高次圏におけるモノドロミー表現と反復積分

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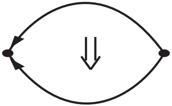
Outline

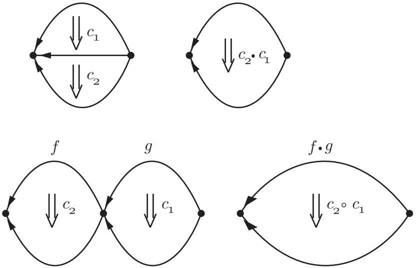
- 高次圏とは何か？
- モノドロミー表現はどのように高次圏に拡張されるか？
- 反復積分を用いた高次ホロノミー函手の構成
- 組みひもコボルディスムなどへの応用
- 有理ホモトピー理論の高次圏での定式化

2-categories

A **2-category** consists of
objects, morphisms and 2-morphisms



There are horizontal and vertical compositions:



with suitable coherency conditions.

Holonomy

M : smooth manifold (or differentiable space)
$\pi : E \longrightarrow M$ topologically trivial vector bundle with fiber V
A : 1-form with values in $\mathrm{End}(V)$ considered as a connection of E

$\gamma : [0,1] \longrightarrow M$: smooth path with $\gamma(0) = \mathbf{x}_0, \gamma(1) = \mathbf{x}_1$
Horizontal sections of the connection A give a linear map

$$Hol(\gamma) : V_{\mathbf{x}_0} \longrightarrow V_{\mathbf{x}_1}$$

called the **holonomy**.

Holonomy via iterated path integrals

The holonomy $Hol(\gamma)$ is expressed as

$$I + \int_\gamma A + \int_\gamma AA + \cdots$$

Put $\gamma^* A = A(t)dt$. The iterated path integral of 1-forms is

$$\int_\gamma \underbrace{A \cdots A}_{k} = \int_{\Delta_k} A(t_1) \cdots A(t_k)\, dt_1 \cdots dt_k$$

where Δ_k is the k-simplex defined by

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \mid 0 \le t_1 \le \cdots \le t_k \le 1\}.$$

Flat connections

The connection A is **flat** if the curvature form vanishes, i.e.,

$$dA + A \wedge A = 0$$

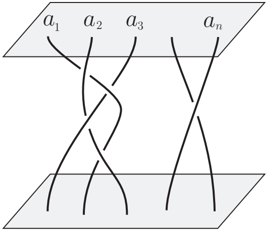
If A is flat, the holonomy $Hol(\gamma)$ depends only on the homotopy class of γ. There is a one-to-one correspondence

$$\{\text{flat connection of } E\} \Longleftrightarrow \{\text{representations } \pi_1(M, \mathbf{x}_0) \to GL(V)\}$$

This describes monodromy representations.

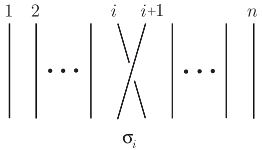
Braid groups

Braid groups were studied by E. Artin in the 1920's.



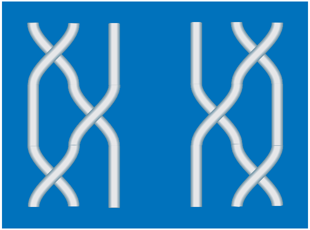
The isotopy classes of geometric braids as above form a group by composition. This is the braid group with n strands denoted by B_n.

Braid relations



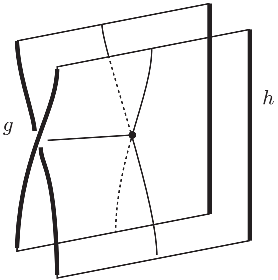
B_n is generated by σ_i, $1 \leq i \leq n-1$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1$$



Braid cobordisms

Surface in $\mathbf{R}^4$ bounding braids expressed as a branched covering with simple branched points over $[0,1]^2$



braided surface, 2-dimensional braid (Kamda, Carter and Saito)

the category of braid cobordisms $\mathcal{B}C_n$:

- objects : geometric braids with n strands
- morphisms : relative isotopy classes of cobordisms between braids

KZ connections

$\mathfrak{g}$: complex semi-simple Lie algebra.
$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \leq i \leq n$ representations.

Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over X_n (the configuration space of ordered distinct n points in $\mathbf{C}$) with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$d\omega + \omega \wedge \omega = 0$$

Grenoble, 1987

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J. L. KOSZUL

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(*) La conférence de Yu. I. Manin, absent, a été présentée par D. Luna.

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37, 4 (1987), 139-160

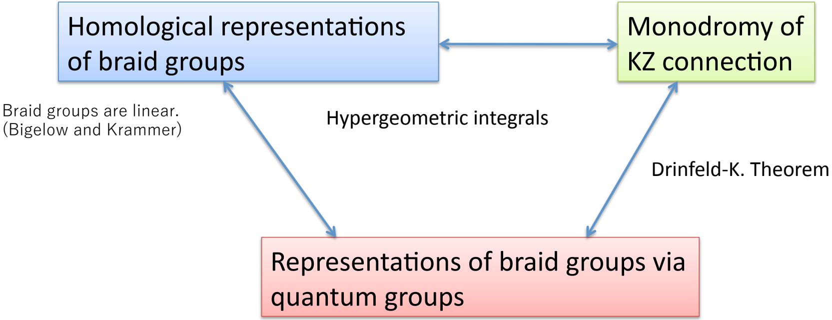
MONODROMY REPRESENTATIONS OF BRAID GROUPS AND YANG-BAXTER EQUATIONS

by Toshitake KOHNO

INTRODUCTION

The purpose of this paper is to give a description of the monodromy of integrable connections over the configuration space arising from classical Yang-Baxter equations. These monodromy representations define a series of linear representations of the braid groups $\theta : B_n \to \mathrm{End}\,(W^{\otimes n})$ with one parameter, associated to any finite dimensional complex simple Lie algebra $\mathfrak{g}$ and its finite dimensional irreducible representations $\rho : \mathfrak{g} \to \mathrm{End}\,(W)$. By means of trigonometric solutions of the quantum Yang-Baxter equations due to Jimbo ([10] and [11]), we give an explicit form of of these representations in the case of a non-exceptional simple Lie algebra and its vector representation (Theorem 1.2.8) and in the case of $\mathfrak{sl}(2,\mathbf{C})$ and its arbitrary finite dimensional irreducible representations (Theorem 2.2.4).

Quantum symmetry in representations of braid groups



Problem: Extend the above constructions to higher categories.

KZ connection and homological representations

$Y_{n,m} = \mathrm{Conf}_m(\mathbf{C} \setminus \{p_1, \cdots, p_n\})$
Homological representation is the action of B_n on the homology $H_m(Y_{n,m}, \mathcal{L})$ where $\mathcal{L}$ is a rank one local system.

Theorem (K, 2011)

The homological representation of B_n is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}$$

where M_λ is the Verma module with highest weight λ.

Suggestion by Manin for higher structures

Prof. T. Kohno
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Nagoya Univ., Japan

September 9, 1987

Dear Professor Kohno,

thank you for your preprints. I send you via Professor Cartier a preprint of mine written in collaboration with Schechtman which you might find interesting. In particular, we have defined there higher braid groups and the Lie algebra (see Th. 5, page 7)

$$\circledast \quad \begin{aligned} &[h_{J_1}, h_{J_2}] = 0 \text{ if } |J_1 \cup J_2| \geq k+3, \text{ where } J \subset (1,\dots,n),\ |J| = k+1 \\ &[h_J, \sum_{\substack{I \subset K \\ |I| = k+1}} h_I] = 0 \text{ if } J \subset K \subset \{1,\dots,n\},\ |K| = k+2 \end{aligned}$$

which for $k=1$ becomes what you call "infinitesimal pure braid relations" in your preprint "Linear representations of braid groups and classical Yang-Baxter equations".

Our work essentially stems from the realization, that such classical objects as

~~discriminant arr~~

- hyperplane arrangement $\{z_i - z_j = 0 \mid i \neq 0\}$, $1 \leq i, j \leq n$
- symmetric group S_n
- braid group B_n (or P_n)
- Yang-Baxter equations

are all but first members of a hierarchy, depending on an additional parameter $1 \leq k \leq n$. In fact, for YB equations this hierarchy was devised by Zamolodchikov and we understood what should stand for the other three cases.

I am sure also that there should exist "higher Hecke algebras" but did not yet worked up a precise definition. It seems that an approach via the monodromy in local systems will give the right answer.

The combinatorial part of our paper is a bit complicated. Probably, a right point of view is that we define a sort of "noncommutative resolvent" of S_n in the following sense. We construct a series of posets $B(n,k)$ organized in a table (read from right to left!)

$$\begin{array}{ccccc} & \text{admissible total orders on 3-element subsets of } (1,\dots,n) & \text{admissible total orders (see §2) on 2-element subsets of } (1,\dots,n) & \text{total orders on } (1,\dots,n) \\ & \| & \| & \| \\ \cdots\cdots & A(n,3) & A(n,2) & S_n = A(n,1) \\ & \downarrow & \downarrow & \downarrow\wr \\ \cdots\cdots & B(n,3) & B(n,2) & S_n = B(n,1) \\ & & & \text{with weak Bruhat order} \end{array}$$

$A(n,3)$ = maximal chains in $B(n,2)$; $A(n,2)$ = maximal chains in $S_n = B(n,1)$

[START READING HERE]

which essentially embodies an idea of constructing

$$\to \begin{bmatrix} \text{relations among} \\ \text{~~second order r~~} \\ \text{relations among} \\ \text{Coxeter relations} \\ \text{modulo trivial} \\ \text{ones} \cdots \end{bmatrix} \to \begin{bmatrix} \text{relations among} \\ \text{Coxeter relations} \\ \text{modulo trivial ones:} \\ \sigma_{(ij)}\sigma_{(kl)} = \sigma_{(kl)}\sigma_{(ij)} \\ \text{if } \#(ijkl) = 4 \end{bmatrix} \to \begin{bmatrix} \text{Coxeter relations} \\ \text{among } \sigma_{(ij)} \end{bmatrix}$$

Here the map $A \twoheadrightarrow B$ is just the factorization "modulo trivial relations".

A lot of things remains to be done to understand better all of this. In particular, it would be nice to construct representations of $\circledast$ by your methods.

Yours sincerely

(Manin Yu. I.)
Steklov Math. Inst.

P.S. The discriminantal arrangements can be also defined by dualization of Bob MacPherson' construction: start with $E^{n-k} \subset \mathbb{C}^n$, intersect E^{n-k} with coordinate planes of dimension $k+1$. You get $\binom{n}{k+1}$ lines in E^{n-k}. Generate a minimal system of linear subspace stable with r. to sums starting from this set. Now take a dual system in a dual space. You get the discriminantal arrangement.

Strategy

- Extend the notion of holonomy as functors from path n-groupoid
- Express the holonomy functors by iterated integrals
- Formulate the flatness conditions in higher categories
- Construct representations of the homotopy n-groupoid $\Pi_n(M)$

Path groupoid

Path groupoid $\mathcal{P}_1(M)$

- Objects : points in M
- Morphisms : piecewise smooth paths between points up to parametrization (thin homotopy)

A smooth homotopy $H : [0,1]^2 \to M$ is called a thin homotopy if it sweeps out a surface with zero area, i.e.,

$$\operatorname{rank} dH_p < 2$$

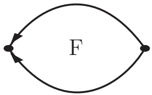
at every point $p \in [0,1]^2$.

Considering up to homotopy, one can define

- Homotopy path groupoid $\Pi_1(M)$

Path 2-groupoids

Path 2-groupoid $\mathcal{P}_2(M)$ is a 2-category whose 2-morphisms are discs (2-fold homotpies) $F : [0,1]^2 \to M$ spanning 2 paths up to parametrization (thin homotopy).



thin homotopy: smooth homotopy $H : [0,1]^3 \to M$ between $F_1, F_2 : [0,1]^2 \to M$ such that $\mathrm{rank}\, dH_p < 3$ holds at every point $p \in [0,1]^3$.

Considering 2-morphisms up to homotopy, one can define

- Homotopy 2-groupoid $\Pi_2(M)$

Path n-groupoids

Path n-groupoid $\mathcal{P}_n(M)$ is an n-category consisting of

- Objects : points in M
- 1-morphisms : piecewise smooth paths up to thin homotopy
- 2-morphisms : 2-fold homotopies between 1-morphisms up to thin homotopy

  $\cdots$
- $(n-1)$-morphisms : $(n-1)$-fold homotopies between $(n-2)$-morphisms up to thin homotopy
- n-morphisms : n-fold homotopies between $(n-1)$-morphisms up to thin homotopy

Considering n-morphisms up to homotopy, one can define

- Homotopy n-groupoid $\Pi_n(M)$

Holonomy as functors

Considering the horizontal sections of a connection on trivial vector bundles on M, we obtain a functor

$$Hol : \mathcal{P}_1(M) \longrightarrow \mathbf{R}\text{-Vect}$$

$\mathbf{R}$-Vect is the category of vector spaces and linear maps over $\mathbf{R}$.

For flat connections we obtain

$$Hol : \Pi_1(M) \longrightarrow \mathbf{R}\text{-Vect}$$

2-connections

We start with a trivial vector bundle over M with fiber V and a connection 1-form A.

- A : $\mathrm{End}(V)$-valued 1-form

A **2-connection** consists of

- an extra vector space W
- B : $\mathrm{End}(W)$-valued 2-form
- homomorphism $\delta : \mathrm{End}(W) \to \mathrm{End}(V)$ such that

$$\delta(B) = dA + A \wedge A$$

- homomorphism $\rho : \mathrm{End}(V) \to \mathrm{End}(W)$

with the compatibility conditions

$$\delta(\rho(x) \cdot v) = x \cdot \delta(v), \quad \delta(v \cdot \rho(x)) = \delta(v) \cdot x$$

Differential forms with values in graded algebras

We put

$$\mathcal{G}_0 = \mathrm{End}(V),\ \mathcal{G}_1 = \mathrm{End}(W)$$

- A : $\mathcal{G}_0$-valued 1-form
- B : $\mathcal{G}_1$-valued 2-form

Graded algebra structure with non-commutative product

$$\mathcal{G}_0 \times \mathcal{G}_1 \longrightarrow \mathcal{G}_1$$

by means of ρ
There is a derivation $\delta : \mathcal{G}_1 \to \mathcal{G}_0$ with $\delta(B) = dA + A \wedge A$.
This suggests a construction of higher graded algebra

$$\oplus_{k \geq 0} \mathcal{G}_k$$

with derivations $\delta_k : \mathcal{G}_k \to \mathcal{G}_{k-1}$.

Approaches for higher holonomies

- The idea of fundamental 2-groupoid due to J. H. C. Whitehead (fundamental 2-group)
- Homotopy ∞-groupoid $\Pi_\infty(M)$ due to Grothendieck - homotopy hypothesis

- Parallel transport for flat 2-connections with values in crossed modules (Baez-Schreiber, Martins-Pickens)
- Construction of A_∞ functor from the dg-category of flat superconnections on M to the category of representations of ∞-groupoid $\Pi_\infty(M)$ (Igusa, Block-Smith, Arias Abad-Schätz)

We describe an approach based on K.-T. Chen's iterated integrals.

$\omega_1, \cdots, \omega_k$: differential forms on M
$\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$: path space of M

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

$$\varphi : \Delta_k \times \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1) \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$

The **iterated integral** of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

Iterated integrals as differential forms on loop space

The expression

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

is the integration along the fiber with respect to the projection $p : \Delta_k \times \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1) \to \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$.

differential form on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$
with degree $p_1 + \cdots + p_k - k$, where $p_j = \deg \omega_j$

Composition of paths

For families of paths

$$\alpha : U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad \beta : U \longrightarrow \mathcal{P}(M; \mathbf{x}_1, \mathbf{x}_2)$$

we have the following composition rule :

$$\left(\int \omega_1 \cdots \omega_k\right)_{\alpha\beta} = \sum_{0 \le i \le k} \left(\int \omega_1 \cdots \omega_i\right)_{\alpha} \wedge \left(\int \omega_{i+1} \cdots \omega_k\right)_{\beta}$$

Here $\left(\int \omega_1 \cdots \omega_k\right)_{\alpha}$ is a differential form on U obtained by pulling back by the iterated integral $\int \omega_1 \cdots \omega_k$ by α.

Differentiation on the path space

As a differential form on the path space $d \int \omega_1 \cdots \omega_k$ is

$$\sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \; \omega_{j+1} \cdots \omega_k$$

$$+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k$$

where $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$.

Let us go back to the situation of 2-connections:

- $A : \mathrm{End}(V)$-valued 1-form
- $B : \mathrm{End}(W)$-valued 2-form
- homomorphism $\delta : \mathrm{End}(W) \to \mathrm{End}(V)$ such that

$$\delta(B) = dA + A \wedge A$$

- homomorphism $\rho : \mathrm{End}(V) \to \mathrm{End}(W)$

We put $\omega = \rho(A) + B$, differential form with values in $\mathrm{End}(W)$.

2-holonomy of a 2-connection (2)

For $\gamma \in \mathcal{P}_1(M)$ we have 1-holonomy

$$Hol_1(\gamma) = I + \int_\gamma A + \int_\gamma AA + \cdots \ \in \mathrm{End}(V)$$

2-morphism $c \in \mathcal{P}_2(M)$ between γ_1, γ_2 considered as a 1-chain on the path space of M.
For $\omega = \rho(A) + B$ the iterated integral

$$T = I + \int \omega + \int \omega\omega + \cdots$$

is considered as a differential form on the the path space of M with values in $\mathrm{End}(W)$.

We put $Hol_2(c) = \langle T, c\rangle$. This defines a 2-functor and we have a representation of the path 2-groupoid $\mathcal{P}_2(M)$ such that

$$\delta Hol_2(c) = Hol_1(\gamma_2) - Hol_1(\gamma_1).$$

If the 2-connection is 2-flat, this gives a representation of the homotopy path 2-groupoid $\Pi_2(M)$
For a vertical composition we have

$$Hol_2(c_1 \cdot c_2) = Hol_2(c_1) + Hol_2(c_2).$$

2-flatness condition

The 2-connection (A, B) is 2-flat if

$$dB - \rho(A) \wedge B + B \wedge \rho(A) = 0.$$

In this case we have a representation of the homotopy 2-groupoid $\Pi_2(M)$.

Chen's formal homology connections (1)

Set $H_+(M) = \bigoplus_{q>0} H_q(M; \mathbf{R})$.
$TH_+(M)$: tensor algebra generated by $H_+(M)$
$\{X_i\}$: basis of $H_+(M)$
Put $\deg x_i = p_i - 1$ for $x_i \in H_{p_i}(M)$.
$\widehat{TH_+(M)}$: completion of $TH_+(M)$ with respect to the augmentation ideal J generated by $\{X_i\}$.

Formal homology connection is

$$\omega \in \Omega^*(M) \otimes \widehat{TH_+(M)}$$

$$\omega = \sum \omega_i \otimes X_i + \sum_{i_1, \cdots, i_k} \omega_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k}$$

with the following properties:
$[\omega_i]$: dual basis of $\{X_i\}$
$\deg \omega_{i_1 \cdots i_k} = \deg X_{i_1} \cdots X_{i_k} + 1$

Chen's formal homology connections (2)

We define a generalized curvature as
$\kappa = d\omega - \epsilon(\omega) \wedge \omega, \ \epsilon(\omega) = \pm\omega$ (parity)

Conditions for Chen's formal homology connection

- $\delta\omega + \kappa = 0$
- δ is a derivation of degree -1
- $\delta X_j \in J^2$, J : the augmentation ideal

Here we suppose that the derivation δ satisfies the Leibniz rule

$$\delta(uv) = (\delta u)v + (-1)^{\deg u} u(\delta v).$$

We have $\delta \circ \delta = 0$ and $(\widehat{TH_+(M)}, \delta)$ forms a complex.
The formal homology connection can be written in the sum

$$\omega = \omega^{(1)} + \omega^{(2)} + \cdots + \omega^{(p)} + \cdots$$

with the p-form part $\omega^{(p)} \in \Omega^p(M) \otimes \widehat{TH_+(M)}_{p-1}$.
The 2-form part of κ is the curvature form for $\omega^{(1)}$. We have the equation

$$\delta\omega^{(2)} + d\omega^{(1)} + \omega^{(1)} \wedge \omega^{(1)} = 0.$$

The 3-form part of κ

$$\kappa^{(3)} = d\omega^{(2)} - \omega^{(1)} \wedge \omega^{(2)} + \omega^{(2)} \wedge \omega^{(1)}$$

is the 2-curvature of the pair $\omega^{(1)}$ and $\omega^{(2)}$.

Transport

Consider the complex $(T\widehat{H_+(M)}, \delta)$.

$T\widehat{H_+(M)}_k$: the degree k part of $T\widehat{H_+(M)}$

$$\delta : T\widehat{H_+(M)}_k \to T\widehat{H_+(M)}_{k-1}$$

For the formal homology connection ω define its transport by

$$T = 1 + \sum_{k=1}^{\infty} \int \underbrace{\omega \cdots \omega}_{k}.$$

Formal homology connections and holonomy

Proposition

Given a formal homology connection (ω, δ) for a manifold M the transport T satisfies $dT = \delta T$.

Proof.

We have

$$\begin{aligned} dT &= -\int \kappa + \left(-\int \kappa\omega + \int \varepsilon(\omega)\kappa \right) + \cdots \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-1)^{i+1} \int \underbrace{\varepsilon(\omega) \cdots \varepsilon(\omega)}_{i} \ \kappa \ \underbrace{\omega \cdots \omega}_{k-i-1}. \end{aligned}$$

Substituting $\kappa = -\delta\omega$ in the above equation and applying the Leibniz rule for δ, we obtain the equation $dT = \delta T$. $\square$

Representations of path groupoids

By taking the degree 0 part of the transport

$$T = 1 + \sum_{k=1}^{\infty} \int \underbrace{\omega \cdots \omega}_{k}$$

we obtain a representation of the path groupoid

$$Hol_1 : \mathcal{P}_1(M) \longrightarrow T\widehat{H_+(M)}_0.$$

For the homotopy path groupoid there is a representation

$$Hol_1 : \Pi_1(M) \longrightarrow T\widehat{H_+(M)}_0/\mathcal{I}_0$$

where $\mathcal{I}_0$ is the ideal generated by the image of the derivation

$$\delta_1 : T\widehat{H_+(M)}_1 \longrightarrow T\widehat{H_+(M)}_0.$$

$T\widehat{H_+(M)}_{\leq n-1}$: subalgebra of $T\widehat{H_+(M)}$ generated by elements of degree $\leq n-1$.
$\mathcal{I}_{n-1}$: the ideal of $T\widehat{H_+(M)}_{\leq n-1}$ generated by the image of

$$\delta_n : T\widehat{H_+(M)}_n \longrightarrow T\widehat{H_+(M)}_{n-1}$$

as 2-sided module over $T\widehat{H_+(M)}_0$.

Theorem

The above construction defines a functor

$$Hol : \Pi_n(M) \longrightarrow T\widehat{H_+(M)}_{\leq n-1}/\mathcal{I}_{n-1}.$$

such that a k-morphism f between $(k-1)$-morphisms between g_0 and g_1 we have

$$\delta Hol(f) = Hol(g_1) - Hol(g_0).$$

The ideal $\mathcal{I}_{n-1}$ corresponds to the n-flatness condition.
The key fact is $dT = \delta T$. If $c_1 - c_2 = \partial y$, then

$$Hol_k(c_1) - Hol_k(c_2) = Hol_k(\partial y) = \langle T, \partial y \rangle = \langle dT, y \rangle.$$

For each k the k-holonomy satisfies

$$\delta Hol_k(c) = Hol_k(\partial c) = Hol_{k-1}(\gamma_2) - Hol_{k-1}(\gamma_1)$$

if $\partial c = \gamma_2 - \gamma_1$.

Representations of the 2-category of braid cobordisms

Theorem

The 2-holonomy map gives a representation of the 2-category of braid cobordisms

$$Hol : \mathcal{B}C_n \longrightarrow T\widehat{H_+(X_n)}_{\leq 1}/\mathcal{I}_1$$

We consider the integration of the transport T on one-parameter deformation family of singular braids with double points associated with a braid cobordism. We need to study the asymptotics

$$\int_\gamma \underbrace{\omega \cdots \omega}_{k} \sim \frac{1}{k!}(\log \varepsilon)^k.$$

and regularize the divergent part.

This is a 2-category extension of Kontsevich integrals.

Further problems

- Study the extension of the holonomy of braids to 2-category by formal homology connection and the associated invariants for 2-dimensional braids.
- Formulate a higher category version of de Rham homotopy theory by means of higher holonomy functors.
- Describe a relation to an algebraic counterpart – categerification of quantum groups, algebra due to Khovanov, Rouquier and Lauda etc.