_1t_1}\Gamma(p_1)^n\left(e^{(-j+p_1)t_2}\Gamma(-j+p_1)+e^{-jt_2}\Gamma(-j-p_1)\right)\right| \leq O\left(\frac{e^{-jt_2}}{(j-1)!}\right).$$

This means that the function of p_1 in the integral is uniformly absolutely-convergent when $\text{Re}\, p_1 = \varepsilon_1$. Therefore, the order of summation $\sum_{j=0}^{\infty}$ and integration $\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}$ is interchangeable. By the similar way we have done in first integral, we proved that the LHS equals to the same residue as the RHS.

$\Psi_\tau(\mathcal{O})$ is a reflection vector

The function $g_{\tau_1,\tau_2}(x) := f(x,\tau_1,\tau_2) = x_1+\cdots+x_n+\frac{x_1\cdots x_n}{Q_1}+\frac{Q_1Q_2}{x_1\cdots x_n}$ defines a real-valued function on $\mathbb{R}^n_{>0}$ with minimal value $u(\tau_1,\tau_2)$. Put $\alpha_\lambda = \{x\in\mathbb{R}^n_{>0}|g_{\tau_1,\tau_2}(x)\leq\lambda\}$. For all $m\in Q$ let us define $\mathcal{I}^{(-m)}(\tau_1,\tau_2,\lambda) := \int_{\alpha_\lambda}\frac{(\lambda-f(x,\tau_1,\tau_2))^{m+\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)}\omega$.

Lemma 3.8 If λ is sufficient close to $u(\tau_1,\tau_2)$, then Morse lemma for f applies

$$\mathcal{I}^{(-m)}(\tau_1,\tau_2,\lambda) = (\lambda-u(\tau_1,\tau_2))^{\frac{n-1}{2}+m}(c_0(\tau_1,\tau_2)+c_1(\tau_1,\tau_2)(\lambda-u(\tau_1,\tau_2))+\cdots).$$

Lemma 3.9 We have

$$\int_{u(\tau_1,\tau_2)}^{\infty} e^{\frac{\lambda}{z}}\mathcal{I}^{(-m)}(\tau_1,\tau_2,\lambda)\,d\lambda = (-z)^{m+\frac{1}{2}}\int_{\mathbb{R}^n_{>0}} e^{\frac{f(x,\tau_1,\tau_2)}{z}}\omega,$$

where τ_1, $\tau_2\in\mathbb{R}$, $z\in\mathbb{R}_{<0}$.

Let us denote $I_E^{(-m-1)} = I^{(-m-1)}E$. The period vectors for quantum cohomology.

Lemma 3.10 Exists $E_0\in H^*(X;\mathbb{C})$ independent of τ_1, τ_2 and λ such that

$$(I_E^{(-m-1)},\Phi_{i,j}) = (-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^{j-1}\mathcal{I}^{(-m-i-j+1+\frac{n}{2})}(\tau,\lambda)$$

where $\tau=(\tau_1,\tau_2)$, $E=e^{-\tau_1p_1-\tau_2p_2}E_0$ and $Q=(Q_1,Q_2)=(e^{\tau_1},e^{\tau_1})$.

By Theorem 1.3, Lemma 3.9 and Lemma 3.10, we get Lemma 3.11.

Lemma 3.11 $E_0 = (2\pi)^{\frac{n-1}{2}}\Psi(\mathcal{O})$.

Therefore, E is proportional to a reflection vector. Since $E = (2\pi)^{\frac{n-1}{2}}\Psi_\tau(\mathcal{O})$, in order to prove that $\Psi_\tau(\mathcal{O})$ is a reflection vector, we need only to check that $(\Psi_\tau(\mathcal{O})|\Psi_\tau(\mathcal{O}))=2$.

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