The Gauss-Bonnet type formulas for surfaces with singular points

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最近の筆者と

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との共同研究に関連した内容.

1. Gaussian curvature K

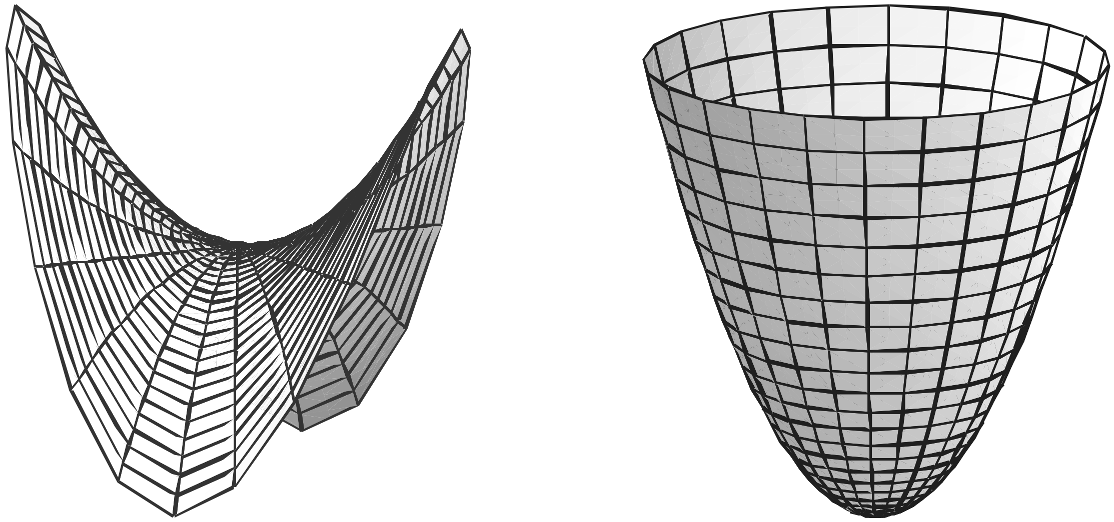
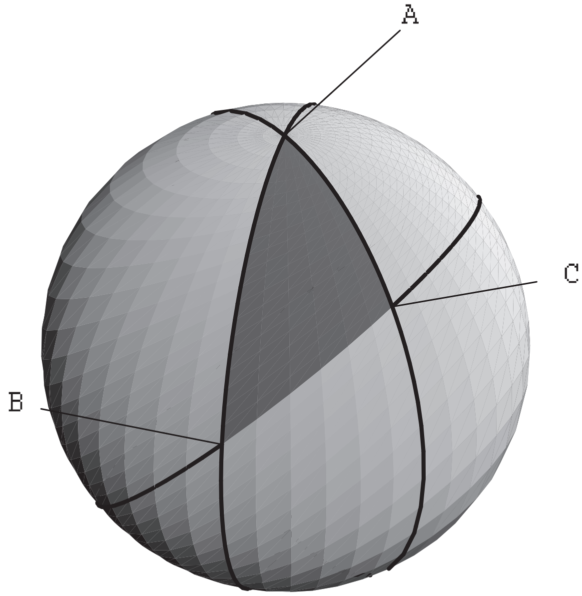


FIGURE 1. (Surfaces of $K < 0$ and $K > 0$)

$L_p(r)$ =the length of the geod. circle of radius r at p

$$K(p) = \lim_{r \to 0} \frac{3}{\pi} \left(\frac{2\pi r - L_p(r)}{r^3} \right).$$

2. The Gauss-Bonnet formula (local version)



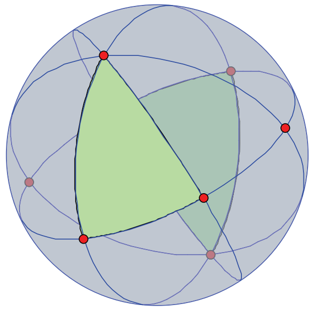
$$\int_{\triangle ABC} K\,dA = \angle A + \angle B + \angle C - \pi,$$

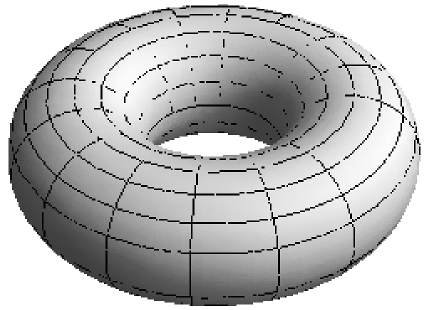
$$\angle A + \angle B + \angle C < \pi \quad (\text{if } K < 0),$$

$$\angle A + \angle B + \angle C > \pi \quad (\text{if } K > 0).$$

3. The Gauss-Bonnet formula (global version)

Polygonal division of closed surfaces.





(The Gauss-Bonnet formula)

$$\int_{M^2} K\,dA = 2\pi\chi(M^2), \tag{3.1}$$

where

$$\chi(M^2) = V - E + F$$

is the Euler number of the surface M^2.

4. Parallel surfaces

An immersion

$$f = f(u,v) : (U; u, v) \to \mathbf{R}^3 \qquad (U \subset \mathbf{R}^2),$$

the unit normal vector $\nu(u,v) := \dfrac{f_u(u,v) \times f_v(u,v)}{|f_u(u,v) \times f_v(u,v)|}$.

For each real number t,

$$f^t(u,v) = f(u,v) + t\nu(u,v)$$

is called a **parallel surface** of f.

p is a singular pt of $f^t \iff (f^t)_u(p) \times (f^t)_v(p) = 0.$

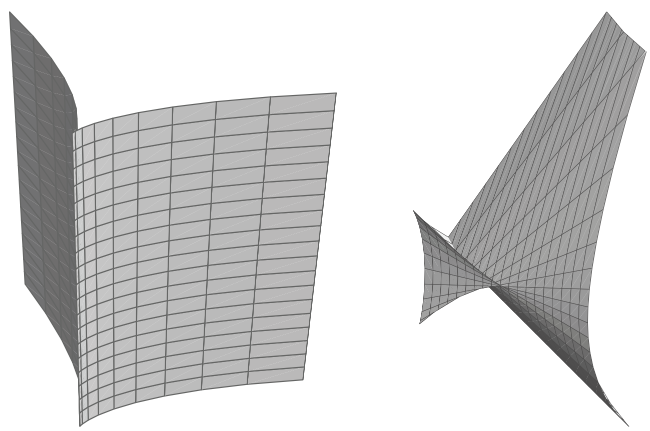


Figure 2. a cuspidal edge and a swallowtail

Cuspidal edges and **swallowtails** are generic singular points appeared on parallel surfaces.

$$f_C = (u^2, u^3, v), \qquad f_S = (3u^4 + u^2v, 4u^3 + 2uv, v).$$

An ellipsoid:

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1.$$

Parallel surfaces of the ellipsoid are given as follows:

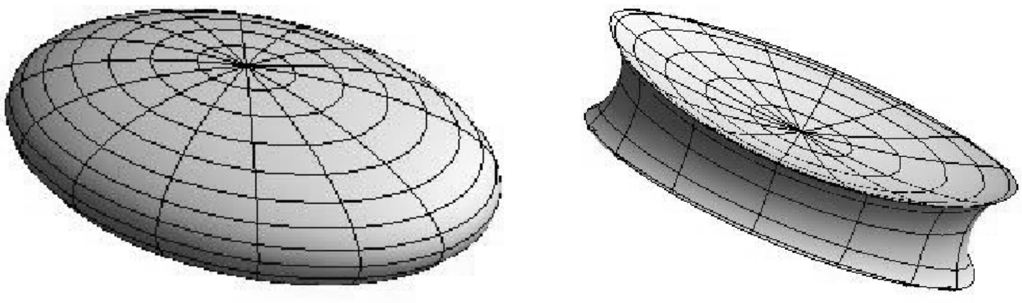


FIGURE 3. the cases of $t = 0$ and $t = 1.2$

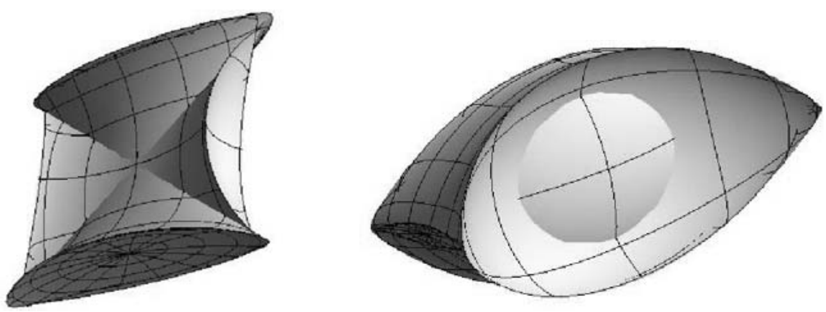


FIGURE 4. the cases of $t = 2$ and $t = 3.2$

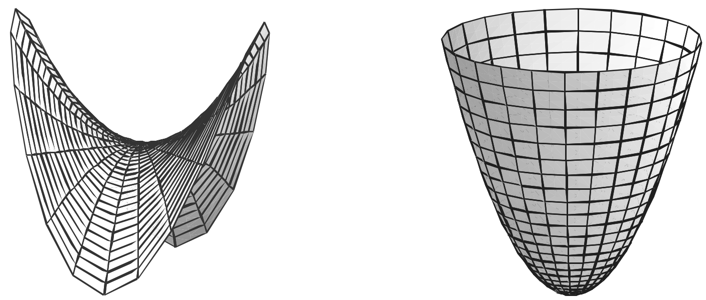


FIGURE 5. Surfaces of $K < 0$ and $K > 0$

The image of cuspidal edges consists of regular curves in $\mathbf{R}^3$. We denote it by $\gamma(s)$, where s is the arclength parameter.

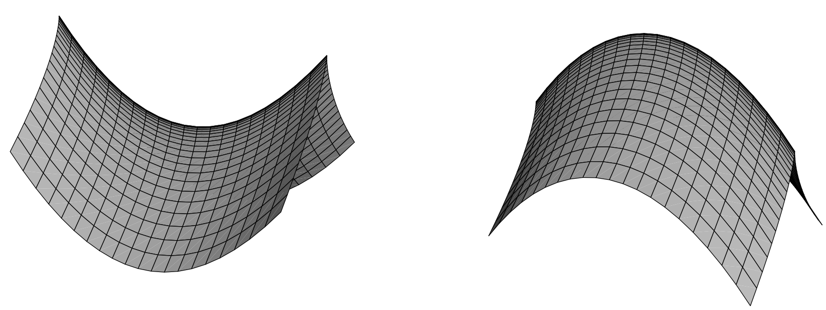


FIGURE 6. a $(-)$-cuspidal edge and a $(+)$-cuspidal edge

the singular curvature $\kappa_s(s) = \varepsilon(s)(\text{geodesic curvature})$

$$= \varepsilon(s)|\det(\nu(s), \gamma'(s), \gamma''(s))|$$

, where

$$\varepsilon(s) = \begin{cases} 1 & (\text{if the surface is bounded by a plane at } \gamma(s)), \\ -1 & (\text{otherwise}). \end{cases}$$

6. Generic cuspidal edges

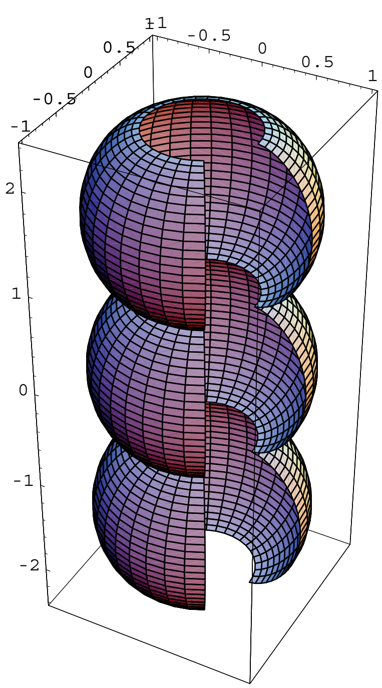
A generic cuspidal edge:

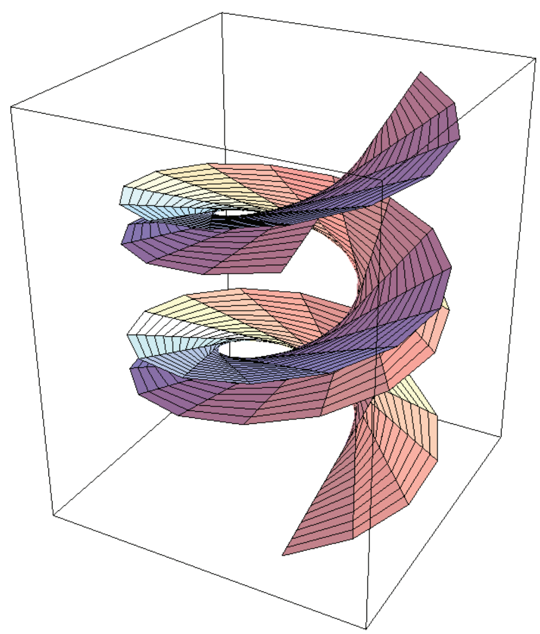
the osculating plane $\neq$ the tangent plane.

In this case, $K \to \pm\infty$ at the same time.

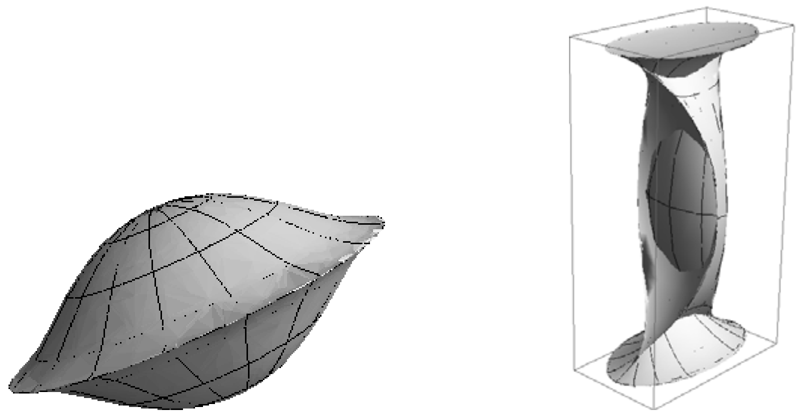
Saji-Yamada-U. [7]

$$K \geq 0 \implies \kappa_s \leq 0.$$





7. Gauss-Bonnet formula of surfaces with singularity



M^2; a compact oriented 2-manifold

$f : M^2 \to \mathbf{R}^3$; **a C^∞-map**

having only cuspidal edges and swalowtails

$\nu : M^2 \to S^2$; **the unit normal vector field.**

$(U; u, v)$; a $(+)$-oriented local coordinate M^2.

the area density function $\lambda := \det(f_u, f_v, \nu)$

$$\lambda(p) = 0 \Longleftrightarrow p \text{ is a singular point}$$

area element $dA := |\lambda| du \wedge dv,$

signed area element $d\hat{A} := \lambda du \wedge dv.$

8. Two Gauss-Bonnet formulas

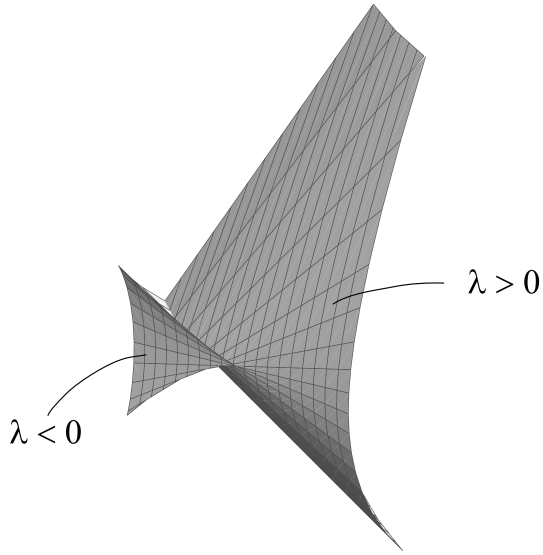


Figure 7. a positive swallowtail

Formulas given by Kossowski(02)and Langevin-Levitt-Rosenberg(95)

$$\int_{M^2 \setminus \Sigma_f} K dA + 2 \int_{\Sigma_f} \kappa_s ds = 2\pi\chi(M^2),$$

$$2\deg(\nu) = \frac{1}{2\pi} \int_{M^2 \setminus \Sigma_f} K d\hat{A}$$
$$= \chi(M_+) - \chi(M_-) + \#SW_+ - \#SW_-,$$

$$M_+ := \{p \in M^2\,;\, dA_p = d\hat{A}_p\},$$
$$M_- := \{p \in M^2\,;\, dA_p = -d\hat{A}_p\}.$$

Maps between planes

$$\mathbf{R}^2 \ni (u, v) \mapsto f(u, v) = (x(u, v), y(u, v)) \in \mathbf{R}^2,$$

$$\text{Singular points of } f \iff \det \begin{pmatrix} x_u(u, v) & x_v(u, v) \\ y_u(u, v) & y_v(u, v) \end{pmatrix} = 0.$$

Generic singular points

a fold $\mathbf{R}^2 \ni (u, v) \longmapsto (u^2, v) \in \mathbf{R}^2,$

the singular set $u = 0$, $f(0, v) = (0, v)$

a cusp $\mathbf{R}^2 \ni (u, v) \longmapsto (uv + v^3, u) \in \mathbf{R}^2,$

the singular set $u = -3v^2$, $f(-3v^2, v) = (-2v^3, -3v^2)$

Figure 8. a fold and a cusp

$$f = f(u, v)$$

Singular curves of cuspidal edges and folds

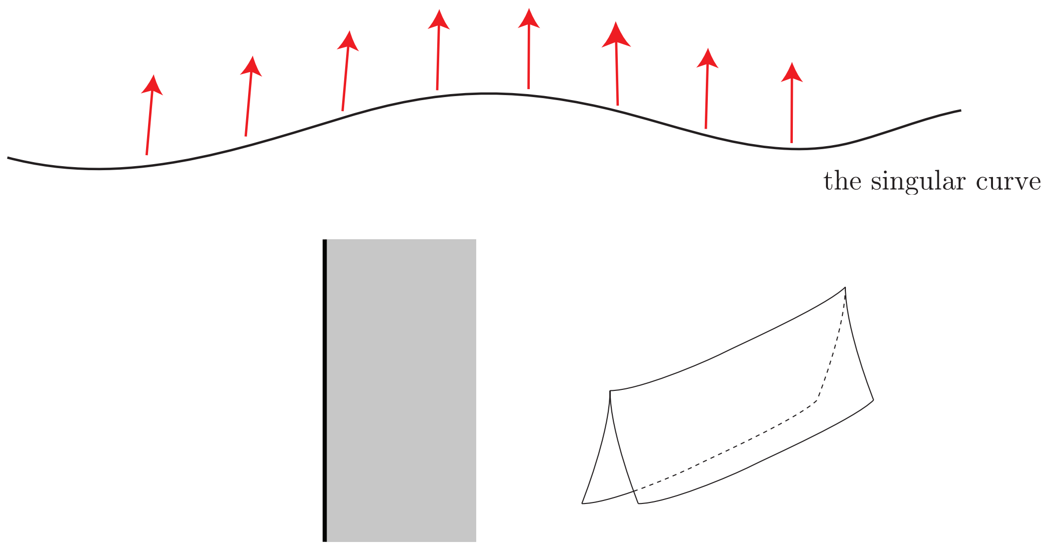


FIGURE 9. a fold and a cuspidal edge

Singular curves of swallowtails and cusps

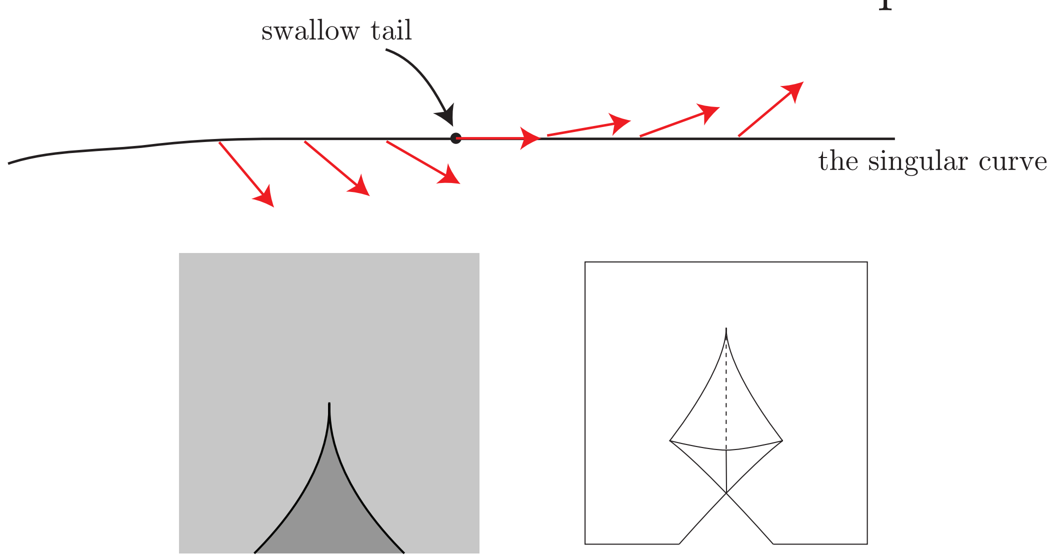


FIGURE 10. a cusp and a swallowtail

10. Singularities of Gauss maps

M^2:an oriented compact manifold.

$f : M^2 \to \mathbf{R}^3$, an immersion.

Singular points of $\nu : M^2 \to S^2 \Longleftrightarrow K_f = 0$.

$$f_t = \frac{1}{t}(f + t\nu), \qquad t \in \mathbf{R}.$$

Then

$$\lim_{t\to\infty} f_t = \nu,$$

$$2\deg(\nu) = \chi(M^t_+) - \chi(M^t_-) + \#SW^t_+ - \#SW^t_-.$$

In particular,

$$2\deg(\nu) = \chi(M^2) = \chi(M^t_+) + \chi(M^t_-).$$

Hence,

$$2\chi(M^t_-) = \#SW^t_+ - \#SW^t_-.$$

Taking the limit $t \to \infty$, we have that

$$2\chi(M^\infty_-) = \#SW^\infty_+ - \#SW^\infty_-.$$

The Gauss map ν satisfies

$$2\chi(M_-^\infty) = \#SW_+^\infty - \#SW_-^\infty.$$

If $t \to \infty$, then the cuspidal edge collapses to a fold, and a swallowtail collapses to a cusp. In particular,

$$\#SW_+^\infty := \#\{(+)\text{-cusps of } \nu\},$$
$$\#SW_-^\infty := \#\{(-)\text{-cusps of } \nu\}.$$

Since $d\hat{A}_\nu = K_f dA_f$, $dA_\nu = |K_f| dA_f$, it holds that

$$M_-^\infty = \{p \in M^2;\ K_f(p) < 0\}.$$

Thus (the Bleecker and Wilson formula [1])

$2\chi(\{K_f < 0\}) = \#\text{positive cusps} - \#\text{negative cusps}.$

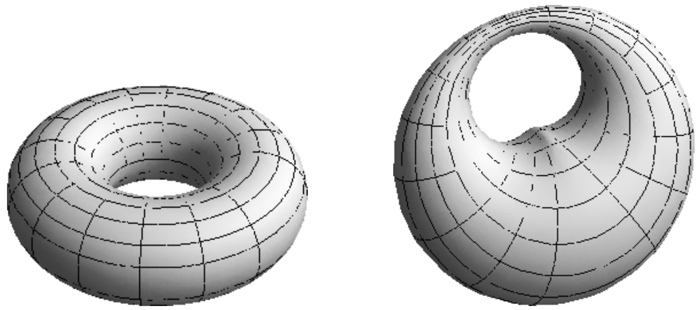


FIGURE 11. a symmetric torus and its perturbation

A deformation of the rotationally symmetric torus

$$f^a(u,v) = (\cos v(2+\varepsilon(v)\cos u), \sin v(2+\varepsilon(v)\cos u),\ \varepsilon(v)\sin u),$$

where

$$\varepsilon(v) := 1 + a\cos v.$$

$a = 0:$ the original torus

$a = 4/5$: $\quad \chi(\{K<0\}) = 1$.

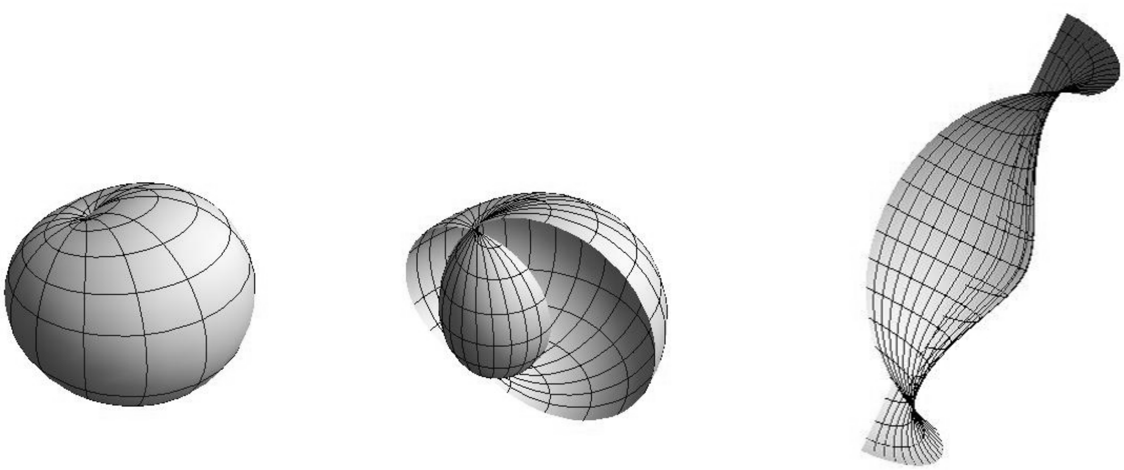


FIGURE 12. A parallel surface of $f^{4/5}$

11. A similar application

The following identity holds for $f_t : M^2 \to \mathbf{R}^3$.

$$\int_{M^2 \setminus \Sigma_{f_t}} K_t dA_t + 2 \int_{\Sigma_{f_t}} \kappa_s ds = 2\pi\chi(M^2).$$

Taking $t \to \infty$, we have that (Saji-Yamada-U. [10])

$$\frac{1}{2\pi} \int_{\{K<0\}} K_f dA_f = \int_{\Sigma_\nu} \kappa_s ds,$$

where

$$\begin{aligned} \kappa_s &:= \text{the singular curvature of } \nu \\ &= \pm \text{the geodesic curvature of } \nu \\ &\begin{cases} > 0 & \text{if } \nu'' \text{ points } \mathrm{Im}(\nu), \\ < 0 & \text{otherwise.} \end{cases} \end{aligned}$$

To prove the formula, we apply

$$K_\nu d\hat{A}_\nu = K dA_\nu, \qquad K_\nu A_\nu = |K_f| dA_f.$$

The intrinsic formulation of wave fronts (Saji-Yamada-U. [10])

The definition $(E, \langle\ ,\ \rangle, D, \varphi)$ of coherent tangent bundle on M^n:

(1) E is a vector bundle of rank n over M^n,

(2) E has a inner product $\langle\ ,\ \rangle$,

(3) D is a metric connection of $(E, \langle\ ,\ \rangle)$,

(4) $\varphi\colon TM^n \to E$ is a bundle homomorphism s.t.

$$D_X\varphi(Y) - D_Y\varphi(X) = \varphi([X, Y]),$$

where X, Y are vector fields on M^n.

The pull-back of the metric $\langle\ ,\ \rangle$

$$ds^2_\varphi := \varphi^* \langle\ ,\ \rangle$$

is called the **first fundamental form** of φ.

$p \in M^n$; φ-singular point $\Longleftrightarrow \mathrm{Ker}(\varphi_p\colon T_pM^n \to E_p) \neq \{0\}$.

coherent tangent bundle = generalized Riemannian manifold

When (M^n, g) is a Riemannian manifold, then

$$E = TM^n, \quad \langle , \rangle := g, \quad D = \nabla^g, \quad \varphi = \mathrm{id}\,.$$

If $f : M^2 \to \mathbf{R}^3$ is a wave front, then

$$E = \nu^\perp, \ \langle , \rangle := g_{\mathbf{R}^3}, \ D = \nabla^T, \ \varphi = df, \ (\psi = d\nu).$$

M^2; a compact oriented 2-manifold,
$(E, \langle , \rangle, D, \varphi)$; an orientable coherent tangent bundle,

$$\exists \mu \in Sec(E^* \wedge E^* \setminus \{0\})$$

such that $\mu(e_1, e_2) = 1$ for (+)-frame $\{e_1, e_2\}$.

The intrinsic definition of the singular curvature

$$\kappa_s := \operatorname{sgn}(d\lambda(\eta(t))) \frac{\mu(D_{\gamma'} n(t), \varphi(\gamma'))}{|\varphi(\gamma')|^3},$$

where $n(t) \in E_{\gamma(t)}$ is the unit vector perpendicular to $\varphi(\gamma')$ on E.

(u, v) ; a (+)-local coordinate on M^2

$$d\hat{A} = \lambda du \wedge dv, \qquad dA = |\lambda| du \wedge dv,$$

$$\lambda := \mu\left(\varphi(\frac{\partial}{\partial u}), \varphi(\frac{\partial}{\partial v})\right).$$

$$(\chi_E =) \frac{1}{2\pi} \int_{M^2} K d\hat{A} = \chi(M_+) - \chi(M_-) + SW_+ - SW_-,$$

$$\int_{M^2} K dA + 2 \int_{\Sigma_\varphi} \kappa_s d\tau = 2\pi\chi(M^2).$$

where

Σ_φ; φ-singular set,

$p \in \Sigma_\varphi$ is non-degenerate $\Leftrightarrow d\lambda(p) \neq 0$,

$p \in \Sigma_\varphi$; A_2-pt (intrinsic cuspidal edge) $\Leftrightarrow \eta \pitchfork \gamma'(0)$ at p,

$p \in \Sigma_\varphi$; A_3-pt (intrinsic swallowtail) $\Leftrightarrow \det(\eta, \gamma') = 0$ and $\det(\eta, \gamma')' = 0$ at p.

Examples of coherent tangent bundle:

(1) Wave fronts as a hypersurface of Riem. manifold,

(2) Smooth maps between n-manifolds.

M^n; an orientable manifold,

(N^n, g); an orientable Riemannian manifold,

$f : M^n \to (N^n, g)$; C^∞-map,

$E_f := f^*TN^n, \quad \langle \ , \ \rangle := g|_{E_f},$

D; induced connection.

$$\varphi := df : TM^n \longrightarrow E_f := f^*TN^n,$$

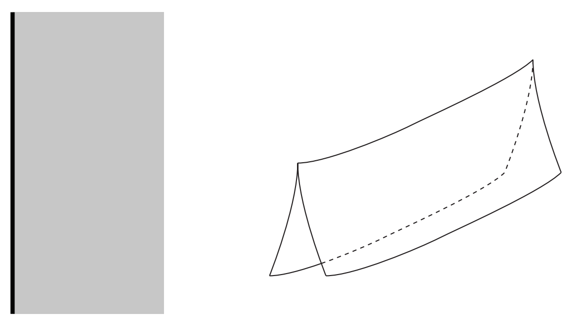


Figure 13. a fold and a cuspidal edge

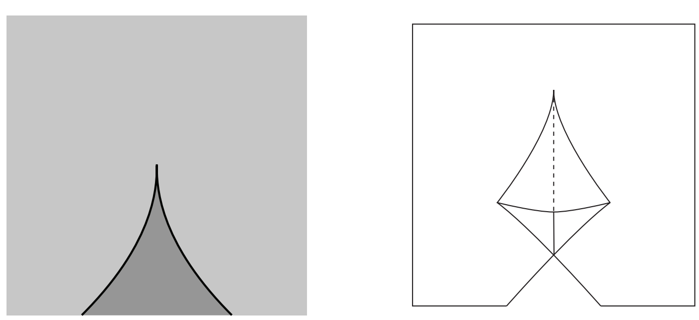


Figure 14. a cusp and a swallowtail

An application of the intrinsic G-B formula

$f : M^2 \to \mathbf{R}^3$; a strictly convex surface,

$\xi : M^2 \to \mathbf{R}^3$; the affine normal map.

$$\nabla_X Y = D_X Y + h(X, Y)\xi,$$

$$D_X \xi = -\alpha(X),$$

where $\alpha : TM^2 \to TM^2$. We set

$$M^2_- := \{p \in M^2 ; \det(\alpha_p) < 0\},$$

then (Saji-Yamada-U. [9])

$$2\chi(M^2_-) = \#SW_+(\xi) - \#SW_-(\xi).$$

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