

Lecture Notes in Mathematical Sciences
The University of Tokyo

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**Carleman Estimates for Anisotropic
Hyperbolic Systems in Riemannian
Manifolds and Applications**

Mourad Bellassoued and Masahiro Yamamoto



Graduate School of Mathematical Sciences

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Preface

In this lecture note, concerning hyperbolic systems in Riemannian manifolds, we give self-contained descriptions on

- derivations of Carleman estimates;
- methods for applications of the Carleman estimates to estimates of solutions and to inverse problems. Moreover limited to equations, we survey the previous and recent results in view of the applicability of the Carleman estimate. We do not intend to pursue any general treatments of the Carleman estimate itself but by showing it in a direct manner, we mainly aim to demonstrate the applicability of the Carleman estimate to the estimation of solutions and inverse problems.

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Basic tools of Riemannian geometry

In this chapter, we will briefly describe the foundations of Riemannian geometry for readers who are unfamiliar with the subject. Our choice of material is made in view of applications to Carleman estimates and inverse problems. Readers who are interested in a more systematic exposition of Riemannian geometry, can consult, for example, [17], [33]. We provide proofs only if they cannot be found in the standards textbooks. Our interest is focused on Riemannian manifolds which are manifolds equipped with metric structure.

1.1 Manifolds

The concept of a manifold is used throughout this lecture note. We start with defining the notion of a coordinate chart.

Definition 1.1.1 *Let \mathcal{M} be a topological space. Then a pair (U, φ) is called a chart (a coordinate system), if*

$$\varphi : U \longrightarrow \varphi(U) \subset \mathbb{R}^n$$

and $\varphi(U)$ is an open set in \mathbb{R}^n . The coordinate functions on U are defined as $x^j : U \rightarrow \mathbb{R}$, and

$$\varphi(a) = (x^1(a), \dots, x^n(a)).$$

Here n is called the dimension of the coordinate system.

Definition 1.1.2 *A topological space \mathcal{M} is called a Hausdorff space if for each two distinct points $a_1, a_2 \in \mathcal{M}$, there are two open sets $U_1, U_2 \subset \mathcal{M}$ such that*

$$a_1 \in U_1, \quad a_2 \in U_2, \quad U_1 \cap U_2 = \emptyset.$$

We now want to consider the case where \mathcal{M} is covered by such charts and satisfies some consistency conditions. We have

Definition 1.1.3 *An n -dimensional atlas on a topological space \mathcal{M} is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ with some index set I such that:*

- \mathcal{M} is covered by $\{U_\alpha\}_{\alpha \in I}$
- $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in \mathbb{R}^n for each $\alpha, \beta \in I$.

- the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

Definition 1.1.4 Two atlases $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ and $\{(V_\beta, \psi_\beta)\}_{\beta \in J}$ are compatible if their union is an atlas.

The set of compatible atlases with a given atlases can be organized by inclusion. The maximal element is called the complete atlas.

Finally we come to the definition of a manifold:

Definition 1.1.5 A smooth manifold \mathcal{M} is called a Hausdorff space with a complete atlas.

Definition 1.1.6 An n -dimensional manifold with boundary is defined as a Hausdorff space together with an open cover $\{U_\alpha\}$ and homeomorphism $\varphi_\alpha : U_\alpha \rightarrow \tilde{U}_\alpha$ such that each \tilde{U}_α is an open set in $\mathbb{R}_+^n := \{x \in \mathbb{R}^n ; x_n \geq 0\}$ and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map whether $U_\alpha \cap U_\beta$ is nonempty.

In this lecture note, we deal with smooth manifolds, i.e., \mathcal{C}^∞ -manifolds, which means that $\varphi_\beta \circ \varphi_\alpha^{-1}, \alpha \in I, \beta \in J$, are \mathcal{C}^∞ mappings. We understand that a compact manifold consists of a finite number of pieces of Euclidean spaces which are glued together. The functions

$$\varphi_\alpha(x) = (x^1(x), \dots, x^n(x)) \in \mathbb{R}^n$$

are called local coordinates on U_α . Sometimes, when there is no danger of misunderstanding, we also write $x = (x^1, \dots, x^n)$ identifying a point $x \in \mathcal{M}$ with its representation in some local coordinates. All manifolds in this book are assumed to be compact and connected.

1.2 Tangent vectors and cotangent vectors

In this section, we introduce the notion of the tangent space $T_a\mathcal{M}$ of a differentiable manifold \mathcal{M} at a point $a \in \mathcal{M}$. This is a vector space of the same dimension as \mathcal{M} .

Definition 1.2.7 A function $f : \mathcal{M} \longrightarrow \mathbb{R}$ is said to be smooth (respectively \mathcal{C}^k) if for every chart $\{(U_\alpha, \varphi_\alpha)\}$ on \mathcal{M} , the function $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \longrightarrow \mathbb{R}$ is smooth (respectively \mathcal{C}^k) for any $\alpha \in I$. The set of all \mathcal{C}^k functions on the manifolds \mathcal{M} will be denoted by $\mathcal{C}^k(\mathcal{M})$.

Definition 1.2.8 Let \mathcal{M} be a differentiable manifold and $a \in \mathcal{M}$ be given. A tangent vector X_a at point $a \in \mathcal{M}$ is defined as a map $X_a : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ such that

- X_a is \mathbb{R} -linear:

$$X_a(\lambda f_1 + f_2) = \lambda X_a(f_1) + X_a(f_2), \quad \text{for all } \lambda \in \mathbb{R}, \quad f_1, f_2 \in \mathcal{C}^\infty(\mathcal{M}),$$

- X_a satisfies the Leibniz rule:

$$X_a(f_1 f_2) = X_a(f_1) f_2(a) + f_1(a) X_a(f_2), \quad \text{for all } f_1, f_2 \in \mathcal{C}^\infty(\mathcal{M}).$$

The set of all tangent vectors at a to \mathcal{M} is denoted by $T_a\mathcal{M}$ and is called the tangent space at a . It is a vector space of dimension n . A basis in this space is given by the coordinate tangent vectors $\left(\frac{\partial}{\partial x_i}\right)_a$ which is defined by

$$\left(\frac{\partial}{\partial x_i}\right)_a(f) = \frac{\partial}{\partial y^i}(f \circ \varphi^{-1})(\varphi(a)), \quad (1.2.1)$$

where $\varphi = (x^1, \dots, x^n)$ is a system of coordinates around a and (y^1, \dots, y^n) is the standard Cartesian coordinate system on \mathbb{R}^n .

In a given coordinate system (x^1, \dots, x^n) , every tangent vector $X \in T_a\mathcal{M}$ can be written as follows:

$$X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i} \quad (1.2.2)$$

where a real-valued function $\alpha^i = X \circ x^i : \mathcal{M} \rightarrow \mathbb{R}$, is called the components of X . Note that X is differentiable if α^i are differentiable.

Looking at (1.2.1) and (1.2.2), we can consider that a tangent vector X gives a directional derivatives of function $f \in C^\infty(\mathcal{M})$:

$$(Xf)(a) = X(a)f(a) = \sum_{i=1}^n \alpha^i(a) \frac{\partial f}{\partial x_i}(a).$$

Now we introduce the tangent bundle $T\mathcal{M}$ of a differentiable manifold \mathcal{M} . Intuitively, this is the object which we obtain by glueing at each point x of \mathcal{M} the corresponding tangent space $T_x\mathcal{M}$. The differentiable structure on \mathcal{M} induces a differentiable structure on the tangent bundle $T\mathcal{M}$ turning it into a differentiable manifold.

Definition 1.2.9 Let \mathcal{M} be differentiable manifold:

- The cotangent space $T_x^*\mathcal{M}$ is the space of linear functionals on $T_x\mathcal{M}$. Its elements are called covectors or one-forms.
- The disjoint union of the tangent spaces

$$T\mathcal{M} = \bigcup_{x \in \mathcal{M}} T_x\mathcal{M}$$

is called the tangent bundle of \mathcal{M} .

- Respectively, the cotangent bundle $T^*\mathcal{M}$ is the union of the spaces $T_x^*\mathcal{M}$, $x \in \mathcal{M}$.
- A one-form ω on the manifold \mathcal{M} is a function that assigns to each point $x \in \mathcal{M}$ a covector $\omega_x \in T_x^*\mathcal{M}$.

Remark 1 The Riemannian metric induces a natural isomorphism $\iota : T_x\mathcal{M} \rightarrow T_x^*\mathcal{M}$ given by $\iota(v) = \langle v, \cdot \rangle$. For $v \in T_x\mathcal{M}$ we denote $v^\flat = \iota(v)$, and similarly for $\varphi \in T_x^*\mathcal{M}$ we denote $\varphi^\sharp = \iota^{-1}(\varphi)$, ι and ι^{-1} are called musical isomorphisms.

An example of a one-form is the differential of a function $f \in C^\infty(\mathcal{M})$, which is defined by

$$df_x(X) = \sum_{i=1}^n \alpha^i \frac{\partial f}{\partial x_i}, \quad X \in T_x \mathcal{M}.$$

Hence f defines the mapping $df : T\mathcal{M} \rightarrow \mathbb{R}$, which is called the differential of f given by

$$df(x, X) = df_x(X).$$

In local coordinates,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i,$$

where (dx^1, \dots, dx^n) is the basis in the space $T_x^* \mathcal{M}$ which is the dual to the basis $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

In general, a one-form in local coordinates can be written as

$$\omega = \sum_{i=1}^n \omega^i dx^i,$$

where $\omega^i = \omega\left(\frac{\partial}{\partial x_i}\right)$.

Now we introduce the notion of vector fields on manifolds which is an assignment of a tangent vector to every point $x \in \mathcal{M}$.

Definition 1.2.10 A vector field X on an n -dimensional manifold \mathcal{M} is a mapping from the manifold to the tangent bundle, $X : \mathcal{M} \rightarrow T\mathcal{M}$, which associates to every point $x \in \mathcal{M}$ a tangent vector $X(x) \in T_x \mathcal{M}$.

The set of all vector fields on \mathcal{M} will be denoted by $\mathfrak{X}(\mathcal{M})$.

Following the interpretation of vector fields as maps that operate on functions in a manifold, we may ask ourselves about applying these operation multiple times. It turns out that not every iteration of vector fields gives another vector field, nonetheless, there is a special combination, given in the following definition, which is very important as we will see later.

Definition 1.2.11 Let $X, Y \in \mathfrak{X}(\mathcal{M})$ be vector fields on a manifold \mathcal{M} , the Lie bracket $[X, Y]$ is defined as the vector field

$$[X, Y] = \sum_{i,j=1}^n \left(\alpha^j \frac{\partial \beta^i}{\partial x_j} - \beta^j \frac{\partial \alpha^i}{\partial x_j} \right) \frac{\partial}{\partial x_i}, \quad X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}, \quad Y = \sum_{j=1}^n \beta^j \frac{\partial}{\partial x_j}.$$

We say that the vector fields X and Y commute if $[X, Y] = 0$.

Lemma 1.2.1 The Lie bracket $[\cdot, \cdot]$ is bilinear over \mathbb{R} . For a differentiable function f , we have

$$[X, Y]f = X(Y(f)) - Y(X(f)).$$

Furthermore the Jacobi identity holds:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for any three vector fields X, Y, Z .

Proof . In local coordinate with $X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^n \beta^i \frac{\partial}{\partial x_i}$, we have

$$[X, Y]f = \sum_{i,j=1}^n \left(\alpha^j \frac{\partial \beta^i}{\partial x_i} \frac{\partial f}{\partial x_i} - \beta^j \frac{\partial \alpha^i}{\partial x_j} \frac{\partial f}{\partial x_i} \right),$$

and this is linear in f , X and Y . This implies the first two claims. The Jacobi identity follows by direct computations. \square

1.3 Riemannian metric

In this section we introduce the Riemannian metric. The metric g provides us with an inner product on each tangent space and can be used to measure the length of curves in the manifold. It defines a distance function and turns the manifold into a metric space in a natural way. A Riemannian metric on a differentiable manifold is an important example of what is called a tensor field.

Definition 1.3.12 *Let \mathcal{M} be a smooth manifold. We call g a Riemannian metric on \mathcal{M} when the function g assigns a non-negative number to smooth vector fields X, Y on \mathcal{M} and satisfies*

$$\begin{aligned} g(X_1 + X_2, Y) &= g(X_1, Y) + g(X_2, Y), & g(X, Y_1 + Y_2) &= g(X, Y_1) + g(X, Y_2), \\ g(fX, Y) &= fg(X, Y) = g(X, fY), & g(X, Y) &= g(Y, X), \end{aligned}$$

and

$$g(X, X) > 0 \quad \text{whenever} \quad X \neq 0$$

for all smooth real-valued functions f and vector fields X, X_1, X_2, Y, Y_1, Y_2 .

A Riemannian manifold (\mathcal{M}, g) is a manifold \mathcal{M} with metric g . We call g a positive definite two-covariant tensor field.

In local coordinates, g is given by a smooth positive definite symmetric matrix function $g = (g_{jk})$:

$$g = \sum_{i,j=1}^n g_{ij} dx^i \otimes dx^j,$$

where g_{ij} are given by

$$g_{ij} = g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

Definition 1.3.13 *For $x \in \mathcal{M}$, the inner product and the norm on the tangent space $T_x \mathcal{M}$ are given by*

$$g(X, Y) = \langle X, Y \rangle = \sum_{j,k=1}^n g_{jk} \alpha^j \beta^k,$$

$$|X| = \langle X, X \rangle^{1/2}, \quad X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta^i \frac{\partial}{\partial x_i}.$$

Lemma 1.3.1 *Let (\mathcal{M}, g) be a Riemannian manifold with Riemannian metric g . Let ω be a transformation which maps a smooth vector field on \mathcal{M} to a smooth functions on \mathcal{M} . Suppose that*

$$\omega(X + Y) = \omega(X) + \omega(Y), \quad \omega(fX) = f\omega(X)$$

for every smooth real-valued function f and vector fields X and Y on \mathcal{M} . Then there exists a unique smooth vector field A on \mathcal{M} with the property that $\omega(Z) = \langle A, Z \rangle$ for all smooth vector fields Z on \mathcal{M} .

Proof. First we verify the uniqueness of the vector field A . Let U be an open set in \mathcal{M} and let A and B be vector fields on U with the property that

$$\langle A, Z \rangle = \omega(Z) = \langle B, Z \rangle$$

for all smooth vector fields Z on U . Then $\langle A - B, Z \rangle = 0$ on U for all vector fields Z . In particular, $\langle A - B, A - B \rangle = 0$ in U . It follows from the definition of a Riemannian metric that $A = B$ in U . This proves the uniqueness of the vector field A .

Now suppose that the open set U is the domain of some smooth coordinate system (x^1, \dots, x^n) . Let

$$g_{ij} = g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle.$$

Then (g_{ij}) is a matrix of smooth functions in U which is positive definite, and hence invertible at each point of U . Let (g^{ij}) be the smooth functions in U characterized by the property that the matrix (g^{ij}) is the inverse of (g_{ij}) at each point in U :

$$\sum_{j=1}^n g^{ij} g_{jk} = \delta_{ik}.$$

Define a smooth vector field A on U by

$$A = \sum_{i,j=1}^n \omega^i g^{ij} \frac{\partial}{\partial x_j},$$

where

$$\omega^i = \omega \left(\frac{\partial}{\partial x_i} \right).$$

Let Z be a smooth vector field on U given by

$$Z = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}.$$

Then

$$\langle A, Z \rangle = \sum_{i,j,k=1}^n \omega^i g^{ij} g_{jk} \alpha^k = \omega(Z).$$

We thus obtain a smooth vector field A over any coordinate patch U with the property that $\langle A, Z \rangle = \omega(Z)$ for all vector fields Z on U . If we are given two overlapping coordinate systems on \mathcal{M} , then the uniqueness result was already proved, and the vector fields over the coordinate patches obtained in the manner just described must agree on the overlap of the coordinate patches. Thus we obtain a smooth vector field A defined over the whole of \mathcal{M} such that $\omega(Z) = \langle A, Z \rangle$ for all smooth vector fields Z on \mathcal{M} , as required. \square

1.4 Connection

For a manifold \mathcal{M} and two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, we want to know how one vector field changes with respect to another. That is, for given two vector fields $X(x), Y(x) \in T_x \mathcal{M}$ for $x \in \mathcal{M}$, how can we understand the differential of $X(x)$ in the direction $Y(x)$? In fact, this is related to the definition of the directional derivative of a vector field at $x \in \mathcal{M}$. However there is a difficulty for the interpretation, and we have to clarify such operations, since vector fields take their value on different tangent spaces. We can clarify by imposing an additional structure on the tangent bundle of our manifold. This structure is called Connection and it allows two different spaces to be compared through, conceptually speaking, a notion of the tangent spaces being infinitesimally rolled or slipped along the manifold in an Euclidean manner and thus preserving the isometry. However, the analytic solution given here is harder to reconcile for intuitive concept and as for details we recommend readers to consult [33].

Definition 1.4.14 *Let \mathcal{M} be a smooth manifold. A connection D_X is defined as a bilinear mapping*

$$\begin{aligned} \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) &\longrightarrow \mathfrak{X}(\mathcal{M}) \\ (X, Y) &\longrightarrow D_X Y \end{aligned}$$

such that

1. D_X is tensorial in X , that is,

$$D_{X+Y}Z = D_X Z + D_Y Z, \quad X, Y, Z \in \mathfrak{X}(\mathcal{M})$$

and

$$D_{fX}Y = f D_X Y, \quad X, Y \in \mathfrak{X}(\mathcal{M}), \quad f \in \mathcal{C}^1(\mathcal{M}).$$

2. $D_X(\cdot)$ is \mathbb{R} -linear in Y , that is,

$$D_X(Y + Z) = D_X Y + D_X Z, \quad X, Y, Z \in \mathfrak{X}(\mathcal{M}).$$

3.

$$D_X(fY) = X(f)Y + fD_XY, \quad X, Y \in \mathfrak{X}(\mathcal{M}), \quad f \in \mathcal{C}^1(\mathcal{M}).$$

Instead of a connection, we also call D_X by a covariant derivative.

Definition 1.4.15 Let \mathcal{M} be a smooth manifold. The torsion tensor of a connection D on $T\mathcal{M}$ is defined by:

$$T(X, Y) = D_XY - D_YX - [X, Y], \quad X, Y \in \mathfrak{X}(\mathcal{M}).$$

We call D torsion free if $T(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(\mathcal{M})$.

On a Riemannian manifold, we can choose the connection called the Levi-Civita connection:

Theorem 1.4.1 (The fundamental theorem of Riemannian geometry). On each Riemannian manifold \mathcal{M} , there exists precisely one torsion free connection, denoted by ∇_X on $T\mathcal{M}$, such that

$$Z(\langle X, Y \rangle) = \langle \nabla_ZX, Y \rangle + \langle X, \nabla_ZY \rangle \quad \text{for all } X, Y, Z \in \mathfrak{X}(\mathcal{M}). \quad (1.4.1)$$

This connection is determined by the formula:

$$\begin{aligned} \langle \nabla_XY, Z \rangle = \frac{1}{2} [& X(\langle Y, Z \rangle) - Z(\langle X, Y \rangle) + Y(\langle Z, X \rangle) \\ & - \langle X, [Y, Z] \rangle + \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle]. \end{aligned} \quad (1.4.2)$$

The connection ∇_X is called the Levi-Civita connection. The proof is found in e.g., Theorem 3.3.1 in [33] but I repeat.

Proof . We shall first prove that each torsion free connection D_X on $T\mathcal{M}$ such that (1.4.1) holds, has to satisfy (1.4.2). This implies the uniqueness. Since ∇_X should satisfy (1.4.1), it has to satisfy

$$\begin{aligned} X(\langle Y, Z \rangle) &= \langle D_XY, Z \rangle + \langle Y, D_XZ \rangle, \\ Y(\langle Z, X \rangle) &= \langle D_YZ, X \rangle + \langle Z, D_YX \rangle, \\ Z(\langle X, Y \rangle) &= \langle D_ZX, Y \rangle + \langle X, D_ZY \rangle. \end{aligned}$$

We shall first prove the existence. For fixed X and Y , we consider the one-form ω defined by

$$\omega(Z) = [X(\langle Y, Z \rangle) - Z(\langle X, Y \rangle) + Y(\langle Z, X \rangle) - \langle X, [Y, Z] \rangle + \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle].$$

Then $\omega(Z)$ is tensorial in Z , because we have

$$\omega(fZ) = f\omega(Z) + \frac{1}{2} [X(f)\langle Y, Z \rangle + Y(f)\langle Z, X \rangle - X(f)\langle Y, Z \rangle - Y(f)\langle X, Z \rangle] = f\omega(Z) \quad (1.4.3)$$

for $f \in \mathcal{C}^\infty(\mathcal{M})$. Moreover the additivity in Z is obvious. Therefore there exists precisely one vector field A which depends on X and Y , such that

$$\omega(Z) = \langle A, Z \rangle.$$

We thus put $D_X Y := A$. It remains to show that this defines a torsion free connection. Let us first verify that D_X defines a connection: The additivity with respect to X and Y is clear. By (1.4.3) we see that it is tensorial, and finally we can show

$$D_X fY = fD_X Y + X(f)Y.$$

On the other hand, (1.4.2) implies

$$\langle D_X Y, Z \rangle - \langle D_Y X, Z \rangle = \langle [X, Y], Z \rangle,$$

and D_X is a torsion free connection. Likewise (1.4.2) follows by adding $\langle D_X Y, Z \rangle$ and $\langle D_X Z, Y \rangle$. \square

For computations, we use, however, local expressions known as the Christoffel symbols which fully describe the connection, and in other words, we want to express the covariant derivative in a local coordinate. Given $X, Y \in \mathfrak{X}(\mathcal{M})$ in a local coordinate system, we can write

$$X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}, \quad Y = \sum_{j=1}^n \beta^j \frac{\partial}{\partial x_j},$$

where α^i, β^j are real-valued functions. Therefore in local coordinate system, we have

$$\nabla_X Y = \sum_{i,j=1}^n \nabla_{\alpha^i \frac{\partial}{\partial x_i}} \left(\beta^j \frac{\partial}{\partial x_j} \right) = \sum_{j=1}^n X(\beta^j) \frac{\partial}{\partial x_j} + \sum_{i,j=1}^n \alpha^i \beta^j \nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial}{\partial x_j} \right). \quad (1.4.4)$$

The last term on the right-hand side can be written as

$$\nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial}{\partial x_j} \right) = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \quad (1.4.5)$$

where $\Gamma_{ij}^k : \mathcal{M} \rightarrow \mathbb{R}$ are real-valued functions for $i, j, k = 1, 2, \dots, n$, and Γ_{ij}^k are called the Christoffel symbols. If we know what these are, then we can compute the connection of any two vector fields. The Christoffel symbols depend on the choice of coordinate system and therefore are not tensorial. However they contain all the information about the behavior of the connection which is tensorial and one can express them in a different coordinate system.

Then the question is how we can compute these Christoffel symbols. They are easily expressed in terms of the coefficients of the metric when this is expressed in a coordinate system. From the definition of the Levi-Civita connection, we have:

$$\begin{aligned} \Gamma_{ij}^k(x) &= \sum_{m,p=1}^n g^{kp} \Gamma_{ij}^m \left\langle \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_p} \right\rangle = \sum_{p=1}^n g^{kp} \left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_p} \right\rangle \\ &= \frac{1}{2} \sum_{p=1}^n g^{kp}(x) \left(\frac{\partial g_{jp}}{\partial x_i} + \frac{\partial g_{ip}}{\partial x_j} - \frac{\partial g_{ik}}{\partial x_p} \right). \end{aligned} \quad (1.4.6)$$

It follows from the definition of connection that

$$\nabla_X Y = \sum_{l,p=1}^n \alpha^p \left(\frac{\partial \beta^l}{\partial x^p} + \sum_{q=1}^n \Gamma_{pq}^l \beta^q \right) \frac{\partial}{\partial x_l}. \quad (1.4.7)$$

Definition 1.4.16 *The covariant differential DZ of a vector fields Z is the bilinear form given by*

$$DZ(X, Y) = \langle \nabla_X Z, Y \rangle, \quad X, Y \in \mathfrak{X}(\mathcal{M}).$$

Lemma 1.4.1 *Let X and Z be smooth real vectors fields. The following identity holds true:*

$$X(\langle Z, X \rangle) = DZ(X, X) + \frac{1}{2}Z(|X|^2).$$

Proof . By (1.4.1), we obtain

$$\begin{aligned} X(\langle Z, X \rangle) &= \langle \nabla_Z, X \rangle + \langle \nabla_Z X, X \rangle \\ &= DZ(X, X) + \frac{1}{2} [\langle \nabla_Z X, X \rangle + \langle X, \nabla_Z X \rangle] \\ &= DZ(X, X) + \frac{1}{2} Z(|X|^2). \end{aligned}$$

The proof is completed. □

1.5 Laplace-Beltrami operator

Definition 1.5.17 *Let (\mathcal{M}, g) be a Riemannian manifold and $f \in \mathcal{C}^1(\mathcal{M})$. The gradient of f , denoted by ∇f , is defined by a vector field on \mathcal{M} metrically equivalent to df :*

$$\langle \nabla f, X \rangle = df(X) = X(f) \quad \text{for all } X \in \mathfrak{X}(\mathcal{M}). \quad (1.5.1)$$

This reads in local coordinates:

$$\nabla f = \sum_{j=1}^n (\nabla f)^j \frac{\partial}{\partial x_j}. \quad (1.5.2)$$

By

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i,$$

equation (1.5.1) yields

$$\sum_{i,j=1}^n g_{ij} (\nabla f)^j \alpha^i = \sum_{i,j=1}^n \frac{\partial f}{\partial x_j} \alpha^i \quad \text{for all } X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathcal{M}).$$

The components of the gradient are

$$(\nabla f)^j = \sum_{i=1}^n \frac{\partial f}{\partial x_i}$$

and then

$$\nabla f = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}.$$

In physics, a force vector field is called conservative if it is of a certain potential energy. This definition can be extended to any vector field on manifolds as follows:

Definition 1.5.18 Let $X \in \mathfrak{X}(\mathcal{M})$ be a vector field on \mathcal{M} . We say that X is provided by a potential ψ if there exists a differentiable function ψ such that $X = \nabla\psi$.

Definition 1.5.19 Let X be a vector field on \mathcal{M} . By $\operatorname{div}X$ we denote a scalar function given by

$$\operatorname{div}X = \sum_{i=1}^n \langle D_{e_i}X, e_i \rangle$$

if $x \in \mathcal{M}$ and (e_1, \dots, e_n) is an orthonormal basis of $T_x\mathcal{M}$. We call it the divergence of a vector field X on \mathcal{M} .

In local coordinates,

$$\operatorname{div}X = \sum_{i=1}^n \left(\frac{\partial \alpha^i}{\partial x_i} + \sum_{j=1}^n \Gamma_{ij}^i \alpha^j \right) = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \partial_i \left(\sqrt{\det g} \alpha^i \right), \quad X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}. \quad (1.5.3)$$

If $f \in \mathcal{C}^1$ and $X \in \mathfrak{X}(\mathcal{M})$, then we have

$$\operatorname{div}(fX) = X(f) + f \operatorname{div}X. \quad (1.5.4)$$

Definition 1.5.20 Let (\mathcal{M}, g) be a Riemannian manifold. The Laplace-Beltrami operator is given by

$$\Delta_g f = \operatorname{div}(\nabla f), \quad f \in \mathcal{C}^2.$$

In local coordinates, Δ_g is given by

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{\det g} g^{jk} \frac{\partial}{\partial x_k} \right). \quad (1.5.5)$$

Here (g^{jk}) is the inverse of the metric $g = (g_{jk})$ and $\det g = \det(g_{jk})$.

Let ψ and f be smooth functions on \mathcal{M} . Applying (1.5.4) with $X = \nabla\psi$, we obtain

$$\operatorname{div}(f\nabla\psi) = f \Delta_g \psi + \langle \nabla f, \nabla \psi \rangle. \quad (1.5.6)$$

1.6 Green's formulas

The metric tensor g induces the Riemannian volume form: this is an n -form defined locally by

$$dv_g = (\det g)^{1/2} dx_1 \wedge \cdots \wedge dx_n.$$

We denote by $L^2(\mathcal{M})$ the completion of $C^\infty(\mathcal{M})$ endowed with the usual inner product

$$\langle f_1, f_2 \rangle = \int_{\mathcal{M}} f_1(x) \overline{f_2(x)} dv_g, \quad f_1, f_2 \in C^\infty(\mathcal{M}).$$

Here and henceforth $\bar{\eta}$ denotes the complex conjugate of $\eta \in \mathbb{C}$.

The Sobolev space $H^1(\mathcal{M})$ is the completion of $C^\infty(\mathcal{M})$ with respect to the norm $\|\cdot\|_{H^1(\mathcal{M})}$:

$$\|f\|_{H^1(\mathcal{M})}^2 = \|f\|_{L^2(\mathcal{M})}^2 + \|\nabla_g f\|_{L^2(\mathcal{M})}^2.$$

The normal derivative is

$$\partial_\nu u := \nabla u \cdot \nu = \sum_{j,k=1}^n g^{jk} \nu_j \frac{\partial u}{\partial x_k}, \quad (1.6.1)$$

where ν is the unit outward vector field to $\partial\mathcal{M}$. Also consult [17].

For a vector field X , the divergence formula reads

$$\int_{\mathcal{M}} \operatorname{div} X dv_g = \int_{\partial\mathcal{M}} \langle X, \nu \rangle d\sigma_g, \quad (1.6.2)$$

where $\partial\mathcal{M}$ is the boundary of \mathcal{M} and $d\sigma_g$ is the area element on $\partial\mathcal{M}$.

For $f \in H^1(\mathcal{M})$, Green's formula reads

$$\int_{\mathcal{M}} \operatorname{div} X f dv_g = - \int_{\mathcal{M}} \langle X, \nabla f \rangle dv_g + \int_{\partial\mathcal{M}} \langle X, \nu \rangle f d\sigma_g. \quad (1.6.3)$$

Then if $f \in H^1(\mathcal{M})$ and $w \in H^2(\mathcal{M})$, then the following identity holds:

$$\int_{\mathcal{M}} \Delta_g w f dv_g = - \int_{\mathcal{M}} \langle \nabla w, \nabla f \rangle dv_g + \int_{\partial\mathcal{M}} \partial_\nu w f d\sigma_g. \quad (1.6.4)$$

Lemma 1.6.1 *Let (\mathcal{M}, g) be a smooth Riemannian manifold with compact boundary $\partial\mathcal{M}$. Then there exists a smooth vector fields N such that*

$$N(x) = \nu(x), \quad x \in \partial\mathcal{M} \quad \text{and} \quad |N| \leq 1, \quad x \in \mathcal{M}, \quad (1.6.5)$$

where ν is the unit normal of $\partial\mathcal{M}$ pointing towards the exterior of \mathcal{M} in terms of the Riemannian metric g .

Proof . Since $\partial\mathcal{M}$ is smooth, for every $x_* \in \partial\mathcal{M}$ there exist an open neighborhood \mathcal{V} of x_* in \mathbb{R}^n and a function $\theta \in C^\infty(\mathcal{V})$ such that

$$\nabla\theta(x) \neq 0, \quad \forall x \in \mathcal{V}, \quad \theta(x) = 0, \quad \forall x \in \mathcal{V} \cap \partial\mathcal{M}.$$

Replacing θ by $-\theta$ if needed, we can assume that

$$\langle \nu(x_*), \nabla\theta(x_*) \rangle > 0.$$

Then the function $\mu : \mathcal{V} \rightarrow \mathbb{R}^n$ given by

$$\mu(x) = \frac{1}{|\nabla\theta(x)|} \nabla\theta(x), \quad x \in \mathcal{V}$$

is smooth. We show that $\mu = \nu$ on $\mathcal{V} \cap \partial\mathcal{M}$. In fact, since $\theta = 0$ on $\mathcal{V} \cap \partial\mathcal{M}$, we have

$$\nabla\theta(x) = \langle \nabla\theta, \nu \rangle \nu + \langle \nabla\theta, \tau \rangle \tau = (\partial_\nu\theta)\nu,$$

which implies that μ , $\nabla\theta$ and ν are parallel each other on $\mathcal{V} \cap \partial\mathcal{M}$. This together with $|\mu| = |\nu| = 1$ shows that $\mu = \nu$ on $\mathcal{V} \cap \partial\mathcal{M}$.

Since $\partial\mathcal{M}$ is compact, $\partial\mathcal{M}$ can be covered with a finite number of neighborhoods $\mathcal{V}_1, \dots, \mathcal{V}_m$. Each of them plays the role of \mathcal{V} in the earlier reasoning. By $\mu_i, i = 1, \dots, m$ we denote the corresponding functions of \mathcal{V}_i , and we have

$$\partial\mathcal{M} \subset \mathcal{V}_1 \cup \dots \cup \mathcal{V}_m$$

and

$$\mu_i = \nu \quad \text{on} \quad \mathcal{V}_i \cap \partial\mathcal{M}, \quad i = 1, \dots, m.$$

Fix an open set \mathcal{V}_0 such that

$$\mathcal{M} \subset \mathcal{V}_0 \cup \mathcal{V}_1 \cup \dots \cup \mathcal{V}_m \quad \text{and} \quad \mathcal{V}_0 \cap \partial\mathcal{M} = \emptyset$$

and define $\mu_0 : \mathcal{V}_0 \rightarrow \mathbb{R}^n$ by $\mu_0(x) = 0$ in \mathcal{V}_0 . Let ψ_0, \dots, ψ_m be a smooth partition of unity corresponding to the covering $\mathcal{V}_0, \dots, \mathcal{V}_m$ of \mathcal{M} :

$$\psi_i \in C_0^\infty(\mathcal{V}_i), \quad \text{and} \quad 0 \leq \psi_i \leq 1, \quad i = 0, 1, \dots, m$$

and

$$\psi_0 + \psi_1 + \dots + \psi_m = 1, \quad \text{on} \mathcal{M}.$$

It is obvious that

$$N = \left(\sum_{i=0}^m \psi_i \mu_i \right) \Big|_{\mathcal{M}}$$

is the required vector field. □

1.7 The Hessian in Riemannian manifolds

Definition 1.7.21 Define the Hessian of $\psi \in C^\infty(\mathcal{M})$ as the second fundamental form:

$$D^2\psi(X, Y) := \nabla_X d\psi(X, Y) = \nabla_X(d\psi)(Y) = X(Y(\psi)) - (\nabla_X Y)(\psi),$$

where ∇_X stands for the Levi-Civita connection.

Since ∇_X is a symmetric connection, we have

$$\nabla_X d\psi(X, Y) - \nabla_X d\psi(Y, X) = [X, Y]\psi + (\nabla_Y X - \nabla_X Y)\psi = 0,$$

so that $D^2\psi$ is a symmetric tensor field on \mathcal{M} . Notice that

$$D^2\psi(X, Y) = \langle \nabla_Y(d\psi), X \rangle \quad \text{for all } X, Y \in T_x\mathcal{M}, \quad x \in \mathcal{M}.$$

In local coordinates, the Hessian of $\psi \in C^2(\mathcal{M})$ with respect to the metric g is defined by

$$D^2\psi(X, X) = \sum_{i,j=1}^n \alpha^i \left(\sum_{l=1}^n \frac{\partial \psi_l}{\partial x_i} g_{lj} + \sum_{k,l=1}^n \psi_k g_{lj} \Gamma_{ik}^l \right) \alpha^j, \quad \forall X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}, \quad (1.7.1)$$

where we recall that $\psi_l(x) = (\nabla\psi(x))_l$ is the l -th coordinate of $\nabla\psi(x)$ and

$$(\nabla\psi(x))_l = \psi_l(x) = \sum_{j=1}^n g^{jl}(x) \frac{\partial \psi}{\partial x_j}(x), \quad l = 1, \dots, n \quad (1.7.2)$$

and Γ_{ik}^l is the connection coefficient (Cristoffel symbol) of the Levi-Civita connection ∇_X to the metric g .

Moreover we have

$$\langle \nabla_X(\nabla\psi), Y \rangle = X(\langle \nabla\psi, Y \rangle) - \langle \nabla\psi, \nabla_X Y \rangle = X(Y(\psi)) - (\nabla_X Y)(\psi) = D^2\psi(X, Y). \quad (1.7.3)$$

Lemma 1.7.1 Let ψ be a $C^2(\mathcal{M})$ function. Then

$$D^2\psi(X, X) = X(\langle X, \nabla\psi \rangle) - \frac{1}{2} \nabla\psi(|X|^2) \quad (1.7.4)$$

for any vector field X .

Proof . Applying Lemma 1.4.1 with $Z = \nabla\psi$, we obtain

$$X(\langle X, \nabla\psi \rangle) = D(\nabla\psi)(X, X) + \frac{1}{2} \nabla\psi(|X|^2).$$

This completes the proof. \square

1.8 The Riemannian curvature

Roughly speaking, by the curvature of the space, we can easily see in \mathbb{R}^3 how much a manifold is diverted or curved away from a straight line or plane, while in higher dimensions, the description of curvature at one point cannot be done by a scalar, and instead we need the tensor. However this tensor is not easy to be understood and we need other simpler indices extracting the underlying information hidden in the curvature tensor. One of the indices is the sectorial curvature.

Definition 1.8.22 *The curvature tensor of the Levi-Civita connection ∇_X is defined by $R : TM \times TM \times TM \rightarrow TM$:*

$$R(X, Y)Z = \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

In local coordinates, we have

$$R \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_k} = \sum_{\ell=1}^n R_{kij}^{\ell} \frac{\partial}{\partial x_{\ell}}.$$

We put

$$R_{\ell kij} := \sum_{m=1}^n g_{\ell m} R_{kij}^m,$$

i.e.,

$$R_{\ell kij} := \left\langle R \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{\ell}} \right\rangle.$$

Lemma 1.8.1 *For vector fields X, Y, Z, W , we have*

$$R(X, Y)Z = -R(Y, X)Z, \quad (1.8.1)$$

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0, \quad (1.8.2)$$

$$\langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle \quad (1.8.3)$$

and

$$\langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle. \quad (1.8.4)$$

Proof . See Jost [33], Lemma 3.3.1. \square

1.8.1 The sectional curvature

The sectional curvature is a measure, for example, about the behavior of the space along a two dimensional subspace. By means of the knowledge of the sectional curvatures along sufficient amount of planes, we can expect to obtain the whole information of the curvature of the space.

Definition 1.8.23 For a Riemannian manifold (\mathcal{M}, g) , we define the sectional curvature at a point $x \in \mathcal{M}$ along the plane Π spanned by linearly independent tangent vectors X and Y in $T_x\mathcal{M}$ as follows:

$$K_x(\Pi) = \frac{\langle R(X, Y)Y, X \rangle}{|X|^2 |Y|^2 - \langle X, Y \rangle^2}.$$

Note that if $X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^n \beta^j \frac{\partial}{\partial x_j}$, then we can easily see that

$$K_x(\Pi) = \frac{\sum_{ijkl=1}^n R_{ijkl} \alpha^i \beta^j \alpha^k \beta^\ell}{\sum_{ijkl=1}^n (g_{ik}g_{jl} - g_{ij}g_{kl}) \alpha^i \beta^j \alpha^k \beta^\ell}.$$

Lemma 1.8.2 Let Π be the plane spanned by linearly independent tangent vectors X and Y and let $K(X, Y) = K_x(\Pi) (|X|^2 |Y|^2 - \langle X, Y \rangle^2)$. Then, for any W, Z , we have

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= K(X + W, Y + Z) - K(X + W, Y) - K(X + W, Z) \\ &\quad - K(X, Y + Z) - K(W, Y + Z) + K(X, Z) + K(W, Y) \\ &\quad - K(Y + W, X + Z) + K(Y + W, X) + K(Y + W, Z) \\ &\quad + K(Y, X + Z) + K(W, X + Z) - K(Y, Z) - K(W, X). \end{aligned}$$

Thus the sectional curvature determines the whole curvature tensor.

Proof . The proof follows from direct computations. □

For 2-dimensional manifold \mathcal{M} , the curvature tensor is simply given by

$$R_{ijkl} = K(x) (g_{ik}g_{jl} - g_{ij}g_{kl}),$$

since $T_x\mathcal{M}$ contain only one plane, namely $T_x\mathcal{M}$ itself. The function $K(x)$ is called the Gauss curvature.

Definition 1.8.24 The Riemannian manifold (\mathcal{M}, g) is called a space of constant sectional curvature or space form if $K_x(\Pi) = k \equiv \text{const}$ for all plane Π spanned by the (linearly independent) tangent vectors X and Y in $T_x\mathcal{M}$ and all $x \in \mathcal{M}$. A space form is called spherical, flat, or hyperbolic if $K_x > 0, = 0, < 0$, respectively. We call (\mathcal{M}, g) an Einstein manifold if

$$R_{ik} = c g_{ik}, \quad c \equiv \text{const}.$$

Lemma 1.8.3 Let (\mathcal{M}, g) be a Riemannian manifold with $\dim \mathcal{M} \geq 3$. If the sectional curvature of \mathcal{M} is constant at each point x , i.e., for all plane Π spanned by the (linearly independent) tangent vectors X and Y in $T_x\mathcal{M}$

$$K_x(\Pi) = f(x),$$

then $f(x) \equiv \text{const}$, and \mathcal{M} is a space form.

Proof . See Jost [33], Theorem 3.3.2. □

1.8.2 Covariant derivative along a curve

We now describe the restriction of the connection D to a curve on a manifold \mathcal{M} . Let $\gamma : I \rightarrow \mathcal{M}$ be a smooth curve defined over some interval I of \mathbb{R} on the manifold \mathcal{M} and $\tilde{X} \in \mathfrak{X}(\mathcal{M})$ be a vector field on \mathcal{M} . Then we say that X is a vector field along a curve if it is given by the restriction of \tilde{X} to the curve, i.e., $X = \tilde{X} \circ \gamma : I \rightarrow \mathcal{M} \rightarrow T\mathcal{M}$. Note that for given any vector field along a curve Y , there always exists an extension of \tilde{Y} .

Definition 1.8.25 Let $\gamma : I \rightarrow \mathcal{M}$ be a smooth curve on \mathcal{M} . For any $t \in I$, the tangent vector

$$\dot{\gamma}(t) = \gamma_* \left(\frac{d}{dt}(t) \right) \in T_{\gamma(t)}\mathcal{M}, \quad \dot{\gamma}(t)f = \frac{d(f \circ \gamma)}{dt}(t), \quad f \in C^\infty(\mathcal{M})$$

is called the velocity vector of γ at the point $\gamma(t)$.

If x is a coordinate system around $\gamma(t_0)$ and $x(\gamma(t)) = (\gamma_1, \dots, \gamma_n)$, then

$$\dot{\gamma}(t) = \sum_{i=1}^n \frac{d\gamma_i}{dt} \frac{\partial}{\partial x_i}(\gamma(t)).$$

Definition 1.8.26 Let $\gamma : I \rightarrow \mathcal{M}$ be a smooth curve on \mathcal{M} . A vector field along γ is a smooth map $X : I \rightarrow T\mathcal{M}$ such that

$$X(t) \in T_{\gamma(t)}\mathcal{M}$$

for all $t \in I$. A vector field X along γ is extendible if $X(t) = \tilde{X}(\gamma(t))$ for some vector field \tilde{X} on a neighborhood of $\gamma(I) \subset \mathcal{M}$.

The velocity field $\dot{\gamma}(t)$ is an example of a vector field along γ . Then we define:

Definition 1.8.27 The covariant derivative of a vector field X along γ in \mathcal{M} is given by

$$D_{\dot{\gamma}}X(t) := D_{\dot{\gamma}(t)}\tilde{X}(\gamma(t)) := \dot{X}(t),$$

where \tilde{X} is a local extension of X and $\dot{\gamma}(t) \in T_{\gamma(t)}\mathcal{M}$.

The properties of the covariant derivative D are the same as the ones defined for connection earlier and is well defined, i.e. does not depend on the choice of local extension, and for any vector field X along γ , we have

$$D_{\dot{\gamma}}(fX) = \frac{df}{dt}X + fD_{\dot{\gamma}}X, \quad f \in C^\infty(I).$$

If x is a coordinate system around $\gamma(t_0)$, $x(\gamma(t)) = (\gamma^1(t), \dots, \gamma^n(t))$ and $X = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x_i}$, then

$$D_{\dot{\gamma}}X = \sum_{k=1}^n \left(\frac{d\gamma^k}{dt} + \Gamma_{ij}^k \frac{d\gamma^i}{dt} \alpha^j \right) \frac{\partial}{\partial x_k} \quad (1.8.5)$$

for t near t_0 .

Remark 2 (Euclidean space). Let $\gamma : I \rightarrow \mathbb{R}^n$. Then $D_{\dot{\gamma}}X = \dot{X}$ and in particular $D_{\dot{\gamma}}\dot{\gamma} = \ddot{\gamma}$. In fact, let \tilde{X} be some extension of X such that $\tilde{X} \circ \gamma(t) = X$. Then

$$D_{\dot{\gamma}}X(t) = D_{\dot{\gamma}(t)}\tilde{X} = (\tilde{X} \circ \gamma)'(t) = \dot{X}(t).$$

Definition 1.8.28 A vector field X along a curve $\gamma : I \rightarrow \mathcal{M}$ is called parallel if

$$\dot{X}(t) := D_{\dot{\gamma}}X = 0.$$

1.9 Geodesics in Riemannian manifolds and the exponential map

We know that in an Euclidean space, a geodesic curve is one whose tangent is constant, that is, a curve $\gamma : I \rightarrow \mathbb{R}^n$ is a geodesic if and only if $|\dot{\gamma}(t)| = \text{constant}$ or $\ddot{\gamma} = 0$. In other words, we see that in an Euclidean space, a geodesic curve is one whose covariant derivative is identically zero:

$$D_{\dot{\gamma}}\dot{\gamma} = \ddot{\gamma} = 0.$$

Motivated by this observation, we extend the notion to manifolds and define geodesics.

Definition 1.9.29 A parametrized curve $\gamma : I \rightarrow \mathcal{M}$ is said to be a geodesic curve of \mathcal{M} if

$$D_{\dot{\gamma}}\dot{\gamma} := \ddot{\gamma} = 0. \quad (1.9.1)$$

Next we want to show that at every point and in every direction, there exists a geodesic. That is, for all $x \in \mathcal{M}$ and $V \in T_x\mathcal{M}$, there exists $\gamma : I \rightarrow \mathcal{M}$ with $\gamma(t_0) = x$ and $\dot{\gamma}(t_0) = V$ for some $t_0 \in I$ such that $D_{\dot{\gamma}}\dot{\gamma} = 0$.

In local coordinates, by (1.9.1), we consider the differential equation

$$D_{\dot{\gamma}}\dot{\gamma} = \sum_{i,j,k=1}^n \left(\frac{d^2\gamma^k}{dt^2} + \Gamma_{ij}^k \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right) \frac{\partial}{\partial x_k} = 0$$

with the initial values $\gamma(t_0) = x$ and $\dot{\gamma}(t_0) = V$. From the theory of ordinary differential equation, we know that solutions exist and so we have the following:

Theorem 1.9.2 (Local existence and uniqueness of geodesics). Let $x \in \mathcal{M}$. Then there exists an open neighborhood $x \in U \subset \mathcal{M}$ and $\varepsilon > 0$ such that for $V \in T_x\mathcal{M}$ with $|V| < \varepsilon$, there exists a unique geodesic $\gamma_V : (-\varepsilon; \varepsilon) \rightarrow \mathcal{M}$ with $\gamma_V(0) = x$ and $\dot{\gamma}_V(0) = V$.

Definition 1.9.30 A Riemannian manifold \mathcal{M} is geodesically complete if all maximal geodesics are defined for all of \mathbb{R} .

Lemma 1.9.1 Let $\gamma : [a, b] \rightarrow \mathcal{M}$ be a curve on \mathcal{M} . Suppose that $p = \gamma(a)$ is a point on γ and $V \in T_p\mathcal{M}$ is a tangent vector at that point. There exists precisely one parallel vector field X along γ such that $X(a) = V$.

Lemma 1.9.2 *Let $\gamma : [a, b] \rightarrow \mathcal{M}$ be a geodesic. Then $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = \text{constant}$, namely, γ is of constant speed.*

Proof . We have

$$\frac{d}{dt} (|\dot{\gamma}(t)|^2) = 2 \langle D_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle = 0.$$

This completes the proof. \square

Let (\mathcal{M}, g) be a Riemannian manifold with its connection D . Then we know that at every point $x \in \mathcal{M}$, every $V \in T_x \mathcal{M}$ defines a unique geodesic at x with V as its tangent vector. Put

$$\mathcal{E} = \{V \in T\mathcal{M}; \gamma_V \text{ is defined at } 1\},$$

where γ_V with $V \in T_x \mathcal{M}$, is the geodesic through $\gamma_V(0) = p$ with velocity $\dot{\gamma}(0) = V$.

Definition 1.9.31 *The exponential map $\exp : \mathcal{E} \subset T\mathcal{M} \rightarrow \mathcal{M}$ is defined by*

$$\begin{aligned} \exp_x : T_x \mathcal{M} &\longrightarrow \mathcal{M}, \\ \exp_x(V) &= \gamma_V(1) \end{aligned}$$

at $x \in \mathcal{M}$. Here γ_V is a geodesic curve with tangent V at x such that $\gamma_V(0) = x$.

Note that this map is defined only locally and its importance comes from the fact that it maps straight lines in the tangent space, since this has the structure of an Euclidean space and so these lines are geodesics. In other words, a local sphere in the tangent space centered at $0 \in T_x \mathcal{M}$ is mapped to a local sphere centered at $x \in \mathcal{M}$ which is perpendicular to all geodesics through the point x . This is known as the Gauss lemma and is stated more explicitly for $t \in \mathbb{R}$ as

$$\gamma_V(t) = \gamma_{tV}(1) = \exp_x(tV).$$

Lemma 1.9.3 *Let (\mathcal{M}, g) be a Riemannian manifold and $\psi \in \mathcal{C}^2(\mathcal{M})$. Let $X \in T_p \mathcal{M}$ and a geodesic $\gamma : [0, \tau] \rightarrow \mathcal{M}$ with $\gamma(0) = p$, $\dot{\gamma}(0) = X$ be chosen. Then*

$$D^2 \psi(X, X) = \left. \frac{d^2}{dt^2} \psi(\gamma(t)) \right|_{t=0}. \quad (1.9.2)$$

Proof . By (1.5.1), we have

$$\begin{aligned} X(X(\psi)) &= \dot{\gamma}(0) \langle \nabla \psi(p), \dot{\gamma}(0) \rangle \\ &= \dot{\gamma}(0) \left(\left. \frac{d}{dt} \psi(\gamma(t)) \right|_{t=0} \right) \\ &= \left. \frac{d^2}{dt^2} \psi(\gamma(t)) \right|_{t=0} \end{aligned} \quad (1.9.3)$$

and $D_{\dot{\gamma}} \dot{\gamma} = 0$, since γ is a geodesic, so that (1.9.2) follows from (1.9.3) and Definition 1.7.21. \square

1.10 The Riemann distance function

1.10.1 Curves and variations of curves

Definition 1.10.32 A regular curve on \mathcal{M} is a smooth map $\gamma : [a, b] \longrightarrow \mathcal{M}$ such that $\dot{\gamma}(t) \neq 0$ for all $t \in [a, b]$. The real number

$$\ell(\gamma) = \int_a^b |\dot{\gamma}(t)| dt$$

is called the length of the regular curve γ .

A piecewise regular curve on \mathcal{M} is continuous map $\gamma : [a, b] \longrightarrow \mathcal{M}$ such that $\gamma|_{[a_{i-1}, a_i]}$ is regular for some subdivision $a = a_0 < a_1 < \dots < a_n = b$. The real number

$$\ell(\gamma) = \sum_{i=1}^n \ell\left(\gamma|_{[a_{i-1}, a_i]}\right)$$

is understood as the length of the piecewise regular curve γ .

For a piecewise regular curve γ on \mathcal{M} , there exists the velocity $\dot{\gamma}(t)$ at each point t which is not any break point a_i . At a break point a_i ,

$$[\dot{\gamma}]_i = \dot{\gamma}(a_i^+) - \dot{\gamma}(a_i^-)$$

denotes the jump of the velocity between $\dot{\gamma}(a_i^-) \in T_{\gamma(a_i)}\mathcal{M}$ and $\dot{\gamma}(a_i^+)$.

Lemma 1.10.1 We have the following properties:

1. The length of a (piecewise) regular curve is invariant under parametrization.
2. Any regular curve has a unit speed parameterization.

Proof . Let $\gamma : [a, b] \longrightarrow \mathcal{M}$ be a regular curve.

1. Let $\sigma : [c, d] \longrightarrow [a, b]$ be a bijective smooth map with $\sigma'(s) \neq 0$ for all $s \in [c, d]$. Then

$$(\gamma \circ \sigma)(s) = \sigma'(s)\dot{\gamma}(\sigma(s))$$

and

$$\int_c^d |(\gamma \circ \sigma)(s)| ds = \pm \int_c^d |\dot{\gamma}(\sigma(s))| |\sigma'(s)| ds = \int_a^b |\dot{\gamma}(t)| dt,$$

where the + applies if $\sigma' > 0$ and the - applies if $\sigma' < 0$.

2. Let $\alpha : [a, b] \longrightarrow [0, \ell(\gamma)]$ be the smooth map given by

$$\alpha(t) = \int_a^t |\dot{\gamma}(s)| ds.$$

Then $\alpha'(t) = |\dot{\gamma}(t)|$. Let σ be the inverse function of α . Then

$$(\gamma \circ \sigma)(\alpha) = \sigma'(\alpha)\dot{\gamma}(\sigma(\alpha)) = \frac{1}{\alpha'(t)}\dot{\gamma}(t) = \frac{1}{|\dot{\gamma}(t)|}\dot{\gamma}(t)$$

is a unit speed curve. This completes the proof.

□

Lemma 1.10.2 *Any two points in \mathcal{M} can be connected by a piecewise regular curve.*

Proof . Since connected and locally path-connected spaces are path-connected, \mathcal{M} is path-connected. Given any two points p and q in \mathcal{M} , there exists a continuous curve $\gamma : [0, 1] \rightarrow \mathcal{M}$ connecting them. Then there is a subdivision $0 = a_0 < a_1 < \dots < a_n = 1$ of $[0, 1]$ such that $\gamma([a_{i-1}, a_i])$ is contained in a coordinate neighborhood $x : U \rightarrow \mathbb{R}^n$ such that $x(U)$ is a ball. Replace $\gamma([a_{i-1}, a_i])$ by a smooth curve within this coordinate neighborhood between the two end-points. □

Definition 1.10.33 *Let (\mathcal{M}, g) be a Riemannian manifold and $\gamma : I \rightarrow \mathcal{M}$ be a curve on \mathcal{M} . A variation of γ is a differential map*

$$\Phi : (-\eta, \eta) \times [a, b] \rightarrow \mathcal{M}$$

such that for all $t \in [a, b]$,

$$\Phi_0(t) = \Phi(0, t) = \gamma(t).$$

The variation Φ is called *proper* if $\Phi_s(a) := \Phi(s, a) = \gamma(a)$ and $\Phi_s(b) = \Phi(s, b) = \gamma(b)$ for all $s \in (-\eta, \eta)$. We denote

$$\Phi_s(t) = \Phi(s, t) = \Phi^t(s), \quad s \in (-\eta, \eta), \quad t \in [a, b],$$

so that Φ_s is a curve in the t -direction (a main curve) and Φ^t is a curve in the s -direction (a transverse curve).

It follows that

$$\partial_t \Phi(s, t) = \frac{\partial}{\partial t} \Phi_s(t) = \dot{\Phi}_s(t) = \Phi_* \left(\frac{\partial}{\partial t} \right), \quad \partial_s \Phi(s, t) = \frac{\partial}{\partial s} \Phi^t(s) = \dot{\Phi}^t(s) = \Phi_* \left(\frac{\partial}{\partial s} \right)$$

are the velocities of the main curve and the transverse curves, respectively. We may view the main velocity field $\partial_t \Phi$ as a vector field along a transverse curve Φ^t and consider its covariant derivative $D_{\dot{\Phi}_s} \dot{\Phi}^t$ along Φ^t . Similarly we may view the transverse velocity field $\partial_s \Phi$ as a vector field along a main curve Φ_s and consider its covariant derivative $D_{\dot{\Phi}^t} \dot{\Phi}_s$.

Lemma 1.10.3 (*symmetry lemma*) *Let $\Phi : (-\eta, \eta) \times [a, b] \rightarrow \mathcal{M}$ be a variation of the curve $\gamma : [a, b] \rightarrow \mathcal{M}$. Then following identity holds true:*

$$D_{\dot{\Phi}^t} \dot{\Phi}_s = D_{\dot{\Phi}_s} \dot{\Phi}^t.$$

Proof . We consider locally. Choose a coordinate system x around $\Phi(s_0, t_0)$. In local coordinates, we have $x(\Phi(s, t)) = (\Phi_1(s, t), \dots, \Phi_n(s, t))$ and

$$\frac{\partial \Phi_s}{\partial t} = \dot{\Phi}_s = \sum_{i=1}^n \frac{\partial \Phi_i}{\partial t} \frac{\partial}{\partial x_i}, \quad \frac{\partial \Phi^t}{\partial s} = \dot{\Phi}^t = \sum_{i=1}^n \frac{\partial \Phi_i}{\partial s} \frac{\partial}{\partial x_i}.$$

By (1.4.7), we obtain

$$\begin{aligned}
D_{\dot{\Phi}^t} \dot{\Phi}_s &= \sum_{ijk=1}^n \left(\frac{d}{ds} \dot{\Phi}_s^k + \Gamma_{ij}^k \frac{d\dot{\Phi}^{it}}{ds} \frac{d\dot{\Phi}_s^j}{dt} \right) \frac{\partial}{\partial x_k} \\
&= \sum_{ijk=1}^n \left(\frac{\partial^2 \Phi^k}{\partial s \partial t} + \Gamma_{ij}^k \frac{\partial \Phi^i}{\partial s} \frac{\partial \Phi^j}{\partial t} \right) \frac{\partial}{\partial x_k} \\
&= \sum_{ijk=1}^n \left(\frac{\partial^2 \Phi^k}{\partial t \partial s} + \Gamma_{ij}^k \frac{\partial \Phi^i}{\partial t} \frac{\partial \Phi^j}{\partial s} \right) \frac{\partial}{\partial x_k} = D_{\dot{\Phi}_s} \dot{\Phi}^t, \tag{1.10.1}
\end{aligned}$$

where we have used $\Gamma_{ij}^k = \Gamma_{ji}^k$. \square

Lemma 1.10.4 *Suppose that X is smooth vector field along a smooth variation of γ through geodesics $\Phi : (-\eta, \eta) \times [a, b] \rightarrow \mathcal{M}$. Then we have*

$$D_{\dot{\Phi}^t} D_{\dot{\Phi}_s} X - D_{\dot{\Phi}_s} D_{\dot{\Phi}^t} X = R(\dot{\Phi}^t, \dot{\Phi}_s) X.$$

Proof. Choose local coordinates x . Let $x(\Phi(s, t)) = (\Phi^1, \dots, \Phi^n)$ and $X = \sum_{i=1}^n \alpha^i(s, t) \frac{\partial}{\partial x_i}$ be a local expressions for Φ and X . We have

$$D_{\dot{\Phi}_s} X = \sum_{i=1}^n D_{\dot{\Phi}_s} \left(\alpha^i \frac{\partial}{\partial x_i} \right) = \sum_{i=1}^n \left(\frac{\partial \alpha^i}{\partial t} \frac{\partial}{\partial x_i} + \alpha^i D_{\dot{\Phi}_s} \left(\frac{\partial}{\partial x_i} \right) \right)$$

and

$$\begin{aligned}
D_{\dot{\Phi}^t} D_{\dot{\Phi}_s} X &= \sum_{i=1}^n D_{\dot{\Phi}^t} \left(\frac{\partial \alpha^i}{\partial t} \frac{\partial}{\partial x_i} + \alpha^i D_{\dot{\Phi}_s} \left(\frac{\partial}{\partial x_i} \right) \right) \\
&= \sum_{i=1}^n \left(\left(\frac{\partial^2 \alpha^i}{\partial s \partial t} \right) \frac{\partial}{\partial x_i} + \frac{\partial \alpha^i}{\partial t} D_{\dot{\Phi}^t} \left(\frac{\partial}{\partial x_i} \right) + \frac{\partial \alpha^i}{\partial s} D_{\dot{\Phi}_s} \frac{\partial}{\partial x_i} + \alpha^i D_{\dot{\Phi}^t} D_{\dot{\Phi}_s} \frac{\partial}{\partial x_i} \right). \tag{1.10.2}
\end{aligned}$$

When we compute the difference $D_{\dot{\Phi}^t} D_{\dot{\Phi}_s} X - D_{\dot{\Phi}_s} D_{\dot{\Phi}^t} X$, many terms cancel and we have

$$D_{\dot{\Phi}^t} D_{\dot{\Phi}_s} X - D_{\dot{\Phi}_s} D_{\dot{\Phi}^t} X = \sum_{i=1}^n \alpha^i (D_{\dot{\Phi}^t} D_{\dot{\Phi}_s} - D_{\dot{\Phi}_s} D_{\dot{\Phi}^t}) \frac{\partial}{\partial x_i} = R(\dot{\Phi}^t, \dot{\Phi}_s) X.$$

At the last equality we used

$$[\dot{\Phi}^t, \dot{\Phi}_s] = \left[\Phi_* \left(\frac{\partial}{\partial s} \right), \Phi_* \left(\frac{\partial}{\partial t} \right) \right] = \Phi_* \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0.$$

This completes the proof. \square

1.10.2 The distance function

Definition 1.10.34 The function $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ given by

$$d(p, q) = \inf \{ \ell(\gamma) ; \gamma \text{ is a piecewise regular curve from } p \text{ to } q \}$$

is called the Riemann distance function.

Lemma 1.10.5 d is a metric on the topological space \mathcal{M} .

Lemma 1.10.6 Let γ be a unit speed smooth curve and $\Phi : (-\eta, \eta) \times [a, b] \rightarrow \mathcal{M}$ be any smooth variation of γ . Then

$$\frac{d}{ds}(\ell(\Phi_s))(0) = \langle v, \dot{\gamma}(t) \rangle \Big|_a^b - \int_a^b \langle V, D_{\dot{\gamma}} \dot{\gamma} \rangle dt,$$

when $V(t)$ is the variational field given by

$$V(t) = \dot{\Phi}^t(0).$$

Proof . We differentiate the function $s \mapsto \ell(\Phi_s)$ and then evaluate the result at $s = 0$. Using that the connection is compatible with the metric and the symmetry Lemma 1.10.3, we obtain

$$\begin{aligned} \frac{d}{ds}(\ell(\Phi_s)) &= \frac{d}{ds} \int_a^b |\dot{\Phi}_s(t)| dt = \int_a^b \frac{\partial}{\partial s} \left(\langle \dot{\Phi}_s(t), \dot{\Phi}_s(t) \rangle^{1/2} \right) dt \\ &= \int_a^b \frac{1}{|\dot{\Phi}_s|} \langle D_{\dot{\Phi}_s} \dot{\Phi}_s, \dot{\Phi}_s \rangle dt = \int_a^b \frac{1}{|\dot{\Phi}_s|} \langle D_{\dot{\Phi}_s} \dot{\Phi}^t, \dot{\Phi}_s \rangle dt. \end{aligned} \quad (1.10.3)$$

When $s = 0$, $\dot{\Phi}_s = \dot{\Phi}_0 = \dot{\gamma}$, $|\dot{\gamma}| = 1$, and $\dot{\Phi}^t(0) = V(t)$, we obtain

$$\frac{d}{ds}(\ell(\Phi_s)) \Big|_{s=0} = \int_a^b \frac{d}{dt} \langle V, \dot{\gamma} \rangle dt - \int_a^b \langle V, D_{\dot{\gamma}} \dot{\gamma} \rangle dt.$$

This completes the proof of the lemma. □

Lemma 1.10.7 Let γ be a unit speed piecewise regular curve and Φ be any piecewise smooth variation of γ . Then

$$\frac{d}{ds}(\ell(\Phi_s))(0) = - \int_a^b \langle V, D_{\dot{\gamma}} \dot{\gamma} \rangle dt - \sum_{i=1}^{n-1} \langle [\dot{\gamma}]_i, V(a_i) \rangle,$$

where $V(t) = \dot{\Phi}^t(0)$ is the variational field along γ .

Proof . We can complete the proof because

$$\ell(\Phi_s) = \sum_{i=1}^{n-1} \ell\left(\Phi_s|_{[a_{i-1}, a_i]}\right),$$

where we take the sum over $[a_{i-1}, a_i]$ of smooth intervals of Φ_s , and the endpoints are fixed under the variation, so that the variational field is 0 at the endpoints. \square

Lemma 1.10.8 *Any vector field V which vanishes at the endpoints of a piecewise smooth curve $\gamma : [a, b] \rightarrow \mathcal{M}$, is the variational field of some variation $\Phi : (-\eta, \eta) \times [a, b] \rightarrow \mathcal{M}$ of the curve γ with fixed endpoints.*

Proof . Thanks to the compactness, we can find $\eta > 0$ so that $\pm\eta V(t) \in \mathcal{E} \subset T\mathcal{M}$ for all $t \in [a, b]$. Let

$$\Phi(s, t) = \exp_{\gamma(t)}(sV(t)), \quad (s, t) \in (-\eta, \eta) \times [a, b].$$

Then Φ is a piecewise smooth variation whose transverse curves are geodesics with velocity

$$\frac{\partial}{\partial s}\Phi(s, t)|_{s=0} = \dot{\Phi}^t(0) = V(t).$$

This completes the proof of the lemma. \square

Corollary 1 *1. Every piecewise regular minimizing curve of constant speed is a geodesic.
2. Every geodesic is a locally minimizing curve.*

Proof . Let γ be a minimizing curve. For any vector field V along γ , we have $\frac{\partial}{\partial s}\Phi(s, t)|_{s=0} = 0$ because $\ell(\Phi_s)$ has a minimum at $s = 0$. Therefore $V(t) = 0$. Use this to show first that $D_{\dot{\gamma}}\dot{\gamma}(t) = 0$ at any non-breaking points. This is a piecewise geodesic. Next we can show that there are no break points, and it is in fact a geodesic. \square

What we showed was in fact that geodesics are critical points of the functional ℓ . Is there a minimizing curve between any two points of \mathcal{M} ? Is a minimizing curve unique? The answer is not affirmative in general. For example, on the earth surface, there are uncountably many minimizing curves between the North Pole and the South Pole. Or consider the situation where you want to go to the other shore side of a lake, and there are usually two possibilities. Only if two points are sufficiently close, then there exists in fact a unique minimizing curve between them.

1.11 Jacobi fields

Let (\mathcal{M}, g) be a Riemannian manifold and let $\gamma : [a, b] \rightarrow \mathcal{M}$ be a geodesic from $p = \gamma(a)$ to $q = \gamma(b)$.

Definition 1.11.35 A smooth vector field J along γ is called a Jacobi field if the following equation holds:

$$D_{\dot{\gamma}}D_{\dot{\gamma}}J + R(J, \dot{\gamma})\dot{\gamma} = 0.$$

This is called the Jacobi equation. As an abbreviation, we shall sometime write

$$\ddot{J} + R(J, \dot{\gamma})\dot{\gamma} = 0, \quad D_{\dot{\gamma}}J = \dot{J}.$$

We denote by \mathcal{J}_{γ} the set of the Jacobi fields along γ .

Lemma 1.11.1 (existence and uniqueness of Jacobi field). Let $\gamma : [a, b] \rightarrow \mathcal{M}$ be a geodesic. For any $V, W \in T_p\mathcal{M}$, there exists a unique Jacobi field J along γ such that

$$J(a) = V, \quad \dot{J}(a) = W.$$

Proof . Let $\{V_1, \dots, V_n\}$ be an orthonormal basis of $T_p\mathcal{M}$. By Lemma 1.9.1, we can take parallel vector fields $\{X_1, \dots, X_n\}$ along γ with $X_j(a) = V_j, j = 1, \dots, n$. Then for each $t \in [a, b]$, it follows that $\{X_1(t), \dots, X_n(t)\}$ is an orthonormal base of $T_{\gamma(t)}\mathcal{M}$. An arbitrary vector fields X along γ is written as

$$X = \sum_{i=1}^n \alpha^i X_i, \quad \alpha^i(t) = \langle X(t), X_i(t) \rangle.$$

Since the vector fields X_i along γ are parallel, that is, $D_{\dot{\gamma}}X_i = 0$, we have

$$D_{\dot{\gamma}}X = \sum_{i=1}^n D_{\dot{\gamma}}(\alpha^i X_i) = \sum_{i=1}^n \left(\frac{d\alpha^i}{dt} X_i + \alpha^i D_{\dot{\gamma}}X_i \right) = \sum_{i=1}^n \frac{d\alpha^i}{dt} X_i$$

and we deduce

$$D_{\dot{\gamma}}D_{\dot{\gamma}}X = \sum_{i=1}^n \frac{d^2\alpha^i}{dt^2} X_i.$$

Moreover we write the curvature term as a linear combination of $X_k, k = 1, \dots, n$:

$$R(X_i, \dot{\gamma})\dot{\gamma} = \sum_{k=1}^n \rho_i^k X_k,$$

and then

$$R(X, \dot{\gamma})\dot{\gamma} = \sum_{i,k=1}^n \alpha^i \rho_i^k X_k.$$

Let $(\alpha_j^1, \dots, \alpha_j^n)$ be a solution to an n -system of linear second-order ordinary differential equations:

$$\frac{d^2\alpha}{dt^2} + \sum_{i=1}^n \alpha^1(t) \rho_i^k(t) = 0, \quad k = 1, \dots, n$$

with Cauchy data

$$\alpha^k(a) = \langle V, V_k \rangle, \quad \frac{d\alpha^k}{dt}(a) = \langle W, V_k \rangle.$$

For such a system, the desired existence and uniqueness result is known. Let J be the vector field along the geodesic γ given by: $J = \sum_{k=1}^n \alpha^k J_k$. Then we have:

$$D_{\dot{\gamma}} D_{\dot{\gamma}} J + R(J, \dot{\gamma})\dot{\gamma} = \sum_{k=1}^n \frac{d^2 \alpha^k}{dt^2} X_k + \sum_{i,k=1}^n \alpha^i \rho_i^k X_k = 0.$$

Furthermore

$$J(a) = V, \quad \dot{J}(a) = D_{\dot{\gamma}} J(a) = W.$$

Thus the proof is completed. \square

Lemma 1.11.2 *Let $\gamma : [a, b] \rightarrow \mathcal{M}$ be geodesic and $\lambda, \mu \in \mathbb{R}$. Then the Jacobi field J along γ with $J(a) = \lambda \dot{\gamma}(a)$ and $J(b) = \mu \dot{\gamma}(b)$ is given by*

$$J(t) = (\lambda + (t - a)\mu) \dot{\gamma}(t).$$

Proof . The proof directly follow from Lemma 1.11.1, since $R(\dot{\gamma}, \dot{\gamma}) = 0$. \square

Lemma 1.11.3 *Let $\gamma : [0, \tau] \rightarrow \mathcal{M}$ be a geodesic in a Riemannian manifold (\mathcal{M}, g) and let $\Phi : (-\eta, \eta) \times [0, \tau] \rightarrow \mathcal{M}$ be a smooth variation of γ through geodesics. Let V be the variational field given by*

$$V(t) = \Phi^t(0) = \left. \frac{\partial}{\partial s} \Phi(s, t) \right|_{s=0}.$$

Then the vector field V along the geodesic γ satisfies the Jacobi equation. (Here for all s , Φ_s is a geodesic in \mathcal{M}).

Conversely every Jacobi field along γ can be obtained in this way, i.e., by a variation of γ through geodesics.

Proof . We apply Lemma 1.10.4 to the vector field $\dot{\Phi}_s$ along Φ to have

$$D_{\dot{\Phi}^t} D_{\dot{\Phi}_s} \dot{\Phi}_s - D_{\dot{\Phi}_s} D_{\dot{\Phi}^t} \dot{\Phi}_s = R(\dot{\Phi}^t, \dot{\Phi}_s) \dot{\Phi}_s.$$

Since for all $s \in (-\eta, \eta)$, $t \mapsto \Phi_s$ is a geodesic, we have $D_{\dot{\Phi}_s} \dot{\Phi}_s = 0$. Lemma 1.10.3 yields

$$D_{\dot{\Phi}^t} \dot{\Phi}_s = D_{\dot{\Phi}_s} \dot{\Phi}^t.$$

Thus

$$D_{\dot{\Phi}_s} D_{\dot{\Phi}_s} \dot{\Phi}^t + R(\dot{\Phi}^t, \dot{\Phi}_s) \dot{\Phi}_s = 0.$$

Since $\dot{\Phi}^t(0) = V(t)$ and $\dot{\Phi}_0(t) = \dot{\gamma}(t)$ at $s = 0$, we have the Jacobi equation:

$$D_{\dot{\gamma}}D_{\dot{\gamma}}V + R(V, \dot{\gamma})\dot{\gamma} = 0.$$

This completes the proof of the first part.

Conversely let V be a Jacobi field along γ . Let $c : (-\eta, \eta) \rightarrow \mathcal{M}$ be a geodesic such that $c(0) = \gamma(0)$ and $\dot{c}(0) = V(0)$. Let X and Y be parallel vector fields along the curve $c(s)$ (Lemma 1.9.1) with

$$X(0) = \dot{c}(0) = V(0), \quad Y(0) = (D_{\dot{c}}V)(0) = \dot{V}(0).$$

We put

$$\Phi(s, t) = \exp_{c(s)}(t(X(s) + sY(s))), \quad s \in (-\eta, \eta), \quad t \in [0, \tau].$$

Then all curves $s \in (-\eta, \eta)$, $t \mapsto \Phi_s(t)$ are geodesics by the definition of the exponential map and $\Phi_0(t) = \exp_{\gamma(0)}(t\dot{\gamma}(0)) = \gamma(t)$. Thus $\Phi(s, t)$ is a variation of $\gamma(t)$ through geodesics. By the first part of the proof, the vector field given by

$$Z(t) = \left. \frac{\partial}{\partial s} \Phi(s, t) \right|_{s=0} = \dot{\Phi}^t(s)|_{s=0} = \dot{\Phi}^t(0),$$

is a Jacobi field along γ . Finally

$$Z(0) = \dot{\Phi}^0(0) = \dot{c}(0) = V(0)$$

and

$$D_{\dot{\gamma}}Z(0) = D_{\dot{\Phi}_s} \dot{\Phi}^t|_{s,t=0} = D_{\dot{\Phi}_t} \dot{\Phi}_s|_{s,t=0} = D_{\dot{c}(s)}(X(s) + sY(s))|_{s=0} = Y(0) = \dot{V}(0).$$

Thus Z is a Jacobi field along γ with the same initial data values $V(0)$, $\dot{V}(0)$ as V .

The uniqueness implies $Z = V$. We have thus shown that V can be obtained from a variation of $\gamma(t)$ through geodesics. \square

Corollary 2 *Let $\gamma : [0, \tau] \rightarrow \mathcal{M}$ be a geodesic on the Riemannian manifold \mathcal{M} such that $\gamma(0) = p \in \mathcal{M}$, i.e.,*

$$\gamma(t) = \exp_p(t\dot{\gamma}(0)).$$

For $Y \in T_p\mathcal{M}$, the Jacobi field J along γ with $J(0) = 0$, $\dot{J}(0) = D_{\dot{\gamma}}J(0) = Y$ is given by

$$J(t) = (D \exp_p)(t\dot{\gamma}(0))(tY).$$

That is, $J(t)$ is given by the value applied to tY of the derivative of $\exp_p : T_p\mathcal{M} \rightarrow \mathcal{M}$ at the point $t\dot{\gamma}(0) \in T_p\mathcal{M}$.

Proof . Let

$$\Phi(s, t) = \exp_p(t(\dot{\gamma}(0) + sY))$$

be a variation of $\gamma(t)$ through geodesic. By Lemma 1.11.3, the corresponding Jacobi field is

$$J(t) = \left. \frac{\partial}{\partial s} \Phi(s, t) \right|_{s=0} = (D \exp_p)(t\dot{\gamma}(0))(tY)$$

and

$$J(0) = (D \exp_p)(0)(0) = 0, \quad \dot{J}(0) = Y.$$

Consequently the derivative of the \exp_p mapping can be computed from Jacobi fields along radial geodesic. This completes the proof. \square

Definition 1.11.36 Let $\gamma : [0, \tau] \rightarrow \mathcal{M}$ be a geodesic. For each $Y \in T_p\mathcal{M}$ by J_Y we denote the Jacobi field along γ with

$$J_Y(0) = 0, \quad \text{and} \quad \dot{J}(0) = D_{\dot{\gamma}}J_Y(0) = Y.$$

Definition 1.11.37 Let $\gamma : [0, \tau] \rightarrow \mathcal{M}$ be a geodesic on \mathcal{M} . A Jacobi field J along γ is called normal if

$$\langle J(t), \dot{\gamma}(t) \rangle = 0, \quad \text{for all } t.$$

Lemma 1.11.4 Let J be a Jacobi field along a geodesic $\gamma : [0, \tau] \rightarrow \mathcal{M}$. Then J is normal if and only if $J(0)$ and $\dot{J}(0)$ are orthogonal to $\dot{\gamma}(0)$.

Proof . We have

$$\frac{d^2}{dt^2} \langle J(t), \dot{\gamma}(t) \rangle = \frac{d}{dt} \langle D_{\dot{\gamma}}J, \dot{\gamma} \rangle = \langle D_{\dot{\gamma}}D_{\dot{\gamma}}J, \dot{\gamma} \rangle = -\langle R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma} \rangle = 0.$$

Then

$$\langle J(t), \dot{\gamma}(t) \rangle = at + b.$$

If J vanishes at two points, then we obtain $\langle J(t), \dot{\gamma}(t) \rangle = 0$ for all t , that is, J is normal. \square

In Riemannian manifold with constant sectional curvature, we can describe the normal Jacobi field more explicitly.

Lemma 1.11.5 (Jacobi field in constant sectional curvature manifold) Suppose that \mathcal{M} is a Riemannian manifold of constant sectional curvature k . Let γ be a unit speed geodesic on \mathcal{M} with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$, where V is a unit vector in $T_p\mathcal{M}$. For any vector field Y orthogonal to V , the normal Jacobi field J_Y is:

$$J_Y(t) = S_k(t)Y(t),$$

where $Y(t)$ is the parallel vector field along γ with $Y(0) = Y$ and

$$S_k(t) = \begin{cases} R \sin(t/R) & k = 1/R^2 \\ t & k = 0 \\ R \sinh(t/R) & k = -1/R^2 \end{cases}$$

is the solution to

$$\ddot{S} + kS = 0, \quad S(0) = 0, \quad \dot{S}(0) = 1.$$

Proof . Put $J(t) = S_k(t)Y(t)$. Then

$$D_{\dot{\gamma}}J = \dot{S}_k(t)Y \quad \text{and} \quad D_{\dot{\gamma}}D_{\dot{\gamma}}J = \ddot{S}_kY = -kJ$$

as $Y(t)$ is parallel. Thus $J(0) = 0$ and $\dot{J}(0) = D_{\dot{\gamma}}J(0) = Y$. By the construction, J is orthogonal to γ . It is a Jacobi because

$$R(J, \dot{\gamma})\dot{\gamma} = k(\langle \dot{\gamma}, \dot{\gamma} \rangle J - \langle J, \dot{\gamma} \rangle \dot{\gamma}) = kJ = -D_{\dot{\gamma}}D_{\dot{\gamma}}J,$$

as γ has unit speed and J is normal to γ . We conclude that $J = J_Y$. \square

1.12 Convexity of Riemannian distance function

Definition 1.12.38 (*conjugate points*) We say that two points p and q on a geodesic segment γ are conjugate if there exists a nonzero Jacobi (necessarily normal) field along γ which vanishes at p and q .

Lemma 1.12.1 Consider the smooth function $\exp_p : \mathcal{E} \rightarrow \mathcal{M}$. Let $q = \exp_p(rv) \neq p$ be a point on the geodesic $\gamma(t) = \exp_p(tv)$. Then

\exp_p is not a local diffeomorphism at $rv \Leftrightarrow p$ and q are conjugate points.

Proof . By Corollary 2, if $J(0) = 0$ and $\dot{J}(0) = D_{\dot{\gamma}}J(0) = Y$, then

$$J_Y(t) = (D \exp_p)(t\dot{\gamma}(0))(tY).$$

The value of such a Jacobi field at $q = \exp_p(rv)$ is

$$J_Y(r) = (D \exp_p)(r\dot{\gamma}(0))(rY).$$

Thus there is a nonzero Jacobi field which vanishes at p and q if and only if there is a nonzero vector Y in the kernel of $(D \exp_p)(t\dot{\gamma}(0))$. \square

For abbreviation, we put for $\rho \in \mathbb{R}$

$$c_{\rho}(t) = \begin{cases} \cos(\sqrt{\rho}t), & \text{if } \rho > 0, \\ 1, & \text{if } \rho = 0, \\ \cosh(\sqrt{-\rho}t), & \text{if } \rho < 0, \end{cases} \quad ; \quad s_{\rho}(t) = \begin{cases} \frac{1}{\sqrt{\rho}} \sin(\sqrt{\rho}t), & \text{if } \rho > 0, \\ t, & \text{if } \rho = 0, \\ \frac{1}{\sqrt{-\rho}} \sinh(\sqrt{-\rho}t), & \text{if } \rho < 0. \end{cases}$$

These functions are the solutions of the Jacobi equation for constant sectional curvature ρ :

$$\ddot{f}(t) + \rho f(t) = 0$$

with initial values $f(0) = 1$, $\dot{f}(0) = 0$ and $f(0) = 0$, $\dot{f}(0) = 1$, respectively.

Let (\mathcal{M}, g) be a Riemannian manifold and let $\gamma : [0, \tau] \rightarrow \mathcal{M}$ be a geodesic on \mathcal{M} . We denote by $J(t)$ a Jacobi field along γ and let $\beta(t) = |J(t)|$.

For $\mu \geq 0$ we denote

$$f_\mu(t) = \beta(0)c_\mu(t|\dot{\gamma}|) + |\dot{\gamma}|^{-1}\dot{\beta}(0)s_\mu(t|\dot{\gamma}|),$$

which solves

$$\ddot{f}(t) + \mu|\dot{\gamma}|^2 f(t) = 0, \quad f_\mu(0) = \beta(0), \quad \dot{f}_\mu(0) = \dot{\beta}(0).$$

Theorem 1.12.3 *Assume that the sectional curvature $K_x(\Pi)$ of the manifold \mathcal{M} satisfies $K_x(\Pi) \leq \mu$ for all plane sections Π , where $\mu \geq 0$. If $f_\mu(t) > 0$ and $J(t) \neq 0$ for $t \in (0, \tau)$, then we have*

$$f_\mu(t) \leq \beta(t), \quad t \in [0, \tau].$$

Proof . By direct computations, we obtain

$$\begin{aligned} \ddot{\beta}(t) + \mu|\dot{\gamma}|^2 \beta(t) &= \frac{d^2}{dt^2} \left(\langle J(t), J(t) \rangle^{1/2} \right) + \mu|\dot{\gamma}|^2 \langle J(t), J(t) \rangle^{1/2} \\ &= \frac{d}{dt} \left(\langle D_{\dot{\gamma}} J, J \rangle \frac{1}{|J|} \right) + \mu|\dot{\gamma}|^2 \langle J(t), J(t) \rangle^{1/2} \\ &= \frac{1}{|J|} \left(-\langle R(J, \dot{\gamma})\dot{\gamma}, J \rangle + \mu|\dot{\gamma}|^2 \langle J, J \rangle \right) \\ &\quad + \frac{1}{|J|^3} \left(|j|^2 |J|^2 - \langle J, j \rangle^2 \right) \geq 0, \end{aligned} \tag{1.12.1}$$

because $K_x(\Pi) \leq \mu$ for $t \in (0, \tau)$, provided J has no zero on $(0, \tau)$. We then also have

$$\frac{d}{dt} \left(\dot{\beta}(t)f_\mu(t) - \beta(t)\dot{f}_\mu(t) \right) = \ddot{\beta}(t)f_\mu(t) - \beta(t)\ddot{f}_\mu(t) = (\ddot{\beta}(t) + \mu|\dot{\gamma}|^2 \beta(t))f_\mu(t) \geq 0,$$

which implies that

$$\dot{\beta}(t)f_\mu(t) - \beta(t)\dot{f}_\mu(t) \geq \dot{\beta}(0)f_\mu(0) - \beta(0)\dot{f}_\mu(0) = 0, \quad \forall t \in (0, \tau).$$

Next we see that

$$\frac{d}{dt} \left(\frac{\beta(t)}{f_\mu(t)} \right) = \frac{1}{f_\mu^2} (\dot{\beta}(t)f_\mu(t) - \beta(t)\dot{f}_\mu(t)) \geq 0.$$

Therefore

$$\beta(t)f_\mu(0) \geq \beta(0)f_\mu(t).$$

This completes the proof. \square

Theorem 1.12.4 *Let (\mathcal{M}, g) be a Riemannian manifold and $p \in \mathcal{M}$. We assume that the exponential map $\exp_p : T_p \mathcal{M} \rightarrow \mathcal{M}$ is a diffeomorphism on $\mathcal{E}_\rho = \{V \in T_p \mathcal{M} : |V| \leq \rho\}$. Let the curvature of \mathcal{M} in the ball*

$$B(p, \delta) := \{x \in \mathcal{M} : d(p, x) \leq \delta\}$$

satisfy $K_x(\Pi) \leq \mu$ for all plane sectional Π with $\mu \geq 0$, and suppose that

$$\delta < \frac{\pi}{2\sqrt{\mu}} \quad \text{in case } \mu > 0.$$

Let

$$\psi(x) := \frac{1}{2}d^2(p, x), \quad x \in B(p, \delta).$$

Then ψ is smooth on $B(p, \delta)$ and satisfies

$$\nabla\psi(x) = -\exp_x^{-1} p, \quad |\nabla\psi(x)| = r(x) := d(p, x).$$

Furthermore

$$D^2\psi(X, X) \geq \begin{cases} \sqrt{\mu}r(x) \cot(\sqrt{\mu}r(x)) |X|^2, & \text{if } \mu > 0, \\ |X|^2, & \text{if } \mu = 0, \quad \forall x \in B(p, \delta), X \in T_x\mathcal{M}. \end{cases}$$

Proof. We have

$$\nabla\psi(x) = -\exp_x^{-1} p.$$

Let $\gamma : [0, 1] \rightarrow \mathcal{M}$ be the geodesic from x to p given by

$$\gamma(t) = \exp_x(-t\nabla\psi)$$

Let $X \in T_x\mathcal{M}$. Let $c(s)$ be the curve in \mathcal{M} with $c(0) = x$, $\dot{c}(0) = X$. Let

$$\Phi(s, t) = \exp_{c(s)}(t \exp_{c(s)}^{-1} p).$$

Thus Φ is a variation of the geodesic γ and we have

$$\Phi(s, 0) = \Phi^0(s) = c(s), \quad \Phi(0, t) = \Phi_0(t) = \exp_x(-t\nabla\psi(x)) = \gamma(t), \quad \Phi(s, 1) = p.$$

Then

$$\nabla\psi(c(s)) = -\exp_{c(s)}^{-1} p = -\frac{\partial}{\partial t}\Phi(s, t)\Big|_{t=0} = -\dot{\Phi}_s(t)|_{t=0} = -\dot{\Phi}_s(0).$$

Furthermore, by Definition 1.7.21, we find

$$\begin{aligned} D^2\psi(\dot{\Phi}^t, \dot{\Phi}^t) &= \dot{\Phi}^t \langle \dot{\Phi}^t, \nabla\psi \rangle - \langle D_{\dot{\Phi}^t} \dot{\Phi}^t, \nabla\psi \rangle \\ &= \langle D_{\dot{\Phi}^t} \dot{\Phi}^t, \nabla\psi \rangle + \langle \dot{\Phi}^t, D_{\dot{\Phi}^t} \nabla\psi \rangle - \langle D_{\dot{\Phi}^t} \dot{\Phi}^t, \nabla\psi \rangle \\ &= \langle \dot{\Phi}^t, D_{\dot{\Phi}^t} \nabla\psi \rangle. \end{aligned} \tag{1.12.2}$$

Now let $J(t) = \frac{\partial}{\partial s}\Phi(s, t)\Big|_{s=0}$ be the Jacobi field along a geodesic $\gamma : [0, 1] \rightarrow \mathcal{M}$ from x to p with

$$J(0) = \dot{\Phi}^0(0) = \dot{c}(0) = X, \quad J(1) = 0.$$

Therefore

$$\begin{aligned} D^2\psi(X, X) &= D^2\psi(\dot{\Phi}^t, \dot{\Phi}^t)|_{s,t=0} = -\left\langle \dot{\Phi}^t, D_{\dot{\Phi}^t}\dot{\Phi}_s \right\rangle|_{s,t=0} \\ &= -\left\langle \dot{\Phi}^t, D_{\dot{\Phi}_s}\dot{\Phi}^t \right\rangle|_{s,t=0} \\ &= -\left\langle \dot{c}(0), \dot{J}(0) \right\rangle \\ &= -\left\langle J(0), \dot{J}(0) \right\rangle. \end{aligned} \tag{1.12.3}$$

Since $J(1) = 0$, and there are no conjugate points on the geodesic segment γ , we see that for any $t \in [0, 1)$, $J(t) \neq 0$. On the other hand, for $t \in (0, 1)$ and $x \in B(p, \delta)$, by the assumptions we have $t\sqrt{\mu}r(x) \in (0, \pi/2)$.

- If there exists $t_0 \in (0, 1)$ such that $f_\mu(t_0) \leq 0$, then

$$\beta(0)c_\mu(t_0r(x)) + (t_0r(x))^{-1}\dot{\beta}(0)s_\mu(r(x)) \leq 0.$$

On the other hand, since

$$\beta(0) = |J(0)| = |X|, \quad \dot{\beta}(t) = \frac{1}{|J(t)|} \left\langle \dot{J}(t), J(t) \right\rangle,$$

we have $\beta(0) = -\frac{1}{|X|}D^2\psi(X, X)$. Then

$$|X|^2 r(x)c_\mu(t_0r(x)) \leq s_\mu(t_0r(x))D^2\psi(X, X).$$

Thus we conclude that

$$|X|^2 \sqrt{\mu}r(x) \cot(\sqrt{\mu}r(x)) \leq D^2\psi(X, X) \quad \text{if } \mu > 0$$

and

$$|X|^2 r(x) \leq t_0r(x)D^2\psi(X, X) \leq r(x)D^2\psi(X, X) \quad \text{if } \mu = 0.$$

- Let $f_\mu(t) > 0$ for all $t \in (0, 1)$. Applying Theorem 1.12.3 with $t = 1$, we obtain

$$\beta(0)c_\mu(r(x)) + (r(x))^{-1}\dot{\beta}(0)s_\mu(r(x)) \leq 0.$$

Then

$$|X|^2 r(x)c_\mu(r(x)) \leq s_\mu(r(x))D^2\psi(X, X).$$

This completes the proof of the theorem. \square

Corollary 3 *Let \mathcal{M} be a Riemannian manifold with sectional curvature ≤ 0 . Let $p \in \mathcal{M}$ and let*

$$\psi(x) = \frac{1}{2}d^2(p, x).$$

If the exponential map \exp_p is a diffeomorphism on the ball $B(p, \rho)$, then we have

$$D^2\psi(X, X) \geq |X|^2, \quad \forall x \in B(p, \rho), \quad X \in T_x\mathcal{M}.$$

Corollary 4 *Let \mathcal{M} be a simply connected and geodesically complete manifold with non-positive curvature and $p \in \mathcal{M}$. Then for*

$$\psi(x) = \frac{1}{2}d^2(p, x),$$

we have

$$D^2\psi(X, X) \geq |X|^2, \quad \forall x \in \mathcal{M}, \quad X \in T_x\mathcal{M}.$$

Lemma 1.12.2 *Let \mathbb{R}^2 have the metric*

$$g = g_1 dx_1 dx_1 + g_2 dx_2 dx_2,$$

where $g_1 > 0, g_2 > 0$ are C^∞ functions on \mathbb{R}^2 . Then the Gaussian curvature is

$$k = \frac{1}{4g_1^2 g_2^2} \left(g_2 \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_1} + g_1 \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_2} + g_1 \left(\frac{\partial g_2}{\partial x_1} \right)^2 + g_2 \left(\frac{\partial g_1}{\partial x_2} \right)^2 - 2g_1 g_2 \left(\frac{\partial^2 g_1}{\partial x_2^2} + \frac{\partial^2 g_2}{\partial x_1^2} \right) \right). \quad (1.12.4)$$

Proof. Let

$$X = \frac{1}{\sqrt{g_1}} \frac{\partial}{\partial x_1}, \quad Y = \frac{1}{\sqrt{g_2}} \frac{\partial}{\partial x_2}.$$

Then $\{X, Y\}$ is an orthonormal basis on (\mathbb{R}^2, g) . We thus have

$$K = \langle R_{XY}X, Y \rangle = \frac{1}{g_1 g_2} \left\langle R_{\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle, \quad (1.12.5)$$

where

$$R_{XY}X = -\nabla_X \nabla_Y X + \nabla_Y \nabla_X X + \nabla_{[X, Y]} X$$

and R is the curvature tensor. Let Γ_{ij}^k be the coefficients of the connection D . Then

$$\Gamma_{12}^1 = \frac{1}{2g_1} \frac{\partial g_1}{\partial x_2}, \quad \Gamma_{11}^2 = -\frac{1}{2g_2} \frac{\partial g_1}{\partial x_2}, \quad \Gamma_{12}^2 = \frac{1}{2g_2} \frac{\partial g_2}{\partial x_1}.$$

From the properties of connection, we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1} &= \nabla_{\frac{\partial}{\partial x_1}} \left(\Gamma_{12}^1 \frac{\partial}{\partial x_1} + \Gamma_{12}^2 \frac{\partial}{\partial x_2} \right) \\ &= \frac{\partial \Gamma_{12}^1}{\partial x_1} \frac{\partial}{\partial x_1} + \Gamma_{12}^1 \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} + \frac{\partial \Gamma_{12}^2}{\partial x_1} \frac{\partial}{\partial x_2} + \Gamma_{12}^2 \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} \\ &= \frac{\partial \Gamma_{12}^1}{\partial x_1} \frac{\partial}{\partial x_1} + \left(\Gamma_{12}^1 \Gamma_{11}^2 + \frac{\partial \Gamma_{12}^2}{\partial x_1} + \Gamma_{12}^2 \Gamma_{12}^2 \right) \frac{\partial}{\partial x_2}. \end{aligned} \quad (1.12.6)$$

Thus

$$\left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle = \left(\Gamma_{12}^1 \Gamma_{11}^2 + \frac{\partial \Gamma_{12}^2}{\partial x_1} + \Gamma_{12}^2 \Gamma_{12}^2 \right) g_2$$

$$= \frac{1}{2} \frac{\partial^2 g_2}{\partial x_1^2} - \frac{1}{4g_1} \left(\frac{\partial g_1}{\partial x_2} \right)^2 - \frac{1}{4g_2} \left(\frac{\partial g_2}{\partial x_1} \right)^2, \quad (1.12.7)$$

since $\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle = 0$. Calculations similar to (1.12.7) give

$$\left\langle \nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle = -\frac{1}{2} \frac{\partial^2 g_1}{\partial x_2^2} + \frac{1}{4g_1} \left(\frac{\partial g_1}{\partial x_1} \right) \left(\frac{\partial g_2}{\partial x_2} \right) + \frac{1}{4g_2} \left(\frac{\partial g_1}{\partial x_2} \right) \left(\frac{\partial g_2}{\partial x_2} \right). \quad (1.12.8)$$

Equations (1.12.7) and (1.12.8), together with (1.12.5), yield (1.12.4). \square

Example 1.1. We consider a Riemannian manifold (\mathbb{R}^2, g) where g is given by

$$g = e^{-(x_1+x_2)} dx_1 dx_1 + e^{-(x_1+x_2)} dx_2 dx_2.$$

We obtain from Lemma 1.12.2 that $K_x = 0$ for any $x = (x_1, x_2) \in \mathbb{R}^2$; that is, (\mathbb{R}^2, g) is of zero curvature. Then the function $\psi(x) = \frac{1}{2} d^2(p, x)$ is strictly convex and satisfies (A.2) in $\mathcal{M} \subset \mathbb{R}^2$ for any $p \notin \mathcal{M}$.

Example 1.2. Consider a Riemannian manifold (\mathbb{R}^2, g) where g is given by

$$g = e^{(x_1+x_2)} dx_1 dx_1 + e^{-(x_1+x_2)} dx_2 dx_2.$$

By Lemma 1.12.2 we obtain

$$K_x = -\frac{1}{2} (e^{-(x_1+x_2)} + e^{x_1+x_2}) < 0.$$

Then the function $\psi(x) = \frac{1}{2} d^2(p, x)$ is strictly convex on $\mathcal{M} \subset \mathbb{R}^2$ for any $p \notin \mathcal{M}$.

Well-posedness and regularity of the wave equation with variable coefficients

2.1 Preliminaries

In this chapter, we will consider the initial-boundary value problem for the wave equation on a manifold with boundary. This initial boundary value problem corresponds to an elliptic operator $-\Delta_g$ given in Chapter 1. We will develop a widely applicable approach to prove existence and uniqueness of solutions and to study their regularity properties. The materials are picked up from Lions and Magenes [47].

Let us consider the following initial boundary value problem for the wave equation with potential $q \in L^\infty(\mathcal{M})$:

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) u = F, & \text{in } (0, T) \times \mathcal{M}, \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 & \text{in } \mathcal{M}, \\ u = f, & \text{on } (0, T) \times \partial\mathcal{M}, \end{cases} \quad (2.1.1)$$

with various assumptions on F, u_0, u_1 and f .

Our primary interest is the study of initial-boundary value problem (2.1.1) in a certain class, for example,

$$u = u(t) \in \mathcal{C}([0, T]; H^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M})).$$

We remind the reader that u belongs to this class when u is a continuous function of $t \in [0, T]$ with value in $H^1(\mathcal{M})$ and is continuously differentiable with respect to $t \in [0, T]$ in $L^2(\mathcal{M})$. We state two main results for the unique existence of the solution with a priori estimates to problem (2.1.1) in two choices of the spaces of data F, u_0, u_1, f (Theorems 2.2.5 and 2.2.6). Moreover we prove the regularity of the Neumann derivatives in both two cases.

If H is a Banach space, then we denote by $L^1(0, T; H)$ the space of measurable functions $h : (0, T) \rightarrow H$, where the norm is defined by

$$\int_0^T \|h(t)\|_H dt = \|h\|_{L^1(0, T; H)} < \infty.$$

It is known that the space $L^1(0, T; H)$ is complete.

We show a classical inequality and use frequently.

Lemma 2.1.1 (*Gronwall's inequality*) *Let an interval I be $[a, +\infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let α , β and u be real-valued functions defined on I . Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I .*

1. *If β is non-negative and u satisfies the integral inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \quad \forall t \in I,$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)dr\right) ds, \quad \forall t \in I.$$

2. *If, in addition, the function α is non-decreasing, then*

$$u(t) \leq \alpha(t) \exp\left(\int_a^t \beta(s)ds\right).$$

Proof .

1. Define:

$$v(s) = \exp\left(-\int_a^s \beta(r)dr\right) \int_a^s \beta(r)u(r)dr, \quad \forall s \in I.$$

We obtain for the derivative

$$v'(s) = \underbrace{\left(u(s) - \int_a^s \beta(r)u(r)dr\right)}_{\leq \alpha(s)} \beta(s) \exp\left(-\int_a^s \beta(r)dr\right), \quad \forall s \in I.$$

Since β and the exponential are non-negative, this gives an upper estimate for the derivative of v . Since $v(a) = 0$, integration of this inequality from a to t gives:

$$v(t) \leq \int_a^t \alpha(s)\beta(s) \exp\left(-\int_a^s \beta(r)dr\right) ds, \quad \forall t \in I.$$

Using the definition of $v(t)$ for the first step, and then this inequality and the functional equation of the exponential function, we obtain

$$\begin{aligned} \int_a^t \beta(s)u(s)ds &= v(t) \exp\left(\int_a^t \beta(r)dr\right) \\ &\leq \int_a^t \alpha(s)\beta(s) \exp\left(\underbrace{\int_a^t \beta(r)dr - \int_a^s \beta(r)dr}_{= \int_s^t \beta(r)dr}\right) ds \end{aligned}$$

Substituting this result into the assumed integral inequality gives Gronwall's inequality.

2. If the function α is non-decreasing, then using the part 1 and $\alpha(s) \leq \alpha(t)$ for $s \leq t$, we obtain

$$u(t) \leq \alpha(t) + \left[-\alpha(t) \exp \left(\int_s^t \beta(r) dr \right) \right]_{s=a}^{s=t} = \alpha(t) \exp \left(\int_a^t \beta(r) dr \right), \quad \forall t \in I.$$

This completes the proof of the lemma. \square

2.2 Principal results

Theorem 2.2.5 *Let $T > 0$ be given. Suppose that*

$$F \in L^1(0, T; L^2(\mathcal{M})), \quad u_0 \in H^1(\mathcal{M}), \quad u_1 \in L^2(\mathcal{M}) \quad \text{and} \quad f \in H^1((0, T) \times \partial\mathcal{M}).$$

Assume, in addition, that the following compatibility condition is valid:

$$f(0, \cdot) = u_0|_{\partial\mathcal{M}}. \quad (2.2.1)$$

Then there exists a unique solution u of (2.1.1) satisfying

$$u \in \mathcal{C}([0, T]; H^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M})),$$

and there exists $C > 0$ such that for any $t \in (0, T)$, we have

$$\begin{aligned} & \|u(t)\|_{H^1(\mathcal{M})} + \|\partial_t u(t)\|_{L^2(\mathcal{M})} \\ & \leq C \left(\|f\|_{H^1((0, T) \times \partial\mathcal{M})} + \|u_0\|_{H^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0, T; L^2(\mathcal{M}))} \right). \end{aligned} \quad (2.2.2)$$

Furthermore

$$\partial_\nu u \in L^2((0, T) \times \partial\mathcal{M}) \quad (2.2.3)$$

and there exists a constant $C = C(T, \mathcal{M}) > 0$ such that

$$\|\partial_\nu u\|_{L^2((0, T) \times \partial\mathcal{M})} \leq C \left(\|f\|_{H^1((0, T) \times \partial\mathcal{M})} + \|u_0\|_{H^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0, T; L^2(\mathcal{M}))} \right). \quad (2.2.4)$$

Theorem 2.2.6 *Let $T > 0$ be given. Suppose that*

$$F \in L^1(0, T; H^{-1}(\mathcal{M})), \quad u_0 \in L^2(\mathcal{M}) \quad u_1 \in H^{-1}(\mathcal{M}), \quad \text{and} \quad f \in L^2((0, T) \times \partial\mathcal{M}).$$

Then there exists a unique solution u of (2.1.1) such that

$$u \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})), \quad (2.2.5)$$

and there exists $C > 0$ such that for any $t \in (0, T)$ we have

$$\begin{aligned} & \|u(t)\|_{L^2(\mathcal{M})} + \|\partial_t u(t)\|_{H^{-1}(\mathcal{M})} \\ & \leq C \left(\|f\|_{L^2((0,T) \times \partial\mathcal{M})} + \|u_0\|_{L^2(\mathcal{M})} + \|u_1\|_{H^{-1}(\mathcal{M})} + \|F\|_{L^1(0,T;H^{-1}(\mathcal{M}))} \right). \end{aligned} \quad (2.2.6)$$

Furthermore,

$$\partial_\nu u \in H^{-1}((0, T) \times \partial\mathcal{M}), \quad (2.2.7)$$

and there exists a constant $C = C(T, \mathcal{M}) > 0$ such that

$$\begin{aligned} & \|\partial_\nu u\|_{H^{-1}((0,T) \times \partial\mathcal{M})} \\ & \leq C \left(\|f\|_{L^2((0,T) \times \partial\mathcal{M})} + \|u_0\|_{L^2(\mathcal{M})} + \|u_1\|_{H^{-1}(\mathcal{M})} + \|F\|_{L^1(0,T;H^{-1}(\mathcal{M}))} \right). \end{aligned} \quad (2.2.8)$$

Theorem 2.2.5 gives a rather comprehensive regularity result for (2.1.1) with $f \in H^1((0, T) \times \partial\mathcal{M})$, while Theorem 2.2.6 is another regularity result with weaker regularity condition on F, u_0, u_1, f .

In order to prove Theorems 2.2.5-2.2.6, in Sections 2.3 and 2.4, we first prove regularity results for the wave equation with the homogenous boundary condition. Then on the basis of the transposition, we establish Theorems 2.2.5-2.2.6.

2.3 Homogenous boundary condition

We start with the case $f = 0$. Then compatibility condition (2.2.1) implies that $u_0 \in H_0^1(\mathcal{M})$. Let us consider the following initial and homogenous boundary value problem for the wave equation:

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) u(t, x) = F(t, x) & \text{in } (0, T) \times \mathcal{M}, \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 & \text{in } \mathcal{M}, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (2.3.1)$$

2.3.1 Existence and uniqueness of a solution

Let H be a separable real Hilbert space, and let V be another separable Hilbert space, which is continuously and densely embedded in H . By $(\cdot, \cdot)_{V', V}$, we denote the dual pairing between V' and V . Moreover let $A \in \mathcal{C}([0, T]; \mathcal{L}(V, V'))$, and let

$$a(t, u, v) = -(A(t)u, v)_{V', V}$$

be the associated quadratic form. We assume that a is symmetric:

$$a(t, u, v) = a(t, v, u)$$

and there exist positive constants α and β such that

$$a(t, u, v) \geq \alpha \|u\|_V^2 - \beta \|u\|_H^2.$$

We consider the evolution equation

$$u''(t) = A(t)u + F(t), \quad u(0) = u_0, \quad u'(0) = u_1 \quad (2.3.2)$$

Let us recall the following classical result (see Lions and Magenes [47]).

Theorem 2.3.7 *Assume that $F \in L^1(0, T; H)$, $u_0 \in V$, $u_1 \in H$ and that A is as described above. Then there exists a unique weak solution*

$$u \in \mathcal{C}([0, T]; V) \cap \mathcal{C}^1([0, T]; H)$$

to the evolution problem (2.3.2).

Let A be the positive self-adjoint operator induced by the bilinear form $a(\cdot, \cdot)$, that is, A is defined by

$$(Au, v)_{H^{-1}, H_0^1} = a(u, v) = \int_{\mathcal{M}} (\langle \nabla u, \nabla v \rangle + quv) \, dv_g, \quad \forall u, v \in H_0^1(\mathcal{M}).$$

Then A is an operator from $V = H_0^1(\mathcal{M})$ into $V' = H^{-1}(\mathcal{M})$, and there exist positive constants α and β such that

$$a(u, v) \geq \alpha \|u\|_{H_0^1(\mathcal{M})}^2 - \beta \|u\|_{L^2(\mathcal{M})}^2.$$

2.3.2 Regularity of solutions

Lemma 2.3.1 *Let $T > 0$ and $q \in L^\infty(\mathcal{M})$ be given. Suppose that*

$$F \in L^1(0, T; L^2(\mathcal{M})), \quad u_0 \in H_0^1(\mathcal{M}), \quad u_1 \in L^2(\mathcal{M}), \quad \text{and} \quad f \equiv 0.$$

Then the unique solution u of (2.3.1) satisfies

$$u \in \mathcal{C}([0, T]; H_0^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M})). \quad (2.3.3)$$

Furthermore there exists a constant $C > 0$ such that we have

$$\|u(t)\|_{H^1(\mathcal{M})} + \|\partial_t u(t)\|_{L^2(\mathcal{M})} \leq C \left(\|u_0\|_{H^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0, T; L^2(\mathcal{M}))} \right). \quad (2.3.4)$$

Proof . Using the classical result for the existence and uniqueness of weak solutions given by Theorem 2.3.7 in abstract evolution equations setting, we obtain

$$u \in \mathcal{C}([0, T]; H_0^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M})). \quad (2.3.5)$$

Multiplying the first equation of (2.3.1) by $\partial_t u$ and using Green's formula, we obtain

$$\frac{d}{dt} \left[\int_{\mathcal{M}} (|\partial_t u(t)|^2 + |\nabla u(t)|^2) \, dv_g \right] + \int_{\mathcal{M}} qu \partial_t u \, dv_g = \int_{\mathcal{M}} F(t, x) \partial_t u(t) \, dv_g. \quad (2.3.6)$$

Let $e(t) = \left(\|\nabla u(t)\|_{L^2(\mathcal{M})}^2 + \|\partial_t u(t)\|_{L^2(\mathcal{M})}^2 \right)^{1/2}$ for $t \in (0, T)$. Then, by (2.3.6), we obtain

$$\frac{d}{dt} (e^2(t)) \leq C \left(\|F(t, \cdot)\|_{L^2(\mathcal{M})} e(t) + e^2(t) \right), \quad \forall t \in (0, T), \quad (2.3.7)$$

which implies that $e'(t) \leq C \left(\|F(t, \cdot)\|_{L^2(\mathcal{M})} + e(t) \right)$. By Gronwall's lemma we find

$$e(t) \leq C_T \left(e(0) + \int_0^T \|F(t, \cdot)\|_{L^2(\mathcal{M})} \, dt \right), \quad \forall t \in (0, T). \quad (2.3.8)$$

The proof of (2.3.4) is completed. \square

Lemma 2.3.2 *Let $T > 0$ and $q \in L^\infty(\mathcal{M})$ be given. Suppose that*

$$F \in L^1(0, T; H^{-1}(\mathcal{M})), \quad u_0 = 0, \quad u_1 = 0, \quad \text{and} \quad f \equiv 0.$$

Then the unique solution u of (2.3.1) satisfies

$$u \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})). \quad (2.3.9)$$

Furthermore there exists a constant $C > 0$ such that

$$\|u(t)\|_{L^2(\mathcal{M})} + \|\partial_t u(t)\|_{H^{-1}(\mathcal{M})} \leq C \|F\|_{L^1(0, T; H^{-1}(\mathcal{M}))}. \quad (2.3.10)$$

Proof . Fix $\lambda > 0$ large. Let A be the positive self-adjoint operator in $H^{-1}(\mathcal{M})$ induced by the bilinear form $a(\cdot, \cdot)$, that is,

$$(A\varphi, \psi)_{H^{-1}, H_0^1} = a(\varphi, \psi) = \int_{\mathcal{M}} (\langle \nabla \varphi, \nabla \psi \rangle + (q + \lambda)\varphi\psi) \, dv_g, \quad \forall \varphi, \psi \in H_0^1(\mathcal{M}).$$

By means of the Lax-Milgram theorem, A is an isomorphism from $\mathcal{D}(A) = H_0^1$ into $H^{-1}(\mathcal{M})$, and $A\varphi = (-\Delta_g + q + \lambda)\varphi$ whenever $\varphi \in H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$, and $A^{-1}\psi = (-\Delta_g + q + \lambda)^{-1}\psi$ for any $\psi \in L^2(\mathcal{M})$. Moreover $A^{1/2}$ is an isomorphism from $H^{-1}(\mathcal{M})$ onto $L^2(\mathcal{M})$. Define

$$w = A^{-1/2}u.$$

Then w satisfies the following boundary value problem:

$$\begin{cases} (\partial_t^2 + A - \lambda)w = A^{-1/2}F, & \text{in } (0, T) \times \mathcal{M}, \\ w(0, \cdot) = 0, \quad \partial_t w(0, \cdot) = 0 & \text{in } \mathcal{M}, \\ w = 0, & \text{on } (0, T) \times \partial\mathcal{M}, \end{cases} \quad (2.3.11)$$

equivalently

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x))w = A^{-1/2}F, & \text{in } (0, T) \times \mathcal{M}, \\ w(0, \cdot) = 0, \quad \partial_t w(0, \cdot) = 0 & \text{in } \mathcal{M}, \\ w = 0, & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (2.3.12)$$

On the other hand, we have $A^{-1/2}F \in L^1(0, T; L^2(\mathcal{M}))$. Thus by Lemma 2.3.1, we see that

$$w \in \mathcal{C}([0, T]; H_0^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M})). \quad (2.3.13)$$

Furthermore there exists a constant $C > 0$ such that

$$\|w(t)\|_{H^1(\mathcal{M})} + \|\partial_t w(t)\|_{L^2(\mathcal{M})} \leq C \|A^{-1/2}F\|_{L^1(0, T; L^2(\mathcal{M}))}. \quad (2.3.14)$$

This implies

$$u \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})). \quad (2.3.15)$$

Furthermore there exists a constant $C > 0$ such that

$$\|u(t)\|_{L^2(\mathcal{M})} + \|\partial_t u(t)\|_{H^{-1}(\mathcal{M})} \leq C \|F\|_{L^1(0, T; H^{-1}(\mathcal{M}))}. \quad (2.3.16)$$

This completes the proof. \square

Lemma 2.3.3 *Let $T > 0$ and $q \in L^\infty(\mathcal{M})$ be given. Suppose that*

$$F \in L^1(0, T; H_0^1(\mathcal{M})), \quad u_0 \in H^2(\mathcal{M}) \cap H_0^1(\mathcal{M}), \quad u_1 \in H_0^1(\mathcal{M}) \quad \text{and} \quad f \equiv 0.$$

Then the unique solution u of (2.3.1) satisfies

$$u \in \mathcal{C}([0, T]; H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H_0^1(\mathcal{M})) \quad (2.3.17)$$

and there exists $C > 0$ such that for any $t \in (0, T)$ we have

$$\|u(t)\|_{H^2(\mathcal{M})} + \|\partial_t u(t)\|_{H^1(\mathcal{M})} \leq C \left(\|u_0\|_{H^2(\mathcal{M})} + \|u_1\|_{H^1(\mathcal{M})} + \|F\|_{L^1(0, T; H_0^1(\mathcal{M}))} \right). \quad (2.3.18)$$

Proof . By the duality argument, the proof follows from Lemma 2.3.2. \square

Lemma 2.3.4 *Let $T > 0$ and $q \in L^\infty(\mathcal{M})$ be given. Suppose that*

$$F \in L^1(0, T; L^2(\mathcal{M})), \quad u_0 \in H_0^1(\mathcal{M}), \quad u_1 \in L^2(\mathcal{M}) \quad \text{and} \quad f \equiv 0.$$

Then the unique solution u of (2.3.1) satisfies

$$u \in \mathcal{C}([0, T]; H_0^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M})) \quad (2.3.19)$$

and there exists $C > 0$ such that for any $t \in (0, T)$ we have

$$E(t) \leq C \left(E(s) + \|F\|_{L^1(0,T;L^2(\mathcal{M}))} \right) e^{C\|q\|_{L^\infty(\mathcal{M})}}, \quad \text{for all } t, s \in [0, T] \quad (2.3.20)$$

for some constant $C = C(T, \mathcal{M})$, where

$$E(t) = \frac{1}{2} \left(\|u(t)\|_{H^1(\mathcal{M})} + \|\partial_t u(t)\|_{L^2(\mathcal{M})} \right).$$

Proof . Let $\delta = \|q\|_{L^\infty(\mathcal{M})}$ and let us set

$$E_\delta(t) = E(t) + \frac{\delta}{2} \|u(t)\|_{L^2(\mathcal{M})}^2.$$

We have

$$E'_\delta(t) = \int_{\mathcal{M}} (\delta - q(x))u(x, t)u'(x, t)dx + \int_{\mathcal{M}} F(x, t)u'(x, t)dx.$$

Then

$$\begin{aligned} E'_\delta(t) &\leq 2\delta \|u(t)\|_{L^2(\mathcal{M})} \|u'(t)\|_{L^2(\mathcal{M})} + \|F(t)\|_{L^2(\mathcal{M})} \|u'(t)\|_{L^2(\mathcal{M})} \\ &\leq (1 + 2\sqrt{\delta})E_\delta(t) + \frac{1}{2} \|F(t)\|_{L^2(\mathcal{M})} \end{aligned}$$

and therefore

$$E_\delta(t) \leq C \left(E_\delta(s) + \|F\|_{L^1(0,T;L^2(\mathcal{M}))} \right) e^{C\|q\|_{L^\infty(\mathcal{M})}}, \quad \text{for all } t, s \in [0, T]$$

from where (2.3.20) easily follows. \square

2.4 Regularity of the normal derivative

Lemma 2.4.1 *Let $T > 0$, $q \in L^\infty(\mathcal{M})$ be given. Assume that $f \equiv 0$. Then for a unique solution u to (2.3.1), the mapping*

$$(u_0, u_1, F) \mapsto \partial_\nu u,$$

is linear and continuous from $H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \times L^1(0, T; L^2(\mathcal{M}))$ to $L^2((0, T) \times \partial\mathcal{M})$. Furthermore there exists a constant $C > 0$ such that

$$\|\partial_\nu u\|_{L^2((0,T) \times \partial\mathcal{M})} \leq C \left(\|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0,T;L^2(\mathcal{M}))} \right). \quad (2.4.1)$$

Proof . Let N be a C^2 vector field on $\overline{\mathcal{M}}$ such that

$$N(x) = \nu(x), \quad x \in \partial\mathcal{M}; \quad |N(x)| \leq 1, \quad x \in \mathcal{M}. \quad (2.4.2)$$

Multiply both sides of the first equation in (2.3.1) by $\langle N, \nabla u \rangle$ and integrate over $(0, T) \times \mathcal{M}$, and we have

$$\begin{aligned} I &:= \int_0^T \int_{\mathcal{M}} F(t, x) \langle N, \nabla u \rangle \, dv_g dt = \int_0^T \int_{\mathcal{M}} \partial_t^2 u \langle N, \nabla u \rangle \, dv_g dt \\ &\quad - \int_0^T \int_{\mathcal{M}} \Delta_g u \langle N, \nabla u \rangle \, dv_g dt + \int_0^T \int_{\mathcal{M}} q(x) u \langle N, \nabla u \rangle \, dv_g dt \\ &:= I_1 + I_2 + I_3. \end{aligned} \quad (2.4.3)$$

Integrating by parts with respect to t , we obtain

$$\begin{aligned} I_1 &= \int_0^T \int_{\mathcal{M}} \partial_t^2 u \langle N, \nabla u \rangle \, dv_g dt = \left[\int_{\mathcal{M}} \partial_t u \langle N, \nabla u \rangle \, dv_g \right]_0^T \\ &\quad - \frac{1}{2} \int_0^T \int_{\mathcal{M}} \langle N, \nabla(|\partial_t u|^2) \rangle \, dv_g dt. \end{aligned} \quad (2.4.4)$$

Then, by the divergence theorem, we obtain

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} \partial_t^2 u \langle N, \nabla u \rangle \, dv_g dt &= \left[\int_{\mathcal{M}} \partial_t u \langle N, \nabla u \rangle \, dv_g \right]_0^T \\ &\quad + \frac{1}{2} \int_0^T \int_{\mathcal{M}} \operatorname{div}(N) |\partial_t u|^2 \, dv_g dt - \frac{1}{2} \left[\int_0^T \int_{\partial\mathcal{M}} |\partial_t u|^2 \, d\sigma_g dt \right]. \end{aligned} \quad (2.4.5)$$

Since the last term is 0, using (2.3.4) in Lemma 2.3.1, we conclude that

$$|I_1| \leq C \left(\|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0,T;L^2(\mathcal{M}))} \right)^2. \quad (2.4.6)$$

On the other hand, Green's theorem yields

$$\begin{aligned} I_2 &= - \int_0^T \int_{\mathcal{M}} \Delta_g u \langle N, \nabla u \rangle \, dv_g dt = \int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla(\langle N, \nabla u \rangle) \rangle \, dv_g dt \\ &\quad - \int_0^T \int_{\partial\mathcal{M}} |\partial_\nu u|^2 \, d\sigma_g dt. \end{aligned} \quad (2.4.7)$$

Thus by Lemma 1.4.1, we deduce

$$\begin{aligned} I_2 &= - \int_0^T \int_{\partial\mathcal{M}} |\partial_\nu u|^2 \, d\sigma_g dt + \frac{1}{2} \int_0^T \int_{\partial\mathcal{M}} |\nabla u|^2 \, d\sigma_g dt \\ &\quad - \int_0^T \int_{\mathcal{M}} DN(\nabla u, \nabla u) \, dv_g dt + \frac{1}{2} \int_0^T \int_{\mathcal{M}} |\nabla u|^2 \operatorname{div}(N) \, dv_g dt. \end{aligned} \quad (2.4.8)$$

Using the fact

$$|\nabla u|^2 = |\partial_\nu u|^2 + |\nabla_\tau u|^2 = |\partial_\nu v|^2, \quad x \in \partial\mathcal{M}$$

and ∇_τ is the gradient of the tangential on $\partial\mathcal{M}$, we obtain

$$\begin{aligned} I_2 = & -\frac{1}{2} \int_0^T \int_{\partial\mathcal{M}} |\partial_\nu u|^2 \, d\sigma_g dt - \int_0^T \int_{\mathcal{M}} DN(\nabla u, \nabla u) \, dv_g dt \\ & + \frac{1}{2} \int_0^T \int_{\mathcal{M}} |\nabla u|^2 \operatorname{div}(N) \, dv_g dt + \underbrace{\left[\frac{1}{2} \int_0^T \int_{\partial\mathcal{M}} |\nabla_\tau u|^2 \, d\sigma_g dt \right]}_{=0}. \end{aligned} \quad (2.4.9)$$

Consequently we deduce

$$\begin{aligned} \left| \int_0^T \int_{\partial\mathcal{M}} |\partial_\nu u|^2 \, d\sigma_g dt \right| & \leq C \left(|I_2| + \left(\|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0,T;L^2(\mathcal{M}))} \right)^2 \right) \\ & \leq C \left(|I_1| + |I_3| + |I| + \left(\|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0,T;L^2(\mathcal{M}))} \right)^2 \right). \end{aligned} \quad (2.4.10)$$

Finally by Lemma 2.3.1, we have

$$|I| + |I_3| \leq C \left(\|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0,T;L^2(\mathcal{M}))} \right)^2. \quad (2.4.11)$$

Collecting (2.4.11), (2.4.10) and (2.4.6), we obtain

$$\int_0^T \int_{\partial\mathcal{M}} |\partial_\nu v|^2 \, d\sigma_g dt \leq C \left(\|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|F\|_{L^1(0,T;L^2(\mathcal{M}))} \right)^2. \quad (2.4.12)$$

This completes the proof of (2.4.1). \square

2.5 Non-homogenous boundary condition

We now turn to the non-homogenous case of the wave problem (2.1.1). Let $\mathcal{H} = L^1(0, T; L^2(\mathcal{M}))$. By $(\cdot, \cdot)_{\mathcal{H}', \mathcal{H}}$, we denote the dual pairing between \mathcal{H}' and \mathcal{H} .

Definition 2.5.39 *Let $T > 0$, $q \in L^\infty(\mathcal{M})$ be given, and let*

$$F \equiv 0, \quad u_0 \in L^2(\mathcal{M}), \quad u_1 \in H^{-1}(\mathcal{M}), \quad \text{and} \quad f \in L^2((0, T) \times \partial\mathcal{M}).$$

Then we say that $u \in \mathcal{H}'$ is a solution of (2.1.1) in the transposition sense if for any $\phi \in \mathcal{H}$, we have

$$(u, \phi)_{\mathcal{H}', \mathcal{H}} = \left[(u_1, v(0))_{H^{-1}, H_0^1} - (u_0, v'(0))_{L^2, L^2} \right] - \int_0^T \int_{\partial\mathcal{M}} f(t, x) \partial_\nu v(t, x) \, d\sigma_g \, dt, \quad (2.5.1)$$

where $v = v(t, x)$ is the solution of the homogenous boundary value problem:

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) v(t, x) = \phi(t, x) & \text{in } (0, T) \times \mathcal{M}, \\ v(T, x) = 0, \quad v'(T, x) = 0 & \text{in } \mathcal{M}, \\ v(t, x) = 0, & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (2.5.2)$$

Henceforth we always interpret the solution to (2.1.1) in the transposition sense.

Then we see the following lemma.

Lemma 2.5.1 *Let $T > 0$, $q \in L^\infty(\mathcal{M})$ be given. Assume that*

$$F \equiv 0, \quad u_0 \in L^2(\mathcal{M}), \quad u_1 \in H^{-1}(\mathcal{M}), \quad \text{and} \quad f \in L^2((0, T) \times \partial\mathcal{M}).$$

There exists a unique solution to (2.1.1)

$$u \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})). \quad (2.5.3)$$

Furthermore there exists a constant $C > 0$ such that

$$\|u(t)\|_{L^2(\mathcal{M})} + \|u'(t)\|_{H^{-1}(\mathcal{M})} \leq C \left(\|u_0\|_{L^2(\mathcal{M})} + \|u_1\|_{H^{-1}(\mathcal{M})} + \|f\|_{L^2((0, T) \times \partial\mathcal{M})} \right). \quad (2.5.4)$$

Proof . Let $\phi \in \mathcal{H} := L^1(0, T; L^2(\mathcal{M}))$. Let $v \in \mathcal{C}([0, T]; H_0^1(\mathcal{M}))$ be a solution of the final boundary value problem for the wave equation (2.5.2). By Lemmata 2.4.1 and 2.3.1, the mapping $\phi \mapsto \partial_\nu v$ is linear and continuous from \mathcal{H} to $L^2((0, T) \times \mathcal{M})$ and there exists $C > 0$ such that

$$\|v(t)\|_{H_0^1(\mathcal{M})} + \|v'(t)\|_{L^2(\mathcal{M})} \leq C \|\phi\|_{\mathcal{H}} \quad (2.5.5)$$

and

$$\|\partial_\nu v\|_{L^2((0, T) \times \partial\mathcal{M})} \leq C \|\phi\|_{\mathcal{H}}. \quad (2.5.6)$$

We define a linear functional ℓ on the linear space \mathcal{H} as follows:

$$\ell(\phi) = \left[(u_1, v(0))_{H^{-1}, H_0^1} - (u_0, v'(0))_{L^2, L^2} \right] - \int_0^T \int_{\partial\mathcal{M}} f(t, x) \partial_\nu v(t, x) \, d\sigma_g \, dt,$$

where v solves (2.5.2). By (2.5.5)-(2.5.6), we obtain

$$|\ell(\phi)| \leq C \left(\|u_0\|_{L^2(\mathcal{M})} + \|u_1\|_{H^{-1}(\mathcal{M})} + \|f\|_{L^2((0, T) \times \partial\mathcal{M})} \right) \|\phi\|_{\mathcal{H}}.$$

It is known that any linear bounded functional on the space \mathcal{H} can be written as

$$\ell(\phi) = (u, \phi)_{\mathcal{H}', \mathcal{H}},$$

where u is some element form the space \mathcal{H}' . Thus system (2.1.1) admits a solution $u \in \mathcal{H}'$ in the transposition sense and we see

$$\|u\|_{\mathcal{H}'} \leq C \left(\|u_0\|_{L^2(\mathcal{M})} + \|u_1\|_{H^{-1}(\mathcal{M})} + \|f\|_{L^2((0,T) \times \partial\mathcal{M})} \right).$$

This completes the proof of the lemma. \square

Next we need the following estimate for non-homogenous elliptic boundary value problem (see Lions and Magenes [47]).

Lemma 2.5.2 *Let $\psi \in H^{-1}(\mathcal{M})$ and $\phi \in H^1(\partial\mathcal{M})$. Let $w \in H^1(\mathcal{M})$ be a unique solution of the following boundary value problem:*

$$\begin{cases} \Delta_g w = \psi & \text{in } \mathcal{M}, \\ w = \phi & \text{on } \partial\mathcal{M}. \end{cases} \quad (2.5.7)$$

Then the following estimate holds true:

$$\|w\|_{H^1(\mathcal{M})} \leq C \left(\|\psi\|_{H^{-1}(\mathcal{M})} + \|\phi\|_{H^1(\partial\mathcal{M})} \right). \quad (2.5.8)$$

2.6 Completion of Proofs of Theorems 2.2.5 and 2.2.6

First we complete the proof of Theorem 2.2.5. First we decompose the solution u of (2.1.1) as

$$u = y + z,$$

where y and z are the solutions respectively to

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) y(t, x) = F & \text{in } (0, T) \times \mathcal{M}, \\ y(0, x) = 0, \quad y'(0, x) = 0 & \text{in } \mathcal{M}, \\ y(t, x) = 0 & \text{on } (0, T) \times \partial\mathcal{M} \end{cases} \quad (2.6.1)$$

and

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) z(t, x) = 0 & \text{in } (0, T) \times \mathcal{M}, \\ z(0, x) = u_0(x), \quad z'(0, x) = u_1(x) & \text{in } \mathcal{M}, \\ z(t, x) = f(t, x) & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (2.6.2)$$

By Lemma 2.3.1, we have

$$y \in \mathcal{C}([0, T]; H_0^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M})) \quad (2.6.3)$$

and there exists a constant $C > 0$ such that we have

$$\|y(t)\|_{H^1(\mathcal{M})} + \|y'(t)\|_{L^2(\mathcal{M})} \leq C \|F\|_{L^1(0,T;L^2(\mathcal{M}))}. \quad (2.6.4)$$

Furthermore, by Lemma 2.3.3, there exists a constant $C > 0$ such that

$$\|\partial_\nu y\|_{L^2((0,T)\times\partial\mathcal{M})} \leq C \|F\|_{L^1(0,T;L^2(\mathcal{M}))}. \quad (2.6.5)$$

Next put $z' = \theta$. Then

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x))\theta(t, x) = 0 & \text{in } (0, T) \times \mathcal{M}, \\ \theta(0, x) = u_1(x), \quad \theta'(0, x) = (\Delta_g - q(x))u_0(x) & \text{in } \mathcal{M}, \\ \theta(t, x) = f'(t, x) & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (2.6.6)$$

Since $f' \in L^2((0, T) \times \partial\mathcal{M})$, $u_1 \in L^2(\mathcal{M})$ and $(\Delta_g - q(x))u_0 \in H^{-1}(\mathcal{M})$, by Lemma 2.5.1, we have

$$\theta \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})). \quad (2.6.7)$$

Furthermore there exists a constant $C > 0$ such that

$$\|\theta\|_{\mathcal{C}([0,T];L^2(\mathcal{M}))} + \|\theta'\|_{\mathcal{C}([0,T];H^{-1}(\mathcal{M}))} \leq C \left(\|u_0\|_{H^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|f\|_{H^1(0,T;L^2(\partial\mathcal{M}))} \right). \quad (2.6.8)$$

Thus (2.6.7) implies the following regularity for z :

$$z \in \mathcal{C}^1([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^2([0, T]; H^{-1}(\mathcal{M}))$$

and

$$\Delta_g z \in \mathcal{C}([0, T]; H^{-1}(\mathcal{M})).$$

Moreover there exists $C > 0$ such that

$$\begin{aligned} \|z'(t)\|_{L^2(\mathcal{M})} &\leq C \left(\|u_0\|_{H^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|f\|_{H^1(0,T;L^2(\partial\mathcal{M}))} \right), \\ \|\Delta_g z(t)\|_{H^{-1}(\mathcal{M})} &\leq C \left(\|u_0\|_{H^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|f\|_{H^1(0,T;L^2(\partial\mathcal{M}))} \right). \end{aligned} \quad (2.6.9)$$

Using Lemma 2.5.2, we can find

$$\|z(t)\|_{H^1(\mathcal{M})} + \|z'(t)\|_{L^2(\mathcal{M})} \leq C \left(\|u_0\|_{H^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|f\|_{H^1((0,T)\times\partial\mathcal{M})} \right). \quad (2.6.10)$$

Collecting (2.6.10) and (2.6.4), we obtain

$$\|u(t)\|_{H^1(\mathcal{M})} + \|u'(t)\|_{L^2(\mathcal{M})} \leq C \left(\|u_0\|_{H^1(\mathcal{M})} + \|u_1\|_{L^2(\mathcal{M})} + \|f\|_{H^1((0,T)\times\partial\mathcal{M})} \right). \quad (2.6.11)$$

The proof of (2.2.4) is similar to Lemma 2.4.1. If one multiplies (2.1.1) by $\langle N, \nabla z \rangle$, then the arguments leading to (2.4.3) gives

$$\begin{aligned} 0 &= \int_0^T \int_{\mathcal{M}} \partial_t^2 z \langle N, \nabla z \rangle \, dv_g dt - \int_0^T \int_{\mathcal{M}} \Delta_g z \langle N, \nabla z \rangle \, dv_g dt \\ &\quad + \int_0^T \int_{\mathcal{M}} q(x) u \langle N, \nabla z \rangle_g \, dv_g dt := I'_1 + I'_2 + I'_3 \end{aligned} \quad (2.6.12)$$

with

$$|I'_1| \leq C \left(\|f\|_{H^1((0,T) \times \partial\mathcal{M})}^2 + \|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{H^1(\mathcal{M})} \right)^2. \quad (2.6.13)$$

Furthermore we derive from Green's formula

$$I'_2 = \frac{1}{2} \int_0^T \int_{\partial\mathcal{M}} |\partial_\nu z|^2 \, d\sigma_g dt + \int_0^T \int_{\mathcal{M}} DN(\nabla z, \nabla z) \, dv_g dt \\ - \frac{1}{2} \int_0^T \int_{\mathcal{M}} |\nabla z|^2 \operatorname{div}(N) \, dv_g dt - \frac{1}{2} \int_0^T \int_{\partial\mathcal{M}} |\nabla_\tau f|^2 \, d\sigma_g dt. \quad (2.6.14)$$

By this with

$$|I'_3| \leq C \left(\|f\|_{H^1((0,T) \times \partial\mathcal{M})} + \|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{H^1(\mathcal{M})} \right)^2, \quad (2.6.15)$$

we derive from (2.6.13), (2.6.14) and (2.6.12) that

$$\|\partial_\nu u\|_{L^2((0,T) \times \partial\mathcal{M})} \leq C \left(\|f\|_{H^1((0,T) \times \partial\mathcal{M})} + \|u_0\|_{H_0^1(\mathcal{M})} + \|u_1\|_{H^1(\mathcal{M})} \right). \quad (2.6.16)$$

The proof of Theorem 2.2.5 is now completed.

Next we proceed to the proof of Theorem 2.2.6. We decompose the solution u of (2.1.1) as

$$u = y + z,$$

where y and z the solutions respectively to (2.6.1) and (2.6.2).

Lemma 2.3.2 implies

$$y \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})) \quad (2.6.17)$$

and there exists a constant $C > 0$ such that

$$\|y(t)\|_{L^2(\mathcal{M})} + \|y'(t)\|_{H^{-1}(\mathcal{M})} \leq C \|F\|_{L^1(0,T;H^{-1}(\mathcal{M}))}. \quad (2.6.18)$$

Next, by Lemma 2.5.1, we have

$$z \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})) \quad (2.6.19)$$

and there exists a constant $C > 0$ such that we have

$$\|z(t)\|_{L^2(\mathcal{M})} + \|z'(t)\|_{H^{-1}(\mathcal{M})} \leq C \left(\|u_0\|_{L^2(\mathcal{M})} + \|u_1\|_{H^{-1}(\mathcal{M})} + \|f\|_{L^2((0,T) \times \partial\mathcal{M})} \right). \quad (2.6.20)$$

Combining (2.6.20) and (2.6.19), we see that

$$u \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})) \quad (2.6.21)$$

and there exists a constant $C > 0$ such that

$$\begin{aligned} & \|u(t)\|_{L^2(\mathcal{M})} + \|u'(t)\|_{H^{-1}(\mathcal{M})} \\ & \leq C \left(\|u_0\|_{L^2(\mathcal{M})} + \|u_1\|_{H^{-1}(\mathcal{M})} + \|f\|_{L^2((0,T)\times\partial\mathcal{M})} + \|F\|_{L^1(0,T;H^{-1}(\mathcal{M}))} \right). \end{aligned} \quad (2.6.22)$$

Now it remains to prove $\partial_\nu u \in H^{-1}((0, T) \times \partial\mathcal{M})$ and (2.2.8). Let $\psi \in H^1((0, T) \times \partial\mathcal{M})$ satisfy $\psi(0, \cdot) = \psi(T, \cdot) = 0$ on $\partial\mathcal{M}$. Let w be a solution of

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) w(t, x) = 0 & \text{in } (0, T) \times \mathcal{M}, \\ w(T, x) = 0, \quad w'(T, x) = 0 & \text{in } \mathcal{M}, \\ w(t, x) = \psi(t, x) & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (2.6.23)$$

Then by Theorem 2.2.5, we have

$$\|w(t)\|_{H^1(\mathcal{M})} + \|w'(t)\|_{L^2(\mathcal{M})} \leq C \|\psi\|_{H^1((0,T)\times\partial\mathcal{M})}. \quad (2.6.24)$$

Furthermore there exists a constant $C = C(T, \mathcal{M}) > 0$ such that

$$\|\partial_\nu w\|_{L^2((0,T)\times\partial\mathcal{M})} \leq C \|\psi\|_{H^1((0,T)\times\partial\mathcal{M})}. \quad (2.6.25)$$

Multiplying the first equation of (2.6.23) by u , integrating it on $(0, T) \times \mathcal{M}$ and using Green's formula, we obtain

$$\int_0^T \int_{\partial\mathcal{M}} \partial_\nu u \psi \, d\sigma_g dt = - \int_0^T \int_{\mathcal{M}} F w \, dv_g dt + \int_0^T \int_{\partial\mathcal{M}} \partial_\nu w f \, d\sigma_g dt + \int_{\mathcal{M}} [u_0 w'(0) - u_1 w(0)] \, dv_g.$$

Now, by (2.6.24) and (2.6.25), we see that there exists a constant $C > 0$ such that

$$\begin{aligned} & \left| \int_0^T \int_{\partial\mathcal{M}} \partial_\nu u \psi \, d\sigma_g dt \right| \\ & \leq C \|\psi\|_{H^1((0,T)\times\partial\mathcal{M})} \left(\|u_0\|_{L^2(\mathcal{M})} + \|u_1\|_{H^{-1}(\mathcal{M})} + \|f\|_{L^2((0,T)\times\partial\mathcal{M})} + \|F\|_{L^1(0,T;H^{-1}(\mathcal{M}))} \right), \end{aligned} \quad (2.6.26)$$

which implies (2.2.8).

Carleman estimate of the wave equation in a Riemannian manifold

In this chapter, we prove a Carleman estimate with second large parameter for a second order hyperbolic operator in a Riemannian manifold \mathcal{M} . Our Carleman estimate holds in the whole cylindrical domain $\mathcal{M} \times (0, T)$ independently of the level set generated by a weight function if functions under consideration vanish on boundary $\partial(\mathcal{M} \times (0, T))$. This type of Carleman estimate is called global in $(0, T) \times \mathcal{M}$. The proof is direct by using calculus of tensor fields in a Riemannian manifold.

3.1 What is a Carleman estimate?

Let $P(x; \partial)$ be a differential operator defined on some Riemannian manifold \mathcal{M} . A Carleman estimate for this operator is the following L^2 -weighted a priori estimate:

$$s \|e^{s\varphi} u\|_{L^2(\mathcal{M})} \leq C \|e^{s\varphi} Pu\|_{L^2(\mathcal{M})}, \quad (3.1.1)$$

where the weight function φ is real-valued with non-vanishing gradient, s is a large positive parameter and u is any smooth compactly supported function in \mathcal{M} . We note that in Carleman estimate (3.1.1), the estimate is valid uniformly for all large $s > 0$, i.e., $s \geq s_0$: a fixed constant. In other words, the constant $C > 0$ should be independent of $s > s_0$ and $u \in \mathcal{C}_0^\infty(\mathcal{M})$. For applications, the parameter s plays an essential role and it is also important how to choose a weight function φ in order to adjust given geometric configurations.

A Carleman estimate was first established by Carleman [12] in 1939 for proving the unique continuation for a two-dimensional elliptic equation. Since then, it has remained an essential method for proving the unique continuation properties for partial differential operators with non-analytic coefficients. This tool has been refined, generalized by many authors and plays now a very important role in the control theory and inverse problems. Calderón [11] in 1958 gave very important development of the Carleman method with a proof of an estimate of the form (3.1.1) using a pseudo-differential factorization of the operator and initiated one method by singular-integral in microlocal analysis. In Chapter VIII in [18], Hörmander shows that microlocal methods can provide the same estimates with weaker assumptions on the regularity of the coefficients of the operator.

As for Carleman estimates, we can refer to [1], [2], [5], [14], [15], [16], [20], [21], [22], [29], [30], [31], [32], [38], [42], [52], [54], and the references therein. Here we do not intend a complete list of related works.

3.2 Weight function

In order to state a Carleman estimate, we need to choose a suitable weight function φ . Let (\mathcal{M}, g) be a compact manifold with boundary $\partial\mathcal{M}$. We assume that there exists a positive and smooth function ψ_0 on \mathcal{M} which satisfies the following assumptions:

- **Assumption (A.1):** ψ_0 is strictly convex on \mathcal{M} with respect to the Riemannian metric g . That is, the Hessian of the function ψ_0 in the Riemannian metric g is positive on $\overline{\mathcal{M}}$:

$$D^2\psi_0(X, X)(x) > 0, \quad x \in \mathcal{M}, \quad X \in T_x\mathcal{M} \setminus \{0\}.$$

Since \mathcal{M} is compact, it follows that there exists a positive constant $\varrho > 0$ such that

$$D^2\psi_0(X, X)(x) > 2\varrho |X|^2, \quad x \in \overline{\mathcal{M}}, \quad X \in T_x\mathcal{M} \setminus \{0\}. \quad (3.2.1)$$

- **Assumption (A.2):** We assume that $\psi_0(x)$ has no critical points on \mathcal{M} :

$$\min_{x \in \mathcal{M}} |\nabla\psi_0(x)| > 0. \quad (3.2.2)$$

- **Assumption (A.3):** Under assumption (A.1)-(A.2), let a subboundary $\Gamma_0 \subset \partial\mathcal{M}$ satisfy

$$\{x \in \partial\mathcal{M}; \partial_\nu\psi_0 \geq 0\} \subset \Gamma_0. \quad (3.2.3)$$

Let us define

$$Q = \mathcal{M} \times (0, T), \quad \Sigma = \partial\mathcal{M} \times (0, T), \quad \Sigma_0 = \Gamma_0 \times (0, T)$$

and

$$\psi(t, x) = \psi_0(x) - \beta(t - t_0)^2 + \beta_0, \quad 0 < \beta < \varrho, \quad 0 < t_0 < T, \quad \beta_0 \geq 0, \quad (3.2.4)$$

where the constant ϱ is given in (3.2.1). We choose a parameter β_0 such that the function ψ given by (3.2.4) is positive. We define the weight function $\varphi : \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(x, t) = e^{\gamma\psi(x, t)}, \quad (3.2.5)$$

where $\gamma > 0$ is a second large parameter and set

$$\sigma(t, x) = s\gamma\varphi(t, x), \quad (3.2.6)$$

where s is a real number, and is considered as the first large parameter. As a preparation, we shall establish a number of elementary properties of the weight function φ which will be useful in the succeeding parts.

Lemma 3.2.1 *Let φ be the weight function given by (3.2.5). Then*

$$\varphi' = \gamma\varphi\psi', \quad \nabla\varphi = \gamma\varphi\nabla\psi, \quad (3.2.7)$$

$$\varphi'' = \gamma\varphi(\psi'' + \gamma|\psi'|^2), \quad \Delta_g\varphi = \gamma\varphi(\Delta_g\psi + \gamma|\nabla\psi|^2), \quad (3.2.8)$$

$$D^2\varphi(\nabla z, \nabla z) = \gamma\varphi(D^2\psi(\nabla z, \nabla z) + \gamma|\langle\nabla z, \nabla\psi\rangle|^2). \quad (3.2.9)$$

Furthermore there exists a constant $C > 0$ such that

$$|(\partial_t^2 - \Delta_g)^2\varphi(t, x)| \leq C\gamma^3\varphi(t, x), \quad \text{for all } (t, x) \in Q. \quad (3.2.10)$$

Proof . Direct computations show (3.2.7) and (3.2.8). Applying Lemma 1.7.1, we obtain for any vector field X :

$$\begin{aligned} D^2\varphi(X, X) &= X(\langle X, \nabla(e^{\gamma\psi}) \rangle) - \frac{1}{2}\nabla(e^{\gamma\psi})(|X|^2) \\ &= X(\gamma\varphi\langle X, \nabla\psi \rangle) - \frac{1}{2}\gamma\varphi\nabla\psi(|X|^2) \\ &= \gamma\varphi\left(X(\langle X, \nabla\psi \rangle) - \frac{1}{2}\nabla\psi(|X|^2)\right) + \gamma\langle X, \nabla\psi \rangle X(\varphi) \\ &= \gamma\varphi D^2\psi(X, X) + \gamma^2\varphi|\langle X, \nabla\psi \rangle|^2. \end{aligned} \quad (3.2.11)$$

For $X = \nabla z$, we obtain (3.2.9). This completes the proof.

Finally, by direct computations, we show (3.2.10). \square

3.3 Conjugate operator

Let us consider the second-order hyperbolic operator $P(x, D)$ given by

$$P(x, \partial) = \partial_t^2 - \Delta_g. \quad (3.3.1)$$

In order to prove a Carleman estimates, the first step is to conjugate the operator P by the exponential weight function. The standard approach to a Carleman estimate of the form (3.1.1) starts from the observation

$$e^{s\varphi}P(x, \partial)u = P_s(t, x, \partial)z, \quad (3.3.2)$$

where P_s is the second-order differential operator given by

$$P_s(t, x, \partial) = e^{s\varphi}P(x, \partial)e^{-s\varphi}, \quad (3.3.3)$$

and the new function z is given by

$$z(t, x) = e^{s\varphi}u(x, t), \quad (t, x) \in Q. \quad (3.3.4)$$

Observing that

$$e^{s\varphi} \partial_t (e^{-s\varphi} z) = z' - s\varphi' z \quad \text{and} \quad e^{s\varphi} \nabla (e^{-s\varphi} z) = \nabla z - s z \nabla \varphi, \quad (3.3.5)$$

we easily obtain

$$P_s(t, x, \partial) z = P_s^+ z + P_s^- z = G_s, \quad (3.3.6)$$

where P_s^+ and P_s^- are two partial differential operator given by:

$$\begin{aligned} P_s^+ z &= z'' - \Delta_g z + s^2 \left(|\varphi'|^2 - |\nabla \varphi|^2 \right) z, \\ P_s^- z &= -2s (z' \varphi' - \langle \nabla z, \nabla \varphi \rangle) - s (\varphi'' - \Delta_g \varphi) z \end{aligned} \quad (3.3.7)$$

and

$$G_s = e^{s\varphi} F. \quad (3.3.8)$$

For obtaining an estimate such as (3.1.1), it suffices to argue for the operator P_s .

With the previous notations, we have

$$\|P_s^+ z\|^2 + \|P_s^- z\|^2 + 2(P_s^+ z, P_s^- z) = \|G_s\|^2. \quad (3.3.9)$$

Now we will make the computation of $2(P_s^+ z, P_s^- z)$. For this, we will develop the six terms appearing in $(P_s^+ z, P_s^- z)$ and integrate by parts several times with respect to the space and time variables.

Lemma 3.3.1 *Let φ be a smooth function in Q . Then for any $z \in H^2(Q)$ such that*

$$z(x, \tau) = z'(x, \tau) = 0, \quad \text{for } \tau = 0, T, \quad (3.3.10)$$

the following identity holds true:

$$\begin{aligned} (P_s^+ z, P_s^- z) &= 2s \int_Q \left(\varphi'' |z|^2 - 2z' \langle \nabla z, \nabla \varphi' \rangle + D^2 \varphi (\nabla z, \nabla z) \right) dv_g dt \\ &\quad + 2s^3 \int_Q |z|^2 \left(|\varphi'|^2 \varphi'' + D^2 \varphi (\nabla \varphi, \nabla \varphi) - 2\varphi' \langle \nabla \varphi, \nabla \varphi' \rangle \right) dv_g dt \\ &\quad - \frac{s}{2} \int_Q |z|^2 (\partial_t^2 - \Delta_g)^2 \varphi dv_g dt + \mathcal{B}_0, \end{aligned}$$

where \mathcal{B}_0 is a boundary term given by:

$$\begin{aligned} \mathcal{B}_0 &= s \int_{\Sigma} (\partial_\nu \varphi |\nabla z|^2 - 2 \langle \nabla z, \nabla \varphi \rangle \partial_\nu z) d\sigma_g dt + s \int_{\Sigma} \left(2\varphi' z' \partial_\nu z - |z'|^2 \partial_\nu \varphi \right) d\sigma_g dt \\ &+ s \int_{\Sigma} \left(z \partial_\nu z (\varphi'' - \Delta_g \varphi) + s^2 \partial_\nu \varphi |z|^2 \left(|\varphi'|^2 - |\nabla \varphi|^2 \right) - \frac{1}{2} |z|^2 \partial_\nu (\varphi'' - \Delta_g \varphi) \right) d\sigma_g dt. \end{aligned} \quad (3.3.11)$$

Proof . By (3.3.7), we see that

$$\begin{aligned}
(P_s^+ z, P_s^- z) &= -2s \int_Q z'' (z' \varphi' - \langle \nabla z, \nabla \varphi \rangle) \, dv_g dt - s \int_Q z'' (\varphi'' - \Delta_g \varphi) z \, dv_g dt \\
&\quad + 2s \int_Q \Delta_g z (z' \varphi' - \langle \nabla z, \nabla \varphi \rangle) \, dv_g dt + s \int_Q \Delta_g z (\varphi'' - \Delta_g \varphi) z \, dv_g dt \\
&\quad - 2s^3 \int_Q (|\varphi'|^2 - |\nabla \varphi|^2) z (z' \varphi' - \langle \nabla z, \nabla \varphi \rangle) \, dv_g dt \\
&\quad - s^3 \int_Q (|\varphi'|^2 - |\nabla \varphi|^2) (\varphi'' - \Delta_g \varphi) |z|^2 \, dv_g dt := \sum_{j=1}^6 \mathcal{I}_j. \quad (3.3.12)
\end{aligned}$$

First one easily see that

$$\begin{aligned}
\mathcal{I}_1 &= -s \int_Q \varphi' \partial_t (|z'|^2) \, dv_g dt - s \int_Q \langle \nabla (|z'|^2), \nabla \psi \rangle \, dv_g dt - 2s \int_Q z' \langle \nabla \varphi', \nabla z \rangle \, dv_g dt \\
&= s \int_Q |z'|^2 (\varphi'' + \Delta_g \varphi) \, dv_g dt - 2s \int_Q z' \langle \nabla \varphi', \nabla z \rangle \, dv_g dt - \left[s \int_{\Sigma} |z'|^2 \partial_{\nu} \varphi \, d\sigma_g dt \right]. \quad (3.3.13)
\end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned}
\mathcal{I}_2 &= -s \int_Q z'' (\varphi'' - \Delta_g \varphi) z \, dv_g dt \\
&= s \int_Q (\varphi'' - \Delta_g \varphi) |z'|^2 \, dv_g dt + \frac{s}{2} \int_Q \partial_t (|z|^2) (\partial_t^2 - \Delta_g) \varphi \, dv_g dt \\
&= s \int_Q (\varphi'' - \Delta_g \varphi) |z'|^2 \, dv_g dt - \frac{s}{2} \int_Q |z|^2 (\partial_t^2 - \Delta_g) \varphi'' \, dv_g dt. \quad (3.3.14)
\end{aligned}$$

Furthermore, by Green's formula and integration by parts, we obtain

$$\begin{aligned}
\mathcal{I}_3 &= 2s \int_Q \Delta_g z (z' \varphi' - \langle \nabla z, \nabla \varphi \rangle) \, dv_g dt \\
&= -2s \int_Q (\langle \nabla z, \nabla z' \rangle \varphi' + \langle \nabla z, \nabla \varphi' \rangle z' - \langle \nabla z, \nabla (\langle \nabla z, \nabla \varphi \rangle) \rangle) \, dv_g dt \\
&\quad + 2s \left[\int_{\Sigma} \partial_{\nu} z (z' \varphi' - \langle \nabla z, \nabla \varphi \rangle) \, d\sigma_g dt \right] \\
&= s \int_Q (|\nabla z|^2 \varphi'' - 2 \langle \nabla z, \nabla \varphi' \rangle z' + 2 \langle \nabla z, \nabla (\langle \nabla z, \nabla \varphi \rangle) \rangle) \, dv_g dt \\
&\quad + 2s \left[\int_{\Sigma} \partial_{\nu} z (z' \varphi' - \langle \nabla z, \nabla \varphi \rangle) \, d\sigma_g dt \right].
\end{aligned}$$

Applying Lemma 1.4.1 with the vector fields $Z = \nabla z$, we obtain

$$\langle \nabla z, \nabla (\langle \nabla z, \nabla \varphi \rangle) \rangle = D^2 \varphi (\nabla z, \nabla z) + \frac{1}{2} \langle \nabla \varphi, \nabla (|\nabla z|^2) \rangle.$$

Therefore, we conclude that

$$\begin{aligned} \mathcal{I}_3 = s \int_Q (|\nabla z|^2 (\varphi'' - \Delta_g \varphi) - 2z' \langle \nabla z, \nabla \varphi' \rangle + 2D^2 \varphi(\nabla z, \nabla z)) \, dv_g dt \\ + \left[s \int_{\Sigma} (2\partial_\nu z (z' \varphi' - \langle \nabla z, \nabla \varphi \rangle) + \partial_\nu \varphi |\nabla z|^2) \, d\sigma_g dt \right]. \end{aligned} \quad (3.3.15)$$

On the other hand, we compute

$$\begin{aligned} \mathcal{I}_4 = s \int_Q \Delta_g z (\varphi'' - \Delta_g \varphi) z \, dv_g dt \\ = -s \int_Q \left(|\nabla z|^2 (\varphi'' - \Delta_g \varphi) + \frac{1}{2} \langle \nabla(|z|^2), \nabla(\varphi'' - \Delta_g \varphi) \rangle \right) \, dv_g dt \\ + \left[s \int_{\Sigma} \partial_\nu z (\varphi'' - \Delta_g \varphi) z \, d\sigma_g dt \right] \\ = -s \int_Q \left(|\nabla z|^2 (\varphi'' - \Delta_g \varphi) - \frac{1}{2} |z|^2 \Delta_g (\varphi'' - \Delta_g \varphi) \right) \, dv_g dt \\ + \left[s \int_{\Sigma} \left(\partial_\nu z (\varphi'' - \Delta_g \varphi) z - \frac{1}{2} |z|^2 \partial_\nu (\varphi'' - \Delta_g \varphi) \right) \, d\sigma_g dt \right]. \end{aligned} \quad (3.3.16)$$

Next we have also

$$\begin{aligned} \mathcal{I}_5 = -2s^3 \int_Q (|\partial_t \varphi|^2 - |\nabla \varphi|^2) z (\varphi' z' - \langle \nabla \varphi, \nabla z \rangle) \, dv_g dt \\ = -s^3 \int_Q (\partial_t (|z|^2) \varphi' - \langle \nabla (|z|^2), \nabla \varphi \rangle) (|\varphi'|^2 - |\nabla \varphi|^2) \, dv_g dt \\ = s^3 \int_Q |z|^2 (\varphi'' - \Delta_g \varphi) (|\varphi'|^2 - |\nabla \varphi|^2) \, dv_g dt \\ + s^3 \int_Q |z|^2 \left(\varphi' \partial_t (|\varphi'|^2 - |\nabla \varphi|^2) - \langle \nabla \varphi, \nabla (|\varphi'|^2 - |\nabla \varphi|^2) \rangle \right) \, dv_g dt \\ + \left[s^3 \int_{\Sigma} \partial_\nu \varphi |z|^2 (|\varphi'|^2 - |\nabla \varphi|^2) \, d\sigma_g dt \right] \\ = s^3 \int_Q |z|^2 (\varphi'' - \Delta_g \varphi) (|\varphi'|^2 - |\nabla \varphi|^2) \, dv_g dt \\ + 2s^3 \int_Q |z|^2 \left(|\varphi'|^2 \varphi'' + D^2 \varphi(\nabla \varphi, \nabla \varphi) - 2\varphi' \langle \nabla \varphi, \nabla \varphi' \rangle \right) \, dv_g dt \\ + \left[s^3 \int_{\Sigma} \partial_\nu \varphi |z|^2 (|\varphi'|^2 - |\nabla \varphi|^2) \, d\sigma_g dt \right]. \end{aligned} \quad (3.3.17)$$

Finally we see that

$$\mathcal{I}_6 = -s^3 \int_Q |z|^2 (\varphi'' - \Delta_g \varphi) (|\varphi'|^2 - |\nabla \varphi|^2) \, dv_g dt. \quad (3.3.18)$$

Then adding (3.3.13)–(3.3.18), we see that

$$\begin{aligned} (P_s^+ z, P_s^- z) &= 2s \int_Q \left(\varphi'' |z|^2 - 2z' \langle \nabla z, \nabla \varphi' \rangle + D^2 \varphi (\nabla z, \nabla z) \right) dv_g dt \\ &\quad 2s^3 \int_Q |z|^2 \left(|\varphi'|^2 \varphi'' + D^2 \varphi (\nabla \varphi, \nabla \varphi) - 2\varphi' \langle \nabla \varphi, \nabla \varphi' \rangle \right) dv_g dt \\ &\quad - \frac{s}{2} \int_Q |z|^2 (\partial_t^2 - \Delta_g)^2 \varphi dv_g dt + \mathcal{B}_0, \end{aligned}$$

where \mathcal{B}_0 is given by (3.3.11). Thus the proof of Lemma 3.3.1 is completed. \square

3.4 Interior estimate

In this section, we want to prove a lower bound of $(P_s^+ z, P_s^- z)$. For this end we decompose this integral in the following form

$$(P_s^+ z, P_s^- z) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{B}_0. \quad (3.4.1)$$

where \mathcal{J}_1 , \mathcal{J}_2 and \mathcal{J}_3 are given by:

$$\mathcal{J}_1 = 2s \int_Q \left(\varphi'' |z|^2 - 2z' \langle \nabla z, \nabla \varphi' \rangle + D^2 \varphi (\nabla z, \nabla z) \right) dv_g dt \quad (3.4.2)$$

$$\mathcal{J}_2 = 2s^3 \int_Q |z|^2 \left(|\varphi'|^2 \varphi'' + D^2 \varphi (\nabla \varphi, \nabla \varphi) - 2\varphi' \langle \nabla \varphi, \nabla \varphi' \rangle \right) dv_g dt \quad (3.4.3)$$

$$\mathcal{J}_3 = -\frac{s}{2} \int_Q |z|^2 (\partial_t^2 - \Delta_g)^2 \varphi dv_g dt. \quad (3.4.4)$$

We denote

$$b(\psi) = |\psi'|^2 - |\nabla \psi|^2.$$

In what follows, we use the same letter C in order to denote constants which are independent of s , γ and z , although it may have different values in different contexts.

Lemma 3.4.1 *Let φ be the weight function given by (3.2.5). Assume that (A.1) holds. Then there exists a constant $C > 0$ such that for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that*

$$\begin{aligned} \mathcal{J}_1 + 2(\varrho + \beta) \mathcal{B}_1 &\geq 2(\varrho - \beta) \int_Q \sigma \left(|z'|^2 + |\nabla z|^2 \right) dv_g dt \\ &\quad - C \left(\int_Q \sigma^3 |b(\psi)| |z|^2 dv_g dt + C_\varepsilon \int_Q \sigma^2 |z|^2 dv_g dt + \varepsilon \|P_s^+ z\|^2 \right), \end{aligned} \quad (3.4.5)$$

where \mathcal{B}_1 is a boundary term give by

$$\mathcal{B}_1 = \left[\int_\Sigma \sigma (\gamma \partial_\nu \psi |z|^2 - z \partial_\nu z) d\sigma_g dt \right].$$

Proof . Using Lemma 3.2.1, we obtain

$$\begin{aligned} \mathcal{J}_1 &= 2s \int_Q \gamma \varphi \left((\psi'' + \gamma |\psi'|^2) |z'|^2 - 2\gamma \psi' z' \langle \nabla z, \nabla \psi \rangle + D^2 \psi (\nabla z, \nabla z) + \gamma |\langle \nabla z, \nabla \psi \rangle|^2 \right) dv_g dt \\ &= 2 \int_Q \sigma \left(\psi'' |z'|^2 + D^2 \psi (\nabla z, \nabla z) + \gamma (\psi' z' - \langle \nabla z, \nabla \psi \rangle)^2 \right) dv_g dt. \end{aligned} \quad (3.4.6)$$

Then

$$\mathcal{J}_1 \geq 2 \int_Q \sigma \left(D^2 \psi (\nabla z, \nabla z) + \psi'' |z'|^2 \right) dv_g dt \geq 4\varrho \int_Q \sigma |\nabla z|^2 dv_g dt - 4\beta \int_Q \sigma |z'|^2 dv_g dt. \quad (3.4.7)$$

Next, multiplying the first equation of (3.3.7) by σz and integrating by parts, we have

$$\begin{aligned} \int_Q P_s^+ z(\sigma z) dv_g dt &= - \int_Q \sigma |z'|^2 dv_g dt + \int_Q \sigma |\nabla z|^2 dv_g dt - \frac{1}{2} \int_Q \sigma' (\partial_t (|z|^2)) dv_g dt \\ &\quad + \frac{1}{2} \int_Q \langle \nabla \sigma, \nabla (|z|^2) \rangle dv_g dt + \int_Q \sigma^3 b(\psi) |z|^2 dv_g dt - \int_\Sigma \sigma z \partial_\nu z d\sigma_g dt \\ &= - \int_Q \sigma |z'|^2 dv_g dt + \int_Q \sigma |\nabla z|^2 dv_g dt + \frac{1}{2} \int_Q (\sigma'' - \Delta \sigma) |z|^2 dv_g dt \\ &\quad + \underbrace{\int_Q \sigma^3 b(\psi) |z|^2 dv_g dt + \left[\int_\Sigma \sigma (\gamma \partial_\nu \psi |z|^2 - z \partial_\nu z) d\sigma_g dt \right]}_{\mathcal{B}_1}. \end{aligned} \quad (3.4.8)$$

Using the fact that

$$\sigma'' - \Delta_g \sigma = \gamma \sigma (\psi'' - \Delta_g \psi) + \gamma^2 \sigma b(\psi),$$

we deduce that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} \left| \int_Q \sigma |z'|^2 dv_g dt - \mathcal{B}_1 \right| &\leq \int_Q \sigma^3 |z|^2 |b(\psi)| dv_g dt + \varepsilon \|P_s^+ z\|^2 \\ &\quad + \int_Q \sigma |\nabla z|^2 dv_g dt + C_\varepsilon \int_Q \sigma^2 |z|^2 dv_g dt. \end{aligned} \quad (3.4.9)$$

Combining (3.4.9) and (3.4.7), we obtain

$$\begin{aligned} \mathcal{J}_1 + 4\beta \mathcal{B}_1 &\geq 4(\varrho - \beta) \int_Q \sigma |\nabla z|^2 dv_g dt \\ &\quad - C \left(\int_Q \sigma^3 |z|^2 |b(\psi)| dv_g dt + \varepsilon \|P_s^+ z\|^2 + C_\varepsilon \int_Q \sigma^2 |z|^2 dv_g dt \right) \end{aligned} \quad (3.4.10)$$

Using (3.4.9) again, we have

$$\begin{aligned} 2(\varrho - \beta) \int_Q \sigma |z'|^2 dv_g dt - C \left(\int_Q \sigma^3 |z|^2 |b(\psi)| dv_g dt + \varepsilon \|P_s^+ z\|^2 + C_\varepsilon \int_Q \sigma^2 |z|^2 dv_g dt \right) \\ \leq 2(\varrho - \beta) \int_Q \sigma |\nabla z|^2 dv_g dt + 2(\varrho - \beta) \mathcal{B}_1. \end{aligned} \quad (3.4.11)$$

Combining (3.4.11) and (3.4.10), we obtain (3.4.4). \square

Lemma 3.4.2 *Let φ be the weight function given by (3.2.5). Assume that (A.1) holds. Then there exists a constant $C > 0$ such that the following estimate holds true:*

$$\mathcal{J}_2 \geq 2\gamma \int_Q \sigma^3 (b(\psi))^2 |z|^2 \, dv_g dt + 4 \int_Q \sigma^3 \left(\varrho |\nabla \psi|^2 - \beta |\psi'|^2 \right) |z|^2 \, dv_g dt. \quad (3.4.12)$$

Proof . Using Lemma 3.2.1, we find

$$\begin{aligned} \mathcal{J}_2 &= 2s^3 \int_Q (\gamma\varphi)^3 |\psi'|^2 (\psi'' + \gamma |\partial_t \psi|^2) |z|^2 \, dv_g dt - 4s^3 \int_Q \gamma^4 \varphi^3 |\psi'|^2 |\nabla \psi|^2 \, dv_g dt \\ &\quad + 2s^3 \int_Q \gamma^3 \varphi^3 (D^2 \psi (\nabla \psi, \nabla \psi) + \gamma |\nabla \psi|^4) |z|^2 \, dv_g dt \\ &= 2 \int_Q \sigma^3 \left(\psi'' |\psi'|^2 |z|^2 + D^2 \psi (\nabla \psi, \nabla \psi) |z|^2 \right) \, dv_g dt + 2\gamma \int_Q \sigma^3 (b(\psi))^2 |z|^2 \, dv_g dt \\ &\geq 2\gamma \int_Q \sigma^3 (b(\psi))^2 |z|^2 \, dv_g dt + 4 \int_Q \sigma^3 \left(\varrho |\nabla \psi|^2 - \beta |\psi'|^2 \right) |z|^2 \, dv_g dt. \end{aligned} \quad (3.4.13)$$

This completes the proof of the lemma. \square

On the other hand, by (3.2.10), we obtain

$$|\mathcal{J}_3| \leq C\gamma^2 \int_Q \sigma |z|^2 \, dv_g dt \leq C\gamma \int_Q \sigma^2 |z|^2 \, dv_g dt. \quad (3.4.14)$$

By Lemma 3.4.2, Lemma 3.4.1 and (3.4.14), we obtain the following lemma.

Lemma 3.4.3 *Let φ be the weight function given by (3.2.5). Assume that (A.1) holds. Then there exists a constant $C > 0$ such that for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*

$$\begin{aligned} \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + 2(\varrho + \beta) \mathcal{B}_1 &\geq 2(\varrho - \beta) \int_Q \sigma \left(|\nabla z|^2 + |z|^2 \right) \, dv_g dt \\ &\quad + 2\gamma \int_Q \sigma^3 (b(\psi))^2 |z|^2 \, dv_g dt + 4 \int_Q \sigma^3 \left(\varrho |\nabla \psi|^2 - \beta |\psi'|^2 \right) |z|^2 \, dv_g dt \\ &\quad - C \left(\int_Q \sigma^3 |z|^2 |b(\psi)| \, dv_g dt + \varepsilon \|P_s^+ z\|^2 + C_\varepsilon \gamma \int_Q \sigma^2 |z|^2 \, dv_g dt \right). \end{aligned} \quad (3.4.15)$$

3.5 Carleman estimate

Theorem 3.5.8 *Let*

$$\sigma(t, x) = s\gamma\varphi(t, x).$$

Assume (A.1) and (A.2). Then there exist constants $C > 0$ and $\gamma_* > 0$ such that for any $\gamma > \gamma_*$ there exists $s_* = s_*(\gamma)$ such that for all $s \geq s_*$ the following Carleman estimate holds:

$$C \int_Q \sigma \left(|\nabla z|^2 + |z'|^2 + \sigma^2 |z|^2 \right) dv_g dt \leq \int_Q |P_s z|^2 dv_g dt - \mathcal{B}, \quad (3.5.1)$$

for $z \in H^2(Q)$ satisfying $z(\tau, \cdot) = z'(\tau, \cdot) = 0$ at $\tau = 0, T$. Here \mathcal{B} is a boundary term given by:

$$\begin{aligned} \mathcal{B} = & \int_{\Sigma} \sigma \left(\partial_\nu \psi_0 |\nabla z|^2 - 2 \langle \nabla z, \nabla \psi_0 \rangle \partial_\nu z \right) d\sigma_g dt + \int_{\Sigma} \sigma \left(2\psi' z' \partial_\nu z - |z'|^2 \partial_\nu \psi_0 \right) d\sigma_g dt \\ & + \int_{\Sigma} \sigma \left(z \partial_\nu z (-2\beta - \Delta_g \psi + \gamma b(\psi)) + \sigma^2 \partial_\nu \psi_0 |z|^2 b(\psi) + \frac{1}{2} |z|^2 \partial_\nu (\Delta_g \psi_0 + \gamma |\nabla \psi_0|^2) \right) d\sigma_g dt \\ & - 2(\varrho + \beta) \left[\int_{\Sigma} \sigma \left(\gamma \partial_\nu \psi |z|^2 - z \partial_\nu z \right) d\sigma_g dt \right]. \end{aligned} \quad (3.5.2)$$

There are two features in our Carleman estimate:

- it is attached with the second large parameter γ . The Carleman estimate was considered in [14], [15], [30], [31] for functions with compact supports. The dependency on the second large parameter is, however, automatically derived if one prove the Carleman estimate by the method stated below. As such direct derivation of Carleman estimate, see also [42].
- Our Carleman estimate does not assume compact supports for functions under consideration.

The proof is based on Bellassoued and Yamamoto [7].

Proof . Since $\beta < \varrho$, for $\eta > 0$ small we have

$$\beta(1 + \eta) < \varrho. \quad (3.5.3)$$

Let us consider

$$Q^\eta = \{ (x, t) \in Q; |b(\psi)| \leq \eta |\nabla \psi|^2 \}.$$

Then

$$\begin{aligned} \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + 2(\varrho + \beta)\mathcal{B}_1 \geq & 2(\varrho - \beta) \int_Q \sigma \left(|\nabla z|^2 + |z'|^2 \right) dv_g dt \\ & + 2\gamma \int_{Q \setminus Q^\eta} \sigma^3 (b(\psi))^2 |z|^2 dv_g dt + 4(\varrho - \beta(1 + \eta)) \int_{Q^\eta} \sigma^3 |z|^2 |\nabla \psi|^2 dv_g dt \\ - C \left(\eta \int_{Q^\eta} \sigma^3 |z|^2 dv_g dt + \int_{Q \setminus Q^\eta} \sigma^3 |z|^2 dv_g dt + \varepsilon \|P_s^+ z\|^2 + C_\varepsilon \gamma \int_Q \sigma^2 |z|^2 dv_g dt \right). \end{aligned} \quad (3.5.4)$$

Using (3.5.4) and assumption (A.2), we obtain

$$\begin{aligned}
\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + 2(\varrho + \beta)\mathcal{B}_1 &\geq \delta \int_Q \sigma \left(|\nabla z|^2 + |z|^2 \right) dv_g dt + 2\gamma\eta^2 C_1 \int_{Q \setminus Q^\eta} \sigma^3 |z|^2 dv_g dt \\
&\quad + C_2(\varrho - \beta(1 + \eta)) \int_{Q^\eta} \sigma^3 |z|^2 dv_g dt \\
&\quad - C \left(\eta \int_{Q^\eta} \sigma^3 |z|^2 dv_g dt + \int_{Q \setminus Q^\eta} \sigma^3 |z|^2 dv_g dt + \varepsilon \|P_s^+ z\|^2 + \gamma \int_Q \sigma^2 |z|^2 dv_g dt \right) \\
&\quad \geq \delta \int_Q \sigma \left(|\nabla z|^2 + |z|^2 \right) dv_g dt + (2\gamma\eta^2 C_1 - C) \int_{Q \setminus Q^\eta} \sigma^3 |z|^2 dv_g dt \\
&\quad + (C_2(\varrho - \beta(1 + \eta)) - \eta C) \int_{Q^\eta} \sigma^3 |z|^2 dv_g dt - C \left(\varepsilon \|P_s^+ z\|^2 + \gamma \int_Q \sigma^2 |z|^2 dv_g dt \right).
\end{aligned} \tag{3.5.5}$$

Then for small η , large $\gamma \geq \gamma_*$ and $s \geq s_*(\gamma)$, we obtain

$$\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + 2(\varrho + \beta)\mathcal{B}_1 \geq \delta \int_Q \sigma \left(|\nabla z|^2 + |z|^2 + \sigma^2 |z|^2 \right) dv_g dt - \frac{1}{4} \|P_s^+ z\|^2. \tag{3.5.6}$$

By (3.4.1) we find

$$2(P_s^+ z, P_s^- z) - 2\mathcal{B} \geq 2\delta \int_Q \sigma \left(|\nabla z|^2 + |z|^2 + \sigma^2 |z|^2 \right) dv_g dt - \frac{1}{2} \|P_s^+ z\|^2, \tag{3.5.7}$$

where $\mathcal{B} = \mathcal{B}_0 - 2(\varrho + \beta)\mathcal{B}_1$. Then there exists $s_*(\gamma) > 0$ such that for any $s \geq s_*$, we have

$$\|G_s\|^2 - 2\mathcal{B} \geq C \int_Q \sigma \left(|\nabla z|^2 + |z|^2 + \sigma^2 |z|^2 \right) dv_g dt. \tag{3.5.8}$$

The proof is completed. \square

Corollary 5 *Assume (A.1) and (A.2). Then there exist constants $C > 0$ and $\gamma_* > 0$ such that for any $\gamma > \gamma_*$ there exists $s_* = s_*(\gamma)$ such that for all $s \geq s_*$ the following Carleman estimate holds:*

$$\begin{aligned}
C \int_Q e^{2s\varphi} \sigma \left(|\nabla u|^2 + |u'|^2 + \sigma^2 |u|^2 \right) dv_g dt &\leq \int_Q e^{2s\varphi} |(\partial_t^2 - \Delta_g)u|^2 dv_g dt \\
&\quad + \int_\Sigma \sigma \left(|\nabla u|^2 + |u'|^2 + \sigma^2 |u|^2 \right) e^{2s\varphi} d\sigma_g dt
\end{aligned} \tag{3.5.9}$$

for $u \in H^2(Q)$ satisfying $u(\tau, \cdot) = u'(\tau, \cdot) = 0$, $\tau = 0, T$.

Corollary 6 *Assume (A.1), (A.2) and (A.3). Then there exist constants $C > 0$ and $\gamma_* > 0$ such that for any $\gamma > \gamma_*$ there exists $s_* = s_*(\gamma)$ such that for all $s \geq s_*$ the following Carleman estimate holds:*

$$C \int_Q e^{2s\varphi} \sigma \left(|\nabla u|^2 + |u'|^2 + \sigma^2 |u|^2 \right) dv_g dt \leq \int_Q e^{2s\varphi} |(\partial_t^2 - \Delta_g)u|^2 dv_g dt + \int_{\Sigma_0} \sigma |\partial_\nu u|^2 e^{2s\varphi} d\sigma_g dt \quad (3.5.10)$$

for $u \in H^2(Q)$ satisfying $u(\tau, \cdot) = u'(\tau, \cdot) = 0$ at $\tau = 0, T$ and $u(t, x) = 0$ on Σ .

Proof . Noting that if $u(t, x) = 0$ on Σ , then we have $z(t, x) = 0$ on Σ and the boundary term \mathcal{B} is given by

$$\mathcal{B} = - \int_{\Sigma} \sigma |\partial_\nu z|^2 \partial_\nu \psi_0 d\sigma_g dt.$$

□

Corollary 7 Assume (A.1), (A.2) and (A.3). Let ω be a neighborhood of Γ_0 . Then there exist constants $C > 0$ and $\gamma_* > 0$ such that for any $\gamma > \gamma_*$ there exists $s_* = s_*(\gamma)$ such that for all $s \geq s_*$ the following Carleman estimate holds:

$$C \int_Q e^{2s\varphi} \sigma \left(|\nabla u|^2 + |u'|^2 + \sigma^2 |u|^2 \right) dv_g dt \leq \int_Q e^{2s\varphi} |(\partial_t^2 - \Delta_g)u|^2 dv_g dt + \int_{\omega^T} \sigma \left(|u'|^2 + \sigma^2 |u|^2 \right) e^{2s\varphi} dv_g dt \quad (3.5.11)$$

for $u \in H^2(Q)$ satisfying $u(\tau, \cdot) = u'(\tau, \cdot) = 0$ at $\tau = 0, T$ and $u(t, x) = 0$ on Σ . Here $\omega^T = \omega \times (0, T)$.

Proof . Let $V_\varepsilon = \{x \in \omega; \text{dist}(x, \partial\omega \cap \mathcal{M}) \leq \varepsilon\}$ and $\omega_\varepsilon = \omega \setminus V_\varepsilon$. We take a smooth cut-off function θ such that $\theta(x) = 1$ for $x \in \mathcal{M} \setminus \omega_{\varepsilon/2}$ and $\theta(x) = 0$ for $x \in \omega_{3\varepsilon/4}$. The function $w := \theta u$ satisfies the equation

$$(\partial_t^2 - \Delta_g)w = \theta(\partial_t^2 - \Delta_g)u + (u\Delta_g\theta + 2\langle \nabla\theta, \nabla u \rangle), \quad \text{in } Q$$

with the boundary conditions:

$$w(t, x) = 0 \quad \text{in } \Sigma; \quad \partial_\nu w = 0 \quad \text{in } \Sigma_0.$$

Furthermore, we have $w(\tau, \cdot) = w'(\tau, \cdot) = 0$, $\tau = 0, T$. Thus applying Theorem 3.5.8 and keeping in mind that $\Delta_g\theta, \nabla\theta$ are supported in ω_ε , we have

$$C \int_Q e^{2s\varphi} \sigma \left(|\nabla u|^2 + |u'|^2 + \sigma^2 |u|^2 \right) dv_g dt \leq \int_Q e^{2s\varphi} |(\partial_t^2 - \Delta_g)u|^2 dv_g dt + \int_{\omega_\varepsilon^T} \sigma \left(|\nabla u|^2 + |u'|^2 + \sigma^2 |u|^2 \right) e^{2s\varphi} dv_g dt. \quad (3.5.12)$$

Let $\rho \in \mathcal{C}^2(\omega)$ be a function such that $\text{supp}\rho \subset \omega$ and $\rho(x) = 1$ for all $x \in \omega_\varepsilon$. Taking the scalar product of $(\partial_t^2 - \Delta_g)u$ with $\sigma\rho u e^{2s\varphi}$, we have:

$$\begin{aligned}
\int_Q (\partial_t^2 - \Delta_g)u (\sigma \rho u e^{2s\varphi}) \, dv_g dt &= - \int_Q \sigma \rho |u'|^2 e^{2s\varphi} \, dv_g dt + \int_Q \sigma \rho |\nabla u|^2 e^{2s\varphi} \, dv_g dt \\
&+ \int_Q \sigma (\gamma \rho \langle \nabla \psi, \nabla u \rangle + \langle \nabla \rho, \nabla u \rangle) u e^{2s\varphi} \, dv_g dt - \int_Q \gamma \sigma \rho \psi' u' u e^{2s\varphi} \, dv_g dt \\
&+ 2 \int_Q \sigma^2 \rho (\langle \nabla \psi, \nabla u \rangle - \psi' u') u e^{2s\varphi} \, dv_g dt. \quad (3.5.13)
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_Q \sigma \rho |\nabla u|^2 e^{2s\varphi} \, dv_g dt &\leq \int_Q |(\partial_t^2 - \Delta_g)u|^2 \, dv_g dt \\
&+ \int_Q \rho(x) \sigma (|u'|^2 + \sigma^2 |u|^2) e^{2s\varphi} \, dv_g dt \quad (3.5.14)
\end{aligned}$$

Estimating the integral $\int_{\omega_\varepsilon^T} \sigma |\nabla u|^2 e^{2s\varphi} \, dv_g dt$ in (3.5.12) by the right-hand side of (3.5.14), we obtain (3.5.11). \square

3.6 Unique continuation and the observability inequality

Originally the Carleman estimate has been invented for proving the uniqueness in a Cauchy problem for an elliptic equation by Carleman [12], and as the first application of the Carleman estimate in this section, we will discuss the methodology for the uniqueness and the conditional stability for a hyperbolic equation by a local version of the Carleman estimate: theorem 3.5.8. Contrast to our Carleman estimate, there is so-called a local Carleman estimate which holds locally in a subdomain defined by the level set of the weight function, not in the whole domain $\mathcal{M} \times (0, T)$. We emphasize that whenever we apply a local version of Carleman estimate, it is essential to introduce a cut-off function and apply the Carleman estimate to the product of a solution to a partial differential equation by the cut-off function.

3.6.1 Conditional stability for the Cauchy problem

Let $\Gamma_1 \subset \partial\mathcal{M}$ be an arbitrary and non-empty sub-boundary of $\partial\mathcal{M}$. We consider a Cauchy problem for a hyperbolic equation:

$$(\partial_t^2 - \Delta_g + q(x)) u = F, \quad (x, t) \in Q \quad (3.6.1)$$

and

$$u(x, t) = f(x, t), \quad \partial_\nu u(x, t) = h(x, t), \quad (x, t) \in \Sigma_1 = \Gamma_1 \times (0, T). \quad (3.6.2)$$

Cauchy problem

Let u satisfy (3.6.1) and (3.6.2). Then determine u in some domain $Q_0 \subset Q$ by f and h .

As for the uniqueness results, we can refer to a lot of works, for example, [2], [14], [18], [19], [29], [34], [41], [50], [49], [51], [52]. Therefore we will not list them comprehensively even though we restrict ourselves to hyperbolic equations. In this section, we give accounts of methods for applying Carleman estimates to prove stability results in the Cauchy problem. One introduces a suitable cut-off function and extends Cauchy data in a suitable Sobolev space to reduce the problem to functions with compact supports and then one can apply a local Carleman estimate to obtain a stability estimate of u by data on Σ_1 . This argument is quite traditional and is valid for other types of partial differential equations.

We define $Q(r)$ by

$$Q(r) = \{(x, t) \in Q; \varphi(x, t) \geq r\}.$$

Theorem 3.6.9 *Let φ be a weight function satisfying (A.1) and (A.2) and let $\Gamma_1 \subset \partial\mathcal{M}$. Let us assume that*

$$Q(r_1) \subset Q \cup \Sigma_1, \quad \Sigma_1 = \Gamma_1 \times (0, T).$$

Then for any $0 < r_1 < r_2 < r_3$, there exist constants C , depending on \mathcal{M} , Γ_1 , φ and r_j such that for a solution u to the Cauchy problem (3.6.1) (3.6.2) we have

$$\|u\|_{H^1(Q(r_2))} \leq C (A + B^{1-\kappa} A^\kappa), \quad (3.6.3)$$

where

$$A = \|F\|_{L^2(Q)} + \|f\|_{H^1(\Sigma_1)} + \|h\|_{L^2(\Sigma_1)}, \quad B = \|u\|_{H^1(Q)}, \quad \kappa = \frac{r_2 - r_1}{r_3 - r_1}.$$

Proof . Let $\theta \in C^\infty(\mathbb{R})$ satisfy $\theta = 1$ in $[r_1 + \varepsilon, +\infty)$ and $\theta = 0$ in $(-\infty, r_1]$. Let

$$w(x, t) = \theta(\varphi(x, t))u(x, t), \quad (x, t) \in Q.$$

Then

$$\begin{aligned} (\partial_t^2 - \Delta_g + q(x)) w &= \theta(\varphi)F + 2\theta'(\varphi) ((\varphi' u' - \langle \nabla \varphi, \nabla u \rangle) + (\varphi'' - \Delta_g \varphi)u) \\ &\quad + \theta''(\varphi)(|\varphi'|^2 - |\nabla \varphi|^2)u. \end{aligned} \quad (3.6.4)$$

Applying the Carleman estimate (Corollary 1) to w , we obtain

$$\begin{aligned} C \int_Q e^{2s\varphi} s \left(|\nabla w|^2 + |w'|^2 + s^2 |w|^2 \right) dv_g dt &\leq \int_Q e^{2s\varphi} |\theta(\varphi)F|^2 dv_g dt \\ &\quad + \int_\Sigma s \left(|\nabla w|^2 + |w'|^2 + s^2 |w|^2 \right) e^{2s\varphi} d\sigma_g dt \\ &\quad + \int_{Q(r_1) \setminus Q(r_1+\varepsilon)} e^{2s\varphi} s \left(|\nabla u|^2 + |u'|^2 + s^2 |u|^2 \right) dv_g dt. \end{aligned} \quad (3.6.5)$$

Therefore

$$C e^{2sr_2} \int_{Q(r_2)} \left(|\nabla u|^2 + |u'|^2 + |u|^2 \right) dv_g dt \leq e^{2sr_3} \left(\|F\|_{L^2(Q(r_3))}^2 + \|f\|_{H^1(\Sigma_1)}^2 + \|h\|_{L^2(\Sigma_1)}^2 \right) + s^3 e^{2s(r_1+\varepsilon)} \|u\|_{H^1(Q_{r_2})}^2 \quad (3.6.6)$$

which implies

$$C \|u\|_{H^1(Q(r_2))} \leq e^{s(r_3-r_2)} A + e^{-s(r_2-r_1-\frac{\varepsilon}{2})} B \quad (3.6.7)$$

for $s \geq s_*$. Now minimizing the right-hand side of (3.6.7) with respect to s , we obtain (3.6.3). \square

3.6.2 Observability inequality

In section 3.5, we consider Cauchy problems where we are not given boundary values on some part of the boundary Σ_1 . In this section, assuming that we know the boundary condition on the whole lateral boundary Σ , but not an initial value, we discuss the estimation of the solution by extra boundary data or interior data of the solution.

Such an estimate is called an observability inequality. As for the derivation of an observability inequality by Carleman estimate, see e.g., [35], [38], and for related works, see [3], [40].

Let us consider the following initial boundary value problem for the wave equation with bounded potential $q \in L^\infty(\mathcal{M})$:

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) u = 0, & \text{in } \mathcal{M} \times (0, T), \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 & \text{in } \mathcal{M}, \\ u = 0, & \text{on } \partial\mathcal{M} \times (0, T). \end{cases} \quad (3.6.8)$$

Let

$$T_0 = \frac{2}{\sqrt{\varrho}} \left(\max_{x \in \mathcal{M}} \psi_0(x) \right)^{\frac{1}{2}}. \quad (3.6.9)$$

Theorem 3.6.10 *Let (\mathcal{M}, g) be a Riemannian manifold such that assumptions (A.1)-(A.2) and (A.3) hold and let $T > T_0$ and $q \in L^\infty(\mathcal{M})$. Then there exists a unique solution u to (3.6.8) with $u_0 \in H_0^1(\mathcal{M})$, $u_1 \in L^2(\mathcal{M})$ such that*

$$u \in \mathcal{C}([0, T]; H_0^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M}))$$

and we can choose a constant $C > 0$ such that

$$\|u_0\|_{H_0^1(\mathcal{M})}^2 + \|u_1\|_{L^2(\mathcal{M})}^2 \leq C \|\partial_\nu u\|_{L^2(\Sigma_0)}^2. \quad (3.6.10)$$

Proof . For $T > T_0$, let us define

$$\psi(x, t) = \psi_0(x) - \beta \left(t - \frac{T}{2} \right)^2 + \beta_0. \quad (3.6.11)$$

We fix $\delta > 0$ and $\beta > 0$ such that

$$\varrho T^2 > 4 \max_{x \in \mathcal{M}} \psi_0(x) + 4\delta \quad (3.6.12)$$

and

$$\beta T^2 > 4 \max_{x \in \mathcal{M}} \psi_0(x) + 4\delta, \quad 0 < \beta < \varrho, \quad (3.6.13)$$

where β is given by (3.2.5). Then $\psi(x, t)$ verifies the following properties:

(i) $\psi(x, 0) < \beta_0 - \delta$ and $\psi(x, T) < \beta_0 - \delta$ for all $x \in \mathcal{M}$. Then there exists $\varepsilon > 0$ such that

$$\psi(x, t) \leq \beta_0 - \frac{\delta}{2}, \quad \forall x \in \mathcal{M}, \quad t \in (0, 2\varepsilon) \cup (T - 2\varepsilon, T).$$

(ii) $\psi\left(x, \frac{T}{2}\right) = \psi_0(x) \geq 0$ for all $x \in \mathcal{M}$. Then there exists $\varepsilon_1 > 0$ such that

$$\psi(x, t) \geq \beta_0 - \frac{\delta}{4}, \quad \forall x \in \mathcal{M}, \quad \left|t - \frac{T}{2}\right| \leq \varepsilon_1.$$

We introduce a cut-off function η satisfying $0 \leq \eta \leq 1$, $\eta \in \mathcal{C}^\infty(\mathbb{R})$, $\eta = 1$ in $(2\varepsilon, T - 2\varepsilon)$ and $\text{Supp } \eta \subset (\varepsilon, T - \varepsilon)$.

Let u be a solution of (3.6.8). Put

$$w(x, t) = \eta(t)u(x, t), \quad (x, t) \in Q.$$

We note

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x))w = 2\eta'(t)u'(x, t) + \eta''(t)u(x, t), & \text{in } (0, T) \times \mathcal{M}, \\ w(0, \cdot) = u_0, \quad \partial_t w(0, \cdot) = u_1 & \text{in } \mathcal{M}, \\ w = 0, & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (3.6.14)$$

Furthermore we have

$$w(\tau, x) = \partial_t w(\tau, x) = 0, \quad \tau = 0, T \quad \text{for all } x \in \mathcal{M}.$$

Applying the Carleman estimate Theorem 3.5.8 to the function w , we obtain

$$\begin{aligned} C \int_Q e^{2s\varphi} \sigma (|\nabla w|^2 + |\partial_t w|^2 + \sigma^2 |w|^2) \, dV_g dt &\leq \int_Q e^{2s\varphi} |\eta' \partial_t u + \eta'' u|^2 \, dV_g dt \\ &+ \int_{\Sigma_0} \sigma |\partial_\nu w|^2 e^{2s\varphi} \, d\sigma_g dt \end{aligned} \quad (3.6.15)$$

for any $\gamma \geq \gamma_*$ and $s \geq s_*(\gamma)$. Fixing $\gamma = \gamma_*$, for any $s \geq s_*$ we have

$$C \int_{T/2-\epsilon_1}^{T/2+\epsilon_1} \int_{\mathcal{M}} e^{2s\varphi} s (|\nabla u|^2 + |\partial_t u|^2) \, dv_g dt \leq \int_Q e^{2s\varphi} |\eta' u' + \eta'' u|^2 \, dv_g dt + \int_{\Sigma_0} s |\partial_\nu u|^2 e^{2s\varphi} \, d\sigma_g dt. \quad (3.6.16)$$

Since η' and η'' are supported in $(0, 2\epsilon) \cup (T - 2\epsilon, T)$, by (i) and (ii) we conclude

$$C e^{2d_1 s} \int_{T/2-\epsilon_1}^{T/2+\epsilon_1} E(t) dt \leq e^{C s} \int_{\Sigma_0} |\partial_\nu u|^2 + e^{2d_0} \int_0^T E(t) dt + e^{2d_0 s} E(0),$$

where

$$d_1 := \exp\left(\gamma\left(\beta_0 - \frac{\delta}{4}\right)\right), \quad d_0 := \exp\left(\gamma\left(\beta_0 - \frac{\delta}{2}\right)\right).$$

On the other hand, by Lemma 2.3.4, we arrive at

$$E(0) \leq C \left(e^{C s} \|\partial_\nu u\|_{L^2(\Sigma_0)}^2 + e^{-2(d_1-d_0)s} E(0) \right).$$

It is easy to find s large such that

$$C e^{-2(d_1-d_0)s} \leq \frac{1}{2}.$$

Thus

$$E(0) \leq C \|\partial_\nu u\|_{L^2(\Sigma_0)}^2,$$

which is exactly the desired inequality (3.6.10). □

Inverse problem and exact controllability for the wave equation in a Riemannian manifold

4.1 Introduction

The main interest of this chapter lies in an inverse problem of identifying unknown coefficients of the wave equation from measurement on lateral boundary. The problem is attractive to many researchers, since it is a mathematical model in geophysics of finding properties of geophysical media by observation of wave fields on a part of the surface of the Earth. We wish to know condition for the uniqueness of solutions, but the uniqueness has not been shown for the case of anisotropic media. Proofs of uniqueness theorems of multidimensional inverse problems for differential equations are based on the following two points;

- the Bukhgeim-Klibanov method presented in [10].
- Carleman estimates near the boundary for boundary value problems.

We remark that the Bukhgeim-Klibanov method is an application of Carleman estimate to inverse problems and effective for various inverse problems of determining coefficients in the equations for which a Carleman estimate holds. Since the Carleman estimate essentially depends on the type of differential equation and the shape of the domain, several serious difficulties may arise in particular for hyperbolic systems with variable coefficients. Stability estimates play a special role in the theory of inverse problems of mathematical physics that are ill-posed in the classical sense (e.g., [42]). They determine the choice of regularization parameters and the rate at which solutions of regularized problems converge to an exact solution (e.g., [13].)

Originally the method by Carleman estimates for inverse problems, was introduced simultaneously and independently by Bukhgeim and Klibanov in 1981 as a powerful tool for proofs of global uniqueness results for multidimensional inverse problems with a single or a finite number of measurements. Gradually different authors have started to successfully modify and apply their method to establish the Lipschitz stability as well as the Hölder stability for hyperbolic and parabolic ill-posed Cauchy problem as well as inverse problem.

Here we recall the following assumptions. Let (\mathcal{M}, g) be a Riemannian compact manifold we assume that:

- **Assumption (A.1):** There exists a function ψ_0 which is strictly convex on \mathcal{M} with respect to the Riemannian metric g . That is, the Hessian of the function ψ_0 in g is positive on \mathcal{M} :

$$D^2\psi_0(X, X)(x) > 0, \quad x \in \mathcal{M}, \quad X \in T_x\mathcal{M} \setminus \{0\}.$$

Since \mathcal{M} is compact, it follows that there exists a positive constant $\varrho > 0$ such that

$$D^2\psi_0(X, X)(x) > 2\varrho |X|^2, \quad x \in \mathcal{M}, \quad X \in T_x\mathcal{M} \setminus \{0\}. \quad (4.1.1)$$

- **Assumption (A.2):** We assume that $\psi_0(x)$ has no critical points on \mathcal{M} :

$$\min_{x \in \mathcal{M}} |\nabla\psi_0(x)| > 0. \quad (4.1.2)$$

- **Assumption (A.3):** Under assumption (A.1)-(A.2), let a subboundary $\Gamma_0 \subset \partial\mathcal{M}$ satisfy

$$\{x \in \partial\mathcal{M}; \partial_\nu\psi_0 \geq 0\} \subset \Gamma_0. \quad (4.1.3)$$

Let us define

$$\tilde{Q} = \mathcal{M} \times (-T, T), \quad \tilde{\Sigma} = \partial\mathcal{M} \times (-T, T), \quad \tilde{\Sigma}_0 = \Gamma_0 \times (-T, T)$$

and

$$\psi(t, x) = \psi_0(x) - \beta t^2 + \beta_0, \quad 0 < \beta < \varrho, \quad \beta_0 \geq 0, \quad (4.1.4)$$

where the constant ϱ is given in (4.1.1). We choose a parameter β_0 such that the function ψ given by (4.1.4) is positive. We define the weight function $\varphi : \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(x, t) = e^{\gamma\psi(x, t)}, \quad (4.1.5)$$

where $\gamma > 0$

Henceforth we assume that (A.1)-(A.2) and (A.3) hold true. Let $\varphi(x, t)$ be the function defined by (4.1.5). Then

$$\varphi(x, t) = e^{\gamma\psi(x, t)} =: \varphi_0(x)\mu(t), \quad (4.1.6)$$

where $\varphi_0(x) \geq 1$ and $\mu(t) \leq 1$ are defined by

$$\varphi_0(x) = e^{\gamma(\psi_0(x) + \beta_0)} \geq e^{\gamma\beta_0} \equiv d_0, \quad \forall x \in \mathcal{M} \quad \text{and} \quad \mu(t) = e^{-\gamma\beta t^2} \leq 1, \quad \forall t \in (-T, T). \quad (4.1.7)$$

Let

$$T_0 = \frac{1}{\sqrt{\varrho}} \left(\max_{x \in \mathcal{M}} \psi_0(x) \right)^{1/2}$$

and we fix $\delta > 0$ and $\beta > 0$ such that

$$\varrho T^2 > \max_{x \in \mathcal{M}} \psi_0(x) + 4\delta$$

and

$$\beta T^2 > \max_{x \in \mathcal{M}} \psi_0(x) + 4\delta.$$

Then $\psi(x, t)$ verifies the following properties:

- $\psi(x, \pm T) \leq \beta_0 - 4\delta$ for all $x \in \mathcal{M}$.
- Then there exists $\varepsilon > 0$ such that

$$\varphi(x, t) = e^{\gamma\psi(x,t)} \leq e^{\gamma(\beta_0 - 2\delta)} \equiv d_1 < d_0, \quad \text{for all } x \in \mathcal{M}, \quad |t| \geq T - 2\varepsilon. \quad (4.1.8)$$

Finally we consider the following Banach spaces $L^2_{s,\varphi}(Q)$ and $H^1_{s,\varphi}(Q)$ which are the space $L^2(Q)$ and $H^1(Q)$ equipped with the following norms:

$$\begin{aligned} \|u\|_{L^2_{s,\varphi}(Q)}^2 &= \int_Q e^{2s\varphi} |u|^2 \, dv_g dt, \\ \|u\|_{H^1_{s,\varphi}(Q)}^2 &= \int_Q e^{2s\varphi} (|\nabla u|^2 + |u'|^2 + s^2 |u|^2) \, dv_g dt. \end{aligned}$$

4.2 Inverse source problem

4.2.1 Preliminaries

Let us consider the following wave equation

$$\begin{cases} \partial_t^2 u - \Delta_g u + q(x)u = h(x)R(x, t) & \text{in } Q = \mathcal{M} \times (-T, T), \\ u(x, 0) = \partial_t u(x, 0) = 0 & \text{in } \mathcal{M}, \\ u(x, t) = 0 & \text{in } \Sigma = \Gamma \times (-T, T), \end{cases} \quad (4.2.1)$$

where h and R are given by

$$h \in L^2(\mathcal{M}), \quad R \in L^1(-T, T; L^\infty(\mathcal{M})), \quad R' \in L^1(-T, T; L^\infty(\mathcal{M})).$$

By Theorem 2.2.5 the unique solution u_h of (4.2.1) satisfies

$$u_h \in \mathcal{C}^2([-T, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([-T, T]; H^1(\mathcal{M})) \cap \mathcal{C}(-T, T; H^2(\mathcal{M}))$$

and

$$\partial_\nu u'_h \in L^2(\Sigma).$$

Let us consider the linear map:

$$\begin{aligned} \mathcal{L}_R : L^2(\mathcal{M}) &\longrightarrow L^2(\Sigma_0) \\ h &\longrightarrow \mathcal{L}_R(h) := (\partial_\nu u'_h)|_{\Sigma_0}. \end{aligned} \quad (4.2.2)$$

Lemma 4.2.1 *There exists a positive constant C such that the following estimate hold true:*

$$\|u'(t)\|_{L^2(\mathcal{M})} + \|\nabla u(t)\|_{L^2(\mathcal{M})} \leq C \|h\|_{L^2(\mathcal{M})}, \quad t \in (-T, T) \quad (4.2.3)$$

and moreover

$$\|u''(t)\|_{L^2(\mathcal{M})} + \|\Delta_g u(t)\|_{L^2(\mathcal{M})} + \|\nabla u'(t)\|_{L^2(\mathcal{M})} \leq C \|h\|_{L^2(\mathcal{M})} \quad t \in (-T, T). \quad (4.2.4)$$

Furthermore we have

$$\|\mathcal{L}_R(h)\|_{L^2(\Sigma)} \leq C \|h\|_{L^2(\mathcal{M})} \quad \text{for all } h \in L^2(\mathcal{M}),$$

Proof . Applying Theorem 2.2.5 to the solution u of (4.2.1), we obtain

$$\begin{aligned} \|u'(t)\|_{L^2(\mathcal{M})} + \|u(t)\|_{H_0^1(\mathcal{M})} &\leq \|hR\|_{L^1(-T,T;L^2(\mathcal{M}))} \\ &\leq \|h\|_{L^2(\mathcal{M})} \|R\|_{L^1(-T,T;L^\infty(\mathcal{M}))} \leq C \|h\|_{L^2(\mathcal{M})}. \end{aligned} \quad (4.2.5)$$

In order to prove (4.2.2), set $v = u'$. Then we have

$$\begin{cases} v'' - \Delta_g v + q(x)v = h(x)R'(x, t), & \text{in } \mathcal{M} \times (-T, T), \\ v(0, x) = 0, \quad v'(0, x) = h(x)R(x, 0), & \text{in } \mathcal{M}, \\ v(x, t) = 0, & \text{in } \Sigma = \Gamma \times (-T, T). \end{cases} \quad (4.2.6)$$

Theorem 2.2.5 yields

$$\begin{aligned} \|v'(t)\|_{L^2(\mathcal{M})} + \|\nabla v(t)\|_{L^2(\mathcal{M})} &\leq C \left(\|v'(0)\|_{L^2(\mathcal{M})} + \|hR'\|_{L^1(-T,T;L^2(\mathcal{M}))} \right) \\ &\leq C \left(\|h\|_{L^2(\mathcal{M})} + \|h\|_{L^2(\mathcal{M})} \|R'\|_{L^1(-T,T;L^\infty(\mathcal{M}))} \right). \end{aligned} \quad (4.2.7)$$

□

We need the following lemmas, which are simple consequences of energy identity.

Lemma 4.2.2 *Let us consider $F \in L^1(-T, T; L^2(\mathcal{M}))$, $z_1 \in L^2(\mathcal{M})$. Let z be a given solution of the second-order hyperbolic system:*

$$\begin{cases} z'' - \Delta_g z + q(x)z = F(x, t) \text{ in } Q = \mathcal{M} \times (-T, T), \\ z(0, x) = 0, \quad z'(0, x) = z_1 \text{ in } \mathcal{M}, \\ z(x, t) = 0 \quad \text{on } \Sigma = \Gamma \times (-T, T). \end{cases} \quad (4.2.8)$$

Then the following estimate holds true:

$$\|z_1\|_{L^2(\mathcal{M})}^2 \leq C \left(\|z\|_{H^1(Q)}^2 + \|Fz'\|_{L^1(Q)} \right) \quad (4.2.9)$$

for some positive constant $C > 0$.

4.2.2 Uniqueness and stability estimate

Let v be a given solution to

$$\begin{cases} v'' - \Delta_g v + q(x)v = h(x)R'(x, t), & \text{in } \mathcal{M} \times (-T, T), \\ v(0, x) = 0, \quad v'(0, x) = f(x)R(x, 0), & \text{in } \mathcal{M}, \\ v(x, t) = 0, & \text{on } \Sigma = \Gamma \times (-T, T), \end{cases} \quad (4.2.10)$$

and we introduce a cut-off function η satisfying $0 \leq \eta \leq 1$, $\eta \in C^\infty(\mathbb{R})$, $\eta = 1$ in $\{t, |t| \leq T - 2\varepsilon\}$ and $\text{supp}\eta \subset \{t, |t| \leq T - \varepsilon\}$. Put

$$w(x, t) = \eta(t)v(x, t), \quad (x, t) \in Q.$$

Noting that w satisfies

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) w = \eta h R' + 2\eta' v' + \eta'' v, & \text{in } (-T, T) \times \mathcal{M}, \\ w(0, \cdot) = 0, \quad w'(0, \cdot) = h(x)R(x, 0), & \text{in } \mathcal{M}, \\ w = 0, & \text{on } (-T, T) \times \partial\mathcal{M}. \end{cases} \quad (4.2.11)$$

Furthermore we have

$$w(\tau, x) = w'(\tau, x) = 0, \quad \tau = \pm T, \quad \text{for all } x \in \mathcal{M}.$$

Applying the Carleman estimate to the function w , we obtain

$$Cs \|w\|_{H_{s,\varphi}^1(Q)}^2 \leq \|hR'\|_{L_{s,\varphi}^2(Q)}^2 + \|\eta'v'\|_{L_{s,\varphi}^2(Q)}^2 + \|\eta''v\|_{L_{s,\varphi}^2(Q)}^2 + s \|\partial_\nu w\|_{L_{s,\varphi}^2(\Sigma_0)}^2$$

for any $s \geq s_*$

Lemma 4.2.3 *Let w be a given solution of (4.2.11). Then there exists a constants $C > 0$ such that for all $s > 0$ large enough, the following estimate holds true:*

$$Cs \|w\|_{H_{s,\varphi}^1(Q)}^2 \leq \|hR'\|_{L_{s,\varphi}^2(Q)}^2 + e^{2d_1s} \|h\|_{L^2(\mathcal{M})}^2 + s \|\partial_\nu w\|_{L_{s,\varphi}^2(\Sigma_0)}^2.$$

Proof . It follows from (4.1.8) that

$$\int_Q e^{2s\varphi} (|\eta'v'|^2 + |\eta''v|^2) \, dv_g dt \leq Ce^{2d_1s} \|h\|_{L^2(\mathcal{M})}^2, \quad (4.2.12)$$

where $C > 0$ is a generic constant. This completes the proof. \square

Lemma 4.2.4 *Let $\Phi \in L^2(-T, T; L^\infty(\mathcal{M}))$. Then there exists a positive function $k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{s \rightarrow +\infty} k(s) = 0$ and*

$$\|h\Phi\|_{L_{s,\varphi}^2(Q)}^2 \leq k(s) \|e^{s\varphi_0} h\|_{L^2(\mathcal{M})}^2, \quad \forall h \in L^2(\mathcal{M}).$$

Proof . We have

$$\int_Q e^{2s\varphi} |h(x)\Phi(x, t)|^2 \, dx dt \leq \int_{\mathcal{M}} e^{2s\varphi_0(x)} |h(x)|^2 \left(\int_0^T e^{-2s(\varphi_0 - \varphi)} \|\Phi(t, \cdot)\|_{L^\infty(\mathcal{M})}^2 \, dt \right) \, dx. \quad (4.2.13)$$

On the other hand, by the Lebesgue theorem, we obtain

$$\begin{aligned} \int_0^T e^{-2s(\varphi_0 - \varphi)} \|\Phi(t, \cdot)\|_{L^\infty(\mathcal{M})}^2 dt &= \int_0^T e^{-2s\varphi_0(x)(1-\mu(t))} \|\Phi(t, \cdot)\|_{L^\infty(\mathcal{M})}^2 dt \\ &\leq \underbrace{\int_0^T e^{-2s(1-\mu(t))} \|\Phi(t, \cdot)\|_{L^\infty(\mathcal{M})}^2 dt}_{=k(s)=s \rightarrow \infty o(1)}. \end{aligned} \quad (4.2.14)$$

This completes the proof. \square

Theorem 4.2.11 *Assume that (A.1), (A.2) (A.3) hold true, $T > T_0$ and that*

$$R(t, x), R'(t, x) \in L^2(-T, T, L^\infty(\mathcal{M})), \quad (4.2.15)$$

$$|R(x, 0)| \geq m_0 > 0 \quad \text{almost everywhere on } \mathcal{M} \quad (4.2.16)$$

with some constant $m_0 > 0$. Then

$$C^{-1} \|h\|_{L^2(\mathcal{M})} \leq \|\mathcal{L}_R(h)\|_{L^2(\Sigma_0)} \leq C \|h\|_{L^2(\mathcal{M})} \quad \text{for all } h \in L^2(\mathcal{M}). \quad (4.2.17)$$

Proof . Let $z = e^{z\varphi} w$. Then we have

$$z'' - \Delta_g z + q(x)z = e^{s\varphi} P_s w.$$

Now decompose the conjugate operator P_s as follows:

$$P_s w = P_s^+ w + P_s^- w, \quad (4.2.18)$$

where P_s^+ and P_s^- are two partial differential operator given by:

$$\begin{aligned} P_s^+ w &= w'' - \Delta_g w + q(x)w - s^2 \left(|\varphi'|^2 - |\nabla\varphi|^2 \right) w, \\ P_s^- w &= 2s (w'\varphi' - \langle \nabla w, \nabla\varphi \rangle) + s (\varphi'' - \Delta_g \varphi) w. \end{aligned} \quad (4.2.19)$$

Therefore we obtain

$$\begin{cases} z'' - \Delta_g z + q(x)z = e^{s\varphi} F(x, t) + e^{s\varphi} A_s w, & \text{in } \mathcal{M} \times (-T, T), \\ z(x, 0) = 0, \quad z'(x, 0) = h(x)R(x, 0)e^{s\varphi(x)}, & \text{in } \mathcal{M}, \\ z(x, t) = 0, & \text{on } \Sigma = \Gamma \times (-T, T), \end{cases} \quad (4.2.20)$$

where

$$A_s w = s^2 \left(|\varphi'|^2 - |\nabla\varphi|^2 \right) w + 2s (z'\varphi' - \langle \nabla z, \nabla\varphi \rangle) + s (\varphi'' - \Delta_g \varphi) w \quad (4.2.21)$$

and

$$F(x, t) = \eta h R'(x, t) + 2\eta' v' + \eta'' v$$

Next we use assumption (4.2.16) and (4.2.9) with $z = e^{s\varphi} w$, and we obtain

$$C \|e^{s\varphi_0} h\|_{L^2(\mathcal{M})}^2 \leq s \|w\|_{H_{s,\varphi}^1(Q)}^2 + \|hR'\|_{L_{s,\varphi}^2(Q)}^2 + e^{2d_1 s} \|h\|_{L^2(\mathcal{M})}^2 + \|e^{s\varphi} A_s w z'\|_{L^1(Q)} + s \|\partial_\nu w\|_{L_{s,\varphi}^2(\Sigma_0)}^2. \quad (4.2.22)$$

Since

$$e^{s\varphi} A_s w z' = e^{2s\varphi} \left[s^2 \left(|\varphi'|^2 - |\nabla \varphi|^2 \right) w + P_s^- w \right] \left[s\varphi' w + w' \right], \quad (4.2.23)$$

we obtain by the Schwartz inequality

$$\|e^{s\varphi} A_s w z'\|_{L^1(Q)} \leq C s \int_Q e^{2s\varphi} \left(s^2 |w|^2 + |\nabla w|^2 + |w'|^2 \right) dx dt = C s \|w\|_{H_{s,\varphi}^1(Q)}^2. \quad (4.2.24)$$

Inserting (4.2.24) into the right-hand side of (4.2.23), we obtain.

$$C \|e^{s\varphi_0} h\|_{L^2(\mathcal{M})}^2 \leq s \|w\|_{H_{s,\varphi}^1(Q)}^2 + \|hR'\|_{L_{s,\varphi}^2(Q)}^2 + e^{2d_1 s} \|h\|_{L^2(\mathcal{M})}^2 + s \|\partial_\nu w\|_{L_{s,\varphi}^2(\Sigma_0)}^2. \quad (4.2.25)$$

We will now complete the proof of Theorem 4.2.11. Using Lemma 4.2.4, we obtain

$$\|e^{s\varphi_0} h\|_{L^2(\mathcal{M})}^2 \leq k(s) \|e^{s\varphi_0} h\|_{L^2(\mathcal{M})}^2 + C e^{2sd_1} \|h\|_{L^2(\mathcal{M})}^2 + C_s \|\partial_\nu v\|_{L^2(\Sigma_0)}^2. \quad (4.2.26)$$

Here we note that the first term on the right-hand side of (4.2.26) can be absorbed into the left-hand side if we take large $s > 0$. On the other hand, since $\varphi_0(x) \geq d_0 > d_1$ for all $x \in \mathcal{M}$, we have for s sufficiently large

$$\|h\|_{L^2(\mathcal{M})}^2 \leq C \|\partial_\nu u'\|_{L^2(\Sigma_0)}^2 = C \|\mathcal{L}_R(h)\|_{L^2(\Sigma_0)}^2. \quad (4.2.27)$$

The proof of Theorem 4.2.11 is completed. \square

4.3 Determination of coefficient

The main interest of this section is an inverse problem of determining unknown coefficients of the wave equation from measurement data on lateral boundary. Physically speaking, we are required to determine a coefficient of a restore force from measurements of boundary displacements. We wish to know conditions for the uniqueness of solutions, but the uniqueness has not been shown for the case where observation is done on arbitrary part of a boundary. We shall address our inverse problem precisely. Let (\mathcal{M}, g) be a compact Riemannian manifold. We consider the Dirichlet mixed problem for a second-order wave equation:

$$\begin{cases} u''(x, t) - \Delta_g u(x, t) + q(x)u(x, t) = 0, & \text{in } \mathcal{M} \times (-T, T) \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), & \text{in } \mathcal{M} \\ u(x, t) = f(x, t) & \text{on } \Gamma \times (-T, T). \end{cases} \quad (4.3.1)$$

We denote the solution to (4.3.1) by u_q .

Let $\Gamma_1 \subset \Gamma$ be a part of the boundary $\Gamma = \partial\Omega$ which is given a priori. A question of our inverse problem is how to conclude $q_1(x) = q_2(x)$ $x \in \mathcal{M}$ under the observation

$$\partial_\nu u_{q_1}(x, t) = \partial_\nu u_{q_2}(x, t); \quad (x, t) \in \widetilde{\Sigma}_1 = \Gamma_1 \times (-T, T) \quad (4.3.2)$$

When Γ_1 is the whole boundary Γ , a strong affirmation result is known for the uniqueness in multidimensional inverse problems with a single observation (Bukhgeim and Klivanov [10]).

In the case where Γ_1 be an arbitrary part of Γ , the condition for unique identification had been an open problem. In recent years, several works (e.g., Isakov and Yamamoto [32], Imanuvilov and Yamamoto [24]) on this subject have appeared, and mainly concerned with the uniqueness and stability in determining a coefficient of the zeroth-order term when the part Γ_1 is given by $\Gamma_1 = \{x \in \Gamma, (x - x_0) \cdot \nu(x) \geq 0\}$. This subboundary can correspond to the geometric optics condition for the observability (see Bardos, Lebeau and Rauch [3]). Kubo [39] gives some Carleman estimates including boundary conditions to show the uniqueness across a lateral boundary for hyperbolic equations, and he shows the uniqueness in a hyperbolic inverse problem by the above unique continuation, provided that $\Gamma \setminus \Gamma_0$ contain a flat part of the boundary.

Imanuvilov and Yamamoto [24] establishes the uniqueness and stability in an inverse problem of determining a potential by the Dirichlet data and the Neumann data on a sufficiently large part of the boundary Γ over a sufficiently long time interval. In particular, their stability result is global in Ω and both-sided Lipschitz stability estimate.

Stability estimates play a special role in the theory of inverse problems and for example determine the choice of regularization parameters and the rate at which solutions of regularized problems converge to an exact solution ([13]).

As related uniqueness and/or stability estimates for inverse problems, we refer for example [4], [5], [6], [8], [9], [23], [25], [29],[36], [37], [55].

The above-mentioned works discuss inverse problems in the case where \mathcal{M} is a bounded domain in an Euclidean space. We shall next consider the stability for our inverse hyperbolic problem in the case of a Riemannian manifold \mathcal{M} . For example, in (4.3.1), assuming that (u_0, u_1) is given, we are concerned with the stability. Our main interest is the Lipschitz stability in the inverse problem, that is, an estimate $\|q_1 - q_2\| \leq [\text{a suitable norm of } (u_{q_1} - u_{q_2})_{|\widetilde{\Sigma}_1}]$.

Throughout this section, let us set

$$\mathcal{Q}(M_0) = \left\{ q \in W^{1,\infty}(\mathcal{M}); \|q\|_{W^{1,\infty}(\mathcal{M})} \leq M_0 \right\} \quad (4.3.3)$$

for any fixed $M_0 > 0$. Let us take the Hilbert space $\mathcal{H}(\mathcal{M}) = (H^3(\mathcal{M}) \cap H_0^1(\mathcal{M})) \oplus H^2(\mathcal{M})$ as the state space of our system. The norm in $\mathcal{H}(\mathcal{M})$ is chosen as follows:

$$\|(u_0, u_1)\|_{\mathcal{H}(\mathcal{M})}^2 = \|u_0\|_{H^3(\mathcal{M})}^2 + \|u_1\|_{H^2(\mathcal{M})}^2, \quad \text{for any } (u_0, u_1) \in \mathcal{H}(\mathcal{M}). \quad (4.3.4)$$

Before stating the main results, we recall the following lemma on the unique existence of a weak solution to problem (4.3.1), which we shall use repeatedly in the sequel. The proof is based on [47], for example. We can also refer to [24].

Lemma 4.3.1 *Let $(u_0, u_1) \in \mathcal{H}(\mathcal{M})$ and let $q \in \mathcal{Q}(M)$. Then there exists a unique solution $u = u_q$ to (4.3.1) starting from (u_0, u_1) within the following class*

$$u \in \mathcal{C}([-T, T]; H^3(\mathcal{M})) \cap \mathcal{C}^1([-T, T]; H^2(\mathcal{M})) \cap \mathcal{C}^2([-T, T]; H^1(\mathcal{M})). \quad (4.3.5)$$

Moreover there exists a positive constant $C = C(M_0)$ such that

$$\|u_q\|_{\mathcal{C}([-T, T]; H^3(\mathcal{M}))} + \|u_q\|_{\mathcal{C}^1([-T, T]; H^2(\mathcal{M}))} + \|u_q\|_{\mathcal{C}^2([-T, T]; H^1(\mathcal{M}))} \leq C \|(u_0, u_1)\|_{\mathcal{H}(\mathcal{M})}. \quad (4.3.6)$$

Furthermore

$$\partial_\nu u'_q \in L^2(\widetilde{\Sigma}).$$

Let us consider the linear mapping

$$\begin{aligned} \mathcal{N}_f : L^\infty(\mathcal{M}) &\longrightarrow L^2(\widetilde{\Sigma}_0) \\ q &\longrightarrow \mathcal{N}_f(q) := (\partial_\nu u'_q)|_{\widetilde{\Sigma}_0}. \end{aligned} \quad (4.3.7)$$

The main result of this section can be stated as follows:

Theorem 4.3.12 *Let $T > T_0$. Let $(u_0, u_1) \in \mathcal{H}(\mathcal{M})$. We assume that*

$$|u_0(x)| \geq m_0 > 0, \quad x \in \mathcal{M}, \quad \text{and} \quad u_{q_2} \in H^1(-T, T; L^2(\mathcal{M})). \quad (4.3.8)$$

Then there exists a constant $C > 0$ such that

$$\|q_1 - q_2\|_{L^2(\mathcal{M})} \leq C \|\mathcal{N}_f(q_1) - \mathcal{N}_f(q_2)\|_{L^2(\widetilde{\Sigma}_0)}, \quad \forall q_1, q_2 \in \mathcal{Q}(M_0). \quad (4.3.9)$$

Here the constant C is dependent on \mathcal{M} , T , M_0 , $\|(u_0, u_1)\|_{\mathcal{H}(\mathcal{M})}$ and independent of $q_1, q_2 \in \mathcal{Q}(M)$.

Proof . We consider the difference $u = u_{q_1} - u_{q_2}$ and have:

$$\begin{cases} u'' - \Delta_g u + q_1(x)u = h(x)R(x, t) & \text{in } \mathcal{M} \times (-T, T), \\ u(x, 0) = u'(x, 0) = 0 & \text{in } \mathcal{M}, \\ u(x, t) = 0 & \text{on } \Gamma \times (-T, T), \end{cases} \quad (4.3.10)$$

where h and R are given by

$$h(x) = q_1(x) - q_2(x), \quad \text{and} \quad R(x, t) = u_{q_2}(x, t). \quad (4.3.11)$$

We have

$$R, R' \in L^2(-T, T, L^\infty(\mathcal{M})).$$

Moreover we have

$$|R(x, 0)| = |u_0(x)| \geq m_0, \quad \text{almost everywhere on } \mathcal{M}.$$

Then by theorem 4.2.1, we obtain

$$\|q_1 - q_2\|_{L^2(\mathcal{M})} = \|h\|_{L^2(\mathcal{M})} \leq C \|\mathcal{L}_R(h)\|_{L^2(\widetilde{\Sigma}_0)} = C \|\partial_\nu u'_{q_1} - \partial_\nu u'_{q_2}\|_{L^2(\widetilde{\Sigma}_0)}.$$

This completes the proof. \square

4.4 Hilbert Uniqueness Method (HUM)

We here present the Hilbert Uniqueness Method (HUM). As for more details, see e.g., Lions [46]. As in the preceding chapter, we assume that there exists a positive and a smooth function ψ_0 on \mathcal{M} which satisfy the following assumptions:

- **Assumption (A.1):** ψ_0 is strictly convex on \mathcal{M} with respect to the Riemannian metric g . That is, the Hessian of the function ψ_0 in g is positive on $\overline{\mathcal{M}}$:

$$D^2\psi_0(X, X)(x) > 0, \quad x \in \mathcal{M}, \quad X \in T_x\mathcal{M} \setminus \{0\}.$$

Since \mathcal{M} is compact, it follows that there exists a positive constant $\varrho > 0$ such that

$$D^2\psi_0(X, X)(x) > 2\varrho|X|^2, \quad x \in \overline{\mathcal{M}}, \quad X \in T_x\mathcal{M} \setminus \{0\}. \quad (4.4.1)$$

- **Assumption (A.2):** We assume that $\psi_0(x)$ has no critical points on \mathcal{M} :

$$\min_{x \in \mathcal{M}} |\nabla\psi_0(x)| > 0. \quad (4.4.2)$$

- **Assumption (A.3):** Under assumption (A.1)-(A.2), let a subboundary $\Gamma_0 \subset \partial\mathcal{M}$ satisfy

$$\{x \in \partial\mathcal{M}; \partial_\nu\psi_0 \geq 0\} \subset \Gamma_0. \quad (4.4.3)$$

Let us define

$$Q = \mathcal{M} \times (0, T), \quad \Sigma = \partial\mathcal{M} \times (0, T), \quad \Sigma_0 = \Gamma_0 \times (0, T), \quad \Gamma_1 = \Gamma \setminus \Gamma_0, \quad \Sigma_1 = \Gamma_1 \times (0, T).$$

We consider the following initial boundary value problem for the wave equation with bounded potential $q \in L^\infty(\mathcal{M})$:

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x))u = 0, & \text{in } (0, T) \times \mathcal{M}, \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 & \text{in } \mathcal{M}, \\ u = f, & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (4.4.4)$$

It follows from Theorem 2.2.6 that there exists a unique solution u of (4.4.4) satisfying

$$u \in \mathcal{C}([0, T]; L^2(\mathcal{M})) \cap \mathcal{C}^1([0, T]; H^{-1}(\mathcal{M})) \quad (4.4.5)$$

for any given:

$$u_0 \in L^2(\mathcal{M}), \quad u_1 \in H^{-1}(\mathcal{M}) \quad \text{and} \quad f \in L^2((0, T) \times \partial\mathcal{M}).$$

Definition 4.4.40 *The problem (4.4.4) is called to be exactly controllable if for any given $(u_0, u_1), (v_0, v_1) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$, there exists $f \in L^2(0, T; L^2(\Gamma))$ such that the unique solution of (4.4.4) starting from (u_0, u_1) satisfies*

$$u(T, \cdot) = v_0, \quad \text{and} \quad u'(T, \cdot) = v_1.$$

Let

$$T_0 = \frac{2}{\sqrt{\varrho}} \left(\max_{x \in \mathcal{M}} \psi_0(x) \right)^{\frac{1}{2}}. \quad (4.4.6)$$

Theorem 4.4.13 *Assume that $T > T_0$. Then for any given $(u_0, u_1) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$ there exists $f \in L^2(0, T; L^2(\Gamma))$ such that*

$$f(x, t) = 0, \quad \text{on} \quad \Sigma_1$$

and the unique solution of (4.4.4) starting from (u_0, u_1) satisfies

$$u(T, \cdot) = 0, \quad \text{and} \quad u'(T, \cdot) = 0.$$

The first idea of HUM is to seek a control f in the special form $f = \partial_\nu w$ where w is solution of the homogenous problem:

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x)) w = 0, & \text{in } (0, T) \times \mathcal{M}, \\ w(0, \cdot) = w_0, \quad \partial_t w(0, \cdot) = w_1 \text{ in } \mathcal{M}, \\ w = 0, & \text{on } (0, T) \times \partial\mathcal{M} \end{cases} \quad (4.4.7)$$

for a suitable choice of initial values $(w_0, w_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$. Let us recall that for any given $(w_0, w_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, the problem (4.4.7) has a unique solution w satisfying

$$w \in \mathcal{C}([0, T]; H_0^1(\mathcal{M})) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M}))$$

and there exists $C > 0$ such that for any $t \in (0, T)$ we have

$$\|w(t)\|_{H^1(\mathcal{M})} + \|\partial_t w(t)\|_{L^2(\mathcal{M})} \leq C \left(\|w_0\|_{H_0^1(\mathcal{M})} + \|w_1\|_{L^2(\mathcal{M})} \right). \quad (4.4.8)$$

Furthermore

$$\partial_\nu w \in L^2((0, T) \times \partial\mathcal{M}) \quad (4.4.9)$$

and there is a constant $C = C(T, \mathcal{M}) > 0$ such that

$$\|\partial_\nu w\|_{L^2((0, T) \times \partial\mathcal{M})} \leq C \left(\|w_0\|_{H_0^1(\mathcal{M})} + \|w_1\|_{L^2(\mathcal{M})} \right). \quad (4.4.10)$$

We recall that the non-homogeneous boundary value problem for a wave equation:

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x))v = 0, & \text{in } (0, T) \times \mathcal{M}, \\ v(T, \cdot) = 0, \quad \partial_t v(T, \cdot) = 0 & \text{in } \mathcal{M}, \\ v = \begin{cases} \partial_\nu w, & \text{on } (0, T) \times \Gamma_0, \\ 0, & \text{on } (0, T) \times \Gamma_1, \end{cases} \end{cases} \quad (4.4.11)$$

possesses a unique solution v satisfying

$$v(0, \cdot) \in L^2(\mathcal{M}), \quad v'(0, \cdot) \in H^{-1}(\mathcal{M})$$

and that

$$\begin{aligned} \|v(0)\|_{L^2(\mathcal{M})} + \|v'(0)\|_{H^{-1}(\mathcal{M})} &\leq C \|v\|_{L^2(\Sigma)} \leq C \|\partial_\nu w\|_{L^2((0, T) \times \partial\mathcal{M})} \\ &\leq C \left(\|w_0\|_{H_0^1(\mathcal{M})} + \|w_1\|_{L^2(\mathcal{M})} \right). \end{aligned}$$

Let the mapping

$$J : \mathcal{H} = H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \longrightarrow H^{-1}(\mathcal{M}) \times L^2(\mathcal{M}) = \mathcal{H}'$$

be defined by

$$J(w_0, w_1) := (v'(0), -v(0)).$$

Lemma 4.4.1 *Assume (A.1)-(A.2) and (A.3). Then J is an isomorphism of \mathcal{H} onto \mathcal{H}' .*

Proof . Let $(w_0, w_1) \in \mathcal{C}_0^\infty \times \mathcal{C}_0^\infty$. Multiplying equation (4.4.11) by w and integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_{\mathcal{M}} w(v'' - \Delta v + qv) \, dv_g dt = \left[\int_{\mathcal{M}} (wv' - w'v) \, dv_g \right]_0^T \\ &\quad + \int_0^T \int_{\mathcal{M}} (w'' - \Delta w + qw)v \, dv_g dt + \int_0^T \int_{\Gamma} (v\partial_\nu w - w\partial_\nu v) \, d\sigma_g dt \\ &= \int_{\mathcal{M}} (w_1 v(0) - w_0 v'(0)) \, dv_g + \int_{\Sigma_0} |\partial_\nu w|^2 \, d\sigma_g dt. \end{aligned} \quad (4.4.12)$$

Hence

$$\langle J(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}, \mathcal{H}'} = \int_0^T \int_{\Gamma_0} |\partial_\nu w|^2 \, d\sigma_g dt.$$

The observability inequality yields

$$\langle J(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}, \mathcal{H}'} \geq C \|(w_0, w_1)\|_{\mathcal{H}}^2, \quad \forall (w_0, w_1) \in \mathcal{C}_0^\infty \times \mathcal{C}_0^\infty.$$

Applying the Lax-Milgram theorem to the linear mapping J , we obtain

$$\langle J(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}, \mathcal{H}'} \geq C \|(w_0, w_1)\|_{\mathcal{H}}^2, \quad \forall (w_0, w_1) \in \mathcal{H}.$$

This completes the proof. \square

Remark As for topics related to the exact controllability, see [3], [43], [44], [45].

Carleman estimates for some thermoelasticity systems

5.1 Introduction

As an important application of our Carleman estimate with second large parameter, we consider thermoelasticity systems. To our best knowledge, there are not many works concerning Carleman estimates for strongly coupled systems of partial differential equations where the principal parts are coupled. Indeed, no general method is available for proving Carleman estimates for systems, except that by the multiplication of the system by the cofactors matrix, we can use the machinery of scalar Carleman estimates for the determinant. Unfortunately this method needs high regularity assumptions on the coefficients. Especially in the case of the boundary problem, since this method increases the multiplicity of real characteristics near the boundary, the Lopatinskii condition is not easily satisfied.

In deriving a Carleman estimate for the thermoelasticity systems, there is another difficulty coming from the coupling of two equations and we have to keep the dependency on the second large parameter in the weight function. Thanks to the second large parameter γ in our Carleman estimate Theorem 3.5.8 for the scalar hyperbolic equation, we will derive Carleman estimates for some strongly coupled systems. Isakov and Kim [30], [31] apply Carleman estimates with second large parameter to a linear elastic system with residual stress, and we can refer to Eller [14], Eller and Isakov [15] as related works. In this section, thanks to Theorem 3.5.8, we establish Carleman estimates for

- a coupled parabolic-hyperbolic system related to the thermoelasticity
- a thermoelasticity plate system
- a thermoelasticity system with residual stress.

The arguments in this chapter are adaptations of [7]. We do not assume that functions to be estimated have compact supports in some cases, while in [14], [15], [30] and [31], functions are always assumed to have compact supports.

5.2 Carleman estimates for elliptic/parabolic operators

In order to prove Carleman estimates for some thermoelasticity systems, we need Carleman estimates with second large parameter also for a second-order parabolic or elliptic operator

$\epsilon \partial_t - \Delta_g$, $\epsilon = 0, 1$. As for Carleman estimates for parabolic equations with singular weight functions, we can refer to Fursikov and Imanuvilov [16], Imanuvilov and Yamamoto [?]. In this section we give a Carleman estimate for parabolic equations with the regular weight function φ .

We use usual functions spaces, $C_0^\infty(Q)$, $H^k(Q)$ and we set

$$H^{2,1}(Q) = H^1(0, T; L^2(\mathcal{M})) \cap L^2(0, T; H^2(\mathcal{M})).$$

We recall that

$$\sigma = s\gamma\varphi,$$

and we set $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

Let (\mathcal{M}, g) be a Riemannian compact manifold. We recall the following assumptions:

- **Assumption (A.1):** There exists a function ψ_0 which is strictly convex on \mathcal{M} with respect to the Riemannian metric g . That is, the Hessian of the function ψ_0 in the Riemannian metric g is positive on \mathcal{M} :

$$D^2\psi_0(X, X)(x) > 0, \quad x \in \mathcal{M}, \quad X \in T_x\mathcal{M} \setminus \{0\}.$$

Since \mathcal{M} is compact, it follows that there exists a positive constant $\varrho > 0$ such that

$$D^2\psi_0(X, X)(x) > 2\varrho |X|^2, \quad x \in \mathcal{M}, \quad X \in T_x\mathcal{M} \setminus \{0\}. \quad (5.2.1)$$

- **Assumption (A.2):** We assume that $\psi_0(x)$ has no critical points in \mathcal{M} :

$$\min_{x \in \mathcal{M}} |\nabla \psi_0(x)| > 0. \quad (5.2.2)$$

- **Assumption (A.3):** Under assumption (A.1)-(A.2), let a subboundary $\Gamma_0 \subset \partial\mathcal{M}$ satisfy

$$\{x \in \partial\mathcal{M}; \partial_\nu \psi_0 \geq 0\} \subset \Gamma_0. \quad (5.2.3)$$

As in the previous chapter, we set

$$\tilde{Q} = \mathcal{M} \times (-T, T), \quad \tilde{\Sigma} = \partial\mathcal{M} \times (-T, T), \quad \tilde{\Sigma}_0 = \Gamma_0 \times (-T, T)$$

and

$$\psi(t, x) = \psi_0(x) - \beta t^2 + \beta_0, \quad 0 < \beta < \varrho, \quad \beta_0 \geq 0, \quad (5.2.4)$$

where the constant ϱ is given in (5.2.1). We choose a parameter β_0 such that the function ψ given by (5.2.4) is positive. We define the weight function $\varphi : \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(x, t) = e^{\gamma\psi(x, t)}, \quad (5.2.5)$$

where $\gamma > 0$ is a parameter.

Then the following parabolic/elliptic Carleman estimate with second large parameter holds:

Lemma 5.2.1 *Let $\epsilon = 0, 1$. There exist three positive constants γ_* , s_* and C such that, for any $\gamma \geq \gamma_*$ and any $s \geq s_*$, the following inequality holds:*

$$\begin{aligned} C\gamma \int_Q \left(\sigma^{-1} \sum_{|\alpha|=2} |\partial^\alpha y(x, t)|^2 + \sigma |\nabla y(x, t)|^2 + \sigma^3 |y(x, t)|^2 \right) e^{2s\varphi} dv_g dt \\ \leq \int_Q |(\epsilon \partial_t - \Delta_g)y(x, t)|^2 e^{2s\varphi} dv_g dt + \int_{\Sigma_0} \sigma |\partial_\nu y|^2 e^{2s\varphi} d\sigma_g dt \end{aligned} \quad (5.2.6)$$

for any $y \in H^{2,1}(Q)$ such that $\epsilon y(x, 0) = \epsilon y(x, T) = 0$ in \mathcal{M} and $y(x, t) = 0$ on Σ .

Proof. For $s > 0$, let us introduce the new functions $z(x, t) = e^{s\varphi}y(x, t)$ and $h_{s,\gamma} = e^{s\varphi}h_0$, where $h_0 = (\epsilon \partial_t - \Delta_g)y$. The standard approach to Carleman estimate (5.2.6) starts from the observation:

$$e^{s\varphi}(\epsilon \partial_t - \Delta_g)y(x, t) = \mathcal{L}_{\gamma,s}(t, x, D)z(x, t) \quad (5.2.7)$$

where

$$\begin{aligned} \mathcal{L}_{\gamma,s}(t, x, D)z(x, t) = \epsilon \partial_t z - \Delta_g z + s \left(\gamma (\Delta_g \psi) \varphi + \gamma^2 |\nabla \psi|^2 \varphi \right) z \\ + 2s\gamma \varphi \langle \nabla \psi, \nabla z \rangle - s^2 \gamma^2 |\nabla \psi|^2 \varphi^2 z - s\epsilon (\partial_t \varphi) z. \end{aligned} \quad (5.2.8)$$

We set

$$Az(x, t) + Bz(x, t) = h_{s,\gamma}(x, t) = h_0(x, t) - \left(\sigma (\Delta_g \psi) + \gamma \sigma |\nabla \psi|^2 \right) z + \epsilon \sigma (\partial_t \varphi) z, \quad (5.2.9)$$

where

$$Az(x, t) = -\Delta_g z - \sigma^2 |\nabla \psi|^2 z \quad (5.2.10)$$

and

$$Bz(x, t) = \epsilon \partial_t z + 2\sigma \langle \nabla \psi, \nabla z \rangle. \quad (5.2.11)$$

With the previous notations, we have

$$\|h_{s,\gamma}\|_{L^2(Q)}^2 = \|Az\|_{L^2(Q)}^2 + \|Bz\|_{L^2(Q)}^2 + 2(Az, Bz)_{L^2(Q)}. \quad (5.2.12)$$

Next we will compute $2(Az, Bz)_{L^2(Q)}$ to first look for lower bounds for $(Az, Bz)_{L^2(Q)}$. We decompose $(Az, Bz)_{L^2(Q)} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3$ with

$$\begin{aligned} \mathcal{K}_1 &:= -\epsilon \int_Q (\Delta_g z + \sigma^2 |\nabla \psi|^2 z) \partial_t z dv_g dt \\ \mathcal{K}_2 &:= -\int_Q \Delta_g z (2\sigma \langle \nabla \psi, \nabla z \rangle) dv_g dt \\ \mathcal{K}_3 &:= -2 \int_Q \sigma^3 |\nabla \psi|^2 z \langle \nabla \psi, \nabla z \rangle dv_g dt. \end{aligned} \quad (5.2.13)$$

We first deal with \mathcal{K}_1 :

$$\mathcal{K}_1 = \epsilon \int_Q \frac{\partial}{\partial t} (|\nabla z|^2) dv_g dt - \frac{1}{2} \epsilon \int_Q \sigma^2 |\nabla \psi_0|^2 \frac{\partial}{\partial t} (|z|^2) dv_g dt$$

$$\begin{aligned}
&= \epsilon \int_Q \sigma \partial_t \sigma |\nabla \psi_0|^2 |z|^2 dx dt \\
&= \epsilon \gamma \int_Q \sigma^2 \partial_t \psi |\nabla \psi_0|^2 |z|^2 dv_g dt,
\end{aligned} \tag{5.2.14}$$

where we have used $\nabla \psi = \nabla \psi_0$.

For the term \mathcal{K}_3 , the integration by parts in x yields

$$\begin{aligned}
\mathcal{K}_3 &= -2 \int_Q \sigma^3 |\nabla \psi_0|^2 z \langle \nabla \psi_0, \nabla z \rangle dv_g dt \\
&= - \int_Q \sigma^3 |\nabla \psi_0|^2 \langle \nabla \psi_0, \nabla (|z|^2) \rangle dv_g dt \\
&= \int_Q |z|^2 \operatorname{div} (\sigma^3 |\nabla \psi_0|^2 \nabla \psi_0) dv_g dt \\
&= \int_Q \sigma^3 |z|^2 (|\nabla \psi_0|^2 \Delta_g \psi_0 + 3\gamma |\nabla \psi_0|^4 + \langle \nabla \psi_0, \nabla (|\nabla \psi_0|^2) \rangle) dv_g dt.
\end{aligned} \tag{5.2.15}$$

We now turn to the term \mathcal{K}_2 . By a similar way to in (3.3.15), we also have

$$\begin{aligned}
\mathcal{K}_2 &= -2 \int_Q \sigma \Delta_g z \langle \nabla \psi_0, \nabla z \rangle dv_g dt \\
&= 2\gamma \int_Q \sigma |\langle \nabla z, \nabla \psi_0 \rangle|^2 dv_g dt - \int_Q \sigma |\nabla z|^2 \Delta_g \psi_0 dv_g dt \\
&\quad - \gamma \int_Q \sigma |\nabla z|^2 |\nabla \psi_0|^2 dv_g dt + 2 \int_Q \sigma D^2 \psi_0 (\nabla z, \nabla z) dv_g dt \\
&\quad - \left[\int_\Sigma \sigma |\partial_\nu z|^2 (\nabla \psi_0 \cdot \nu) d\sigma_g dt \right].
\end{aligned} \tag{5.2.16}$$

From (5.2.16), (5.2.15) and (5.2.14), we have

$$\begin{aligned}
(Az, Bz)_{L^2(Q)} &= 3\gamma \int_Q \sigma^3 |\nabla \psi_0|^4 |z|^2 dv_g dt + 2\gamma \int_Q \sigma |\langle \nabla z, \nabla \psi_0 \rangle|^2 dv_g dt \\
&\quad - \gamma \int_Q \sigma |\nabla z|^2 |\nabla \psi_0|^2 dx dt + \mathcal{Q}_1(z, \nabla z) \\
&\quad - \left[\int_\Sigma \sigma |\partial_\nu z|^2 (\nabla \psi_0 \cdot \nu) d\sigma_g dt \right],
\end{aligned} \tag{5.2.17}$$

where $\mathcal{Q}_1(z, \nabla z)$ satisfies

$$|\mathcal{Q}_1(z, \nabla z)| \leq C \left(\int_Q \sigma |\nabla z|^2 dv_g dt + \int_Q \sigma^3 |z|^2 dv_g dt + \gamma \int_Q \sigma^2 |z|^2 dv_g dt \right). \tag{5.2.18}$$

Multiply (5.2.9) by $\gamma \sigma z |\nabla \psi_0|^2$ and integrate by parts, and we obtain

$$\begin{aligned}
\gamma \int_Q \sigma z |\nabla \psi_0|^2 h_{s,\gamma}(x, t) \, dv_g \, dt &= \gamma \int_Q \sigma z |\nabla \psi_0|^2 (Bz) \, dv_g \, dt \\
&\quad - \gamma \int_Q \sigma z |\nabla \psi_0|^2 \Delta_g z \, dv_g \, dt - \gamma \int_Q \sigma^3 |z|^2 |\nabla \psi_0|^4 \, dv_g \, dt \\
&= \gamma \int_Q \sigma z |\nabla \psi_0|^2 (Bz) \, dv_g \, dt + \gamma \int_Q \sigma |\nabla z|^2 |\nabla \psi_0|^2 \, dv_g \, dt \\
&\quad + \gamma^2 \int_Q \sigma \langle \nabla \psi_0, \nabla z \rangle |\nabla \psi_0|^2 z \, dv_g \, dt + \gamma \int_Q \sigma z \langle \nabla z, \nabla (|\nabla \psi_0|^2) \rangle \, dv_g \, dt \\
&\quad - \gamma \int_Q \sigma^3 |z|^2 |\nabla \psi_0|^4 \, dv_g \, dt. \quad (5.2.19)
\end{aligned}$$

Hence

$$2\gamma \int_Q \sigma^3 |z|^2 |\nabla \psi_0|^4 \, dv_g \, dt = 2\gamma \int_Q \sigma |\nabla z|^2 |\nabla \psi_0|^2 \, dv_g \, dt + 2\mathcal{Q}_2(z, \nabla z), \quad (5.2.20)$$

where $\mathcal{Q}_2(z, \nabla z)$ satisfies

$$\begin{aligned}
|\mathcal{Q}_2(z, \nabla z)| &\leq C \left(\gamma^2 \int_Q \sigma^2 |z|^2 \, dv_g \, dt + s^{-1} \gamma \int_Q \sigma |\nabla z|^2 \, dv_g \, dt \right) \\
&\quad + \frac{1}{16} \|Bz\|^2 + \frac{1}{2} \|h_{s,\gamma}\|^2 \quad (5.2.21)
\end{aligned}$$

and we have used $\varphi \geq 1$ in Q .

As a consequence, we have

$$\begin{aligned}
(Az, Bz)_{L^2(Q)} &= 3\gamma \int_Q \sigma^3 |\nabla \psi_0|^4 |z|^2 \, dv_g \, dt + 2\gamma \int_Q \sigma |\langle \nabla z, \nabla \psi_0 \rangle|^2 \, dv_g \, dt \\
&\quad - \gamma \int_Q \sigma |\nabla z|^2 |\nabla \psi_0|^2 \, dv_g \, dt + \mathcal{Q}_1(z, \nabla z) - \left[\int_\Sigma \sigma |\partial_\nu z|^2 (\nabla \psi_0 \cdot \nu) \, d\sigma_g \, dt \right] \\
&= \gamma \int_Q \sigma^3 |\nabla \psi_0|^4 |z|^2 \, dv_g \, dt + 2\gamma \int_Q \sigma |\langle \nabla z, \nabla \psi_0 \rangle|^2 \, dv_g \, dt \\
&\quad + \gamma \int_Q \sigma |\nabla z|^2 |\nabla \psi_0|^2 \, dv_g \, dt - \left[\int_\Sigma \sigma |\partial_\nu z|^2 (\nabla \psi_0 \cdot \nu) \, d\sigma_g \, dt \right] \\
&\quad + \mathcal{Q}_1(z, \nabla z) + 2\mathcal{Q}_2(z, \nabla z). \quad (5.2.22)
\end{aligned}$$

Now, combining (5.2.22), (5.2.21) and (5.2.18), we have

$$\begin{aligned}
2(Az, Bz) + 2 \left[\int_\Sigma \sigma |\partial_\nu z|^2 (\nabla \psi_0 \cdot \nu) \, d\sigma_g \, dt \right] &\geq 2\gamma \int_Q \sigma^3 |\nabla \psi_0|^4 |z|^2 \, dv_g \, dt \\
&\quad + 2\gamma \int_Q \sigma |\nabla z|^2 |\nabla \psi_0|^2 \, dv_g \, dt - C \left(\int_Q \sigma |\nabla z|^2 \, dv_g \, dt + \int_Q \sigma^3 |z|^2 \, dv_g \, dt \right) \\
&\quad - C \left(\gamma^2 \int_Q \sigma^2 |z|^2 \, dv_g \, dt + s^{-1} \gamma \int_Q \sigma |\nabla z|^2 \, dv_g \, dt \right) - \frac{1}{4} \|Bz\|^2 - 2 \|h_{s,\gamma}\|^2. \quad (5.2.23)
\end{aligned}$$

Now, since $\nabla\psi_0 \neq 0$ in $\overline{\mathcal{M}}$, we conclude that for any $s \geq s_*$ and $\gamma \geq \gamma_*$, we obtain

$$\begin{aligned} 2(Az, Bz) + 2 \left[\int_{\Sigma} \sigma |\partial_{\nu} z|^2 (\nabla\psi_0 \cdot \nu) \, d\sigma_g \, dt \right] \\ \geq C\gamma \left(\int_Q \sigma^3 |z|^2 \, dv_g \, dt + \int_Q \sigma |\nabla z|^2 \, dv_g \, dt \right) - \frac{1}{4} \|Bz\|^2 - 2 \|h_{s,\gamma}\|^2. \end{aligned} \quad (5.2.24)$$

Thus we have also

$$\begin{aligned} \|h_{s,\gamma}\|^2 + \left[\int_{\Sigma} \sigma |\partial_{\nu} z|^2 (\nabla\psi_0 \cdot \nu) \, d\sigma_g \, dt \right] \geq \\ C \left(\gamma \int_Q (\sigma^3 |z|^2 + \sigma |\nabla z|^2) \, dv_g \, dt + \|Bz\|^2 + \|Az\|^2 \right). \end{aligned} \quad (5.2.25)$$

Next we will estimate $|\Delta_g z|$. Since

$$|Az|^2 \geq C |\Delta_g z|^2 - \sigma^4 |\nabla\psi_0|^4 |z|^2 \quad \text{in } Q, \quad (5.2.26)$$

by $\varphi \geq 1$ and (5.2.25) we obtain

$$\begin{aligned} C\gamma \int_Q \sigma^{-1} |\Delta_g z(x, t)|^2 \, dx \, dt \leq \|Az\|^2 + C\gamma \int_Q \sigma^3 |z|^2 \, dv_g \, dt \\ \leq C \left(\|h_{s,\gamma}\|^2 + \int_{\Sigma} \sigma |\partial_{\nu} z|^2 (\nabla\psi_0 \cdot \nu) \, d\sigma_g \, dt \right). \end{aligned} \quad (5.2.27)$$

By (5.2.27) and (5.2.25), we deduce

$$\begin{aligned} \|h_{s,\gamma}\|^2 + \left[\int_{\Sigma} \sigma |\partial_{\nu} z|^2 (\nabla\psi_0 \cdot \nu) \, dv_g \, dt \right] \geq \\ C\gamma \int_Q (\sigma^3 |z(x, t)|^2 + \sigma |\nabla z(x, t)|^2 + \sigma^{-1} |\Delta_g z(x, t)|^2) \, dv_g \, dt. \end{aligned} \quad (5.2.28)$$

The final step is to add integral of $\sum_{|\alpha|=2} |\partial^{\alpha} y(x, t)|^2$ to the right-hand side of (5.2.28). This can be made using the following computation

$$\Delta_g(\sigma^{-1/2} z) = \sigma^{-1/2} \Delta_g z + \left(\frac{\gamma^2}{4} \sigma^{-1/2} |\nabla\psi|^2 - \frac{\gamma}{2} \sigma^{-1/2} \Delta_g \psi \right) z - \gamma \sigma^{-1/2} \langle \nabla z, \nabla \psi \rangle. \quad (5.2.29)$$

We deduce from (5.2.29) and the elliptic estimates that

$$C \sum_{|\alpha|=2} \int_Q |\partial^{\alpha}(\sigma^{-1/2} z)|^2 \, dv_g \, dt \leq \int_Q (\sigma^{-1} |\Delta_g z|^2 + \sigma^3 |z|^2 + \sigma |\nabla z|^2) \, dv_g \, dt, \quad (5.2.30)$$

where we have used $z(\cdot, t) = 0$ on $\partial\mathcal{M}$ for all $t \in (0, T)$.

On the other hand, we can find

$$C \sum_{|\alpha|=2} \int_Q \sigma^{-1} |\partial^\alpha z|^2 \, dv_g \, dt \leq \sum_{|\alpha|=2} \int_Q |\partial^\alpha (\sigma^{-1/2} z)|^2 \, dv_g \, dt + \int_Q (\sigma^3 |z|^2 + \sigma |\nabla z|^2) \, dv_g \, dt. \quad (5.2.31)$$

By (5.2.31) and (5.2.30), we obtain

$$C\gamma \sum_{|\alpha|=2} \int_Q \sigma^{-1} |\partial^\alpha z|^2 \, dv_g \, dt \leq \gamma \int_Q (\sigma^{-1} |\Delta_g z|^2 + \sigma^3 |z|^2 + \sigma |\nabla z|^2) \, dv_g \, dt. \quad (5.2.32)$$

Substituting $z(x, t) = e^{s\varphi} y(x, t)$ and noting (5.2.9), we can complete the proof of (5.2.6). \square

From Lemma 5.2.1 we can derive the following type of Carleman estimates:

Lemma 5.2.2 *Let $\epsilon = 0, 1$ and $k \in \mathbb{N}$. There exist three positive constants γ_* , s_* and C such that, for any $\gamma \geq \gamma_*$ and any $s \geq s_*$, the following inequality holds:*

$$\begin{aligned} C\gamma \int_Q \left(\sigma^{k-1} \sum_{|\alpha|=2} |\partial^\alpha y(x, t)|^2 + \sigma^{k+1} |\nabla y(x, t)|^2 + \sigma^{k+3} |y(x, t)|^2 \right) e^{2s\varphi} \, dv_g \, dt \\ \leq \int_Q \sigma^k |(\epsilon \partial_t - \Delta_g) y(x, t)|^2 e^{2s\varphi} \, dv_g \, dt + \int_{\Sigma_0} \sigma^{k+1} |\partial_\nu y|^2 e^{2s\varphi} \, d\sigma_g \, dt \end{aligned} \quad (5.2.33)$$

for any $y \in H^{2,1}(Q)$ such that $\epsilon y(x, 0) = \epsilon y(x, T) = 0$ in \mathcal{M} and $y(x, t) = 0$ on Σ .

Proof . In order to apply Lemma 5.2.1, we introduce $y_1 = \varphi^{k/2} y$ with $\varphi^{k/2} = e^{\frac{k\gamma}{2}\psi}$. Let us remark that

$$\left| \nabla \varphi^{\frac{k}{2}} \right| \leq C\gamma \varphi^{\frac{k}{2}}, \quad \left| \partial_t \varphi^{\frac{k}{2}} \right| \leq C\gamma \varphi^{\frac{k}{2}}, \quad \left| \Delta_g \varphi^{\frac{k}{2}} \right| \leq C\gamma^2 \varphi^{\frac{k}{2}}. \quad (5.2.34)$$

Then

$$\begin{aligned} |\Delta_g y_1|^2 &\geq C\varphi^k |\Delta_g y|^2 - C\gamma^4 \varphi^k |y|^2 - C\gamma^2 \varphi^k |\nabla y|^2, \\ |\partial^\alpha y_1|^2 &\geq C\varphi^k |\partial^\alpha y|^2 - C\gamma^4 \varphi^k |y|^2 - C\gamma^2 \varphi^k |\nabla y|^2, \quad |\alpha| = 2 \end{aligned} \quad (5.2.35)$$

and

$$|\nabla y_1|^2 \geq C\varphi^k |\nabla y|^2 - C\gamma^2 \varphi^k |y|^2. \quad (5.2.36)$$

After computations we also see that

$$\begin{aligned} \gamma \left(\sigma^{-1} \left(|\Delta_g y_1|^2 + \sum_{|\alpha|=2} |\partial^\alpha y_1|^2 \right) + \sigma |\nabla y_1|^2 + \sigma^3 |y_1|^2 \right) \\ \geq C s^{-1} \varphi^{k-1} \left(|\Delta_g y|^2 + \sum_{|\alpha|=2} |\partial^\alpha y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) \\ - C s^{-1} \varphi^{k-1} \left(s^{-2} \sigma^2 |\nabla y|^2 + \gamma^2 \sigma^2 |y|^2 \right). \end{aligned} \quad (5.2.37)$$

Then for $s \geq s_*$ and $\gamma \geq \gamma_*$, we obtain

$$\begin{aligned} & \gamma \left(\sigma^{-1} \left(|\Delta_{\mathbb{g}} y_1|^2 + \sum_{|\alpha|=2} |\partial^\alpha y_1|^2 \right) + \sigma |\nabla y_1|^2 + \sigma^3 |y_1|^2 \right) \\ & \geq C s^{-1} \varphi^{k-1} \left(|\Delta_{\mathbb{g}} y|^2 + \sum_{|\alpha|=2} |\partial^\alpha y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right). \end{aligned} \quad (5.2.38)$$

On the other hand, we have

$$|(\epsilon \partial_t - \Delta_{\mathbb{g}}) y_1|^2 \leq \varphi^k (|(\epsilon \partial_t - \Delta_{\mathbb{g}}) y|^2 + \gamma^4 |y|^2 + \gamma^2 |\nabla y|^2). \quad (5.2.39)$$

From (5.2.39), (5.2.38) and (5.2.37), we deduce

$$\begin{aligned} & \gamma \int_Q \sigma^{k-1} \left(|\Delta_{\mathbb{g}} y|^2 + \sum_{|\alpha|=2} |\partial^\alpha y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) e^{2s\varphi} \, dv_{\mathbb{g}} \, dt \leq \\ & C \left(\int_Q \sigma^k |(\epsilon \partial_t - \Delta_{\mathbb{g}}) y|^2 e^{2s\varphi} \, dv_{\mathbb{g}} \, dt + \int_{\Sigma_0} \sigma^{k+1} |\partial_\nu y|^2 e^{2s\varphi} \, d\sigma_{\mathbb{g}} \, dt \right) \end{aligned} \quad (5.2.40)$$

provided that $s \geq s_*$ and $\gamma \geq \gamma_*$. This completes the proof. \square

5.3 Carleman estimate for a coupled parabolic-hyperbolic system

Let \mathcal{M} be an n -dimensional compact connected C^∞ Riemannian manifold and let $T > 0$ be given. In this subsection we will use the scalar Carleman estimate with second large parameter (Theorem 3.5.8) to prove Carleman estimates for a parabolic-hyperbolic strongly coupled system arising in the thermoelasticity theory. In order to formulate our Carleman type estimates, we introduce some notations:

$$\begin{aligned} & (\partial_t^2 - c \Delta_{\mathbb{g}}) u(x, t) + a \Delta_{\mathbb{g}} y(x, t) = f_h(x, t) \quad \text{in } Q \equiv \mathcal{M} \times (0, T), \\ & (\partial_t - \Delta_{\mathbb{g}}) y(x, t) + a \partial_t u = f_p(x, t) \quad \text{in } Q, \\ & u(x, t) = 0, \quad y(x, t) = 0 \quad \text{on } \Sigma \equiv \Gamma \times (0, T). \end{aligned} \quad (5.3.1)$$

The coupling parameter a and the velocity c are assumed to be positive constants. The boundary of \mathcal{M} may be empty, and in the case of no boundary, the Dirichlet boundary condition in (5.3.1) is neglected.

Let (u, y) satisfy the linear coupled parabolic-hyperbolic system (5.3.1) such that

$$y(x, 0) = y(x, T) = 0, \quad \partial_t^j u(x, 0) = \partial_t^j u(x, T) = 0, \quad \text{for all } x \in \mathcal{M}, j = 0, 1. \quad (5.3.2)$$

Furthermore we assume that there exists a positive function ϑ satisfying the assumptions (A.1), (A.2) and (A.3) with respect to the metric g .

The following theorem is a Carleman estimate with second large parameter for the coupled parabolic-hyperbolic system (5.3.1).

Theorem 5.3.14 *There exist $\gamma_* > 0$ and $C > 0$ such that for any $\gamma > \gamma_*$, there exists $s_* = s_*(\gamma) > 0$ such that the following estimate holds:*

$$\begin{aligned} & C \int_Q (|\Delta_g y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 + \sigma (|\nabla u|^2 + |\partial_t u|^2) + \sigma^2 |u|^2) e^{2s\varphi} dv_g dt \\ & \leq \int_Q (\gamma^{-1} \sigma |f_p|^2 + |f_h|^2) e^{2s\varphi} dv_g dt + \int_{\Sigma_0} \sigma (\gamma^{-1} \sigma |\partial_\nu y|^2 + |\partial_\nu u|^2) e^{2s\varphi} d\sigma_g dt \end{aligned} \quad (5.3.3)$$

for any solution $(u, y) \in H^2(Q) \times H^{2,1}(Q)$ to problem (5.3.1) satisfying (5.3.2) and any $s \geq s_*$.

System (5.3.1) arises in three spatial dimensions when analyzing the linear system of thermoelasticity:

$$\begin{aligned} (\partial_t^2 - \Delta_{\mu,\lambda}) \mathbf{w}(x, t) + a \nabla y(x, t) &= \mathbf{f}_e(x, t) & \text{in } Q, \\ (\partial_t - \Delta) y(x, t) + a \operatorname{div} \partial_t \mathbf{w} &= f_p(x, t) & \text{in } Q, \\ \mathbf{w}(x, t) = 0, \quad y(x, t) &= 0 & \text{on } \Sigma, \end{aligned} \quad (5.3.4)$$

where $\Delta_{\mu,\lambda}$ is the elliptic second-order linear differential operator given by

$$\Delta_{\mu,\lambda} \mathbf{v}(x) \equiv \mu \Delta \mathbf{v}(x) + (\mu + \lambda) (\nabla \operatorname{div} \mathbf{v}(x)) \quad x \in \mathcal{M} \quad (5.3.5)$$

for $\mathbf{v} = (v_1, v_2, v_3)^\top$, where \cdot^\top denotes the transpose of matrix. Here t and $x = (x_1, x_2, x_3)$ denote the time variable and the spatial variable respectively, and $\mathbf{w} = (w_1, w_2, w_3)^\top$ denotes the displacement at the location x and the time t , and $y = y(x, t)$, the temperature, is a scalar function, $f_p \in L^2(Q)$ is a heat source and $\mathbf{f}_e \in H^1(Q)$ is a body force, and for simplicity, we here assume that $\lambda > 0$ and $\mu > 0$ are constants. Setting $\mathbf{v} = \operatorname{curl} \mathbf{w}$, $u = \operatorname{div} \mathbf{w}$, $\Delta_g = \Delta$, $c = 2\mu + \lambda$ and $f_h = \operatorname{div} \mathbf{f}_e$, we can change the first equation in (5.3.4) into a diagonal system of hyperbolic equations in $\mathbf{w}, v, \mathbf{v}$ with principal parts $\partial_t^2 - \mu \Delta$ and $\partial_t^2 - (\lambda + 2\mu) \Delta$. Subsection 5.5.1 gives the detailed arguments of such diagonalization in a bit general case. Thanks to the diagonalization, we can apply Lemma 5.2.1 and Theorem 3.5.8 to establish a Carleman estimate for (5.3.4) provided that \mathbf{w} has compact supports. For Carleman estimates for Lamé system for functions without compacy supports, see [26], [27] and [28].

In the case where \mathcal{M} is a bounded domain in \mathbb{R}^n , that is, \mathcal{M} has boundary, see [14] and [15]. See also [1] as for a Carleman estimate in a bounded domain \mathcal{M} with singular weight function which was introduced in Fursikov and Imanuvilov [16]. **Proof.** Let (u, y) satisfy the parabolic-hyperbolic system (5.3.1) and (5.3.2). Applying the first version of the parabolic Carleman estimate (5.2.6) with $\epsilon = 1$ to the second equation in (5.3.1), we obtain

$$\begin{aligned} C\gamma \int_Q (\sigma^{-1} |\Delta_g y|^2 + \sigma |\nabla y|^2 + \sigma^3 |y|^2) e^{2s\varphi} dv_g dt &\leq \int_Q |f_p|^2 e^{2s\varphi} dv_g dt \\ &+ \int_Q |\partial_t u|^2 e^{2s\varphi} dv_g dt + \int_{\Sigma_0} \sigma |\partial_\nu y|^2 e^{2s\varphi} d\sigma_g dt. \end{aligned} \quad (5.3.6)$$

Furthermore Theorem 3.5.8 yields

$$\begin{aligned}
C \int_Q \sigma (|\nabla u|^2 + |\partial_t u|^2 + \sigma^2 |u|^2) e^{2s\varphi} dv_g dt &\leq \int_Q |f_h|^2 e^{2s\varphi} dv_g dt \\
&+ \int_Q |\Delta_g y|^2 e^{2s\varphi} dv_g dt + \int_{\Sigma_0} \sigma |\partial_\nu u|^2 e^{2s\varphi} d\sigma_g dt. \quad (5.3.7)
\end{aligned}$$

Adding (5.3.7) and (5.3.6), we arrive at

$$\begin{aligned}
C \int_Q (\gamma(\sigma^{-1}|\Delta_g y|^2 + \sigma|\nabla y|^2 + \sigma^3|y|^2) + \sigma(|\nabla u|^2 + |\partial_t u|^2 + \sigma^2|u|^2)) e^{2s\varphi} dv_g dt \\
\leq \int_Q (|f_p|^2 + |f_h|^2) e^{2s\varphi} dv_g dt + \int_Q |\Delta_g y|^2 e^{2s\varphi} dv_g dt \\
+ \int_{\Sigma_0} \sigma (|\partial_\nu u|^2 + |\partial_\nu y|^2) e^{2s\varphi} d\sigma_g dt, \quad (5.3.8)
\end{aligned}$$

provided that $\gamma \geq \gamma_*$ and $s \geq s_*(\gamma)$.

By the second version of the parabolic Carleman estimate (5.2.33) with $k = 1$ and $\epsilon = 1$, we have

$$\begin{aligned}
C \int_Q (|\Delta_g y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2) e^{2s\varphi} dv_g dt &\leq \int_Q \gamma^{-1} \sigma |f_p|^2 e^{2s\varphi} dv_g dt \\
&+ \int_Q \gamma^{-1} \sigma |\partial_t u|^2 e^{2s\varphi} dv_g dt + \int_{\Sigma_0} \gamma^{-1} \sigma^2 |\partial_\nu y|^2 e^{2s\varphi} d\sigma_g dt. \quad (5.3.9)
\end{aligned}$$

By (5.3.9) and (5.3.8), we find

$$\begin{aligned}
C \int_Q (\gamma\sigma^{-1}|\Delta_g y|^2 + \gamma\sigma|\nabla y|^2 + \gamma\sigma^3|y|^2 + \sigma (|\nabla u|^2 + |\partial_t u|^2 + \sigma^2|u|^2)) e^{2s\varphi} dv_g dt \\
\leq \int_Q (\gamma^{-1}\sigma|f_p|^2 + |f_h|^2 + \gamma^{-1}\sigma|\partial_t u|^2) e^{2s\varphi} dv_g dt \\
+ \int_{\Sigma_0} \sigma (|\partial_\nu u|^2 + \gamma^{-1}\sigma|\partial_\nu y|^2) e^{2s\varphi} d\sigma_g dt, \quad (5.3.10)
\end{aligned}$$

provided that $\gamma \geq \gamma_*$ and $s \geq s_*(\gamma)$. Thanks to the second large parameter γ , we can absorb the term $\gamma^{-1}\sigma|\partial_t u|^2$ into the left-hand side. Adding now (5.3.10) and (5.3.9), we obtain (5.3.3).

This completes the proof of Theorem 5.3.14. \square

5.4 Carleman estimate for thermoelasticity plate system

In this subsection we will prove Carleman estimate for a thermoelasticity plate system. Let us consider a bounded and isotropic body occupying an open and bounded domain $\mathcal{M} \subset \mathbb{R}^2$ with C^∞ boundary $\partial\mathcal{M}$. Given $T > 0$, we consider the following linear system of

thermoelasticity plate which describes the small vibrations of a thin isotropic thermoelastic plate in presence of exterior forces and heat sources

$$\begin{aligned} (1 - \mu(x)\Delta) \partial_t^2 u(x, t) + \Delta^2 u + a\Delta y(x, t) &= f_b(x, t) \text{ in } Q = \mathcal{M} \times (0, T) \\ \partial_t y(x, t) - \Delta y(x, t) - a\Delta \partial_t u &= f_p(x, t) \text{ in } Q = \mathcal{M} \times (0, T) \\ u(x, t) = \Delta u(x, t) = 0, \quad y(x, t) &= 0 \text{ on } \Sigma = \partial\mathcal{M} \times (0, T). \end{aligned} \quad (5.4.1)$$

By u and y we denote the vertical displacement and the temperature of the plate, respectively. The coupling parameter a is assumed to be positive and the coefficient $\mu(x) > 0$ on $\overline{\mathcal{M}}$. Furthermore we assume that ψ_0 satisfy (A.1)-(A.2)-(A.3) with respect to the metric $g = \mu^{-1}Id$.

Let (u, y) satisfy the linear coupled system (5.4.1) such that

$$y(x, 0) = y(x, T) = 0, \quad \partial_t^j u(x, 0) = \partial_t^j u(x, T) = 0 \text{ for all } x \in \mathcal{M}, j = 0, 1. \quad (5.4.2)$$

The following theorem is a Carleman estimate with a second larger parameter for the thermoelasticity plate system (5.4.1).

Theorem 5.4.15 *There exist two constants $\gamma_* > 0$ and $C > 0$ such that for any $\gamma > \gamma_*$, there exists $s_* = s_*(\gamma) > 0$ such that the following estimate holds:*

$$\begin{aligned} C \int_Q \left(|\Delta y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 + \sum_{|\alpha| \leq 2} \sigma^{2(2-|\alpha|)} (\sigma^2 |\partial^\alpha u|^2 + |\partial^\alpha \partial_t u|^2) \right) e^{2s\varphi} dxdt \\ + \int_Q \sigma (\sigma^2 |\Delta u|^2 + |\Delta \partial_t u|^2 + |\nabla(\Delta u)|^2) e^{2s\varphi} dxdt \leq \int_Q (\gamma^{-1} \sigma |f_p|^2 + |f_b|^2) e^{2s\varphi} dxdt \\ + \int_{\Sigma_0} \sigma (\sigma^3 |\partial_\nu u|^2 + \sigma |\partial_\nu \partial_t u|^2 + |\partial_\nu(\Delta u)|^2 + \sigma |\partial_\nu y|^2) e^{2s\varphi} d\omega dt \end{aligned} \quad (5.4.3)$$

for any solution $(u, y) \in H^3(Q) \times H^{2,1}(Q)$ to problem (5.4.1) satisfying (5.4.2) and any $s \geq s_*$.

In order to prove Theorem 5.4.15, we need a Carleman estimate for a scalar plate equation.

5.4.1 Carleman estimate for the plate equation

In this subsection, we derive a global Carleman estimate for a solutions of the plate equation. We consider the non-stationary plate equation

$$\begin{aligned} \partial_t^2 w - \mu(x)\Delta \partial_t^2 w + \Delta^2 w + P_1(x, \partial)\Delta w + P_2(x, \partial)w &= f \text{ in } Q, \\ w(x, t) = \Delta w(x, t) = 0 &\text{ on } \Sigma, \end{aligned} \quad (5.4.4)$$

where P_1 and P_2 are first-order and second-order differential operators in x respectively and $f \in L^2(Q)$ is a source term.

The following Carleman estimate holds:

Lemma 5.4.1 *There exist two constants $C > 0$ and $\gamma_* > 0$ such that for any $\gamma > \gamma_*$ there exists $s_* = s_*(\gamma)$ such that for all $s \geq s_*$ the following Carleman estimate holds:*

$$\begin{aligned} C\gamma \int_Q \sum_{|\alpha| \leq 2} \sigma^{2(2-|\alpha|)} (\sigma^2 |\partial^\alpha w|^2 + |\partial^\alpha \partial_t w|^2) e^{2s\varphi} dxdt \\ + C \int_Q (\sigma^3 |\Delta w|^2 + \sigma |\Delta \partial_t w|^2 + \sigma |\nabla(\Delta w)|^2) e^{2s\varphi} dxdt \leq \int_Q |f|^2 e^{2s\varphi} dxdt \\ + \int_{\Sigma_0} (\sigma^4 |\partial_\nu w|^2 + \sigma^2 |\partial_\nu \partial_t w|^2 + \sigma |\partial_\nu(\Delta w)|^2) e^{2s\varphi} d\omega dt \quad (5.4.5) \end{aligned}$$

for any solution $w \in H^3(Q)$ to problem (5.4.3) satisfying $\partial_t^j w(x, 0) = \partial_t^j w(x, T) = 0$, $j = 0, 1$, and any $s \geq s_*$.

Proof . Let us introduce the following new function z which is given by

$$z = w - \mu(x)\Delta w. \quad (5.4.6)$$

Then we have

$$\Delta z = \Delta w - \mu(x)\Delta^2 w - (\Delta\mu)\Delta w - 2\nabla\mu \cdot \nabla(\Delta w).$$

Moreover we have

$$\partial_t^2 z(x, t) = \partial_t^2 w(x, t) - \mu(x)\Delta \partial_t^2 w(x, t).$$

Thus z satisfies the following second-order hyperbolic equation

$$\partial_t^2 z(x, t) - \frac{1}{\mu} \Delta z + \tilde{P}_1(x, \partial)z = f(x, t) + \tilde{P}_2(x, \partial)w \quad \text{in } Q, \quad (5.4.7)$$

where \tilde{P}_1, \tilde{P}_2 are first-order and second-order differential operators in x respectively. The Carleman estimate (Theorem 3.5.8) for the hyperbolic operator, yields

$$\begin{aligned} C \int_Q (\sigma |\nabla z|^2 + \sigma |\partial_t z|^2 + \sigma^3 |z|^2) e^{2s\varphi} dxdt \leq \int_Q |f|^2 e^{2s\varphi} dxdt \\ + \sum_{|\alpha| \leq 2} \int_Q |\partial^\alpha w|^2 e^{2s\varphi} dxdt + \int_{\Sigma_0} \sigma |\partial_\nu z|^2 e^{2s\varphi} d\omega dt. \quad (5.4.8) \end{aligned}$$

On the other hand, since w solves the elliptic equation

$$-\mu(x)\Delta w + w = z \quad \text{in } Q,$$

the elliptic Carleman estimates (5.2.6) with $\epsilon = 0$, yields

$$\begin{aligned} \gamma \int_Q \sigma^{k-1} \left(\sum_{|\alpha|=2} |\partial^\alpha w|^2 + \sigma^2 |\nabla w|^2 + \sigma^4 |w|^2 \right) e^{2s\varphi} dxdt \leq C \int_Q \sigma^k |z|^2 e^{2s\varphi} dxdt \\ + C \int_{\Sigma_0} \sigma^{k+1} |\partial_\nu w|^2 e^{2s\varphi} d\omega dt. \quad (5.4.9) \end{aligned}$$

Thus

$$C \sum_{|\alpha| \leq 2} \int_Q |\partial^\alpha w|^2 e^{2s\alpha} dxdt \leq \int_Q \sigma |z|^2 e^{2s\varphi} dxdt + \int_{\Sigma_0} \sigma^2 |\partial_\nu w|^2 e^{2s\varphi} d\omega dt. \quad (5.4.10)$$

Therefore by (5.4.8), choosing s and γ large, we see that

$$\begin{aligned} \int_Q (\sigma |\nabla z|^2 + \sigma |\partial_t z|^2 + \sigma^3 |z|^2) e^{2s\varphi} dxdt &\leq C \int_Q |f|^2 e^{2s\varphi} dxdt \\ &+ C \int_{\Sigma_0} (\sigma^2 |\partial_\nu w|^2 + \sigma |\partial_\nu(\Delta w)|^2) e^{2s\varphi} d\omega dt. \end{aligned} \quad (5.4.11)$$

Furthermore by (5.4.9) with $k = 3$, (5.4.11) and (5.4.8), we deduce

$$\begin{aligned} C\gamma \int_Q \sigma^2 \left(\sum_{|\alpha|=2} |\partial^\alpha w|^2 + \sigma^2 |\nabla w|^2 + \sigma^4 |w|^2 \right) e^{2s\varphi} dxdt \\ + C \int_Q (\sigma |\nabla(\Delta w)|^2 + \sigma |\Delta \partial_t w|^2 + \sigma^3 |\Delta w|^2) e^{2s\varphi} dxdt \\ \leq \int_Q |f|^2 e^{2s\varphi} dxdt + \int_Q \sigma |\partial_t w|^2 e^{2s\varphi} dxdt \\ + \int_{\Sigma_0} (\sigma^4 |\partial_\nu w|^2 + \sigma |\partial_\nu(\Delta w)|^2) e^{2s\varphi} d\omega dt. \end{aligned} \quad (5.4.12)$$

Taking into account $\partial_t z = \partial_t w - \mu \Delta \partial_t w$, applying again the elliptic Carleman estimate (5.2.6) with $k = 1$ and using (5.4.11), we arrive at

$$\begin{aligned} C\gamma \int_Q \left(\sum_{|\alpha|=2} |\partial^\alpha \partial_t w|^2 + \sigma^2 |\nabla \partial_t w|^2 + \sigma^4 |\partial_t w|^2 \right) e^{2s\varphi} dxdt \\ \leq \int_Q \sigma |\partial_t z|^2 e^{2s\varphi} dxdt + \int_{\Sigma_0} \sigma^2 |\partial_\nu \partial_t w|^2 e^{2s\varphi} d\omega dt \\ \leq C \int_Q |f|^2 e^{2s\varphi} dxdt + C \int_{\Sigma_0} (\sigma^2 |\partial_\nu w|^2 + \sigma |\partial_\nu(\Delta w)|^2 + \sigma^2 |\partial_\nu \partial_t w|^2) e^{2s\varphi} d\omega dt. \end{aligned} \quad (5.4.13)$$

Combining (5.4.12) and (5.4.13), we obtain (5.4.5).

This provides the desired conclusion to Lemma 5.4.1. \square

5.4.2 Proof of the Carleman estimate for the thermoelasticity plate system

Now we proceed to the proof of Theorem 5.4.15. Let (u, y) be a solution of the linear system of the thermoelasticity plate equation (5.4.1). Applying the first version of the parabolic Carleman estimate (5.2.1) with $\epsilon = 1$ to the second equation in (5.4.1), we obtain

$$\begin{aligned}
C\gamma \int_Q (\sigma^{-1} |\Delta y|^2 + \sigma |\nabla y|^2 + \sigma^3 |y|^2) e^{2s\varphi} dxdt &\leq \int_Q |f_p|^2 e^{2s\varphi} dxdt \\
&+ \int_Q |\Delta \partial_t u|^2 e^{2s\varphi} dxdt + \int_{\Sigma_0} \sigma |\partial_\nu y|^2 e^{2s\varphi} d\omega dt. \quad (5.4.14)
\end{aligned}$$

By the Carleman estimate (5.4.5) for the plate equation, we have

$$\begin{aligned}
C \int_Q \left(\sum_{|\alpha| \leq 2} \gamma \sigma^{2(2-|\alpha|)} (\sigma^2 |\partial^\alpha u|^2 + |\partial^\alpha \partial_t u|^2) + (\sigma^3 |\Delta u|^2 + \sigma |\Delta \partial_t u|^2 + \sigma |\nabla(\Delta u)|^2) \right) e^{2s\varphi} dxdt \\
\leq \int_Q |f_b|^2 e^{2s\varphi} dxdt + \int_Q |\Delta y|^2 e^{2s\varphi} dxdt \\
+ \int_{\Sigma_0} (\sigma^4 |\partial_\nu u|^2 + \sigma^2 |\partial_\nu \partial_t u|^2 + \sigma |\partial_\nu(\Delta u)|^2) e^{2s\varphi} d\omega dt. \quad (5.4.15)
\end{aligned}$$

Adding (5.4.14) and (5.4.15), we find that

$$\begin{aligned}
\gamma \int_Q \sum_{|\alpha| \leq 2} \sigma^{2(2-|\alpha|)} (\sigma^2 |\partial^\alpha u|^2 + |\partial^\alpha \partial_t u|^2) e^{2s\varphi} dxdt \\
+ \int_Q (\sigma^3 |\Delta u|^2 + \sigma |\Delta \partial_t u|^2 + \sigma |\nabla(\Delta u)|^2) e^{2s\varphi} dxdt \\
+ \gamma \int_Q (\sigma^{-1} |\Delta y|^2 + \sigma |\nabla y|^2 + \sigma^3 |y|^2) e^{2s\varphi} dxdt \\
\leq C \int_Q (|f_p|^2 + |f_b|^2) e^{2s\varphi} dxdt + C \int_Q |\Delta y|^2 e^{2s\varphi} dxdt \\
+ \int_{\Sigma_0} (\sigma^4 |\partial_\nu u|^2 + \sigma^2 |\partial_\nu \partial_t u|^2 + \sigma |\partial_\nu(\Delta u)|^2 + \sigma |\partial_\nu y|^2) e^{2s\varphi} d\omega dt, \quad (5.4.16)
\end{aligned}$$

provided that $\gamma \geq \gamma_*$ and $s \geq s_*(\gamma)$.

By the second version of the parabolic Carleman estimate (5.2.33) with $k = 1$ and $\epsilon = 1$, we have

$$\begin{aligned}
C \int_Q (|\Delta y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2) e^{2s\varphi} dxdt &\leq \int_Q \gamma^{-1} \sigma |f_p|^2 e^{2s\varphi} dxdt \\
&+ \int_Q \gamma^{-1} \sigma |\Delta \partial_t u|^2 e^{2s\varphi} dxdt + \int_{\Sigma_0} \gamma^{-1} \sigma^2 |\partial_\nu y|^2 e^{2s\varphi} d\omega dt. \quad (5.4.17)
\end{aligned}$$

Inserting (5.4.17) into (5.4.16), thanks to the second large parameter γ , we see

$$\begin{aligned}
& \gamma \int_Q \sum_{|\alpha| \leq 2} \sigma^{2(2-|\alpha|)} (\sigma^2 |\partial^\alpha u|^2 + |\partial^\alpha \partial_t u|^2) e^{2s\varphi} dxdt \\
& \quad + \int_Q (\sigma^3 |\Delta u|^2 + \sigma |\Delta \partial_t u|^2 + \sigma |\nabla(\Delta u)|^2) e^{2s\varphi} dxdt \\
& \quad + \int_Q (|\Delta y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2) e^{2s\varphi} dxdt \leq C \int_Q (|f_b|^2 + \gamma^{-1} \sigma |f_p|^2) e^{2s\varphi} dxdt \\
& \quad + \int_{\Sigma_0} (\sigma^4 |\partial_\nu u|^2 + \sigma^2 |\partial_\nu \partial_t u|^2 + \sigma |\partial_\nu(\Delta u)|^2 + \sigma^2 |\partial_\nu y|^2) e^{2s\varphi} d\omega dt. \quad (5.4.18)
\end{aligned}$$

This completes the proof of Theorem 5.4.15.

5.5 Carleman estimate for thermoelasticity system with residual stress

In this subsection we will prove a Carleman estimate for the thermoelasticity system with residual stress. Let us consider an isotropic and homogeneous thermoelastic body occupying an open and bounded domain \mathcal{M} of \mathbb{R}^3 with C^∞ boundary $\Gamma = \partial\mathcal{M}$. Given $T > 0$, we consider the following problem for the linear system of thermoelasticity with residual stress:

$$\begin{aligned}
\partial_t^2 \mathbf{u}(x, t) - \Delta_{\mu, \lambda, \mathbf{r}} \mathbf{u}(x, t) + a \nabla y(x, t) &= \mathbf{f}_e(x, t) \quad \text{in } Q = \mathcal{M} \times (0, T), \\
\partial_t y(x, t) - \Delta y(x, t) + a \operatorname{div} \partial_t \mathbf{u} &= f_p(x, t) \quad \text{in } Q,
\end{aligned} \quad (5.5.1)$$

where the coupling parameter a is assumed to be positive constant and $\mathbf{u} = (u_1, u_2, u_3)^T$ denotes the displacement at the location x and the time t , and $y = y(x, t)$, the temperature, is a scalar function, $f_p \in L^2(Q)$ is a heat source and $\mathbf{f}_e \in (L^2(0, T; H^1(\mathcal{M})))^3$, and $\Delta_{\mu, \lambda, \mathbf{r}}$ is the elliptic second-order linear differential operator given by

$$\begin{aligned}
\Delta_{\mu, \lambda, \mathbf{r}} \mathbf{v}(x) &\equiv \mu(x) \Delta \mathbf{v}(x) + (\mu(x) + \lambda(x)) (\nabla \operatorname{div} \mathbf{v}(x)) \\
&\quad + (\operatorname{div} \mathbf{v}(x)) \nabla \lambda(x) + (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \nabla \mu(x) + \nabla \cdot ((\nabla \mathbf{v}) \mathbf{r}), \quad x \in \mathcal{M} \quad (5.5.2)
\end{aligned}$$

for $\mathbf{v} = (v_1, v_2, v_3)^T$,

$$[(\nabla \mathbf{v}) \mathbf{r}]_{jk} = \sum_{\ell=1}^3 (\partial_\ell v_j) r_{k\ell},$$

and $r(x) = (r_{jk}(x))_{jk} \in \mathcal{C}^2(\overline{\mathcal{M}})$ is a residual stress tensor such that $r_{jk} = r_{kj}$ on $\overline{\mathcal{M}}$ (e.g., Man [48]), $\nabla \cdot \mathbf{r}$ is a vector with the j -th component given by

$$(\nabla \cdot \mathbf{r})_j = \sum_{k=1}^3 \frac{\partial r_{jk}}{\partial x_k}.$$

Here we assume that the density in the first equation in (5.5.1) and the thermal coefficients in the second equation in (5.5.1) are normalized to be one. We will assume that the Lamé parameters $\mu, \lambda \in \mathcal{C}^2(\overline{\mathcal{M}})$ satisfy

$$\mu(x) > 0, \quad \lambda(x) + 2\mu(x) > 0, \quad x \in \overline{\mathcal{M}}. \quad (5.5.3)$$

We denote by g_1 and g_2 the two metric tensors given by

$$g_1^{-1} = (\mu\delta_{jk} + r_{jk})_{jk}, \quad g_2^{-1} = ((2\mu + \lambda)\delta_{jk} + r_{jk})_{jk},$$

where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$. Here we assume that there exists a positive function ϑ satisfying the assumptions (A.1) and (A.2) with respect to the metrics g_1 and g_2 . The following theorem is a Carleman estimate with a second larger parameter for the thermoelasticity system (5.5.1):

Theorem 5.5.16 *There exist two constants $\gamma_* > 0$ and $C > 0$ such that for any $\gamma > \gamma_*$ there exists $s_* = s_*(\gamma) > 0$ such that the following estimate holds:*

$$\begin{aligned} \int_Q \left(\left(\sum_{|\alpha|=2} |\partial^\alpha y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) + \sigma (|\operatorname{div} \partial_t \mathbf{u}|^2 + |\nabla_{x,t} \mathbf{u}|^2 + \sigma^2 |\mathbf{u}|^2) \right) e^{2s\vartheta} dxdt \\ \leq C \int_Q (\gamma^{-1} \sigma |f_p|^2 + |\mathbf{f}_\epsilon|^2 + |\nabla \mathbf{f}_e|^2) e^{2s\vartheta} dxdt \end{aligned}$$

for any solution $(\mathbf{u}, y) \in (C_0^\infty(Q))^3 \times C_0^\infty(Q)$ to problem (5.5.1) and any $s \geq s_*$.

Remark 3 For the case where $a = 0$ in (5.5.1), that is, \mathbf{u} and y are not coupled, see Isakov and Kim [30], [31]. If \mathcal{M} has no boundary, we can prove the theorem for all $(\mathbf{u}, y) \in (C^\infty(Q))^n \times C^\infty(Q)$.

In order to prove Theorem 5.5.16, we use a Carleman estimates with a second large parameter for the Lamé system with residual stress.

5.5.1 Carleman estimate for the Lamé system with residual stress

In this subsection, we derive a Carleman estimate for a solutions of the hyperbolic elasticity system with residual stress.

We consider the three dimensional isotropic non-stationary Lamé system with residual stress

$$\partial_t^2 \mathbf{v}(x, t) - \Delta_{\mu, \lambda, \mathbf{r}} \mathbf{v}(x, t) = \mathbf{f}(x, t) \quad \text{in } Q, \quad (5.5.4)$$

where $\mathbf{f} \in [L^2(0, T; H^1(\Omega))]^3$ is a source term.

From Theorem 3.5.8 we derive the following Carleman estimate.

Lemma 5.5.1 *There exist two constants $\gamma_* > 0$ and $C > 0$ such that for any $\gamma > \gamma_*$, there exists $s_* = s_*(\gamma) > 0$ such that the following estimate holds:*

$$\begin{aligned} \int_Q (\sigma (|\nabla_{x,t} \mathbf{v}|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v})|^2 + |\nabla_{x,t}(\operatorname{curl} \mathbf{v})|^2) + \sigma^3 (|\mathbf{v}|^2 + |\operatorname{div} \mathbf{v}|^2 + |\operatorname{curl} \mathbf{v}|^2)) e^{2s\vartheta} dxdt \\ \leq C \int_Q (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2) e^{2s\vartheta} dxdt \end{aligned}$$

for any solution $\mathbf{v} \in (C_0^\infty(Q))^3$ to problem (5.5.4) and any $s \geq s_*$.

Proof . Let $v = \operatorname{div} \mathbf{v}$ and $\mathbf{w} = \operatorname{curl} \mathbf{v}$. We apply curl and div to (5.5.4) and obtain

$$\partial_t^2 \mathbf{v} - \Delta_{g_1} \mathbf{v} + \mathcal{A}_1(\mathbf{v}, v) = \mathbf{f},$$

$$\partial_t^2 v - \Delta_{g_2} v + \mathcal{A}_2(\mathbf{v}, v, \mathbf{w}) = \operatorname{div} \mathbf{f} + \sum_{j,k=1}^3 \nabla(r_{jk}) \cdot \partial_j \partial_k \mathbf{v},$$

$$\partial_t^2 \mathbf{w} - \Delta_{g_1} \mathbf{w} + \mathcal{A}_3(\mathbf{v}, v, \mathbf{w}) = \operatorname{curl} \mathbf{f} + \sum_{j,k=1}^3 \nabla(r_{jk}) \times \partial_j \partial_k \mathbf{v},$$

where \mathcal{A}_j are linear differential operators of the first order with bounded coefficients in \mathcal{M} . Consequently, by the Carleman estimate (Theorem 3.5.8) with second large parameter, we see that for any $\gamma \geq \gamma_*$ and $s \geq s_*$, we have

$$\begin{aligned} & C \int_Q (\sigma(|\nabla_{x,t} v|^2 + |\nabla_{x,t} \mathbf{v}|^2 + |\nabla_{x,t} \mathbf{w}|^2) + \sigma^3(|v|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2)) e^{2s\varphi} dxdt \\ & \leq \int_Q (|\mathbf{f}|^2 + |\operatorname{div}(\mathbf{f})|^2 + |\operatorname{curl}(\mathbf{f})|^2) e^{2s\varphi} dxdt + \left[\sum_{|\alpha|=2} \int_Q |\partial_x^\alpha \mathbf{v}|^2 e^{2s\varphi} dxdt \right]. \end{aligned} \quad (5.5.5)$$

On the other hand, by the elliptic Carleman estimate (5.2.33) with $k = 1$ and $\epsilon = 0$, we obtain

$$\begin{aligned} & C \sum_{|\alpha|=2} \int_Q |\partial_x^\alpha \mathbf{v}|^2 e^{2s\varphi} dxdt \leq \int_Q (|\Delta \mathbf{v}|^2 + \sigma^2 |\nabla \mathbf{v}|^2 + \sigma^4 |v|^2) e^{2s\varphi} dxdt \\ & \leq C\gamma^{-1} \int_Q \sigma |\Delta \mathbf{v}|^2 e^{2s\varphi} dxdt \leq C\gamma^{-1} \int_Q \sigma (|\nabla v|^2 + |\nabla \mathbf{w}|^2) e^{2s\varphi} dxdt, \end{aligned} \quad (5.5.6)$$

where we have used the formula $\Delta \mathbf{v} = \nabla v - \operatorname{curl} \mathbf{w}$. Now, combining (5.5.6) and (5.5.5), thanks to the second large parameter γ , we complete the proof of Lemma 5.5.1. \square

5.5.2 Proof of the Carleman estimate for the thermoelasticity

Now we complete the proof of Theorem 5.5.16. Let (\mathbf{u}, y) be a solution of the linear system of the thermoelasticity (5.5.1). Applying the first version of the parabolic Carleman estimate (5.2.6), we obtain

$$\begin{aligned} & C\gamma \int_Q (\sigma^{-1} \sum_{|\alpha|=2} |\partial^\alpha y|^2 + \sigma |\nabla y|^2 + \sigma^3 |y|^2) e^{2s\varphi} dxdt \\ & \leq \int_Q |f_p|^2 e^{2s\varphi} dxdt + \int_Q |\operatorname{div} \partial_t \mathbf{u}|^2 e^{2s\varphi} dxdt. \end{aligned} \quad (5.5.7)$$

By Lemma 5.5.1, we have

$$C \int_Q (\sigma |\operatorname{div} \partial_t \mathbf{u}|^2 + \sigma |\nabla_{x,t} \mathbf{u}|^2 + \sigma^3 |\mathbf{u}|^2) e^{2s\varphi} dxdt \leq \int_Q (|\mathbf{f}_e|^2 + |\nabla \mathbf{f}_e|^2) e^{2s\varphi} dxdt + \int_Q (|\nabla y|^2 + \sum_{|\alpha|=2} |\partial^\alpha y|^2) e^{2s\varphi} dxdt. \quad (5.5.8)$$

Adding (5.5.7) and (5.5.8), we obtain

$$C \int_Q \left(\gamma \left(\sigma^{-1} \sum_{|\alpha|=2} |\partial^\alpha y|^2 + \sigma |\nabla y|^2 + \sigma^3 |y|^2 \right) + \sigma |\operatorname{div} \partial_t \mathbf{u}|^2 + \sigma |\nabla_{x,t} \mathbf{u}|^2 + \sigma^3 |\mathbf{u}|^2 \right) e^{2s\varphi} dxdt \leq \int_Q (|f_p|^2 + |\mathbf{f}_e|^2 + |\nabla \mathbf{f}_e|^2) e^{2s\varphi} dxdt + \int_Q \sum_{|\alpha|=2} |\partial^\alpha y|^2 e^{2s\varphi} dxdt, \quad (5.5.9)$$

provided that $\gamma \geq \gamma_*$ and $s \geq s_*(\gamma)$.

By the second version of the parabolic Carleman estimate (5.2.33) with $k = 1$, we have

$$C\gamma \int_Q \left(\sum_{|\alpha|=2} |\partial^\alpha y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) e^{2s\varphi} dxdt \leq \int_Q \sigma |f_p|^2 e^{2s\varphi} dxdt + \int_Q \sigma |\operatorname{div} \partial_t \mathbf{u}|^2 e^{2s\varphi} dxdt. \quad (5.5.10)$$

Inserting (5.5.10) into (5.5.9), thanks to the second large parameter γ , we find

$$\int_Q \left(\left(\sum_{|\alpha|=2} |\partial^\alpha y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) + \sigma |\operatorname{div} \partial_t \mathbf{u}|^2 + \sigma |\nabla_{x,t} \mathbf{u}|^2 + \sigma^3 |\mathbf{u}|^2 \right) e^{2s\varphi} dxdt \leq C \int_Q (\gamma^{-1} \sigma |f_p|^2 + |\mathbf{f}_e|^2 + |\nabla \mathbf{f}_e|^2) e^{2s\varphi} dxdt.$$

This completes the proof of Theorem 5.5.16.

Global Carleman estimate for the Laplace-Beltrami operator with an extra elliptic variable and applications

6.1 Introduction

We formulate our Carleman estimate. Let $n \geq 2$ and (\mathcal{M}, g) be an n -dimensional compact Riemannian manifold with smooth boundary $\partial\mathcal{M}$ and smooth metric g . All manifolds under consideration will be assumed smooth (which means C^∞) and oriented. We denote by Δ_g the Laplace-Beltrami operator associated to the metric g . In local coordinates, Δ_g is given by

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{\det g} g^{jk} \frac{\partial}{\partial x_k} \right). \quad (6.1.1)$$

Here (g^{jk}) is the inverse of the metric $g = \{g_{jk}\}$ and $\det g = \det(g_{jk})$. Let us consider the following second-order elliptic operator:

$$P(x, \tau; \partial) = \partial_\tau^2 + \Delta_g + P_1(x, \tau; \partial), \quad (6.1.2)$$

where $P_1(x, \tau; \partial)$ is a first-order partial operator with coefficients in $L^\infty(\mathbb{R} \times \mathcal{M})$.

This partial differential operator is of elliptic type in a non-smooth manifold $\mathbb{R} \times \mathcal{M}$ and require an independent proof (cf. Lemma 5.2.1 with $\epsilon = 0$). The operator (6.1.2) is important when we study the unique continuation for the hyperbolic equation $\partial_t^2 - \Delta_g$ by the Fourier-Bros-Iagolnitzer transform. As related works, we refer to [5], [6], [49], [50], [51].

Throughout this chapter, we use the following notations:

$$a(x, \xi) = \sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k, \quad x \in \mathcal{M}, \xi_1, \dots, \xi_n \in \mathbb{R}. \quad (6.1.3)$$

Given two symbols p and q we define their Poisson bracket by

$$\{p, q\}(x, \xi) = \frac{\partial p}{\partial \xi} \frac{\partial q}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial q}{\partial \xi} = \sum_{j=1}^n \left(\frac{\partial p}{\partial \xi_j} \frac{\partial q}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial q}{\partial \xi_j} \right). \quad (6.1.4)$$

In order to state our Carleman estimate with boundary observation, we need to introduce the following assumptions.

Assumption (A.1): We assume that there exists a positive function $\vartheta : \overline{\mathcal{M}} \rightarrow \mathbb{R}$ which possesses no critical points on $\overline{\mathcal{M}}$:

$$\min_{x \in \overline{\mathcal{M}}} |\nabla \vartheta(x)|^2 > 0. \quad (6.1.5)$$

Assumption (A.2): Under assumption (A.1), let a subboundary $\Gamma_0 \subset \partial \mathcal{M}$ satisfy

$$\{x \in \partial \mathcal{M}; \langle \nabla \vartheta, \nu(x) \rangle \geq 0\} \subset \Gamma_0.$$

Let us define

$$\widetilde{Q} = \mathcal{M} \times (-T, T), \quad \widetilde{\Sigma}_0 = \Gamma_0 \times (-T, T), \quad \widetilde{\Sigma} = \partial \Omega \times (-T, T)$$

and

$$\psi(x, \tau) = \vartheta(x) - \beta \tau^2 + \beta_0, \quad 0 < \beta, \quad \beta_0 \geq 0. \quad (6.1.6)$$

We choose a parameter β_0 such that the function ψ given by (6.1.6) is positive. We define the weight function $\varphi : \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$ by $\varphi(x, \tau) = e^{\gamma \psi(x, \tau)}$, where $\gamma > 0$ is a second large parameter and set

$$\sigma = s \gamma \varphi,$$

where s is a real number. Let us introduce the following notation:

$$\mathcal{H}_0^1(Q) = \{u \in H^1(-T, T; L^2(\mathcal{M})) \cap L^2(-T, T; H_0^1(\mathcal{M})); \partial_t^j u(\cdot, \pm T) = 0, j = 0, 1\}. \quad (6.1.7)$$

The following global Carleman estimate with boundary observation is our first main result:

Theorem 6.1.1 *There exist three positive constants γ_* , s_* and C such that, for any $\gamma \geq \gamma_*$ and any $s \geq s_*$, the following inequality holds:*

$$\begin{aligned} C \gamma \int_{\widetilde{Q}} \left(\sigma^{-1} \sum_{|\alpha|=2} |\partial^\alpha y(x, \tau)|^2 + \sigma (|\nabla y(x, \tau)|^2 + |\partial_\tau y(x, \tau)|^2) + \sigma^3 |y(x, \tau)|^2 \right) e^{2s\varphi} \, dv_g \, d\tau \\ \leq \int_{\widetilde{Q}} |(\partial_\tau^2 + \Delta_g)y(x, \tau)|^2 e^{2s\varphi} \, dv_g \, d\tau + \int_{\widetilde{\Sigma}_0} \sigma |\partial_\nu y|^2 e^{2s\varphi} \, d\sigma_g \, d\tau \end{aligned} \quad (6.1.8)$$

for any $y \in H^2(\widetilde{Q})$ such that $\partial_t^j y(x, \pm T) = 0$, $j = 0, 1$, in \mathcal{M} and $y(x, \tau) = 0$ on $\widetilde{\Sigma}$. Here and henceforth, if \mathcal{M} has no boundary, then the Dirichlet boundary condition $y = 0$ on Σ is not necessary.

6.2 Proof of Theorem 6.1.1

In this section we complete the proof of Theorem 6.1.1. We will divide the proof into three steps.

6.2.1 Change of variables

For $s > 0$, let us introduce the new functions $z(x, \tau) = e^{s\varphi}y(x, \tau)$ and $h_{s,\gamma} = e^{s\varphi}h_0$, where $h_0 = (-\partial_\tau^2 - \Delta_g)y$. The standard approach for the proof (6.1.8) starts from the observation

$$e^{s\varphi}(-\partial_\tau^2 - \Delta_g)y(x, \tau) = \mathcal{L}_{\gamma,s}(\tau, x, D)z(x, \tau), \quad (6.2.1)$$

where

$$\begin{aligned} & \mathcal{L}_{\gamma,s}(\tau, x, D)z(x, \tau) \\ &= -\partial_\tau^2 z - \Delta_g z + s(\gamma(\partial_\tau^2 \psi + \Delta_g \psi) \varphi + \gamma^2(|\partial_\tau \psi|^2 + |\nabla \psi|^2) \varphi) z \\ & \quad + 2s\gamma\varphi(\partial_\tau \psi \partial_\tau z + \langle \nabla \psi, \nabla z \rangle) - s^2\gamma^2(|\partial_\tau \psi|^2 + |\nabla \psi|^2) \varphi^2 z. \end{aligned} \quad (6.2.2)$$

We set

$$\begin{aligned} Az(x, \tau) + Bz(x, \tau) &= h_{s,\gamma}(x, \tau) \\ &:= h_0(x, \tau) - (\sigma(\partial_\tau^2 \psi + \Delta_g \psi) + \gamma\sigma(|\partial_\tau \psi|^2 + |\nabla \psi|^2)) z, \end{aligned} \quad (6.2.3)$$

where

$$Az(x, \tau) = -\partial_\tau^2 z - \Delta_g z - \sigma^2(|\partial_\tau \psi|^2 + |\nabla \psi|^2) z \quad (6.2.4)$$

and

$$Bz(x, t) = 2\sigma(\partial_\tau \psi \partial_\tau z + \langle \nabla \psi, \nabla z \rangle). \quad (6.2.5)$$

With the previous notation, we have

$$\|h_{s,\gamma}\|_{L^2(\tilde{Q})}^2 = \|Az\|_{L^2(\tilde{Q})}^2 + \|Bz\|_{L^2(\tilde{Q})}^2 + 2(Az, Bz)_{L^2(\tilde{Q})}. \quad (6.2.6)$$

Next we will make the computations of $2(Az, Bz)_{L^2(\tilde{Q})}$. We will first look for lower bounds for $(Az, Bz)_{L^2(\tilde{Q})}$. We decompose $(Az, Bz)_{L^2(\tilde{Q})} = \mathcal{K}_1 + \mathcal{K}_2$ with

$$\begin{aligned} \mathcal{K}_1 &= - \int_{\tilde{Q}} (\partial_\tau^2 + \Delta_g) z \left(2\sigma(\partial_\tau \psi \partial_\tau z + \langle \nabla_g \psi, \nabla_g z \rangle_g) \right) dv_g d\tau \\ \mathcal{K}_2 &= -2 \int_{\tilde{Q}} \sigma^3 (|\partial_\tau \psi|^2 + |\nabla_g \psi|_g^2) z (\partial_\tau \psi \partial_\tau z + \langle \nabla_g \psi, \nabla_g z \rangle_g) dv_g d\tau. \end{aligned} \quad (6.2.7)$$

We first deal with \mathcal{K}_2 . For the term \mathcal{K}_2 , integration by parts in x yields

$$\begin{aligned} \mathcal{K}_2 &= -2 \int_{\tilde{Q}} \sigma^3 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) z (\partial_\tau \psi \partial_\tau z + \langle \nabla \vartheta, \nabla z \rangle) dv_g d\tau \\ &= - \int_{\tilde{Q}} \sigma^3 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) (\partial_\tau \psi \partial_\tau (|z|^2) + \langle \nabla \vartheta, \nabla (|z|^2) \rangle) dv_g d\tau \\ &= 3\gamma \int_{\tilde{Q}} \sigma^3 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)^2 |z|^2 dv_g d\tau \\ & \quad + 2 \int_{\tilde{Q}} \sigma^3 (\partial_\tau^2 \psi |\partial_\tau \psi|^2 + D^2 \psi (\nabla \vartheta, \nabla \vartheta)) |z|^2 dv_g d\tau \\ & \quad + \int_{\tilde{Q}} \sigma^3 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) (\partial_\tau^2 \psi + \Delta \psi) |z|^2 dv_g d\tau. \end{aligned} \quad (6.2.8)$$

We now turn to the term \mathcal{K}_1 . We also have

$$\begin{aligned}
\mathcal{K}_1 &= -2 \int_{\tilde{Q}} \sigma (\partial_\tau^2 + \Delta_g) z (\partial_\tau \psi \partial_\tau z + \langle \nabla \vartheta, \nabla z \rangle) \, dv_g \, d\tau \\
&= -2 \int_{\tilde{Q}} \sigma \partial_\tau^2 z (\partial_\tau \psi \partial_\tau z + \langle \nabla \vartheta, \nabla z \rangle) \, dv_g \, d\tau \\
&\quad -2 \int_{\tilde{Q}} \sigma \Delta_g z (\partial_\tau \psi \partial_\tau z + \langle \nabla \vartheta, \nabla z \rangle) \, dv_g \, d\tau \\
&= \mathcal{K}_{11} + \mathcal{K}_{12},
\end{aligned} \tag{6.2.9}$$

where

$$\begin{aligned}
\mathcal{K}_{11} &= - \int_{\tilde{Q}} \sigma \partial_\tau \psi \partial_\tau (|\partial_\tau z|^2) \, dv_g \, d\tau + \int_{\tilde{Q}} \sigma \langle \nabla \vartheta, \nabla (|\partial_\tau z|^2) \rangle \, dv_g \, d\tau \\
&\quad + 2\gamma \int_{\tilde{Q}} \sigma \partial_\tau \psi \partial_\tau z \langle \nabla \vartheta, \nabla z \rangle \, dv_g \, d\tau \\
&= \gamma \int_{\tilde{Q}} \sigma |\partial_\tau \psi|^2 |\partial_\tau z|^2 \, dv_g \, d\tau + \int_{\tilde{Q}} \sigma \partial_\tau^2 \psi |\partial_\tau z|^2 \, dv_g \, d\tau \\
&\quad - \gamma \int_{\tilde{Q}} \sigma |\nabla \vartheta|^2 |\partial_\tau z|^2 \, dv_g \, d\tau - \int_{\tilde{Q}} \sigma \Delta_g \psi |\partial_\tau z|^2 \, dv_g \, d\tau \\
&\quad + 2\gamma \int_{\tilde{Q}} \sigma \partial_\tau \psi \partial_\tau z \langle \nabla \vartheta, \nabla z \rangle \, dv_g \, d\tau.
\end{aligned} \tag{6.2.10}$$

Furthermore

$$\begin{aligned}
\mathcal{K}_{12} &= -2 \int_{\tilde{Q}} \sigma \Delta_g z ((\partial_\tau z) \partial_\tau \psi + \langle \nabla z, \nabla \psi \rangle) \, dv_g \, d\tau \\
&= 2\gamma \int_{\tilde{Q}} \sigma \langle \nabla \psi, \nabla z \rangle ((\partial_\tau z) \partial_\tau \psi + \langle \nabla z, \nabla \psi \rangle) \, dv_g \, d\tau \\
&\quad + 2 \int_{\tilde{Q}} \sigma \langle \nabla z, (\partial_\tau \nabla z \partial_\tau \psi) \rangle \, dv_g \, d\tau + 2 \int_{\tilde{Q}} \sigma \langle \nabla z, \nabla (\langle \nabla z, \nabla \psi \rangle) \rangle \, dv_g \, d\tau \\
&\quad - 2 \left[\int_{\tilde{\Sigma}} \sigma \partial_\nu z ((\partial_\tau z) \partial_t \psi + \langle \nabla z, \nabla \psi \rangle) \, d\sigma_g \, d\tau \right] \\
&= 2\gamma \int_{\tilde{Q}} \sigma \langle \nabla z, \nabla \psi \rangle (\partial_\tau \psi \partial_\tau z) \, dv_g \, d\tau + 2\gamma \int_{\tilde{Q}} \sigma |\langle \nabla \psi, \nabla z \rangle|^2 \, dv_g \, d\tau \\
&\quad + \int_{\tilde{Q}} \sigma \frac{\partial}{\partial \tau} (|\nabla z|^2) \partial_\tau \psi \, dv_g \, d\tau + 2 \int_{\tilde{Q}} \sigma \langle \nabla z, \nabla (\langle \nabla z, \nabla \psi \rangle) \rangle \, dv_g \, d\tau \\
&\quad - 2 \left[\int_{\tilde{\Sigma}} \sigma (\partial_\nu z) ((\partial_\tau z) \partial_\tau \psi + \langle \nabla z, \nabla \psi \rangle) \, d\sigma_g \, d\tau \right].
\end{aligned} \tag{6.2.11}$$

Applying Theorem 1.4.1 with $Z = \nabla z$, we obtain

$$\begin{aligned}
\langle \nabla z, \nabla (\langle \nabla z, \nabla \psi \rangle) \rangle &= \nabla z (\langle \nabla z, \nabla \psi \rangle) \\
&= \langle D_{\nabla z} \nabla z, \nabla \psi \rangle + \langle \nabla z, D_{\nabla z} \nabla \psi \rangle \\
&= D^2 \psi (\nabla z, \nabla z) + D^2 z (\nabla z, \nabla \psi).
\end{aligned} \tag{6.2.12}$$

Moreover

$$\begin{aligned} \langle \nabla \psi, \nabla (|\nabla z|^2) \rangle &= \nabla \psi (\langle \nabla z, \nabla z \rangle) \\ &= \langle D_{\nabla \psi} \nabla z, \nabla z \rangle + \langle \nabla z, D_{\nabla \psi} \nabla z \rangle \\ &= 2D^2 z (\nabla z, \nabla \psi). \end{aligned}$$

Hence

$$\langle \nabla z, \nabla (\langle \nabla z, \nabla \psi \rangle) \rangle = D^2 \psi (\nabla z, \nabla z) + \frac{1}{2} \langle \nabla \psi, \nabla (|\nabla z|^2) \rangle. \quad (6.2.13)$$

Therefore

$$\begin{aligned} \mathcal{K}_{12} &= 2\gamma \int_{\tilde{Q}} \sigma \langle \nabla \psi, \nabla z \rangle (\partial_\tau \psi) (\partial_\tau z) \, dv_g d\tau + 2\gamma \int_{\tilde{Q}} \sigma |\langle \nabla \psi, \nabla z \rangle|^2 \, dv_g d\tau \\ &\quad - \gamma \int_{\tilde{Q}} \sigma |\nabla z|^2 |\partial_\tau \psi|^2 \, dv_g d\tau - \int_{\tilde{Q}} \sigma |\nabla z|^2 \partial_\tau^2 \psi \, dv_g d\tau \\ &\quad + 2 \int_{\tilde{Q}} \sigma D^2 \psi (\nabla z, \nabla z) \, dv_g d\tau - \gamma \int_{\tilde{Q}} \sigma |\nabla z|^2 |\nabla \psi|^2 \, dv_g d\tau - \int_{\tilde{Q}} \sigma |\nabla z|^2 \Delta_g \psi \, dv_g d\tau \\ &\quad - \int_{\tilde{\Sigma}} \sigma \langle \nabla \psi, \nu \rangle |\partial_\nu z|^2 \, d\sigma_g d\tau. \end{aligned} \quad (6.2.14)$$

Collecting (6.2.14) and (6.2.10), we obtain

$$\begin{aligned} \mathcal{K}_1 &= 2\gamma \int_{\tilde{Q}} \sigma (\partial_\tau \psi \partial_\tau z + \langle \nabla \vartheta, \nabla z \rangle)^2 \, dv_g d\tau \\ &\quad - \gamma \int_{\tilde{Q}} \sigma (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) (|\partial_\tau z|^2 + |\nabla z|^2) \, dv_g d\tau \\ &\quad - \int_{\tilde{Q}} \sigma (\partial_\tau^2 \psi + \Delta_g \psi) (|\partial_\tau z|^2 + |\nabla z|^2) \, dv_g d\tau \\ &\quad + 2 \int_{\tilde{Q}} \sigma (\partial_\tau^2 \psi |\partial_\tau z|^2 + D^2 \psi (\nabla z, \nabla z)) \, dv_g d\tau \\ &\quad - \int_{\tilde{\Sigma}} \sigma \langle \nabla \psi, \nu \rangle |\partial_\nu z|^2 \, d\sigma_g d\tau. \end{aligned} \quad (6.2.15)$$

6.2.2 Interior estimate

From (6.2.15) and (6.2.8), we have

$$\begin{aligned} (Az, Bz)_{L^2(\tilde{Q})} &= \mathcal{K}_1 + \mathcal{K}_2 \\ &= 3\gamma \int_{\tilde{Q}} \sigma^3 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)^2 |z|^2 \, dv_g d\tau + 2\gamma \int_{\tilde{Q}} \sigma (\partial_\tau \psi \partial_\tau z + |\langle \nabla z, \nabla \vartheta \rangle|)^2 \, dv_g d\tau \\ &\quad - \gamma \int_{\tilde{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) \, dv_g d\tau + \mathcal{Q}_1(z, \partial_\tau z, \nabla z) \\ &\quad - \left[\int_{\tilde{\Sigma}} \sigma |\partial_\nu z|^2 \langle \nabla \vartheta, \nu \rangle \, d\sigma_g d\tau \right], \end{aligned} \quad (6.2.16)$$

where $\mathcal{Q}_1(z, \nabla z)$ satisfies

$$|\mathcal{Q}_1(z, \partial_\tau z, \nabla z)| \leq C \left(\int_{\tilde{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) \, dv_g \, d\tau + \int_{\tilde{Q}} \sigma^3 |z|^2 \, dv_g \, d\tau \right). \quad (6.2.17)$$

Multiply (6.2.3) by $\gamma \sigma z (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)$ and integrate by parts, and we obtain

$$\begin{aligned} & \gamma \int_{\tilde{Q}} \sigma z (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) h_{s,\gamma}(x, \tau) \, dv_g \, d\tau = \gamma \int_{\tilde{Q}} \sigma z (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) (Bz) \, dv_g \, d\tau \\ & - \gamma \int_{\tilde{Q}} \sigma z (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) (\partial_\tau^2 + \Delta_g) z \, dv_g \, d\tau - \gamma \int_{\tilde{Q}} \sigma^3 |z|^2 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)^2 \, dv_g \, d\tau \\ & = \gamma \int_{\tilde{Q}} \sigma z (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) (Bz) \, dv_g \, d\tau + \gamma \int_{\tilde{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) \, dv_g \, d\tau \\ & \quad + \gamma^2 \int_{\tilde{Q}} \sigma (\partial_\tau \psi \partial_\tau z + \langle \nabla \vartheta, \nabla z \rangle) (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) z \, dv_g \, d\tau \\ & \quad + 2\gamma \int_{\tilde{Q}} \sigma z (\partial_\tau \psi \partial_\tau z \partial_\tau^2 \psi + D^2 \psi (\nabla \psi, \nabla z)) \, dv_g \, d\tau \\ & \quad - \gamma \int_{\tilde{Q}} \sigma^3 |z|^2 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)^2 \, dv_g \, d\tau. \end{aligned} \quad (6.2.18)$$

Hence

$$\begin{aligned} 2\gamma \int_{\tilde{Q}} \sigma^3 |z|^2 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)^2 \, dv_g \, d\tau & = 2\gamma \int_{\tilde{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) \, dv_g \, d\tau \\ & \quad + 2\mathcal{Q}_2(z, \partial_\tau z, \nabla z), \end{aligned} \quad (6.2.19)$$

where $\mathcal{Q}_2(z, \partial_\tau z, \nabla z)$ satisfies

$$\begin{aligned} |\mathcal{Q}_2(z, \partial_\tau z, \nabla z)| & \leq C \left(\gamma^2 \int_{\tilde{Q}} \sigma^2 |z|^2 \, dv_g \, d\tau + s^{-1} \gamma \int_{\tilde{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) \, dv_g \, d\tau \right) \\ & \quad + \frac{1}{16} \|Bz\|^2 + \frac{1}{2} \|h_{s,\gamma}\|^2 \end{aligned} \quad (6.2.20)$$

and we have used $\varphi \geq 1$ in Q .

As a consequence

$$\begin{aligned} (Az, Bz)_{L^2(\tilde{Q})} & = \gamma \int_{\tilde{Q}} \sigma^3 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)^2 |z|^2 \, dv_g \, d\tau + 2\gamma \int_{\tilde{Q}} \sigma (\partial_\tau \psi \partial_\tau z + \langle \nabla z, \nabla \vartheta \rangle)^2 \, dv_g \, d\tau \\ & \quad + \gamma \int_{\tilde{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) \, dv_g \, d\tau - \left[\int_{\tilde{\Sigma}} \sigma |\partial_\nu z|^2 \langle \nabla \vartheta, \nu \rangle \, d\sigma_g \, d\tau \right] \\ & \quad + \mathcal{Q}_1(z, \partial_\tau z, \nabla z) + 2\mathcal{Q}_2(z, \partial_\tau z, \nabla z). \end{aligned} \quad (6.2.21)$$

Now, combining (6.2.21), (6.2.20) and (6.2.17), we have

$$\begin{aligned}
2(Az, Bz) + 2 \left[\int_{\bar{\Sigma}} \sigma |\partial_\nu z|^2 \langle \nabla \vartheta, \nu \rangle d\sigma_g d\tau \right] &\geq 2\gamma \int_{\bar{Q}} \sigma^3 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)^2 |z|^2 dv_g d\tau \\
&\quad + 2\gamma \int_{\bar{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2) dv_g d\tau \\
&\quad - C \left(\int_{\bar{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) dv_g d\tau + \int_{\bar{Q}} \sigma^3 |z|^2 dv_g d\tau \right) \\
&\quad - C \left(\gamma^2 \int_{\bar{Q}} \sigma^2 |z|^2 dv_g d\tau + s^{-1} \gamma \int_{\bar{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) dv_g d\tau \right) \\
&\quad - \frac{1}{4} \|Bz\|^2 - 2 \|h_{s,\gamma}\|^2.
\end{aligned} \tag{6.2.22}$$

Now, since $\nabla \vartheta \neq 0$ on $\overline{\mathcal{M}}$, we conclude that for any $s \geq s_*$ and $\gamma \geq \gamma_*$, we obtain

$$\begin{aligned}
2(Az, Bz) + 2 \left[\int_{\bar{\Sigma}} \sigma |\partial_\nu z|^2 \langle \nabla \vartheta, \nu \rangle d\sigma_g d\tau \right] \\
\geq C\gamma \left(\int_{\bar{Q}} \sigma^3 |z|^2 dv_g d\tau + \int_{\bar{Q}} \sigma (|\partial_\tau z|^2 + |\nabla z|^2) dv_g d\tau \right) - \frac{1}{4} \|Bz\|^2 - 2 \|h_{s,\gamma}\|^2.
\end{aligned} \tag{6.2.23}$$

Thus we have also

$$\begin{aligned}
\|h_{s,\gamma}\|^2 + \left[\int_{\bar{\Sigma}} \sigma |\partial_\nu z|^2 \langle \nabla \vartheta, \nu \rangle d\sigma_g d\tau \right] &\geq \\
C \left(\gamma \int_{\bar{Q}} (\sigma^3 |z|^2 + \sigma (|\partial_\tau z|^2 + |\nabla z|^2)) dv_g d\tau + \|Bz\|^2 + \|Az\|^2 \right).
\end{aligned} \tag{6.2.24}$$

6.2.3 Completion of the proof

Next we will estimate $|(\partial_t^2 + \Delta_g)z|$. Since

$$|Az|^2 \geq C |(\partial_\tau^2 + \Delta_g)z|^2 - \sigma^4 (|\partial_\tau \psi|^2 + |\nabla \vartheta|^2)^2 |z|^2 \quad \text{in } Q, \tag{6.2.25}$$

by $\varphi \geq 1$ and (6.2.24) we obtain

$$\begin{aligned}
C\gamma \int_{\bar{Q}} \sigma^{-1} |(\partial_\tau^2 + \Delta_g)z(x, \tau)|^2 dx d\tau &\leq \|Az\|^2 + C\gamma \int_{\bar{Q}} \sigma^3 |z|^2 dv_g d\tau \\
&\leq C \left(\|h_{s,\gamma}\|^2 + \int_{\bar{\Sigma}} \sigma |\partial_\nu z|^2 \langle \nabla \vartheta, \nu \rangle d\sigma_g d\tau \right).
\end{aligned} \tag{6.2.26}$$

By (6.2.26), (6.2.24) and Assumption (A.2), we deduce

$$\|h_{s,\gamma}\|^2 + \left[\int_{\bar{\Sigma}} \sigma |\partial_\nu z|^2 (\nabla \vartheta \cdot \nu) \, dv_g \, d\tau \right] \geq$$

$$C\gamma \int_{\bar{Q}} \left(\sigma^3 |z(x, \tau)|^2 + \sigma (|\partial_\tau z(x, \tau)|^2 + |\nabla z(x, \tau)|^2) + \sigma^{-1} |(\partial_\tau^2 + \Delta_g)z(x, \tau)|^2 \right) \, dv_g \, d\tau. \quad (6.2.27)$$

The final step is to add integral of $\sum_{|\alpha|=2} |\partial^\alpha y(x, \tau)|^2$ to the right-hand side of (6.2.27). This can be made by the following computation

$$(\partial_\tau^2 + \Delta_g)(\sigma^{-1/2}z) = \sigma^{-1/2}(\partial_\tau^2 + \Delta_g)z + \left(\frac{\gamma^2}{4} \sigma^{-1/2} (|\partial_\tau \psi|^2 + |\nabla \psi|^2) - \frac{\gamma}{2} \sigma^{-1/2} (\partial_\tau^2 + \Delta_g)\psi \right) z - \gamma \sigma^{-1/2} (\partial_\tau z \partial_\tau \psi + \langle \nabla z, \nabla \psi \rangle). \quad (6.2.28)$$

We deduce from (6.2.28) and the elliptic estimates that

$$C \sum_{|\alpha|=2} \int_{\bar{Q}} |\partial^\alpha (\sigma^{-1/2}z)|^2 \, dv_g \, d\tau \quad (6.2.29)$$

$$\leq \int_{\bar{Q}} \left(\sigma^{-1} |\partial_\tau^2 + \Delta_g z|^2 + \sigma^3 |z|^2 + \sigma (|\partial_\tau z|^2 + |\nabla z|^2) \right) \, dv_g \, d\tau,$$

where we have used $z = 0$ on ∂Q .

On the other hand, we can find

$$C \sum_{|\alpha|=2} \int_{\bar{Q}} \sigma^{-1} |\partial^\alpha z|^2 \, dv_g \, d\tau \quad (6.2.30)$$

$$\leq \sum_{|\alpha|=2} \int_{\bar{Q}} |\partial^\alpha (\sigma^{-1/2}z)|^2 \, dv_g \, dt + \int_{\bar{Q}} (\sigma^3 |z|^2 + \sigma (|\partial_\tau z|^2 + |\nabla z|^2)) \, dv_g \, d\tau.$$

By (6.2.29) and (6.2.30), we obtain

$$C\gamma \sum_{|\alpha|=2} \int_{\bar{Q}} \sigma^{-1} |\partial^\alpha z|^2 \, dv_g \, d\tau \quad (6.2.31)$$

$$\leq \gamma \int_{\bar{Q}} \left(\sigma^{-1} |(\partial_\tau^2 + \Delta_g)z|^2 + \sigma^3 |z|^2 + \sigma (|\partial_\tau z|^2 + |\nabla z|^2) \right) \, dv_g \, d\tau.$$

Substituting $z(x, \tau) = e^{s\varphi} y(x, \tau)$ and noting (6.2.27), we can complete the proof of (6.3.1).

6.3 Interpolation inequality

By Theorem 6.1.1, we can prove the following estimate of solution in $\mathcal{M} \times (-T/2, T/2)$ to

$$\partial_\tau^2 u + \Delta_g u + P_1 u = f \quad \text{in } \mathcal{M} \times (-T/2, T/2)$$

and

$$u = 0 \quad \text{on } \partial\mathcal{M} \times (-T/2, T/2)$$

by Neumann data on $\tilde{\Gamma}_0$. This estimate is useful for establishing estimates for inverse problems by combining Fourier-Gros-Iagolnitzer transform.

Lemma 6.3.1 *Let $\tilde{Q}_0 = \mathcal{M} \times (-T/2, T/2)$. Then there exist constants $C > 0$ and $\mu \in (0, 1)$ such that the following estimate holds:*

$$\|u\|_{H^1(\tilde{Q}_0)} \leq C \left(\|(\partial_\tau^2 + \Delta) u\|_{L^2(\tilde{Q})} + \|\partial_\nu u\|_{L^2(\tilde{\Sigma}_0)} \right)^\mu \|u\|_{H^1(\tilde{Q})}^{1-\mu} \quad (6.3.1)$$

for any $u \in H^1(\tilde{Q})$, $u(x, \tau) = 0$ on $\tilde{\Sigma}$ and the right-hand side of (6.3.1) is finite.

Proof . First we choose the parameter $\beta = \beta_T$ such that

$$\beta^{-1} \max_{x \in \mathcal{M}} \vartheta(x) \leq \frac{T^2}{4}, \quad (6.3.2)$$

and let $\beta_0 > 0$ satisfy

$$\beta_0 > \beta T^2.$$

Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function defined by

$$\chi(\tau) = \begin{cases} 1 & \text{if } |\tau| \leq \frac{T}{2} \\ 0 & \text{if } |\tau| \geq \frac{3T}{4}. \end{cases}$$

For any $u \in H^2(\tilde{Q})$ satisfying $u(x, \tau) = 0$ for $(x, \tau) \in \tilde{\Sigma}$, we set $y(x, \tau) = \chi(\tau)u(x, t)$. Applying Theorem 6.1.1, we obtain

$$\begin{aligned} C\gamma \int_{\tilde{Q}_0} \left(\sigma^{-1} \sum_{|\alpha|=2} |\partial^\alpha u(x, \tau)|^2 + \sigma (|\nabla_g u(x, \tau)|_g^2 + |\partial_\tau u(x, \tau)|^2) + \sigma^3 |u(x, \tau)|^2 \right) e^{2s\varphi} dv_g d\tau \\ \leq \int_{\tilde{Q}} |(\partial_\tau^2 + \Delta_g)y(x, \tau)|^2 e^{2s\varphi} dv_g d\tau + \int_{\tilde{\Sigma}_0} \sigma |\partial_\nu u|^2 e^{2s\varphi} d\sigma_g d\tau. \end{aligned} \quad (6.3.3)$$

Moreover we have

$$\begin{aligned} \int_{\tilde{Q}} |(\partial_\tau^2 + \Delta_g)y(x, \tau)|^2 e^{2s\varphi} dv_g d\tau \leq C \int_{\tilde{Q}} |(\partial_\tau^2 + \Delta_g)u(x, \tau)|^2 e^{2s\varphi} dv_g d\tau \\ + \int_{\tilde{Q} \setminus \tilde{Q}_1} (|u(x, \tau)|^2 + |\partial_\tau u(x, \tau)|^2) e^{2s\varphi} dv_g d\tau, \end{aligned} \quad (6.3.4)$$

where $\tilde{Q}_1 = \mathcal{M} \times (-3T/4, 3T/4)$. Then

$$\begin{aligned}
& C\gamma \int_{\widetilde{Q}_0} \left(\sigma (|\nabla u(x, \tau)|^2 + |\partial_\tau u(x, \tau)|^2) + \sigma^3 |u(x, \tau)|^2 \right) e^{2s\varphi} \, dv_g \, d\tau \\
& \leq C \int_{\widetilde{Q}} |(\partial_\tau^2 + \Delta_g)u(x, \tau)|^2 e^{2s\varphi} \, dv_g \, d\tau + \int_{\widetilde{Q} \setminus \widetilde{Q}_1} (|u(x, \tau)|^2 + |\partial_\tau u(x, \tau)|^2) e^{2s\varphi} \, dv_g \, d\tau \\
& \quad + \int_{\widetilde{\Sigma}_0} \sigma |\partial_\nu u|^2 e^{2s\varphi} \, d\sigma_g \, d\tau. \quad (6.3.5)
\end{aligned}$$

Since

$$\min_{(x, \tau) \in \widetilde{Q}_0} \varphi(x, \tau) \geq e^{\gamma(\beta_0 - \beta T^2/4)} \equiv d_1$$

and

$$\max_{(x, \tau) \in \widetilde{Q} \setminus \widetilde{Q}_1} \varphi(x, \tau) \leq e^{\gamma(\max_{x \in \mathcal{M}} \vartheta(x) - \beta 9T^2/16)} \leq e^{\gamma(\beta_0 - \beta 5T^2/16)} \equiv d_2 < d_1,$$

we have

$$\begin{aligned}
& C e^{2d_1 s} \int_{\widetilde{Q}_0} \left(|\nabla u(x, \tau)|^2 + |\partial_t u(x, \tau)|^2 + |u(x, \tau)|^2 \right) \, dv_g \, d\tau \\
& \leq C e^{Ds} \left(\int_{\widetilde{Q}} |(\partial_\tau^2 + \Delta_g)u(x, \tau)|^2 \, dv_g \, d\tau + \int_{\widetilde{\Sigma}_0} \sigma |\partial_\nu u|^2 e^{2s\varphi} \, d\sigma_g \, d\tau \right) \\
& \quad + e^{2d_2 s} \int_{\widetilde{Q} \setminus \widetilde{Q}_1} (|u(x, \tau)|^2 + |\partial_\tau u(x, \tau)|^2) \, dv_g \, d\tau. \quad (6.3.6)
\end{aligned}$$

Then

$$\begin{aligned}
& C \int_{\widetilde{Q}_0} \left(|\nabla u(x, \tau)|^2 + |\partial_\tau u(x, \tau)|^2 + |u(x, \tau)|^2 \right) \, dv_g \, d\tau \\
& \leq C e^{Ds} \left(\int_{\widetilde{Q}} |(\partial_\tau^2 + \Delta_g)u(x, \tau)|^2 \, dv_g \, d\tau + \int_{\widetilde{\Sigma}_0} \sigma |\partial_\nu u|^2 e^{2s\varphi} \, d\sigma_g \, d\tau \right) \\
& \quad + e^{-2ds} \int_{\widetilde{Q} \setminus \widetilde{Q}_1} (|u(x, \tau)|^2 + |\partial_\tau u(x, \tau)|^2) \, dv_g \, d\tau, \quad (6.3.7)
\end{aligned}$$

where $d = d_1 - d_2 > 0$. Minimizing in s , we obtain (6.3.1). \square

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