**Lecture Notes in Mathematical Sciences**
**The University of Tokyo**

東京大学数理科学レクチャーノート　17

# TEICHMÜLLER SPACES AND CRYSTALLOGRAPHIC GROUPS

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2019

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# Part I The Deligne-Mumford Compactification

## 1. Introduction

These notes are based on the lectures which the author delivered in June of 2012 at the Graduate School of Mathematical Sciences of the University of Tokyo, and in June of 2016 at Fudan University, China.

The primary purpose of the lectures is to construct a "tautological orbifold-atlas[1]" on the Deligne-Mumford compactification of the moduli space of Riemann surfaces. Our method is based on the mapping class groups, the curve complexes, and Fenchel-Nielsen coordinates. As a by-product, we show that at maximally degenerate frontier points of the augmented Teichmüller spaces there arise certain Euclidean crystallographic groups.

The final version of these notes will appear in two papers separately, one [53] in *Handbook of Teichmüller Theory* Vol. VII, and the other [54] in *Tokyo Journal of Mathematics*.

The author is grateful to Professor Toshitake Kohno for his invitation to post these notes in Lecture Notes in Mathematical Sciences, The University of Tokyo.

February 2019
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Throughout the discussion, we will fix a topological surface $\Sigma_{g,n}$ obtained by removing distinct $n$ points from a closed connected oriented surface $\Sigma_g$ of genus $g$. A surface with no punctures $\Sigma_{g,0}$ is nothing but $\Sigma_g$. We will assume that the Euler characteristic $\chi(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is negative. The *mapping class group* $\Gamma_{g,n}$ of $\Sigma_{g,n}$ is defined by

$$\Gamma_{g,n} = \{f : \Sigma_{g,n} \to \Sigma_{g,n} \mid \text{orientation preserving homeomorphisms}\}/\simeq$$

where $f \simeq g$ means that the homeomorphisms $f$ and $g$ are *isotopic*. Note that in this situation $f$ is isotopic to $g$ if and only if $f$ is *homotopic* to $g$ (see [22]). The isotopy class of $f$ (the mapping class) is denoted by $[f]$. The group structure of $\Gamma_{g,n}$ is defined by composition of maps: $[f][g] = [f \circ g]$.

A *marked Riemann surface* is a pair $(S, w)$ consisting of a Riemann surface $S$ and an orientation preserving homeomorphism $w : \Sigma_{g,n} \to S$ called a *marking*. Since we are assuming $\chi(\Sigma_{g,n}) < 0$, a Riemann surface modeled on $\Sigma_{g,n}$ has a hyperbolic metric (the Poincaré metric).

Marked Riemann surfaces $(S, w)$ and $(S', w')$ are *equivalent* and denoted by $(S, w) \sim (S', w')$ if there exists a biholomorphic mapping $h : S \to S'$ such that

[1]In the case of manifolds, an example of a "tautological atlas" is that of the $n$-dimensional projective space $\mathbf{P}^n = \{(x_0 : x_1 : \ldots : x_n)\}$ consisting of the $n$-dimensional affine spaces $A_i = \{x_0, x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n\}$, $i = 0, 1, \ldots, n$. We do not give any technical definition of "tautological atlas", but the meaning would be understandable.

the following diagram is homotopically commutative (i.e., $w' \simeq h \circ w$):

$$\begin{array}{ccc} \Sigma_{g,n} & \xrightarrow{w} & S \\ {\scriptstyle id.}\downarrow & & \downarrow{\scriptstyle h} \\ \Sigma_{g,n} & \xrightarrow[w']{} & S'. \end{array}$$

The equivalence class of $(S, w)$ is denoted by $[S, w]$. The *Teichmüller space* $T_{g,n}$ modeled on $\Sigma_{g,n}$ is the set of equivalence classes of marked Riemann surfaces. It is a metric space endowed with the *Teichmüller metric*, and by Teichmüller's theorem, it is homeomorphic to $\mathbf{R}^{6g-6+2n}$ (see §8.2 of the present notes, and Chapter 5 of [33] or Chapter 6 of [31]).

In what follows, we assume $3g - 3 + n \geqq 1$.

By the Ahlfors-Bers theory ([4],[6],[7],[8],[9]), Teichmüller space $T_{g,n}$ is a complex manifold of complex dimension $3g - 3 + n$. The mapping class group $\Gamma_{g,n}$ acts on $T_{g,n}$ by the rule:

$$(1) \qquad [f][S, w] = [S, w \circ f^{-1}] \quad \text{for} \quad \forall [f] \in \Gamma_{g,n}, \ \forall [S, w] \in T_{g,n}.$$

Note that the action changes the marking $w$ but not the Riemann surface $S$. The action is properly discontinuous. It preserves the Teichmüller metric and the complex structure. (See [33], [31].) The *moduli space* $M_{g,n}$ is defined to be the quotient

$$M_{g,n} = T_{g,n}/\Gamma_{g,n}.$$

This is a normal complex variety ([19]).

The moduli space $M_{g,n}$ parametrizes all the isomorphism classes of Riemann surfaces homeomorphic to $\Sigma_{g,n}$. It is known that $M_{g,n}$ is not compact. But by adding "frontier points"which parametrize Riemann surfaces with nodes, it can be compactified. This is the *Deligne-Mumford compactification* of the moduli space (the *DM-compactification* for short), [21]. We will denote the DM-compactification by $\overline{M}_{g,n}$.

Unfortunately, the arguments of Deligne and Mumford [21] are not easy to understand. Because of its importance, several authors have tried to understand the compactification by analytic methods. L. Bers [12], [13] was one of the authors who started an analytic approach to the DM-compactification, but his project was not completed. An analytic construction was given only in the 21st century by J. H. Hubbard and S. Koch [32]. (See [45], p.459, for Kra's comment on their work.)

The DM-compactification of the moduli space has a structure of a complex orbifold. The primary purpose of the present lectures is to construct natural orbifold-charts of $\overline{M}_{g,n}$ which are indexed by simplexes of Harvey's *curve complex* $\mathcal{C}_{g,n}$ (see §2.2 below and [28]). Note that our orbifold-charts $\{(D_\varepsilon(\sigma), W(\sigma))\}_{\sigma \in \mathcal{C}_{g,n}}$ are *in a generalized sense*, in which the groups $W(\sigma)$ are not necessarily finite, but they act on the complex manifolds $D_\varepsilon(\sigma)$ *properly discontinuously.* See §2 for a formal definition of generalized orbifold-charts.

Moreover, strictly speaking, our orbifold-charts are not indexed by the curve complex $\mathcal{C}_{g,n}$ itself but by the quotient of $\mathcal{C}_{g,n}$ under the action of $\Gamma_{g,n}$. The quotient complex $\mathcal{C}_{g,n}/\Gamma_{g,n}$ is a finite simplicial complex. The index set of our orbifold-charts is a finite set of simplexes, each simplex chosen from a coset of $\mathcal{C}_{g,n}/\Gamma_{g,n}$ as a representative.

For each $\sigma \in \mathcal{C}_{g,n}$, the chart $(D_\varepsilon(\sigma), W(\sigma))$ is defined as follows. Let $\Gamma(\sigma)$ be the free abelian group generated by right-handed Dehn twists about the simple closed curves belonging to $\sigma$, and let $N\Gamma(\sigma)$ be the normalizer of $\Gamma(\sigma)$ in $\Gamma_{g,n}$. The group $W(\sigma)$ is the quotient group $N\Gamma(\sigma)/\Gamma(\sigma)$, called the "Weyl group"(see §4, Definition 5). Note that our Weyl group $W(\sigma)$ is a discrete group, but not necessarily a finite group. The manifold $D_\varepsilon(\sigma)$ is a certain complex manifold of complex dimension $3g-3+n$, called the "controlled deformation space"(see §6, Definition 9). Topologically, $D_\varepsilon(\sigma)$ is homeomorphic to an open cell of (real) dimension $6g-6+2n$ (see Lemma 14). The group $W(\sigma)$ acts on $D_\varepsilon(\sigma)$ properly discontinuously.

Our main theorem states the following:

**Theorem 1.** *The finite family* $\{(D_\varepsilon(\sigma), W(\sigma))\}_{\sigma \in \mathcal{C}_{g,n}/\Gamma_{g,n}}$ *is a ("tautological") atlas of complex orbifold-charts of the DM-compactification* $\overline{M}_{g,n}$ *of moduli space.*

We admit the empty set $\emptyset$ to be a member of $\mathcal{C}_{g,n}$. The chart corresponding to $\emptyset$ is nothing but the pair of Teichmüller space and the mapping class group: $(D_\varepsilon(\emptyset), W(\emptyset)) = (T_{g,n}, \Gamma_{g,n})$.

Essentially the same orbifold charts appear in many places in the literature. In fact, the discussion of Hubbard-Koch [32] contains them as an important part. Precisely speaking, however, to get an orbifold chart such as $(D_\varepsilon(\sigma), W(\sigma))$, we must be a little bit careful: the definition of orbifold chart requires that *the quotient* $D_\varepsilon(\sigma)/W(\sigma)$ *should be an open subset of* $\overline{M}_{g,n}$. (See §2, Definition 1.) This is rather a delicate condition, especially in constructing a "tautological atlas". Our $(D_\varepsilon(\sigma), W(\sigma))$ (see §6, Definition 9) meets the requirement, but the similar pair $(D^*_\varepsilon(\sigma), W(\sigma))$ (see §6, Definition 8) does not. This is the reason why we modified $(D^*(\sigma), W(\sigma))$ to $(D_\varepsilon(\sigma), W(\sigma))$. See Remark after Lemma 15.

Theorem 1 was stated in [51] with the idea of the proof. It was stated again in [52] together with an argument on the $\varepsilon$-thick part of Teichmüller space. In these lectures, we will present the full discussions.

An interesting feature of our construction is that certain Euclidean crystallographic groups appear in connection with maximal simplexes of $\mathcal{C}_{g,n}$. In fact, our second purpose is to explain this appearance. Let $\hat{T}_{g,n}$ be the *augmented Teichmüller space* or equivalently the *metric completion of* $T_{g,n}$ *with respect to the Weil-Petersson metric.* (See §3, and §8.5.) The action of the mapping class group $\Gamma_{g,n}$ on $T_{g,n}$ extends continuously to an action on the completion $\hat{T}_{g,n}$. Let $Aut(T_{g,n})$ be the group of self-biholomorphisms of $T_{g,n}$. A natural homomorphism $\alpha : \Gamma_{g,n} \to Aut(T_{g,n})$ is defined by the action of $\Gamma_{g,n}$ on $T_{g,n}$. By Royden's theorem and its generalization ([67], [23], see also [31]), the homomorphism $\alpha$ is onto, provided $3g-3+n > 1$. Thus the group $Aut(T_{g,n})$ acts on the completion $\hat{T}_{g,n}$. Let $\sigma_0$ be a maximal simplex of $\mathcal{C}_{g,n}$. Then we have a unique frontier point $p(\sigma_0) \in \hat{T}_{g,n} - T_{g,n}$ corresponding to the degenerate surface obtained by pinching each curve of $\sigma_0$ to a node. (Here we follow [83] Definition 1, and call any point belonging to $\hat{T}_{g,n} - T_{g,n}$ a *frontier point.*) Our second main theorem states the following:

**Theorem 2.** *If* $\sigma_0$ *is a maximal simplex, then the Weyl group* $W(\sigma_0)$ *is a finite group. Furthermore, if* $3g-3+n > 1$*, then for the action of* $Aut(T_{g,n})$ *on* $\hat{T}_{g,n}$*, the isotropy group of the frontier point* $p(\sigma_0)$ *is a crystallographic group acting on Euclidean* $(3g-3+n)$*-space* $\mathbf{E}^{3g-3+n}$.

The author believes that this result is new. For details, see §10.

The maximal simplex $\sigma_0$ determines a *pants decomposition* of the surface $\Sigma_{g,n}$ and *vice versa*. (See §9.1.) If $3g-3+n>1$ and $(g,n)\neq(2,0),(1,2)$, the homomorphism $\alpha:\Gamma_{g,n}\to Aut(T_{g,n})$ is an isomorphism. (See [67], [23], [31], and Lemma 20.) Thus as a corollary to Theorem 2, we have the following

**Corollary 2.1.** *Suppose* $3g-3+n>1$ *and* $(g,n)\neq(2,0),(1,2)$. *Then to each pants decomposition of* $\Sigma_{g,n}$ *is attached a subgroup of* $\Gamma_{g,n}$ *which is a crystallographic group acting on Euclidean* $(3g-3+n)$*-space* $\mathbf{E}^{3g-3+n}$.

These results remind us of the similar property of hyperbolic geometry that the isotropy group of a cusp point consists of parabolic transformations and the horospheres based at the cusp point have the Euclidean structure. (See [64] §4.6.)

Conversely, given a flat manifold with the Bieberbach fundamental group, is it realized as the cusp cross-section of a complete finite volume one-cusped hyperbolic manifold? A. Szczepański attacks this problem from the viewpoint of fumdamental groups ([70]) and eta invariants ([71]).

These lectures are divided in two parts.

**Part I** is concerned with **the Deligne-Mumford compactification** of moduli spaces of Riemann surfaces, and **Part II** with **crystallographic groups**.

Our plan after §1 is the following:

In §2, we will recall the notion of orbifold and curve complex.

In §3, we will review the augmented Teichmüller spaces $\hat{T}_{g,n}$.

In §4, we will study the $\varepsilon$-thick part $T^{\varepsilon}_{g,n}$ of $T_{g,n}$ and the action of the mapping class group $\Gamma_{g,n}$ on it.

In §5, we will give our proof of the well-known fact that the quotient space $\hat{T}_{g,n}/\Gamma_{g,n}$ is compact and topologically identified with the DM-compactification of the moduli space.

In §6, we will introduce the *deformation spaces* $D^*_{\varepsilon}(\sigma)$ and the *controlled deformation spaces* $D_{\varepsilon}(\sigma)$, as refinements of Bers' deformation spaces [13]. Theorem 1 will be proved in this section.

**Part II** begins at §7.

In §7, which is short, we will recall the definition of crystallographic groups.

In §8, we will review basic facts on Teichmüller spaces and the Weil-Petersson metric.

In §9, we will review Wolpert's results on Fenchel-Nielsen deformations.

In §10, we will prove our second main theorem (Theorem 2).

In §11, we will give some examples.

**The bibliography** at the end is common to Parts I and II.

## 2. **Orbifolds and curve complexes**

In this section, we will recall the formal definition of orbifolds and curve complexes.

2.1. **Orbifolds.** Orbifolds were first introduced by I. Satake [68] under the name of *V-manifolds*, and were re-discovered by W. Thurston as an essential tool in the geometry and topology of 3-manifolds ([72]). The name of *orbifold* is due to Thurston. "Orbifolds conveniently blend the theory of manifolds with that of finite group actions"[15], p.441. The definition given here is based on the simplified version due to F. Bonahon and L. C. Siebenmann [15] (see also [55]). Let $\tilde{U}$ be a finite dimensional smooth manifold which is acted on by a (not necessarily finite) group $G$. Suppose the action of $G$ is smooth and properly discontinuous. Then the isotropy group of each point of $\tilde{U}$ is finite, and the quotient space $\tilde{U}/G$ is a good example of an orbifold. The following definition is a somewhat generalized one, having this example in mind.

**Definition 1.** *A smooth $m$-*dimensional orbifold *(briefly, an $m$-*orbifold*) is a $\sigma$-compact Hausdorff space $M$ which is locally modeled on a quotient space of a finite group action on a smooth $m$-dimensional manifold. More precisely,*
*(i) an $m$-*orbifold *$M$ is covered by an atlas of* orbifold-charts $\{(\tilde{U}_i, G_i, \varphi_i, U_i)\}_{i\in I}$, *each chart consisting of a smooth $m$-manifold $\tilde{U}_i$, a (not necessarily finite) group $G_i$ acting on $\tilde{U}_i$ smoothly and* properly discontinuously, *an open set $U_i$ of $M$ and a* folding map $\varphi_i : \tilde{U}_i \to U_i$ *which induces a natural homeomorphism $\tilde{U}_i/G_i \to U_i$.*
*(ii) (*compatibility condition*) for $x \in \tilde{U}_i$ and $y \in \tilde{U}_j$ such that $\varphi_i(x) = \varphi_j(y) \in U_i \cap U_j$, there exists a diffeomorphism $\psi : \tilde{V}_x \to \tilde{V}_y$ from an open neighborhood of $x$ in $\tilde{U}_i$ to an open neighborhood of $y$ in $\tilde{U}_j$ such that $\psi(x) = y$ and $\varphi_j\psi = \varphi_i$.*

**Remark.** The usual definition of orbifold requires the group action of $G_i$ on $\tilde{U}_i$ to be *effective*. In our discussion, however, we do not assume the effectiveness, aiming at the uniform treatment to cover certain sporadic cases where the actions of the mapping class groups on the Teichmüller spaces are not effective (for example the case $(g, n) = (2, 0)$). Note that Hinich and Vaintrob [30] §3 also consider non-effective orbifolds. □

In what follows, we will always consider *complex $m$-orbifolds*, in which the manifolds $\tilde{U}_i$ are complex manifolds of complex dimension $m$, the gluing diffeomorphisms $\psi$ are biholomorphic, and the groups $G_i$ act on $\tilde{U}_i$ holomorphically. We will often use the simplified notation $\{(\tilde{U}_i, G_i)\}_{i\in I}$ to represent the orbifold-charts $\{(\tilde{U}_i, G_i, \varphi_i, U_i)\}_{i\in I}$.

2.2. **Curve complexes.** Given a surface $\Sigma_{g,n}$, W. Harvey [28] introduced an abstract simplicial complex called the *complex of curves* (or *curve complex*) $\mathcal{C}_{g,n} = \mathcal{C}(\Sigma_{g,n})$.

**Definition 2.** *A* vertex *of the* curve complex *$\mathcal{C}_{g,n}$ is an isotopy class of an essential simple closed curve on $\Sigma_{g,n}$, where a closed curve is said to be* essential *if it is neither null-homotopic nor homotopic to a puncture. A* simplex *$\sigma$ of $\mathcal{C}_{g,n}$ is a collection of disjoint, mutually non-isotopic essential simple closed curves on $\Sigma_{g,n}$: $\sigma = \langle C_1, \ldots, C_k\rangle$.*

The number $k$ of simple closed curves contained in $\sigma$ will be denoted by $|\sigma|$. It is known that $|\sigma| \leqq 3g-3+n$. Clearly we have $\dim \sigma = |\sigma|-1$. Harvey [28] introduced the curve complex to study the "boundary structure" of Teichmüller space and the action of $\Gamma_{g,n}$ on the boundary.

The mapping class group $\Gamma_{g,n}$ naturally acts on $\mathcal{C}_{g,n}$ as automorphisms. N. V. Ivanov [36] proved the converse, assuming $g \geqq 2$. M. Korkmaz [43] and F. Luo [47] extended Ivanov's results to cover the cases $g = 0, 1$. The final form of their results is the following:

**Theorem 3** ([47])**.** *(a) If $3g-3+n \geqq 2$ and $(g,n) \neq (1,2)$, then any automorphism of $\mathcal{C}_{g,n}$ is induced by a self-homeomorphism of the surface $\Sigma_{g,n}$.*
*(b) Any automorphism of $\mathcal{C}_{1,2}$ preserving the set of vertices represented by separating loops is induced by a self-homeomorphism of the surface.*
*(c) There is an automorphism of $\mathcal{C}_{1,2}$ which is not induced by any homeomorphisms.*

**Remark** (see Lemma 20 in §10 below). Suppose $3g - 3 + n \geqq 1$ and $(g, n) \neq (2, 0), (1, 2), (1, 1), (0, 4)$, then the natural homomorphism $\Gamma_{g,n} \to \mathrm{Aut}(\mathcal{C}_{g,n})$ is injective. If $(g, n) = (2, 0), (1, 2)$ or $(1, 1)$, the kernel of $\Gamma_{g,n} \to \mathrm{Aut}(\mathcal{C}_{g,n})$ is isomorphic to $\mathbf{Z}_2$. If $(g, n) = (0, 4)$, the kernel of $\Gamma_{0,4} \to \mathrm{Aut}(\mathcal{C}_{0,4})$ is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. $\square$

Combining Remark and Theorem 3, we have

**Corollary 3.1.** *If $3g - 3 + n \geqq 2$ and $(g, n) \neq (1, 2)$, then*

$$\mathrm{Aut}(\mathcal{C}_{g,n}) = \begin{cases} \Gamma^*_{g,n} & \textit{for} \quad (g,n) \neq (2,0) \\ \Gamma^*_{2,0}/\mathbf{Z}_2 & \textit{for} \quad (g,n) = (2,0), \end{cases}$$

*where $\Gamma^*_{g,n}$ denotes the extended mapping class group containing orientation reversing homeomorphisms.*

## 3. Augmented Teichmüller spaces

L. Bers [12] [13] and W. Abikoff [1] [3] introduced the *augmented Teichmüller space* $\hat{T}_{g,n}$ by attaching to $T_{g,n}$ a special type of Kleinian groups called *regular b-groups.* Regular $b$-groups are a disguised form of Riemann surfaces with nodes. Thus the augmented Teichmüller space is the Teichmüller space to which Riemann surfaces with nodes are attached. The most natural construction of the augmented Teichmüller space $\hat{T}_{g,n}$ would be to take the metric completion of Teichmüller space with respect to the *Weil-Petersson metric* ([83], See [82] Theorem 4.4.)

As mentioned in §1, the Teichmüller metric is a natural metric on $T_{g,n}$ from the viewpoint of quasi-conformal deformations. But if we view $T_{g,n}$ as a Riemannian manifold, it is not so natural a metric, because *the Teichmüller metric is not defined by* $ds^2$ ([73], p.384). The Weil-Petersson metric is a natural metric from this point of view ([73], [4] §4). S. Wolpert [74] and T. Chu [20] proved, however, that Teichmüller space equippd with the Weil-Petersson metric is not complete. In fact, there exist geodesics which terminate in a finite length. H. Masur [49], S. Wolpert [80], [81], [82], and S. Yamada [83] studied in detail the behavior of the metric tensors on this metric completion $\hat{T}_{g,n}$.

Let $\sigma = \langle C_1, \ldots, C_k \rangle$ be a simplex in the curve complex $\mathcal{C}_{g,n}$. Then a singular surface, denoted by $\Sigma_{g,n}(\sigma)$, is obtained from $\Sigma_{g,n}$ by pinching each curve $C_i \in \sigma$ to a point. On a marked Riemann surface $(S, w)$, there is a disjoint union of simple closed geodesics $\{c_1, \ldots, c_k\}$ such that $c_i$ represents the isotopy class of $w(C_i)$ $(i = 1, \ldots, k)$. If each $c_i$ shrinks to a point as $[S, w]$ approaches the "boundary"of $\hat{T}_{g,n}$, then in the limit we have a Riemann surface with nodes. (See [83], Proposition 1.) This nodal Riemann surface is modeled on $\Sigma_{g,n}(\sigma)$.

Here we will recall from Bers' paper [12] the formal definition of a Riemann surface with nodes.

**Definition 3** ([12])**.** *A* Riemann surface with nodes *(or a* nodal Riemann surface *for short) $S$ is a connected complex space such that every point $p \in S$ has arbitrarily small neighborhood isomorphic either to the set $|z| < 1$ in $\mathbb{C}$ or to the set $|z| < 1$, $|w| < 1$, $zw = 0$ in $\mathbb{C}^2$. In the second case, $p$ is called a* node*. Every component of the complement of the nodes is called a* part *of $S$. Each part is assumed to have negative Euler characteristic. (Thus each part has a hyperbolic metric.)*

If $\Sigma_{g,n}(\sigma)$ has no complex structures (in [57], we called such a singular topological surface a *chorizo space*) , the meaning of a "part"or a "node"of $\Sigma_{g,n}(\sigma)$ will be clear. A *marking* of a nodal Riemann surface $S$ (modeled on $\Sigma_{g,n}(\sigma)$) is a homeomorphism $w : \Sigma_{g,n}(\sigma) \to S$ which is orientation preserving on each part.

Teichmüller space of marked Riemann surfaces with nodes, $T(\sigma)$, is defined similarly to $T_{g,n}$. This is isomorphic to the product of Teichmüller spaces modeled on the parts of $\Sigma_{g,n}(\sigma)$ ([49], p.624). Its complex dimension is given as follows (see [49], [83]):

$$\dim_{\mathbb{C}} T(\sigma) = 3g - 3 + n - |\sigma|. \tag{2}$$

**Remark.** Teichmüller space $T_{g,n}$ is a bounded domain of $\mathbb{C}^{3g-3+n}$ homeomorphic to an open cell. (See [33], §6.1.4.) Being a product of the Teichmüller spaces of its parts, Teichmüller space $T(\sigma)$ of $\Sigma_{g,n}(\sigma)$ is again a bounded domain of $\mathbb{C}^{3g-3+n-|\sigma|}$, and homeomorphic to an open cell. □

Following [83], we will call $\hat{T}_{g,n} \setminus T_{g,n}$ the *frontier set* or the *boundary* and will denote it by $\partial T_{g,n}$. $T(\sigma)$ has its own Weil-Petersson metric, and its metric completion $\hat{T}(\sigma)$ is meaningful ([83], and Wolpert [81]). Each component of the boundary Teichmüller spaces is totally geodesic (Theorem 2 of [83]), and we have

$$\hat{T}_{g,n} = \bigcup_{\sigma \in \mathcal{C}_{g,n}} T(\sigma) \quad \text{and} \quad \partial T_{g,n} = \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}} T(\sigma), \tag{3}$$

where $T(\emptyset) = T_{g,n}$.

The equalities (3) are essentially the same as those on p.330 of [83], because we have the following equation:

$$\hat{T}(\sigma) = \bigcup_{\sigma < \tau} T(\tau), \tag{4}$$

where $\sigma < \tau$ means that $\sigma$ is a face of $\tau$. See Lemma 6 of §5.

As was earlier pointed out by Weil himself [73], p.389, the Weil-Petersson metric is invariant under the action of $\Gamma_{g,n}$. The action of $\Gamma_{g,n}$ is extended to a continuous action on the augmented Teichmüller space $\hat{T}_{g,n}$. It is seen (Remark at the end of §4) that

$$[f](T(\sigma)) = T(f(\sigma)) \quad \text{for} \quad \forall [f] \in \Gamma_{g,n}, \ \ \forall \sigma \in \mathcal{C}_{g,n}. \tag{5}$$

Let $\Gamma(\sigma)$ be the free abelian subgroup of $\Gamma_{g,n}$ generated by the right-handed Dehn twists $\tau(C_i)$ about the simple closed curves $C_i$ belonging to $\sigma$. The following theorem is proved in [83] (Theorem 3 and Remark after that).

**Theorem 4.** *The fixed point set of the action of* $\Gamma(\sigma)$ *on* $\hat{T}_{g,n}$ *is* $\hat{T}(\sigma)$.

**Corollary 4.1.** *The action of* $\Gamma_{g,n}$ *on* $\hat{T}_{g,n}$ *is not properly discontinuous.*

*Proof.* If $\sigma \neq \emptyset$, each point in $\hat{T}(\sigma)$ has the infinite isotropy subgroup which contains $\Gamma(\sigma)$. □

## 4. The $\varepsilon$-thick part

Let $C$ be an essential simple closed curve on $\Sigma_{g,n}$. For any point $p = [S, w] \in T_{g,n}$, let $l_C(p)$ be the length of the simple closed geodesic $c$ on $S$ homotopic to $w(C)$.

**Lemma 1.** *For any mapping class $[f]$, we have*

$$l_{f(C)}([f](p)) = l_C(p).$$

*Proof.* Let $p = [S, w]$ be any point of $T_{g,n}$. By (1), $[f](p)$ is the equivalence class of $(S, w \circ f^{-1})$. By the definition of the length function, $l_{f(C)}([f](p))$ is the length of the simple closed geodesic on $S$ homotopic to $w \circ f^{-1}(f(C)) = w(C)$. Thus $l_{f(C)}([f](p))$ is equal to $l_C(p)$. $\square$

Define $L : T_{g,n} \to \mathbf{R}$ by

$$L(p) \stackrel{\text{def}}{=} \min_{C \subset \Sigma_{g,n}} l_C(p).$$

Following [38], we will call $L$ the *systole function*. By Lemma 1, the systole function $L$ is invariant under the action of $\Gamma_{g,n}$. It is piecewise real analytic on $T_{g,n}$ (see [2], or [33] Lemma 3.7).

**Lemma 2** ([41], [2], [64] p.655)**.** *There exists a universal constant $M(> 0)$ such that two distinct simple closed geodesics on any Riemann surface $S$ are disjoint if their length are smaller than $M$.*

We will call such a constant $M$ a *(2-dimensional) Margulis constant*. It is not unique, of course, because any positive number smaller than $M$ has again the same property.

Take a positive real number $\varepsilon$ smaller than a Margulis constant $M$. The *$\varepsilon$-thick part* $T^{\varepsilon}_{g,n}$ is a subspace of $T_{g,n}$ defined as follows:

$$T^{\varepsilon}_{g,n} \stackrel{\text{def}}{=} \{p \in T_{g,n} \mid L(p) \geqq \varepsilon\}.$$

This space $T^{\varepsilon}_{g,n}$ is a real analytic manifold with corners. It has many essential features in common with $T_{g,n}$. L. Ji and S. Wolpert [38] proved that $T^{\varepsilon}_{g,n}$ is an equivariant deformation retract of $T_{g,n}$ with respect to the action of $\Gamma_{g,n}$. (They call $T^{\varepsilon}_{g,n}$ the *truncated Teichmüller space*.) N. V. Ivanov [35] used $T^{\varepsilon}_{g,n}$ in his cohomological study of the mapping class groups.

**Definition 4** (Facet)**.** *Let $\sigma = \langle C_1, \ldots, C_k\rangle$ be a (non-empty) simplex of $\mathcal{C}_{g,n}$. The* facet $F^{\varepsilon}(\sigma)$ *corresponding to $\sigma$ is defined to be the set of those points $p \in T_{g,n}$ that satisfy the following conditions:*
*(1)* $l_{C_i}(p) = \varepsilon, \quad i = 1, \ldots, k,$ *and*
*(2)* $l_C(p) > \varepsilon$ *for any other essential simple closed curve* $C \subset \Sigma_{g,n}$.

By the definition,

$$F^{\varepsilon}(\sigma) \cap F^{\varepsilon}(\tau) = \emptyset, \text{ if } \sigma \neq \tau. \tag{6}$$

We call the set

$$\partial T^{\varepsilon}_{g,n} \stackrel{\text{def}}{=} \{p \in T_{g,n} \mid L(p) = \varepsilon\}$$

the *boundary* of $T^{\varepsilon}_{g,n}$. The boundary is a disjoint union of the facets:

$$\partial T^{\varepsilon}_{g,n} = \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}} F^{\varepsilon}(\sigma). \tag{7}$$

In what follows, we will put $m = 3g-3+n$, for simplicity. By adding suitable $m-k$ simple closed curves $C_{k+1}, \dots, C_m$ to the members of $\sigma$, we get a maximal simplex of $\mathcal{C}_{g,n}$:

$$\tilde{\sigma} = \langle C_1, \dots, C_k, C_{k+1}, \dots, C_m \rangle.$$

Recall that with $\tilde{\sigma}$ are associated the *Fenchel-Nielsen coordinates*. (See §9 for a detailed explanation.) In fact, $\tilde{\sigma}$ decomposes the surface $\Sigma_{g,n}$ into a union of (generalized) pairs of pants $P_1, \dots, P_{2g-2+n}$. Here a *generalized pair of pants* means a surface homeomorphic to a 2-sphere with the interiors of three disjoint disks removed (this is an ordinary one), or a once punctured annulus, or twice punctured disk. A pair of pants has the Euler characteristic $-1$. A marked Riemann surface $(S, w)$ representing $p \in T_{g,n}$ is decomposed into a union of hyperbolic pants by the system of $m$ simple closed geodesics $\langle c_1, \dots, c_m \rangle$ on $S$, where $c_i$ is homotopic to $w(C_i)$, $i = 1, \dots, m$. The Fenchel-Nielsen coordinate $l_i(p) (> 0)$ is the hyperbolic length of $c_i$, and $\tau_i(p)$ is the amount of the "twist"on gluing pairs of pants along $c_i$. Following [81] [82] we measure the magnitude of twist not by the angle but by the hyperbolic length of the movement along $c_i$.

The Fenchel-Nielsen coordinates $(l_1, \dots, l_m, \tau_1, \dots, \tau_m)$ are real analytic global coordinates of $T_{g,n}$ (see [2], [33]), which give a real analytic isomorphism $T_{g,n} \cong \mathbf{R}_+^m \times \mathbf{R}^m$. With respect to these coordinates, $F^\varepsilon(\sigma)$ is written by

$$l_1 = \cdots = l_k = \varepsilon, \;\; l_{k+1} > \varepsilon, \cdots, l_m > \varepsilon. \tag{8}$$

It immediately follows that

$$\dim_{\mathbf{R}} F^\varepsilon(\sigma) = 2m - k, \quad \text{where} \quad k = |\sigma|.$$

Note that the facet $F^\varepsilon(\sigma)$ always contains a factor of twist-coordinate space $\mathbf{R}^m = \{(\tau_1, \dots, \tau_m)\}$, because the coordinates $(\tau_1, \dots, \tau_m)$ are free from any constraints.

If another simplex $\tau \in \mathcal{C}_{g,n}$ (with $|\tau| = l$) contains $\sigma$ as a face, then with suitable simple closed curves $C_{k+1}, \dots, C_l$ on $\Sigma_{g,n}$, we have

$$\tau = \langle C_1, \dots, C_k, C_{k+1}, \dots, C_l \rangle.$$

Choosing further simple closed curves $C_{l+1}, \dots, C_m$ on $\Sigma_{g,n}$, we have a maximal simplex $\tilde{\sigma}' = \langle C_1, \dots, C_k, \dots, C_l, \dots, C_m \rangle$. With respect to the associated Fenchel-Nielsen coordinates, $F^\varepsilon(\tau)$ is written as

$$l_1 = \cdots = l_k = l_{k+1} = \cdots = l_l = \varepsilon, \;\; l_{l+1} > \varepsilon, \cdots, l_m > \varepsilon. \tag{9}$$

Thus we have proved the following

**Lemma 3.** *If $|\sigma| = k$, the facet $F^\varepsilon(\sigma)$ is real analytically isomorphic to $\mathbf{R}_{>\varepsilon}^{m-k} \times \mathbf{R}^m$. Furthermore, the closure $\overline{F^\varepsilon(\sigma)}$ contains $F^\varepsilon(\tau)$ in its boundary if and only if $\sigma \subsetneqq \tau$ (i.e. $\sigma$ is a proper face of $\tau$).*

Given a simplex $\sigma$ with $|\sigma| < m$, there are infinitely many simplexes $\tau$ with $\sigma < \tau$. Thus $F^\varepsilon(\sigma)$ is surrounded by infinitely many facets $F^\varepsilon(\tau)$. In this sense, a facet $F^\varepsilon(\sigma)$ itself is an infinite polyhedron (unless the simplex $\sigma$ is maximal).

The following theorem is regarded as a toy analogue of Royden-Earle-Kra's theorem [67], [23], or of Masur-Wolf's theorem [50]. This theorem is not quite necessary in order to prove our main theorem, but it gives a global perspective to our construction of orbifold-charts.

**Theorem 5.** *Suppose* $3g-3+n \geqq 2$ *and* $(g,n) \neq (2,0), (1,2)$. *Considering* $T^{\varepsilon}_{g,n}$ *as an infinite polyhedron, we have*

$$\mathrm{Aut}(T^{\varepsilon}_{g,n}) = \Gamma_{g,n}, \tag{10}$$

*where* $\mathrm{Aut}(T^{\varepsilon}_{g,n})$ *is the orientation preserving automorphism group of the polyhedron.*

*Proof.* The totality of the closed facets on $\partial T^{\varepsilon}_{g,n}$ makes a complex (*the facet complex*) in the following sense:
(i) Two closed facets $\overline{F^{\varepsilon}(\sigma)}$ and $\overline{F^{\varepsilon}(\tau)}$ are disjoint, or else intersect in a common closed facet $\overline{F^{\varepsilon}(\rho)}$, where $\rho = \langle \sigma, \tau \rangle$.
(ii) Given a closed facet $\overline{F^{\varepsilon}(\tau)}$, there are only a finite number of closed facets $\overline{F^{\varepsilon}(\sigma)}$ such that $\overline{F^{\varepsilon}(\sigma)} \supset \overline{F^{\varepsilon}(\tau)}$.

These properties are easily proved by Lemma 3.

A *flag* in the facet complex means a finite sequence of the closed facets:

$$\overline{F^{\varepsilon}(\sigma_1)} \supset \overline{F^{\varepsilon}(\sigma_2)} \supset \cdots \supset \overline{F^{\varepsilon}(\sigma_u)}.$$

This flag corresponds uniquely to a flag in $\mathcal{C}_{g,n}$:

$$\sigma_1 < \sigma_2 < \cdots < \sigma_u.$$

Now an automorphism of $T^{\varepsilon}_{g,n}$ induces an automorphism of the facet complex on the boundary $\partial T^{\varepsilon}_{g,n}$, and that of the flag complex of the facet complex. Since the latter complex is (inversely) isomorphic to the flag complex of $\mathcal{C}_{g,n}$ (which is in turn isomorphic to the barycentric subdivision of $\mathcal{C}_{g,n}$), we get an automorphism of $C_{g,n}$. By Theorem 3, this automorphism of $\mathcal{C}_{g,n}$ is induced by a unique element of $\Gamma_{g,n}$.

Conversely an element of $\Gamma_{g,n}$ induces an automorphism of $T_{g,n}$ and that of $T^{\varepsilon}_{g,n}$. The argument is closed, and the proof is complete. □

Essentially the same arguments have been done in A. Papadopoulos [63] and K. Ohshika [62].

**Remark.** If $(g,n) = (2,0)$, we have

$$\mathrm{Aut}(T^{\varepsilon}_{2,0}) = \Gamma_{2,0}/\mathbf{Z}_2. \tag{11}$$

This is proved by the same arguments as of Theorem 5, by using Corollary 3.1 instead of Theorem 3. □

Recall that, for a non-empty simplex $\sigma = \langle C_1, \dots, C_k \rangle$ of $\mathcal{C}_{g,n}$, $\Gamma(\sigma)$ is the free abelian subgroup of $\Gamma_{g,n}$ generated by the right-handed Dehn twists $\tau(C_i)$ about $C_i$, $i = 1, \dots, k$. Let $N\Gamma(\sigma)$ be the *normalizer* of $\Gamma(\sigma)$ in $\Gamma_{g,n}$.

**Theorem 6.** *Suppose* $3g-3+n \geqq 1$. *A mapping class* $[f]$ *belongs to the normalizer* $N\Gamma(\sigma)$ *if and only if* $[f]$ *permutes the isotopy classes of the curves* $C_i$, $i = 1, \dots, k$, *of* $\sigma$.

*Proof.* Let $[f]$ be an element of $N\Gamma(\sigma)$, $C_i$ any curve taken from $\sigma$. Since

$$\tau(f(C_i)) = [f]\tau(C_i)[f]^{-1} \ \in \ \Gamma(\sigma),$$

the Dehn twist $\tau(f(C_i))$ commutes with every $\tau(C_j)$ ($C_j \in \sigma$). It is known that if two simple closed curves $C$ and $C'$ cannot be separated by any isotopy of $\Sigma_{g,n}$, then the Dehn twists $\tau(C)$ and $\tau(C')$ do not commute in $\Gamma_{g,n}$ (see Ishida [34]). Applying this to $f(C_i)$, we may assume that $f(C_i) \cap C_j = \emptyset$, for $j = 1, \ldots, k$.

Let $\Sigma'$ denote the component of the (possibly non-connected) surface $\Sigma_{g,n} \setminus \sigma$ that contains $f(C_i)$, where $\Sigma_{g,n} \setminus \sigma$ denotes the surface obtained by cutting open $\Sigma_{g,n}$ along the simple closed curves $C_j,\ j = 1, \ldots, k$. We need the following

**Claim**. *If an essential simple closed curve $C_0$ in a compact, connected, oriented surface $\Sigma'$ is not peripheral, that is, not isotopic in $\Sigma'$ to any boundary curve, then there is a simple closed curve $C'$ in $\Sigma'$ which intersects $C_0$ and such that no isotopy of $\Sigma'$ can separate it from $C_0$.*

*Proof of Claim.* If $C_0$ is a non-separating curve in $\Sigma'$, then there exists a simple closed curve $C'$ which transversely intersects $C_0$ in a point. This curve $C'$ has the required property. On the other hand, if $C_0$ is a separating curve in $\Sigma'$, then both the components $\Sigma'_1$, $\Sigma'_2$ of the cut open surface $\Sigma' \setminus C_0$ have negative Euler characteristic (because $C_0$ is not peripheral). Take two points $a, b$ on $C_0$, then we can find an embedded arc on each of $\Sigma'_1$ and $\Sigma'_2$ which has $a, b$ as end points and is not isotopic (fixing $\{a, b\}$) into $C_0$ by any isotopy of the component. Joining the two arcs, we get a simple closed curve $C'$ in $\Sigma'$ which intersects $C_0$ in the two points $\{a, b\}$. From the construction, $C'$ cannot be separated from $C_0$ by any isotopy of $\Sigma'$. This completes the proof of Claim. □

Now we return to the proof of Theorem 6. By the claim above, if $f(C_i)$ were not peripheral in $\Sigma'$, then there would be a simple closed curve $C'$ in $\Sigma'$ which intersects $f(C_i)$ and such that no isotopy of $\Sigma'$ can separate it from $f(C_i)$. By [34], the Dehn twist $\tau(f(C_i))$ does not commute with the Dehn twist $\tau(C')$. Every Dehn twist $\tau(C_j)$, however, commutes with $\tau(C')$, because $C_j$ is disjoint from $C'$. This contradicts the assumption that $\tau(f(C_i))$ belongs to $\Gamma(\sigma)$. Thus $f(C_i)$ must be isotopic in $\Sigma'$ to a boundary curve, which is a copy of some $C_j$. Since we took $C_i$ from $\sigma$ arbitrarily, this proves that $[f]$ permutes the isotopy classes of curves $C_j,\ j = 1, \ldots, k$.

Conversely, if $[f]$ permutes the isotopy classes of curves $C_j,\ j = 1, \ldots, k$, $[f]$ clearly belongs to the normalizer $N\Gamma(\sigma)$. The proof of Theorem 6 is complete. □

**Corollary 6.1.** *When $\Gamma_{g,n}$ acts on $T^{\varepsilon}_{g,n}$, the subgroup which preserves a facet $F^{\varepsilon}(\sigma)$ is precisely the normalizer $N\Gamma(\sigma)$.*

*Proof.* Recall from (1) of §1 that a mapping class $[f]$ maps a point $p = [S, w]$ to $[f](p) = [S, w \circ f^{-1}]$, and that this action does not change the Riemann surface $S$ but changes the marking. If $p$ belongs to the facet $F^{\varepsilon}(\sigma)$, the set of closed geodesics $\{c_1, \ldots, c_k\}$ on $S$, each $c_i$ being homotopic to $w(C_i)$, is precisely the set of closed geodesics of hyperbolic length $\varepsilon$.

If as assumed the image $[f](p)$ belongs to the same facet $F^{\varepsilon}(\sigma)$, then the set of closed geodesics $\{c'_1, \ldots, c'_k\}$ on $S$, each $c'_i$ being homotopic to $w \circ f^{-1}(C_i)$, is precisely the set of closed geodesics of hyperbolic length $\varepsilon$. On the same Riemann surface $S$, the two sets of closed geodesics must coincide:

$$\{c_1, \ldots, c_k\} = \{c'_1, \ldots, c'_k\}.$$

The set of isotopy classes of simple closed curves $\{f^{-1}(C_1), \ldots, f^{-1}(C_k)\}$ on $\Sigma_{g,n}$ must coincide with $\{C_1, \ldots, C_k\}$, and by Theorem 6, $[f]$ belongs to the normalizer $N\Gamma(\sigma)$.

Let us prove the converse. By Lemma 1, we have a general formula

$$[f](F^{\varepsilon}(\sigma)) = F^{\varepsilon}(f(\sigma)). \tag{12}$$

where $f(\sigma) = \langle f(C_1), \dots, f(C_k)\rangle$. If $[f]$ belongs to $N\Gamma(\sigma)$, it permutes $C_1, \dots, C_k$ by Theorem 6, and $[f]$ satisfies $f(\sigma) = \sigma$. Thus we have

$$[f](F^{\varepsilon}(\sigma)) = F^{\varepsilon}(f(\sigma)) = F^{\varepsilon}(\sigma),$$

hence $[f]$ preserves $F^{\varepsilon}(\sigma)$.

The proof of Corollary 6.1 is complete. □

For a simplex $\sigma = \langle C_1, \dots, C_k\rangle$ of $\mathcal{C}_{g,n}$, let us introduce the following group:

**Definition 5** ("Weyl group")**.** *We denote the quotient group $N\Gamma(\sigma)/\Gamma(\sigma)$ by $W(\sigma)$, and for the formal resemblance would like to call it the* Weyl group *associated with the simplex $\sigma$.*

Our $W(\sigma)$ is not necessarily a finite group.

**Corollary 6.2.** *The group $W(\sigma)$ is the mapping class group of the nodal surface $\Sigma_{g,n}(\sigma)$.*

*Proof.* Let $[f]$ be any element of $N\Gamma(\sigma)$. By Theorem 6, $[f]$ induces a permutation $\pi$ on the set of the isotopy classes of $C_i,\ i = 1, \dots, k$. Thus we may assume

$$f(C_i) = C_{\pi(i)}, \quad i = 1, \dots, k.$$

The nodal surface $\Sigma_{g,n}(\sigma)$ is obtained from $\Sigma_{g,n}$ by pinching each curve $C_i$ to a point (§3). Thus clearly, $f$ induces an orientation preserving self-homeomorphism $\tilde{f}$ of $\Sigma_{g,n}(\sigma)$.

Conversely, given an orientation preserving self-homeomorphism of $\Sigma_{g,n}(\sigma)$, we can find a self-homeomorphism $f$ of $\Sigma_{g,n}$ which permutes the isotopy classes of $C_i,\ i = 1, \dots, k$, and such that the homeomorphism $\tilde{f}$ induced on $\Sigma_{g,n}(\sigma)$ is isotopic to the given one. Let $g$ be a self-homeomorphism of $\Sigma_{g,n}$ which permutes the isotopy classes of $C_i,\ i = 1, \dots, k$, then $\tilde{f}$ and $\tilde{g}$ are isotopic on $\Sigma_{g,n}(\sigma)$, if and only if $f$ is isotopic to $g$ on $\Sigma_{g,n}$ up to the Dehn twists about $C_i,\ i = 1, \dots, k$, in other words, if and only if $[f] = [g]\ \in N\Gamma(\sigma)/\Gamma(\sigma)$.

The proof of Corollary 6.2 is complete. □

**Definition 6** (Fringe)**.** *Let $\sigma = \langle C_1, \dots, C_k\rangle$ be a (non-empty) simplex of $\mathcal{C}_{g,n}$. The* fringe $FR^{\varepsilon}(\sigma)$ *bounded by the facet $F^{\varepsilon}(\sigma)$ is defined by*

$$FR^{\varepsilon}(\sigma) \overset{\text{def}}{=} \bigcup_{0<\delta<\varepsilon} F^{\delta}(\sigma). \tag{13}$$

By the definition,

$$FR^{\varepsilon}(\sigma) \cap FR^{\varepsilon}(\tau) = \emptyset, \quad \text{if} \quad \sigma \neq \tau. \tag{14}$$

For a point $p = [S, w] \in T_{g,n}$, let $c_i$ be the closed geodesic on $S$ which is homotopic to $w(C_i),\ i = 1, \dots, k$ (here we take $\sigma = \langle C_1, \dots, C_k\rangle$). A point $p = [S, w]$ belongs to $FR^{\varepsilon}(\sigma)$ if and only if $L(p) < \varepsilon$ and the $k$ geodesics $\{c_1, \dots, c_k\}$ (and only these geodesics) have the length $L(p)$. Since a point $p$ satisfying $L(p) < \varepsilon$ must belong to some $F^{\delta}(\sigma')$ $(0 < \delta < \varepsilon)$, we have the following partition of $T_{g,n}$:

$$T_{g,n} = T^{\varepsilon}_{g,n} \bigcup_{\emptyset\neq\sigma\in\mathcal{C}_{g,n}} FR^{\varepsilon}(\sigma). \tag{15}$$

By (14), this is a disjoint union.

**Lemma 4.** *As $\varepsilon \to 0$, the facet $F^\varepsilon(\sigma)$ approaches the augmented frontier Teichmüller space $\hat{T}(\sigma)$.*

*Proof.* Take a small positive numer $\eta$ satisfying $\varepsilon < \eta < M$, where $M$ is a Margulis contant. Let $\tilde{\sigma} = \langle C_1, \ldots, C_k, C_{k+1}, \ldots, C_m\rangle$ be a maximal simplex of $\mathcal{C}_{g,n}$ which contains $\sigma = \langle C_1, \ldots, C_k\rangle$ as a face. In terms of the Fenchel-Nielsen coordinates $(l_1, \ldots, l_m, \tau_1, \ldots, \tau_m)$ associated with $\tilde{\sigma}$, the fringe $FR^\eta(\sigma)$ is described as

$$\exists \xi \quad \text{such that} \quad 0 < \xi < \eta,\ \ l_1 = \cdots = l_k = \xi,\ \ l_{k+1} > \xi, \cdots, l_m > \xi.$$

The facet $F^\varepsilon(\sigma)$ which is written as

$$l_1 = \cdots = l_k = \varepsilon,\ \ l_{k+1} > \varepsilon, \cdots, l_m > \varepsilon$$

moves in $FR^\eta(\sigma)$, and converges to $T(\sigma)$, or at worst to $\hat{T}(\sigma)$, as $\varepsilon \to 0$.

This rather intuitive explanation is also justified metrically, because near $T(\sigma)$ the Weil-Petersson metric tensor $G$ is "almost product"of two metrics: the metric along $T(\sigma)$ and the metric along Earle-Marden's *plumbing coordinates* [48], [24]. The plumbing coordinates of the facet $F^\varepsilon(\sigma)$ converge to 0 as $\varepsilon \to 0$. (See [49], and Proposition 3 of [83].)

A sharper estimate is provided by Wolpert. Let $d_{T(\sigma)}$ be the distance on $\hat{T}_{g,n}$ to $\hat{T}(\sigma)$. Wolpert proved ([81], Corollary 4.10)

$$d_{T(\sigma)} = \left(2\pi \sum_{i=1}^{k} l_i\right)^{1/2} + O\left(\sum_{i=1}^{k} l_i^{5/2}\right). \tag{16}$$

Thus as $\varepsilon \to 0$, points on $F^\varepsilon(\sigma)$ (for which $l_1 = \cdots = l_k = \varepsilon$) converge to points in $\hat{T}(\sigma)$.

To be more specific, we have to use in advance the argument which will be given in the proof of Lemma 12 below (in §6). The argument starts with the fact that $\hat{T}_{g,n}$ is a CAT(0) space. (For the definition of CAT(0) space, see [17], [81], [82], [83], [85] §5.1.) By Yamada [83], given a $\sigma \in \mathcal{C}_{g,n}$ there exists a family of geodesics in $\hat{T}_{g,n}$ with the property that any point $p \in T_{g,n}$ is connected by a geodesic in the family to a unique point $\pi_\sigma(p) \in \hat{T}(\sigma)$ such that the length of the geodesic segment between $p$ and $\pi_\sigma(p)$ is equal to the distance $d_{T(\sigma)}(p)$. This property together with Wolpert's formula (16) assures that the family of geodesics are transverse to $F^\varepsilon(\sigma)$, and that as $\varepsilon \to 0$ $F^\varepsilon(\sigma)$ converges along the geodesic family to $\hat{T}(\sigma)$. $\square$

Furthermore, if we use an argument supplied by Yamada (see the proof of Lemma 12), we can prove the following more precise statement

**Addendum to Lemma 4.** $F^\varepsilon(\sigma)$ approaches $T(\sigma)$ as $\varepsilon \to 0$.

By Lemma 4 and its Addendum, $FR^\varepsilon(\sigma) \cup T(\sigma)$ is a metric space embedded in the augmented Teichmüller space $\hat{T}_{g,n}$. We have the following partition extending (15):

$$\hat{T}_{g,n} = T^\varepsilon_{g,n} \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}} (FR^\varepsilon(\sigma) \cup T(\sigma)). \tag{17}$$

This is a disjoint union.

**Remark.** The action of the mapping class group $\Gamma_{g,n}$ on $\hat{T}_{g,n}$ preserves the partition (17). A mapping class $[f] \in \Gamma_{g,n}$ sends $FR^\varepsilon(\sigma) \cup T(\sigma)$ to $FR^\varepsilon(f(\sigma)) \cup$

$T(f(\sigma))$. This follows from the rule (12). In particular, we have the following formula:

$$[f](T(\sigma)) = T(f(\sigma)). \tag{18}$$

## 5. The compactness theorem

Although the following theorem is well-known (cf. [26], [27], [3], [46], [66], [30], [32]), we will give our own proof in this section:

**Theorem 7.** *As a topological space, the quotient $\hat{T}_{g,n}/\Gamma_{g,n}$ is compact, and it is identical with the Deligne-Mumford compactification $\overline{M}_{g,n}$.*

Hinich-Vaintrob [30], Earle-Marden [24], and Hubbard-Koch [32] gave a complex structure to $\hat{T}_{g,n}/\Gamma_{g,n}$, and established the analytic identification of $\hat{T}_{g,n}/\Gamma_{g,n}$ with the algebraic variety $\overline{M}_{g,n}$.

For the sake of exposition, we will give somewhat explicit description of the moduli spaces, including facets and fringes, and will use this to derive the result.

We start with the partition (17). By Definition 6, the fringe $FR^{\varepsilon}(\sigma)$ is foliated by the facets $\{F^{\delta}(\sigma)\}_{0<\delta<\varepsilon}$. From Corollary 6.1, it follows that a mapping class $[f] \in \Gamma_{g,n}$ preserves $FR^{\varepsilon}(\sigma)$ if and only if $[f]$ belongs to $N\Gamma(\sigma)$. Thus $N\Gamma(\sigma)$ acts on $FR^{\varepsilon}(\sigma)$. This action preserves the foliated structure (13). By Lemma 4 together with its Addendum, this action is extended to the action on $FR^{\varepsilon}(\sigma) \cup T(\sigma)$.

Take a maximal simplex $\tilde{\sigma} \in \mathcal{C}_{g,n}$ containing $\sigma = \langle C_1, \ldots, C_k \rangle$ as a face. Let us describe the action of the free abelian subgroup $\Gamma(\sigma)$ of $N\Gamma(\sigma)$ on $FR^{\varepsilon}(\sigma) \cup T(\sigma)$ in terms of the Fenchel-Nielsen coordinates associated with the maximal simplex $\tilde{\sigma}$

$$(l_1, \ldots, l_k, \ldots, l_m, \tau_1, \ldots, \tau_k, \ldots, \tau_m).$$

If a point $p$ of $FR^{\varepsilon}(\sigma)$ belongs to a facet $F^{\delta}(\sigma)$ $(0 < \delta < \varepsilon)$, its Fenchel-Nielsen coordinates satisfy

$$l_1(p) = \cdots = l_k(p) = \delta, \quad l_{k+1}(p) > \delta, \cdots, l_m(p) > \delta.$$

The generater $\tau(C_i)$ $(1 \leqq i \leqq k)$ of $\Gamma(\sigma)$ sends the point $p$ to the point

$$\tau(C_i)(p) = (\underbrace{\delta, \ldots, \delta}_{k}, l_{k+1}(p), \ldots, l_m(p), \tau_1(p), \ldots, \tau_i(p)+\delta, \ldots, \tau_k(p), \tau_{k+1}(p), \ldots, \tau_m(p)),$$

(see (33) in §7). In other words, $\Gamma(\sigma)$ acts on $F^{\delta}(\sigma)$ as the $\delta$-lattice group on the twist coordinate $k$-space $\mathbf{R}^k = \{(\tau_1, \ldots, \tau_k)\}$. The quotient $\mathbf{R}^k/\Gamma(\sigma)$ is a $k$-dimensional torus $T^k_{\delta} = \underbrace{S^1_{\delta} \times \cdots \times S^1_{\delta}}_{k}$, where $S^1_{\delta}$ denotes a circle with peripheral length $\delta$. Thus $F^{\delta}(\sigma)/\Gamma(\sigma)$ is identified with

$$(\mathbf{R}_{>\delta})^{m-k} \times T^k_{\delta} \times \mathbf{R}^{m-k}.$$

Since $\Gamma(\sigma)$ preserves the foliation $FR^{\varepsilon}(\sigma) = \bigcup_{0<\delta<\varepsilon} F^{\delta}(\sigma)$, we can identify the quotient

$$FR^{\varepsilon}(\sigma)/\Gamma(\sigma)$$

with

$$\bigcup_{0<\delta<\varepsilon} (\mathbf{R}_{>\delta})^{m-k} \times T^k_{\delta} \times \mathbf{R}^{m-k}.$$

This is identified with

$$\mathbf{R}_{+}^{m-k} \times \mathrm{Cone}^{\varepsilon}_0(T^k) \times \mathbf{R}^{m-k}, \tag{19}$$

where $\mathrm{Cone}^{\varepsilon}_0(T^k)$ denotes an open cone (whose rays have the length $\varepsilon$) over the $k$-dimensional torus $T^k$ with the cone vertex $\mathbf{0}$ deleted. If the cone vertex $\mathbf{0}$ is filled in, then by Lemma 4 and its Addendum, the "central axis "$\mathbf{R}_{+}^{m-k} \times \{\mathbf{0}\} \times \mathbf{R}^{m-k}$

in $\mathbf{R}_+^{m-k} \times \mathrm{Cone}^\varepsilon(T^k) \times \mathbf{R}^{m-k}$ corresponds to the frontier Teichmüller space $T(\sigma)$, on which the group $\Gamma(\sigma)$ acts as the identity (Theorem 4).

We have proved the following

**Lemma 5.** *As a topological space, the quotient*

$$(FR^\varepsilon(\sigma) \cup T(\sigma))/\Gamma(\sigma)$$

*is identified with*

$$\mathbf{R}_+^{m-k} \times \mathrm{Cone}^\varepsilon(T^k) \times \mathbf{R}^{m-k},$$

*where* $\mathrm{Cone}^\varepsilon(T^k)$ *denotes an open cone over* $T^k$ *whose rays have length* $\varepsilon$*. Via this identification, the central axis* $\mathbf{R}_+^{m-k} \times \{\mathbf{0}\} \times \mathbf{R}^{m-k}$ *corresponds to the frontier Teichmüller space* $T(\sigma)$*.*

The group $N\Gamma(\sigma)$ acts on $FR^\varepsilon(\sigma) \cup T(\sigma)$. Since $\Gamma(\sigma)$ is a normal subgroup of $N\Gamma(\sigma)$, the quotient group $W(\sigma) = N\Gamma(\sigma)/\Gamma(\sigma)$ acts on the quotient space $(FR^\varepsilon(\sigma) \cup T(\sigma))/\Gamma(\sigma)$. By the construction of $(FR^\varepsilon(\sigma) \cup T(\sigma))/\Gamma(\sigma)$, the action of $W(\sigma)$ preserves the frontier Teichmüller space $T(\sigma)$. This action of $W(\sigma)$ on $T(\sigma)$ is just like the action of $\Gamma_{g,n}$ on $T_{g,n}$, because by Corollary 6.2, $W(\sigma)$ is the mapping class group of the nodal surface $\Sigma_{g,n}(\sigma)$, and the frontier Teichmüller space $T(\sigma)$ is the Teichmüller space modeled on the nodal surface $\Sigma_{g,n}(\sigma)$. The frontier Teichmüller space $T(\sigma)$ has its own Weil-Petersson metric, and is embedded in $\hat{T}_{g,n}$ totally geodesically ([83], Theorem 2). We can consider the Weil-Petersson completion $\hat{T}(\sigma)$ of $T(\sigma)$ just like the completion $\hat{T}_{g,n}$ of the ordinary Teichmüller space.

**Lemma 6.** *We have*

$$\hat{T}(\sigma) = \bigcup_{\sigma < \tau} T(\tau). \tag{20}$$

*Proof.* On taking the metric completion $\hat{T}(\sigma)$, we attach to $T(\sigma)$ the Teichmüller spaces of singular surfaces obtained from $\Sigma_{g,n}(\sigma)$ by pinching further essential simple closed curves on $\Sigma_{g,n}(\sigma)$. Each of the resulting surfaces is of the form

$$\Sigma_{g,n}(\tau), \quad \text{where} \quad \sigma \lneqq \tau.$$

Thus the attached Teichmüller spaces are $T(\tau)$, $\sigma \lneqq \tau$. Hence Lemma follows.

The second half of the proof of Lemma 4 together with its Addendum provides a more specific argument: As remarked above, $T(\sigma)$ has its own Weil-Petersson metric, and it is totally geodesically embedded in $\hat{T}_{g,n}$ ([83]). The completion $\hat{T}(\sigma)$ is a CAT(0) space. We can repeat the same argument as in Lemma 4 on $T(\sigma)$: For any $\tau(\gneqq \sigma)$, we can consider the facet $F_\sigma^\varepsilon(\tau)$ of $T(\sigma)$. As $\varepsilon \to 0$, $F_\sigma^\varepsilon(\tau)$ moves in $T(\sigma)$ and approaches $T(\tau)$. □

**Proof of Theorem 7.**

We will prove the compactness of the quotient space $\hat{T}_{g,n}/\Gamma_{g,n}$ by induction on the dimension of $T_{g,n}$.

**Starting case:** $\dim_{\mathbb{C}} T_{g,n} = 1$**.**

Since $\dim_{\mathbb{C}} T_{g,n} = 3g - 3 + n$, we have two subcases $(g,n) = (0,4)$ or $(1,1)$.

**Subcase (i)** $(g,n) = (0,4)$**.**

According to Birman [14], pp.205–207, the mapping class group $\Gamma_{0,4}$ has a presentation with generators $\omega_1, \omega_2, \omega_3$ and definig relations

$$\begin{aligned}
&\omega_1\omega_2\omega_1 = \omega_2\omega_1\omega_2\\
&\omega_2\omega_3\omega_2 = \omega_3\omega_2\omega_3\\
&\omega_1\omega_3 = \omega_3\omega_1\\
&(\omega_1\omega_2\omega_3)^4 = 1\\
&\omega_1\omega_2\omega_3^2\omega_2\omega_1 = 1.
\end{aligned}$$

To understand the geometric meaning of these generators, consider $\Sigma_{0,4}$ as a square pillow with the four corner points deleted. Let $A, B, C, D$ be the resulting punctures at the corners in this cyclic order. Then $\omega_1$ is the positive half twist about the edge $\overline{AB}$, $\omega_2$ the positive half twist about the edge $\overline{BC}$, and $\omega_3$ the positive half twist about the edge $\overline{CD}$. (See [39]). Birman defines two elements $a = \omega_1\omega_3^{-1}$ and $b = \omega_2\omega_1\omega_3^{-1}\omega_2^{-1}$. Geometrically, the element $a$ represents a 180° rotation on the axis through the middle points of the edges $\overline{AB}$ and $\overline{CD}$, while $b$ represents a 180° rotation on the axis through the centers of the back and front squares $ABCD$.

Let $G$ be the subgroup of $\Gamma_{0,4}$ generated by $\omega_1$ and $\omega_2$, and let $N$ be the subgroup generated by $a$ and $b$. Then $\Gamma_{0,4}$ is the semi-direct product of the normal subgroup $N$ and the subgroup $G$ (Lemma 5.4.1 of [14]). The group $N$ is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, while the group $G$ is isomorphic to $PSL(2, \mathbf{Z})$ under the mapping

$$\omega_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \omega_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Thus we have an exact sequence (Lemma 5.4.3 of [14])

$$1 \to N \to \Gamma_{0,4} \to PSL(2, \mathbf{Z}) \to 1.$$

Teichmüller space of $\Sigma_{0,4}$ is the upper half plane $\mathbf{H}$. The group $\Gamma_{0,4}$ acts on $\mathbf{H}$ as linear fractions:

$$\omega_1(z) = z + 1, \quad \omega_2(z) = \frac{z}{-z+1}.$$

The subgroup $N$ acts trivially. The moduli space $\mathbf{H}/PSL(2, \mathbf{Z})$ is a real 2-orbifold (or a complex 1-orbifold) homeomorphic to the punctured sphere. It has two cone points of indices 2 and 3. (§1.2.1 of [33].)

The augmented Teichmüller space $\hat{T}_{0,4}$ is the union $\mathbf{H} \cup (\mathbf{Q} \cup \{\infty\})$, and the quotient $\hat{T}_{0,4}/PSL(2, \mathbf{Z})$ is obtained by filling the puncture of $\mathbf{H}/PSL(2, \mathbf{Z})$ with a point. Thus $\hat{T}_{0,4}/\Gamma_{0,4} (= \hat{T}_{0,4}/PSL(2, \mathbf{Z}))$ is homeomorphic to $S^2$, which is compact. Subcase (i) is complete.

**Subcase (ii)** $(g, n) = (1, 1)$.

This subcase is dealt with in Hubbard-Koch [32] as Example 2.3. The result is completely analogous to Subcase (i). We will add some explanations.

It is easy to see that the mapping class group $\Gamma_{1,1}$ is isomorphic to $SL(2, \mathbf{Z})$. Its center ($\cong \mathbf{Z}_2$) corresponds to the hyperelliptic involution of $\Sigma_{1,1}$ which fixes the puncture and three other points. We have the exact sequence

$$1 \to \mathbf{Z}_2 \to \Gamma_{1,1} \to PSL(2, \mathbf{Z}) \to 1.$$

The Teichmüller space of $\Sigma_{1,1}$ is again the upper half plane $\mathbf{H}$, and $\Gamma_{1,1}$ acts on $\mathbf{H}$ as linear fractions. (The center $\mathbf{Z}_2$ acts trivially.) The augmented Teichmüller space

$\hat{T}_{1,1}$ is the union $\mathbf{H} \cup (\mathbf{Q} \cup \{\infty\})$, and the quotient space $\hat{T}_{1,1}/\Gamma_{1,1}$ is homeomorphic to $S^2$, thus compact. Subcase (ii) is complete.

Subcases (i) and (ii) complete Starting case where $\dim_{\mathbb{C}} T_{g,n} = 1$.

**Inductive step:** The inductive hypothesis is that $\hat{T}_{g,n}/\Gamma_{g,n}$ is compact if $\dim_{\mathbb{C}} T_{g,n} < N$, $N$ being a certain integer. Assuming this hypothesis, we will prove the compactness of $\hat{T}_{g,n}/\Gamma_{g,n}$ in the case where $\dim_{\mathbb{C}} T_{g,n} = N$.

Recall the partition (17)

$$\hat{T}_{g,n} = T^{\varepsilon}_{g,n} \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}} (FR^{\varepsilon}(\sigma) \cup T(\sigma)).$$

From the above partition of $\hat{T}_{g,n}$, we get the following finite partition of the quotient $\hat{T}_{g,n}/\Gamma_{g,n}$:

$$\hat{T}_{g,n}/\Gamma_{g,n} = T^{\varepsilon}_{g,n}/\Gamma_{g,n} \bigcup_{\emptyset \neq \sigma \in \, \mathcal{C}_{g,n}/\Gamma_{g,n}} (FR^{\varepsilon}(\sigma) \cup T(\sigma))/N\Gamma(\sigma). \tag{21}$$

Here we insert a lemma, which will be used later. This lemma is a general principle and is independent of the inductive step. We refer the reader to the closely related results of [46] and [30].

**Lemma 7.** *Let* $\Gamma$ *be a subgroup of* $\Gamma_{g,n}$ *of* finite index*. If* $\hat{T}_{g,n}/\Gamma_{g,n}$ *is compact, then* $\hat{T}_{g,n}/\Gamma$ *is compact, too.*

*Proof.* From (17), we get

$$\hat{T}_{g,n}/\Gamma = T^{\varepsilon}_{g,n}/\Gamma \bigcup_{\emptyset \neq \sigma \in \, \mathcal{C}_{g,n}/\Gamma} (FR^{\varepsilon}(\sigma) \cup T(\sigma))/N\Gamma'(\sigma), \tag{22}$$

where $N\Gamma'(\sigma) = \Gamma \cap N\Gamma(\sigma)$. Comparing (22) with (21), we see that there is a finite branched covering

$$\hat{T}_{g,n}/\Gamma \to \hat{T}_{g,n}/\Gamma_{g,n}.$$

Then Lemma 7 is proved by a standard argument. □

We return to the inductive step. We will prove the compactness of $\hat{T}_{g,n}/\Gamma_{g,n}$ in the case where $\dim_{\mathbb{C}} T_{g,n} = N$, assuming the compactness for the cases where $\dim_{\mathbb{C}} < N$. We will examine each component of (21):

The quotient

$$T^{\varepsilon}_{g,n}/\Gamma_{g,n} \tag{23}$$

is compact by the Mumford compactness theorem ([60], [10]). See [31] §7.3, in particular Theorems 7.3.1 and 7.3.3.

To analyze the component $(FR^{\varepsilon}(\sigma) \cup T(\sigma))/N\Gamma(\sigma)$, we will prove the following

**Lemma 8.** *The quotient space* $\hat{T}(\sigma)/W(\sigma)$ *is compact.*

*Proof.* Recall that $T(\sigma)$ is the frontier Teichmüller space modeled on the nodal surface $\Sigma_{g,n}(\sigma)$, and that $W(\sigma)$ is the mapping class group of $\Sigma_{g,n}(\sigma)$ (Corollary 6.2). Let

$$\Sigma_{g_j,n_j}, \quad j = 1, \dots, u$$

be the totality of the parts of $\Sigma_{g,n}(\sigma)$. The mapping class group $W(\sigma)$ may permute the parts. We define the *pure mapping class group*

$$PW(\sigma)$$

to be the subgroup of $W(\sigma)$ that does not permute the parts, and that does not permute the punctures on each part $\Sigma_{g_j,n_j}$. The subgroup $PW(\sigma)$ has a finite index in $W(\sigma)$. $PW(\sigma)$ induces on each part $\Sigma_{g_j,n_j}$ a subgroup $P_jW(\sigma)$ of the mapping class group $\Gamma_{g_j,n_j}$. The subgroup $P_jW(\sigma)$ has a finite index in $\Gamma_{g_j,n_j}$. Since $T(\sigma)$ is a product of the Teichmüller spaces $T_{g_j,n_j}$ of the parts ([49]), we have

$$\hat{T}(\sigma) = \prod_{j=1}^{u} \hat{T}_{g_j,n_j}.$$

Thus

$$\hat{T}(\sigma)/PW(\sigma) = \prod_{j=1}^{u} \hat{T}_{g_j,n_j}/P_jW(\sigma). \tag{24}$$

Note that $\dim_{\mathbb{C}} T_{g_j,n_j} \leqq \dim_{\mathbb{C}} T(\sigma) < \dim_{\mathbb{C}} T_{g,n}$. Here we use the inductive hypothesis that $\hat{T}_{g_j,n_j}/\Gamma_{g_j,n_j}$ is compact for $j = 1, \dots, u$. Since the subgroup $P_jW(\sigma)$ has a finite index in $\Gamma_{g_i,n_i}$, we conclude by Lemma 7 that $\hat{T}_{g_j,n_j}/P_jW(\sigma)$ is compact for $j = 1, \dots, u$. By (24), $\hat{T}(\sigma)/PW(\sigma)$ is compact, and the existence of a continuous surjection

$$\hat{T}(\sigma)/PW(\sigma) \to \hat{T}(\sigma)/W(\sigma)$$

assures that $\hat{T}(\sigma)/W(\sigma)$ is compact. The proof of Lemma 8 is complete. □

**Definition 7** (Augmented fringe)**.** *Put*

$$\widehat{FR^{\varepsilon}}(\sigma) \stackrel{\text{def}}{=} FR^{\varepsilon}(\sigma) \cup \hat{T}(\sigma).$$

*We call* $\widehat{FR^{\varepsilon}}(\sigma)$ *the* augmented fringe*.*

Notice that $T(\sigma)$ in $FR^{\varepsilon}(\sigma) \cup T(\sigma)$ is replaced by $\hat{T}(\sigma)$ in the augmented fringe $\widehat{FR^{\varepsilon}}(\sigma)$.

Since $\hat{T}(\sigma)$ is the fixed point set of the action of $\Gamma(\sigma)$ by Theorem 4, $\hat{T}(\sigma)$ remains unaffected in the quotient $\widehat{FR^{\varepsilon}}(\sigma)/\Gamma(\sigma)$. From the partition (21), we get

$$\hat{T}_{g,n}/\Gamma_{g,n} = T^{\varepsilon}_{g,n}/\Gamma_{g,n} \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}/\Gamma_{g,n}} \widehat{FR^{\varepsilon}}(\sigma)/N\Gamma(\sigma). \tag{25}$$

Note that this decomposition is no longer a disjoint union.

**Completion of the inductive step.**

To prove the compactness of $\hat{T}_{g,n}/\Gamma_{g,n}$, we take an infinite sequence of points $\{p_i\}_{i=1}^{\infty}$ in $\hat{T}_{g,n}/\Gamma_{g,n}$, and will show that we can find a subsequence convergent in $\hat{T}_{g,n}/\Gamma_{g,n}$. Since the decomposition (25) is a finite union, $T^{\varepsilon}_{g,n}/\Gamma_{g,n}$ or at least one component $\widehat{FR^{\varepsilon}}(\sigma)/N\Gamma(\sigma)$ contains an infinite subsequence. If $T^{\varepsilon}_{g,n}/\Gamma_{g,n}$ contains an infinite subsequence, then by the Mumford compactness theorem, it contains a subsequence convergent in $T^{\varepsilon}_{g,n}/\Gamma_{g,n}$. If a component $\widehat{FR^{\varepsilon}}(\sigma)/N\Gamma(\sigma)$ contains an infinite subsequence, the subsequence may be assumed not to stay above any systole level $L > \delta$ (otherwise an infinite subsequence would stay within the set $T^{\delta}_{g,n}/\Gamma_{g,n}$, and by the Mumford compactness theorem it would have a convergent subsequence in this set). Therefore, the sequence escapes from all the subsets $L > \varepsilon/n$, $n = 1, 2, \ldots$, and it approaches the set $\hat{T}(\sigma)/W(\sigma)$. By Lemma 8, $\hat{T}(\sigma)/W(\sigma)$ is compact. Thus there is an infinite subsequence which converges to a point of $\hat{T}(\sigma)/W(\sigma)$. This completes the inductive step, and we have proved the compactness of $\hat{T}_{g,n}/\Gamma_{g,n}$.

Note that the Deligne-Mumford compactification is the moduli space for all stable curves, and stable curves are the same as nodal Riemann surfaces. Since $\hat{T}_{g,n}/\Gamma_{g,n}$ is compact and contains all the frontier points corresponding to nodal Riemann surfaces, $\hat{T}_{g,n}/\Gamma_{g,n}$ is topologically identical with the DM compactification $\overline{M}_{g,n}$ of the moduli space $M_{g,n}$. This completes the proof of Theorem 7. $\square$

## 6. **Construction of the orbifold-charts**

In this section, we will construct our "tautological"orbifold-charts of the DM-compactification $\overline{M}_{g,n}$. We identify $\overline{M}_{g,n}$ with the quotient space $\hat{T}_{g,n}/\Gamma_{g,n}$ by Theorem 7.

Let $m$ denote the complex dimension $3g-3+n$ of Teichmüller space $T_{g,n}$. Let $M$ be a Margulis constant. Take a number $\varepsilon$ with $0<\varepsilon<M$, and fix it throughout the discussion.

Let $\sigma=\langle C_1,\dots,C_k\rangle$ be a simplex of $\mathcal{C}_{g,n}$. Define open subsets $\mathcal{U}^*_\varepsilon(\sigma)$ and $\mathcal{U}_\varepsilon(\sigma)$ of $T_{g,n}\bigcup_{\rho<\sigma}T(\rho)$ as follows:

$$\mathcal{U}^*_\varepsilon(\sigma)\overset{\text{def}}{=}\{p=[S,w]\in T_{g,n}\bigcup_{\rho<\sigma}T(\rho)\mid 0\leqq l_{C_i}(p)<\varepsilon,\ i=1,\dots,k\}\tag{26}$$

and

$$\begin{aligned}\mathcal{U}_\varepsilon(\sigma)\overset{\text{def}}{=}\{p=[S,w]\in T_{g,n}\bigcup_{\rho<\sigma}T(\rho)\mid 0\leqq l_{C_i}(p)<\varepsilon,\ \ i=1,\dots,k,\\ \text{and other simple closed geodesics on } S \text{ are}\\ \text{longer than } \max\{l_{C_1}(p),\cdots,l_{C_k}(p)\}\}.\end{aligned}\tag{27}$$

Note that $T_{g,n}\bigcup_{\rho<\sigma}T(\rho)$ is a finite union, because there are only finitely many faces $\rho$ of $\sigma$. Of course, $\mathcal{U}_\varepsilon(\sigma)\subset\mathcal{U}^*_\varepsilon(\sigma)$. If $\sigma=\emptyset$, then $\mathcal{U}^*_\varepsilon(\emptyset)=\mathcal{U}_\varepsilon(\emptyset)=T_{g,n}$.

Let $\tilde{\sigma}=\langle C_1,\dots,C_k,\dots,C_m\rangle$ be a maximal simplex of $\mathcal{C}_{g,n}$ containing $\sigma$. Let $(l_1,\dots,l_m,\tau_1,\dots,\tau_m)$ be the associated Fenchel-Nielsen coordinates. Wolpert [81] proves that these coordinates, except for the first $k$ twist coordinates $\tau_1,\dots,\tau_k$, extend to $T_{g,n}\cup T(\sigma)$ continuously. More precisely, the map

$$((l_j,\tau_j),l_i): T_{g,n}\bigcup_{\rho<\sigma}T(\rho)\to\prod_{j=k+1}^{m}(\mathbf{R}_+\times\mathbf{R})\times\prod_{i=1}^{k}(\mathbf{R}_{\geq 0})\tag{28}$$

is continuous ([81], §4). With these extended Fenchel-Nielsen coordinates, the open set $\mathcal{U}^*_\varepsilon(\sigma)$ is described as follows:

$$0\leqq l_i<\varepsilon,\quad i=1,\dots,k.\tag{29}$$

In this expression (29), the points satisfying $l_i(p)=0$ for some $i\in\{1,\dots,k\}$ are frontier points $\in\bigcup_{\rho<\sigma}T(\rho)$. In particular, we have

$$T(\sigma)\subset\mathcal{U}^*_\varepsilon(\sigma).\tag{30}$$

The description for $\mathcal{U}_\varepsilon(\sigma)$ is the following:

$$0\leqq l_i<\varepsilon,\ \ i=1,\dots,k.\quad\text{and}\quad\max\{l_1,\dots,l_k\}<l_j,\ \ j=k+1,\dots,m.\tag{31}$$

We have a similar inclusion:

$$T(\sigma)\subset\mathcal{U}_\varepsilon(\sigma).\tag{32}$$

**Lemma 9.** *(i) For a mapping class $[f]$, we have*

$$[f](\mathcal{U}^*_\varepsilon(\sigma))=\mathcal{U}^*_\varepsilon(f(\sigma))\ \ \text{and}\ \ [f](\mathcal{U}_\varepsilon(\sigma))=\mathcal{U}_\varepsilon(f(\sigma)).$$

*If $[f]$ belongs to the normalizer $N\Gamma(\sigma)$, then we have*

$$[f](\mathcal{U}^*_\varepsilon(\sigma))=\mathcal{U}^*_\varepsilon(\sigma)\ \ \text{and}\ \ [f](\mathcal{U}_\varepsilon(\sigma))=\mathcal{U}_\varepsilon(\sigma).$$

*(ii) A mapping class $[f]$ satisfies $[f](\mathcal{U}_\varepsilon(\sigma))\cap\mathcal{U}_\varepsilon(\sigma)\neq\emptyset$, if and only if $[f]\in N\Gamma(\sigma)$.*

*Proof.* (i) By Lemma 1, it follows that $[f](\mathcal{U}^*_\varepsilon(\sigma)) = \mathcal{U}^*_\varepsilon(f(\sigma))$ and $[f](\mathcal{U}_\varepsilon(\sigma)) = \mathcal{U}_\varepsilon(f(\sigma))$ . If $[f] \in N\Gamma(\sigma)$, then $[f]$ permutes the isotopy classes of $\{C_1, \ldots, C_k\}$ by Theorem 6. Hence $f(\sigma) = \sigma$, and we have the result.
(ii) Suppose a mapping class $[f]$ sends a point $p = [S, w]$ of $\mathcal{U}_\varepsilon(\sigma)$ to another point $[f](p) = [S, w \circ f^{-1}]$ of $\mathcal{U}_\varepsilon(\sigma)$. By the definition of $\mathcal{U}_\varepsilon(\sigma)$, on the Riemann surface (or Riemann surface with nodes) $S$, the closed geodesics (or nodes) $c_1, \ldots, c_k$ which are homotopic to $w(C_1), \cdots, w(C_k)$ are shorter than any other closed geodesics. If $p = [S, w\circ f^{-1}]$ belongs to the same $\mathcal{U}_\varepsilon(\sigma)$, then on the same surface $S$, the geodesics (or nodes) $c'_1, \ldots, c'_k$ which are homotopic to $w(f^{-1}(C_1)), \cdots, w(f^{-1}(C_k))$ have the same property. The two sets of geodesics (or nodes) must coincide $\{c_1, \ldots, c_k\} = \{c'_1 \ldots, c'_k\}$. This implies that the sets of isotopy classes of the simple closed curves on $\Sigma_{g,n}$ coincide: $\{C_1, \cdots, C_k\} = \{f^{-1}(C_1), \cdots, f^{-1}(C_k)\}$, and that $f$ permutes the isotopy classes of $C_1, \cdots, C_k$. Thus $[f]$ belongs to $N\Gamma(\sigma)$.

The converse follows from (i). The proof of Lemma 9 is complete. $\square$

**Lemma 10.** *$\mathcal{U}_\varepsilon(\sigma) \cap \mathcal{U}_\varepsilon(\tau) \neq \emptyset$, if and only if $\sigma < \tau$ or $\sigma > \tau$.*

*Proof.* Assume $\sigma = \langle C_1, \ldots, C_k\rangle$, and $\tau = \langle C'_1, \ldots, C'_l\rangle$. Suppose that there is a point $p = [S, w]$ in the intersection $\mathcal{U}_\varepsilon(\sigma) \cap \mathcal{U}_\varepsilon(\tau)$.

Since $p \in \mathcal{U}_\varepsilon(\sigma)$, the closed geodesics (or nodes) $c_1, \ldots, c_k$ which are homotopic to $w(C_1), \ldots, w(C_k)$ are the first $k$ shortest geodesics such that other simple closed geodesics on $S$ are longer than $\max\{l_{C_1}(p), \ldots, l_{C_k}(p)\}$. Likewise, since $p \in \mathcal{U}_\varepsilon(\tau)$, the closed geodesics (or nodes) $c'_1, \ldots, c'_l$ which are homotopic to $w(C'_1), \ldots, w(C'_l)$ are the first $l$ shortest geodesics such that other simple closed geodesics on $S$ are longer than $\max\{l_{C'_1}(p)\ldots, l_{C'_l}(p)\}$.

If $k \leqq l$, the two conditions above are compatible on the same $S$, only if $\{c_1, \ldots, c_k\}$ is a subset of $\{c'_1, \ldots, c'_l\}$. In this case, $\{C_1, \ldots, C_k\}$ is a subset of $\{C'_1, \ldots, C'_l\}$, or equivalently, $\sigma < \tau$.

If $l \leqq k$, we can prove $\tau < \sigma$ similarly.

Conversely, suppose $\sigma < \tau$. We may assume

$$\sigma = \langle C_1, \ldots, C_k\rangle, \text{ and } \tau = \langle C_1, \ldots, C_k, C_{k+1}, \ldots, C_l\rangle.$$

Then those points $p = [S, w] \in T_{g,n}$ that satisfy the condition

$$l_{C_1}(p) = \cdots = l_{C_k}(p) = \frac{1}{3}\varepsilon, \quad l_{C_{k+1}}(p) = \cdots = l_{C_l}(p) = \frac{2}{3}\varepsilon.$$

and other simple closed geodesics on $S$ are longer than $\frac{2}{3}\varepsilon$,

belong to the intersection $\mathcal{U}_\varepsilon(\sigma) \cap \mathcal{U}_\varepsilon(\tau)$. Thus $\mathcal{U}_\varepsilon(\sigma) \cap \mathcal{U}_\varepsilon(\tau) \neq \emptyset$.

If $\sigma > \tau$, we have the same conclusion. The proof of Lemma 10 is complete. $\square$

By Lemma 9, the normalizer $N\Gamma(\sigma)$ acts on $\mathcal{U}^*_\varepsilon(\sigma)$ and on $\mathcal{U}_\varepsilon(\sigma)$. In particular, $\Gamma(\sigma)$ acts on them.

**Definition 8** (Deformation space)**.** *The quotient space*

$$D^*_\varepsilon(\sigma) \stackrel{\text{def}}{=} \mathcal{U}^*_\varepsilon(\sigma)/\Gamma(\sigma)$$

*is called the* deformation space *associated with the simplex $\sigma$.*

**Definition 9** (Controlled deformation space)**.** *The quotient space*

$$D_\varepsilon(\sigma) \stackrel{\text{def}}{=} \mathcal{U}_\varepsilon(\sigma)/\Gamma(\sigma)$$

*is called the* controlled deformation space *associated with the simplex $\sigma$.*

These spaces are considered as refinements of Bers' deformation spaces [12], [13].

In our previous papers [51], [52], we constructed controlled deformation spaces in a more complicated manner, depending on $2m$ numbers $0 < \varepsilon_1 < \eta_1 < \varepsilon_2 < \eta_2 < \cdots < \varepsilon_m < \eta_m < M$. This was because certain similarity between controlled deformation spaces and handle-body decompositions of manifolds was taken into account. In the present discussion, we have adopted simpler construction without noticing the similarity.

**Lemma 11.** *The deformation space $D^*_\varepsilon(\sigma)$ is an open $2m$-cell homeomorphic to*

$$(\Delta_\varepsilon)^k \times T(\sigma),$$

*where $\Delta_\varepsilon$ is the open 2-disk of radius $\varepsilon$.*

*Proof.* In terms of the Fenchel-Nielsen coordinates $(l_1, \dots, l_m, \tau_1, \dots, \tau_m)$ associated with a maximal simplex $\tilde{\sigma} = \langle C_1, \dots, C_k, C_{k+1}, \dots, C_m\rangle$ which contains $\sigma = \langle C_1, \dots, C_k\rangle$, the open set $\mathcal{U}^*_\varepsilon(\sigma) \cap T_{g,n}$ is described as

$$S_1 \times \cdots \times S_k \times \prod_{j=k+1}^{m} (\mathbf{R}_+ \times \mathbf{R}),$$

where $S_i$ denotes the strip region $S_i = \{(l_i, \tau_i) \mid 0 < l_i < \varepsilon\}$. The Dehn twist $\tau(C_i)$ (with $i \in \{1, \dots, k\}$) leaves $\prod_{j=k+1}^{m}(\mathbf{R}_+ \times \mathbf{R})$ invariant and acts on $S_i$ by

$$\tau(C_i)(l_i, \tau_i) = (l_i, \tau_i + l_i). \tag{33}$$

The quotient space $S_i/\langle\tau(C_i)\rangle$ is identified with the punctured $\varepsilon$-disk $\Delta^0_\varepsilon := \Delta_\varepsilon \setminus \{0\}$ by the correspondence: $(l_i, \tau_i) \mapsto l_i e^{2\pi\sqrt{-1}\tau_i/l_i}$. Thus the quotient $(\mathcal{U}^*_\varepsilon(\sigma) \cap T_{g,n})/\Gamma(\sigma)$ is identified with

$$(\Delta^0_\varepsilon)^k \times \prod_{j=k+1}^{m} (\mathbf{R}_+ \times \mathbf{R}).$$

Filling the punctures of $\Delta^0_\varepsilon$'s by the continuous extension (28), we get

$$\mathcal{U}^*_\varepsilon(\sigma)/\Gamma(\sigma) = (\Delta_\varepsilon)^k \times \prod_{j=k+1}^{m} (\mathbf{R}_+ \times \mathbf{R}). \tag{34}$$

The continuous extension (28) assures that the factor $\prod_{j=k+1}^{m}(\mathbf{R}_+ \times \mathbf{R})$ gives coordinates of $\{\mathbf{0}\} \times T(\sigma)$.

Since $T(\sigma)$ is homeomorphic to an open $(2m-2k)$-cell, $D^*_\varepsilon(\sigma)$ is homeomorphic to an open $2m$-cell. The proof of Lemma 11 is complete. □

**Remark.** Comparing Lemmas 11 and 5 we see that the factor $\mathrm{Cone}^\varepsilon(T^k)$ of the quotient of $(FR^\varepsilon(\sigma) \cup T(\sigma))/\Gamma(\sigma)$ is embedded in the factor $(\Delta_\varepsilon)^k$ of the deformation space $D^*_\varepsilon(\sigma)$ as the cone over the "corner torus" $(\partial\Delta_\varepsilon)^k$. □

The quotient group $W(\sigma) = N\Gamma(\sigma)/\Gamma(\sigma)$ acts on $D^*_\varepsilon(\sigma)$ and on $D_\varepsilon(\sigma)$.

**Lemma 12.** *The Weyl group $W(\sigma)$ acts on $D^*_\varepsilon(\sigma)$ properly discontinuously.*

**Proof.** A natural argument would be to appeal to Yamada's result [83] that the Weil-Petersson completion $\hat{T}_{g,n}$ is a CAT(0) space and $\hat{T}(\sigma)$ is a complete convex subset in $\hat{T}_{g,n}$. (For the definition of CAT(0) space, see [17] Chapter II.1, [81] §4, [82] Chapter 5, and [85] §5.1.) This implies the following ([83] p.342, [17] Proposition 2.4):

(i) For every $p$, there exists a unique point $\pi_\sigma(p) \in \hat{T}(\sigma)$ such that $d(p, \pi_\sigma(p)) = d(p, \hat{T}(\sigma))$, where $d(\ ,\ )$ is the distance.
(ii) If $p'$ belongs to the geodesic segment connecting $p$ and $\pi_\sigma(p)$, then $\pi_\sigma(p') = \pi_\sigma(p)$.
(iii) Given $p \notin \hat{T}(\sigma)$ and $q \in \hat{T}(\sigma)$, if $q \neq \pi_\sigma(p)$ then the Alexandrov angle (as defined in [17] p.9, or see [85] p.95) $\angle_{\pi_\sigma(p)}(p, q) \geqq \pi/2$.

These properties provide us with a "brush"of geodesics with the core $\hat{T}(\sigma)$. This brush structure is invariant under the action of $N\Gamma(\sigma)$. Confining our attention to geodesic segments of short lengths, say $\delta > 0$, attached to $T(\sigma)$, and passing to the quotient space by $\Gamma(\sigma)$, we obtain a region $\mathcal{B}_\delta$ in $D^*_\varepsilon(\sigma)$ which has the structure of a brush with short geodesic fibers attached to $T(\sigma)$. The region $\mathcal{B}_\delta$ is equipped with the projection $\pi_\sigma : \mathcal{B}_\delta \to T(\sigma)$ along the geodesic fibers, which is equivariant under the action of $W(\sigma)$. Being the mapping class group of the nodal surface $\Sigma_{g,n}(\sigma)$ (Corollary 6.2), the group $W(\sigma)$ acts on $T(\sigma)$ properly discontinuously. The existence of the equivariant projection $\pi_\sigma : B_\delta \to T(\sigma)$ assures that its action on $\mathcal{B}_\delta$ is also properly discontinuous. (In the above argument, we changed the core of the brush from $\hat{T}(\sigma)$ to $T(\sigma)$. The geometric justification of this passage is provided by the fact that in $\hat{T}_{g,n}$ the stratum $T(\sigma)$ "meets"another $T(\sigma')$ along $\hat{T}(\sigma \cup \sigma')$ $(\subset \hat{T}(\sigma) \cap \hat{T}(\sigma'))$ at right angles if $\sigma \not< \sigma'$ nor $\sigma' \not< \sigma$. See [82], Chapter 6, §4. The shortest geodesics attached to the points of $T(\sigma)$ meet $T(\sigma)$ at right angles. See property (iii) above. Thus the domain of the geodesic projection $\pi_\sigma : \hat{T}_{g,n}/\Gamma(\sigma) \to \hat{T}(\sigma)$ is "stratified"according to the stratification of the target $\hat{T}(\sigma)$, and we can regard $D^*_\varepsilon(\sigma)$ as contained in the domain of the projection whose target is $T(\sigma)$. The author learned this argument from Sumio Yamada.)

Under the free action of $\Gamma(\sigma)$, the complement $(D^*_\varepsilon(\sigma) \setminus \mathcal{B}_{\delta/2})$ is a quotient of a sub-region of $\mathcal{U}^*_\varepsilon(\sigma)$ on which $N\Gamma(\sigma)$ acts properly discontinuously. Thus the group $W(\sigma) = N\Gamma(\sigma)/\Gamma(\sigma)$ acts on $(D^*_\varepsilon(\sigma) \setminus \mathcal{B}_{\delta/2})$ properly discontinuously. Since $D^*_\varepsilon(\sigma) = (D^*_\varepsilon(\sigma) \setminus \mathcal{B}_{\delta/2}) \cup \mathcal{B}_\delta$, we get the assertion of Lemma 12 by combining the two results above. □

**Theorem 8** (Hubbard and Koch [32]). *The deformation space $D^*_\varepsilon(\sigma)$ is a complex $m$-manifold homeomorphic to an open $2m$-cell.*

This theorem is essentially Theorem 10.1 of Hubbard and Koch [32], because their space $\mathcal{Q}_\Gamma$ (see §7 of [32]) is almost the same as our deformation space $D^*_\varepsilon(\sigma)$. The only difference is that their $\mathcal{Q}_\Gamma$ has no restriction on the "size"in the directions which are normal to the core $T(\sigma)$, while our $D^*_\varepsilon(\sigma)$ has the size restriction $\varepsilon$. Hubbard and Koch proved that $\mathcal{Q}_\Gamma$ is a complex manifold. Their main concern is to define a complex structure in a neighborhood of a point $p = (\mathbf{0}, \mathbf{u})$ of $\{\mathbf{0}\} \times T(\sigma)$ (in our notation). Their complex coordinates near the point $p$ are (like [49] §2, [76] §4, [80] §2, [24] §13) the product of the *plumbing coordinates* (cf. Earle and Marden [48], [24]) and the coordinates in an open set $U \subset T(\sigma)$.

More specifically, they started from a $\sigma$-marked family[2] of nodal surfaces $Y_\sigma(\mathbf{0}, \mathbf{u})$ over $\{\mathbf{0}\} \times U$, and constructed a *plumbed family* $Y_\sigma = Y_\sigma(t_1, \ldots, t_k, \mathbf{u})$ over $(\Delta_\delta)^k \times U$, where $(t_1, \ldots, t_k)$ are plumbing coordinates, each $t_i$ belonging to an open $\delta$-disk $\Delta_\delta$ (with a small $\delta > 0$). Let $(l_1, \ldots, l_m, \tau_1, \ldots, \tau_m)$ be the Fenchel-Nielsen coordinates associated with a maximal simplex $\tilde{\sigma}$ containing $\sigma \in \mathcal{C}_{g,n}$. The fiber of

[2] A *$\sigma$-marking* means a marking modulo $\Gamma(\sigma)$. See the proof of Lemma 13 below.

$Y_\sigma \to (\Delta_\delta)^k \times U$ over a point $(t_1, \dots, t_k, \mathbf{u})$ has its position in $D^*_\varepsilon(\sigma)$ described by the "extended" Fenchel-Nielsen coordinates (see Proof of Lemma 11)

$$(l_1 e^{2\pi\sqrt{-1}\tau_1/l_1}, \dots, l_k e^{2\pi\sqrt{-1}\tau_k/l_k}, l_{k+1}, \dots, l_m, \tau_{k+1}, \dots, \tau_m).$$

The map $\Phi_{\delta,U} : (\Delta_\delta)^k \times U \to D^*_\varepsilon(\sigma)$ which sends $(t_1, \dots, t_k, \mathbf{u})$ to

$$(l_1 e^{2\pi\sqrt{-1}\tau_1/l_1}, \dots, l_k e^{2\pi\sqrt{-1}\tau_k/l_k}, l_{k+1}, \dots, l_m, \tau_{k+1}, \dots, \tau_m)$$

is continuous ([32], Proposition 9.1). Hubbard-Koch ([32] §9) proved that if we take $\delta$ and $U$ sufficiently small, $\Phi_{\delta,U}$ is a topological embedding, and stratumwise it is analytic. Using $\Phi_{\delta,U}$ as a local chart, they put the complex coordinates $(t_1, \dots, t_k, \mathbf{u})$ to the open neighborhood $\Phi_{\delta,U}((\Delta_\delta)^k \times U)$ of $p$. Note that the restriction $\Phi_{\delta,U}|\{\mathbf{0}\} \times U$ is the identity $id_{\{\mathbf{0}\}\times U}$. Therefore their complex structure about $p = (\mathbf{0}, \mathbf{u})$ is a product $(\Delta_\delta)^k \times U$. The point $p$ was arbitrarily taken from $\{\mathbf{0}\} \times T(\sigma)$, thus as a consequence, $D^*_\varepsilon(\sigma)$ is a complex $m$-manifold, and $T(\sigma)$ is a complex submanifold of $D^*_\varepsilon(\sigma)$.

On the other hand, we already proved that $D^*_\varepsilon(\sigma)$ is homeomorphic to an open $2m$-cell (Lemma 11).

**Remark.** (Bers' Conjecture) Bers ([12] p.47, [13]) made an announcement to the effect that, in our notation, $D^*_\varepsilon(\sigma)$ is a bounded domain in $\mathbb{C}^m$. But he did not give any proof. We would like to call this statement *Bers' Conjecture*.

When introducing the complex structure to $D^*_\varepsilon(\sigma)$ as above, if we could take $U = T(\sigma)$ and thus $D^*_\varepsilon(\sigma)$ would contain an open submanifold biholomorphic to $(\Delta_\delta)^k \times T(\sigma)$, then by taking an $\varepsilon' > 0$ smaller than $\varepsilon$, we would be able to show that $D^*_{\varepsilon'}(\sigma)$ is a bounded domain. Hubbard and Koch, however, gave a warning that we could not take $U = T(\sigma)$, ([32], Remark 8.1). This implies that their construction does not give a bounded domain $D^*_\varepsilon(\sigma)$.

Earle and Marden ([24] §13, Theorem II) states that $D^*_\varepsilon(\sigma)$ can be topologically embedded in $(\Delta_\delta)^k \times T(\sigma)$ by using the plumbing coordinates. If this is the case, $D^*_\varepsilon(\sigma)$ would again be a bounded domain. But unfortunately, any proof of their Theorem II has not yet been published (except for the case of a maximal simplex $\sigma \in \mathcal{C}_{g,n}$, for which Kra [44] has proved the corresponding result: if $\sigma$ is maximal, $D^*_\varepsilon(\sigma)$ is a bounded domain). There is a bad news: Hinich [29], p.152, claimed that his result contradicts a partial consequence of Earle-Marden's theorem stated as a corollary on p.346 of [48].

Wolpert [80] proved the following estimate: at a point $p = (\mathbf{0}, \mathbf{u}) \in \{\mathbf{0}\} \times T(\sigma)$ and at a node $p_i$ on the singular fiber $X_p$ of $Y_\sigma(\mathbf{0}, \mathbf{u})$, let $t_i$ be the plumbing coordinate with which we open up the node $p_i$, and let $l_i$ be the length of a simple closed geodesic which appears on the opened up smooth fiber $X_{t_i}$, then Wolpert ([80]) gave the estimate

$$l_i = 2\pi^2(-\log|t_i|)^{-1} + O(\log|t_i|)^{-2}.$$

If the second term $O(\log|t_i|)^{-2}$ is a uniform estimate w.r.t. the position of $p \in \{\mathbf{0}\} \times T(\sigma)$, then we would find some $\delta$ such that

$$\Phi_{\delta,T(\sigma)}((\Delta_\delta)^k \times T(\sigma)) \subset D^*_\varepsilon(\sigma).$$

Furthermore, in this case, if we take smaller $\varepsilon'$ such that $D^*_{\varepsilon'}(\sigma) \subset \Phi_{\delta,T(\sigma)}((\Delta_\delta)^k \times T(\sigma))$, $D^*_{\varepsilon'}(\sigma)$ would be a bounded domain, and Bers' Conjecture would follow. But in a discussion with the author, Wolpert ascertained that the second term $O(\log|t_i|)^{-2}$ is not a uniform estimate w.r.t. $p$. Thus the above argument fails.

In conclusion, it seems that at present there is no complete proof of Bers' Conjecture. □

**Lemma 13.** *The action of $W(\sigma)$ on $D^*_\varepsilon(\sigma)$ is holomorphic.*

*Proof.* This follows from the universality of the marked family of stable curves constructed by Hubbard and Koch [32]: They constructed a $\Gamma$-marked proper flat family of stable curves with sections $Y_\Gamma \to \mathcal{Q}_\Gamma$ ([32], §10). Their $\Gamma$ is nothing but our $\sigma = \langle C_1, \ldots, C_k \rangle$ (see Definition 1.4 of [32]), and they defined $\mathcal{Q}_\Gamma$ by

$$\mathcal{Q}_\Gamma := U_\Gamma/\Delta_\Gamma, \text{ where } U_\Gamma := \bigcup_{\Gamma' \subset \Gamma} S_{\Gamma'} \subset \hat{T}_{g,n}.$$

(See §7 of [32].) Their $\Delta_\Gamma$ is our $\Gamma(\sigma)$ ([32] §0.1), and $S_\Gamma$ is nothing but our $T(\sigma)$ ([32] §2.1).

Since discussions with two different systems of notation are inconvenient, let us follow their arguments using our own notation.

Given a simplex $\sigma = \langle C_1, \ldots, C_k \rangle \in \mathcal{C}_{g,n}$, we will define

$$\mathcal{U}^*(\sigma) \stackrel{\text{def}}{=} T_{g,n} \bigcup_{\rho < \sigma} T(\rho) \subset \hat{T}_{g,n}. \tag{35}$$

This is exactly Hubbard and Koch's $U_\Gamma$, and comparing this definition with (26), we see that $\mathcal{U}^*(\sigma)$ is an analog of $\mathcal{U}^*_\varepsilon(\sigma)$ without size restriction $\varepsilon$. Let us define

$$D^*(\sigma) \stackrel{\text{def}}{=} \mathcal{U}^*(\sigma)/\Gamma(\sigma). \tag{36}$$

Compared with Definition 8, $D^*(\sigma)$ is the deformation space *without* size restriction $\varepsilon$. This $D^*(\sigma)$ is precisely the same as Hubbard and Koch's $\mathcal{Q}_\Gamma$. Thus $D^*(\sigma)$ is a complex m-manifold (see Theorem 10.1 of [32]). One more thing to add is the strata of $\mathcal{Q}_\Gamma$. (See §7.1 of [32].) Their stratum $\mathcal{Q}_\Gamma^{\Gamma'}$ $(\Gamma' \subset \Gamma)$ of $\mathcal{Q}_\Gamma$ is the quotient of $S_{\Gamma'}$ by $\Delta_\Gamma$. We will denote this stratum by $T_\sigma(\rho)$, where $\rho < \sigma$. Thus the stratum $T_\sigma(\rho)$ of $D^*(\sigma)$ is the quotient of $T(\rho)$ by $\Gamma(\sigma)$.

By the above construction, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{U}^*(\rho) & \xrightarrow{\subset} & \mathcal{U}^*(\sigma) \\ {\scriptstyle /\Gamma(\rho)}\downarrow & & \downarrow{\scriptstyle /\Gamma(\sigma)} \\ D^*(\rho) & \xrightarrow[\psi^\rho_\sigma]{} & D^*(\sigma), \end{array} \quad \text{where} \quad \rho < \sigma. \tag{37}$$

The map $\psi^\rho_\sigma$ is Hubbard and Koch's $\Psi_\Gamma^{\Gamma'}$ in Step 1 of their proof of Theorem 10.1, which corresponds to taking quotient by $\Gamma(\sigma - \rho)$. They have proved that $\psi^\rho_\sigma : D^*(\rho) \to D^*(\sigma)$ is a holomorphic covering map of its image. Since $\Gamma(\rho)$ acts on $T(\rho)$ trivially, $T(\rho)$ remains in $D^*(\rho)$ untouched, and the image of $T(\rho)$ under the quotient map $\psi^\rho_\sigma$ is the stratum $T_\sigma(\rho)$.

The Weyl group $W(\sigma)$ acts on $D^*(\sigma)$ properly discontinuously. The proof is the same as Lemma 12. Before completing the proof of Lemma 13, we will prove the following Claim which has no size restriction $\varepsilon$:

**Claim.** *The action of $W(\sigma)$ on $D^*(\sigma)$ is holomorphic.*

*Proof of Claim.* Hubbard and Koch's $\Gamma$-marked family $p_\Gamma : X_\Gamma \to \mathcal{Q}_\Gamma$ (Theorem 10.1 of [32]) is in our terminology a $\sigma$-marked family, and we would like to denote it by $p_\sigma : X(\sigma) \to D^*(\sigma)$. Recall that they introduced the notion of a $\Gamma$-*marking* as a marking modulo $\Delta_\Gamma$. See [32], §5, Definitions 5.1 and 5.2. Thus in our terminology, a *$\sigma$-marking* is a marking modulo $\Gamma(\sigma)$.

Take a point $p$ in $T(\sigma)$. Then there is an open neighborhood $U$ in $T(\sigma)$ and a small disk $(\Delta_\delta)^k$ (where $k = |\sigma|$) representing the plumbing coordinates, such that the product $U \times (\Delta_\delta)^k$ gives the complex structure[3] of $D^*(\sigma)$ near $p$. (See [32], §8 and Theorem 9.11.) Hubbard and Koch's family $p_\sigma : X(\sigma) \to D^*(\sigma)$ restricted to $U \times (\Delta_\delta)^k$ is a plumbed family ([32], §8.3). The fiber over a point $(u, \mathbf{t}) \in U \times (\Delta_\delta)^k$ is a Riemann surface $(S, w)$ constructed from the "central nodal fiber"over $(u, \mathbf{0})$ by plumbing. (See [32], §8.3.) This family has a natural $\sigma$-marking (i.e., a marking modulo $\Gamma(\sigma)$), because the central fiber over $(u, 0)$ has a natural $\sigma$-marking as a nodal surface parametrized by $T(\sigma)$. The fiber over a nearby point $(u, \mathbf{t})$ is obtained by opening up some nodes, and the ambiguity of Dehn twists arising at this opening process causes no difficulty if one considers a $\sigma$-marking.

Now take an element $[f] \in W(\sigma)$. We change the natural $\sigma$-marking of $X(\sigma)|(U \times (\Delta_\delta)^k) \to U \times (\Delta_\delta)^k$ from $(S, w)$ to $(S, w \circ f^{-1})$. Since $W(\sigma)$ is the group $N\Gamma(\sigma)$ modulo $\Gamma(\sigma)$, this change of the $\sigma$-marking makes sense, and we get a new $\sigma$-marked family which we denote by $X'(\sigma) \to U \times (\Delta_\delta)^k$. By the universality of Hubbard and Koch's family $p_\sigma : X(\sigma) \to D^*(\sigma)$, the family with the new $\sigma$-marking $X'(\sigma) \to U \times (\Delta_\delta)^k$ is pulled back from $p_\sigma : X(\sigma) \to D^*(\sigma)$ by a unique analytic map $g : U \times (\Delta_\delta)^k \to D^*(\sigma)$. See [32], §10 and §12. On the other hand, the action of $[f] \in W(\sigma)$ on $D^*(\sigma)$ maps $p = [S, w]$ to $[f](p) = [S, w \circ f^{-1}]$. Thus the analytic map $g : U \times (\Delta_\delta)^k \to D^*(\sigma)$ coincides with the action of $[f]$ on $U \times (\Delta_\delta)^k$. This proves that the action of $[f]$ is holomorphic locally on $U \times (\Delta_\delta)^k$.

So far we have considered a neighborhood of a point $p$ in $T(\sigma)$.

If we take a point $p \notin T(\sigma)$, there are two cases: $p$ is in a stratum $T_\sigma(\rho)$ for some face $\rho < \sigma$, or $p$ is not in such a stratum.

In the first case, choose a point $\tilde{p}$ on $T(\rho)$ such that $\psi^\rho_\sigma(\tilde{p}) = p$. Then as before, we find a neighborhood $V \times (\Delta_\delta)^l$ (where $V \subset T(\rho)$ and $l = |\rho|$) of $\tilde{p}$ in $D^*(\rho)$ which gives the complex structure of $D^*(\rho)$ near $\tilde{p}$, and such that the restriction of Hubbard and Koch's family $p_\rho : X(\rho) \to D^*(\rho)$ to $V \times (\Delta_\delta)^l$ is a plumbed family with a natural $\rho$-marking. Since $\rho < \sigma$, the natural $\rho$-marking of the restricted family $X(\rho)|(V \times (\Delta_\delta)^l) \to V \times (\Delta_\delta)^l$ gives a natural $\sigma$-marking of the restriction of Hubbard and Koch's family over $\psi^\rho_\sigma(V \times (\Delta_\delta)^l)$, $X(\sigma)|(\psi^\rho_\sigma(V \times (\Delta_\delta)^l)) \to \psi^\rho_\sigma(V \times (\Delta_\delta)^l)$. This natural $\sigma$-marking is independent of the choice of the lift $\tilde{p}$ of $p$. In fact, if $\tilde{p}'$ is another lift of $p$, the natural $\rho$-marking of the restriction of $p_\rho : X(\rho) \to D^*(\rho)$ to a neighborhood $V' \times (\Delta_\delta)^l$ of $\tilde{p}'$ differs from the natural $\rho$-marking of the restriction to the neighborhood $V \times (\Delta_\delta)^l$ of $\tilde{p}$ by the action of $\Gamma(\sigma - \rho)$ (see (37)). Thus both $\rho$-markings give the same $\sigma$-marking of $X(\sigma)|(\psi^\rho_\sigma(V \times (\Delta_\delta)^l) \to \psi^\rho_\sigma(V \times (\Delta_\delta)^l)$. By changing this natural $\sigma$-marking from $(S, w)$ to $(S, w \circ f^{-1})$, and using the universality of Hubbard and Koch's family $X(\sigma) \to D^*(\sigma)$, we can repeat the same argument as before. This proves that the action of $[f] \in W(\sigma)$ is holomorphic on the open set $\psi^\rho_\sigma(V \times (\Delta_\delta)^l)$.

In the second case where $p$ is not in any stratum $T_\sigma(\rho)$, we can take a lift $\tilde{p}$ in Teichmüller space $T_{g,n}$. Then the natural marked family of Riemann surfaces over a neighborhood of $\tilde{p}$ (see [11], [44] §4.6, [51] §1) gives a family of Riemann surfaces

[3] Hubbard and Koch's order of the factors of the product $U \times (\Delta_\delta)^k$ is different from ours $(\Delta_\delta)^k \times U$ which we used previously. But in this proof we follow their order.

with a natural $\sigma$-marking over a neighborhood of $p$ in $D^*(\sigma)$, and we can repeat the same argument.

Thus the action of $[f] \in W(\sigma)$ on $D^*(\sigma)$ is globally holomorphic. This completes the proof of Claim. $\square$

We return to the proof of Lemma 13. But Lemma 13 follows immediately from Claim above, because $D^*_\varepsilon(\sigma)$ is an open subset of $D^*(\sigma)$ which is preserved by the action of $W(\sigma)$. This completes the proof of Lemma 13. $\square$

The controlled deformation space $D_\varepsilon(\sigma)$ is an open subset of $D^*_\varepsilon(\sigma)$ which is preserved by the action of $W(\sigma)$. A closer look proves the following.

**Lemma 14.** *The controlled deformation space $D_\varepsilon(\sigma)$ is a complex $m$-manifold, and homeomorphic to an open $2m$-cell.*

*Proof.* Since $D_\varepsilon(\sigma)$ is an open subset of $D^*_\varepsilon(\sigma)$, the first assertion follows from Theorem 8. To prove the second assertion, let $\Sigma_{g_1,n_1}, \dots, \Sigma_{g_u,n_u}$ be the parts of $\Sigma_{g,n}(\sigma)$. Their Teichmüller spaces have their own systole functions $L_j : T_{g_j,n_j} \to \mathbf{R}$, $j = 1, \dots, u$. Then the systole function $L_\sigma$ of $T(\sigma)$ is given by

$$L_\sigma(p) = \min\{L_1(p_1), \cdots, L_u(p_u)\}.$$

Here we consider $T(\sigma)$ as a product $T_{g_1,n_1} \times \cdots \times T_{g_u,n_u}$ and write $p \in T(\sigma)$ as $p = (p_1, \dots, p_u)$, $p_j \in T_{g_j,n_j}$.

Put $l_\sigma = \min\{L_\sigma, \varepsilon\}$. Fixing the topological product structure $D^*_\varepsilon(\sigma) = (\Delta_\varepsilon)^k \times T(\sigma)$ of Lemma 11, and using the condition (27) for $\mathcal{U}_\varepsilon(\sigma)$, we see that the controlled deformation space $D_\varepsilon(\sigma)$ is given as follows:

$$D_\varepsilon(\sigma) = \bigcup_{p \in T(\sigma)} (\Delta_{l_\sigma(p)})^k \times \{p\} \subset (\Delta_\varepsilon)^k \times T(\sigma).$$

Thus $D_\varepsilon(\sigma)$ is the total space of an open $2k$-disk bundle over $T(\sigma)$. Since $T(\sigma)$ is homeomorphic to an open $(2m - 2k)$-cell, we get the result. $\square$

By Lemmas 12 and 13, the action of $W(\sigma)$ on $D_\varepsilon(\sigma)$ is properly discontinuous and holomorphic.

**Lemma 15.** *The quotient space $D_\varepsilon(\sigma)/W(\sigma)$ is an open subset of $\hat{T}_{g,n}/\Gamma_{g,n}$.*

*Proof.* Let $\varphi_\sigma : D_\varepsilon(\sigma) \to D_\varepsilon(\sigma)/W(\sigma)$ be the projection map. Suppose that two points $p$ and $q$ of $D_\varepsilon(\sigma)$ represent the same point of $\hat{T}_{g,n}/\Gamma_{g,n}$. Then there are lifts $\tilde{p}$, $\tilde{q} \in \mathcal{U}_\varepsilon(\sigma)$ such that $\tilde{p}$ is mapped to $\tilde{q}$ by a certain mapping class $[f] \in \Gamma_{g,n}$. By Lemma 9 (ii), $[f]$ belongs to $N\Gamma(\sigma)$. Thus $p$ is mapped to $q$ by an element of $W(\sigma)$ which is represented by $[f]$. This proves $\varphi_\sigma(p) = \varphi_\sigma(q)$. Thus $D_\varepsilon(\sigma)/W(\sigma)$ is identified with an open subset of $\hat{T}_{g,n}/\Gamma_{g,n}$. This completes the proof of Lemma 15. $\square$

**Remark.** In contrast to Lemma 15, the quotient space $D^*_\varepsilon(\sigma)/W(\sigma)$ is not necessarily an open subset of $\hat{T}_{g,n}/\Gamma_{g,n}$: Consider the following example: Suppose we have two simplexes $\sigma = \langle C_1, C_2\rangle$ and $\tau = \langle C_3, C_4\rangle$ of $\mathcal{C}_{g,n}$ which are disjoint in $\Sigma_{g,n}$. Suppose an involutive mapping class $[f]$ interchanges $C_1$ and $C_3$, and interchanges $C_2$ and $C_4$. By Theorem 6, the mapping class $[f]$ does not belong to the normalizer $N\Gamma(\sigma)$ nor to $N\Gamma(\tau)$, but it belongs to $N\Gamma(\langle\sigma,\tau\rangle)$. The mapping class $[f]$ represents an element of $W(\langle\sigma,\tau\rangle)$ acting on $D^*_\varepsilon(\langle\sigma,\tau\rangle)$, and it has fixed points on the "diagonal set" $l_1 = l_3, l_2 = l_4$. Since $D^*_\varepsilon(\langle\sigma,\tau\rangle) \setminus T(\langle\sigma,\tau\rangle) = D^*_\varepsilon(\sigma) \cap D^*_\varepsilon(\tau)$, $D^*_\varepsilon(\sigma)$

contains these fixed points, and nearby points are folded by $[f]$ in $\hat{T}_{g,n}/\Gamma_{g,n}$. But $W(\sigma)$ does not contain the mapping class $[f]$, and the quotient space $D^*_\varepsilon(\sigma)/W(\sigma)$ does not have such a folded region. Thus $D^*_\varepsilon(\sigma)/W(\sigma)$ is not an open subset of $\hat{T}_{g,n}/\Gamma_{g,n}$. This implies that $(D^*_\varepsilon(\sigma), W(\sigma))$ is not adequate to be an orbifold chart. On the other hand, the controlled deformation spaces in this example satisfy $D_\varepsilon(\sigma) \cap D_\varepsilon(\tau) = \emptyset$, so they do not cause such difficulty. □

Let $M_\varepsilon(\sigma)$ denote the quotient space $D_\varepsilon(\sigma)/W(\sigma)$, which is considered as an open subset of $\overline{M}_{g,n} = \hat{T}_{g,n}/\Gamma_{g,n}$ by Lemma 15.

**Lemma 16.** *We have*

$$\overline{M}_{g,n} = \bigcup_{\sigma \in \mathcal{C}_{g,n}/\Gamma_{g,n}} M_\varepsilon(\sigma),$$

*where the simplex $\sigma$ can be empty set $\emptyset$.*

*Proof.* Because of the inclusion $T(\sigma) \subset \mathcal{U}_\varepsilon(\sigma)$, we have

$$\partial \hat{T}_{g,n} = \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}} T(\sigma) \subset \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}} \mathcal{U}_\varepsilon(\sigma).$$

Passing to the quotient under the action of $\Gamma_{g,n}$, we have

$$\partial \hat{T}_{g,n}/\Gamma_{g,n} \subset \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}} D_\varepsilon(\sigma)/W(\sigma).$$

Since $[f](D_\varepsilon(\sigma)/W(\sigma)) = D_\varepsilon(f(\sigma))/W(f(\sigma))$ for $[f] \in \Gamma_{g,n}$, the right-hand side of the above inclusion can be replaced by the finite union:

$$\partial \hat{T}_{g,n}/\Gamma_{g,n} \subset \bigcup_{\emptyset \neq \sigma \in \mathcal{C}_{g,n}/\Gamma_{g,n}} D_\varepsilon(\sigma)/W(\sigma).$$

Taking the union with $T_{g,n}/\Gamma_{g,n} (= D_\varepsilon(\emptyset)/W(\emptyset))$ on both sides, we get

$$\hat{T}_{g,n}/\Gamma_{g,n} = \bigcup_{\sigma \in \mathcal{C}_{g,n}/\Gamma_{g,n}} D_\varepsilon(\sigma)/W(\sigma).$$

Since $\hat{T}_{g,n}/\Gamma_{g,n} = \overline{M}_{g,n}$ and $D_\varepsilon(\sigma)/W(\sigma) = M_\varepsilon(\sigma)$, the proof of Lemma 16 is complete. □

Here are some intersectional properties of the family $\{M_\varepsilon(\sigma)\}_{\sigma \in \mathcal{C}_{g,n}/\Gamma_{g,n}}$. To state the results, we have to use the precise notation

$$\{M_\varepsilon([\sigma])\}_{[\sigma] \in \mathcal{C}_{g,n}/\Gamma_{g,n}},$$

where $[\sigma]$ stands for the image of a simplex $\sigma$ of $\mathcal{C}_{g,n}$ under the projection $\mathcal{C}_{g,n} \to \mathcal{C}_{g,n}/\Gamma_{g,n}$. $[\sigma]$ is considered as a simplex of the finite simplicial complex $\mathcal{C}_{g,n}/\Gamma_{g,n}$. Thus $[\sigma] = [\tau]$ if and only if there is a mapping class $[f]$ such that $f(\sigma) = \tau$. The relation $[\sigma] < [\tau]$ means that the simplex $[\sigma]$ is a face of the simplex $[\tau]$ in the complex $\mathcal{C}_{g,n}/\Gamma_{g,n}$, or equivalently, that there is a mapping class $[f] \in \Gamma_{g,n}$ such that $f(\sigma) < \tau$.

In the following lemma, $V([\sigma])$ stands for the quotient $T([\sigma])/W([\sigma])$. Note that $\dim_{\mathbb{C}} V([\sigma]) = m - |\sigma|$ (see Lemma 11). $V([\sigma])$ is a complex subvariety of $M_\varepsilon([\sigma])$, because $T(\sigma)$ is a complex submanifold of $D_\varepsilon(\sigma)$ (see [32], §8.2, and §9).

**Lemma 17.** *(i)* $M_\varepsilon([\sigma]) \cap M_\varepsilon([\tau]) \neq \emptyset$*, if and only if* $[\sigma] < [\tau]$ *or* $[\sigma] > [\tau]$*.*
*(ii)* $V([\sigma]) \cap V([\tau]) \neq \emptyset$ *if and only if* $[\sigma] = [\tau]$*.*
*(iii) Let* $\overline{M_\varepsilon([\rho])}$ *be the topological closure of* $M_\varepsilon([\rho])$ *in* $\overline{M}_{g,n}$*. Then* $V([\sigma]) \subset \overline{M_\varepsilon([\rho])}$ *if and only if* $[\sigma] > [\rho]$*.*
*(iv) Let* $\overline{V([\sigma])}$ *be the topological closure of* $V([\sigma])$ *in* $\overline{M}_{g,n}$*, and let* $\partial\overline{V([\sigma])} = \overline{V([\sigma])} \setminus V([\sigma])$ *be the "boundary" of* $\overline{V([\sigma])}$*. Then we have*

$$\partial\overline{V([\sigma])} = \bigcup_{[\sigma]\lneqq[\tau]} V([\tau]).$$

*Proof.* (i) Since $M_\varepsilon([\sigma]) = D_\varepsilon(\sigma)/W(\sigma) = \mathcal{U}_\varepsilon(\sigma)/N\Gamma(\sigma)$ and likewise $M_\varepsilon([\tau]) = \mathcal{U}_\varepsilon(\tau)/N\Gamma(\tau)$, the result follows from Lemma 10.
(ii) First suppose $\sigma \neq \emptyset$, and put $\sigma = \langle C_1, \dots, C_k\rangle$. Suppose $V([\sigma]) \cap V([\tau]) \neq \emptyset$, then there is a lift $\tilde{p} = [S, w]$ of $p \in V([\sigma]) \cap V([\tau])$ which is in the intersection $T(f(\sigma)) \cap T(\tau)$ for a certain mapping class $[f]$. On the Riemann surface with nodes $S$, the nodes are obtained by pinching the simple closed geodesics homotopic to $\{w(f(C_1)), \dots, w(f(C_k))\}$. If $\tau = \langle C'_1, \dots, C'_l\rangle$, the nodes of the same $S$ are obtained by pinching the curves $\{w(C'_1), \dots, w(C'_l)\}$. Thus the two sets of the isotopy classes of curves, $\{f(C_1), \dots, f(C_k)\}$ and $\{C'_1, \dots, C'_l\}$, must coincide. This implies $f(\sigma) = \tau$, namely $[\sigma] = [\tau]$.

If $\sigma = \emptyset$, then $V(\emptyset) = T(\emptyset)/W(\emptyset) = T_{g,n}/\Gamma_{g,n}$. Thus $V(\emptyset) \cap V([\tau]) \neq \emptyset$ if and only if $\tau = \emptyset$.

The proof of (ii) is complete.
(iii) Suppose $V([\sigma]) \subset \overline{M_\varepsilon([\rho])}$. Then $M_\varepsilon([\sigma]) \cap M_\varepsilon([\rho]) \neq \emptyset$. From (i), we have $[\sigma] < [\rho]$ or $[\rho] < [\sigma]$. We may assume $\sigma < \rho$ or $\sigma > \rho$. If $\sigma \lneqq \rho$, then we may assume $\sigma = \langle C_1, \dots, C_k\rangle$ and $\rho = \langle C_1, \dots, C_k, C_{k+1}, \dots, C_l\rangle$. Take a small positive number $\eta$ with $\varepsilon < \varepsilon + \eta < M$. Then those points $p = [S, w] \in \hat{T}_{g,n}$ that satisfy the condition

$$\begin{aligned} &l_{C_1}(p) = \cdots = l_{C_k}(p) = 0, \\ &\qquad\text{and other simple closed geodesics on } S \text{ are longer than } \varepsilon + \eta, \end{aligned}$$

belong to $T(\sigma) \setminus \overline{\mathcal{U}_\varepsilon(\rho)}$. This implies $V([\sigma]) \not\subset \overline{M_\varepsilon([\rho])}$, a contradiction. Thus we have $[\sigma] > [\rho]$.

Conversely suppose $[\sigma] > [\rho]$. We may assume $\sigma > \rho$. Then we have $T(\sigma) \subset \hat{T}(\rho)$, because

$$\hat{T}(\rho) = \bigcup_{\rho<\tau} T(\tau), \qquad \text{see Lemma 6.}$$

It follows that

$$V([\sigma]) = T(\sigma)/W(\sigma) \subset \hat{T}(\rho)/N\Gamma(\rho) \subset \overline{\mathcal{U}_\varepsilon(\rho)}/N\Gamma(\rho) = \overline{M_\varepsilon([\rho])}.$$

The proof of (iii) is complete.
(iv) By Lemma 6, we have

$$\partial\overline{T(\sigma)} = \partial\hat{T}(\sigma) = \bigcup_{\sigma\lneqq\tau} T(\tau).$$

Dividing the both sides by the action of $N\Gamma(\sigma)$, we have

$$\partial\overline{V([\sigma])} = \partial\overline{T(\sigma)}/N\Gamma(\sigma) = \bigcup_{[\sigma]\lneqq[\tau]} T(\tau)/N\Gamma(\tau) = \bigcup_{[\sigma]\lneqq[\tau]} V([\tau]).$$

The proof of (iv) is complete. □

As a consequence of Lemmas 12, 13, 14, 15, and 16, we have the following (see Definition 1):

**Theorem 9.** *In the Deligne-Mumford compactification $\overline{M}_{g,n}$ of moduli space, the finite family $\{(D_\varepsilon(\sigma), W(\sigma), \varphi_\sigma, M_\varepsilon(\sigma))\}_{\sigma\in\mathcal{C}_{g,n}/\Gamma_{g,n}}$ forms an atlas of orbifold-charts of a complex $m$-orbifold.*

Our main Theorem 1 is a restatement of this theorem.

# Part II Crystallographic groups

## 7. Crystallographic groups

We will give here the definition of crystallographic groups.

**Definition 10** (See [37], §I.3)**.** *A* crystallographic group *in Euclidean $m$-space* $\mathbf{E}^m$ *is a discrete group $G$ of isometries of* $\mathbf{E}^m$ *whose translation vectors form a lattice* $L \subset \mathbf{E}^m$.

Recall that a *lattice* $L$ is a discrete subgroup of $\mathbf{E}^m$ which generates $\mathbf{E}^m$ as a real vector space. Any lattice $L$ is generated by a basis for the vector space $\mathbf{E}^m$ ([37], §I.1). The image of $G$ under linearization $Isom(\mathbf{E}^m) \to O(\mathbf{E}^m)$ is called the *point group* of $G$ and denoted by $\overrightarrow{G}$. This is a finite subgroup of $O(\mathbf{E}^m)$. There is a canonical exact sequence

$$1 \to T \to G \to \overrightarrow{G} \to 1 \tag{38}$$

where $T$ is the translation subgroup of $G$, isomorphic to the lattice $L$ through the map which assigns a vector $x \in L$ to the translation $\tau_x$ by $x$. ([37], §I.3.) See also [64], §7.5, and [69], Ch.3.

A beautiful, enjoyable treatment of 2-, 3-dimensional crystallographic groups can be found in [59] (in Spanish). Also the author recommends two books written in Japanese, [40], [42], to learn crystallographic groups in dimensions 2 and 3.

**Remark.** Affine Weyl groups $W_a$ defined in [16], Ch.VI, §2.1, are a special type of crystallographic groups, which are generated by reflections with respect to certain hyperplanes associated with a root system. The point group of an affine Weyl group is the Weyl group $W$ of the root system. Bourbaki ([16], Ch.VI, §2.5, Definition 2) calls this Weyl group a crystallographic group, which is a finite group. Thus crystallographic groups in Bourbaki's sense are different from ours defined above. See [37], §V.8.

## 8. **Teichmüller spaces and the Weil-Petersson metric**

In this section, we review basic facts on Teichmüller spaces and the Weil-Petersson metric. Our main references are Ahlfors [6], Imayoshi and Taniguchi [33], and Nag [61]. For a differential geometric treatment of the same subjects with a much wider perspective, see [84], [85].

We begin by recalling the definition of Teichmüller space from scratch.

8.1. **Definitions.** By a *marked Riemann surface*, we mean a pair $(R, f)$ consisting of a Riemann surface $R$ of finite type and an orientation preserving homeomorphism called a *marking* $f : \Sigma_{g,n} \to R$. Marked Riemann surfaces $(R, f)$ and $(S, g)$ are said to be *equivalent* (and denoted by $(R, f) \sim (S, g)$) if there exists a conformal isomorphism $h : R \to S$ such that the diagram

$$\begin{array}{ccc} \Sigma_{g,n} & \xrightarrow{f} & R \\ {\scriptstyle id.(=)}\downarrow & & \downarrow{\scriptstyle h} \\ \Sigma_{g,n} & \xrightarrow[g]{} & S \end{array}$$

is homotopically commutative (i.e. $h \circ f \simeq g$). The *Teichmüller space* $T_{g,n}$ is defined to be the totality of the equivalence classes, $\{(R, f)\}/\sim$, equipped with the "Teichmüller distance" explained below.

An orientation preserving homeomorphism of Riemann surfaces $w : R \to S$ is called *quasiconformal*, provided it is $ACL$ (absolutely continuous on lines, see [6] p.23, and [33], p.77), and provided

$$\left\| \frac{w_{\overline{z}}}{w_z} \right\|_\infty < k, \quad \text{for some } k \ (0 \leqq k < 1).$$

We call $\mu(z) = \frac{w_{\overline{z}}}{w_z}$ the *Beltrami differential.* The *dilatation* of $w$, $K(w)$, is defined to be

$$K(w) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} \qquad (\geqq 1).$$

In general, $w$ is called $K$*-quasiconformal* if $K(w) \le K$. Let $p = [R, f]$ and $q = [S, g]$ be two points of $T_{g,n}$. Then the *Teichmüller distance* is defined by

$$d(p, q) = \frac{1}{2} \log \inf_w K(w) \tag{39}$$

where $w : R \to S$ runs over all quasiconformal mappings homotopic to $g \circ f^{-1}$. Teichmüller space $T_{g,n}$ is a complete metric space with respect to this distance ([33], Theorem 5.4).

**Remark.** In the definition of the Teichmüller distance (39), the factor $\frac{1}{2}$ is sometimes omitted. (Cf. [6] p.120, [33], p.125.) However, H. Miyachi [58], Theorem 1 (iv), has proved the following beautiful identity

$$i(p, q) = e^{d(p,q)} \quad \text{for} \quad p, q \in T_{g,n},$$

where $i(p, q)$ is a certain generalized intersection number and $d(p, q)$ is the Teichmüller distance as defined in (39). Note that the factor $\frac{1}{2}$ is essential for his identity to hold. □

The *mapping class group* $\Gamma_{g,n}$ is defined as follows:

$$\Gamma_{g,n} = \{h : \Sigma_{g,n} \to \Sigma_{g,n} \mid \text{orientation preserving homeomorphisms}\}/\text{isotopy}.$$

The group $\Gamma_{g,n}$ acts on $T_{g,n}$ via

$$[h]_*([R,f]) = [R, f\circ h^{-1}]. \tag{40}$$

The action is properly discontinuous, and preserves the Teichmüller distance ([33], Propositions 5.5 and 6.18). Note that the action of $\Gamma_{g,n}$ changes only markings, but not Riemann sufaces.

8.2. **Teichmüller's theorem.** Let $p = [R,f] \in T_{g,n}$ be any point. We fix $p$ as a "base point"for our discussion below. By the uniformization theorem, the universal covering space $\tilde{R}$ is identified with the upper half plane $\mathbf{H}$ $(= \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\})$. The covering translations $\tilde{R} \to \tilde{R}$ make a *Fuchsian group* $\Gamma$, *i.e.*, a discrete subgroup of $\mathrm{Aut}(\mathbf{H})$. By choosing the identification $\tilde{R} = \mathbf{H}$ appropriately, we may assume that the following condition $(*)$ is satisfied:

$(*)$ *Each point of* $0, 1, \infty$ *is fixed by some element of* $\Gamma - \{id\}$.
Clearly, we have $R = \mathbf{H}/\Gamma$.

**Definition 11.** *A holomorphic function* $\varphi$ *on* $\mathbf{H}$ *is called a* quadratic differential (with respect to $\Gamma$)*, if for each* $\gamma \in \Gamma$ *it satisfies*

$$\varphi(\gamma(z))(\gamma'(z))^2 = \varphi(z), \quad \forall z \in \mathbf{H}.$$

If $\varphi$ is a quadratic differential on $\mathbf{H}$, it descends to a holomorphic quadratic differential on $R$. In the case where $R$ has punctures, we always assume that the descended quadratic differential has at most simple poles at the punctures. We denote the $\mathbb{C}$-vector space of quadratic differentials (with respect to $\Gamma$) by

$$Q(\Gamma) \quad \text{or more precisely, by} \;\; Q(\mathbf{H}, \Gamma).$$

By the Riemann-Roch theorem, we have

$$\dim_{\mathbb{C}} Q(\Gamma) = 3g - 3 + n.$$

We can introduce a norm $\|\varphi\|$ on $Q(\Gamma)$ by defining

$$\|\varphi\| = 2\int_\Delta |\varphi| dxdy, \quad z = x + iy,$$

where $\Delta$ denotes a fundamental domain ($\subset \mathbf{H}$) of the action of $\Gamma$. Let $Q(\Gamma)_1$ denote the open unit ball in $Q(\Gamma)$:

$$Q(\Gamma)_1 = \{\varphi \in Q(\Gamma) \mid \|\varphi\| < 1\}.$$

Teichmüller proved that to each $\varphi \in Q(\Gamma)_1$ uniquely corresponds an "extremal quasiconformal mapping" $w : R \to S$ of $R$ to a certain Riemann surface $S$. Thus one has a mapping

$$\mathcal{T} : Q(\Gamma)_1 \to T_{g,n}, \quad \varphi \mapsto [S, w\circ f].$$

(Here $f$ is the marking of the base point $p = [R,f] \in T_{g,n}$.) Teichmüller's theorem states the following

**Theorem 10** ([33], Theorem 5.15)**.** *The mapping* $\mathcal{T} : Q(\Gamma)_1 \to T_{g,n}$ *is a surjective homeomorphism.*

As a corollary, $T_{g,n}$ is homeomorphic to $\mathbf{R}^{6g-6+2n}$.

8.3. **The complex structure on $T_{g,n}$.** We use the same notation, and fix a base point $[R, f]$ as above. We can put a natural complex structure on $T_{g,n}$ via Ahlfors-Bers theory ([4], [6], [7], [8]). Let $[S, g] \in T_{g,n}$ be another point of $T_{g,n}$, $h : R \to S$ being a quasiconformal mapping in the homotopy class of $g \circ f^{-1}$. By choosing the identification $\tilde{S} = \mathbf{H}$ properly, we may assume that the lift $\tilde{h} : \mathbf{H}(= \tilde{R}) \to \mathbf{H}(= \tilde{S})$ is *normalized* in the sense that it fixes 0, 1 and $\infty$.

Since two lifts $\mathbf{H} \to \mathbf{H}$ of $h : R \to S$ differ only by the multiplication of conformal automorphisms of $\mathbf{H}$ from the left, the normalized lift is uniquely determined. We call this lift $\tilde{h}$ the *canonical lift of $h : R \to S$ with respect to* $\Gamma$.

Set $\Gamma' = \tilde{h}\Gamma\tilde{h}^{-1}$. Then $\Gamma'$ is a Fuchsian group, and $S = \mathbf{H}/\Gamma'$. $\Gamma'$ is said to be a *quasiconformal deformation* of $\Gamma$ deformed by the quasiconformal mapping $\tilde{h}$.

Recalling Teichmüller's theorem, we may consider $T_{g,n}$ as the totality of quasiconformal deformations of $\Gamma$.

In what follows, we only consider those quasiconformal homeomorphisms $w : \mathbf{H} \to \mathbf{H}$ that satisfy
(i) $w$ is normalized, and
(ii) $w\Gamma w^{-1}$ is a Fuchsian group.

Let $QuasiConf(\Gamma)$ denote the set of such quasiconformal mappings. A quasiconformal mapping $w \in QuasiConf(\Gamma)$ deforms the Fuchsian group $\Gamma$, and it descends to a quasiconformal mapping $h : R \to \mathbf{H}/w\Gamma w^{-1}$. It is known that the descendants $h_1$ and $h_2$ of $w_1$ and $w_2$ give the same point of Teichmüller space $T_{g,n}$

$$[\mathbf{H}/w_1\Gamma w_1^{-1}, h_1 \circ f] = [\mathbf{H}/w_2\Gamma w_2^{-1}, h_2 \circ f]$$

if and only if the restrictions of $w_1$ and $w_2$ on the real line $\mathbf{R}$ coincide[4]. See [33], §5.1.2. With this in mind, we say that $w_1$ and $w_2$ are *equivalent* ($w_1 \sim w_2$) if $w_1|\mathbf{R} = w_2|\mathbf{R}$, and denote the equivalence class by $[w]$. Then we define a set $T(\Gamma)$ as follows:

$$T(\Gamma) = \{[w] \mid w \in QuasiConf(\Gamma)\} \; (= QuasiConf(\Gamma)/\sim). \tag{41}$$

$T(\Gamma)$ is called the *Teichmüller space of* $\Gamma$. It is naturally identified with the Teichmüller space $T_{g,n}$, but the notation $T(\Gamma)$ is sometimes preferable in order to record the distinguished base point $p = [R, f]$.

Let $B(\mathbf{H})_1$ (or $B(\mathbb{C})_1$) be the set of all measurable ($\mathbb{C}$-valued) functions $\mu$ on $\mathbf{H}$ (or $\mathbb{C}$) satisfying $\|\mu\|_\infty < 1$. Then for $\mu \in B(\mathbf{H})_1$ (or $\mu \in B(\mathbb{C})_1$), the *Beltrami equation*

$$w_{\overline{z}} = \mu w_z$$

can be solved, and a solution gives a quasiconformal homeomorphism $w : \mathbf{H} \to \mathbf{H}$ (or $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$). This solution $w$ is uniquely determined under the condition that it is normalized. ([33], Theorem 4.30, Proposition 4.33. See also [6], Ch. V.) Moreover, if $\mu$ depends on real parameters analytically, differentiably, or continuously, the same is true for the solution $w$ [7]. This unique solution is denoted by $w^\mu$.

For $\mu$ in $B(\mathbf{H})_1$, it is shown ([33], p.124) that $w^\mu : \mathbf{H} \to \mathbf{H}$ belongs to $QuasiConf(\Gamma)$ if and only if $\mu$ satisfies

$$\mu(\gamma(z))\frac{\overline{(\gamma'(z))}}{\gamma'(z)} = \mu(z), \quad \text{for a. e. } z \in \mathbf{H}, \; \gamma \in \Gamma. \tag{42}$$

[4]Precisely speaking, quasiconformal mappings $w_1, w_2 : \mathbf{H} \to \mathbf{H}$ are uniquely extended to homeomorphisms $\overline{\mathbf{H}} \to \overline{\mathbf{H}}$, where $\overline{\mathbf{H}} = \mathbf{H} \cup \mathbf{R}$, and the relation $w_1|\mathbf{R} = w_2|\mathbf{R}$ is actually talking about the extended homeomorphisms.

Let $B(\mathbf{H},\Gamma)$ denote the complex vector space of all bounded measurable functions on $\mathbf{H}$ satisfying (42), and $B(\mathbf{H},\Gamma)_1$ the open unit ball of $B(\mathbf{H},\Gamma)$ defined by

$$B(\mathbf{H},\Gamma)_1 = B(\mathbf{H},\Gamma) \cap B(\mathbf{H})_1.$$

Clearly there is a bijection

$$B(\mathbf{H},\Gamma)_1 \to QuasiConf(\Gamma), \quad \mu \mapsto w^\mu. \tag{43}$$

(The inverse is given by $w \mapsto w_{\overline{z}}/w_z$.)

Introduce an equivalence relation of Beltrami differentials in $B(\mathbf{H},\Gamma)_1$ by saying that $\mu$ and $\nu$ are *equivalent* (denoted by $\mu \sim \nu$) if $w^\mu \sim w^\nu$. Then the bijection (43) induces the following bijection of quotient spaces

$$B(\mathbf{H},\Gamma)_1/\sim \to QuasiConf(\Gamma)/\sim \overset{(41)}{=} T(\Gamma), \quad [\mu] \mapsto [w^\mu], \tag{44}$$

where $[\mu]$ is the equivalence class of $\mu$.

Let $\mathbf{H}^*$ denote the lower half plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$. We extend a function $\mu \in B(\mathbf{H},\Gamma)_1$ to $\mathbb{C}$ by setting

$$\tilde{\mu}(z) = \begin{cases} \mu(z) & \text{on } \mathbf{H} \\ 0 & \text{on } \mathbf{H}^* \cup \mathbf{R}. \end{cases}$$

The normalized solution of the Beltrami equation $w_{\overline{z}} = \tilde{\mu} w_z$ gives a quasiconformal homeomorphism $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. To distinguish from $w^\mu : \mathbf{H} \to \mathbf{H}$, we denote it by $w_\mu$. Though $w_\mu$ fixes $0, 1, \infty$, it does not necessarily preserves $\mathbf{H}$ $(\subset \hat{\mathbb{C}})$.

**Lemma 18** ([33], Lemma 6.1)**.** *For any two elements $\mu, \nu \in B(\mathbf{H},\Gamma)_1$, the following two conditions are equivalent:*
*(i) $w^\mu|\mathbf{R} = w^\nu|\mathbf{R}$.*
*(ii) $w_\mu|\mathbf{H}^* = w_\nu|\mathbf{H}^*$.*

Since $\mathrm{Aut}(\mathbf{H}) = \mathrm{PSL}(2,\mathbf{R})$ and $\mathrm{Aut}(\hat{\mathbb{C}}) = \mathrm{PSL}(2,\mathbb{C})$, the discrete group $\Gamma$ (which is a Fuchsian group in $\mathrm{Aut}(\mathbf{H})$) can be considered as a discrete subgroup of $\mathrm{Aut}(\hat{\mathbb{C}})$ (a *Kleinian group*). The compatibility condition (42) (interpreted as a condition for the extended Beltrami coefficient $\tilde{\mu}$) assures that $w_\mu \Gamma (w_\mu)^{-1}$ is a Kleinian group. Since $\tilde{\mu}|\mathbf{H}^* = 0$, the restriction $w_\mu|\mathbf{H}^* : \mathbf{H}^* \to w_\mu(\mathbf{H}^*)$ is a conformal mapping (Weyl's lemma, see Lemma 4.6 of [33].) Then $w_\mu|\mathbf{H}^*$ descends to a conformal isomorphism $R^* = \mathbf{H}^*/\Gamma \to R^*_\mu (:= w_\mu(\mathbf{H}^*)/w_\mu\Gamma(w_\mu)^{-1})$.

The difference between conformal mappings and Möbius transformations is measured by the Schwarzian derivative. For a conformal mapping $f$ on a domain $D$ in $\mathbb{C}$, we define the *Schwarzian derivative* $\{f, z\}$ of $f$ by

$$\{f,z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

A conformal mapping $f : D \to f(D)$ is a Möbius transformation if and only if $\{f, z\} = 0$ on $D$. ([33], Lemma 6.3.)

In our case, considering the Schwarzian derivative of $w_\mu|\mathbf{H}^*$, we set

$$\varphi_\mu(z) = \{w_\mu|\mathbf{H}^*, z\}, \quad z \in \mathbf{H}^*.$$

Then from the formula for the Schwarzian derivative of composite maps we can derive

$$\varphi_\mu(\gamma(z))\gamma'(z)^2 = \varphi_\mu(z), \quad z \in \mathbf{H}^*, \quad \gamma \in \Gamma.$$

(See [33], Lemma 6.4.) This means that $\varphi_\mu$ descends to a holomorphic quadratic differential (denoted by the same letter $\varphi_\mu$) on the Riemann surface $R^* = \mathbf{H}^*/\Gamma$. Let $Q(\mathbf{H}^*, \Gamma)$ denote the $\mathbb{C}$-vector space of quadratic differentials (with respect to $\Gamma$) on $\mathbf{H}^*$. Then $\varphi_\mu \in Q(\mathbf{H}^*, \Gamma)$.

Lemma 18 and the bijection (44) assure that the map $[w^\mu] \mapsto \varphi_\mu$ is a well-defined injective map (For the proof of the injectivity, see [33], Lemma 6.4.)

$$\mathcal{B} : T(\Gamma) \to Q(\mathbf{H}^*, \Gamma). \tag{45}$$

This map is called *Bers' embedding*. Combining $\mathcal{B}$ with the projection $B(\mathbf{H}, \Gamma)_1 \to T(\Gamma)$ (see (44)), we get a projection $\Phi : B(\mathbf{H}, \Gamma)_1 \to Q(\mathbf{H}^*, \Gamma)$. This is called *Bers' projection*.

**Proposition 1** (See [33], Proposition 6.5)**.** *Bers' projection* $\Phi : B(\mathbf{H}, \Gamma)_1 \to Q(\mathbf{H}^*, \Gamma)$ *and Bers' embedding* $\mathcal{B} : T(\Gamma) \to Q(\mathbf{H}^*, \Gamma)$ *are continuous.*

We know that $T(\Gamma) = T_{g,n}$ is homeomorphic to $\mathbf{R}^{6g-6+2n}$ and (by the Riemann-Roch theorem) that $Q(\mathbf{H}^*, \Gamma)$ is a $(3g-3+n)$-dimensional complex vector space. Thus Brouwer's theorem implies that the $\mathcal{B} : T(\Gamma) \to Q(\mathbf{H}^*, \Gamma)$ is a homeomorphism onto its image. The image is denoted by $T_B(\Gamma)$. In fact, $T_B(\Gamma)$ is a bounded domain of $Q(\mathbf{H}^*, \Gamma)$. (See [33], Lemma 6.7.) Identify $T_{g,n}$, $T(\Gamma)$ and $T_B(\Gamma)$. Then all of these Teichmüller spaces inherit the *complex structure* of $Q(\mathbf{H}^*, \Gamma)$. So far the arguments depend on the basepoint $p = [R, f] \in T_{g,n}$, and it might seem that the complex structure on $T_{g,n}$ has the same dependence. But actually, it can be shown that the complex structure on $T_{g,n}$ is independent of the choice of the basepoint ([33], Theorem 6.12). Moreover, it can be shown that with respect to this complex structure the action of the mapping class group $\Gamma_{g,n}$ on $T_{g,n}$ is biholomorphic ([33], Theorem 6.18).

8.4. **The exponential map.** Take an arbitrary point $p = [R, f] \in T_{g,n}$, and suppose that $R = \mathbf{H}/\Gamma$ as above. In the previous subsection, we considered Bers' projection $\Phi : B(\mathbf{H}, \Gamma)_1 \to Q(\mathbf{H}^*, \Gamma)$, and we saw that $T_{g,n}$ inherits a complex structure form $Q(\mathbf{H}^*, \Gamma)$ by identifying $T_{g,n}$ with $T_B(\Gamma) = \mathrm{Image}(\Phi)$. On the other hand $B(\mathbf{H}, \Gamma)_1$ has a natural complex structure inherited from the complex linear space $B(\mathbf{H}, \Gamma)$ (see [4], §3.2). We will always assume these complex structures on $B(\mathbf{H}, \Gamma)_1$ and $T_{g,n}$. By Proposition 1, the map

$$\Phi : B(\mathbf{H}, \Gamma)_1 \to T_{g,n}$$

is continuous. More strongly, we have

**Proposition 2** (See [4], [61] p.190)**.** *The map* $\Phi : B(\mathbf{H}, \Gamma)_1 \to T_{g,n}$ *is a surjective complex submersion.*

Wolpert [79] suggestively calls this map the *exponential map*.

The differential of $\Phi$ at $0 \in B(\mathbf{H}, \Gamma)_1$ is a complex linear map

$$d\Phi_0 : B(\mathbf{H}, \Gamma) \to Q(\mathbf{H}^*, \Gamma),$$

which is calculated by the following method:

First recall Ahlfors' general integral formula for "infinitesimal quasiconformal mappings". Let $\mu_t = t\nu(z) + o(t)$ be a short arc in $B(\mathbb{C})_1$, and $w_{\mu_t} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the normalized solution of the Beltrami equation

$$w_{\bar{z}} = \mu_t w_z.$$

Then $w_{\mu_t}$ converges to the identity as $t \to 0$, and the infinitesimal quasiconformal mapping, or the first variation of $w_{\mu_t}$

$$\dot{w}(z) = \lim_{t\to 0} \frac{w_{\mu_t}(z) - z}{t}$$

is represented by the integral formula

$$\dot{w}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \nu(\zeta)\frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}d\xi d\eta. \tag{46}$$

(See Ahlfors [6], p.104, where $\zeta$ and $z$ are interchanged, see also [33] Theorem 4.37.) If

$$\nu \mid \mathbf{H}^* = 0,$$

formula (46) becomes

$$\dot{w}(z) = -\frac{1}{\pi}\int_{\mathbf{H}} \nu(\zeta)\frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}d\xi d\eta, \quad z \in \mathbb{C}. \tag{47}$$

In this case, $w_{\mu_t}$ is holomorphic on $\mathbf{H}^*$, and by definition of Bers' projection $\Phi : B(\mathbf{H},\Gamma)_1 \to Q(\mathbf{H}^*,\Gamma)$, we have[5]

$$\begin{aligned}\Phi(\mu_t) &= \{w_t, z\} \\ &= \frac{w_t'''}{w_t'} - \frac{3}{2}\left(\frac{w_t''}{w_t'}\right)^2 \\ &= t\dot{w}''' + o(t),\end{aligned}$$

here primes $(w_t', w_t'', w_t''')$ indicate the differentiation w.r.t. $z \in \mathbf{H}^*$.

Thus

$$d\Phi_0(\nu)(z) = (\dot{w}(z))'''. \tag{48}$$

Substituting the right hand side of (47) for $\dot{w}$ in (48), we get

$$d\Phi_0(\nu)(z) = -\frac{6}{\pi}\int_{\mathbf{H}} \frac{\nu(\zeta)}{(\zeta-z)^4}d\xi d\eta, \quad z \in \mathbf{H}^*. \tag{49}$$

(See [4][6] (1.18), [33] Theorem 6.10.)

The following properties of $\dot{w}(z)$ will be used later in §4.2.

**Proposition 3.** *We have*
*(i)* $(\dot{w})_{\overline{z}} = \nu$,
*(ii)* $\dot{w}(z) = 0$, *for z=0,1, and*
*(iii)* $\dot{w}(z) = o(|z^2|)$ *as* $z \to \infty$.

Property (i) is observed by Ahlfors [4] (1.5). Properties (ii) and (iii) immediately follow from (46).

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[5] Here we have simplified the notation $w_{\mu_t}$ to $w_t$.

[6] Ahlfors [4] calculates on the unit disk instead of the upper half plane $\mathbf{H}$, and his formula does not coincide with (49).

8.5. **Weil-Petersson metric.** In this subsection, we will discuss the Weil-Petersson metric, which is a natural Riemannian metric on $T_{g,n}$. First we will observe that there is a natural pairing of Beltrami differentials $\mu \in B(\mathbf{H}, \Gamma)$ and quadratic differntials $\varphi \in Q(\mathbf{H}, \Gamma)$. By Definition 11, a quadratic differentail $\varphi \in Q(\mathbf{H}, \Gamma)$ is a holomorphic function on $\mathbf{H}$ satisfying

$$\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z), \quad z \in \mathbf{H}, \quad \gamma \in \Gamma.$$

This together with (42) implies

$$\mu(\gamma(z))\varphi(\gamma(z))\gamma'(z)\overline{\gamma'(z)} = \mu(z)\varphi(z), \quad z \in \mathbf{H}, \quad \gamma \in \Gamma.$$

Thus the 2-form $\mu(z)\varphi(z)dz \wedge d\overline{z}$ is $\Gamma$-invariant, and descends to a 2-form on $R$, which we denote by $\mu\varphi$. Note that the value of the integral

$$\int_R \mu\varphi$$

is independent of the choice of the coordinate on $R$, and gives a pairing of $\mu$ and $\varphi$. (See [82] p.15.) Since $dz \wedge d\overline{z} = -2idx \wedge dy$, the 2-form $\mu(z)\varphi(z)dx \wedge dy$ is $\Gamma$-invariant too. For the sake of simplicity, we use the latter 2-form $\mu(z)\varphi(z)dx \wedge dy$ to define a complex valued bilinear pairing $(\mu, \varphi)_{\mathbb{C}}$:

$$(\mu, \varphi)_{\mathbb{C}} = \int_\Delta \mu(z)\varphi(z)dxdy, \quad z = x + iy,$$

where $\Delta$ is a fundamental domain of $\Gamma$. Looking at Wolpert's papers, this choice seems to match his calculations. ( See the equation in the proof of Theorem 2.10 of [75], for example.) Its real part gives a real valued pairing

$$(\mu, \varphi) = \mathrm{Re}\left((\mu, \varphi)_{\mathbb{C}}\right). \tag{50}$$

We denote the kernel of $d\Phi_0$ by $N(\Gamma)$. This is a complex vector subspace of $B(\mathbf{H}, \Gamma)$.

**Theorem 11** ([4], Lemma 8. See [79], Theorem 1.2)**.** *A Beltrami differential $\mu$ belongs to $N(\Gamma)$ if and only if $(\mu, \varphi)_{\mathbb{C}} = 0$ for all quadratic differentials $\varphi \in Q(\mathbf{H}, \Gamma)$.*

Since $d\Phi_0$ is surjective, the quotient $B(\mathbf{H}, \Gamma)/N(\Gamma)$ is isomorohic to $Q(\mathbf{H}^*, \Gamma)$, whose complex dimension is $3g - 3 + n$, the same dimension as $Q(\mathbf{H}, \Gamma)$. Thus the pairing $(\mu, \varphi)_{\mathbb{C}}$ induces the *non-singular* pairing

$$(\cdot, \cdot)_{\mathbb{C}} : B(\mathbf{H}, \Gamma)/N(\Gamma) \times Q(\mathbf{H}, \Gamma) \to \mathbb{C}. \tag{51}$$

Also the real valued pairing is non-singular:

$$(\cdot, \cdot) : B(\mathbf{H}, \Gamma)/N(\Gamma) \times Q(\mathbf{H}, \Gamma) \to \mathbf{R}. \tag{52}$$

The quotient $B(\mathbf{H}, \Gamma)/N(\Gamma)$ is naturally regarded as the *tangent space* of $T_{g,n}$ at $p = [R, f]$. Thus via the pairing (52), $Q(\mathbf{H}, \Gamma)$ is the *cotangent space* of $T_{g,n}$ at $p$. The pairing (52) is the *tangent-cotangent pairing*

As we shall explain below, there are natural representatives of $B(\mathbf{H}, \Gamma)/N(\Gamma)$ by *harmonic Beltrami differentials*: For any $\varphi \in Q(\mathbf{H}, \Gamma)$, we define the *associated harmonic Beltrami differential* $\mu_\varphi$ as follows [7] :

$$\mu_\varphi(z) = (\mathrm{Im}\ z)^2\ \overline{\varphi(z)}. \tag{53}$$

It is easy to see that $\mu_\varphi(z)$ satisfies (42), and $\mu_\varphi$ belongs to $B(\mathbf{H}, \Gamma)$.

[7] Imayoshi and Taniguchi [33] defines $\mu_\varphi = -2(\mathrm{Im}\ z)^2\ \overline{\varphi(z)}$, while Wolpert [79] defines $\mu_\varphi(z) = (z - \overline{z})^2\overline{\varphi(z)} = -4(\mathrm{Im}\ z)^2\ \overline{\varphi(z)}$. In the present paper, we follow Wolpert[81], and define $\mu_\varphi$ by (53). This definition makes calculations transparent.

Introduce the $\infty$-norm in quadratic differentials $Q(\mathbf{H}^*,\Gamma)$ by defining:

$$\|\psi\|_\infty = \sup_{z\in\mathbf{H}^*} (\text{Im } z)^2|\psi(z)|.$$

The Ahlfors-Weill theorem ([33], Theorem 6.9) asserts that if the norm of a quadratic differential $\psi \in Q(\mathbf{H}^*,\Gamma)$ satisfies $\|\psi\|_\infty < \frac{1}{2}$, then the correspondence

$$\psi(z) \mapsto -2(\text{Im } z)^2\psi(\overline{z}), \quad z \in \mathbf{H}^*$$

is a local cross-section of the exponential map

$$\Phi : B(\mathbf{H},\Gamma)_1 \to T_{g,n}(\subset Q(\mathbf{H}^*,\Gamma)).$$

This implies that a quadratic differential $\psi \in Q(\mathbf{H}^*,\Gamma)$ with $\|\psi\|_\infty < \frac{1}{2}$ belongs to $T_{g,n}$ and that the linear map

$$\varphi(z) \mapsto -2\mu_\varphi(z) : Q(\mathbf{H},\Gamma) \to B(\mathbf{H},\Gamma)$$

is an injective map, which makes the diagram commute.

$$\begin{array}{ccc}
Q(\mathbf{H},\Gamma) & \xrightarrow{\varphi(z)\mapsto -2\mu_\varphi(z)} & B(\mathbf{H},\Gamma) \\
{\scriptstyle \varphi(z)\mapsto\psi(z)=\overline{\varphi(\overline{z})}}\big\downarrow & & \big\downarrow{\scriptstyle id.(=)} \\
Q(\mathbf{H}^*,\Gamma) & \xrightarrow[\psi(z)\mapsto -2(\text{Im}z)^2\psi(\overline{z})]{} & B(\mathbf{H},\Gamma).
\end{array}$$

(Recall $\mu_\varphi(z) = (\text{Im}z)^2\overline{\varphi(z)}$ and note that $\psi(z) = \overline{\varphi(\overline{z})}$ belongs to $Q(\mathbf{H}^*,\Gamma)$ for any $\varphi \in Q(\mathbf{H},\Gamma)$.) It follows that the map

$$\varphi(z) \mapsto \mu_\varphi(z) : Q(\mathbf{H},\Gamma) \to B(\mathbf{H},\Gamma)$$

is injective and its image $\mathcal{H}(\Gamma)$, consisting of harmonic Beltrami differentials, is a linear subspace of $B(\mathbf{H},\Gamma)$ transverse to $N(\Gamma)$. Thus we can identify $\mathcal{H}(\Gamma)$ with the quotient space $B(\mathbf{H},\Gamma)/N(\Gamma)$, which is the tangent space of $T_{g,n}$ at the base point $p = [R,f]$.

Now take $\varphi$ and $\psi$ in $Q(\mathbf{H},\Gamma)$. Then

$$(\text{Im } z)^2\varphi\overline{\psi} = \left((\text{Im } z)^2\overline{\psi}\right)\varphi = \mu_\psi\varphi$$

is a $\Gamma$-invariant 2-form, and the pairing

$$\langle\varphi,\psi\rangle_{\mathbb{C}} = \int_\Delta (\text{Im } z)^2\varphi\overline{\psi}dxdy \tag{54}$$

is well-defined. We call this pairing *the Weil-Petersson cometric* (see [82], Definition 2.4.). Since $\langle\varphi,\psi\rangle_{\mathbb{C}} = (\mu_\psi,\varphi)_{\mathbb{C}}$, the Weil-Petersson commetric is a non-singular Hermitian pairing on $Q(\mathbf{H},\Gamma)$.

For any $\mu,\nu \in B(\mathbf{H},\Gamma)$, we can verify by using (42)

$$\mu(\gamma(z))\overline{\nu(\gamma(z))} = \mu(z)\overline{\nu(z)}.$$

Thus $\mu\overline{\nu}$ defines a $\Gamma$-invariant function on $\mathbf{H}$, which we integrate by the invariant area form $dA = \frac{dxdy}{(\text{Im } z)^2}$ to get

$$\langle\mu,\nu\rangle_{\mathbb{C}} = \int_\Delta \mu\overline{\nu}(\text{Im } z)^{-2}dxdy.$$

Taking $\mu = \mu_\varphi = (\text{Im } z)^2\overline{\varphi}$ and $\nu = \mu_\psi = (\text{Im } z)^2\overline{\psi}$, we have

$$\langle\mu,\nu\rangle_{\mathbb{C}} = \int_\Delta (\text{Im } z)^4\overline{\varphi}\psi(\text{Im } z)^{-2}dxdy = \int_\Delta (\text{Im } z)^2\overline{\varphi}\psi dxdy = \langle\psi,\varphi\rangle_{\mathbb{C}}. \tag{55}$$

This shows that the restriction of the pairing to the harmonic differentials $\mathcal{H}(\Gamma)$ gives a non-singular Hermitian pairing $\langle \mu, \nu \rangle_{\mathbb{C}}$, which is called the (complex) *Weil-Petersson* metric on $\mathcal{H}(\Gamma)$. This is invariant under the action of the mapping class group $\Gamma_{g,n}$ on $T_{g,n}$ (see [33] Theorem 7.14).

Its real part

$$\langle \mu, \nu \rangle = \mathrm{Re}(\langle \mu, \nu \rangle_{\mathbb{C}}) \tag{56}$$

is the *Weil-Petersson Riemannian metric*[8] [81]. Teichmüller space $T_{g,n}$ with this metric is Kähler ([4], Theorem 4) and has negative sectional curvature ([5], [81]). It is non-complete ([74], [20]). By [83] its metric completion $\hat{T}_{g,n}$ is CAT(0)-space and is identified with Abikoff's *augmented Teichmüller space* ([1], [2]). The *Kähler form* is given[9] by

$$\omega(\mu, \nu) = -\mathrm{Im}(\langle \mu, \nu \rangle_{\mathbb{C}}). \tag{57}$$

Clearly we have $\omega(\mu, \nu) = \langle i\mu, \nu \rangle$.

[8] In [79], Wolpert defines the WP Riemannian metric by $2\mathrm{Re}(\langle \mu, \nu \rangle_{\mathbb{C}})$. Imayoshi-Taniguchi [33] follows this definition. Here we follow [81] and define the WP Riemannian metric by (56).

[9] In [79], Wolpert defines the Kähler form $\omega$ by $-2\mathrm{Im}(\langle \mu, \nu \rangle_{\mathbb{C}})$, and [33] follows this. While we follow [81] and define $\omega$ by (57).

## 9. The Fenchel-Nielsen deformation

In this section, we will review Wolpert's results ([75], [77], [78]) which are relevant to our purpose.

9.1. **Fenchel-Nielsen coordinates.** By *a (generalized) pair of pants* we shall mean a surface homeomorphic to a 2-sphere with three disjoint open disks (whose closures are also disjoint) deleted, or to a once punctured annulus, or to a twice punctured disk. A pair of pants has Euler characteristic $-1$. They are assumed to be oriented.

The surface $\Sigma_{g,n}$ has a system of disjoint simple closed curves

$$\mathcal{D} = \{C_1, C_2, \cdots, C_m\}, \quad m = 3g - 3 + n$$

such that the closure of each connected component $P_j$ of $\Sigma_{g,n} - \bigcup_{i=1}^m C_i$ is a pair of pants. The number $k$ of the connected components is equal to $2g - 2 + n$, and we call the collection of these pants $\mathcal{P} = \{P_1, P_2, \cdots, P_k\}$ *the pants decomposition* of $\Sigma_{g,n}$ corresponding to *the system $\mathcal{D}$ of decomposing curves.*

Of course, a system $\mathcal{D}$ of decomposing curves is identified with a maximal simplex (which will be denoted by the same symbol $\mathcal{D}$) of the curve complex $\mathcal{C}_{g,n}$.

**Remark.** If $\mathcal{D}' = \{C_1', \cdots, C_m'\}$ is another system of decomposing curves, and if $C_i$ is isotopic to $C_i'$ on $\Sigma_{g,n}$, for each $i = 1, \cdots, m$, then we consider $\mathcal{D}$ and $\mathcal{D}'$ are the *same* systems. Precisely speaking, $\mathcal{D}$ is the system consisting of the *isotopy classes* of $C_i$, $i = 1, \cdots, m$. □

Let $p = [R, f]$ be any point of $T_{g,n}$, $\mathcal{D} = \{C_1, C_2, \cdots, C_m\}$ a system of decomposing curves of $\Sigma_{g,n}$. Then for each $i = 1, 2, \cdots, m$, there exists a simple closed geodesic $c_i$ of $R$ freely homotopic to $f(C_i)$. The closed geodesics $\{c_1, c_2, \cdots, c_m\}$ are pairwise disjoint.

Let $l_i(p)$ denote the hyperbolic length of $c_i$ measured by the Poincaré metric of $R$. Then we have $m$ functions

$$l_i : T_{g,n} \to \mathbf{R}^+, \quad i = 1, 2, \cdots, m.$$

The functions $l_i$ are real analytic (see [2] p.87, or [33], Lemma 3.7).

We may (and will) assume that $f(C_i) = c_i$, $i = 1, 2, \cdots, m$. Then the image of the pants decomposition $\mathcal{P}$ of $\Sigma_{g,n}$ gives a pants decomposition $f(\mathcal{P})$ of $R$. The members of $f(\mathcal{P})$ are hyperbolic pants. It is known that every hyperbolic pair of pants $P$ admits an anti-holomorphic involution $\sigma_P : P \to P$. The fixed point set of $\sigma_P$ consists of three disjoint geodesic arcs with the property that when they intersect the boundary (circle) components of $P$ the intersections are perpendicular. Let $P_{i,1}, P_{i,2}$ $(\in f(\mathcal{P}))$ be the hyperbolic pairs of pants which are glued together along $c_i$. (It is possible that $P_{i,1} = P_{i,2}$.) The closed geodesic $c_i$ as a boundary of $P_{i,1}$ has two points $p_{i,1}, q_{i,1}$ which are the intersection of $c_i$ and the geodesic arcs fixed by the anti-holomorphic involution of $P_{i,1}$. The points $p_{i,1}$ and $q_{i,1}$ divide the circle $c_i$ into two arcs of the same length $l_i(p)/2$. Likewise, the closed geodesic $c_i$ as a boundary of $P_{i,2}$ has two points $p_{i,2}, q_{i,2}$ of the same nature.

In the special case where $p_{i,1} = p_{i,2}$ (hence $q_{i,1} = q_{i,2}$), the anti-holomorphic involutions of $P_{i,1}$ and $P_{i,2}$ extend to make an anti-holomorphic involution of $P_{i,1} \cup P_{i,2}$. In general, we have $p_{i,1} \neq p_{i,2}$, and the hyperbolic length of an arc on $c_i$ with terminal points $\{p_{i,1}, p_{i.2}\}$ measures the amount of *twist*, i.e, the deviation from the above matching situation. We want to consider the length as a real number with sign ($\pm$). Note that a Riemann surface has a natural orientation compatible with the complex coordinates. We give the orientation to $c_i$ as the boundary of the Riemann surface $P_{i,1}$, and measure the hyperbolic length from $p_{i,1}$ to $p_{i,2}$. More precisely, we trace the geodesic arc (that is fixed by the anti-holomorphic involution of $P_{i,1}$) to the point $p_{i,1}$ then turn to the *left* and trace the subarc of $c_i$ of hyperbolic length $t_i$ to the point $p_{i,2}$. We get in this way the *positive* length $t_i$ of the subarc. While if we turn to the *right* at the point $p_{i,1}$ and trace the subarc of $c_i$ of hyperbolic length $l_i - t_i$ to the point $p_{i,2}$, then we get the *negative* length $t_i - l_i$.

More generally, we consider "multiple arcs"on $c_i$ with terminal points $p_{i,1}, p_{i,2}$. Then similar measurement gives various sensed hyperbolic lengths $t_i + nl_i$ mutually different by integral multiples of $l_i$. If we interchange the roles of $P_{i,1}$ and $P_{i,2}$, the explanation is the same, and we get the same set of sensed hyperbolic lengths $t_i + nl_i,\ n \in \mathbf{Z}$. Finally, if we interchange the labeling of the points $p_{i,1}$ and $q_{i,1}$, then we get the set of sensed hyperbolic lengths $t_i + nl_i/2,\ n \in \mathbf{Z}$.

Note that the quantities $l_i$ and $t_i$ depend on the point $p = [R, f] \in T_{g,n}$. Thus we get a continuous function $t_i$ at least locally which has additive ambiguity of integral multiples of $l_i/2$. This function $t_i$ is real analytic.

Usually, we use the function

$$\theta_i = \frac{2\pi t_i}{l_i}$$

instead of $t_i$ itself. This function $\theta_i$ measures the amount of the twist by radians.

Fixing an arbitrary base point $p_0 = [R_0, f_0]$, we get a globally defined continuous function

$$\theta_i : T_{g,n} \to \mathbf{R}$$

by analytic continuation starting from $p_0$. The function $\theta_i$ has additive ambiguity by constant functions $n\pi,\ n \in \mathbf{Z}$.

The functions $\theta_i,\ i = 1, 2, \cdots, m$, are real analytic (see [2], p.87, or [33], Lemma 3.8).

**Theorem 12** ([2], p.91, or [33], Theorem 3.10)**.** *The map*

$$(l_1, \cdots, l_m, \theta_1, \cdots, \theta_m) : T_{g,n} \to (\mathbf{R}^+)^m \times (\mathbf{R})^m$$

*is a real analytic diffeomorphism.*

We call these coordinates the *Fenchel-Nielsen coordinates* of $T_{g,n}$ associated with the pants decomposition $\mathcal{P}$ (or with the system $\mathcal{D}$ of decomposing curves).

Since $l_i$ is constant along $\theta_j$-axes $(i, j = 1, \cdots, m)$, we have

**Corollary 12.1.** *The map*

$$(l_1, \cdots, l_n, t_1, \cdots, t_m) : T_{g,n} \to (\mathbf{R}^+)^m \times (\mathbf{R})^m$$

*is a real analytic diffeomorphism.*

These coordinates are called the *Fenchel-Nielsen coordinates*, too.

9.2. **The Fenchel-Nielsen deformtion.** Let $p = [R, f]$ be a point of $T_{g,n}$, $c$ a simple closed geodesic on $R$. The *Fenchel-Nielsen deformation* of $R$ with respect to $c$ means the family $\{R_t \mid t \in \mathbf{R}\}$ of marked Riemann surfaces $R_t$ obtained by cutting the surface $R$ along $c$, rotating one side of the cut relative to the other and attaching the sides in the new position. (See [75], p.501, or [33], §8.1). The number $t$ measures the magnitude of the twist just as in §9.1. Let us repeat here an explanation similar to the one done in the previous subsection.

Let $S_1$ and $S_2$ be the Riemann sub-surfaces in $R$ each of which has $c$ as a boundary. One bank of $c$ belongs to $S_1$ and the other to $S_2$. Possibly $S_1 = S_2$. Let $A$ be a short geodesic arc on $R$ which cuts $c$ transversely. Then $c$ divides $A$ into $A_1$ and $A_2$, $A_i$ being in $S_i$, $i = 1, 2$. The intersection point $p$ $(= c \cap A)$ is duplicated into $p_1, p_2$ ($p_i$ being a terminal point of $A_i$, $i = 1, 2$). If the magnitude of the twist $t$ is sufficiently small, we can determine $t$ without ambiguity as a real number. For this, give the orientation to $c$ as the boundary of $S_1$. We trace $A_1$ to $p_1$ then turn to the left or to the right according as the short subarc on $c$ whose terminal points are $p_1$ and $p_2$ lies left or right of the point $p_1$, then trace the short subarc to the point $p_2$. The hyperbolic length of this subarc is $|t|$. (Note that $2|t|$ is smaller than the length of $c$.) If we turned to the left (or to the right) at the point $p_1$, then $t > 0$ (or $t < 0$). If we interchange the roles of $S_1$ and $S_2$, we obtain the same result.

Let $p = [R, f]$ be an arbitrary point of $T_{g,n}$, $C$ an essential simple closed curve on $\Sigma_{g,n}$. Let $\{R_t \mid t \in \mathbf{R}\}$ be the Fenchel-Nielsen deformation of $R$ with respect to a simple closed geodesic $c$ freely homotopic to $f(C)$. If we construct a family of quasi-conformal mappings $v_t : R \to R_t$ depending continuously on $t$, $p_t = [R_t, v_t \circ f]$ draws a real analytic curve on $T_{g,n}$ passing $p$ when $t = 0$. Let $\left(\frac{\partial}{\partial t}\right)_p$ be the tangent vector to this curve at $t = 0$. Varying $p$, we get the deformation vector field

$$\left\{ \left( \frac{\partial}{\partial t} \right)_p \right\}_{p \in T_{g,n}}$$

on $T_{g,n}$ representing the infinitesimal Fenchel-Nielsen twist. Wolpert [75] studied this vector field from analytic and geometric viewpoint.

He started with a quasiconformal homeomorphism of $\mathbf{H}$:

$$w_0(z) = \begin{cases} z & \arg z < \theta_1 \\ z e^{\varepsilon(\theta - \theta_1)} & \theta_1 < \arg z = \theta < \theta_2 \\ z e^{\varepsilon(\theta_2 - \theta_1)} & \theta_2 < \arg z, \end{cases} \tag{58}$$

where $0 < \theta_1 < \frac{\pi}{2} < \theta_2 < \pi$ are arbitrarily chozen. We assume $R = \mathbf{H}/\Gamma$, and for simplicity that the imaginary axis $i\mathbf{R}^+$ covers the simple closed geodesic $c$ on $R$. The homeomorphism $w_0$ covers a Fenchel-Nielsen twist about $c$. Note that the twist in (58) is *right-handed* if the magnitude $\varepsilon$ of twist is positive.

It seems that there is no general consensus on whether a right-handed twist is a positive twist or not. The answer depends on the author. The present author prefers to consider a *left-handed twist* to be positive. This standpoint has already been adopted in our previous work [56], [57]. Also the way of determining the sign of the twist coordinate $t$ explained above is consistent with this. The standpoint is in a sense arbitrary, but should be consistent in an author. Thus in the present lecture the author considers as before left-handed twist to be positive. This will cause some discrepancies in the sign between Wolpert's formulas and ours.

Imayoshi and Taniguchi adopt the same standpoint as ours, and in their explanation of Wolpert's work (§8.1 of [33]) they construct a quasiconformal mapping

$w^t$ as follows:

(59) $$w^t = \begin{cases} z & 0 < \theta < \frac{\pi}{2} - \theta_0 \\ z\exp(\frac{-t}{2\theta_0}(\theta - \frac{\pi}{2} + \theta_0)) & \frac{\pi}{2} - \theta_0 \le \theta \le \frac{\pi}{2} + \theta_0 \\ z\exp(-t) & \frac{\pi}{2} + \theta_0 < \theta < \pi, \end{cases}$$

here $\theta = \arg z$. Note that $w^t$ is a left-handed twist if the magnitude $t$ of twist is positive.

Following [75] and [33], we calculate the complex dilatation $\tau_t$ of $w^t$:

(60) $$\tau_t(z) = \frac{-it}{4\theta_0 + it}\chi_I(\theta)\frac{z}{\overline{z}}, \quad z \in \mathbf{H}.$$

Here $\chi(\theta)$ is the characteristic function of $[\frac{\pi}{2} - \theta_0, \frac{\pi}{2} + \theta_0]$.

It is not quite true that a Fenchel-Nielsen twist with respect to $c$ is lifted to $w_0$, because $w_0$ is not $\Gamma$-invariant. To fix this point, Wolpert [75] constructed a $\Gamma$-invariant quasiconformal mapping $v$ by moving $w_0$ by various elements of $\Gamma$, and taking the limit of the finite compositions of these (see [75], p.504). He showed that the complex dilatation of $v$ is the sum of the Beltrami differentials obtained by "moving" the complex dilatation of $w_0$ ([75], p.504-505).

Imayoshi-Taniguchi [33] follows this recipe and obtains the $\Gamma$-invariant Beltrami differential:

(61) $$\mu_t = \sum_{\gamma \in \langle \gamma_0 \rangle \backslash \Gamma} (\tau_t \circ \gamma)\frac{\overline{\gamma'}}{\gamma'},$$

where we assume that $\gamma_0(z) = \lambda z$ ($\lambda > 1$) belongs to $\Gamma$ and covers $c$. Note that the complex dilatation $\tau_t$ is invariant under the action of $\gamma_0$. The Beltrami differential $\{\mu_t\}$ belongs to $B(\mathbf{H}, \Gamma)_1$, and draws a curve $\{\mu_t \mid t \in \mathbf{R}\}$. This curve represents the Fenchel-Nielsen deformation of $R$ with respect to $c$. Then by applying the method of §8.4, we can calculate the tangent vector to the curve $\Phi(\mu_t)$ at $t = 0$.

By differentiation, we get

$$\frac{d}{dt}\tau_t|_{t=0} = -\frac{i}{4\theta_0}\chi_I(\arg z)\frac{z}{\overline{z}}.$$

Set $\nu_0(z) =$ (the right-hand side of the above equality), and set

(62) $$\nu(z) = \sum_{\gamma \in \langle \gamma_0 \rangle \backslash \Gamma} \nu_0(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)}, \quad z \in \mathbf{H}.$$

Then we have

$$\lim_{t \to 0} \left\| \frac{\mu_t}{t} - \nu \right\|_\infty = 0.$$

Ahlfors' formula (47) gives the integral representation of the first variation of $w_{\mu_t}$ ($w_{\mu_t}$ being the normalized solution of the Beltrami equation $w_{\overline{z}} = \mu_t w_z$):

(63) $$\dot{w}(z) = -\frac{1}{\pi}\int_{\mathbf{H}} \nu(\zeta)\frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}d\xi d\eta, \quad z \in \mathbb{C}.$$

Substituting (62) for $\nu$ in the right-hand side, and calculating, we can rewrite (63) as follows:

(64)
$$\dot{w}(z) = \sum_{\gamma \in \langle \gamma_0 \rangle \backslash \Gamma} \left( -\frac{\gamma(z)}{\gamma'(z)}\Big\{ \int_0^{\arg \gamma(z)} \frac{\chi_I(t)}{2\theta_0}dt + \frac{i}{2\pi}\log \gamma(z)\Big\} + \frac{i}{2\pi}P_\gamma(z) \right) \quad z \in \mathbb{C},$$

where $P_\gamma(z)$ is a polynomial of degree at most two. In the calculation, we use the properties of $\dot{w}$ stated in Proposition 3. For details, see [75], pp.513–514, or [33], pp.222–223.

In particular, if $z \in \mathbf{H}^*$, the integral term in (64) vanishes, and we have

$$\dot{w}(z) = -\frac{i}{2\pi} \sum_{\gamma \in \langle \gamma_0 \rangle \backslash \Gamma} \left( \frac{\gamma}{\gamma'} \log \gamma - P_\gamma \right), \quad z \in \mathbf{H}^*. \tag{65}$$

Using (48) and applying Bol's formula

$$\left( \frac{\gamma}{\gamma'} \log \gamma \right)''' = -\left( \frac{\gamma'}{\gamma} \right)^2$$

we have obtained

$$d\Phi_0(\nu) = \frac{i}{2\pi} \sum_{\gamma \in \langle \gamma_0 \rangle \backslash \Gamma} \left( \frac{\gamma'}{\gamma} \right)^2. \tag{66}$$

Both sides belong to $Q(\mathbf{H}^*, \Gamma)$.

So far we have assumed that the simple closed geodesic $c$ is covered by the imaginary axis $i\mathbf{R}^+$. If we take a general $c$ which is covered by a geodesic in $\mathbf{H}$ whose end points are reals, say $\alpha$ and $\beta$, the transformation $B$ ( $\in PSL(2, \mathbf{R})$) which sends this geodesic to $i\mathbf{R}^+$ pulls back the $\langle \gamma_0 \rangle$-invariant quadratic differential $\theta_0(z) = \frac{1}{z^2}$ to

$$\theta_0(B(z))(B'(z))^2 = \frac{(\alpha - \beta)^2}{(z-\alpha)^2 (z-\beta)^2}. \tag{67}$$

For simplicity, we denote the right-hand side of (67) by $\omega_C(z)$. Though the aspect is different, this $\omega_C$ is the same thing as Wolpert's $\omega_C$ in Definition 2.5 of [75]. Put $\gamma_C(z) = B^{-1} \circ \gamma_0 \circ B(z)$. Then $\omega_C$ is invariant under $\gamma_C$.

**Definition 12** (See [75], Definition 2.6)**.** *(The Petersson series)*

$$\Theta_C(z) = \sum_{\gamma \in \langle \gamma_C \rangle \backslash \Gamma} \omega_C(\gamma(z)) \gamma'(z)^2. \tag{68}$$

Since the coefficients of $\Theta_C$ are real numbers, $\Theta_C$ can be considered as an element of $Q(\mathbf{H}, \Gamma)$, and at the same time, as an element of $Q(\mathbf{H}^*, \Gamma)$.

The result for a general $c$ is the following:

**Theorem 13.** *Let $\nu$ be the first order term in the expansion of the Beltrami differential for the Fenchel-Nielsen deformation with respect to $c$, then*

$$d\Phi_0(\nu) = \frac{i}{2\pi} \Theta_C. \tag{69}$$

Note that the sign of the right-hand side is opposite to Wolpert's formula in Theorem 2.7 of [75].

Recall from §8.5 that for any $\psi \in Q(\mathbf{H}^*, \Gamma)$, $\mu_\psi(z) = (\operatorname{Im} z)^2 \psi(\overline{z})$ is a harmonic Beltrami differential which belongs to $B(\mathbf{H}, \Gamma)$. Also $\psi \mapsto -2\mu_\psi$ is a cross section for $d\Phi_0 : B(\mathbf{H}, \Gamma) \to Q(\mathbf{H}^*, \Gamma)$. Namely,

$$d\Phi_0(-2\mu_\psi) = \psi.$$

Applying this to $\Theta_C \in Q(\mathbf{H}^*, \Gamma)$, we get

$$d\Phi_0(-2\mu_{\Theta_C}) = \Theta_C.$$

Since the coefficiens of $\Theta_C$ are reals, we have

$$\Theta_C(\overline{z}) = \overline{\Theta_C(z)}, \quad z \in \mathbf{H}.$$

Thus

$$-2\mu_{\Theta_C}(z) = -2(\operatorname{Im} z)^2\Theta_C(\overline{z}) = -2(\operatorname{Im} z)^2\overline{\Theta_C(z)}, \quad z \in \mathbf{H}$$

and we have

$$d\Phi_0(-2(\operatorname{Im} z)^2\overline{\Theta_C}) = \Theta_C. \tag{70}$$

Combining (69) and (70), we get

$$d\Phi_0\left(\nu + 2(\operatorname{Im} z)^2(\frac{i}{2\pi}\overline{\Theta_C})\right) = 0.$$

Since the kernel of $d\Phi_0 : B(\mathbf{H},\Gamma) \to Q(\mathbf{H}^*,\Gamma)$ is $N(\Gamma)$ we have the following corollary.

**Corollary 13.1.** *The first order term $\nu$ of the Fenchel-Nielsen deformation about $C$ is equivalent to $-\frac{i}{\pi}(\operatorname{Im} z)^2\overline{\Theta_C}$ modulo $N(\Gamma)$.*

This is Corollary 2.8 of [75]. The sign there differs from the above formula.

Miraculously, the quadratic differential $\Theta_C$ appears again in the variational formula of length function. Let $C$ be an essential simple closed curve on $\Sigma_{g,n}$. Let $p = [R, f]$ be an arbitrary point of $T_{g,n}$, where $R = \mathbf{H}/\Gamma$ and $f : \Sigma_{g,n} \to R$ being the marking as before. The *length function* $l_C(p)$ gives the length of the simple closed geodesic $c$ on $R$ freely homotopic to $f(C)$. The differential $dl_C$ is given by the following theorem. Wolpert[75] attributes this formula to Gardiner [25].

**Theorem 14.** *(See* [75]*, Theorem 2.9, and* [33]*, Theorem 8.3.)*

$$(dl_C)_p(\mu) = \left(\mu, \frac{2}{\pi}\Theta_C\right).$$

Here, $\mu$ is any element of $B(\mathbf{H},\Gamma)$ which we consider representing a tangent vector at $p \in T_{g,n}$. The pairing $\left(\mu, \frac{2}{\pi}\Theta_C\right)$ is the tangent-cotangent pairing (see §3.5). Imayoshi and Taniguchi [33], pp.225–226, gives a detailed proof of this theorem.

Recall that in Theorem 13, $\nu(\in B(\mathbf{H},\Gamma))$ was the first term in the expansion of the Beltrami differential for the Fenchel-Nielsen deformation about $c$: $\nu$ is the initial velocity vector of the deformation. Corollary 13.1 says that if the tangent space $T_p(T_{g,n}) = B(\mathbf{H},\Gamma)/N(\Gamma)$ is identified with the harmonic Beltrami differentials $\mathcal{H}$, $\nu$ is identified with the tangent vector $-\frac{i}{\pi}(\operatorname{Im} z)^2\overline{\Theta_C}$. Following Wolpert [81], we denote this tangent vector by $\mathbf{t}_C$, hence

$$\mathbf{t}_C = -\frac{i}{\pi}(\operatorname{Im} z)^2\overline{\Theta_C}. \tag{71}$$

Teichmüller space $T_{g,n}$ is a Riemannian manifold equipped with the Weil-Petersson metric $\langle\cdot,\cdot\rangle$, and we can consider the gradient vector grad $l_C$ of the length function $l_C$. As an element of $\mathcal{H}$, the gradient vector is represented by the following formula:

$$\operatorname{grad} l_C = \frac{2}{\pi}(\operatorname{Im} z)^2\overline{\Theta_C}. \tag{72}$$

*Proof of (72).* For any $\mu \in \mathcal{H}$, we have

$$\begin{aligned}
&\langle \mu, \mathrm{grad}\, l_C \rangle \quad \text{Weil-Petersson product} \\
&= (\mu, dl_C) \quad \text{tangent-cotangent pairing} \\
&= \mathrm{Re} \int_\Delta \mu \, \frac{2}{\pi} \Theta_C \, dxdy \quad \text{by Theorem 14} \\
&= \mathrm{Re} \int_\Delta \mu \, (\mathrm{Im}\, z)^2 \frac{2}{\pi} \Theta_C \, dA \quad dA = \tfrac{dxdy}{(\mathrm{Im}\, z)^2} \text{ is the invariant area form on } \mathbf{H} \\
&= \mathrm{Re} \int_\Delta \mu \, \overline{(\mathrm{Im}\, z)^2 \frac{2}{\pi} \overline{\Theta_C}} \, dA \\
&= \langle \mu, \frac{2}{\pi} (\mathrm{Im}\, z)^2 \overline{\Theta_C} \rangle \quad \text{Weil-Petersson product.}
\end{aligned}$$

Thus $(\mathrm{grad}\, l_C)_p = \frac{2}{\pi} (\mathrm{Im}\, z)^2 \overline{\Theta_C}$. The proof of (72) is complete. □

Combining (71) and (72), we have the following *twist-length duality formula* ([75], Theorem 2.10) :

$$\mathrm{grad}\, l_C = 2i\mathbf{t}_C. \tag{73}$$

The sign of the right-hand side is opposite to the formula stated in [81], p.277.

## 10. **Proof of Theorem** 2

In this section, we will prove Theorem 2 of §1.

10.1. **Riera's formula and Wolpert's evaluation.** Let $p = [R, f]$ be an arbitrary point $\in T_{g,n}$, where $f : \Sigma_{g,n} \to R$ is the marking. Let $C_\alpha$ and $C_\beta$ be disjoint, mutually non-isotopic essential simple closed curves on $\Sigma_{g,n}$. For simplicity, we will denote the geodesic length functions $l_{C_\alpha}$ and $l_{C_\beta}$ by $l_\alpha$ and $l_\beta$ respectively, and the quadratic differentials $\Theta_{C_\alpha}$ and $\Theta_{C_\beta}$ by $\Theta_\alpha$ and $\Theta_\beta$. G. Riera [65] calculated the real part of the Weil-Petersson coproduct of $\Theta_\alpha$ and $\Theta_\beta$. When $c_\alpha$ and $c_\beta$ are disjoint as in the present case, his general formula turns to the following (see [65], Theorem 2):

If $c_\alpha \neq c_\beta$, we have

$$\mathrm{Re}(\langle \Theta_\alpha, \Theta_\beta \rangle_{\mathbb{C}}) = \frac{\pi}{2} \sum_{\langle A \rangle \backslash \Gamma / \langle B \rangle} \left( c \log \left( \frac{c+1}{c-1} \right) - 2 \right), \tag{74}$$

where $c = \cosh \delta$, $\delta$ being the hyperbolic distance from the axis of $A$ to each disjoint axis congruent to the axis of $B$.

If $c_\alpha = c_\beta$, then

$$\mathrm{Re}(\langle \Theta_\alpha, \Theta_\alpha \rangle_{\mathbb{C}}) = \frac{\pi}{2} \left( l_\alpha + \sum_{\langle A \rangle \backslash \Gamma / \langle A \rangle} c \log \left( \frac{c+1}{c-1} \right) - 2 \right), \tag{75}$$

where $c = \cosh \delta$, $\delta$ being the hyperbolic distance from the axis of $A$ to each disjoint axis congruent to it.

Using (72), we get

$$\langle \mathrm{grad}\, l_\alpha, \mathrm{grad}\, l_\beta \rangle = \frac{4}{\pi^2} \mathrm{Re}(\langle \Theta_\alpha, \Theta_\beta \rangle_{\mathbb{C}}). \tag{76}$$

See (55) and (56).

Via (76), we can convert formulas (74) and (75) to the Weil-Petersson product formula of grad $l_\alpha$ and grad $l_\beta$. Wolpert [81] evaluated the expansion of the right-hand sides of (74) and (75), and obtained the following formula valid for sufficiently small $l_\alpha$ and $l_\beta$ (see [81], Lemma 3.12, [82], Theorem 3.7).

$$\langle \mathrm{grad}\, l_\alpha, \mathrm{grad}\, l_\beta \rangle = \frac{2}{\pi} l_\alpha \delta_{\alpha\beta} + O(l_\alpha^2 l_\beta^2). \tag{77}$$

Applying the duality formula (73) to (77), we get

$$\langle \mathbf{t}_\alpha, \mathbf{t}_\beta \rangle = \frac{1}{2\pi} l_\alpha \delta_{\alpha\beta} + O(l_\alpha^2 l_\beta^2). \tag{78}$$

10.2. **Facet** $F^\varepsilon(\mathcal{D})$**.** Let $\mathcal{P}$ be a pants decomposition of $\Sigma_{g,n}$ corresponding to a system of decomposing curves $\mathcal{D} = \{C_1, C_2, \cdots, C_m\}$. Let

$$(l_1, \cdots, l_m, t_1, \cdots, t_m)$$

be the Fenchel-Nielsen coordinates associated with $\mathcal{P}$ (or equivalently with $\mathcal{D}$).

Recall that there exists a universal constant $M > 0$ (*a 2-dimensional Margulis constant*, not unique) such that two distinct simple closed geodesics on a Riemann surface $R$ are disjoint if their lengths are shorter than $M$ (see §4, Lemma 2, and [41], [2], [64] p.655).

Facets $F^\varepsilon(\sigma)$ were defined in Definition 4 of §4, where $\sigma \in \mathcal{C}_{g,n}$. The facets for the maximal simplexes play an essential role in this section, so we will recall the definition in the special case $\sigma = \mathcal{D}$:

**Definition 13.** *Let $\varepsilon$ be a positive number smaller than a Margulis constant $M$. We define* the facet $F^\varepsilon(\mathcal{D})$ *as follows:*

$$F^\varepsilon(\mathcal{D}) = \{p \in T_{g,n} \mid l_i(p) = \varepsilon,\ i = 1, 2, \cdots, m\}$$

*where $l_1, \cdots, l_m$ are the Fenchel-Nielsen length coordinates associated with the maximal simplex $\mathcal{D}$.*

The facet $F^\varepsilon(\mathcal{D})$ is an $m$-dimensional real analytic submanifold. Moreover, recalling Wolpert's formula[10] which represents the Kähler form $\omega$ of $T_{g,n}$ (Wolpert [78], [82])

$$\omega = \frac{1}{2}\sum_{i=1}^{m} dt_i \wedge dl_i, \tag{79}$$

we know that $F^\varepsilon(\mathcal{D})$ is a *Lagrangian submanifold* of $T_{g,n}$, because $\omega | F^\varepsilon(\mathcal{D}) = 0$. The facet $F^\varepsilon(\mathcal{D})$ is parametrized by the Fenchel-Nielsen twist coordinates $(t_1, \cdots, t_m)$ , and is homeomorphic to $\mathbf{R}^m$.

For an essential simple closed curve $C$ on $\Sigma_{g,n}$, let $\tau_C$ denote the right-handed Dehn twist (i.e. negative Dehn-twist in our sense) about $C$. Let $\Gamma(\mathcal{D})$ denote the subgroup of the mapping class group $\Gamma_{g,n}$ generated by the Dehn twists $\tau_i = \tau_{C_i}$ about the simple closed curves $C_i,\ i = 1, \cdots, m$ of $\mathcal{D}$. $\Gamma(\mathcal{D})$ is a free abelian group of rank $m$. By (40), $\tau_i$ acts on $T_{g,n}$ by

$$\tau_i([R, f]) = [R, f \circ \tau_i^{-1}].$$

Since $\tau_i^{-1}$ is a left-handed Dehn twist (i.e., positive Dehn twist from our viewpoint) about $\mathcal{C}_i$, we have

$$t_i(\tau_i(p)) = t_i(p) + \varepsilon, \quad t_j(\tau_i(p)) = t_j(p),\ \ j \neq i, \quad \text{where } \ p = [R, f]. \tag{80}$$

The action of $\tau_i$ preserves $F^\varepsilon(\mathcal{D})$ because it preserves the length functions, $l_j,\ j = 1, \cdots, m$.

Here we remark that the action of $\Gamma(\mathcal{D})$ extends to the augmented Teichmüller space $\hat{T}_{g,n}$ , and that it fixes a unique boundary point $p(\mathcal{D})$ corresponding to the maximally degenerate Riemann surface, obtained by pinching each curve of $\mathcal{D}$ to a node (see [83], Theorem 3 and Remark on the same page). The facet $F^\varepsilon(\mathcal{D})$ shrinks to $p(\mathcal{D})$ as $\varepsilon \to 0$.

Let $N\Gamma(\mathcal{D})$ be the normalizer of $\Gamma(\mathcal{D})$ in the mapping class group $\Gamma_{g,n}$. The following theorem is nothing but a restatement of Theorem 6 in §4, in the special case where the simplex $\sigma$ is a maximal one $\mathcal{D}$.

**Theorem 15.** *Suppose $m = 3g - 3 + n \geqq 1$. A mapping class $[h]$ belongs to the normalizer $N\Gamma(\mathcal{D})$ if and only if $[h]$ permutes the isotopy classes of the curves $C_i, i = 1, \cdots, m$ of $\mathcal{D}$.* □

[10]The formula displayed in (79) differs in the sign from [78] and [82] Theorem 3.14, as was explained in §9.2 (between the formulas (58) and (59)). It also differs in the coefficients from [78], because [78] defined the Weil-Petersson cometric by $\langle \phi, \psi \rangle = \frac{1}{2}\int_\Delta (\text{Im } z)^2 \phi\overline{\psi} dxdy$, while [82] Definition 2.4 defined it by $\langle \phi, \psi \rangle = \int_\Delta (\text{Im } z)^2 \phi\overline{\psi} dxdy$ and we followed the latter definition. See (54).

**Corollary 15.1.** *The action of $N\Gamma(\mathcal{D})$ preserves the facet $F^\varepsilon(\mathcal{D})$.*

This corollary is a special case of Corollary 6.1.

Let $\Sigma_{g,n}(\mathcal{D})$ denote as before the topological space obtained from $\Sigma_{g,n}$ by pinching each curve $C_i$ of $\mathcal{D}$ to a point (node). $\Sigma_{g,n}(\mathcal{D})$ is a surface with nodes (a *chorizo space*, [57] ).

Let $W(\mathcal{D})$ denote the quotient group $N\Gamma(\mathcal{D})/\Gamma(\mathcal{D})$.

The following corollary is a special case of Corollary 6.2.

**Corollary 15.2.** *The group $W(\mathcal{D})$ is the mapping class group of $\Sigma_{g,n}(\mathcal{D})$.*

It is known that the mapping class group of $\Sigma_{g,n}(\mathcal{D})$ is a finite group. Thus we have

**Corollary 15.3.** *The quotient $N\Gamma(\mathcal{D})/\Gamma(\mathcal{D})$ is a finite group.*

$N\Gamma(\mathcal{D})$ acts on the Lagrangian submanifold $F^\varepsilon(\mathcal{D})$ isometrically. Its normal subgroup $\Gamma(\mathcal{D})$ acts as "translations", and the quotient $N\Gamma(\mathcal{D})/\Gamma(\mathcal{D})$ is finite. At this stage, we might well call $N\Gamma(\mathcal{D})$ a *Lagrangian analogue of a crystallographic group.*

A general result is known that *if an abstract group $G_0$ has a free abelian normal subgroup $T_0$ of finite rank which is maximal abelian and which has finite index, then $G_0$ is isomorphic to a Euclidean crystallographic group.* (See Theorem on p.31 of [69]). Applying this, we know that the group $N\Gamma(\mathcal{D})$ is isomorphic to some Euclidean crystallographic group. But the following theorem *directly* shows that the Lagrangian crystallographic group $N\Gamma(\mathcal{D})$ acting on $F^\varepsilon(\mathcal{D})$ becomes a Euclidean crystallographic group in the limit of $\varepsilon \to 0$ :

**Theorem 16.** *Assume $m = 3g - 3 + n > 1$. Then the group $N\Gamma(\mathcal{D})$ is a crystallographic group acting on a certain "ideal"Euclidean $m$-space $\mathbf{E}^m$.*

**Remark** (See Lemma 20 below). In the case of $(g, n) = (2, 0)$ or $(1, 2)$, we consider $N\Gamma(\mathcal{D})$ as a subgroup of $Aut(T_{g,n}) = \Gamma_{g,n}/\mathbf{Z}_2$. $\square$

*Proof of Theorem 16.* We use the twist-twist pairing (78), where $\mathbf{t}_i$ stands for $\frac{\partial}{\partial t_i}$. $F^\varepsilon(\mathcal{D})$ becomes a Riemannian manifold with the Weil-Petersson metric:

$$\langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \rangle = \frac{\varepsilon}{2\pi}\delta_{ij} + O(\varepsilon^4). \tag{81}$$

Let $(x_1, \cdots, x_m)$ denote the natural coordinates of $\mathbf{R}^m$. We define a diffeomorphism $f_\varepsilon : \mathbf{R}^m \to F^\varepsilon(\mathcal{D})$ by setting

$$\begin{aligned}(t_1, \cdots, t_m) &= f_\varepsilon(x_1, \cdots, x_m)\\ &= (\varepsilon x_1, \cdots, \varepsilon x_m).\end{aligned}$$

Let us define a metric $\langle \cdot, \cdot \rangle_\varepsilon$ on $\mathbf{R}^m$ by rescaling the metric induced from the Weil-Petersson metric under $f_\varepsilon$:

$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_\varepsilon \stackrel{\text{def}}{=} \frac{2\pi}{\varepsilon^3} \langle df_\varepsilon(\frac{\partial}{\partial x_i}), df_\varepsilon(\frac{\partial}{\partial x_j}) \rangle. \tag{82}$$

Since

$$df_\varepsilon(\frac{\partial}{\partial x_i}) = \varepsilon \frac{\partial}{\partial t_i},$$

we have

$$\begin{aligned}\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle_\varepsilon &= \frac{2\pi}{\varepsilon^3}\langle \varepsilon\frac{\partial}{\partial t_i}, \varepsilon\frac{\partial}{\partial t_j}\rangle \\ &= \frac{2\pi}{\varepsilon}\left(\frac{\varepsilon}{2\pi}\delta_{ij} + O(\varepsilon^4)\right) \quad \text{by (81)} \\ &= \delta_{ij} + O(\varepsilon^3).\end{aligned}$$

This implies the following

**Lemma 19.** *As $\varepsilon$ approaches 0, the metric $\langle\cdot,\cdot\rangle_\varepsilon$ on $\mathbf{R}^m$ converges to the Euclidean metric.*

We consider the action of $N\Gamma(\mathcal{D})$ on $\mathbf{R}^m$ induced by $f_\varepsilon$:

$$\gamma(p) := f_\varepsilon^{-1}(\gamma(f_\varepsilon(p))), \quad \forall p \in \mathbf{R}^m, \forall \gamma \in N\Gamma(\mathcal{D}). \tag{83}$$

The group $N\Gamma(\mathcal{D})$ acts on $F^\varepsilon(\mathcal{D})$ as isometries with respect to the Weil-Petersson metric. Thus the induced action of $N\Gamma(\mathcal{D})$ on $\mathbf{R}^m$ is isometric with respect to the induced metric $\langle\cdot,\cdot\rangle_\varepsilon$. As we saw above, the metric $\langle\cdot,\cdot\rangle_\varepsilon$ converges to the Euclidean metric as $\varepsilon \to 0$. Therefore, in the limit, the action of $N\Gamma(\mathcal{D})$ on $\mathbf{R}^m$ becomes an isometric action on Euclidean $m$-space $\mathbf{E}^m$. Note that the induced action of the abelian subgroup $\Gamma(\mathcal{D})$ is coincident with the natural action of the lattice $\mathbf{Z}^m$. Let $T$ be the translation subgroup of $N\Gamma(\mathcal{D})$. Then

$$\mathbf{Z}^m \subset T \subset N\Gamma(\mathcal{D}).$$

Since $N\Gamma(\mathcal{D})/\mathbf{Z}^m$ is finite by Corollary 15.3, we conlude that $N\Gamma(\mathcal{D})/T$ is finite. Thus $N\Gamma(\mathcal{D})$ is a crystallographic group acting on $\mathbf{E}^m$. The proof of Theorem 16 is complete. □

**Proposition 4.** *Suppose $3g-3+n > 1$. For the action of $\Gamma_{g,n}$ on $\hat{T}_{g,n}$, the isotropy subgroup of the frontier point $p(\mathcal{D})$ is $N\Gamma(\mathcal{D})$.*

*Proof.* For a (not necessarily maximal) simplex $\sigma \in \mathcal{C}_{g,n}$, let $T(\sigma)$ be the Teichmüller space of nodal Riemann surfaces modeled on $\Sigma_{g,n}(\sigma)$. (See §3.) For two simplexes, $\sigma, \tau \in \mathcal{C}_{g,n}$, Lemma 17 states that $T(\sigma) \cap T(\tau) \neq \emptyset$ in $\hat{T}_{g,n}$ if and only if $\sigma = \tau$. (See Lemma 17 (ii) in §6 and its proof.)

Now in the present case, we have a maximal simplex $\mathcal{D}$, and in this case the Teichmüller space $T(\mathcal{D})$ is a point (because $\Sigma_{g,n}(\mathcal{D})$ admits no deformation), and this point is nothing but the point denoted by $p(\mathcal{D})$.

Let $[h] \in \Gamma_{g,n}$ be a mapping class. Using the rule (18) of §4, we have

$$[h](p(\mathcal{D})) = [h](T(\mathcal{D})) \overset{(18)}{=} T(h(\mathcal{D})) = p(h(\mathcal{D})).$$

Thus $[h](p(\mathcal{D})) = p(\mathcal{D})$ if and only if $h(\mathcal{D}) = \mathcal{D}$. By Theorem 15, this last condition is equivalent to that $[h] \in N\Gamma(\mathcal{D})$. This proves Proposition 4. □

*Proof of Theorem 2.* Now Theorem 2 is obtained by combining Theorem 16 and Proposition 4. □

Before proving Corollary 2.1 to Theorem 2 (of §1), we will prove the following (well-known) result.

**Lemma 20.** *Suppose $3g-3+n > 0$ and $(g,n) \neq (2,0), (1,2), (1,1), (0,4)$. Then the natural homomorphism $\alpha : \Gamma_{g,n} \to \mathrm{Aut}(T_{g,n})$ is injective. If $(g,n) = (2,0), (1,2)$*

*or* $(1,1)$*, the kernel of* $\alpha : \Gamma_{g,n} \to \mathrm{Aut}(T_{g,n})$ *is isomorphic to* $\mathbf{Z}_2$*. If* $(g,n) = (0,4)$*, the kernel of* $\alpha : \Gamma_{0,4} \to \mathrm{Aut}(T_{0,4})$ *is isomorphic to* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$*.*

*Proof.* Suppose that the kernel of $\alpha : \Gamma_{g,n} \to \mathrm{Aut}(T_{g,n})$ is non-trivial. Let $\gamma$ be a non-trivial element of $Ker(\alpha)$. Take an arbitrary pants decomposition $\mathcal{P}$ of $\Sigma_{g,n}$ with decomposing curve system $\mathcal{D} = \{C_1, \cdots, C_m\}$.

Since the action of $\gamma$ on $T_{g,n}$ is the identity $id$, it preserves the facet $F^{\varepsilon}(\mathcal{D})$. Let $p = [R, f]$ be a point of $F^{\varepsilon}(\mathcal{D})$. The simple closed geodesics $c_1, \cdots, c_m$ on $R$ which are freely homotopic to $f(C_1), \cdots, f(C_m)$ have the same geodesic length $\varepsilon$. Since $\gamma(p) = p$, $\gamma$ maps $R$ to itself, and we may assume $\gamma$ preserves the Poincaré metric on $R$. Let $c_i$ be a closed geodesic from $\{c_1, \cdots, c_m\}$. Then $\gamma(c_i)$ has the length $\varepsilon$. We took $\varepsilon$ smaller than a Margulis constant $M$. Thus $\gamma(c_i)$ is disjoint of $c_1, \cdots, c_m$, unless it coincides with one of them. Recall that $m$ is the maximal number of disjoint, mutually non-isotopic essential simple closed curves on $R$. Then the simple closed geodesic $\gamma(c_i)$ cannot be disjoint of all the curves $c_1, \cdots, c_m$, and it must coincide with one of the decomposing curves $c_j$. We took $i$ arbitrarily from $\{1, \cdots, m\}$, thus $\gamma$ permutes the curves of $\mathcal{D}$. But $\gamma$ acts as the identity of $T_{g,n}$. Thus $\gamma$ induces the trivial permutation of $\mathcal{D}$ (Otherwise, $\gamma$ would induce a non-trivial permutation of the Fenchel-Nielsen axes $l_i, t_i$).

Let $P$ be a pair of pants arbitrarily chosen from $\mathcal{P}$.

(1) First, suppose $P$ is an ordinary pair of pants. Let $c_1, c_2, c_3$ be the boundary curves of $P$.

Recall that $\gamma$ is a non-trivial element of $Ker(\alpha)$. Two subcases are possible:

(1-i) $\gamma$ maps the pants $P$ to another pair of pants $P'$, or

(1-ii) $\gamma$ maps $P$ to itself, and interchanges two boundary curves, say $c_1$ and $c_2$.

In case (1-i), the image $P'$ of $P$ under $\gamma$ must have the same boundary curves as $P$, because $\gamma$ induces the trivial permutation of $\mathcal{D}$. As a consequence, $\Sigma_{g,n}$ is the double $P \cup P'$, which is a closed surface of genus two $\Sigma_{2,0}$, and $\gamma$ has order 2: it interchanges $P$ and $P'$.

In case (1-ii), $\gamma$ interchanges $c_1$ and $c_2$, and induces a $180°$ rotation on $c_3$. The curves $c_1$ and $c_2$ were the same curve before the pants decomposition, and through the curve $c_3$, $P$ must be connected to another pair of pants $P'$. The $180°$ rotation of $c_3$ is extended to $P'$.

If $P'$ is again an ordinary pair of pants, with two boundary curves $c_1'$ and $c_2'$ other than $c_3$, $\gamma$ again interchanges $c_1'$ and $c_2'$, which were the same curve before the pants decomposition. Therefore, the original surface $\Sigma_{g,n}$ is obtained by pasting $P$ and $P'$ and gluing together $c_1$ and $c_2$, and $c_1'$ and $c_2'$, respectively. The resulting surface is again a closed surface of genus 2, and $\gamma$ acts as a hyperelliptic involution. (Note that $\gamma$ does not cause any twist about $c_3$. Otherwise, $\gamma$ would induce a nontrivial action on $T_{g,n}$ along the twist axis corresponding to $c_3$.)

If $P'$ is a twice punctured disk, the $180°$ rotation of $c_3$ is extended to an involution of $P'$ which interchanges the punctures. The original $\Sigma_{g,n}$ is obtained by pasting $P$ and $P'$ along $c_3$, and pasting together $c_1$ and $c_2$. The resulting surface is a twice punctured torus, and $\gamma$ acts as an involution interchanging the two punctures.

(2) Secondly, we consider the case where $P$ is a once punctured annulus. Let $c_1$ and $c_2$ be the boundary curves of $P$. There are two cases to be considered.

(2-i) $\gamma$ maps $P$ to another once punctured annulus $P'$, or

(2-ii) $\gamma$ maps $P$ to itself, and permutes $c_1$ and $c_2$.

In case (2-i), $P'$ has the same boundary curves $c_1$ and $c_2$. The surface $\Sigma_{g,n}$ is obtained by pasting $P$ and $P'$ along $c_1$ and $c_2$. The resulting surface is a twice punctured torus $\Sigma_{1,2}$, and $\gamma$ acts as an involution interchanging $P$ and $P'$.

In case (2-ii), $\gamma$ acts on the once punctured annulus $P$ as an involution interchanging the boundary curves $c_1$ and $c_2$.This time, $c_1$ and $c_2$ had to be the same curve before the decomposition, and $\Sigma_{g,n}$ is obtained by gluing together $c_1$ and $c_2$. The resulting surface is a once punctured torus $\Sigma_{1,1}$ and $\gamma$ acts as a non-trivial involution which induces a reflection on the curve $c_1 = c_2$.

(3) Finally, suppose $P$ is a twice punctured disk. There are two cases to be considered:

(3-i) $\gamma$ maps $P$ to another twice punctured disk $P'$.

(3-ii) $\gamma$ maps $P$ to itself and induces a $180^\circ$ rotation to the boundary curve $c_1$ of $P$.

In case (3-i), the surface $\Sigma_{g,n}$ is obtained by pasting $P$ and $P'$ along $c_1$, and the resulting surface is a four times punctured sphere $\Sigma_{0,4}$. $\gamma$ acts as an involution interchanging $P$ and $P'$.

In case (3-ii), $\gamma$ acts as an involution of $P$ which induces a $180^\circ$ rotation on the boundary curve $c_1$. Let $P'$ be the neighbor of $P$ having the same boundary curve $c_1$. The involution $\gamma$ is extended to an involution of $P'$. If $P'$ is an ordinary pair of pants, the involution interchanges the remaining two boundary curves of $P'$ as in the case (1-ii). Thus the original surface $\Sigma_{g,n}$ is obtained by attaching a twice punctured disk to a boundary curve of a pair of ordinary pants whose remaining boundary curves are pasted together. The resulting surface is a twice punctured torus $\Sigma_{1,2}$. $\gamma$ acts on $\Sigma_{1,2}$ as an involution which interchanges the two punctures.

If the neighbor $P'$ is a twice punctured disk, the involution $\gamma$ of $P$ is extended to $P'$ as an involution which interchanges the two punctures of $P'$. The original surface is obtained by pasting $P$ to $P'$ along $c_1$. The resulting surface is a four times punctured sphere $\Sigma_{0,4}$. The involution $\gamma$ induces a $180^\circ$ rotation on $c_1$, and interchanges two punctures on $P$ and $P'$, respectively. This involution on $\Sigma_{0,4}$ is of different type from the involution on $\Sigma_{0,4}$ discussed in case (3-i). Thus the kernel of $\alpha : \Gamma_{0,4} \to \mathrm{Aut}(T_{0.4})$ is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ generated by two mutually commutative involutions.

Summarizing the above argument, we get Lemma 20. $\square$

*Proof of Corollary 2.1.* Suppose $3g - 3 + n > 1$, and $(g, n) \neq (2, 0), (1, 2)$. Then by Lemma 20, $\alpha : \Gamma_{g,n} \to \mathrm{Aut}(T_{g,n})$ is injective. On the other hand, by Royden's theorem [67] and its generalization [23], the homomorphism $\alpha : \Gamma_{g,n} \to \mathrm{Aut}(T_{g,n})$ is surjective. Thus under the condition of Corollary 2.1, $\Gamma_{g,n}$ is isomorphic to $\mathrm{Aut}(T_{g,n})$. This fact together with Theorem 2 proves Corollary 2.1. $\square$

10.3. **A natural representation of the point group.** In this subsection, we will show that there is another convergent process giving a natural representation of the point group $\overrightarrow{N\Gamma(\mathcal{D})}$. Let us define a diffeomorphism $f_{\sqrt{\varepsilon}} : \mathbf{R}^m \to F^\varepsilon(\mathcal{D})$ by setting

$$\begin{aligned}(t_1, \cdots, t_m) &= f_{\sqrt{\varepsilon}}(x_1, \cdots, x_m)\\ &= (\sqrt{\varepsilon}x_1, \cdots, \sqrt{\varepsilon}x_m)\end{aligned}$$

Also define a metric $\langle \cdot, \cdot \rangle_{\sqrt{\varepsilon}}$ on $\mathbf{R}^m$ as follows:

$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_{\sqrt{\varepsilon}} \stackrel{\text{def}}{=} \frac{2\pi}{\varepsilon^2} \langle df_{\sqrt{\varepsilon}} \left( \frac{\partial}{\partial x_i} \right), df_{\sqrt{\varepsilon}} \left( \frac{\partial}{\partial x_j} \right) \rangle. \tag{84}$$

As before, the right-hand side is the Weil-Petersson metric on $F^{\varepsilon}(\mathcal{D})$.

Since

$$df_{\sqrt{\varepsilon}} \left( \frac{\partial}{\partial x_i} \right) = \sqrt{\varepsilon} \frac{\partial}{\partial t_i},$$

we have

$$\begin{aligned} \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_{\sqrt{\varepsilon}} &= \frac{2\pi}{\varepsilon^2} \langle \sqrt{\varepsilon} \frac{\partial}{\partial t_i}, \sqrt{\varepsilon} \frac{\partial}{\partial t_j} \rangle \\ &= \frac{2\pi}{\varepsilon} \left( \frac{\varepsilon}{2\pi} \delta_{ij} + O(\varepsilon^4) \right) \quad \text{by } (81) \\ &= \delta_{ij} + O(\varepsilon^3). \end{aligned}$$

Thus the induced metric on $\mathbf{R}^m$ converges to the Euclidean metric.

We define the induced action of $N\Gamma(\mathcal{D})$ on $\mathbf{R}^m$ via $f_{\sqrt{\varepsilon}}$:

$$\gamma(p) = f_{\sqrt{\varepsilon}}^{-1}(\gamma(f_{\sqrt{\varepsilon}}(p))) \quad \forall p \in \mathbf{R}^m, \forall \gamma \in N\Gamma(\mathcal{D}).$$

This action is isometric with respect to the induced metric $\langle \cdot, \cdot \rangle_{\sqrt{\varepsilon}}$. Thus as before, the action converges to a Euclidean isometry group as $\varepsilon \to 0$. But this time, the action of $\tau_i$ is pulled back to the following translation:

$$(x_1, \cdots, x_i, \cdots, x_m) \mapsto (x_1, \cdots, x_i + \sqrt{\varepsilon}, \cdots, x_m),$$

and the free abelian group $\Gamma(\mathcal{D})$ is pulled back to a "fine" lattice $(\sqrt{\varepsilon}\mathbf{Z})^m$. This implies that in the limit of $\varepsilon \to 0$ the action of $\Gamma(\mathcal{D})$ becomes stationary. Let $T$ be the translation subgroup of $N\Gamma(\mathcal{D})$. Then for $\varepsilon > 0$, the fundamental region of $(\sqrt{\varepsilon}\mathbf{Z})^m$ is covered by a finite number of copies of the fundamental region of $T$. Thus as $\varepsilon$ tends to 0, the action of $T$ becomes stationary too. This implies that in the limit we naturally get an isometric action of the quotient group $N\Gamma(\mathcal{D})/T$ on $\mathbf{E}^m$, and this quotient is nothing but the point group $\overrightarrow{N\Gamma(\mathcal{D})}$.

## 11. **Examples**

11.1. **Closed surface of genus 2.** Let us consider an oriented closed surface $\Sigma_2$ of genus 2. There are two kinds of pants decompositions of $\Sigma_2$. (See Fig. 1.)

These decompositions are represented by trivalent graphs, whose vertices represent pairs of pants, and the edges decomposing curves. (See Fig. 2.)

We denote the corresponding crystallographic groups by $G_0$ (left one), and $G_1$ (right one).

The translation subgroup $T(G_0)$ of $G_0$ is generated by $\langle a\rangle$, $\langle b\rangle$ and $\frac{1}{2}\langle c\rangle$, where $\langle a\rangle$ denotes the Dehn twist about the decomposing curve $(a)$ corresponding to the edge $a$. Similarly for $\langle b\rangle$ and $\langle c\rangle$. The coefficient $\frac{1}{2}$ of $\langle c\rangle$ stands for the half-twist around the decomposing curve $(c)$. Now the left-right reflection of the graph induces an orientation preserving homeomorphism of $\Sigma_2$ which reverses the orientation of the curve $(c)$. But the Dehn twist does not depend on the orientation of the curve. Thus the left-right reflection of the graph leaves $\langle c\rangle$ fixed. It interchanges $\langle a\rangle$ and $\langle b\rangle$. While the up-down reflection of the graph induces the so-called "hyperelliptic involution"of $\Sigma_2$, which belongs to the kernel of $\Gamma_2 \to Aut(T_2)$. (Lemma 20.)

Therefore, the point group $\overrightarrow{G_0}$ is $D_1$, the dihedral group of order 2. The generator is the reflection of $\mathbf{E}^3$ with respect to the plane containing the axis corresponding to $\langle c\rangle$, which interchages the axes corresponding to $\langle a\rangle$ and $\langle b\rangle$.

Recall that the 3-dimensional crystallographic groups are classified into 32 crystal classes, according to the types of spherical orbifolds produced by the point groups. These 32 crystal classes are divided into 7 crystal systems according to the number of the axes or reflection planes which are contained in the point groups (see [59], [42], [18], and https://en.wikipedia.org/wiki/Crystal_system). The above crystallographic group $G_0$ belongs to the *domatic crystal class* (whose spherical orbifold is a disk with *silvered boundary* in the sense of [55]) which is contained in the *monoclinic crystal system*. More specifically, J. M. Montesinos-Amilibia identified the group $G_0$ as the group 2/2/2/01 in the notation of H. Brown, R. Bülow, J, Neubüser, H. Wondratschek and H. Zassenhaus [18], namely the group 8 in the International Table of Crystallography (IT for short). (See p.62 of [18].) Note that Brown et al. [18] classify the crystallographic groups up to the conjugation of affine groups. An example of an actual mineral crystal representing the group IT 8 is "Tsepinite-Na". (Figs. 3, 4. See also https://www.mindat.org/min-11013.html or

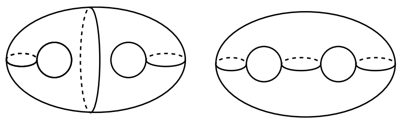

FIGURE 1. Two pants decompositions

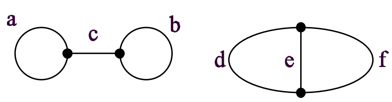


FIGURE 2. Corresponding trivalent graphs

http://www.webmineral.com/data/Tsepinite-Na.shtml#.XBi5nxB7mEl). Tsepinite was named for Anatoliy I. Tsepin (1946–), Russian microprobe analyst.

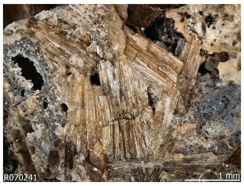


FIGURE 3. Tsepinite-Na

FIGURE 4. Tsepinite-Na, a closer look

In the group $G_1$, the translation subgroup $T(G_1)$ is generated by $\langle d \rangle$, $\langle e \rangle$ and $\langle f \rangle$. The dihedral group of order 6, $D_3$, permutes the edges, thus permutes the axes of $\mathbf{E}^3$ corresponding to $\langle d \rangle, \langle e \rangle$ and $\langle f \rangle$. The point group $\overrightarrow{G_1}$ is isomorphic to $D_3$, and the corresponding spherical orbifold is $D_{\overline{33}}$. The crystallographic group $G_1$ belongs to the *ditrigonal pyramidal crystal class* which is contained in the *trigonal crystal system*. According to Montesinos, $G_1$ is the group 5/4/1/01 in the notation of Brown et al. and is identified as the group IT 160. See [18], p.71. An example of an actual mineral crystal representing this group is "Tourmaline". (See [59], Fig. 52, and https://www.mindat.org/min-4003.html.) The pictures Figs. 5, 6, and 7 were taken by J. M. Montesinos-Amilibia.

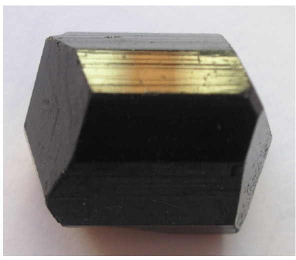

FIGURE 5. Tourmaline: Two ends are different

FIGURE 6. One end of a specimen of tourmaline, a different one from Figure 5

FIGURE 7. The other end of the same specimen as in Figure 6

11.2. **Closed surface of genus 3.** The closed surface $\Sigma_3$ has 5 types of pants decompositions. See Fig. 8.

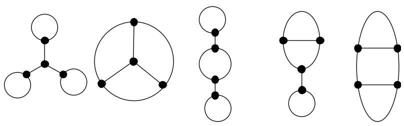

FIGURE 8. Five types of decompositions of $\Sigma_3$

The considerations similar to genus 2 case give the point groups of the corresponding 6-dimensional crystallographic groups. They are from left to right (in Fig. 8) $D_3$, $S_4$ (symmetric group on 4 elements), $D_1 \times D_1$, $D_1 \times D_1$, and $D_1 \times D_1 \times D_1 \times D_1$.

## References


[1] W. Abikoff, *Augmented Teichmüller spaces*, Bull. Amer. Math. Soc., **82** (1976), 333-334.

[2] W. Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Math. Springer-Verlag **820**, 1976.

[3] W. Abikoff, *Degenerating families of Riemann surfaces*, Ann. Math., **105** (1977), 29–44.

[4] L. V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. Math., **74** (1961), 171–191.

[5] L. V. Ahlfors, *Curvature properties of Teichmüller space*, J. d'Analyse Math., **9** (1961), 161–176.

[6] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, D. Van Nostrand, Princeton, N. J. 1966. Wadsworth, Inc., Belmont, California 94002, 1987.

[7] L. V. Ahlfors and L. Bers, *Riemann mapping theorem for variable metrices*, Ann. Math., **72** (1960), 385–404.

[8] L. Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc., **66** (1960), 94–97.

[9] L. Bers, *Correction to "Spaces of Riemann surfaces as bounded domains"*, Bull. Amer. Math. Soc., **67** (1961), 465–466.

[10] L. Bers, *A remark on Mumford's compactness theorem*, Israel J. Math., **12** (1972), 400–407.

[11] L. Bers, *Fiber spaces over Teichmüller spaces*, Acta Math., **130** (1973), 89–126.

[12] L. Bers, *Spaces of degenerating Riemann surfaces*, in "Discontinuous groups amd Riemann surfaces"(Proc. of the 1973 Conf. Univ. of Maryland), ed. by L. Greenberg, Princeton U. P. (1974), 43–55.

[13] L. Bers, *On spaces of Riemann surfaces with nodes*, Bull. Amer. Math. Soc., **80** (1974), 1219–1222.

[14] J. S. Birman, *Braids, Links, and Mapping Class Groups*, Ann. Math. Studies, **82**, Princeton U. P. 1975.

[15] F. Bonahon and L. Siebenmann, *The characteristic toric splitting of irreducible compact 3-orbifolds*, Math. Ann., **278** (1987), 441–479.

[16] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*, Éléments de Mathématique, Hermann, 1968.

[17] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer, 1999.

[18] H. Brown, R. Bülow, J, Neubüser, H. Wondratschek and H. Zassenhaus, *Crystallographic groups of four-dimensional space*, Wiley monographs in crystallography, John Wiley & Sons, Inc., 1978.

[19] H. Cartan, *Quotient d'un espace analytique par un groupe d'automorphismes*, "Algebraic geometry and topology", A Symposium in honor of S. Lefschetz, ed. by R. H. Fox, D. C. Spencer, and A. W. Tucker, Princeton Math. Series, **12**, Princeton U. P., (1957), 90–102.

[20] T. Chu, *The Weil-Peterssen Metric in the moduli space*, Chinese J. Math., **4** (1976), 29–51.

[21] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math., **36** (1969), 75–109.

[22] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math., **115** (1966), 83–107.

[23] C. J. Earle and I. Kra, *On holomorphic mappings between Teichmüller spaces*, in "Contributions to Analysis"(A collection of papers dedicated to Lipman Bers), ed. by L. V. Ahlfors, I. Kra, B. Maskit, and L. Nirenberg, Academic Press, (1974), 107–124.

[24] C. J. Earle and A. Marden, *Holomorphic plumbing coordinates*, in "Quasiconformal Mappings, Riemann Surfaces, and Teichmüller spaces", ed. by Y. Jiang and S. Mitra, Contemporary Math., **575** (2012), 41–52.

[25] F. P. Gardiner, *Schiffer's interior variation and quasiconformal mapping*, Duke Math. J., **42** (1975), 371–380.

[26] W. Harvey, *Chabauty spaces of discrete groups*, in "Discontinuous groups amd Riemann surfaces"(Proc. of the 1973 Conf. Univ. of Maryland), ed. by L. Greenberg, Princeton U. P. (1974), 239–246.

[27] W. Harvey, *Spaces of discrete groups*, in "Discrete groups and automorphic functions"(Proc. Conf., Cambridge, 1975), Academic Press, 1977, 295–348.

[28] W. Harvey, *Geometric structure of surface mapping class groups* in "Homological Group Theory"ed. by C. T. C. Wall, London Math. Soc. Lecture Note Series, **36** (1979), 255–269.

[29] V. Hinich, *Plumbing coordinates on Teichmüller space: A counterexample*, Israel J. of Mathematics, **175** (2010), 151–156.

[30] V. Hinich and A. Vaintrob, *Augmented Teichmüller spaces and orbifolds*, Selecta Mathematica, New Series, **16** (2010), 533–629.

[31] J. H. Hubbard, *Teichmüller theory*, **Vol. 1**, Matrix Editions, 2006.

[32] J. H. Hubbard and S. Koch, *An analytic construction of the Deligne-Mumford ompactification of the moduli space of curves*, J. Diff. Geom., **98** (2014), 261–313.

[33] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer-Verlag, 1992.

[34] A. Ishida, *The structure of Subgroup of Mapping Class Groups Generated by Two Dehn Twists*, Proc. Japan Acad., **72** Ser. A (1996), 240-241.

[35] N. V. Ivanov, *Complexes of curves and the Teichmüller modular groups*, Russian Math. Surveys **42:3** (1987), 55–107.

[36] N. V. Ivanov, *Automorphisms of complex of curves and of Teichmüller spaces*, Int. Math. Research Notes, **14** (1997), 651-666.

[37] B. Iversen, *Lectures on crystallographic groups*, Lecture Notes Series 1990/91 No.60, Matematisk Institut, Aarhus Universitet.

[38] L. Ji and S. A. Wolpert, *A cofinite universal space for proper actions for mapping class groups*, Contemporary Math., **510** (2010), 151–163.

[39] S. Kamada and Y. Matsumoto, *Certain racks associated with the braid groups*, in "Knots in Hellas '98"(Proc. of the International Conf. on Knot Theory and Its Ramifications), ed. by C. McA. Gordon, V. F. R. Jones, L. H. Kauffman. S. Lambropoulou, and J. H. Przytycki. World Scientific. 2000. 118–130.

[40] T. Kawasaki, 文様の幾何学 ー文様における群作用と対称性 (*Geometry of patterns - Group action and symmetry in patterns*), 牧野書店, Tokyo, 2014.

[41] L. Keen, *Collars on Riemann surfaces*, in "Discontinuous Groups and Riemann surfaces", Proc. of the 1973 Conf. at Univ. of Maryland (ed. by L. Greenberg), Ann. Math. Studies, **79** (1974), 263–268.

[42] T. Kohno, 結晶群 (*Crystallographic groups*), 共立出版, Tokyo, 2015.

[43] M. Korkmaz, *Automorphisms of complexes of curves on punctured spheres and on punctured tori*, Topology and its Appl., **95** (1999), 85–111.

[44] I. Kra, *Horocyclic coordinates for Riemann surfaces and moduli spaces. I: Teichmüller and Riemann spaces of Kleinian groups*, J. of the AMS, **3** (1990), 499–578.

[45] I. Kra, *Riemann surfaces, by Simon Donaldson*, Bull. Amer. Math. Soc., **49** (2012), 455–463.

[46] E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Algebraic Geometry, **3** (1994), 283–293.

[47] F. Luo, *Automorphisms of the complex of curves*, Topology, **39** (2000), 283–298.

[48] A. Marden, *Geometric complex coordinates for Teichmüller space*, in "Mathematical Aspects of String Theory"ed. by S. T. Yau, World Scientific, Singapore, 1987, 341–357.

[49] H. Masur, *The extension of the Weil-Petersson metric to the boundary of Teichmüller space*, Duke Math. J., **43** (1976), 623–635.

[50] H. Masur and M. Wolf, *The Weil-Petersson isometry groups*, Geometriae Dedicata, **93** (2002), 177-190.

[51] Y. Matsumoto, *On the universal degenerating family of Riemann surfaces*, IRMA Lectures in Mathematics and Theoretical Physics **20**, Proc. of "Singularities in Geometry and Topology, Strasbourg 2009" ed. by V. Blanlœil and T. Ohmoto, (2012), 71–102.

[52] Y. Matsumoto, *Curve complexes and the DM-compactification of moduli spaces of Riemann surfaces*, RIMS Kôkyûroku **1991**, *Topology, Geometry and Algebra of low dimensional manifolds, May 25∼29, 2015* ed. by T. Kitano, (2016), 109–116.

[53] Y. Matsumoto, *The Deligne-Mumford compactification and crystallographic groups*, to appear in "Handbook of Teichmüller Theory"(ed. by A. Papadopoulos) Vol VII.

[54] Y. Matsumoto, *Crystallographic groups arising from Teichmüller spaces*, to appear in Tokyo J. Math.

[55] Y. Matsumoto and J. M. Montesinos-Amilibia, *A proof of Thurston's uniformization theorem of geometric orbifolds*, Tokyo J. Math., **14** (1991), 181–196.

[56] Y. Matsumoto and J. M. Montesinos-Amilibia, *Pseudo-periodic homeomorphisms and degeneration of Riemann surfaces*, Bull. Amer. Math. Soc., **30** (1994), 70–75.

[57] Y. Matsumoto and J. M. Montesinos-Amilibia, *Pseudo-periodic maps and degeneration of Riemann surfaces*, Lecture Notes in Math., Spinger **2030**, 2011.

[58] H. Miyachi, *Unification of extremal length geometry on Teichmüller space via intersection number*, Math. Zeitschrift, **278** (2014), 1065–1095.

[59] J. M. Montesinos-Amilibia, *Grupos crystalográficos y topología en Escher*, Rev. R. Acad. Cienc. Exact Fis. Nat. (Esp) **104** No.1, (2010), 27–47.
[60] D. Mumford, *A remark on Mahler's compactness theorem*, Proc. Amer. Math. Soc., **28** (1971), 289–294.
[61] S. Nag, *The complex analytic theory of Teichmüller spaces*, Canadian Math. Soc. Series of Monographs and Advanced Texts, A Willey-Interscience Publication, 1988.
[62] K. Ohshika, *Reduced Bers boundaries of Teichmüller spaces*, Annales de L'Institut Fourier, **64** (2014), 145–176.
[63] A. Papadopoulos, *A rigidity theorem for the mapping class group acting on the space of unmeasured foliations on a surface*, Proc. Amer. Math. Soc., **136** (2008), 4453–4460.
[64] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds* (second edition), Graduate Texts in Math., **149**, Springer, 2006, Springer.
[65] G. Riera, *A formula for the Weil-Petersson product of quadratic differentials*, J. Anal. Math., **95** (2005), 105–120.
[66] J. W. Robbin and D. A. Salamon, *A construction of the Deligne-Mumford orbifold*, J. Eur. Math. Soc., **8** (2006), 611–699.
[67] H. L. Royden, *Automorphisms and isometries of Teichmüller space*, in "Advances in the theory of Riemann surfaces"(ed. by L. V. Ahlfors et al. ) Ann. of Math. Studies, **66** (1971), 369–383.
[68] I. Satake, *On a generalization of the notion of manifolds*, Proc. Nat. Acad. Sci. USA, **42** (1956), 359–363.
[69] R. L. E. Schwarzenberger, *$N$-dimensional crystallography*, Pitnam Advanced Publishing Program, 1980.
[70] A. Szczepański, *About the possibilities of application of relative hyperbolization of polyhedra*, Math. Nachr., **172** (1995), 283–290.
[71] A. Szczepański, *Eta invariants for flat manifolds*, Ann. Global Anal. Geom., **41** (2012), no. 2, 125–138.
[72] W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes, Princeton University (1976–79). 1997.
[73] A. Weil, *On the moduli of Riemann surfaces*, [1958b] Œvres scientifiques, Collected Papers II, 381–389.
[74] S. Wolpert, *Noncompleteness of the Weil-Petersson metric for Teichmüller space*, Pacific J. Math., **61** (1975), 573–577.
[75] S. Wolpert, *The Fenchel-Nielsen deformation*, Ann. Math., **115** (1982), 501–528.
[76] S. Wolpert, *On the homology of the moduli space of stable curves*, Ann. Math., **116** (1983), 491–523.
[77] S. Wolpert, *On the symplectic geometry of deformations of a hyperbolic surfaces*, Ann. Math., **117** (1983), 207–234.
[78] S, Wolpert, *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math., **107** (1985), 969–997.
[79] S. Wolpert, *Geodesic length functions and the Nielsen problem*, J. Diff. Geom., **25** (1987), 275–296.
[80] S. Wolpert, *The hyperbolic metric and the geometry of the universal curve*, J. Diff. Geom., **31** (1990), 417–472.
[81] S. Wolpert, *Behavior of geodesic-length functions on Teichmüller space*, J. Diff. Geom., **79** (2008), 277–334.
[82] S. Wolpert, *Families of Riemann surfaces and Weil-Petersson Geometry*, Regional Conference Series in Mathematics, **113**, Amer. Math. Soc., 2010.
[83] S. Yamada, *On the geometry of Weil-Petersson completion of Teichmüller spaces*, Math. Research Lett., **11** (2004), 327–344.
[84] S. Yamada, Teichmüller 空間の Weil-Petersson 凸幾何学 (*Weil-Petersson convex geometry of Teichmüller spaces*), 数学 **65** no. 2 (2013), 174–198.
[85] S. Yamada, *Local and global aspects of Weil-Petersson geometry*, in "Handbook of Teichmüller Theory"(ed. by A. Papadopoulos) Vol. IV (2014), 43–111.

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