On the Pimsner-Popa Index of Central Sequence Subalgebras

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Background and Introduction

The theory of von Neumann algebras was initiated in the 1930s and studies certain algebras of linear operators on Hilbert spaces. The theory has deep interrelations with many other mathematical disciplines such as representation theory and ergodic theory. Moreover, the applications of operator algebras to quantum physics have continued to reveal unexpected connections (as observables are represented by unbounded operators on a Hilbert space), for example, type III von Neumann algebras are one of the main objects of Quantum Field Theory.

Vaughan Jones studied the inclusions of certain von Neumann algebras, called II_1 factors, and their indexes in the 1980s. Jones suggested that the possible values of index of type II_1 factors are quite supernormal and then led to the discovery of Jones Polynomial, which is an important invariant in knot theory. Although the result was axiomatized by topologist without the foundation of theory of von Neumann algebras, the study of index of factors or von Neumann algebras is now of independent interest.

The notions of ultrapower algebras M^ω and central sequence algebras $M_\omega = M' \cap M^\omega$ for type II_1 factors M play an important role in the theory of type II_1 factors. Central sequences were introduced by Murray and von Neumann [MvN36] to distinguish the hyperfinite II_1 factor R and the free group factor $L(\mathbb{F}_n)$. Later, the central sequence algebra and the ultraproduct technique were widely studied especailly in the classification of type II_1 factors and the classification of group actions on type II_1 factors.

We are concerned with the following problem. Let $N \subset M$ be an inclusion of hyperfinite II_1 factors with finite index and let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. What can we say about the (Pimser-Popa) index of $M' \cap N^\omega \subset M' \cap M^\omega$?

Here $M' \cap N^\omega$ is called the central sequence subalgebra of $M' \cap M^\omega$.

The problem can be mainly divided into two parts according to the depth of $N \subset M$. If $N \subset M$ has a finite depth, the central sequence subalgebra is actually a type II_1 factor and the Pimsner-Popa index coincides with the Jones index. The problem has already solved by Kawahigashi [Kaw92]. However, the result is less clear in the infinite-depth case. $M' \cap M^\omega$ may not be a factor and it is intractable to determine the conditional expectation $E_{M' \cap N^\omega}(x)$ of non-scaler elements in $M' \cap M^\omega$. Jones has conjectured the index $[M' \cap M^\omega : M' \cap N^\omega]$ is infinite if the depth of $N \subset M$ is infinite.

Notations and Preliminaries

- Von Neumann algebra: weakly closed, unital $*-$subalgebra $M \subset B(H)$.
- Factor: the center $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.
- Type II_1 factor: infinite-dimensional factor, with a tracail state $\tau\colon M \to \mathbb{C}$.
- Notion of amenability (Connes): existence of a hypertrace on $B(H)$.
- Ultrapower algebra: $M^\omega = \ell^\infty(\mathbb{N}, M)/\mathcal{I}$, where

$$\ell^\infty(\mathbb{N}, M) = \{(x_n) \in M^{\mathbb{N}} \mid \sup \|x_n\| < \infty\},$$

together with its closed ideal

$$\mathcal{I} = \{(x_n) \in \ell^\infty(\mathbb{N}, M) \mid \lim_{n\to\omega} \|x_n\|_2 = 0\}.$$

Connes' Celebrated Theorem ([Con76])

Up to isomorphisms, this is a unique amenable type II_1 factor.

This is the hyperfinite II_1 factor R of Murray and von Neumann: the direct limit of $M_2(\mathbb{C}) \subset M_4(\mathbb{C}) \subset M_8(\mathbb{C}) \subset \cdots$.

If M is the hyperfinite II_1 factor, then M^ω and $M' \cap M^\omega$ are both type II_1 factors.

Subfactors and Index

Although subfactors N of the hyperfinite II_1 factor M are hyperfinite, the way they sit inside can encode extremely rich structures.

- Jones index: $[M : N] = \dim_N L^2(M)$ (the index is always assumed to be finite).
- Jones tunnel and tower of II_1 factors

$$\cdots \subset N_k \subset \cdots \subset N_1 \subset N \subset M \subset M_1 \subset \cdots \subset M_k \subset \cdots,$$

with each one of them being the basic construction of the previous inclusion.
- Finite depth: the central dimensions of $N_k' \cap M$ are bounded for $k \in \mathbb{N}$ (equivalently, the principal graph of $N \subset M$ is finite).

Pimsner-Popa Index ([PP86], Section 2)

Since $M' \cap N^\omega$ is not necessarily a factor, we cannot define its Jones index in $M' \cap M^\omega$. Let N be a subalgebra of a type II_1 factor M. We define its Pimsner-Popa index by

$$[M : N] = (\sup\{\lambda \mid E_N(x) \geq \lambda x, x \in M_+\})^{-1}.$$

Pimsner-Popa index is a generalization of Jones index and they coincide when N is a factor.

Finite-Depth Case

If $N \subset M$ is an inclusion of hyperfinite II_1 factors with finite index and finite depth, then $N \subset M$ admits a generating tunnel, i.e. a tunnel $\{N_k\}$ such that

$$M = \bigvee_k (N_k' \cap M), \qquad N = \bigvee_k (N_k' \cap N).$$

Then we can compute

$$M' \cap N^\omega = \bigcap_k N_k^\omega, \tag{1}$$

and hence $M' \cap N^\omega$ is a II_1 factor.
Applying the following properties of finite-index, finite-depth inclusions

- there is $c > 0$ such that for any $x \in \left(N_j \vee \mathcal{Z}(N_j' \cap M)\right)_+$, $j \in \mathbb{N}$, we have $E_{N_j}(x) \geq cx$ ([Pop90], Theorem 4.3);
- there is $c' > 0$ such that $E_{N_j \vee (N_j' \cap M)}(x) \geq c'x$ holds for all $j \geq 1$, $x \in M_+$ ([Kaw92], Lemma 3.4),

we can prove that the index $M' \cap N^\omega \subset M' \cap M^\omega$ is finite ([Kaw92], Lemma 3.5).

Examples of Infinite-Depth Inclusions

We exhibit the following three examples of infinite-depth inclusions, the third of which is a review of [Bur10].

- The diagonal subfactors.
  Let $\alpha_1 = \mathrm{Id}, \alpha_2, \ldots, \alpha_n$ be n outer automorphisms of the hyperfinite II_1 factor R such that they generates an infinite group in $\mathrm{Out}(R)$. Then
  $$N = \left\{ \begin{pmatrix} x & & & \\ & \alpha_2(x) & & \\ & & \ddots & \\ & & & \alpha_n(x) \end{pmatrix} ; x \in R \right\}$$
  is a finite-index, infinite-depth subfactor of $M = R \otimes M_{n+1}(\mathbb{C})$.
- Bisch-Haagerup subfactors.
  Let H and K be finite groups acting on R by outer actions such that they generates an infinite subgroup in $\mathrm{Out}(R)$. Then
  $$N = R^H \subset R \rtimes K = M$$
  is of finite index and infinite depth.

It can be proved that both two examples satisfy $[M' \cap M^\omega : M' \cap N^\omega] = \infty$.

- Let
  $$\begin{matrix} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{matrix}$$
  be a connected, symmetric and Markov commuting square of finite-dimensional C*-algebras. The Jones towers $A_{00} \subset A_{10} \subset \cdots \subset A_{\infty 0} = N$ and $A_{01} \subset A_{11} \subset \cdots \subset A_{\infty 1} = M$ gives an inclusion of hyperfinite II_1 factors with finit index.
  Burstein([**?**], Theorem 2) showed that if the subfactor consturcted above is of infinite depth, then the Pimsner-Popa index
  $$[M' \cap M^\omega : M' \cap N^\omega] = \infty.$$

Strongly Amenable Subfactors

We are interested in the case where the inclusion $N \subset M$ has an infinite depth but admits a generating tunnel $\{N_k\}_k$. Such hyperfinite II_1 subfactors are called strongly amenable subfactor ([Pop94], Definition 3.1.1, Theorem 4.1.2).
Under the strong amenability assumption, we also have

$$M' \cap N^\omega = \bigcap_k N_k^\omega$$

and $M' \cap N^\omega$ is a subfactor of $M' \cap M^\omega$ just as the finite-depth case.

Question

If the inclusion $N \subset M$ of hyperfinite II_1 factors is strongly amenable and has an infinite depth, then

$$[M' \cap M^\omega : M' \cap N^\omega] = \infty.$$

A Possible Method

We afford a probable way to solve this question. Let $p_k \in \mathcal{Z}(N_k' \cap M)$ be some non-zero projection and $\tilde{p} = (p_k) \in M^\omega$. The generating property implies $\tilde{p} \in M' \cap M^\omega$. It suffices to compute $E_{N_n}(p_k)$ for sufficiently large $k > n$. The commuting square condition of

$$\begin{array}{ccc} N_n & \subset & M \\ \cup & & \cup \\ N_k' \cap N_n & \subset & N_k' \cap M \end{array}$$

implies $E_{N_n}(p_k) = E_{N_k' \cap N_n}(p_k)$. The question can be probably solved by computing $E_{N_k' \cap N_n}(p_k)$ for minimal central projections p_k. In the finite-depth case, a possible computation is contained in [Pop90] Theorem 3.8. Also, we may give a explicit form of $E_{N_k' \cap N}(p_k)$ by using the method of string algebras.

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