$\mathcal{R}$-Boundedness, H^∞-Calculus, Maximal (L^p-) Regularity and Applications to Parabolic PDE's

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Preface

The notion of $\mathcal{R}$-boundedness of operator families and its relation to maximal regularity for linear Cauchy problems has undergone a substantial developement in recent years. The aim of this series of lectures is to summarize some important results in this direction and to demonstrate their strength in applications to linear and nonlinear partial differential equations of parabolic type.

In the first part of the series we state sufficient conditions implying the maximal regularity of a linear operator in an abstract framework. In particular we show how the search for such conditions leads to the notions of bounded imaginary powers or an $\mathcal{H}^\infty$-calculus of a linear operator as well as to the notion of the $\mathcal{R}$-boundedness of operator families.

In the second part we show how these results can be applied to linear and nonlinear parabolic PDE's. Exemplary this is demonstrated for such systems as the Navier-Stokes equations and free boundary value problems of Stefan type.

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Notation and basic definitions

Let X, Y be Banachspaces, A be a linear closed operator in X. We use the following standard notation:

$D(A)$	: domain of A.
$R(A) = A(D(A))$	: range of A.
$\rho(A)$	: resolvent set of A.
$\sigma(A) = \mathbb{C} \setminus \rho(A)$	: spectrum of A.
$\mathscr{L}(X,Y)(\mathscr{L}(X))$	: set of all linear bounded operators from X to Y (X to X).
$Isom(X,Y)$	: set of all (bounded) isomorphisms from X to Y.
$(X,Y)_{\theta,p}$	: real interpolation space for $0 < \theta < 1$ and $1 \le p \le \infty$ (see Triebel [25]).
$[X,Y]_\theta$	: complex interpolation space for $0 \le \theta \le 1$ (see [25]).
$(G, \mathcal{M}, \mu)$	: probability space, where G is a set, $\mathcal{M}$ a σ-algebra on G, and μ a probability measure on $\mathcal{M}$.
$\langle \cdot \vert \cdot \rangle$	: standard scalar product in $\mathbb{R}^n$.
(u,v)	: standard X, X' dual pairing ($\int uv \mathrm{d}x$ if $X = L^p$).
Σ_ϕ	: complex sector $\Sigma_\phi := \{z \in \mathbb{C} \setminus \{0\} : \lvert \arg z \rvert < \phi\}$.

We say that $A : D(A) \to X$ is the generator of a bounded holomorphic C_0-semigroup on X, if there is a $\phi \in (0, \pi/2)$ such that $\{\mathrm{e}^{-zA}\}_{z \in \Sigma_\phi}$ is a family of uniformly bounded operators on X and $z \mapsto \mathrm{e}^{-zA}$ is strongly continuous. The class of all generators on X we denote by $HOL(X)$.

For $\Omega \subseteq \mathbb{R}^n$ open by $L_p(\Omega, X)$ (norm: $\|\cdot\|_p$) and $H^s_p(\Omega, X)$, for $1 \le p \le \infty$, $s \in \mathbb{R}$, we denote X-valued Lebesgue and Bessel potential space of order s, respectively. By $W^s_p(\Omega, X)$, $1 \le p < \infty$, $s \in \mathbb{R} \setminus \mathbb{Z}$, we denote the Sobolev-Slobodeckij space with norm

$$\|g\|_{W^s_p(\Omega,X)} = \|g\|_{H^{[s]}_p(\Omega,X)} + \left(\int_\Omega \int_\Omega \frac{\|g(x) - g(y)\|^p_X}{|x-y|^{n+(s-[s])p}} \mathrm{d}x \mathrm{d}y \right)^{1/p},$$

where $[s]$ denotes the largest integer smaller than s. Let $T \in (0, \infty]$ and $J = (0,T)$. We set

$$ {}_0W^s_p(J,X) := \begin{cases} \{u \in W^s_p(J,X) : u(0) = u'(0) = \ldots = u^{(k)}(0) = 0\}, \\ \text{if} \quad k + \frac{1}{p} < s < k + 1 + \frac{1}{p}, \; k \in \mathbb{N} \cup \{0\}, \\ W^s_p(J,X), \quad \text{if} \quad s < \frac{1}{p}. \end{cases}$$

The spaces ${}_0H^s_p(J,X)$ are defined analogously. Furthermore, $C^\infty_c(\Omega, X)$ denotes the space of smooth functions with compact support in Ω and $C^\infty_{c,\sigma}(\Omega, X) := \{u \in C^\infty_c(\Omega, X) : \operatorname{div} u = 0\}$ its subspace of solenoidal functions. We also set $L^p_\sigma(\Omega, X) := \overline{C^\infty_{c,\sigma}(\Omega, X)}^{\|\cdot\|_p}$. Higher order differentials are denoted by $D^\alpha :=$

$D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multiindex and $|\alpha| := \sum_{j=1}^n \alpha_j$. Here $D_j^{\alpha_j} u(x) := \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} u(x)$ for $j = 1, \dots, n$. As usual we write $W^{k,p}(\Omega, X)$, $k \in \mathbb{N}$, for the Sobolev space and $\widehat{W}^{1,p}(\Omega, X)$ for the homogeneous Sobolev space, which is $\{u \in L^1_{loc}(\Omega, X) : \|\nabla u\|_p < \infty\}$ modulo constants, whereas $\mathrm{BUC}(\Omega, X)$ stands for the space of all bounded uniformly continuous functions on Ω. By $\mathcal{S}(\mathbb{R}^n, X)$ we mean the Schwartz space of rapidly decreasing functions and the Fourier transform defined on $\mathcal{S}(\mathbb{R}^n, X)$ we denote by

$$\hat{u}(\xi) := \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} u(x)\mathrm{d}x, \quad u \in \mathcal{S}(\mathbb{R}^n, X),$$

whereas on $\mathcal{S}'(\mathbb{R}^n, X) := \mathscr{L}(\mathcal{S}(\mathbb{R}^n), X)$ it is defined by duality. Finally, the Laplace transform for $f \in L^\infty((0,\infty), X)$ is denoted by

$$\mathcal{L}u(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} f(t)\mathrm{d}t, \quad \lambda > 0.$$

If $X = \mathbb{R}^n$ (or $\mathbb{C}^n$) we set $L^p(\Omega) := L^p(\Omega, \mathbb{R}^n)$, $W^{k,p}(\Omega) := W^{k,p}(\Omega, \mathbb{R}^n)$, etc. The classical Mikhlin multiplier result:

Let $1 < q < \infty$, $k = \min\{j \in \mathbb{N} : j > n/2\}$ and let $m \in C^k(\mathbb{R}^n \setminus \{0\})$ satisfy

$$\|m\|_M := \max_{|\alpha| \le k} \sup_{x \in \mathbb{R}^n} |\xi|^{|\alpha|} |D^\alpha m(\xi)| < \infty.$$

Then m is a multiplier in $L^q(\mathbb{R}^n)$ and there exists a $C > 0$ such that

$$\|\mathcal{F}^{-1} m \mathcal{F}\|_{\mathscr{L}(L^q(\mathbb{R}^n))} \le C\|m\|_M.$$

Overview of classes defined in the lecture

$\mathscr{S}(X)$	: class of sectorial operators in X.
$\mathcal{R}\mathscr{S}(X)$	: class of $\mathcal{R}$-sectorial operators in X.
MR_p	: class of all operators having maximal regularity on X.
$BIP(X)$	: class of operators having bounded imaginary powers on X.
$\mathcal{H}^\infty(X)$	: class of operators admitting a bounded H^∞-calculus on X.
$\mathcal{RH}^\infty(X)$	: class of operators admitting an $\mathcal{R}$-bounded H^∞-calculus on X.
$\mathcal{HT}$	: class of all Banach spaces X such that the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some $p \in (1, \infty)$.
ϕ_A	: spectral angle of A.
ϕ_A^R	: $\mathcal{R}$-angle of A.
θ_A	: power angle of A.
ϕ_A^∞	: H^∞-angle of A.
$\phi_A^{\infty,R}$	: $\mathcal{R}$-H^∞-angle of A.

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Reference: R. Denk, M. Hieber, J. Prüß, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Memories Amer. Math. Soc., '03, [7].

I Significance of maximal (L^p-) regularity

Motivation: Mean curvature flow

Let $t \mapsto \Gamma(t)$ describe the motion of a hypersurface in $\mathbb{R}^{n+1}$. The evolution is modeled by the mean curvature flow equation:

$$\dot{X} = \kappa\nu, \quad \Gamma(0) = \Gamma_0. \tag{1}$$

Locally: $X(t,x) = (x, u(t,x))^T$, $\kappa = \operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2})$, $\nu = (-\nabla u, 1)^T/\sqrt{1+|\nabla u|^2})$. Then (1) reduces to

$$(MC)\begin{cases} \partial_t u &= \sum_{j,k}(\delta_{jk} + \frac{\partial_j u \partial_k u}{1+|\nabla u|^2})\partial_j\partial_k u =: -F(u) \text{ in } (0,T)\times\mathbb{R}^n, \\ u(0) &= u_0 \text{ in } \mathbb{R}^n. \end{cases}$$

"Quasilinear parabolic evolution equation of second order".

Construction of solutions by a fixed point argument:

Step1: linearize

$$\partial_t u \underbrace{-\Delta u}_{=F'(0)u} = \underbrace{\sum_{j,k} \frac{\partial_j u \partial_k u}{1+|\nabla u|^2}\partial_j\partial_k u}_{=F'(0)u-F(u)} =: G(u).$$

The term $F'(0)$ is the Frechét derivative of F at 0. The left hand side is a "second order linearization of (MC)".

Step 2: solve linearized problem

$$(\text{LMC})\begin{cases} \partial_t u - \Delta u &= f \text{ in } (0,T)\times\mathbb{R}^n, \\ u(0) &= u_0 \text{ in } \mathbb{R}^n, \end{cases}$$

i.e. show that for all $(f, u_0) \in \mathbb{F} = \mathbb{F}_1 \times \mathbb{F}_2 = L^p((0,T), L^p(\mathbb{R}^n)) \times W_p^{2-2/p}(\mathbb{R}^n)$ there exists a unique solution $u \in \mathbb{E} \hookrightarrow \mathbb{F}_1$, where $\mathbb{E}$ is the related space of solutions.

Step 3: apply a fixed point argument to the nonlinear problem

Denote by

$$L^{-1} : \mathbb{F} \to \mathbb{E}$$

the solution operator of (LMC), i.e. $u = L^{-1}(f, u_0)$. Then, formally, (MC) can be rephrased as

$$u = L^{-1}(G(u), u_0) \quad \text{"fixed point equation".}$$

To apply a fixed point argument it is required that

$$G(\mathbb{E}) \hookrightarrow \mathbb{F}_1 \tag{2}$$

i.e.

$$\underbrace{\frac{\partial_j u \partial_k u}{1 + |\nabla u|^2}}_{\in L^\infty \text{ by Sobolev's embed.}} \qquad \underbrace{\partial_j \partial_k u}_{\in L^p} \in L^p((0,T), L^p(\mathbb{R}^n)).$$

<u>Observation:</u> The more regularity we have for $u \in \mathbb{E}$, the "higher" is the chance that (2) is satisfied.

The "maximal" regularity for the solution u of (LCM) we can expect is that $\partial_t u, \Delta u \in \mathbb{F}_1$, i.e.

$$\mathbb{E} = W^{1,p}((0,T), L^p(\mathbb{R}^n)) \cap L^p((0,T), W^{2,p}(\mathbb{R}^n))$$

and that

$$||\partial_t u||_{\mathbb{F}_1} + ||\Delta u||_{\mathbb{F}_1} \leq C(||f||_{\mathbb{F}_1} + ||u_0||_{F_2}). \tag{3}$$

Then, for $p > 1$ large enough s.t. $\mathbb{E} \hookrightarrow L^\infty((0,T) \times \mathbb{R}^n)$ we have that

$$\frac{\partial_j u \partial_k u}{1 + |\nabla u|^2} \in L^\infty((0,T) \times \mathbb{R}^n) \Rightarrow (2) \text{ is satisfied.}$$

Then a fixed point argument can be applied in order to get local-in-time solutions for (MC). This demonstrates the significance of estimate (3) in the treatment of nonlinear PDE's.

<u>Definition 1.1.</u> (Maximal regularity)
Let $T \in (0, \infty], J = (0,T), 1 \leq p \leq \infty$, and X be a Banach space. A linear closed operator $A : D(A) \to X$ in X is said to have maximal (L^p-) regularity, if for each $(f, u_0) \in L^p(J, X) \times (X, D(A))_{1-1/p,p}$ there is a unique function u satisfying

$$\begin{cases} u' + Au &= f, \quad t \in J, \\ \quad u(0) &= u_0, \end{cases}$$

f.a.a. $t \in J$ and the estimate

$$||u'||_{L^p(J,X)} + ||Au||_{L^p(J,X)} \leq C \left(||f||_{L^p(J,X)} + ||u_0||_{(X,D(A))_{1-1/p,p}} \right) \tag{4}$$

with $C > 0$ independent of f and u_0. We denote the class of all such operators by $MR_p(X)$.

Remark 1.2.

(a) $A \in MR_p(X)$ for one $p \in [1,\infty] \Rightarrow A \in MR_p(x)$ for all $p \in (1,\infty)$. (Sobolevskii '64 [23])

(b) If $T < \infty$ or $0 \in \rho(A)$, the term $||u'||_{L^p(J,X)}$ on the left hand side of (4) can be replaced by $||u||_{W^{1,p}(J,X)}$. Then the solution operator $L^{-1} : (f, u_0) \mapsto u$ is an isomorphism i.e.

$$L^{-1} \in \text{ Isom } (L^p(J,X) \times (X, D(A))_{1-1/p,p}, W^{1,p}(J,X) \cap L^p(J, D(A))).$$

In this situation, by the closed graph theorem, (4) is equivalent to $Au \in L^p(J,X)$.

Maximal regularity is a powerful tool in the treatment of nonlinear PDE's. E.g. it is useful in:

— constructing local-in-time strong solutions.

— constructing global weak solutions.

— proving uniqueness of "mild" solutions.

— proving existence of global strong (and therefore uniqueness of weak) solutions for the 2-dimensional Navier-Stokes equations.

— constructing real analytic (classical) solutions.

Note: The search for sufficient conditions on A and X that imply the maximal L^p-regularity leads to the notions of $\mathcal{R}$-boundedness, H^∞-calculus, and spaces of class $\mathcal{HT}$ (UMD spaces).

II Sufficient conditions implying maximal regularity

Definition 2.1. Let X be a Banach space and $A : D(A) \to X$ be a closed operator in X. A is called sectorial, i.e. $A \in \mathscr{S}(X)$, if

(i) A is injective and $\overline{D(A)} = \underbrace{\overline{R(A)}}_{=AD(A)} = X$,

(ii) $(-\infty, 0) \subseteq \rho(A)$ and $\exists M > 0$ s.t. $||\lambda(\lambda + A)^{-1}||_{L(X)} \leq M, \lambda > 0$.

Then, by the Taylor expansion, there exists a $\phi \in (0,\pi)$ and $C_\phi > 0$ such that

$$\Sigma_{\pi-\phi} := \{z \in \mathbb{C}\backslash\{0\} : |\text{arg} z| < \pi - \phi\} \subseteq \rho(-A)$$

and

$$||\lambda(\lambda + A)^{-1}||_{L(X)} \leq C_\phi, \quad \lambda \in \Sigma_{\pi-\phi}. \tag{5}$$

The angle $\phi_A := \inf\{\phi \in (0,\pi) : (5) \text{ holds}\}$ is called spectral angle of A. If $\phi_A < \pi/2$, then A is the generator of a bounded holomorphic C_0-semigroup.

Examples: elliptic operators on $L^q(\Omega)$ (e.g. Dirichlet-Laplacian Δ_D); Stokes operator $A = -P\Delta$ on $L^q_\sigma(\Omega)$, $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$, or bounded and sufficiently smooth (see Lunardi [16]).

2.1 An equivalent condition involving $\mathcal{R}$-boundedness

Let $A \in \mathscr{S}(X), \phi_A < \pi/2, f \in L^p(\mathbb{R}_+, X)$, and consider

$$\begin{cases} u_t + Au &= f, \quad t > 0, \\ \quad u(0) &= 0. \end{cases}$$

The solution to this problem is given by

$$u(t) = \int_0^t e^{-(t-s)A} f(s) ds, \ t > 0.$$

Formally we have (think of A as a matrix!)

$$\begin{aligned} Au(t) &= \int_0^t Ae^{-(t-s)A} f(s) ds \\ &= \int_{-\infty}^{\infty} \underbrace{\chi_{(0,\infty)}(t-s)Ae^{-(t-s)A}}_{=:k_{op}(t-s)} \quad \underbrace{\chi_{(0,\infty)}(s)f(s)}_{=:\tilde{f}(s)} ds \\ &= (k_{op} * \tilde{f})(t) \end{aligned}$$

with an operator-valued kernel k_{op}. On the other hand we obtain by applying Fourier transform that

$$\begin{aligned} \mathcal{F}k_{op}(\lambda) &= \int_{-\infty}^{\infty} e^{i\lambda t} \chi_{(0,\infty)}(t) Ae^{-tA} dt \\ &= A \int_0^{\infty} e^{i\lambda t} e^{-tA} dt \\ &= A(i\lambda - A)^{-1} \end{aligned}$$

This implies that

$$Au = \mathcal{F}^{-1} A(i\lambda - A)^{-1} \mathcal{F} \tilde{f}. \tag{6}$$

Thus the question of maximal regularity, i.e. of whether $Au \in L^p(\mathbb{R}_+, X)$ holds, is reduced to show that $\lambda \mapsto A(i\lambda - A)^{-1}$ is an operator-valued Fourier multiplier on $L^p(\mathbb{R}, X)$.

Problem 2.2: What is a sufficient condition, such that $m : \mathbb{R} \to \mathcal{L}(X)$ is a multiplier (operator-valued), i.e. $\mathcal{F}^{-1} m \mathcal{F} \in \mathcal{L}(L^p(\mathbb{R}, X))$.

2.1.1. The Hilbert space case

Let H be a Hilbert space and $p = 2$. By Plancherel's Theorem we get that

$$\mathcal{F}^{-1} m \mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}, H)) \Leftrightarrow m \in L^\infty(\mathbb{R}, \mathcal{L}(H)).$$

Observation: sufficient condition:

scalar-valued case, $H = \mathbb{C}$	operator-valued case
$\sup_{\lambda \in \mathbb{R}} \lvert m(\lambda) \rvert < \infty$	$\sup_{\lambda \in \mathbb{R}} \lvert\lvert m(\lambda) \rvert\rvert_{\mathcal{L}(H)} < \infty$

$$\text{"just replace } |\cdot| \quad \text{by} \quad ||\cdot||_{\mathcal{L}(H)} \text{"} \tag{7}$$

Hence we have

Proposition 2.3. Let H be a Hilbert space

$$\begin{aligned} &\text{maximal regularity in } L^2(\mathbb{R}_+, H) \\ \Leftrightarrow\ & ||\underbrace{A(i\lambda - A)^{-1}}_{= I + i\lambda(i\lambda - A)^{-1}}||_{\mathcal{L}(H)} < C, \quad \lambda \in \mathbb{R}, \\ \Leftrightarrow\ & A \text{ sectorial and } \phi_A < \frac{\pi}{2}. \end{aligned}$$

2.1.2. The Banach space case

Reminder: A sufficient condition in the scalar-valued case ($X = \mathbb{C}$) is (Mikhlin's result)

$$m \in C^1(\mathbb{R}\backslash\{0\}), \sup_{\lambda \in \mathbb{R}\backslash\{0\}} |m(\lambda)| < \infty, \sup_{\lambda \in \mathbb{R}\backslash\{0\}} |\lambda m'(\lambda)| < \infty.$$

Question 1: Does (7) also work for the Banach space case, i.e. do we have

$$\left.\begin{array}{l} m \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X)),\ \max_{k=0,1} \sup_{\lambda \in \mathbb{R}\backslash\{0\}} ||\lambda^k m^{(k)}(\lambda)||_{\mathcal{L}(X)} < \infty \\ \Rightarrow \mathcal{F}^{-1} m \mathcal{F} \in \mathcal{L}(L^p(\mathbb{R}, X)), 1 < p < \infty \text{ ?} \end{array}\right\} \tag{8}$$

Answer: No! (8) is true if and only if X is a Hilbert space. ('if' part : Schwartz '61 [22] 'only if' part : Pisier, see Lancien, Lancien, Le Merdy '98 [15])

Question 2: Is "at least" Prop 2.3 still valid for Banach spaces ?

Answer : No! Kalton, Lancien '99 [14]: Let X be a Banach space and suppose that

$$A \in MR(X) \Leftrightarrow A \text{ sectorial and } \phi_A < \frac{\pi}{2}.$$

Then X is isomorphic to a Hilbert space. [Note: So far no explicit counterexample is known!]

Conclusion: To solve Problem 2.2 the uniform boundedness of $m(\lambda)$ and $\lambda m'(\lambda)$ in $\mathcal{L}(X)$ is not enough, a stronger property is required.

Idea of L. Weis [26]: (Bourgain '86 X-valued [5]): Replace $\sup_{\lambda \in \mathbb{R}\setminus\{0\}} ||\cdot||_{\mathcal{L}(X)}$ (uniform boundedness) in (8) by the "$\mathcal{R}$-bound" of an operator family.

Definition 2.4. ($\mathcal{R}$-boundedness) (implicitly Bourgain '83 [4], Berkson, Gillespie '94 [2])
Let X be a Banach space. An operator family $\mathcal{T} \subseteq \mathcal{L}(X)$ is called $\mathcal{R}$-bounded, if there exists a $C > 0$ and a $p \in [1, \infty)$ such that

$$||\sum_{j=1}^{N} \varepsilon_j T_j x_j||_{L^p(G,X)} \leq C||\sum_{j=1}^{N} \varepsilon_j X_j||_{L^p(G,X)} \tag{9}$$

for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$, and for all independent symmetric $\{-1, 1\}$-valued random variables ε_j on some probability space (G, M, μ). The number $\inf\{C > 0\text{: (9) is valid}\}$ is called $\mathcal{R}$-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$.

Remark 2.5.

(a) In general, $\mathcal{R}$-boundedness is difficult to verify directly.

(b) $\mathcal{T} \subseteq \mathcal{L}(X)$ $\mathcal{R}$-bounded $\Rightarrow$ $\mathcal{T}$ is uniformly bounded. (Take $N = 1$ and use $||\varepsilon_j||_{L^p(\Omega)} = 1$.)

(c) Def 2.3 is independent of $p \in [1, \infty)$.

(d) If X is a Hilbert space, then: $\mathcal{T}$ $\mathcal{R}$-bounded $\Leftrightarrow$ $\mathcal{T}$ uniformly bounded.

(e) Let $\Omega \subseteq \mathbb{R}^n$ be open and $X = L^q(\Omega), 1 \leq q < \infty$. Then $\mathcal{T} \subseteq \mathcal{L}(L^q(\Omega))$ is $\mathcal{R}$-bounded if and only if the "square function estimate"

$$||(\sum_{j=1}^{N} |T_j f_j|^2)^{1/2}||_{L^q(\Omega)} \leq C||(\sum_{j=1}^{N} |f_j|^2)^{1/2}||_{L^q(\Omega)}$$

holds for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, and $f_j \in L^q(\Omega)$.

For the proof and further useful properties see [7].

In order to demonstrate how delicate it is to verify $\mathcal{R}$-boundedness of an operator family, exemplary we outline the proof of (d). Assume $\mathcal{T} \subseteq \mathcal{L}(X)$ is uniformly bounded. Then we calculate

$$
\begin{aligned}
&||\sum_{j=1}^{N}\varepsilon_j T_j x_j||^2_{L^2(G,X)} \\
&= \sum_{j,k=1}^{N}\int_G (\varepsilon_j(\omega)T_j x_j, \varepsilon_k(\omega)T_k x_k)_x d\mu(\omega) \\
&= \sum_{j,k=1}^{N}[\int_G \varepsilon_j(\omega)\varepsilon_k(\omega)d\mu](T_j x_j, T_k x_k)_X \\
&= \sum_{j=1}^{N}[\int_G \varepsilon_j(\omega)^2 d\mu]||T_j x_j||^2_X \\
&\leq C\sum_{j=1}^{N}[\int_G \varepsilon_j(\omega)d\mu]||x_j||^2_X \\
&= C\sum_{j,k=1}^{N}[\int_G \varepsilon_j(\omega)\varepsilon_k(\omega)d\mu](x_j, x_k)_X \\
&= C||\sum_{j=1}^{N}\varepsilon_j x_j||^2_{L^2(G,X)} \qquad \textbf{q.e.d.}
\end{aligned}
$$

Observe that most steps above only work in a Hilbert space context. By the presence of the random variables ε_j it is clear that the intuitive “pulling in” of norms usually used in the Banach space context is in general not possible. Even for simple examples as e.g. $\lambda(\lambda-\Delta)^{-1}$, where Δ denotes the Laplacian in $\mathbb{R}^n$ one has to be very familiar with the notion of $\mathcal{R}$-boundedness in order to verify this condition. In many situations it is easier to verify sufficient conditions that imply the $\mathcal{R}$-boundedness.

Explicit examples :

— $\{T\}$ is $\mathcal{R}$-bounded.

— $K \subseteq \mathbb{C}$ compact, $F : K \to \mathcal{L}(X)$ holomorphic $\Rightarrow \{F(\lambda); \lambda \in K\}$ $\mathcal{R}$-bounded

— The family $\{\lambda(\lambda-\Delta)^{-1}; \lambda \in \Sigma_{\pi-\phi_0}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\mathbb{R}^n))$ for $\phi_0 \in (0,\pi), 1 < q < \infty$.

Definition 2.6. *A* Banach space X is said to be of class $\mathcal{HT}$ (or a UMD space) if there exists a $p \in (1,\infty)$ such that $H \in \mathcal{L}(L^p(\mathbb{R},X))$, where

$$
Hf(t) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|s|>\varepsilon}\frac{f(t-s)}{s}ds, \quad t \in \mathbb{R},\ f \in S(\mathbb{R},X).
$$

Examples : Hilbert spaces, L^q-spaces for $1 < q < \infty$, $X \in \mathcal{HT} \Rightarrow X$ reflexive (see Amann '95 [1]).

<u>Theorem 2.7.</u> (L. Weis '01 [26]) (Answer to Problem 2.2) Let X be of class $\mathcal{HT}$ and $1 < p < \infty$. Assume $m \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X))$ s.t.

$$\max_{k=0,1} \mathcal{R}(\{\lambda^k m^{(k)}(\lambda) : \lambda \in \mathbb{R}\backslash\{0\}\}) < \infty.$$

Then

$$\mathcal{F}^{-1} m \mathcal{F} \in \mathcal{L}(L^p(\mathbb{R}, X)).$$

<u>Remark :</u>

— Theorem 2.7 is a generalization of the classical Mikhlin multiplier result and the $\mathcal{HT}$-valued version of Bourgain.

— Theorem 2.7 can be generalized to arbitrary dim $n \in \mathbb{N}$ (see Weis, Strkalj '00 [24], Haller, Heck, Noll '01 [12]).

Now consider

$$m(\lambda) = i\lambda(i\lambda - A)^{-1},$$
$$\lambda m'(\lambda) = (i\lambda)^2(i\lambda - A)^{-2}.$$

Exercise :

$$\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{L}(X)\ \mathcal{R}-\text{bounded}$$
$$\Rightarrow \{T_1T_2 : T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2\}\ \mathcal{R}-\text{bounded}.$$

By (6) this implies

$$\{i\lambda(i\lambda - A)^{-1}; \lambda \in \mathbb{R}\backslash\{0\}\}\ \mathcal{R}-\text{bounded} \Rightarrow A \in MR(X).$$

More precisely the following holds.

<u>Theorem 2.8.</u> Let X be a Banach space of class $\mathcal{HT}$, $1 < p < \infty$, and A be sectorial in X with $\phi_A < \frac{\pi}{2}$. Then there are equivalent:

(i) $A \in MR_p(X)$,

(ii) $\{i\lambda(i\lambda - A)^{-1}; \lambda \in \mathbb{R}\backslash\{0\}\}$ is $\mathcal{R}$-bounded,

(iii)
$$\mathcal{R}(\{\lambda(\lambda + A)^{-1}; \lambda \in \Sigma_{\pi-\phi}\}) < \infty \qquad (10)$$
for some $\phi < \frac{\pi}{2}$, i.e. A is "$\mathcal{R}$-sectorial" with "$\mathcal{R}$-angle"
$$\phi_A^R := \inf\{\phi \in (0, \pi) : \text{ (10) holds }\} < \frac{\pi}{2}.$$

pf: Weis '01 [26], Denk, Hieber, Prüß '03 [7].

2.2 Other sufficient conditions.

Let $J = (0,T)$, X be a Banach space, and let A be sectorial in X. We consider again

$$(\text{CP}) \begin{cases} (\frac{d}{dt} + A)u &= f, \quad t > 0, \\ \quad u(0) &= 0. \end{cases}$$

We want to regard "$(\frac{d}{dt} + A)$" as a sum of closed operators in $E := L^p(J, X)$. To this end we define

$$\tilde{A}u := Au \ , \ D(\tilde{A}) := \{u \in L^p(J,X) : Au \in L^p(J,X)\},$$
$$Bu := \frac{d}{dt}u \ , \ D(B) := \{u \in W^{1,p}(J,X) : u(0) = 0\},$$

and consider the sum

$$\tilde{A} + B \text{ defined on } D(\tilde{A} + B) := D(\tilde{A}) \cap D(B).$$

The problem (CP) has maximal regularity means that for all $f \in E$ there exists a unique solution u of (CP) such that

$$||u'||_E + ||Au||_E \leq C||f||_E.$$

But this is equivalent to the fact that $R(\tilde{A}+B) = (\tilde{A}+B)(D(\tilde{A}) \cap D(B)) = E$ and that

$$||\tilde{A}u||_E + ||Bu||_E \leq C||(\tilde{A}+B)u||_E \ , \ u \in D(\tilde{A}) \cap D(B).$$

This in turn is equivalent to say that $\tilde{A}+B$ with $D(\tilde{A}+B) = D(\tilde{A}) \cap D(B)$ is closed and $R(\tilde{A}+B) = E$ ($\Leftrightarrow 0 \in \rho(A+B)$).

Question: What are suitable conditions on $\tilde{A}, B, X$, such that $\tilde{A}+B$ is closed and $R(\tilde{A}+B) = E$?

Theorem 2.9. (Dore-Venni '87 [8]) Let E be Banach space of class $\mathcal{HT}$ and A, B be sectorial in E such that $0 \in \rho(B) \cap \rho(A)$ and

(i) $(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}, \quad \lambda \in \rho(A), \ \mu \in \rho(B)$,

(ii) $A, B \in BIP(E)$, where $A \in BIP(E)$ means that $A^{is} \in \mathcal{L}(E), s \in \mathbb{R}$, and

$$||A^{is}||_{\mathcal{L}(E)} \leq C, \ s \in [-1,1] \quad \text{"bounded imaginary powers"},$$

.

(iii) $\theta_A + \theta_B < \pi$, where

$$\theta_A := \overline{\lim_{|s|\to\infty}} \frac{\log ||A^{is}||}{|s|} (\geq \phi_A) \qquad \text{“power angle of } A\text{”}.$$

Then $A + B$ is closed and $0 \in \rho(A + B)$.

Remark: $0 \in \rho(A) \cap \rho(B)$ can be removed, see Giga, Sohr '91 [11].

Cor 2.10. Let X be a Banach space of class $\mathcal{HT}$, $J = (0, T)$ for $T \in (0, \infty]$, and $A \in BIP(X)$ with $\theta_A < \frac{\pi}{2}$. Then we have $A \in MR(X)$.

proof: First note that $A \in BIP(X)$ implies $\tilde{A} \in BIP(L^p(J, X))$, and the fact X is of class $\mathcal{HT}$ that also $E = L^p(J, X)$ is of class $\mathcal{HT}$. Now let $Bu = u'$, $D(B) = \{u \in W^{1,p}(J, X) : u(0) = 0\}$. It is well known that $B \in BIP(L^p(J, X))$ and $\theta_B = \frac{\pi}{2}$. Theorem 2.9 implies that $\tilde{A} + B$ is closed and that for all $f \in E$ there exists a $u \in D(\tilde{A}) \cap D(B)$ such that

$$||Au||_E + ||u'||_E \leq C||(\tilde{A} + B)u||_E = C||f||_E.$$

Thus we have $A \in MR(X)$. **q.e.d.**

Next, let $0 < \phi < \pi$ and set

$$H^\infty(\Sigma_\phi) := \{h : \Sigma_\phi \to \mathbb{C} : \ h \text{ bounded and holomorphic}\}$$

and

$$H_0^\infty(\Sigma_\phi) := \{h \in H^\infty(\Sigma_\phi) : |h(z)| \leq C \frac{|z|^s}{(1 + |z|)^{2s}} \text{ for some } C, s > 0\}.$$

Let A be sectorial on the Banach space X and $\phi_A < \phi < \pi$. For $h \in H_0^\infty(\Sigma_\phi)$ we set

$$h(A) := \frac{1}{2\pi i} \int_\Gamma h(\lambda)(\lambda - A)^{-1} d\lambda,$$

where $\Gamma := \{te^{i\theta} : \infty > t > 0\} \cup \{te^{-i\theta} : 0 \leq t < \infty\}$ with $\phi_A < \theta < \phi$. This integral is absolutely convergent. Indeed, we have that

$$||h(A)x|| \leq C \int_0^\infty \frac{t^s}{(1 + t)^{2s}} \frac{1}{t} ||x|| dt \leq C_s ||x||, \quad x \in X,$$

which yields $h(A) \in \mathcal{L}(X)$. In fact, $h \mapsto h(A)$ is an algebra homomorphism from $H_0^\infty(\sum_\phi)$ to $\mathcal{L}(X)$.

Definition 2.11 (H^∞-calculus, McIntosh '86 [18]) The operator A admits a bounded H^∞-calculus on a Banach space $X, (A \in \mathcal{H}^\infty(X))$, if

$$||h(A)||_{\mathcal{L}(X)} \leq C||h||_{L^\infty(\Sigma_\phi)}, \quad h \in H_0^\infty(\Sigma_\phi). \tag{11}$$

The angle $\phi_A^\infty := \inf\{\phi \in (0,\pi) : \text{ (11) holds}\}$ is called H^∞-angle of A.

Now put $g(z) := z/(1+z)^2 \in H_0^\infty(\Sigma_\phi)$.

Exercise:

$$g(A) = A(1+A)^{-2} \quad \text{and} \quad g(A)^{-1} = A^{-1}(1+A)^2.$$

For $h \in H^\infty(\Sigma_\phi)$ we set

$$h(A) := \underbrace{(h \cdot g)}_{\in H_0^\infty(\Sigma_\phi)} (A) g(A)^{-1}$$

initially defined on $D(A) \cap R(A) \overset{\text{dense}}{\hookrightarrow} X$. It is known that $A \in \mathcal{H}^\infty(X)$ implies that (11) is valid for all $h \in H^\infty(\Sigma_\phi)$.

Reference for H^∞-calculus:

— McIntosh '86 [18],

— Cowling, Duong, McIntosh, Yagi '96 [6],

— Denk, Hieber, Prüß [7].

Examples

— bounded operators

— selfadjoint operators in Hilbert spaces

— Let $\Omega \subset \mathbb{R}^n$ be of class C^3 bounded, exterior, or a perturbed half-space, and $1 < q < \infty$. Then we have that the Dirichlet-Laplacian $-\Delta_\Omega \in \mathcal{H}^\infty(L^q(\Omega))$ (here C^2 is enough) (see Prüß, Sohr '93 [20]) and the Stokes operator $A \in \mathcal{H}^\infty(L^q_\sigma(\Omega))$, where

$$A = -P\Delta, \; D(A) := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap \underbrace{L^q_\sigma(\Omega)}_{:= \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}}$$

(see Noll, S. '03 [19]).

Relations

Let X be a Banach space and $A \in \mathscr{S}(X)$. Observe that $(z \mapsto z^{is}) \in H^\infty(\Sigma_\phi)$.

$$\Rightarrow \mathcal{H}^\infty(X) \subseteq \text{ BIP}(X).$$

Denote by $HOL(X)$ the class of all sectorial operators generating a holomorphic C_0-semigroup on X. Then we have

$A \in \mathcal{H}^\infty(X) \Rightarrow A \in \text{ BIP}(X) \overset{X\ \mathcal{HT}}{\Rightarrow} A \in \mathcal{R}\mathscr{S}(X) \underset{\phi_A^{\mathcal{R}} < \pi/2}{\overset{X\ \mathcal{HT}}{\Leftrightarrow}} A \in MR_p(X)$
$\Rightarrow A \in HOL(X)$.

In particular,

$$\phi_A^\infty \le \theta_A \le \phi_A^{\mathcal{R}} \le \phi_A.$$

Examples:

$$\begin{aligned} -\Delta \text{ in } L^p(\mathbb{R}^n): &\qquad \phi_{-\Delta}^\infty = \theta_{-\Delta} = \phi_{-\Delta}^{\mathcal{R}} = \phi_{-\Delta} = 0 \\ B = \partial_t \text{ in } L^p((0,T),X): &\qquad \phi_B^\infty = \theta_B = \phi_B^{\mathcal{R}} = \phi_B = \frac{\pi}{2} \end{aligned}$$

Ⅲ Applications to parabolic PDE's

3.1 Navier-Stokes equations (NSE)

Let $1 < q < \infty$, $T \in (0,\infty)$, $J = (0,T)$, and $\Omega \subseteq \mathbb{R}^n$ be open. We consider the system

$$\text{(NSE)} \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p &= f \quad \text{in } J \times \Omega, \\ \text{div} u &= 0 \quad \text{in } J \times \Omega, \\ u &= 0 \quad \text{on } J \times \partial\Omega, \\ u|_{t=0} &= u_0 \quad \text{in } \Omega. \end{cases}$$

By applying the concept described in the previous sections here we want to show the existence of a local real analytic solution for (NSE).

Step 1: Lincarize : $\rightarrow$ Stokes equations

$$\text{(SE)} \begin{cases} \partial_t u - \Delta u + \nabla p &= f \quad \text{in } J \times \Omega, \\ \text{div} u &= 0 \quad \text{in } J \times \Omega, \\ u &= 0 \quad \text{on } J \times \partial\Omega, \\ u|_{t=0} &= u_0 \quad \text{in } \Omega. \end{cases}$$

Step 2: maximal regularity for (SE):

We define the Stokes operator in $L^q_\sigma(\Omega) := \overline{\{u \in C_c^\infty(\Omega) : \text{div} u = 0\}}^{||\cdot||_q}$

$$A := -P\Delta, \quad D(A) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega),$$

where

$$P : L^q(\Omega) \rightarrow L^q_\sigma(\Omega) \quad \text{“Helmholtz projection”}.$$

Then (SE) is formally reduced to

$$\text{(CP)} \begin{cases} u' + Au &= f \quad \text{in } J, \\ u(0) &= u_0. \end{cases}$$

To demonstrate the method from now on let $\Omega = \mathbb{R}^n$, and write $L^q = L^q(\mathbb{R}^n)$, $W^{k,q} = W^{k,q}(\mathbb{R}^n)$, etc. Then we have $P = I + RR^T$, where

$$R = \mathcal{F}^{-1}\left[\frac{i\xi}{|\xi|}\right]\mathcal{F}$$

denotes the "Riesz operator". An application of the classical Mikhlin multiplier result yields

$$P \in \mathcal{L}(L^q, L^q_\sigma).$$

Furthermore we have that $P\Delta = \Delta P$, which implies

$$A = -\Delta|_{L^q_\sigma}, \quad \text{where } D(-\Delta) = W^{2,q}.$$

This also yields $(\lambda + A)^{-1} = (\lambda - \Delta)^{-1}|_{L^q_\sigma}$ or even more general that $h(A) = h(-\Delta)|_{L^q_\sigma}$ for $h \in H_0^\infty(\sum_\phi)$, $\phi \in (0, \pi)$. This means, in order to show $A \in MR(L^q_\sigma)$ it suffices to prove $-\Delta \in MR(L^q)$.

<u>Lemma</u> (exercise) Let $k > \frac{n}{2}$, $\phi \in (0, \pi)$. Then

$$\mathcal{F}h(-\Delta)f = h(|\xi|^2)\mathcal{F}f, \quad f \in L^q,$$

and there exists a $C_\phi > 0$ such that

$$\max_{|\alpha| \leq k} \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{|\alpha|} |D^\alpha h(|\xi|^2)| \leq C_\phi ||h||_{L^\infty(\Sigma_\phi)}, \; h \in H_0^\infty(\Sigma_\phi).$$

Hint : Cauchy's estimate formula for holomorphic functions.

Mikhlin's result now implies

$$-\Delta \in \mathcal{H}^\infty(L^q) \Rightarrow A \in \mathcal{H}^\infty(L^q_\sigma) \subseteq MR(L^q_\sigma).$$

<u>Step 3:</u> Applying a fixed point argument:

We define the class of data as

$$\mathbb{F}_T := \mathbb{F}_T^1 \times \{0\} \times \mathbb{F}_T^2 := L^q(J, L^q_\sigma) \times \{0\} \times \underbrace{(L^q_\sigma, D(A))_{1-1/q,q}}_{=W_q^{2-2/q} \cap L^q_\sigma},$$

and the class of solutions as

$$\mathbb{E}_T := \mathbb{E}_T^1 \times \mathbb{E}_T^2 := W^{1,q}(J, L^q_\sigma) \cap L^q(J, D(A)) \times L^q(J, \widehat{W}^{1,q}).$$

Furthermore, we set

$$L(u,p) := \begin{pmatrix} \partial_t u - \Delta u + \nabla p \\ \operatorname{div} u \\ u|_{t=0} \end{pmatrix}, \quad (u,p) \in \mathbb{E}_T.$$

Now,
$$A \in M\mathbb{R}(L^q_\sigma) \Rightarrow L \in \text{ Isom}(\mathbb{E}_T, \mathbb{F}_T).$$
Next, for $(f, u_0) \in \mathbb{F}_T$ we set
$$(u^*, p^*) := L^{-1}(f, 0, u_0) \in \mathbb{E}_T.$$
and $F(u) := -(u \cdot \nabla)u$. We rephrase the system (NSE) as
$$\begin{aligned} L(u,p) =& (f + F(u), 0, u_0) \\ \underset{\pi=p-p^*}{\overset{v=u-u^*}{\Longleftrightarrow}} L(v,\pi) =& (f + F(u), 0, u_0) - L(u^*, p^*) \\ =& (F(v + u^*), 0, 0) =: H_0(v, \pi). \end{aligned}$$
Thus, our fixed point problem reads as follows:
$$(v, \pi) = L^{-1} H_0(v, \pi), \quad (v, \pi) \in {}_0\mathbb{E}_T,$$
where
$${}_0\mathbb{E}_T := \{(u,p) \in \mathbb{E}_T : u|_{t=0} = 0\}.$$
Observe that the reduction to a fixed point problem in a space with zero time trace at $t = 0$ has the following two advantages:

- The norm $||L^{-1}||_{\mathcal{L}(\mathbb{F}_T, {}_0\mathbb{E}_T)}$ is uniformly bounded in $T \in [0, T_0]$ for $T_0 > 0$ fixed.
- The constant of the Sobolev embedding $W^{1,q}_0(J, X) \hookrightarrow BUC(J, X)$ is independent of $T \leq T_0$.

Assume that $p > n + 2$. By a result of Amann (see [1]) we know that
$$\mathbb{E}^1_T \hookrightarrow BUC(J, W^{2-1/q}_q) \overset{\text{Sobolev}}{\hookrightarrow} BUC(J, BUC^1).$$
Note that the embedding constants of the above two embeddings can be chosen uniformly in $T \in [0, T_0]$, if we assume that time trace is zero at $t = 0$, i.e. if we replace $\mathbb{E}^1_T$ by ${}_0\mathbb{E}^1_T$. This implies that
$$\begin{aligned} ||(u \cdot \nabla)w||_{\mathbb{F}^1_T} &\leq ||u||_{L^\infty(J \times \mathbb{R}^n)} ||\nabla w||_{\mathbb{F}^1_T} \\ &\leq C||u||_{\mathbb{E}^1_T} ||w||_{\mathbb{E}^1_T}, \quad u, w \in \mathbb{E}^1_T. \end{aligned}$$
Next, we calculate the Frechét derivative of F, that is
$$DF(v + u^*)[\overline{v}] = [(v + u^*) \cdot \nabla]\overline{v} + [\overline{v} \cdot \nabla](v + u^*).$$
Now, let $(v, \pi) \in {}_0\mathbb{B}_T(r) = \{(v, p) \in {}_0\mathbb{E}_T : ||(u, p)||_{\mathbb{E}_T} < r\}$ and $(\overline{v}, \overline{\pi}) \in {}_0\mathbb{E}_T$. Then, by the discussion above, we obtain for the Frechét derivative of H_0 that
$$||DH_0(v, \pi)[\overline{v}, \overline{\pi}]||_{\mathbb{F}_T} \leq ||DF(v + u^*)[\overline{v}]||_{\mathbb{F}^1_T}$$

$$
\begin{aligned}
&\leq ||(v\cdot\nabla)\overline{v}||_{\mathbb{F}^1_T} + ||(u^*\cdot\nabla)\overline{v}||_{\mathbb{F}^1_T} + ||(\overline{v}\cdot\nabla)v||_{\mathbb{F}^1_T} + ||(\overline{v}\cdot\nabla)u^*||_{\mathbb{F}^1_T} \\
&\leq ||v||_{L^\infty(J\times\mathbb{R}^n)}||(\overline{v},\overline{\pi})||_{{}_0\mathbb{E}_T} + ||u^*||_{\mathbb{F}^1_T}||(\overline{v},\overline{\pi})||_{{}_0\mathbb{E}_T} \\
&\quad + ||\overline{v}||_{L^\infty(J\times\mathbb{R}^n)}||v,\pi||_{{}_0\mathbb{E}_T} + ||\overline{v}||_{L^\infty(J\times\mathbb{R}^n)}||u^*||_{\mathbb{F}^1_T} \\
&\leq C\Bigg(\underbrace{||(v,\pi)||_{{}_0\mathbb{E}_T}}_{\leq\frac{1}{8C||L^{-1}||}\text{ for small } r} + \underbrace{||u^*||_{\mathbb{F}^1_T}}_{\leq\frac{1}{8C||L^{-1}||}\text{ for small } T} \Bigg)||(\overline{v},\overline{\pi})||_{{}_0\mathbb{E}_T},
\end{aligned}
$$

where $\|L^{-1}\| := \|L^{-1}\|_{\mathcal{L}(\mathbb{F},{}_0\mathbb{E}_T)}$. This implies that

$$
||DH_0(v,\pi)||_{\mathcal{L}({}_0\mathbb{E}_T,\mathbb{F}_T)} \leq \frac{1}{4||L^{-1}||} \tag{12}
$$

for sufficiently small $r, T > 0$. Applying the mean value theorem we therefore get that

$$
\begin{aligned}
||L^{-1}H_0(v,\pi)||_{{}_0\mathbb{E}_T} &\leq ||L^{-1}||(||H_0(v,\pi) - H_0(0,0)||_{\mathbb{F}_T}) + ||H_0(0,0)||_{\mathbb{F}_T} \\
&\leq \frac{1}{4}||(v,\pi)||_{{}_0\mathbb{E}_T} + \underbrace{||(u^*\cdot\nabla)u^*||_{\mathbb{F}^1_T}}_{\leq\frac{r}{4}\text{ for } T \text{ small enough}} \leq \frac{r}{2}.
\end{aligned}
$$

In other words we can find $r_0, T_0 > 0$ such that $L^{-1}H_0({}_0\mathbb{B}_{T_0}(r_0)) \subseteq {}_0\mathbb{B}_{T_0}(r_0)$. Furthemore, we also obtain that

$$
\begin{aligned}
&||L^{-1}H_0(v_1,\pi_1) - L^{-1}H_0(v_2,\pi_2)||_{{}_0\mathbb{E}_T} \\
&\leq ||L^{-1}||\ ||H_0(v_1,\pi_1) - H_0(v_2,\pi_2)||_{\mathbb{F}_T} \leq \frac{1}{4}, \quad (v_1,\pi_1),(v_2,\pi_2) \in {}_0\mathbb{B}_{T_0}(r_0)
\end{aligned}
$$

The contraction mapping principle then implies the existence of a unique fixed point $(v,\pi) \in {}_0\mathbb{B}_{T_0}(r_0)$. Thus, $(u,p) = (v+u^*, \pi+p^*)$ solves (NSE).

Hence we have proved the following result.

Theorem 3.1. Let $q > n+2$, $T \in (0,\infty)$, and $J = (0,T)$. Suppose also that $f \in L^q(J, L^q_\sigma)$ and $u_0 \in (L^q_\sigma, D(A))_{1-1/q,q}$. Then there exists a $T_0 > 0$ and a unique (strong) solution (u,p) of (NSE), such that

$$
u \in W^{1,q}(I, L^q_\sigma) \cap L^q(I, D(A)),
$$

$$
p \in L^q(I, \widehat{W}^{1,q}) \quad (\text{here } I = (0,T_0)).
$$

Remark 3.2.

(1) $q > n+2$ can be improved to $q > \frac{n+2}{3}$.

(2) Basically, step 3 applies to all $\Omega \subseteq \mathbb{R}^n$ such that $A \in MR(L^q_\sigma(\Omega))$.

Step 4: Analyticity :

Let (u,p) the unique solution constructed in step 3 and $f \in \mathbb{F}_T^1$ real analytic in t and x. In order to prove analyticity we employ a "parametertrick", i.e. we introduce

$$u_{\lambda,\mu}(t,x) := \tau_{\lambda,\mu}u(t,x) := u(\lambda t, x+t\mu),$$
$$p_{\lambda,\mu}(t,x) := \tau_{\lambda,\mu}p(t,x) := p(\lambda t, x+t\mu),$$

for $(\lambda,\mu) \in (1-\delta, 1+\delta)\times\mathbb{R}^n$ and show that the dependence on (λ,μ) is analytic. (Idea goes back to Masuda [17], Angenent.) To apply this parametertrick to nonlinear PDE's we need the following.

Basic ingredients:

(1) maximal regularity for the linearization,

(2) $\tau_{\lambda,\mu}F = F\tau_{\lambda,\mu}$, $F \in C^w(\mathbb{G}_T, \mathbb{E}_T^1)$, for certain $\mathbb{G}_T \subseteq \mathbb{E}_T$ and the nonlinearity F (here $\mathbb{G}_T = {}_0\mathbb{B}_T(r) + (u^*, p^*)$),

(3) the implicit function theorem.

For simplicity assume $u_0 = 0$. The couple $(u_{\lambda,\mu}, p_{\lambda,\mu})$ satisfies

$$\begin{cases} \partial_t u_{\lambda,\mu} - \lambda\Delta u_{\lambda,\mu} + \lambda\nabla p_{\lambda,\mu} &= \lambda f_{\lambda,\mu} - \lambda F(u_{\lambda,\mu}) + < \mu|\nabla u > \\ \mathrm{div} u_{\lambda,\mu} &= 0, \\ u_{\lambda,\mu}|_{t=0} &= 0. \end{cases}$$

The "implicit function" is then defined by

$$\Psi((v,\pi)(\lambda,\mu)) := \begin{pmatrix} \partial_t v - \lambda\Delta v + \lambda\nabla\pi - \lambda f_{\lambda,\mu} - \lambda F(v) + < \mu|\nabla v > \\ \mathrm{div}\ v \\ v|_{t=0} \end{pmatrix}$$

Lemma (exercise) Let $T < T_0$. There exists a neighborhood $\Lambda \subseteq (1-\delta, 1+\delta)\times\mathbb{R}^n$ of $(1,0)$ s.t.

(i) $||f_{\lambda,\mu}||_{\mathbb{F}_T^1} \leq 2||f||_{\mathbb{F}_{T_0}^1}, \quad (\lambda,\mu) \in \Lambda,$

(ii) $||u_{\lambda,\mu}||_{{}_0\mathbb{E}_T^1} \leq 2||u||_{{}_0\mathbb{E}_{T_0}^1}, \quad (\lambda,\mu) \in \Lambda,$

(iii) $||p_{\lambda,\mu}||_{\mathbb{E}_T^2} \leq 2||p||_{\mathbb{E}_{T_0}^2}, \quad (\lambda,\mu) \in \Lambda,$

This implies that for $r < \frac{r_0}{2}$ the function

$$\Psi : {}_0\mathbb{B}_T(r) \times \Lambda \to \mathbb{F}_T$$

is well-defined. Next, let $D_1\Psi$ be the Frechét derivative with respect to (v,π). Then

$$D_1\Psi((u,p),(\lambda,\mu))[\overline{v},\overline{\pi}] = L[\overline{v},\overline{\pi}] - DH_0(u,p)[\overline{v},\overline{\pi}].$$

The maximal regularity now yields that

$$L \in \text{ Isom } ({}_0\mathbb{E}_T, \mathbb{F}_T)$$

From step 3 we know that

$$||DH_0(u,p)||_{\mathcal{L}({}_0\mathbb{E}_T,\mathbb{F}_T)} \le \frac{1}{2}, \quad (u,p) \in {}_0\mathbb{B}_T(r).$$

Thus $DH_0(u,p)$ can be regarded as a small perturbation of L. By a standard Neumann series argument we therefore obtain that

$$D_1\Psi((u,p),(1,0)) \in \text{ Isom } ({}_0\mathbb{E}_T, \mathbb{F}_T).$$

Clearly, we also have that $\Psi((u,p),(1,0)) = 0$. The implicit function theorem now implies the existence of a neighborhood $U \subseteq \Lambda$ of $(1,0)$ and a neighborhood $V \subseteq {}_0\mathbb{E}_T$ of (u,p) and a function

$$(g_1, g_2) : U \to V, \quad (\lambda, \mu) \mapsto (g_1(\lambda,\mu), g_2(\lambda,\mu))$$

such that

$$\Psi((g_1,g_2),(\lambda,\mu)) = 0.$$

The uniqueness of (u,p) implies that $(g_1, g_2) = (u_{\lambda,\mu}, p_{\lambda,\mu})$. Moreover, from $F \in C^w({}_0\mathbb{E}_T, \mathbb{F}_T)$ we deduce $\Psi \in C^w({}_0\mathbb{B}_T(r) \times \mathit{\Lambda}, \mathbb{F}_T)$. Hence we also have that

$$((\lambda,\mu) \mapsto (u_{\lambda,\mu}, p_{\lambda,\mu})) \in C^w(\mathit{\Lambda}, {}_0\mathbb{E}_T).$$

In view of

$${}_0\mathbb{E}_T \hookrightarrow (BUC(J \times \mathbb{R}^n))^n \times C(J \times \mathbb{R}^n)$$

we may fix $(t_0, x_0) \in J \times \mathbb{R}^n$ and obtain

$$[(\lambda,\mu) \mapsto (u(\lambda t_0, x_0 + t_0\mu), p(\lambda t_0, x_0 + t_0\mu))] \in C^w(\mathit{\Lambda}, \mathbb{R}^{n+1}).$$

This finally results in

$$(u,p) \in C^w(J \times \mathbb{R}^n, \mathbb{R}^{n+1}),$$

i.e. u, p are real analytic functions. **q.e.d.**

Remark 3.3.

(1) For the general case with $u_0 \neq 0$ one can employ the splitting $(u,p) = (v,\pi) + (u^*, p^*)$ and prove first that $(u^*, p^*) \in C^w(J \times \mathbb{R}^n, \mathbb{R}^{n+1})$. This in turn can be done by the same method, i.e. by applying the parametertrick to (SE).

(2) Since the mean curvature flow (MC) (see Chapter I) has the same linearization as (NSE) we already have maximal regularity for (MC). Therefore, by applying step 3 and 4 to (MC) one can obtain similar results as for (NSE), i.e. local-in-time existence and analyticity of solutions.

IV More on applications of R-boundedness and H^∞-calculus

Once again : The crucial step in the approach to nonlinear PDE's presented in Chapter III is:

"Verification of maximal regularity for the linearized system".

This requires optimal mapping properties (regularity) of the related "principal symbol".

Example (Chapter III): The heat equation in $\mathbb{R}^n$ as a linearization of (NSE) and (MC). Principal symbol:

$$m(\lambda, \xi) = (\lambda + |\xi|^2)^{-1}$$

Formally we have

$$(\partial_t - \Delta)^{-1} f = \mathcal{F}^{-1}\mathcal{L}^{-1} m \mathcal{L}\mathcal{F} f,$$

where $\mathcal{L}$, $\mathcal{F}$ denote Laplace and Fourier transform respectively. Then by $\Delta \in \mathcal{H}^\infty(L^q(\mathbb{R}^n))+$ and the Dore and Venni Theorem (Theorem 2.9) we obtain that

$$(\partial_t - \Delta)^{-1} \in \mathcal{L}(L^q, W^{1,q}(J, L^q) \cap L^q(J, D(-\Delta))).$$

However, if, for instance, $\Omega \subseteq \mathbb{R}^n$ has a boundary, the principal symbol i.g. has a more complicated structure (i.g. it is no sum).

Discussion of some examples: [No explicit derivation, just discussion of the principal symbol]

4.1. Stefan problem with surface tension.

The classical situation of the two-phase Stefan problem is a melting ice cube in water. Mathematically this situation is modeled by

$$(SP)\begin{cases} u_t - c\Delta u &= 0 \quad \text{in } \bigcup_{t>0}(\{t\} \cup \Omega(t)), \\ u^\pm &= \sigma\kappa \quad \text{on } \bigcup_{t>0}(\{t\} \cup \partial\Omega(t)), \\ c\partial_\nu u^+ - c\partial_\nu u^- &= V_\nu \quad \text{on } \bigcup_{t>0}(\{t\} \cup \partial\Omega(t)), \\ u|_{t=0} &= u_0 \quad \text{in } \Omega(0), \\ \Gamma|_{t=0} &= \Gamma_0. \end{cases}$$

Here we have $\Omega = \Omega^+ \cup \Omega^-$ and

σ: surface tension coefficient,

κ: mean curvature,

V_ν: normal velocity of $\Gamma(t)$,

$u^\pm$: temperature phases,

Γ: (Free) interface,

$c > 0$: diffusion coefficient.

By a localization procedure these equations can be reduced to a quasilinear problem on

$$(0,\infty) \times \underbrace{\mathbb{R}^n \times \mathbb{R}\backslash\{0\}}_{=:\dot{\mathbb{R}}^{n+1}}$$

A suitable linearization of this quasilinear problem reads as

$$(LSP) \begin{cases} u_t - c\Delta u &= f \quad \text{in } (0,\infty) \times \dot{\mathbb{R}}^{n+1}, \\ u^\pm + \Delta\rho &= g \quad \text{on } (0,\infty) \times \mathbb{R}^n, \\ \rho_t + c\partial_y u^+ - c\partial_y u^- &= h \quad \text{on } (0,\infty) \times \mathbb{R}^n, \\ u|_{t=0} &= u_0 \quad \text{in } \dot{\mathbb{R}}^{n+1}, \\ \rho|_{t=0} &= \rho_0 \quad \text{in } \mathbb{R}^n, \end{cases}$$

where ρ is a function describing the motion of the free interface, i.e. $\Gamma = \text{graph}(\rho)$ (see Escher, Prüß, Simonett '03 [9] for more details). Since u solves a heat equation, the problem is essentially solved if sufficient regularity for ρ is proved. Now, by applying Fourier and Laplace transform we can obtain the following explicit representation:

$$\mathcal{LF}\rho(\lambda,\xi) = \underbrace{\frac{1}{\lambda + \sqrt{c}|\xi|^2\sqrt{\lambda + c|\xi|^2}}}_{=:m(\lambda,|\xi|^2)\ \text{“principal symbol”}} \mathcal{LF}h(\lambda,\xi).$$

<u>Observation:</u> Here we have no “sum” structure as for the symbol $(\lambda + |\xi|^2)^{-1}$ of the heat equation. Therefore Theorem 2.8 or Theorem 2.9 can not be applied directly. Here we need a corresponding result for a more general class of symbols.

First let us determine the spaces of regularity for h and ρ. Since it solves a heat equation, the space for u is

$${}_0H^1(J, L^q(\dot{\mathbb{R}}^{n+1})) \cap L^q(J, D(-\Delta)).$$

By trace theory we therefore have that

$$\partial_y u^\pm|_{\partial\mathbb{R}^{n+1}_\pm} \in {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J, L^q) \cap L^q(J, W_q^{1-1/q}) =: \mathbb{F}.$$

Since this should also be the space for the right hand side h, we obtain by the equations as desired space for the free interface ρ,

$${}_0W_q^{3/2-1/2_q}(J, L^q) \cap {}_0W_q^{1-1/q}(J, H^2) \cap L^q(J, W^{4-1/q}) =: \mathbb{E}.$$

Now let $Gu := \partial_t u$ be defined on

$$D(G) = {}_0W_q^{3/2-1/2q}(J, L^q) \cap {}_0H_q^1(J, W_q^{1-1/q})$$

and $Bu = -c\Delta u$ be defined on

$$D(B) = {}_0W^{1/2-1/2q}(J, H_q^2) \cap L^q(J, W_q^{3-1/q})$$

Formally, we have to show that

$$\begin{aligned} & m(G,B) = \mathcal{F}^{-1}\mathcal{L}^{-1}[m(\lambda,|\xi|^2)]\mathcal{F}\mathcal{L} \in \mathcal{L}(\mathbb{F},\mathbb{E}) \\ \underset{[9]}{\overset{\text{exercise}}{\Leftrightarrow}} \quad & Gm(G,B), (G+B)^{1/2}m(G,B) \in \mathcal{L}(\mathbb{F},\mathbb{F}). \end{aligned} \tag{13}$$

To this end note that it is well known that

$$G, B \in \mathcal{H}^\infty(\mathbb{F}), \quad \phi_G^\infty = \frac{\pi}{2}, \quad \phi_B^\infty = 0.$$

Lemma 4.1. (Exercise) Let $\varphi_0 \in (0, \frac{\pi}{2}), \varphi \in (0, \frac{\varphi_0}{2})$. Then there exists a $C = C(\varphi_0, \varphi) > 0$ such that

$$||\lambda m(\lambda,z)||_{L^\infty(\Sigma_{\pi-\varphi_0}\times\Sigma_\varphi)} + ||(\lambda+z)^{1/2}m(\lambda,z)||_{L^\infty(\Sigma_{\pi-\varphi_0}\times\Sigma_\varphi)} \leq C.$$

Now, in view of $B \in \mathcal{H}^\infty(\mathbb{F})$, Lemma 4.1 implies that

$$||\lambda m(\lambda,B)||_{\mathcal{L}(\mathbb{F},\mathbb{F})} + ||(\lambda+B)^{1/2}m(\lambda,B)||_{\mathcal{L}(\mathbb{F},\mathbb{F})} < C, \quad \lambda \in \Sigma_{\pi-\varphi_0}.$$

In order to obtain (13), would like to insert G for λ.

But: Corresponding to Problem 2.2 in Chapter I the uniform boundedness is in general not enough. Also here we need the $\mathcal{R}$-boundedness.

Theorem 4.2. (Kalton, Weis '03 [13]) Let X be a Banach space of class $\mathcal{HT}$ and $A \in \mathcal{H}^\infty(X)$. Let $F : \Sigma_\phi \to \mathcal{L}(X)$ such that

(1) $\{F(\lambda) : \lambda \in \Sigma_\phi\}$ is $\mathcal{R}$-bounded.

(2) $F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1}F(\lambda), \quad \lambda \in \Sigma_\phi,\ \mu \in \rho(A)$.

If $\phi_A^\infty < \phi$, then

$$||F(A)||_{\mathcal{L}(X)} \leq C(\phi, \phi_A^\infty)\mathcal{R}\{F(\lambda) : \lambda \in \Sigma_\phi\}.$$

It is well known that $B \in \mathcal{RH}^\infty(\mathbb{F})$, i.e. the operator B admits an $\mathcal{R}$-bounded H^∞-calculus that is $B \in \mathcal{H}^\infty(\mathbb{F})$ and the set $\{h(B) : h \in H^\infty(\Sigma_\phi), ||h||_\infty \leq C\}$ is $\mathcal{R}$-bounded for $\phi > \phi_B^\infty$ and some $C > 0$. This implies that

$$\mathcal{R}\{\lambda m(\lambda,B) : \lambda \in \Sigma_{\pi-\varphi_0}\}, \mathcal{R}\{(\lambda+B)^{1/2}m(\lambda,B) : \lambda \in \Sigma_{\pi-\varphi_0}\} < \infty.$$

Thus, by an application of Theorem 4.2 we deduce (13). This finally results in

$$(h \mapsto \rho) \in \mathcal{L}(\mathbb{F},\mathbb{E}).$$

Based on this fact one can prove maximal regularity for (LSP) and similar to Chapter III the existence of a local-in-time analytic solution (u, Γ) for (SP) (see [9]).

4.2. Other examples

In a similar way as described in Section 4.1 more complicated problems and the related principal symbols can be handled.

Example 2. Stefan problem with two different diffusion coefficients

(This is for example the case in the water-ice situation)
Principle symbol:

$$m(\lambda, |\xi|^2) = \frac{1}{\lambda + \sigma\sqrt{c_+}|\xi|^2\sqrt{\lambda + c_+|\xi|^2} + \sigma\sqrt{c_-}|\xi|^2\sqrt{\lambda + c_-|\xi|^2}}$$

Example 3. Model problem associated to the spin coating process. (A process used in the production of electronic semiconductors).

Principle symbol:

$$m(\lambda, |\xi|^2) = \frac{1}{\lambda\left(\omega(\lambda,\xi)^{3/2} + \lambda|\xi| + 3\omega(\lambda,\xi)|\xi|^3\right) + \left(\omega(\lambda,\xi) + |\xi|\right)|\xi|^3}.$$

where $\omega(\lambda, \xi) := \sqrt{\lambda + |\xi|^2}$.

Bibliography

[1] H. Amann. *Linear and Quasilinear Parabolic Problems. Vol. I.* Birkhäuser, 1995.

[2] E. Berkson and T.A. Gillespie. Spectral decomposition and harmonic analysis on UMD spaces. *Studia Math.*, 112:13–49, 1994.

[3] W. Borchers and T. Miyakawa. L^2 decay for the Navier-Stokes flow in halfspaces. *Math. Ann.*, 282:139–155, 1988.

[4] J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Arkiv Math.*, 21:163–168, 1983.

[5] J. Bourgain. Vector-valued singular integrals and the H^1-BMO duality. In *Probability theory and harmonic analysis*, volume 98 of *Pure Appl. Math.*, pages 1–19. Marcel Dekker, 1986.

[6] M. Cowling, I. Doust, A. McIntosh, and A. Yagi. Banach space operators with a bounded H^∞ functional calculus. *J. Austral. Math. Soc.*, 60:51–89, 1996.

[7] R. Denk, M. Hieber, and J. Prüss. R-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166:viii+114, 2003.

[8] G. Dore and A. Venni. On the closedness of the sum of two closed operators. *Math. Z.*, 196:189–201, 1987.

[9] J. Escher, J. Prüß, and G. Simonett. Analytic solutions for a Stefan problem with Gibbs-Thomson correction. *J. reine angew. Math.*, 563:1–52, 2003.

[10] A. Friedman. *Partial Differential Equations.* Holt, Rinehart and Winston, 1969.

[11] Y. Giga and H. Sohr. Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. *J. Funct. Anal.*, 102:72–94, 1991.

[12] R. Haller, H. Heck, and A. Noll. Mikhlin's theorem for operator-valued Fourier multipliers in n variables. *Math. Nachr.*, 244:110–130, 2002.

[13] N. Kalton and L. Weis. The H^∞-calculus and sums of closed operators. *Math. Ann.*, 321:319–345, 2001.

[14] N.J. Kalton and G. Lancien. A solution to the problem of L^p-maximal regularity. *Math. Z.*, 235(3):559–568, 2000.

[15] F. Lancien, G. Lancien, and C. Le Merdy. A joint functional calculus for sectorial operators with commuting resolvents. *Proc. Lond. Math. Soc., III*, 77(2):387–414, 1998.

[16] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1995.

[17] K. Masuda. On the regularity of solutions of the nonstationary Navier-Stokes equations. *Springer Lecture Notes in Mathematics*, 771:360–370, 1980.

[18] A. McIntosh. Operators which have an H^∞-calculus. *Proc. Centre Math. Analysis*, 14:210–231, 1986.

[19] A. Noll and J. Saal. H^∞-calculus for the Stokes operator on L_q-spaces. *Math.Z.*, 244:651–688, 2003.

[20] J. Prüss and H. Sohr. Imaginary powers of elliptic second order differential operators in L^p-spaces. *Hiroshima Math. J.*, 23:161–192, 1993.

[21] J. Saal. Stokes and Navier-Stokes equations with Robin boundary conditions in a half-space. *J. Math. Fluid Mech.*, 8:211–241, 2006.

[22] J. Schwartz. A remark on inequalities of Calderon-Zygmund type for vector-valued functions. *Comm. Pure Appl. Math.*, 14:785–799, 1961.

[23] P.E. Sobolevskii. Coercivness inequalities for abstract parabolic equations. *Soviet Math. (Doklady)*, 5:894–897, 1964.

[24] Ž. Štrkalj and L. Weis. On operator-valued Fourier multiplier theorems. Transactions AMS, to appear.

[25] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* North Holland, 1978.

[26] L. Weis. Operator-valued multiplier theorems and maximal L^p-regularity. *Math. Ann.*, 319(4):735–758, 2001.