Geometric Structure of Affine Deligne-Lusztig Varieties for GL_3

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Introduction

The Langlands correspondence, which contains class field theory as a special case, is one of the most important topics in number theory. Shimura varieties have been used, with great success, towards applications in the realm of the Langlands program. In this context, geometric and homological properties of affine Deligne-Lusztig varieties have been used to examine Shimura varieties and the local Langlands correspondence.

Definition

Set $k = \mathbb{F}_q, L = \bar{k}((t)), \mathcal{O} = \bar{k}[[t]]$. Fix a split connected reductive group G/k. Let $b \in G(L)$. Let λ be a dominant cocharacter.

Definition

The affine Deligne-Lusztig variety $X_\lambda(b)$ is the locally closed subvariety of the affine Grassmannian given by

$$X_\lambda(b)(\bar{k}) = \{xG(\mathcal{O}) \in G(L)/G(\mathcal{O}) \mid x^{-1}b\sigma(x) \in G(\mathcal{O})t^\lambda G(\mathcal{O})\}.$$

We can also define the affine Deligne-Lusztig varieties associated to arbitrary parahoric subgroups (especially Iwahori subgroups).

- The σ-centralizer J_b of b acts on $X_\lambda(b)$.
- The affine Deligne-Lusztig variety is an analogue of the classical Deligne-Lusztig variety, which plays a crucial role in the representation theory of finite reductive groups.
- The second parameter b in the affine Deligne-Lusztig varieties makes them rather challenging to study.

Motivation

- Simple descriptions (Görtz-He)
  - They showed that certain affine Deligne-Lusztig varieties are naturally a union of classical Deligne-Lusztig varieties.
- Cohomology (Chan-Ivanov)
  - They showed that the homology of a certain affine Deligne-Lusztig variety (at infinite level) gives a geometric realization of the local Langlands correspondence.
  - Along the way, they provide examples of explicitly described Iwahori-level affine Deligne-Lusztig varieties, which is similar to those considered by Görtz and He.
- Application of geometric properties (Xiao-Zhu, ...)
  - The geometric properties of affine Deligne-Lusztig varieties (simple descriptions, non-emptiness, irreducible components, etc.) have been studied and applied by many people.
  - For example, Xiao-Zhu studied irreducible components of affine Deligne-Lusztig varieties and applied the results to the basic locus of Shimura varieties.

Main Theorem

Let $G = \mathrm{GL}_3$ and let b be basic.

Theorem

The irreducible components of $X_\lambda(b)$ are parameterized by the set $\coprod_{\mu \in M} J_b/K_\mu$, where M is a finite set consisting of certain dominant cocharacters μ determined by λ, and K_μ is the stabilizer of a lattice depending on μ under the action of J_b. Each irreducible component is an affine bundle over a simple variety.

- As an immediate corollary, we classify the cases where $X_\lambda(b)$ has such a simple description as Chan-Ivanov pointed out.
- The crucial ingredient of the proof is the method of Kottwitz in "Orbital integrals on GL_3" (so we restricted ourselves to the case $G = \mathrm{GL}_3$).

In the future, we are going to study $X_\lambda(b)$ for general G. As mentioned above, such studies can be applied to number theory. In particular, the case $G = \mathrm{GL}_n$ can be applied to the geometric realization of the local Langlands correspondence.