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Hyperbolic flows

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Anatole Katok, in memoriam

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CHAPTER 0

Introduction

This book presents the theory of flows, that is, continuous-time dynamical systems, with an emphasis on the theory of (uniformly) hyperbolic dynamics for flows. It serves both as an introduction as well as an exposition of recent developments in uniformly hyperbolic dynamics.

While the study of flows historically predates that of discrete-time systems, the literature tends to develop the theory of dynamical systems primarily in the context of discrete-time systems and leaves it to the reader (or unaddressed) to transfer those insights to flows. It is thus often implicit that “things work analogously for flows,” or that “this is different for flows.” This book fills a gap in the introductory literature by giving a “flows-first” introduction to dynamical systems and focusing on continuous-time systems, rather than treating these as afterthoughts or exceptions to methods and theory developed for discrete-time systems.

Even the introductory parts of this text have distinctive features beyond the fact that flows are the subject. Chapter 5 is to our knowledge unique in the literature for the extent to which it implements the Anosov–Katok–Bowen program of developing the dynamical features of hyperbolic sets from shadowing alone.[1] While it is satisfying in itself to see this implemented, it does seem particularly timely as well because recent work of Climenhaga, Thompson and collaborators has put the Bowen approach back in the enter of smooth ergodic theory, and to great effect. We are also happy to provide the reader with a range of examples of hyperbolic flows, of which several are quite recent discoveries. Chapter 5 may furthermore be the first account to provide a proper natural definition of a (uniformly) hyperbolic flow (Definition 5.3.48) based on the equivalence of the 3 popular notions (Theorem 5.3.45 on page 269), which, although not in itself new, does not seem to be as well known as it should. We also call attention to the very end of the book, where Section 12.7 reproduces a clean proof by Abdenur and Viana of absolute continuity of the invariant foliations in the greatest generality for partially hyperbolic dynamical

[1] Specifically, the Shadowing Lemma and the Shadowing Theorem, which include uniqueness, so in terms of customary usage one should say that shadowing and expansivity produce the insights in Chapter 5.

systems. This exceeds what we need but seemed like a most desirable addition to the literature.

As complements to this introductory material the reader may also enjoy a brisk introduction in a similar spirit that focuses on discrete time [**147**], the rather larger book [**181**], and the more example-driven text [**149**].

The second half of the book, from Chapter 7 onwards,[2] includes a range of advanced topics in uniformly hyperbolic dynamics with a focus on the topology and dynamics of Anosov flows and a number of topics in recent research that for the most part have not appeared in any expository literature. These topics are no less accessible than the introductory subjects, but here we take even more opportunities to augment the results we prove with complements whose proofs we do not include, and in Chapters 9 and 10 we more frequently take the liberty of providing outlines of proofs rather than full proofs. This is meant to provide not only a substantial introduction to these subjects with proofs, but further vistas to form a more complete panorama.

There is much to acknowledge that has significantly helped us write this book. We owe a debt to students from Tufts University, Brandeis University, the University of Tokyo and the ***ETH*** Zürich for their forbearance, support and criticism,[3] to colleagues and students who commented helpfully on book drafts from afar, to Manfred Einsiedler and Michael Struwe for arranging the Nachdiplom Lectures at the ***ETH***, and to Takashi Tsuboi and Masahiko Kanai for arranging lectures on hyperbolic flows at the University of Tokyo. It seems highly appropriate and satisfying that thereby the second author was at the last stages of writing a department colleague of Masahiko Kanai, whose work was foundational for substantial parts of the rigidity theory described near the end of the book, as well as of Shuhei Hayashi, who with his proof of the stability theorems for hyperbolic flows placed one of the crowning glories atop hyperbolic dynamics in the 20th century.

Some of the writing in this book owes to earlier books and research articles by one or the other of us, which included text we deemed—in more or less adapted form—to be an excellent fit for this work. This implies a debt to our respective coauthors of such prior works, Anatole Katok notably among them. In some cases, original research papers by others still remain the best exposition of ideas we could not omit from this book, so it will be apparent and often explicit where we followed their ideas; Bowen foremost comes to mind. And occasionally, unpublished lecture notes (such as by Lanford at the ***ETH***) provided the most elegant proofs we know of a needed fact.

[2] Actually, from Section 6.5.

[3] In the Talmud, R. Chanina remarked, ”I have learned much from my teachers, more from my colleagues, and the most from my students” (Ta’anis 7a).

1. Purpose and scope

The book is divided into two parts. The first of these develops the general theory of flows. The second part is about hyperbolicity and includes an introduction as well as a panorama of current topics. This book is self-contained in the technical sense, that is, it includes definitions of all dynamics concepts with which we work, but without any pretense to being comprehensive with introductory material.

It has been written in a way that it can be adapted to a course in a number of different ways depending on the purpose of the course. Starred chapters and starred sections are not necessarily "harder," but they are optional, and the material is not necessary for further sections except for an occasional result that can be used as a black box. Much of this material is hard to find in the literature except for original sources.

The core chapters are Chapter 1, Chapter 5, and Chapter 6. If one wants to emphasize ergodic properties of flows then one could include Chapter 3, Chapter 4, and Chapter 8; or at least portions of them. For a more topological or geometric course one would instead include Chapter 2, and portions from either Chapter 9 and/or Chapter 10. However, there are sections in Chapter 10 that invoke some ergodic theory. The core chapters include exercises.

The appendices contain material on maps that are helpful for certain sections. For those already familiar with the theory for maps this can be omitted or quickly covered. For those not familiar with the discrete case it will be necessary to cover the needed parts of these to understand either the material on ergodic theory in Chapter 3 or the material on invariant foliations in Chapter 6.

To give our selection of flows versus discrete-time systems some context, we describe a few connections between these. Historically, dynamical systems were flows, such as those that arise from differential equations that describe a mechanical system. Poincaré is widely regarded as the founder of the discipline of dynamical systems as we know it, and among the wealth of notions he created is that of a local section, known also as a Poincaré section. This arose in the context of periodic orbits (trajectories) of a continuous-time dynamical system as anchors to study other motions in the system. Such a nearby motion will track the periodic motion for possibly considerable amounts of time, and it is often of less interest whether it lags or leads a little as to how it moves closer to or further from the periodic orbit. To focus on these transverse phenomena Poincaré considered a small hypersurface perpendicular to the periodic orbit on which he could track successive "hits" by a nearby motion. This defines a map on this disk, called the Poincaré (first) return map, see Figure 0.1.1 This is an early way in which discrete-time dynamical systems arose.

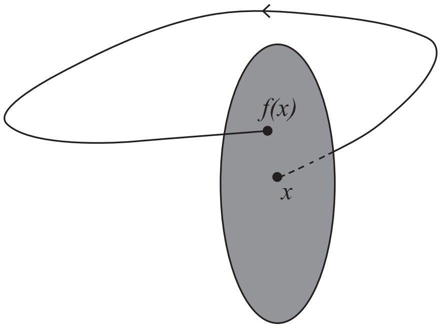


FIGURE 0.1.1. Poincaré section and map

Coming from a different direction, billiard systems illustrate how a similar approach works both naturally and globally. A mathematical billiard system idealizes physical billiards by ignoring the spin and rolling of the balls: a point particle moves along straight lines and is reflected in the boundary with incoming angle equal to outgoing angle. This makes them more like air hockey or a description of light in a mirrored room. (And tables of shapes other than rectangular are of considerable interest.) These are naturally continuous-time systems, but they come with natural discrete moments in time: the moments in which collisions occur. Indeed, all information about the evolution of such a system is contained in the locations and velocities of all balls at the moment of a collision, because this determines the motion until the next collision and the positions and velocities at that subsequent moment. Therefore, the dynamics can be described as a map on

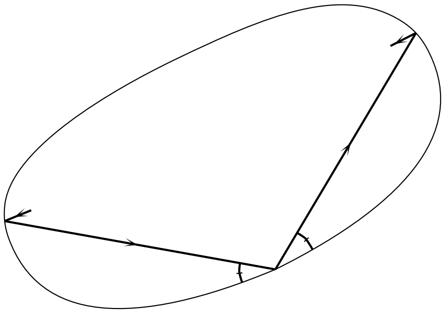


FIGURE 0.1.2. Billiard

the "collision space" that sends each collision configuration to the next one. Once again, a discrete-time system describes the dynamics of a continuous-time system.

This latter process can be reversed: Given the discrete-time system, one more piece of information reconstructs the flow entirely: the "return time" from one collision to the next. We call this assembly of a map and a return-time function a suspension if the return time is constant (Definition 1.2.4), and a special flow or flow under a function otherwise (Definition 1.2.7).

There are also aspects of dynamics in which pronounced differences between flows and discrete-time systems are manifested. On one hand, this occurs when "longitudinal" effects matter, that is, when time-changes make a difference. In the case of a special flow this amounts to properties that are affected by the choice of "roof" or return-time function versus those that are not. For instance, the existence of a dense orbit is unaffected by the choice of roof function, but whether all periodic orbits are commensurate (their periods are various multiples of one positive number) clearly does depend on return times. Another notable feature of flows is that they permit surgery constructions to construct new flows. Accordingly, such a construction establishes that Anosov flows need not have a dense orbit (Section 9.3), but it is a long-open and exceedingly difficult problem to decide whether Anosov *diffeomorphisms* always have a dense orbit. In fact, it is not even known whether every Anosov diffeomorphism has a fixed point.

The theory of continuous-time dynamical systems does not directly reduce to that of discrete-time dynamical systems in the most obvious way: few diffeomorphisms arise as time-t maps of flows (Definition 1.1.1) since (every time-t map of) every flow is isotopic to the identity.[4] Also: time-t maps of flows have "roots" of all orders, being the n^{th} iterate of the time–t/n map. But one might say that a full continuous-time theory yields a full discrete-time theory because every diffeomorphism can be represented as a Poincaré section for some flow via the suspension/special-flow construction—provided one has a comprehensive understanding of the dynamics of a section in terms of that of the flow. This does not work in reverse because that construction is not unique, and many flows generate a given diffeomorphism, with confounding "longitudinal" effects as above.

More to the point in our context: for the study of hyperbolic flows (Chapter 5) it may be useful to know all about hyperbolic maps, but that theory does not apply to time-1 maps of (any) flows since those are never hyperbolic (unless the periodic points for the flow are all hyperbolic equilibria). More specifically, the time-t map

[4] One point of view from which flows produce a "sparse" set of maps of a given manifold is related to the mapping class group. For a manifold M the *mapping class group* is the set of isotopy-classes of homeomorphisms (or diffeomorphisms) of M. Flows are contained in the trivial equivalence class of the mapping class group.

of a hyperbolic flow satisfies a weaker condition called partial hyperbolicity due to the flow direction, in which neither contraction nor expansion occur. Thus, this flows-first book complements the existing literature emphasising discrete-time systems.

Once more, beyond the general theory, our emphasis is on uniformly hyperbolic dynamics. Neither partial nor nonuniform hyperbolicity are themselves subjects in this book. (The sole exception being the proof of absolute continuity of the invariant foliations for partially hyperbolic diffeomorphisms: while it is provided here to be applied to uniformly hyperbolic flows via time-1 maps, the proof covers partially hyperbolic diffeomorphisms in full generality.)

In short, discrete-time dynamics and continuous-time dynamics have closely related toolkits and close interactions, but the discrete-time focus of the existing literature leaves room for an explicit presentation of continuous-time dynamics.[5]

2. Historical sketch

We now outline some of the developments that brought about the theory of hyperbolic flows.[6] There are several intertwined strands of the history of hyperbolic dynamics: Geodesic flows and statistical mechanics on one hand and hyperbolic phenomena ultimately traceable to some application of dynamical systems. Geodesic flows were studied, for example, by Hadamard, Hedlund, Hopf (primarily either on surfaces or in the case of constant curvature) and Anosov–Sinai (negatively curved surfaces and higher-dimensional manifolds). Other hyperbolic phenomena appear in the work of Poincaré (homoclinic tangles in celestial mechanics [**242**]), Perron (differential equations [**233**]), Cartwright, Littlewood (relaxation oscillations in radio circuits [**81**,**82**,**202**]), Levinson (the van der Pol equation, [**201**]) and Smale (horseshoes, [**277**,**278**]), as well as countless others in recent history.

a. Homoclinic tangles and negative curvature. The advent of complicated dynamics took place in the context of Newtonian mechanics, according to which simple underlying rules governed the evolution of the world in clockwork fashion. The successes of classical and especially celestial mechanics in the 18th and 19th century were seemingly unlimited and Pierre Simon de Laplace felt justified in saying (in the opening passage he added to [**191**, p. 2]):

> Nous devons donc envisager l'état présent de l'univers, comme l'effet de son état antérieur, et comme la cause de celui qui va suivre. Une intelligence qui pour un instant donné, connaîtrait toutes les forces dont la nature est animée,

[5] To be clear, the *research* literature does not omit the continuous-time theory altogether, it is among *books* that this work occupies a unique place.

[6] An expanded version can be found in [**147**].

> et la situation respective des êtres qui la composent, si d'ailleurs elle était assez vaste pour soumettre ces données à l'analyse, embrasserait dans la même formule les mouvemens des plus grands corps de l'univers et ceux du plus léger atome: rien ne serait incertain pour elle, et l'avenir comme le passé, serait présent à ses yeux.[7]

The enthusiasm in this passage is understandable and its forceful description of (theoretical) determinism is a good anchor for an understanding of one of the basic aspects of dynamical systems. Moreover, the titanic life's work of Laplace in celestial mechanics earned him the right to make such bold pronouncements. Another bold pronouncement of his, that the solar system is stable, came under renewed scrutiny later in the 19th century, and Henri Poincaré was expected to win a competition to finally establish this fact. However, Poincaré came upon hyperbolic phenomena in revising his prize memoir [**242**] on the three-body problem. He found that homoclinic tangles (which he had initially overlooked) caused great difficulty and necessitated essentially a reversal of the main thrust of that memoir [**30**]. He perceived that there is a highly intricate web of invariant curves and that this situation produces dynamics of unprecedented complexity:

> Que l'on cherche à se représenter la figure formée par ces deux courbes et leurs intersections en nombre infini dont chacune correspond à une solution doublement asymptotique, ces intersections forment une sorte de treillis, de tissu, de réseau à mailles infiniment serrées; chacune des deux courbes ne doit jamais se recouper elle-même, mais elle doit se replier sur elle-même d'une manière trés complexe pour venir recouper une infinité de fois toutes les mailles du réseau. On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer.[8]

This is often viewed as the moment chaotic dynamics was first noticed. He concluded that in all likelihood the prize problem could not be solved as posed: To find series expansions for the motions of the bodies in the solar system that converge uniformly for all time. Indeed, when Birkhoff picked up the study of this situation

[7] We ought then to consider the present state of the universe as the effects of its previous state and as the cause of that which is to follow. An intelligence that, at a given instant, could comprehend all the forces by which nature is animated and the respective situation of the beings that make it up, if moreover it were vast enough to submit these data to analysis, would encompass in the same formula the movements of the greatest bodies of the universe and those of the lightest atoms. For such an intelligence nothing would be uncertain, and the future, like the past, would be open to its eyes.

[8] If one tries to imagine the figure formed by these two curves with an infinite number of intersections, each corresponding to a doubly asymptotic solution, these intersections form a kind of trellis, a fabric, a network of infinitely tight mesh; each of the two curves must not cross itself but it must fold on itself in a very complicated way to intersect all of the meshes of the fabric infinitely many times. One will be struck by the complexity of this picture, which I will not even attempt to draw

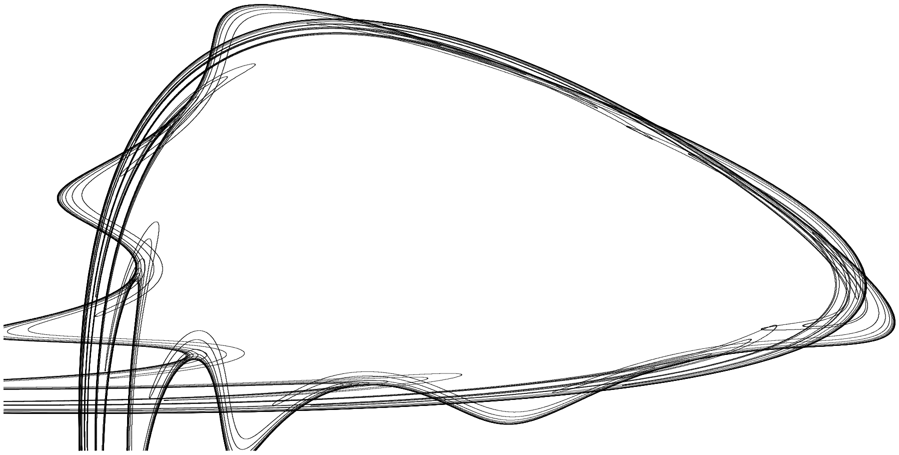


FIGURE 0.2.1. Homoclinic tangles [©Cambridge University Press, reprinted from [**181**] with permission]

in his prize memoir [**48**] for the Papal Academy of Sciences, he noted that and described how this implies complicated dynamics [**48**, p. 184].

b. Geodesic flows. A major class of mathematical examples motivating the development of hyperbolic dynamics is that of geodesic flows (that is, free-particle motion) of Riemannian manifolds of negative sectional curvature. Hadamard considered (noncompact) surfaces in $\mathbb{R}^3$ of negative curvature [**142**] and found, with apparent delight, that if the unbounded parts are "large" (do not pinch to arbitrarily small diameter as you go outward along them) then at any point the initial directions of bounded geodesics form a Cantor set. Since only countably many directions give geodesics that are periodic or asymptotic to a periodic one, this also proves the existence of more complicated bounded geodesics. Hadamard was fully aware of the connection to Cantor's work and to similar sets discovered by Poincaré, and he appreciated the relation between the complicated dynamics in the two contexts. Hadamard also showed that each homotopy class (except for the "waists" of cusps) contains a unique geodesic. Duhem [**109**] seized upon this to describe the dynamics of a geodesic flow in terms of what might now be called deterministic chaos: Duhem used it to illustrate that determinism in classical mechanics does not imply any practical long-term predictability.

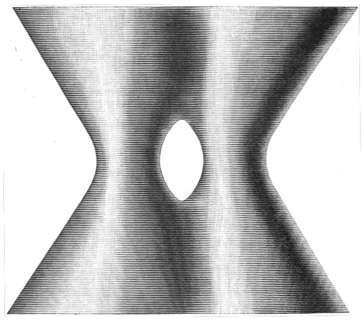


FIGURE 0.2.2. Negatively curved surface [Reproduced from Hadamard [**142**] ©1898 Elsevier Masson SAS. All rights reserved.]

Several authors trace the introduction of symbolic dynamics to the work of Hadamard on geodesic flows. Birkhoff is among them: In his proof of the Birkhoff–Smale Theorem (see Theorem 6.3.2) symbolic sequences appear (as well as a picture that resonates with Figure 6.3.2). It appears, however, that only in 1944 did symbol spaces begin to be seen as dynamical systems, rather than as a coding device [**91**].

c. Boltzmann's Fundamental Postulate. Well before Poincaré's work, James Clerk Maxwell (1831–1879) and Ludwig Boltzmann (1844–1906) had aimed to give a rigorous formulation of the kinetic theory of gases and statistical mechanics. A central ingredient was Boltzmann's Fundamental Postulate, which says that the time and space (phase or ensemble) averages of an observable (a function on the phase space) agree. Apparently because of a misstatement by Maxwell,[9] one often ascribes to him the so-called Ergodic Hypothesis:

The trajectory of the point representing the state of the system in phase space passes through every point on the constant-energy hypersurface of the phase space.

Poincaré and many physicists doubted its validity since no example satisfying it had been exhibited [**243**]. Accordingly, in 1912 Paul and Tatiana Ehrenfest [**112**] proposed the alternative Quasi-Ergodic Hypothesis:

The trajectory of the point representing the state of the system in phase space is dense on the constant energy hypersurface of the phase space.

[9] "…the system, if left to itself in its actual state of motion, will, sooner or later, pass through every phase…"

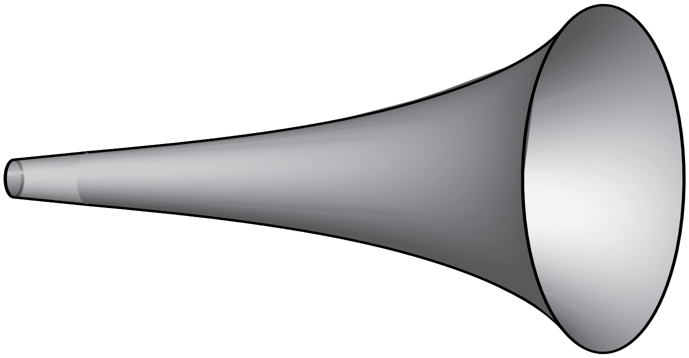


FIGURE 0.2.3. The pseudosphere [©Cambridge University Press, reprinted from [**181**] with permission]

Indeed, within a year proofs (by Rosenthal and Plancherel) appeared that the Ergodic Hypothesis fails [**238**, **259**]. (This is obvious today because a trajectory has measure zero in an energy surface.) These difficulties led to the search for *any* mechanical systems with this second property. The motion of a single free particle (that is, the geodesic flow) in a negatively curved space (beginning with the pseudosphere, Figure 0.2.3) emerged as the first and for a long time sole class of examples with this property. Within a decade, the understanding of the problem led to the pertinent contemporary notion, and this turned out to be probabilistic in nature.[10] The 1931 Birkhoff Ergodic Theorem (Theorem 3.2.16) ("time averages exist a.e.")[11] laid the foundation for the definition of ergodicity now in use, which is: "No proper invariant set has positive measure."[12]

If this is the case, then time averages agree with space averages—Boltzmann's Fundamental Postulate. Furthermore, almost every orbit is dense in the support of the measure.

The 1930s saw a flurry of work in which Artin's 1924 work on the modular surface was duly extended to other manifolds of *constant* negative curvature. For constant curvature, finite volume and finitely generated fundamental group the

[10] This serves to point out that the earlier quote by Laplace about determinism comes from his *Philosophical essay on probabilities*, where he goes on to say that we often do not have sufficiently detailed initial data, and must hence resort to a probabilistic approach. The motion of a molecule of air was a prominent instance he mentioned in that context.

[11] This was proved after the von Neumann Ergodic Theorem 3.2.4 but published earlier [**294**]—and the true foundational paper of ergodic theory is much more likely [**218**].

[12] These two combine to give the Strong Law of Large Numbers.

geodesic flow was shown to be topologically transitive [**186**, **208**], topologically mixing [**156**], ergodic [**161**], and mixing [**157**, **162**]. (In the case of infinitely generated fundamental group the geodesic flow may be topologically mixing without being ergodic [**267**]). If the curvature is allowed to vary between two negative constants then finite volume implies topological mixing [**134**] (see also [**137**, p. 183]). But as Hedlund noted in an address delivered before the New York meeting of the American Mathematical Society on October 27, 1938:

> Outstanding problems remain unsolved, a notable one being the problem of metric transitivity [ergodicity] of the geodesic flow on a closed analytic surface of *variable* negative curvature.

It so happens that Eberhard Hopf was just then working on this problem [**162**]. He considered compact surfaces of nonconstant (predominantly) negative curvature and was able to show ergodicity of the Liouville measure (phase volume).

From Hopf's work there was no progress in the direction of ergodicity of geodesic flows (= free particle motion) for almost 30 years. Hopf's argument had shown roughly that Birkhoff averages of a continuous function must be constant on almost every leaf of the horocycle foliation, and, since these foliations are C^1, the averages are constant a.e. He realized that much of the argument was independent of the dimension of the manifold (indeed, he carried much of the work out in arbitrary dimension), but could not verify the C^1 condition in higher dimension. Dmitri Anosov [**10**] axiomatized Hopf's instability, defining Anosov flows, and he showed that differentiability may indeed fail in higher dimension, but that the Hopf argument can still be used because the invariant laminations have an absolute continuity property [**10**, **12**, **20**, **27**, **65**, **248**]. This extension is interesting because despite the ergodicity paradigm central to statistical mechanics, Boltzmann's Fundamental Postulate, there was a dearth of examples of ergodic Hamiltonian systems. The quintessential model for the Fundamental Postulate, the gas of hard spheres, resisted sustained attempts to prove ergodicity for half a century [**271**–**273**].[13]

The Hopf argument remains the main method for establishing ergodicity of volume in hyperbolic dynamical systems without an algebraic structure (the alternative tool being the theory of equilibrium states, see [**181**, Theorem 20.4.1]).

d. Picking up from Poincaré. Like Hadamard, several mathematicians had begun to pick up some of Poincaré's work during his lifetime. Birkhoff did so soon after Poincaré's death. He addressed issues that arose from the mathematical development of mechanics and celestial mechanics such as Poincaré's Last Geometric

[13] Half a century because Sinai convinced physicists that he had solved this problem in 1963 [**192**].

Theorem and the complex dynamics necessitated by homoclinic tangles [**46**, Section 9]. He was also important in the development of ergodic theory,[14] notably by proving the Pointwise Ergodic Theorem 3.2.16.

The work of Cartwright and Littlewood during World War II on relaxation oscillations in radar circuits [**81**, **82**, **202**] consciously built on Poincaré's work. Further study of the van der Pol equation by Levinson [**201**] contained the first example of a structurally stable diffeomorphism with infinitely many periodic points. Structural stability had originated in 1937 with Andronov and Pontryagin [**9**] (necessary and sufficient conditions on singularities and periodic orbits for structural stability of vector fields on a disk) but began to flourish only 20 years later—thanks in no small part to Pontryagin's favorite student, Anosov. Inspired by Peixoto's work, which generalized [**9**] to any orientable closed surface [**232**], Smale had been after a program of studying diffeomorphisms with a view to classification [**279**], and he proved that Morse–Smale systems (finitely many periodic points with stable and unstable sets in general position) are structurally stable. The Cartwright–Littlewood example was brought to his attention by Levinson just as he conjectured that Morse–Smale systems are the only structurally stable ones [**276**]. He eventually extracted from Levinson's work the horseshoe [**277**, **278**]. Independently, Thom (unpublished) studied hyperbolic toral automorphisms (Example 1.5.23) and their structural stability. Smale in turn was in contact with the Russian school, where Anosov systems (then C- or U-systems) had been shown to be structurally stable, and their ergodic properties were studied by way of further development of the study of geodesic flows in negative curvature.

This book focuses on uniformly hyperbolic flows, and even in this realm there are plenty of new developments. Section 5.2 gives instances of uniformly hyperbolic flows of which several are quite new, and Chapter 9 includes various further constructions of such (notably in Section 9.3 and Section 9.2). Our presentation of these includes results in a range of directions that still await publication.

The initial development of the theory of hyperbolic systems in the 1960s was followed by the founding of the theory of *nonuniformly* hyperbolic dynamical systems in the 1970s, mostly by Pesin [**28**, **224**, **234**] (during which time the hyperbolic theory continued its development). One of the high points in the development of smooth dynamics is the proof by Robbin, Robinson, Mañé and Hayashi [**155**] that structural stability indeed characterizes hyperbolic dynamical systems. For diffeomorphisms this was achieved in the 1980s, for flows in the 1990s. Starting in the 1980s the field of geometric and smooth rigidity came into being and is flourishing now (Chapter 10). At the same time topological and stochastic properties of attractors began to be better understood with techniques that nowadays blend ideas from

[14] The Poincaré Recurrence Theorem 3.2.1 is proved in Poincaré's prize memoir [**242**]

hyperbolic and one-dimensional dynamics. Meanwhile, the theory of *partially* hyperbolic dynamical systems, which goes back to seminal works of Brin and Pesin in the 1970s, has seen explosive development since the last years of the 20th century [**236**], which in turn has entailed renewed interest in the methods of uniformly hyperbolic dynamical systems and their possible extensions to this new realm.

Of course, insights into complicated dynamics have penetrated well beyond pure mathematics. In the sciences, these ideas have fundamentally changed the appreciation of nonlinear behavior and that complex data may arise from simple models; they have also provided terminology for describing complexity [**131**]. Celestial mechanics is the realm where applications have most clearly gone beyond the descriptive; since the 1980s the design of trajectories for space probes has irreversibly moved beyond perturbing the 2-body problem in ways that make entirely new mission designs feasible and economical in astonishing ways [**35**]. This can also be said to have added to the very foundation of how evidence is used to build science [**289**].

Part 1

Flows

CHAPTER 1

Topological dynamics

This chapter introduces flows and develops the basic notions of dynamical behaviors from a topological point of view, and provides a foundation for the remainder of the book. The themes of this chapter are the following: definition and basic properties of flows, properties of individual orbits, techniques for varying speed or time, notions of equivalence for flows, and the interplay between continuous and discrete time. Throughout the chapter we provide a number of examples to refer to in this and subsequent chapters; these examples are chosen to illustrate various notions and phenomena that will be encountered throughout the book.

We also examine the orbit structure of flows by first defining various notions of recurrence (including periodicity) and sensitive dependence. We then turn to a more global approach to the orbit structure of a flow. The chain decomposition and the Conley Theorem can be viewed as pinnacles of organizing recurrent behavior in a global context, and they later turn out to be basic for hyperbolic flows.

With a view to hyperbolic flows, we also look at properties of topological flows that involve even closer entanglement of orbits: transitivity, mixing, and expansivity for flows. Lastly, we describe symbolic flows, which will later provide finite models for hyperbolic flows.

1. Basic properties

We begin by introducing flows, the central concept of this book. The notion of a flow arose from studying solutions to differential equations. Over time mathematicians realized that the notion of a flow could be generalized to the definition we give below. We relate flows to solutions of a differential equation in Subsection 1.1b.

Definition 1.1.1 (Flow)**.** A *flow* on a set X is a mapping $\varphi\colon X\times\mathbb{R}\to X$ such that

- $\varphi(x,0)=x$, and
- $\varphi(\varphi(x,t),s)=\varphi(x,s+t)$.

Here, X is variously referred to as the *phase space* or *state space* of the flow. A flow is C^r for $0\le r\le\infty$ if φ is C^r. When we use the term *smooth flow* we will mean a flow that is at least C^1. We usually assume that either X is a topological (or metric) space and φ is continuous or that X is a measure space and Φ is measure-preserving.

a. Time-t maps and orbits. It is illuminating in different ways to fix one input variable at a time. Fixing t yields self-maps of the space X, where a *self map* is a map from X to X. For $t \in \mathbb{R}$ the *time-t* map is $\varphi^t := \varphi(\cdot, t) \colon X \to X$.[1] We will typically refer to a flow by $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ to avoid confusion with φ^t.

Claim 1.1.2. *If Φ is a flow and $t \in \mathbb{R}$, then the time-t map φ^t of the flow is a bijection with inverse φ^{-t}.*

Proof. Taking $s = -t$ in Definition 1.1.1 gives $\varphi^t \circ \varphi^{-t} = \mathrm{Id} = \varphi^{-t} \circ \varphi^t$. □

Thus, $\varphi^0 = \mathrm{Id}$ and $\varphi^s \circ \varphi^t = \varphi^{s+t}$ for all $s, t \in \mathbb{R}$. Hence, a flow is a group action of the real numbers. One can also study actions of groups other than $\mathbb{R}$, but for the most part we will restrict to group actions by $\mathbb{R}$.

Definition 1.1.3. The *inverse flow* of a flow $t \mapsto \varphi^t$ is the flow $t \mapsto \varphi^{-t}$.

Remark 1.1.4. Note that if $a < b$ where $a, b \in \mathbb{R}$, then the flow is completely determined by the mapping $\varphi\colon X \times [a, b] \to X$. (By inversion and the group law, this determines the flow for $t \in [a, b] - [a, b]$, which contains an interval I around 0, and by iteration, this determines the flow for $t \in \mathbb{Z}I = \mathbb{R}$.)

We now provide a number of simple examples of flows.

Example 1.1.5. If $v \in \mathbb{R}$ and $\varphi^t(x) := x + tv$, then $\varphi^0(x) = x$ and if $s, t \in \mathbb{R}$, then

$$\varphi^{s+t}(x) = x + (s+t)v = (x + sv) + tv = \varphi^t(\varphi^s(x)).$$

So Φ is a flow on $\mathbb{R}$.

This illustrates the contrast to discrete-time dynamical systems: a translation and its iterates constitute an action of $\mathbb{Z}$ (or $\mathbb{N}$), and here we have a family of translations parametrized by a continuous parameter, and in fact, it contains all translations.

Example 1.1.6. By considering $\mathbb{R} \pmod 1 = \mathbb{R}/\mathbb{Z}$ in Example 1.1.5, one projects the flow from the previous example to a flow on a circle—which can also be represented as $(z, t) \mapsto e^{2\pi i t} z$ for $|z| = 1$ in $\mathbb{C}$.

This illustrates that the gap between continuous and discrete time is greater than suggested by the previous example: while any two translations of $\mathbb{R}$ are dynamically the same, circle rotations as maps have quite disparate behaviors. Rotations by a rational number are periodic, while rotations by an irrational number exhibit

[1] Our notations ":=," "=:," ":⇔," and "⇔:" define the quantity/property on the side of the ":"

rather nontrivial dynamics. By contrast, this circle flow is about as simple as a flow can be. There is just a little more complexity in the next example.

Example 1.1.7. If $a \in \mathbb{R}$ and $\varphi^t(x) := x \cdot e^{at}$, then $\varphi^0(x) = x$ and if $s, t \in \mathbb{R}$, then

$$\varphi^{s+t}(x) = x \cdot e^{a(s+t)} = (x \cdot e^{as}) \cdot e^{at} = \varphi^t(\varphi^s(x)).$$

So Φ is a flow on $\mathbb{R}$.

More generally, flows in dimension 1 are dynamically quite simple. For that reason we will typically investigate flows on higher dimensional spaces. We now provide higher-dimensional examples of flows where more interesting dynamics can be present. The next example is a flow on the torus.

Example 1.1.8 ("Asteroids")**.** The linear flow Φ_v on the n-torus $\mathbb{T}^n$ in the direction $v \in \mathbb{R}^n$ is defined by $\varphi^t(x) = \varphi_v^t(x) = (x + tv) \bmod 1$. As in Example 1.1.5, this defines a flow (which generalizes the one in Example 1.1.6). Geometrically, a point moves with constant speed along a straight line and (like in old video games such as Asteroids) reemerges from one side of a fundamental domain after encountering the opposite side.

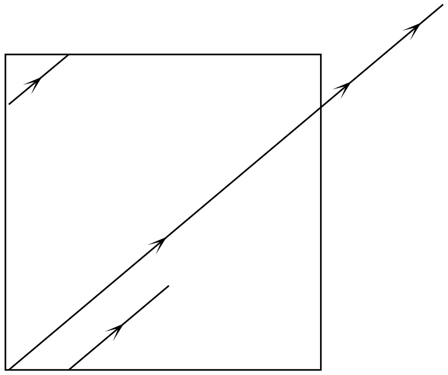


FIGURE 1.1.1. Linear flow

This example is not quite as new as it first seems. Taking n copies of Example 1.1.6 with possibly different speeds, one can construct their cartesian product, described more generally as follows.

Example 1.1.9. If Φ and Ψ are flows on X and Y, respectively, then their cartesian product $\Phi \times \Psi$ on $X \times Y$ defined by $(\varphi \times \psi)^t(x, y) = (\varphi^t, \psi^t)(x, y) := (\varphi^t(x), \psi^t(y))$ is a flow on the product space.

Having explored flows as families of maps, we now take the complementary approach of fixing $x \in X$ and letting t vary to focus attention on the time-evolutions of individual initial conditions, that is, curves in X.

Definition 1.1.10 (Orbits)**.** The *orbit* of $x \in X$ under the flow Φ on X is

$$\mathcal{O}(x) := \varphi^{\mathbb{R}}(x) := \{\varphi^t(x) : t \in \mathbb{R}\} \quad \text{or} \quad \mathcal{O}(x) := \varphi\restriction_{\{x\}\times\mathbb{R}} : t \mapsto \varphi^t(x),$$

depending on whether we wish to keep track of the time parameter or not.[2] Similarly, the *forward orbit of x* is

$$\mathcal{O}^+(x) := \varphi^{[0,\infty)}(x) = \{\varphi^t(x) : t \geq 0\},$$

and the *backward orbit of x* is

$$\mathcal{O}^-(x) := \varphi^{[0,\infty)}(x) = \{\varphi^t(x) : t \leq 0\}.$$

We say that x is a *fixed point* (or *equilibrium* or *singularity*) of Φ if $\mathcal{O}(x) = \{x\}$ (so $\varphi^t(x) = x$ for all $t \in \mathbb{R}$).

A point $x \in X$ is *periodic* for a flow Φ if there exists some $t > 0$ such that $\varphi^t(x) = x$ and a $t > 0$ with $\varphi^t(x) \neq x$. The point x is t-periodic, and its orbit is said to be *closed.* The set of periodic points is denoted by Per(Φ).The (*least* or *prime*) period of x is the infimum of all t such that x is t-periodic.

As a fixed point does not have a flow direction, fixed points and periodic points for flows require somewhat different analysis.

Remark 1.1.11. Example 1.1.5 has only a single orbit, which is $\mathbb{R}$ parametrized with speed $|v|$. Even though the parametrizations differ for different initial points, they differ only by a constant offset of time, so we do not consider these as different orbits even if there is an intent to pay attention to the parametrization. Example 1.1.7 has 3 orbits: the origin and two half-lines. Unless $n = 1$ (in which case there is a single orbit), Example 1.1.8 has uncountably many orbits, each of which is the projection of a line to $\mathbb{T}^n$; these lines are all parallel. If $v = (1, 0, \dots, 0)$ then they all project to circles. If $v = (1, \sqrt{2}, 0, \dots, 0)$ then they all lie in the projection of the xy-plane and fill it densely.

In Example 1.1.8 the existence of a periodic orbit implies that there is a $T \neq 0$ for which $Tv \in \mathbb{Z}^n$, in which case *every* orbit is periodic. (This is the case, for example, if $v \in \mathbb{Q}^n$.) Fixed points occur only if $v = 0$, in which case the flow is trivial. Summarized in slightly different terms, Φ_v has periodic orbits (and is itself periodic) if and only if $v \in \mathbb{R}\mathbb{Z}^n$.

[2] Here, the orbit of x and $\varphi^s(x)$ are the same even in the former case, so parametrized orbits are identified if they differ only by precomposition with a translation of $\mathbb{R}$.

Near a fixed point a flow is going to be very “slow,” and it is plausible that the absence of fixed points implies positive minimum speed by a compactness argument. This is indeed easy when “speed” makes sense, as it does when the flow is described by differential equations as in the next section. Let us demonstrate here that continuity of the flow is sufficient:

Proposition 1.1.12 (“Minimum speed”)**.** *If Φ is a continuous flow without fixed points on a compact space, then there is a $T_0 > 0$ such that for any $t \in (0, T_0)$ there is a $\gamma_t > 0$ with $d(\varphi^t(x), x) \geq \gamma_t$ for all x.*

Proof. If t is such that for all $n \in \mathbb{N}$ there is an x_n with $d(\varphi^t(x_n), x_n) < 1/n$, then an accumulation point x of the x_n satisfies $d(\varphi^t(x), x) = 0$. Thus, if there are no periodic points, take $T_0 = 1$. Otherwise, $T_0 := \inf\{t \mid \varphi^t(x) = x \text{ for some } x \in X\} > 0$ will serve. □

Definition 1.1.13. Let Φ be a flow on a topological space X. A point $x \in X$ has a *flow box* neighborhood if there is a neighborhood U of x and a continuous embedding $h\colon U \to \mathbb{R}^{n+1}$ such that $h \circ \Phi = \Psi \circ h$, where $\psi^t\colon (x, s) \mapsto (x, s + t)$, see Figure 1.1.2.

Proposition 1.1.14 (Flow box)**.** *If Φ is a C^1 flow, then any point where the generating vector field is nonzero admits a flow box.*

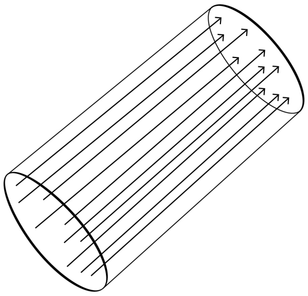


FIGURE 1.1.2. A flow box

Proof. By the Inverse-Function Theorem there is an $\epsilon > 0$ such that if B is an ϵ-ball transverse to the vector field at the point in question, then

$$B \times [-\epsilon, \epsilon] \to M, \quad (x, t) \mapsto \varphi^t(x)$$

is an embedding. □

Figure 1.1.4 below illustrates the obvious fact that this only works away from fixed points. In the purely measurable context, however, something much like this is indeed is not just locally, but *globally* possible; this is the Ambrose–Kakutani–Rokhlin Special-Flow Representation Theorem 3.6.2. In the topological setting there are some flows where there is a global flow box. This is the notion of a suspension flow Definition 1.2.4.

b. Differential equations. The study of flows originated in the field of ordinary differential equations, and smooth flows always arise in this way. We now make this connection explicit. This is meant to serve readers coming to this subject from a background in differential equations, but for others we point out that knowledge of differential equations is not required.

Proposition 1.1.15. *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$ the initial-value problem*

$$\begin{cases} \dfrac{dx}{dt} = f(x) \\ x(0) = \xi \end{cases}$$

has a unique solution $x_\xi(t)$ defined for all $t \in \mathbb{R}$, then $\varphi^t(\xi) := x_\xi(t)$ is a flow.

Proof. Given $s \in \mathbb{R}$ and $y\colon \mathbb{R} \to \mathbb{R}^n$ defined by $y(t) = x_\xi(t+s)$ we write $x' := \frac{dx}{dt}$ and $y' := \frac{dy}{dt}$ and have $y(0) = x_\xi(s)$ and $y'(t) = x'_\xi(t+s) = f(x(t+s,\xi)) = f(y(t))$. So y is a solution to $\frac{dx}{dt} = f(x)$. Since a solution is unique we have

$$\varphi^{t+s}(\xi) = x_\xi(t+s) = y(t) = x_{y(0)}(t) = x_{x_\xi(s)}(t) = (\varphi^t \circ \varphi^s)(\xi)$$

as well as $\varphi^0(\xi) = x_\xi(0) = \xi$. □

Remark 1.1.16. Example 1.1.5 arises in this way from $\frac{dx}{dt} = v$ and Example 1.1.7 from $\frac{dx}{dt} = ax$. In both cases, and in general, $f(x) = \frac{d}{dt}\big|_{t=0}\varphi^t(x) =: \dot{\varphi}(x)$ will serve.

We now examine another class of flows for which we can find explicit formulas for the flow. A flow Φ on a vector space X is *linear* if for all $x, y \in X$, $t \in \mathbb{R}$ and scalars α and β we have

$$\varphi^t(\alpha x + \beta y) = \alpha\varphi^t(x) + \beta\varphi^t(y).$$

An example of a linear flow is the flow generated by the differential equation $x' = Ax$ where A is an $n \times n$ real valued matrix ($A \in \mathcal{M}_n(\mathbb{R})$). If $n = 1$, then solutions of $x' = ax$ are easily seen to be of the form $x(t) = Ce^{at}$. We will see that solutions to $x' = Ax$ have a similar form.

Definition 1.1.17. If $A \in \mathcal{M}_n(\mathbb{R})$, then the *exponential* of A is $e^A = \displaystyle\sum_{k=0}^{\infty} \frac{A^k}{k!}$.

Proposition 1.1.18. *e^A is well-defined.*

Proof. To show that the series converges we use that $\|AB\| \le \|A\|\|B\|$ and hence $\|A^n\| \le \|A\|^n$. If $M < N$ then

$$\left\| \sum_{k=0}^{N} \frac{A^k}{k!} - \sum_{k=0}^{M} \frac{A^k}{k!} \right\| = \left\| \sum_{k=M+1}^{N} \frac{A^k}{k!} \right\| \le \sum_{k=M+1}^{N} \frac{\|A\|^k}{k!}.$$

Since $\sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$ is convergent, hence Cauchy (in $\mathbb{R}$), this shows that $\sum_{k=0}^{m} \frac{A^k}{k!}$ is Cauchy, hence convergent in $\mathscr{M}_n(\mathbb{R})$. □

Analogously to the power series representation of the exponential function on $\mathbb{R}$, this gives $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$, $e^{A0} = \mathrm{Id}$, and

$$\frac{d}{dt} e^{At} = \sum_{k=1}^{\infty} \frac{k(At)^{k-1}}{k!} \cdot A = \sum_{k=0}^{\infty} A \frac{(At)^k}{k!} = Ae^{At}.$$

So for each $x \in \mathbb{R}^n$ the function $e^{At}x$ is the solution to $y' = Ay$ with initial condition $y(0) = v$. The flow $\varphi^t(x) = e^{At}x$ is a linear flow.

If $A \in \mathscr{M}_n(\mathbb{R})$ has n-linearly independent eigenvectors, then we can diagonalize A and explicitly compute e^{At} from the power series. When A does not have n-linearly independent eigenvectors, we can instead use the decomposition into generalized eigenspaces (Theorem 12.2.5). Specifically, we will (as such vacuously) write $e^{At} = e^{\lambda t} e^{(A-\lambda I)t}$ and then find a basis of vectors for each of which the matrix exponential on the right collapses to a polynomial.

Since $(A - \lambda I)$ commutes with λI, $e^{At} = e^{(A-\lambda I)t} e^{\lambda I t}$, so $e^{\lambda I t} = e^{\lambda t} I$. If λ is an eigenvalue of A we let $M(\lambda)$ be the generalized eigenspace (Section 12.2) and $r(\lambda)$ be the natural number at which the nullspace of $(A - \lambda I)^k$ stabilizes. If $v \in M(\lambda)$, then $(A - \lambda I)^{r(\lambda)} v = 0$ and the series for $e^{(A-\lambda I)t} v$ terminates:

$$e^{(A-\lambda I)t} v = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k}{k!} (A - \lambda I)^k v = \sum_{k=0}^{r(\lambda)-1} \frac{t^k}{k!} (A - \lambda I)^k v.$$

Hence,

$$e^{At} v = e^{\lambda t} e^{(A-\lambda I)t} v = e^{\lambda t} \sum_{k=0}^{r(\lambda)-1} \frac{t^k}{k!} (A - \lambda I)^k v.$$

So now we have the following result.

Theorem 1.1.19. *Let λ_1, λ_p be eigenvalues with $M(\lambda_1), \dots, M(\lambda_p)$ the corresponding generalized eigenspaces (Theorem 12.2.5). If $\zeta = v_1 + \cdots v_p$ for $v_j \in M(\lambda_j)$ and*

$1 \le j \le p$, then $x' = Ax$ and $\zeta = x(0)$ has the solution

$$x(t) = \sum_{j=1}^{p} e^{\lambda_j t} \left[\sum_{k=0}^{r(\lambda_j)-1} (A - \lambda_j I)^k \frac{t^k}{k!} \right] v_j.$$

Remark 1.1.20. Note that this could be written as $x(t) = \sum_{j=1}^{p} e^{\lambda_j t} e^{(A-\lambda_j I)t} v_j$, but the point is that the middle term is polynomial rather than exponential, i.e., up to polynomial error we have $x(t) \approx \sum_{j=1}^{p} e^{\lambda_j t} v_j$.

It is rare, however, for a flow to be given in terms of explicit formulas, just like it is rare for a discrete-time dynamical system to admit simple closed-form expressions of arbitrary iterates. Indeed, the insight by Poincaré that founded the discipline of dynamical systems was that one can study the dynamics without expressing solutions, that is, orbits, explicitly in closed form or in terms of power series.

For a smooth manifold M and $1 \le r \le \infty$ let $\mathscr{X}^r = \mathscr{X}^r(M)$ be the space of C^r vector fields on M and $\mathscr{X}^{\mathrm{Lip}}(M)$ the space of Lipschitz-continuous vector fields. That is, if $V \in \mathscr{X}^{\mathrm{Lip}}(M)$, then for each $x \in M$ we have $V(x) \in T_xM$ and the map $x \mapsto V(x)$ is a section of the tangent bundle $TM = \bigcup_{x \in M} T_xM$ and the section varies in a Lipschitz-continuous manner. In a local coordinate chart we can use an existence-and-uniqueness theorem by Picard[3] and Proposition 1.1.15 to show that the flow exists for small values of t. If M is compact without boundary (that is, "closed"), then for arbitrary values of t we can use compositions of maps defined in local coordinates to define the flow on M for all t. Then

$$\frac{dx}{dt} = V(x) \tag{1.1.1}$$

generates a flow (Proposition 1.1.15), which we will denote by Φ_V. Conversely, if Φ is a C^1 flow on a smooth manifold, then

$$V(x) := \dot{\varphi}(x) := \varphi'(x) := \frac{d}{dt}\varphi^t(x)_{|t=0} \tag{1.1.2}$$

defines a continuous vector field on the manifold.

[3] The proof due to Picard considers a Banach space of candidates for solutions of the differential equation and constructs an operator that is a contraction mapping and whose fixed points are solutions of the differential equation. By the Contraction Mapping Theorem (Proposition 12.1.3) there is a unique fixed point, hence a unique solution to the differential equation, and this depends smoothly on initial values and the vector field.

Remark 1.1.21. Equation (1.1.1) extends to *nonautonomous* or time-dependent differential equations, that is, differential equations of the form

$$\begin{cases} \dfrac{dx}{dt} = V(x,t) \\ x(0) = \xi \end{cases}$$

when V is continuous and $x \mapsto V(x,t)$ is Lipschitz-continuous; the aforementioned theorem of Picard gives (local) existence and uniqueness of solutions, which then extend to globally-defined ones as in (1.1.1). The resulting maps $t \mapsto \varphi^t(\cdot)$ may not satisfy $\varphi^s \circ \varphi^t = \varphi^{s+t}$, however, so we may not obtain a flow. This corresponds to having a time-dependent vector field in (1.1.1), and some of the results and techniques presented in this book can be adapted to this context. It is possible, of course to make a nonautonomous differential equation autonomous by treating t as an additional independent variable. The price is that the resulting differential equation on $M \times \mathbb{R}$ no longer "lives" on a compact space. Furthermore, one may lose structural information; for instance, a nonautonomous linear differential equation may in this way become nonlinear because of nonlinearities in the time-dependence.

We now give a classical example of a nonlinear ordinary differential equation.

Example 1.1.22 (The pendulum)**.** Consider a pendulum consisting of a point mass in the plane attached by a rod to a fixed joint. If we take $2\pi x$ to be the angle of deviation from the vertical then (with a suitable choice of units) the pendulum is described by the differential equation

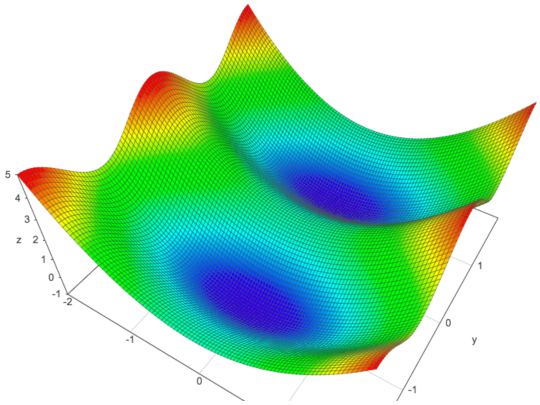
$$\frac{d^2x}{dt^2} + \sin 2\pi x = 0,$$

Writing $v = \frac{dx}{dt}$ (velocity) we obtain the system of first-order differential equations

$$\begin{cases} \dfrac{dx}{dt} = v, \\ \dfrac{dv}{dt} = -\sin 2\pi x \end{cases}$$

for $x \in S^1$, $v \in \mathbb{R}$. The total energy of the system is the kinetic energy plus the potential energy. It is not hard to show that in this case the total energy is given by $H(x,v) = \frac{1}{2}v^2 - \frac{1}{2\pi}\cos 2\pi x$ (see Figure 1.1.3). As this equation is for the undamped (frictionless) pendulum we know that energy is conserved for a solution, and so the function H on the cylinder $S^1 \times \mathbb{R}$ is invariant under the flow:

$$\frac{d}{dt}H(x,v) = v\frac{dv}{dt} + \frac{dx}{dt}\sin 2\pi x = 0.$$



https://academo.org/demos/3d-surface-plotter/?expression=x%5E2%2Bcos(pi*y)&xRange=-2%2C+2&yRange=-2%2C+2&resolution=100

FIGURE 1.1.3. Total energy of the pendulum

We say that H is a *constant of motion* (sometimes also referred to as a (first) integral), and this means that the orbits are on level curves $H =$ const. as shown in Figure 1.1.4. This means that without solving the system of differential equations, we can

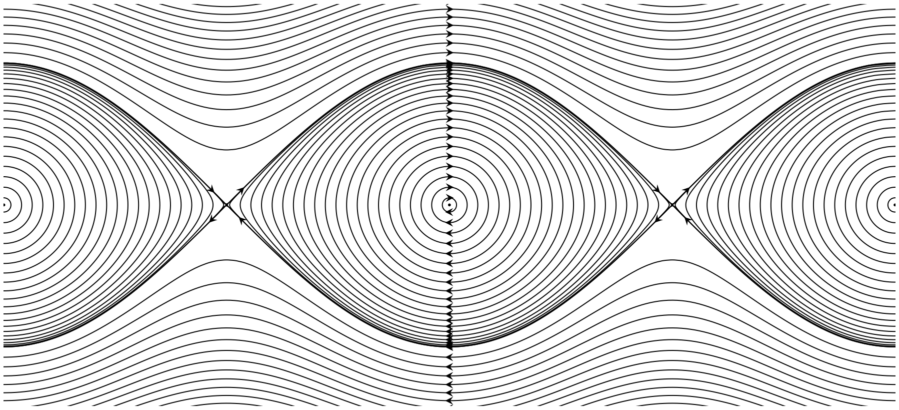


FIGURE 1.1.4. Energy levels of the pendulum rolled out to the plane

describe the solution curves precisely. (It helps that $v = \frac{dx}{dt}$ tells us that in the upper half of the picture the direction of motion is to the right and to the left in the lower half.) For $-1/2\pi < H < 1/2\pi$ each energy level consists of a single closed curve corresponding to oscillations around the stable equilibrium $(x, v) = (0, 0)$; these are periodic orbits, and the period increases monotonically to $+\infty$ as a function of $H \in (-1/2\pi, 1/2\pi)$.

Because of its utility, we formalize the notion of constant of motion:

Definition 1.1.23. A *constant of motion* or *first integral* for a flow is a continuous invariant function. Here, a function $f\colon X \to \mathbb{R}$ is said to be *invariant* for a flow Φ if $f \circ \varphi^t(x) = f(x)$ for all $t \in \mathbb{R}$.

Because a constant function is always trivially a constant of motion, we abuse semantics (omitting "nonconstant") and follow established terminology by saying that *a flow has no constants of motion* if each continuous invariant function is constant.

Those closed periodic orbits are separated from higher-energy orbits corresponding to rotation around the joint by a *homoclinic loop* (an orbit that joins an equilibrium to itself) with $H = 1/2\pi$ containing the equilibrium $(x, v) = (1/2, 0)$. For $H > 1/2\pi$ each energy level consists of two orbits corresponding to rotation in opposite directions.

This is a good moment to note the character of the fixed points that are joined by these homoclinic loops. They satisfy the definition given below.

Definition 1.1.24. A fixed point p of a flow Φ is said to be *hyperbolic* if for any $t \neq 0$ the differential $D_p\varphi^t$ of the time-t map φ^t at p has no eigenvalues on the unit circle (or, more generally, if $D_p\varphi^t$ is a hyperbolic linear map as in Definition 12.4.1).

Figure 1.1.4 is a little less confusing than the picture on the actual phase cylinder. It shows 2 hyperbolic "saddle" points connected by 2 arcs; the upper one consist of points tending to the right saddle in positive time and to the left one in negative time, and pts on the lower one do likewise in reverse. Such arcs are called *saddle connections*. Likewise, these can be seen as unstable sets for the respective other saddle. The full stable set of the saddle on the right consists of the upper of these arcs as well as the corresponding lower arc to its right, of which only half is shown. So its stable set is a arc with the saddle in its interior. Likewise for the unstable set. The points in any of the arcs in Figure 1.1.4 are *heteroclinic*, positively asymptotic to one saddle and negatively asymptotic to another. On the phase cylinder they are *homoclinic* because they are positively and negatively asymptotic to the same saddle, which thus has 2 *homoclinic loops*.

c. Geodesic flows. We now describe flows that arise naturally from differential geometry. As these will be very important in later chapters we have separated these examples into a separate section and revisit them again in Chapter 2.

In the case of a complete Riemannian manifold, flows arise naturally from the geodesics on the manifold. We will restrict ourselves at present to closed connected orientable surfaces. Any such surface is homeomorphic to one of the following: a sphere, a torus, or a higher-genus surface. By the Uniformization Theorem, each of these surfaces admits a metric of constant Gauss curvature, which is either positive, zero, or negative respectively.

We first describe the concept of geodesic flow, and then study the geodesic flow for the torus and sphere. We will wait to discuss the geodesic flow on surfaces of negative curvature until Chapter 2 and show in Theorem 5.2.4 that these provide the classical example of a hyperbolic flow. In the case of negative curvature we will see that the geodesic flow is far more dynamically complicated.

The flow in Example 1.1.8 describes a point moving in a fixed direction with constant speed; this is the motion of a particle that is not subject to any external forces. This flow is connected to acceleration via $F = ma$ and the absence of an external force implies zero acceleration and hence constant velocity.[4]

The motion of a free particle is described as in Example 1.1.8, except that any velocity vector v is allowed. Thus, a state of this system is given by a location and a velocity, that is, by a point on the torus and a tangent vector. We know that the geodesics of $\mathbb{R}^n$ are exactly the straight lines, and so the geodesics on the (flat) torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are exactly the projections of straight lines. Therefore the description of the motion of a free particle on $\mathbb{T}^n$, also known as the *geodesic flow* on $\mathbb{T}^n$ is given as follows. On the tangent bundle $\mathbb{T}^n \times \mathbb{R}^n$ of $\mathbb{T}^n$ the geodesic flow is defined by

$$g^t(x, v) := (x + tv \ (\text{mod } 1), v).$$

The geodesic flow on $\mathbb{T}^n$ is completely integrable, that is, it decomposes into invariant tori carrying linear flows—with frequency vector ω on the invariant torus $\{(x, v) \bigm| x \in \mathbb{T}^n, v = \omega\}$.

In like manner, the motion of a free particle on any Riemannian manifold can be described as motion along geodesics, the "straight lines" for the manifold, except that there usually are no formulas as explicit as in the formula for the torus given above to describe the time-evolution. Indeed, the geodesic equation in differential geometry describes geodesics as having zero acceleration, and for each vector v at a point x of a manifold there is a unique geodesic $\gamma_{(x,v)}$ such that $\gamma_{(x,v)}(0) = x$ and $\dot{\gamma}_{(x,v)}(0) = v$, where $\dot{\gamma}$ denotes the t-derivative or tangent vector (that is, velocity vector). The geodesic flow is defined as follows.

[4] Here, F denotes force, m, mass, and a, acceleration.

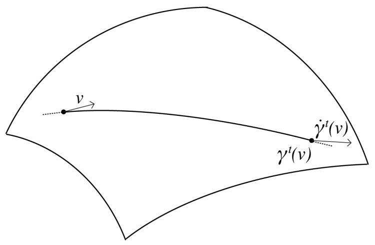


FIGURE 1.1.5. Geodesic flow

Definition 1.1.25 (Geodesic flow)**.** The *geodesic flow* g^t of a Riemannian manifold M is defined on the tangent bundle of M by

$$g^t(x,v) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)),$$

that is, the tangent vector v at x is sent to the tangent vector of the geodesic $\gamma_{(x,v)}$ at time t (that is, at the point $\gamma_{(x,v)}(t)$).

Remark 1.1.26. The fact that the geodesic flow on the flat torus decomposes naturally into linear flows as in Example 1.1.8 is specific to the torus, but the fact that the speed $\|v\|$ is preserved holds generally[5]. Moreover, restricting attention to vectors of a given norm produces a flow that is much like the flow obtained by restricting to vectors of any other norm, except for being uniformly faster or slower. Therefore we will normally (and often implicitly) restrict the geodesic flow to unit vectors, that is, to the *unit tangent bundle* . Note that this is a fixed-point-free flow on a compact space.

Example 1.1.27. The sphere with constant positive curvature has a particularly simple geodesic flow: it involves motion along great circles with unit speed and is hence periodic, that is, the time-2π map is the identity.

Example 1.1.28 (Magnetic flows)**.** That the geodesics are so simple in the case of the sphere makes it easy to explore a slight variation on the theme of free-particle motion. A deformation of this geodesic flow is obtained by modeling a constant magnetic field perpendicular to the sphere. For a charged particle this produces a constant deflection, which means that instead of moving along great circles, such a particle moves along curves with constant and nonzero geodesic curvature, and

[5] This is conservation of kinetic energy.

these happen to be circles of latitude when viewed as a perturbation of the equator. Each vector is tangent to two such circles, one of which "bends right" and the other of which "bends left" according to the sign of the geodesic curvature or the orientation of the magnetic field. This is called the *magnetic flow*. In all cases, we retain the feature that all orbits are periodic with the same period.

Remark 1.1.29 (Reversibility and the flip map)**.** Magnetic flows call attention to a symmetry of geodesic flows that the magnetic flows lack. A geodesic flow can be reversed by reversing vectors, that is, $-\varphi^t(-v) = \varphi^t(v)$. Flows with this property are said to be *reversible*, and because of its importance in this respect, the map $v \mapsto -v$ is called the *flip map*. Magnetic flows lack this symmetry; the closest match would be to flip vectors as well as the magnetic field at the same time. While we concentrate on Riemannian metrics when we study geodesic flows, Finsler metrics (where instead of an inner product, each tangent space is given a norm) give rise to additional examples of flows, but those often lack reversibility.

Beyond surfaces, there are, of course, Riemannian manifolds of higher dimension. To see what makes the geodesic flows of the torus and the sphere tractable it will be useful to put them into a framework that will enable us to study geodesic flows in other cases when a Riemannian manifold possesses a lot of isometries and "symmetries," and therefore, the geodesic flowcan be described without explicitly solving the geodesic equation.

2. Time-change, flow under a function, and sections

In this section we study phenomena that are different in the continuous-time case then in the discrete-time case. Specifically, we investigate reparametrizations of a flow. (Remark 1.1.26 is suggestive of this.) We also look at connections between flows and maps by use of suspensions and sections.

Definition 1.2.1 (Time-change)**.** A flow Ψ on M is a *time-change* of another flow Φ if for each $x \in M$ the orbits $\mathcal{O}_\Phi(x) = \{\varphi^t(x)\}_{t\in\mathbb{R}}$ and $\mathcal{O}_\Psi(x) = \{\psi^t(x)\}_{t\in\mathbb{R}}$ coincide and the orientations given by the change of t in the positive direction are the same.

Equivalently, if Φ and Ψ are smooth flows with generating vector fields V and W respectively and Ψ is a time change of Φ, then $W = \rho V$ for some continuous $\rho\colon M \to [0,\infty)$ with $\rho \neq 0$ away from fixed points. Usually we (implicitly) assume that ρ is as smooth as Φ (in order for Ψ to be equally smooth).

Proposition 1.2.2. *If Ψ is a time-change of Φ then their fixed points coincide, and $\psi^t(x) = \varphi^{\alpha(t,x)}(x)$ for every $x \in M$, where*

$$\alpha(t+s,x) = \alpha(t,x) + \alpha(s,\psi^t(x)), \tag{1.2.1}$$

and

$$\alpha(t,x) \geq 0 \quad \textit{if } t \geq 0 \tag{1.2.2}$$

Indeed, either $\psi^t(x) = x$ *for all* $t \in \mathbb{R}$*, or* $\alpha(t,x) > 0$ *if* $t > 0$*.*

Proof. The "group" property $\psi^{t+s} = \psi^s \circ \psi^t$ gives (1.2.1), while (1.2.2) reflects preservation of orientation. The factor ρ in Definition 1.2.1 generates α as follows: $\alpha(t,x) = \int_0^t \rho(\psi^s(x))\,ds$; this gives the last claim. □

If Φ and Ψ are C^r in Proposition 1.2.2 and x is not a fixed point, then $\alpha(t,x)$ is C^r in both variables by the Implicit-Function Theorem (though at fixed points. α might not even be continuous in x), while $\rho = V\alpha = \frac{\partial}{\partial t}\alpha(t,x)_{|t=0}$ is C^{r-1}.

Sometimes the term "time-change" is used for the flow generated by a scalar multiple of the vector field V even if it vanishes at some points where $V \neq 0$. Interesting examples of time-changes arise in connection with constructions that will be seen to amount to a reversal of finding Poincaré sections (Figure 0.1.1).

The equation (1.2.1) is important beyond time-changes and defines a notion one can consider in greater generality.

Definition 1.2.3. A *cocycle* over a flow Ψ on X is a group-valued function $\alpha \colon \mathbb{R} \times X \to G$ such that the *cocycle equation* (1.2.1) holds (using additive notation).

Note that (1.2.1) with $t = 0$ gives $\alpha(0,x) = 0$. The cocycle equation can be motivated and remembered as saying: to go time $t+s$ go time t and then go time *s from that point*.[6] Another natural example of a cocycle is the differential of a flow, that is, $\alpha(t,x) = D\psi^t(x)$, where the cocycle equation is the chain rule with composition of linear maps as the group operation (so instead of "+" we have composition). Another is the cocycle generated by a function $a \colon X \to \mathbb{R}$ by setting $\alpha(t,x) := \int_0^t a(\psi^s(x))\,ds$; this is how the cocycles in time-changes come about. The one most pertinent here is a real-valued cocycle that defines a time-change—the cocycle equation ensures that the time-changed map is a flow (Proposition 1.2.2). Note that expressing the new time through the old time gives rise to a cocycle over the "new" flow ψ^t.

We now study how flows can arise naturally from a map. Indeed, a number of examples arise in this manner (such as Example 1.5.23 and Definition 6.3.4).

Definition 1.2.4 (Suspension)**.** For a homeomorphism $f \colon M \to M$ of a topological space we define the *suspension flow* $f_\circ$ as the "vertical" flow generated by the vector field $\frac{\partial}{\partial t}$ on the *suspension manifold* (or *mapping torus*) $M_f := (M \times \mathbb{R})/\sim$, where $(x,s) \sim \alpha^n(x,s)$ for all $n \in \mathbb{Z}$ with $\alpha(x,s) := (f(x), s-1)$. (This is well-defined because the vertical flow commutes with α.)

[6] The word "cocycle" rightly hints at a cohomology theory (Definition 1.3.20).

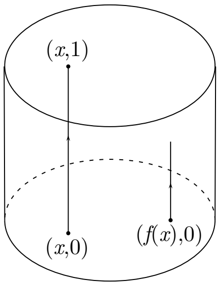


FIGURE 1.2.1. Suspension

The notion of a suspension flow is related to the solution of differential equations with periodic coefficients.

Example 1.2.5. Let $M = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and consider the following situations:

(1) If $f(z) = e^{2\pi i\alpha}z$, then the suspension manifold M_f is homeomorphic to the 2-torus and $f_\circ$ is linear. All orbits are periodic if α is rational, and all orbits are dense if α is irrational; see Example 1.6.2 below.

(2) If $f(z) = \overline{z}$, then M_f is the Klein bottle and $f_\circ$ has two orbits of period 1, and all others have period 2.

Remark 1.2.6 (Metric for a suspension manifold)**.** That Definition 1.2.4 produces a topological space is not surprising, but at times a suitable distance function on M_f is needed if M is a metric space, and the one induced from the suspension construction is not well-defined. To this end it is convenient to think of M_f as $M \times [0,1]$ with $(x,1) \sim (f(x),0)$. Let ρ be a metric (that is, distance function) on M and assume (up to scaling, hence without loss of generality) that the ρ-diameter of M is at most 1. Then

$$\rho_t((y,t),(z,t)) \coloneqq (1-t)\rho(y,z) + t\rho(f(y),f(z)) \geq \min(\rho(y,z),\rho(f(y),f(z))) \eqqcolon \rho'(y,z)$$

defines a metric on $M \times \{t\} \subset M \times [0,1]$. To define the distance between arbitrary $x_1, x_2 \in M \times [0,1]$ consider finite "paths" $x_1 = w_0, w_1, \dots, w_n = x_2$ such that for each i either $w_i, w_{i+1} \in M \times \{t\}$ for some t (in which case we call the pair a horizontal segment of length $\rho_t(w_i, w_{i+1})$) or $w_i = (\alpha, t_1)$ and $w_{i+1} = (\alpha, t_2)$ for some $\alpha \in M$ (in which case we call the pair a vertical segment of length $|t_1 - t_2|$). The length of such a path is the sum of the lengths of its segments, and $d(x_1, x_2)$ is the infimum of such path lengths. This is nondegenerate (since $d((y,t),(z,s)) \geq \rho'(y,z) + |t-s|$) and symmetric, and it satisfies the triangle inequality and induces the given topology.

The next construction is closely related to the idea of a Poincaré section (Figure 0.1.2) where the return time to the section is not necessarily a constant function. (We will make the connection precise after the definition.) In this case, the return time varies continuously with the base point on the section, and this gives a generalization of a suspension flow as defined below.

Definition 1.2.7 (Special flow, flow under a function)**.** Starting with a map $f\colon M \to M$ define the *special flow* or *flow under a function* $r\colon M \to (0,\infty)$ as the flow $\Phi_r = \Phi_{f,r}$ generated by the vector field $\frac{\partial}{\partial t}$ on

$$M_{f,r} \coloneqq M \times \mathbb{R}/\sim, \text{ where } (x,s) \sim \alpha^n(x,s) \text{ for all } n \in \mathbb{Z}$$

with $\alpha(x,s) \coloneqq (f(x), s - r(x))$. (This is well-defined because the vertical flow commutes with α.) The function r is also called a *roof function*.

Topologically the flows on M_f and $M_{f,r}$ are related by a time change (scale the vector field $\frac{\partial}{\partial t}$ on $M_{f,r}$ to $r(x)\frac{\partial}{\partial t}$). Equivalently, consider the manifold $M_{f,r}$ obtained from $M_r \coloneqq \{(x,t) \mid x \in M,\ t \in \mathbb{R},\ 0 \le t \le r(x)\}$ by identifying pairs $(x, r(x))$ and $(f(x), 0)$.

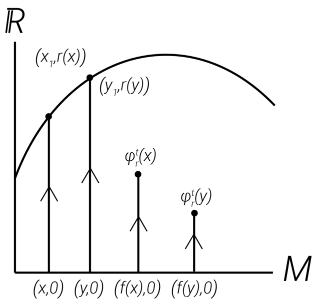


FIGURE 1.2.2. Special flow

Remark 1.2.8. Special flows will arise in Example 1.2.9, Definition 1.8.3, Example 1.5.23, Definition 6.3.4, and Section 6.4. Indeed, in ergodic theory, this is the universal model (Theorem 3.6.2).

From a flow under a function one can recover the original map f in the construction as follows. Identifying X with the projection S of $X \times \{0\}$ (or the graph of any function on X) to the identification space, the desired map sends each point of this *section* to the point of first return. Formally, if Φ denotes the flow, this *first-return map* on S is given by $x \mapsto \varphi^{\min\{t>0 \mid \varphi^t(x)\in S\}}(x) \in S$.

Locally, one can often use this first-return idea to define a map that reflects the local transverse character of a flow near a reference orbit. This is most naturally a way to study a periodic orbit via a small transversal, an idea that goes back to Poincaré's approach of taking periodic orbits as anchors for studying the overall dynamics by working outwards from behavior near them. With a little care this is still possible and useful if a point x and $\varphi^t(x)$ are so close to each other for some t that there is a small hypersurface S through x transverse to $\dot{\varphi}$ (as in Remark 1.1.16 and Figure 0.1.1), called a *local section* or *Poincaré section.* If φ and S are smooth then by the Implicit-Function Theorem so is the *first-return time* $T(x) \coloneqq \min\left\{t > 0 \mid \varphi^t(x) \in S\right\}$ on a neighborhood of x in S, and on a possibly smaller neighborhood the map $f(x) \coloneqq \varphi^{T(x)}(x)$ is well-defined and continuous (hence smooth if φ and S are). Example 1.1.8 illustrates this in a global way: either of the circles $S^1 \times \{0\}$ or $\{0\} \times S^1$ is a section that meets every orbit.

Example 1.2.9 (Billiard flow)**.** Consider a strictly convex region $R \subset \mathbb{R}^2$ with smooth boundary Γ and define a flow on unit vectors as follows. For a vector v at a point $x \in R \smallsetminus \Gamma$ follow the line through x in the direction v with unit speed until it encounters Γ (geodesic flow as in Definition 1.1.25). If $x \in \Gamma$, and v points outside of R, reflect v in Γ according to "angle in=angle out" ("optical" or "specular" reflection) and follow this inward direction as before (see Figure 0.1.2). In this case, there is a *global section* or *global transversal* given by inward-pointing vectors at points of Γ. The induced map on this section is called the *billiard* map, and the billiard flow is the flow over the billiard map under the function given by the free path length (until the next encounter with the boundary).

In this section we have seen differences as well as natural connections between discrete-time and continuous-time dynamics. In summary, there is no discrete-time counterpart to time-changes (other than passing to an iterate), suspensions produce flows from maps, and sections produce (usually local) maps from a flow or part thereof. The flexibility of flows in time adds challenges and a richness to this subject compared to discrete-time dynamics, sections can provide a useful tool for local study of flows, and suspensions (as well as special flows) are on one hand topologically special but on the other hand a source of topologically interesting dynamical systems.

3. Conjugacy and orbit-equivalence

Our ambition is to study classes of flows, and it helps to have effective means to relate or identify different flows with substantially the same or similar features.

They might be naturally equivalent in that they differ only by a global change of coordinates, or one may be a subsystem of another. It turns out that for flows there are several notions of equivalence that correspond to a single notion for maps. This is related to the fact that the notion of conjugacy, which is the main notion of similarity used for maps, is of less use for flows due to longitudinal effects (effects in the flow direction). We now make some of these notions precise.

a. Conjugacy and semi-conjugacy. The first notion we define is topological conjugacy and is an equivalence relation that preserves the topological properties of a flow. The related notion of a semi-conjugacy does not preserve all of the topological properties, but often preserves sufficient properties to be useful.

Definition 1.3.1 (Factor, conjugacy)**.** We say that the flow Φ on M is a *lift* or *extension* of Ψ on N, and Ψ is a *factor* of Φ if there exists a continuous surjection $h\colon M \to N$ such that $h \circ \varphi(t,x) = \psi(t,h(x))$ for all $x \in M$ and all t. In that case we sometimes say that Φ and Ψ are *semiconjugate*.[7] If h is a homeomorphism, then we say that the flows are *topologically conjugate* (or *flow-equivalent*). If, furthermore, h is C^r, then we say that the flows are C^r-conjugate or C^r flow-equivalent.

Remark 1.3.2. Thus, topological conjugacy preserves the entire orbit structure of a flow, and the orbit structure of a factor is naturally "included" in its extension.

This notion was central to Smale's idea of classifying dynamical systems; it provides an equivalence relation for which there is hope to understand the equivalence classes. That it is indeed an equivalence relation is not hard to check (Exercise 1.9).

Proposition 1.3.3. *The circle flow (Example 1.1.6) is a factor of any suspension.*

Proof. The factor map is the projection to the fiber direction. □

Remark 1.3.4. The factor map in Proposition 1.3.3 collapses a lot; the preimage of any point is a copy of the base. This is nonetheless useful, but we will also encounter factors that are definitely not homeomorphisms, but close to it. This is well-illustrated by the surjective continuous map $\sum_{i\in\mathbb{N}} \frac{2a_i}{3^i} \mapsto \sum_{i\in\mathbb{N}} \frac{a_i}{2^i}$ from the ternary Cantor set $\Big\{\sum_{i\in\mathbb{N}} \frac{2a_i}{3^i} \;\Big|\; a_i \in \{0,1\}\Big\}$ (Figure 1.3.1)to $[0,1]$, which is injective on points that are not end-points of complementary intervals of the Cantor set, and 2-to-1 on those end-points, a countable set. The semiconjugacies we obtain from "coding" later on are sometimes conjugacies (Example 1.8.16) and at least also this close to being conjugacies (Example 1.8.18). In those situations, the Cantor model has a useful combinatorial structure.

[7] A drawback to this terminology is that it sounds more symmetric than it is by being unclear about which flow is a factor of which.

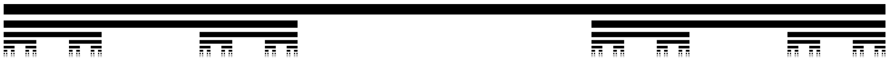


FIGURE 1.3.1. The ternary Cantor set

For future reference, the *Cantor function* $\mathfrak{c}\colon [0,1] \to [0,1]$ is defined by linearly interpolating this map across complementary intervals—which makes it constant on those.

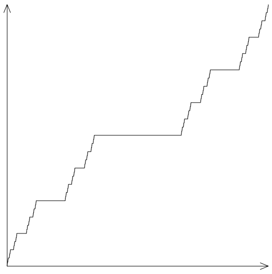


FIGURE 1.3.2. The Cantor function $\mathfrak{c}$

Example 1.3.5. Consider an interval $I = (a,b) \subset \mathbb{R}$ and a flow Φ on it defined by $\frac{dx}{dt} = f(x)$ with $f(x) > 0$ on I and $f \xrightarrow[x\to\{a,b\}]{} 0$ (see Proposition 1.1.15). Then Φ is topologically conjugate to the flow $\psi^t\colon x \mapsto x + t$ on $\mathbb{R}$ (see Example 1.1.5) via $h_{\Phi,c}\colon \mathbb{R} \to I$, $s \mapsto \varphi^s(c)$ for any fixed $c \in (a,b)$ because

$$h_{\Phi,c}(\Psi^t(s)) = h_{\Phi,c}(s+t) = \varphi^{s+t}(c) = \varphi^t(\varphi^s(c)) = \varphi^t(h_{\Phi,c}(s)).$$

It is rare to have an explicit formula for a topological conjugacy; in Example 1.3.5 it helps that the dynamical system in question consists of a single orbit. On the other hand, the conjugacy is not unique, the choices being parametrized by $c \in (a,b)$. This corresponds to the fact that $h^{-1}_{\Phi,c'} \circ h_{\Phi,c}$ is a self-conjugacy by a constant time shift.

Example 1.3.6. The definition of φ in Example 1.3.5 naturally extends to $[a,b]$ by taking a,b to be fixed points, that is, $\varphi^t(a) = a$ and $\varphi^t(b) = b$ for all t (see Definition 1.1.10).

Example 1.3.7 (North-south dynamics)**.** Let $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$ be the standard unit sphere in $\mathbb{R}^3$. We consider the flow that moves every point downward (or “southward”, if we think of S^2 as the surface of the globe and take the earth’s axis to be vertical) along a great circle (meridian) connecting the point $(0,0,1)$ ("the north pole") and $(0,0,-1)$ ("the south pole"). The speed of the motion is equal to the derivative of the vertical coordinate along the meridian. In other words, our flow is generated by integrating the following vector field v on the sphere:

$$v(x,y,z) = (xz, yz, -x^2 - y^2).$$

To see this note that the downward unit vector tangent to the sphere at (x,y,z) is given by $(xz, yz, -(x^2+y^2))/\sqrt{x^2+y^2}$. The absolute value of its z-coordinate $\sqrt{x^2+y^2}$ gives the norm of the gradient vector. The two poles are the only zeroes of this vector field and consequently they are fixed points for the flow. Every point except for the north pole asymptotically approaches the south pole as time goes to plus infinity. In fact this convergence is exponential. Similarly, every point except for the south pole exponentially approaches the north pole as time goes to minus infinity. This example can be extended to gradient flow on any n-sphere for $n \geq 1$ and will have similar dynamics.

Interestingly, the lowest-dimensional case of the preceding example turns out to be an ingredient in the study of an important class of hyperbolic flows (Figure 2.3.2). We revisit gradient flows in Example 1.4.12.

Example 1.3.8 (Uniqueness and smoothness of conjugacies)**.** In the context of Proposition 1.1.15 with $n = 1$ consider a flow on $\mathbb{R}$ generated by $\frac{dx}{dt} = f(x)$ with $x \cdot f(x) > 0$ for $x \neq 0$. Then $\varphi^t(x) \xrightarrow[t\to\infty]{} 0$ monotonically for any $x \in \mathbb{R}$, that is, Φ is contracting, and Φ is conjugate to a linear flow as follows. Example 1.3.5 with $a = 0$, $b = \infty$ shows that $\Phi\restriction_{\mathbb{R}^+}$ is conjugate to the flow $(t,x) \mapsto x + t$ on $\mathbb{R}$, hence to the flow $(t,y) \mapsto ye^{-t}$ on $\mathbb{R}^+$. Similarly on $\mathbb{R}^-$, and setting $h(0) = 0$ gives a conjugacy to the flow $(t,y) \mapsto ye^{-t}$ on $\mathbb{R}$. This further implies that any 2 such flows are topologically conjugate. As noted in Example 1.3.5, the conjugacy is not unique (we freely chose the image of 0), and in this context we can make independent choices on $\mathbb{R}^+$ and $-\mathbb{R}^+$.

That 0 is the fixed point is not central here because the homeomorphism $x \mapsto x + c$ is a conjugacy to a contracting flow that fixes $c \in \mathbb{R}$. Thus, *all contracting flows on* $\mathbb{R}$ *are pairwise topologically conjugate.*

One can also show that the choice of conjugacy we have described is all there is. The first indication is given by linear flows: if h conjugates the linear contracting flow $(t,x) \mapsto \lambda^t x$ on $\mathbb{R}^+$ to itself (that is, it commutes with the linear flow), then $h(\lambda^t x) = \lambda^t(x)$ for all $t \in \mathbb{R}$, so h is linear (on $\mathbb{R}^+$) and hence determined by choosing the image of a single point. Since all contracting flows are conjugate to a linear flow, this gives the complete story: If h_1, h_2 both conjugate Φ to Ψ and h conjugates Φ to a linear flow, then $hh_2h_1h^{-1}$ conjugates the linear flow to itself and is hence unique up to a choice of scale factor.

Lastly, the topological conjugacy between 2 differentiable contracting flows we have obtained here is differentiable, except possibly (indeed, probably) at 0: if it is differentiable at 0, then differentiating the conjugacy relation at 0 shows that $(\varphi^t)'(0) = (\psi^t)'(0)$ for all $t \in \mathbb{R}$.[8] In this case, uniqueness up to a scale factor shows that a conjugacy that is differentiable at 0 is unique once we prescribe 1 as the derivative at 0.

The presence of this obstruction shows that using differentiable conjugacy to define equivalence of flows creates "structural fragility" in the sense that a typical perturbation of a flow would not be equivalent to the flow itself. This is an important reason for preferring continuous conjugacy as the natural notion of equivalence. Indeed, with respect to this notion we will find the opposite of structural "fragility" for hyperbolic flows (Corollary 5.4.7) (though with a weakening of "conjugacy" to "orbit-equivalence"; see Subsection 1.3b). Looking even further ahead we note that the rarity of smooth conjugacy (or indeed, orbit-equivalence) can in this context make it intensely interesting in some rather particular respects; this is central to rigidity theory (Chapter 10).

Example 1.3.9 (South-south dynamics)**.** An example on the circle arises from Example 1.3.6 by identifying a and b; the resulting flow has a single fixed point (Figure 1.3.3). (Note that this is also included in Figure 1.5.4.) With more specific choices one can describe this as generated by the differential equation $\frac{dx}{dt} = f(x)$ on $[0,1]$ mod 1 with $f(0) = 0$ and $f(x) > 0$ otherwise.

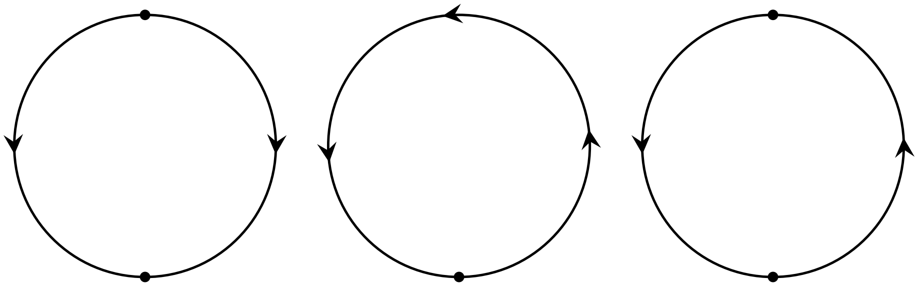


FIGURE 1.3.3. North-south (Example 1.3.7), south-south (Example 1.3.9), south-north-south (Example 1.3.10) dynamics

No two of the flows in Examples 1.3.5, 1.3.6 and 1.3.9 are topologically conjugate because the spaces on which they are defined are not homeomorphic.

[8] This is sufficient for the existence of a conjugacy that is differentiable at 0, but the argument is not elementary. Theorem 10.1.10 implies this, and it might be interesting to simplify its proof for the present situation.

Example 1.3.10 (South-north-south dynamics)**.** A companion example to the preceding circle flows reverses one arrow in the north-south dynamics, giving 2 fixed points with one orbit each connecting them in one direction versus the other (Figure 1.3.3). Note that this dynamics also appears as part of Figure 1.1.4 (Figure 1.3.4). Clearly, no 2 flows in Figure 1.3.3 are topologically conjugate.

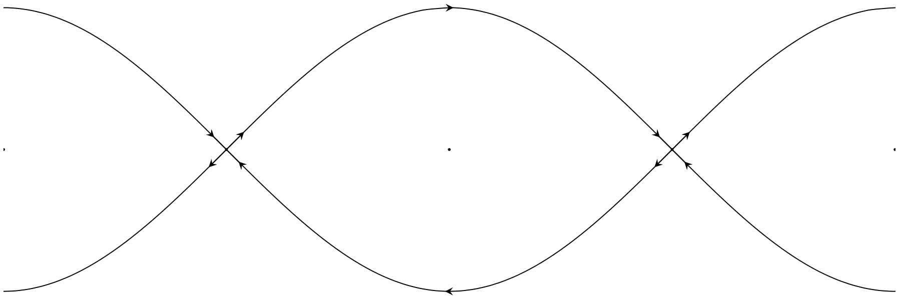


FIGURE 1.3.4. The south-north-south dynamics in the context of Figure 1.1.4

Example 1.3.11. More generally, consider an interval $I = [a, b] \subset \mathbb{R}$ and a flow φ^t on I defined by $\frac{dx}{dt} = f(x)$ with f continuous on I and $f(a) = f(b) = 0$ (see Proposition 1.1.15), that is, we do not assume $f > 0$. The zeros of f are the fixed points of this flow. It is illuminating to prove that two such flows are conjugate if (and clearly only if) there is an increasing homeomorphism that identifies the respective sets where f is zero, positive, and negative.

Example 1.3.12 (Akin)**.** Example 1.3.11 describes a class of flows which could likewise be defined on $S^1 = [0,1]/\mathbb{Z}$, and we specialize this to a pair of examples on $[0,1]$ and S^1. Choose $f\colon [0,1] \to [0,1]$ continuous such that $f^{-1}(\{0\}) = C$, the ternary Cantor set (Remark 1.3.4), and define the Akin flow $A = (\alpha^t)_{t\in\mathbb{R}}$ on $[0,1]$ by $\frac{dx}{dt} = f$, $A_\circ$ its projection to S^1. Note that here we do not only specialize the fixed-point set but also unidirectional motion.

Note that the Cantor function $\mathfrak{c}$ (Remark 1.3.4) is a constant of motion for A, and $\mathfrak{c}(1-\mathfrak{c})$ is a constant of motion for both A and $A_\circ$.

Example 1.3.13. Let $v = (v_1, \dots, v_{n-1}, 1)$. The linear flow Φ_v (Example 1.1.8) on the n-torus is C^∞ conjugate to the suspension of the translation $f_\gamma\colon x \mapsto x + \gamma$ on the $(n-1)$-torus, where $\gamma \coloneqq (v_1, \dots, v_{n-1})$: Consider the map H from the suspension manifold $M = \mathbb{T}^{n-1}_{T_\gamma}$ to the torus $\mathbb{T}^n$ given by

$$H(x_1, \dots, x_{n-1}, t) = (x_1 + v_1 t, x_2 + v_2 t, \dots, x_{n-1} + v_{n-1} t, x_n + t).$$

It is differentiable for $t \neq 0$. Differentiability at $t = 0$ follows from the definition of the smooth structure on the suspension manifold. The differential of H carries the upward vector field $\frac{\partial}{\partial t}$ to the vector field $v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n}$ and hence conjugates the flows generated by those vector fields, which are exactly the suspension flow and the linear flow, respectively.

Example 1.3.14. The cartesian product of two flows has either flow as a factor by the projection.

Example 1.3.15. The flow $(x_1, x_2) \mapsto (x_1, x_2 + tx_1)$ on $\mathbb{T}^2$ has the identity on $S^1 = \mathbb{T}^1$ as a factor (via $h\colon (x_1, x_2) \mapsto x_1$).

More generally, special flows admit a straightforward sufficient criterion for conjugacy that is useful, even though this is viewed as a trivial example of conjugacy.

Definition 1.3.16. For an invertible map $f\colon X \to X$ two functions $r_1, r_2\colon X \to (0,\infty)$ are *cohomologous* via a *transfer function* $g\colon X \to \mathbb{R}$ if $r_1(x) = r_2(x) + g(f(x)) - g(x)$ for all $x \in X$.

Proposition 1.3.17. *Let $f\colon X \to X$ be an invertible map and $r_1, r_2\colon X \to (0,\infty)$ be cohomologous via a transfer function g. Then Φ_{r_1} and Φ_{r_2} are conjugate via a conjugacy with the same regularity as g.*

Proof. The function $\bar{h}\colon X \times \mathbb{R} \to X \times \mathbb{R}$ defined by $(x, s) \mapsto (x, s + g(x))$ is as regular as g and commutes with the vertical flow, while by assumption

$$\bar{h} \circ \alpha_1(x, s) = (f(x), s - r_1(x) + g(f(x))) = (f(x), s + g(x) - r_2(x)) = \alpha_2 \circ \bar{h}(x, s). \quad \square$$

Example 1.3.18 (Trivial time-change)**.** Let Φ be a smooth flow with generating vector field V. Let $h(x) = \varphi^{b(x)}(x)$, where b be a differentiable function with

$$(Vb)(x) = db(V)(x) = \frac{db(\varphi^t(x))}{dt}|_{t=0} > 0 \text{ when } V(x) \neq 0,$$

that is, the derivative in the flow-direction is positive if $V(x) \neq 0$. Then

$$(h \circ \varphi^t \circ h^{-1})(hx) = h(\varphi^t(x)) = \varphi^{b(\varphi^t(x))}(\varphi^t(x)) = \varphi^{t+b(\varphi^t(x))}(x) = \varphi^{t+b(\varphi^t x)-b(x)}(hx),$$

and

$$\beta(t, x) \coloneqq t + b(\varphi^t x) - b(x)$$

satisfies (1.2.2). This kind of time-change is said to be *trivial*. An equivalent way to describe these is as follows.

Proposition 1.3.19 (Trivial time-changes)**.** *Consider a flow Φ generated by the vector field V and a smooth $f\colon M \to \mathbb{R}$ such that $1+df(V) > 0$. Then $h\colon x \mapsto \varphi^{f(x)}(x)$ conjugates the flow generated by the vector field $V_f := \dfrac{V}{1+df(V)}$ to Φ.*

Proof. Smoothness of f and $1 + df(V) > 0$ ensure that h is a diffeomorphism. Now we write $x_t = \varphi^t(x)$ and use the chain rule to compute

$$\begin{aligned} dh(V(x)) = \frac{dh}{dt} = \frac{d}{dt}\varphi^{f(x_t)}(x_t)|_{t=0} &= \frac{d\varphi}{dt}|_{t=0} df(V)(x) + V(\varphi^{f(x)}(x)) \\ &= V(\varphi^{f(x)}(x)) \cdot df(V)(x) + V(\varphi^{f(x)}(x)) = (1 + df(V)(x))V(\varphi^{f(x)}(x)), \end{aligned}$$

which gives $dh(V_f) = V$ upon division by $1 + df(V)(x)$. □

b. Orbit-equivalence. Example 1.3.8 showed that differentiable conjugacy is more restrictive than we want because even small perturbations of a flow can remove it from a given smooth conjugacy class. However, even topological conjugacy is restrictive because even gentle time-changes may render topological conjugacy impossible, that is, the equivalence classes of topological conjugacy are often too small to be helpful for classifying flows. The notion of an orbit-equivalence is often a more natural equivalence relation for flows—although, unlike topological conjugacy, an orbit-equivalence fails to preserve some important topological properties (such as mixing) and quantities (such as entropy).

Before moving on, we introduce the counterpart of Definition 1.3.16 for functions over flows.

Definition 1.3.20. For a flow Φ on X generated by a vector field V, two functions $r_1, r_2\colon X \to \mathbb{R}$ are *cohomologous* via a *transfer function* $g\colon X \to \mathbb{R}$ if $r_1 = r_2 + Vg$, where Vg is the derivative along the flow. If $r_2 \equiv 0$ then r_1 is *null-cohomologous*.

By definition, topological conjugacy preserves topological properties, as is the case with the corresponding notion for maps. However, topological conjugacy for flows is in many contexts too narrow a notion because of its rigidity with respect to the parametrization of orbits. That is to say, the equivalence classes for topological conjugacy are too small in general to be interesting for flows. Therefore one more often encounters the following notion of equivalence for flows, which allows for the possibility of time-changes.

Definition 1.3.21 (Orbit-equivalence)**.** A flow Ψ on N is said to be an *orbit factor* of a flow Φ on M if there exists a continuous surjection $h\colon M \to N$ that sends orbits of Φ to orbits of Ψ. We also say that Ψ and Φ are *semiequivalent*. If h is a homeomorphism, then the flows are *orbit-equivalent*.

Remark 1.3.22. Orbit-equivalence occurs more commonly for flows than conjugacy and therefore tends to be the more prominent concept of these two. However, it does not preserve some topological properties sensitive to "longitudinal" effects, notably topological mixing (Definition 1.6.31) and topological entropy (Definition 4.2.2; see equation (4.3.5)). This is a reason we do not refer to this as topological equivalence. It is, of course, an equivalence relation (Exercise 1.12).

In some simple contexts (Example 1.3.11) there is little distinction between orbit-equivalence and topological conjugacy.

Remark 1.3.23. If a flow Φ without fixed points is topologically orbit-equivalent to a flow Ψ via h (that is, h is a homeomorphism that maps orbits of Φ to orbits of Ψ), then $h^{-1}\circ\psi^t\circ h$ is a flow with the same orbits as Φ, and the reparametrization is homeomorphic: If $x \in X$ is not periodic, then $\sigma_x\colon \mathbb{R}\to\mathbb{R}$ defined by $h^{-1}(\psi^t(h(x))) = \varphi^{\sigma_x(t)}(x)$ is a homeomorphism with $\sigma_x(0)=0$. If $x\in X$ is periodic for Φ with least period ν and μ is its least period for $h^{-1}\circ\psi^t\circ h$, then $h^{-1}(\psi^t(h(x))) = \varphi^{\sigma_x(t)}(x)$ defines a strictly monotone continuous map σ_x on $[0,\mu]$ whose range is an interval of length ν with 0 as an end-point. Extending naturally to $[n\mu,(n+1)\mu]$ gives a homeomorphism of $\mathbb{R}$.

We now begin to study the *relative* behavior of orbits, an important concern for topological dynamics.

Definition 1.3.24. For a flow Φ and point $x\in X$ define the *stable* and *unstable sets* of x by

$$\begin{aligned} W^s(x) = W^{ss}(x) &:= \{y\in X \mid d(\varphi^t(x),\varphi^t(y) \xrightarrow[t\to+\infty]{} 0\},\\ W^u(x) = W^{uu}(x) &:= \{y\in X \mid d(\varphi^t(x),\varphi^t(y) \xrightarrow[t\to-\infty]{} 0\}. \end{aligned} \tag{1.3.1}$$

The sets

$$W^{cs}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^s(x)) \text{ and } W^{cu}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^u(x))$$

are called the *center-stable* and *center-unstable* sets of x.

The triangle inequality gives (Exercise 1.8):

Proposition 1.3.25. *For a flow Φ if $x, y\in X$, then*

- *$W^s(x)\cap W^s(y)\neq\varnothing$ implies $W^s(x)=W^s(y)$;*
- *$W^{cs}(x)\cap W^{cs}(y)\neq\varnothing$ implies $W^{cs}(x)=W^{cs}(y)$;*
- *$W^u(x)\cap W^u(y)\neq\varnothing$ implies $W^u(x)=W^u(y)$; and*
- *$W^{cu}(x)\cap W^{cu}(y)\neq\varnothing$ implies $W^{cu}(x)=W^{cu}(y)$.*

By uniform continuity, conjugacies and orbit-equivalences preserve these sets as follows:

Proposition 1.3.26. *If Φ and Ψ are flows with hyperbolic sets conjugate by h, then $h(W^s(x)) = W^s(h(x))$ and $h(W^u(x)) = W^u(h(x))$ for all x, hence likewise for center-stable and center-unstable sets. For an orbit-equivalence a similar statement holds for the center-stable and center-unstable sets (only).*

Proposition 1.3.27 (Longitudinal regularity of orbit-equivalence)**.** *For $r \in \mathbb{N}$ an orbit-equivalence between fixed-point free C^r flows can be chosen to depend C^r on time.*

PROOF [**126**]**.** If h maps orbits of Φ homeomorphically to orbits of Ψ, then (as in Proposition 1.2.2) there is a continuous cocycle α over Ψ such that

$$h(\varphi^t(x)) = \psi^{\alpha(t,x)}(h(x)),$$

that is, $\alpha(t+s,x) = \alpha(t,\varphi^s(x))+\alpha(s,x)$. For some $T > 0$ set $k(x){:=}\psi^{\frac{1}{T}\int_0^T \alpha(\tau,x)\,d\tau}(h(x))$ to get (reverting to the original notation $\psi(t,x) = \psi^t(x)$):

$$\begin{aligned}
k(\varphi^t(x)) &= \psi\Bigl(\frac{1}{T}\int_0^T \underbrace{\alpha(\tau,\varphi^t(x))}_{=\alpha(t+\tau,x)-\alpha(t,x)}\,d\tau, \underbrace{h(\varphi^t(x))}_{=\psi^{\alpha(t,x)}(h(x))}\Bigr)\\
&= \psi\Bigl(\underbrace{\frac{1}{T}\int_0^T \alpha(t+\tau,x)-\alpha(t,x)\,d\tau + \alpha(t,x)}_{=\frac{1}{T}\int_0^T \alpha(t+\tau,x)\,d\tau=\frac{1}{T}\int_t^{T+t}\alpha(\tau,x)\,d\tau}, \underbrace{h(x)}_{=\psi(-\frac{1}{T}\int_0^T \alpha(\tau,x)\,d\tau,k(x))}\Bigr)\\
&= \psi\Bigl(\underbrace{\frac{1}{T}\int_t^{T+t}\alpha(\tau,x)\,d\tau - \frac{1}{T}\int_0^T \alpha(\tau,x)\,d\tau}_{=:\beta(t,x)=\frac{1}{T}\int_0^t \alpha(T+\tau,x)-\alpha(\tau,x)\,d\tau}, k(x)\Bigr).
\end{aligned}$$

Since Ψ is smooth, differentiability of $t \mapsto k(\varphi^t(x)) = \psi^{\beta(t,x)}(k(x))$ follows because $\dfrac{d}{dt}\beta(t,x) = \dfrac{1}{T}(\alpha(T+t,x)-\alpha(t,x))$ by the Fundamental Theorem of Calculus, so $t \mapsto \beta(t,x)$ is C^1.

Now, if $t \mapsto \alpha(t,x)$ is C^1, then this construction leads to a β such that $t \mapsto \beta(t,x)$ is C^2, that is, recursively, we can make the orbit equivalence C^r in the time direction so long as the flows themselves are C^r. □

Naturally, orbit equivalence fails to distinguish between flows under a function (Definition 1.2.7) and suspensions (Definition 1.2.4).

Proposition 1.3.28. *Let M be a compact differentiable manifold, $f\colon M \to M$ a C^m diffeomorphism, and $r\colon M \to \mathbb{R}_+$ a C^m function on M. Then the special flow on the manifold M_f^r is C^m orbit-equivalent to the suspension flow on M_f.*

PROOF. Let $k := \min r$ and $K := \max r$. Consider a C^∞ function $g\colon [0,1]\times[k,K] \to \mathbb{R}$ such that

(1) $g(t,s) = t$ for $t \in [0,1/4]$,
(2) $g(t,s) = t+s-1$ for $t \in [3/4,1]$,
(3) $\dfrac{\partial}{\partial t}g(t,s) > 0$.

Then the map $(x,t) \mapsto (x, g(t, r(x)))$ is a diffeomorphism between M_f and M_f^r which takes the vertical vector field $\dfrac{\partial}{\partial t}$ on M_f to a vertical vector field on M_f^r, and hence conjugates the suspension flow with a time change of the special flow under r. □

Both orbit-equivalence and conjugacy preserve periodic orbits, although only conjugacy preserves their periods. In the next sections we investigate the structure and relative behavior of orbits for a flow; we specifically examine properties related to stability and recurrence.

4. Attractors and repellers

We begin with the notion of attracting and repelling fixed points before moving to more general attracting and repelling sets for a flow. The notion of an attracting fixed point is related to the notion of having a steady state in an engineered system that is stable in the sense that the system returns to it if it was ever jolted away. In terms of differential equations, a "stable" solution is such that "nearby" solutions either don't go far away from the solution or converge to the stable solution in some sense. The desirability of this is also related to the fact that in practice we don't have infinite precision in starting a time-evolution, so stability can provide the mechanism for settling into the desired state. This notion is of particular interest with respect to fixed or periodic points, but Poincaré focused attention on using these as anchors for the understanding of other orbits.

Definition 1.4.1 (Attracting fixed points)**.** A fixed point p of a flow Φ is *attracting*[9] if it is in the interior of its stable set (Definition 1.3.24), ithat is, given $\epsilon > 0$ there is a $\delta > 0$ such that $d(x,p) < \delta \Rightarrow d(\varphi^t(x), p) < \epsilon$ for all $t \geq 0$ and there is a $\gamma > 0$ such that $d(x,p) < \gamma \Rightarrow \varphi^t(x) \xrightarrow[t\to\infty]{} p$. In other words, the *basin of attraction* is open. A fixed point is *repelling* if it is attracting for φ^{-t}.

A periodic point p is *asymptotically stable* or *attracting* if it is stable and there exists some $\gamma > 0$ such that $d(x,p) < \gamma \Rightarrow \exists\delta\colon\ d(\varphi^t(x), \varphi^{t+\delta}(p)) \xrightarrow[t\to\infty]{} 0$.

In Example 1.3.6 the fixed point b is attracting but a is not. In Example 1.1.7 the origin is attracting if and only if $a < 0$. In Example 1.1.22 the origin is not attracting because all orbits nearby are periodic and hence not asymptotic to the origin. (This

[9] In the theory of differential equations this is called an asymptotically stable fixed point.

changes with damping; see Figure 1.4.1.) In Example 1.3.6 the fixed point b is attracting. The fixed point in Example 1.3.9 is neither attracting nor repelling.

Example 1.4.2. The orbit of $(0,0)$ is periodic and asymptotically stable for the flow $(x,y)\mapsto(x+t \pmod 1), ye^{-t})$ on $S^1\times\mathbb{R}$.

Attracting fixed points have a neighborhood such that each point limits on the fixed point as time approaches infinity. Similarly, for repelling fixed points the same happens as time approaches minus infinity.

a. Linear flows. We now investigate stability and conjugacy for the classical example of a linear flow arising from a matrix. Let $A\in\mathcal{M}_n(\mathbb{R})$ and e^{At} be the flow on $\mathbb{R}^n$ generated by A for the equation $x'=Ax$. To investigate the stability of the origin for these linear constant coefficient flows we will use the closed-form solution that is particularly amenable to discerning asymptotic growth and decay from Theorem 1.1.19. There is an easy criterion for asymptotic stability of 0 for $x'=Ax$.

Theorem 1.4.3. *For $A\in\mathcal{M}_n(\mathbb{R})$ there are $K,\alpha>0$ with $\|e^{At}\|\le Ke^{-\alpha t}$ if and only if all eigenvalues have negative real part.*

Remark 1.4.4. The proof gives a sharper version: if $-\alpha\in(\max_j \operatorname{Re}(\lambda_j),0)$, then there is a K such that $\|e^{At}\|\le Ke^{-\alpha t}$ for all t.

Proof. Let $\zeta\in\mathbb{C}^d$ and $x(t)$ the solution of $x'=Ax$ with $x(0)=\zeta$. Let $\lambda_1,\dots,\lambda_p$ be the eigenvalues of A. For $\lambda_j=\alpha_j+i\beta_j$ and $\zeta=v_1+\cdots+v_p$ where $v_j\in M(\lambda_j)$ for each $1\le j\le p$ we have

$$x(t)=\sum_{j=1}^{p}e^{\lambda_j t}\sum_{k=0}^{r(\lambda_j)-1}(A-\lambda_j I)^k\frac{t^k}{k!}v_j.$$

Let $M=\max\{\|A-\lambda_j I\|^k : 1\le j\le p, 0\le k\le r(\lambda_j)-1$. Then

$$\|x(t)\|\le\sum_{j=1}^{p}e^{\alpha_j t}\sum_{k=0}^{r(\lambda_j)-1}M\frac{|t|^k}{k!}\|v_j\|.$$

We now define a norm $\|\cdot\|_A$ on $\mathbb{C}^d$ by

$$\|\zeta\|_A=\|v_1\|+\cdots+\|v_p\|.$$

Then there exists some $C>0$ such that $\|v_i\|\le\|\zeta\|_A\le C\|\zeta\|$. Then

$$\|x(t)\|\le\left[\sum_{j=1}^{p}e^{\alpha_j t}\sum_{k=0}^{r(\lambda_j)-1}\frac{|t|^k}{k!}\right]MC\|\zeta\|.$$

If all $\alpha_j < 0$, then choose $0 > \alpha > \max_{1\le j\le p} \alpha_j$. So

$$\lim_{t\to\infty} \frac{e^{\alpha_j t} t^k}{e^{\alpha t}} = 0$$

So there exists some $K_0 > 0$ such that

$$\frac{e^{\alpha_j t} t^k}{e^{\alpha t}} \le K_0$$

for all $t \ge 0$, $1 \le j \le p$, and $0 \le k \le r(\lambda_j) - 1$. This implies that $e^{\alpha_j t} t^k \le K_0 e^{\alpha t}$. Define $K = (r(\lambda_1) + r(\lambda_2) + \cdots + r(\lambda_p))K_0$.

The converse is easy (consider eigenvectors). □

The above result helps us establish topological conjugacy for flows arising from matrices if the origins are asymptotically stable.

Proposition 1.4.5. *If all eigenvalues of $A, B \in \mathscr{M}_n(\mathbb{R})$ have negative real part, then the flows e^{At} and e^{Bt} are topologically conjugate.*

Remark 1.4.6. By reversing the flow, the conclusion also holds when all eigenvalues have positive real part.

Proof. By Theorem 1.4.3 there are $C \ge 1$, $a_0 > 0$ such that $\|e^{At}x\| \le Ce^{-a_0 t}\|x\|$ for all $x \in \mathbb{R}^d$ and $t \ge 0$. For $a \in (0, a_0)$ there is a T such that $\|e^{At}x\| \le e^{-at}\|x\|$ for all $t \ge T$. Define a new norm $\|x\|_A = \int_0^T e^{as}\|e^{As}x\|ds$. Then

$$\|e^{At}x\|_A = \int_0^T e^{as}\|e^{As}e^{At}x\|ds.$$

Write $t = nT + \tau$ with $0 \le \tau < T$. Then

$$\begin{aligned}\|e^{At}x\|_A &= \int_0^{T-\tau} e^{as}\|e^{AnT}e^{A(\tau+s)}x\|ds + \int_{T-\tau}^{T} e^{as}\|e^{A(n+1)T}e^{A(\tau-T+s)}x\|ds \\ &\le \int_\tau^T e^{a(u-\tau-nT)}\|e^{Au}x\|du + \int_0^\tau e^{a(u+T-\tau-(n+1)T)}\|e^{Au}x\|du \\ &= e^{-at}\int_0^T e^{au}\|e^{Au}x\|du = e^{-at}\|x\|_A.\end{aligned}$$

Such a norm $\|\cdot\|_A$ is called an *adapted norm* or *Lyapunov norm* and always exists in the hyperbolic case (Proposition 5.1.5 and Proposition 12.3.8). We likewise define an adapted norm $\|\cdot\|_B$ for B. Now let

$$S_A = \{x \mid \|x\|_A = 1\} \text{ and } S_B = \{x \mid \|x\|_B = 1\}.$$

Each nonzero orbit for A crosses S_A exactly once, and likewise for B, so we call these *fundamental domains* of the flows,[10] and

$$h_0 : S_A \to S_B, \quad x \mapsto \frac{x}{\|x\|_B}$$

is a homeomorphism with $h_0^{-1}(y) = \frac{y}{\|y\|_A}$. Extend h_0 to $\mathbb{R}^n$ by using that for $x \in \mathbb{R}^n \smallsetminus \{0\}$ there is a unique $\tau(x)$ such that $e^{A\tau} x \in S_A$, and $\tau(e^{At}x) = \tau(x) - t$ for all $t \in \mathbb{R}$:

$$h(x) \coloneqq \begin{cases} e^{-B\tau(x)} h_0(e^{A\tau(x)}x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a bijection of $\mathbb{R}^n$, continuous on $\mathbb{R}^n \smallsetminus \{0\}$, and satisfies

$$\begin{aligned} h(e^{At}x) &= e^{-B\tau(e^{At}x)} h_0(e^{A\tau(e^{At}x)} e^{At}x) \\ &= e^{-B(\tau(x)-t)} h_0(e^{A(\tau(x)-t)} e^{At}x) \\ &= e^{Bt} e^{-B\tau(x)} h_0(e^{A\tau(x)}x) \\ &= e^{Bt} h(x). \end{aligned}$$

To check that h is continuous at 0, let $x_j \xrightarrow[j\to\infty]{} 0$. Then $\tau_j = \tau(x_j) \xrightarrow[j\to\infty]{} -\infty$. Let $y_j = h_0(e^{A\tau_j} x_j)$. Since $\|y_j\|_B = 1$ for all j we have

$$\|h(x_j)\|_B = \|e^{-B\tau_j} y_j\|_B \le e^{-b|\tau_j|} \xrightarrow[j\to 0]{} 0,$$

so h is continuous at the origin. □

In the present context, we say that a matrix A is *hyperbolic* if none of the eigenvalues have zero real part. The eigenvalues are roots of the characteristic polynomial which vary continuously with the coefficients of the characteristic polynomial, and these coefficients vary continuously with the matrix. Thus, hyperbolicity is an open and can shown to be a dense property among matrices. The more general definition of hyperbolicity (Definition 5.3.48) is similarly shown to be an open property (Theorem 5.4.5). However, in this generality the set of hyperbolic flows is not dense among all smooth flows.

There is a related notion of hyperbolicity for maps (Definition 1.1.24) that can be seen by taking the time-t or time one map of the flow. In this case of the time one map we are looking at e^A and so we take the exponential of the eigenvalues for the flow. The corresponding notion is that the derivative of the map has no eigenvalues with modulus 1 (corresponding to strictly imaginary eigenvalues for the flow).

[10] These are sections, but the terminology is adopted from the discrete-time context, where it is natural.

For a hyperbolic matrix A the stable space for A is E^s_A, or E^s when there is no ambiguity, and consists of all vectors that are in the span of the generalized eigenspaces corresponding to eigenvalues with negative real part. Similarly, the unstable space for A is E^u_A, or E^u when there is no ambiguity, and consists of all vectors that are in the span of the generalized eigenspaces corresponding to eigenvalues with positive real part. Then $\mathbb{R}^n = E^s \oplus E^u$. Furthermore, Theorem 1.1.19 shows that if $x \in E^\sigma$ for $\sigma = u$ or s, then $x(t) \in E^\sigma$ for all $t \in \mathbb{R}$. So these subspaces are invariant for the flow. Remark 1.4.4 shows that there exists some K and $\alpha > 0$ such that

$$\|e^{At}x\| \le Ke^{-\alpha t} \text{ for all } x \in E^s,\ t \ge 0, \text{ and } \|e^{At}x\| \le Ke^{\alpha t} \text{ for all } x \in E^u,\ t \le 0.$$

Beyond the linear context, these rates of exponential contraction in forward or backward time will be the defining feature of hyperbolicity (Definition 5.3.48).

The next result shows that there are topological conjugacies between linear flows with constant coefficients if they are hyperbolic and the dimensions of the splittings into stable and unstable subspaces are equal.

Theorem 1.4.7. *If $A, B \in \mathcal{M}_n(\mathbb{R})$ are hyperbolic with stable/unstable splittings of the same dimension, then their flows e^{At} and e^{Bt} are topologically conjugate.*

Proof. Let $h_\sigma : E^\sigma_A \to E^\sigma_B$ for $\sigma = u$ or s where h is the conjugacy from Proposition 1.4.5. Let π_σ be the projection from $\mathbb{R}^d$ to E^σ for $\sigma = u$ or s. Then $x = \pi_u(x) + \pi_s(x)$. It is now a straightforward calculation to show that $h(x) = h_u(\pi_u(x)) + h_s(\pi_s(x))$ defines the desired conjugacy. □

The Hartman-Grobman Theorem (Theorem 5.6.1) states that if Φ is a flow with fixed point p such that the linear approximation of Φ at p is given by a hyperbolic matrix, then locally the nonlinear flow is topologically conjugate to the linearized flow.

We note that for nonhyperbolic matrices one does not expect, in general, that there is a conjugacy between the nonlinear flow and the linearized flow near a fixed point. In fact, it is not hard to give examples where the linearized flow is not asymptotically stable at the origin, but the nonlinear flow is asymptotically stable at the fixed point.

b. Lyapunov functions and attractors. Until the 1950s, local analysis, that is, the study of asymptotic stability and hyperbolicity, was largely limited to fixed points and periodic points. Attention focused on fixed points whose linearization is hyperbolic and periodic orbits whose return map is hyperbolic as described above. From the late 1950s onward more complicated invariant sets came into view as attractors, that is, possessing an open set of points that asymptotically limit on these sets. These sets are called an *attractor* if this happens as time approaches infinity or *repeller* if this happens as time approaches minus infinity (Definition 1.4.16).

Lyapunov developed a method to determine the basin of attraction for ordinary differential equations that does not require solving the equation, but instead uses something called a *Lyapunov function*—a continuous function that decreases along orbits. The difficulty to this method is in finding a Lyapunov function. In certain physical situations there are ways to do this; for instance, energy will be decreasing in mechanical systems with friction. Differential equations that admit Lyapunov functions sometimes allow heuristic approaches to guessing a Lyapunov function.

Example 1.4.8. If we modify the pendulum in Example 1.1.22 to account for friction, then a possible model is given for some $c > 0$ by the differential equation

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \sin 2\pi x = 0,$$

With $v \coloneqq \frac{dx}{dt}$ we obtain the system of first-order differential equations

$$\begin{cases} \dfrac{dx}{dt} = v, \\ \dfrac{dv}{dt} = -\sin 2\pi x - cv \end{cases}$$

for $x \in S^1$, $v \in \mathbb{R}$. Hence the total energy given by $H(x, v) = \frac{1}{2}v^2 - \frac{1}{2\pi}\cos 2\pi x$ (Figure 1.1.3) on the cylinder $S^1 \times \mathbb{R}$ decreases along orbits of the flow:

$$\frac{d}{dt}H(x, v) = v\frac{dv}{dt} + \frac{dx}{dt}\sin 2\pi x = -cv^2 \le 0,$$

with strict inequality when $v \neq 0$. Therefore, energy is now a Lyapunov function rather than a constant of motion, so orbits no longer lie on the energy level sets in Figure 1.1.4 but cross them “downward” at all times, which gives the phase portrait in Figure 1.4.1. Friction thus changes the character of the stable equilibria: They are now asymptotically stable.

Definition 1.4.9 (Lyapunov function)**.** For a flow φ^t on a space X a continuous function $L\colon X \to \mathbb{R}$ is a *Lyapunov function* if $L(\varphi^t(x)) \le L(x)$ for all $x \in X$ and all $t \ge 0$.

Remark 1.4.10. Note that constant functions are therefore always Lyapunov functions. It is thus tempting to require strict inequality when $t > 0$ and x is not fixed, as is the case in Example 1.4.8. (In that case we could say that L is a *strict* Lyapunov function.) However natural that might be for situations like the damped pendulum, there are important applications in which it is crucial to avoid this restriction.

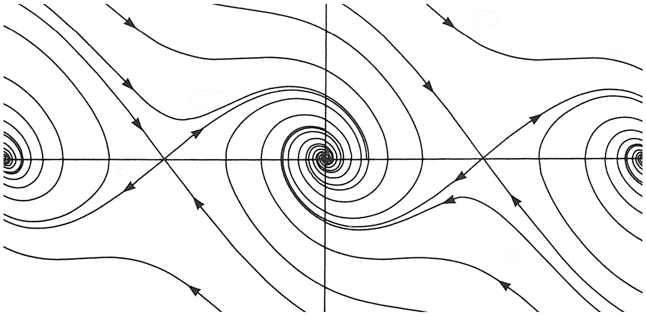


FIGURE 1.4.1. Phase portrait of the damped pendulum

On the other hand, if f is a Lyapunov function, then so are $\arctan f$, $f+c$ for any constant c, and cf for any positive constant c, so one can alway assume without loss of generality that a Lyapunov function takes values in a prescribed closed bounded interval (of positive length).

Example 1.4.11. $x \mapsto \mathfrak{c}(x) - x$ (Figure 1.3.2) is a strict Lyapunov function for the Akin flow (Example 1.3.12) on $[0,1]$.

Aside from Example 1.4.8, other previous instances of flows admit somewhat obvious Lyapunov functions. In Example 1.3.7 the height (that is, the z-coordinate) will do, and in Example 1.3.6 $L(x) = -x$ clearly works because (see Example 1.3.5) $\frac{d}{dt}L(x) = f(x) < 0$ away from the end-points. In Example 1.4.2, $|y|$ is a Lyapunov function whose absolute minimum is attained on the attracting periodic orbit.

Example 1.4.12 (Gradient flows)**.** Example 1.3.7 is a specific manifestation of a class of flows that by design have a Lyapunov function that one can think of as a "height." Consider a Riemannian metric on a compact smooth manifold M and a real-valued function F on M. At each point $x \in M$ that is not a critical point for F one can define the unique direction of fastest increase for F, that is, the unit tangent vector $\zeta(x) \in T_xM$ such that $\mathscr{L}_{\zeta(x)}F = \max_{\eta \in T_xM} \mathscr{L}_\eta F / \|\eta\|$, where $\mathscr{L}_\eta F$ denotes the Lie (directional) derivative of the function F along the vector η.

We define the gradient vector field ∇F by

$$\nabla F(x) = \begin{cases} \mathscr{L}_{\zeta(x)}F \cdot \zeta(x) & \text{if } x \text{ is noncritical,} \\ 0 & \text{if } x \text{ is critical.} \end{cases}$$

Suppose that in local coordinates $(x_1,\dots,x_n)$ the Riemannian metric has the form $ds^2 = \sum g_{ij}(x_1,\dots,x_n)dx_i dx_j$. Then

$$\nabla F(x_1,\dots,x_n) = G^{-1}(x)\left(\frac{\partial F}{\partial x_1},\dots,\frac{\partial F}{\partial x_n}\right),$$

where $G(x) = \{g_{ij}(x)\}$ and G^{-1} is the inverse matrix, so it is a smooth vector field on M. The flow generated by the gradient vector field ∇F is called the *gradient flow* of F.

From calculus we know that the gradient defined in coordinates is orthogonal to level sets of the function. This is still true in this setting because the direction of the gradient vector field is that of the fastest increase of the function F.

Example 1.3.7 is the gradient flow for the function $F(x,y,z) = -z$ on the two-sphere provided with the Riemannian metric induced from the standard Euclidean metric in $\mathbb{R}^3$.

Example 1.4.13 (Toral gradient flows)**.** To consider a less simple instance than Example 1.3.7, let $M \approx \mathbb{T}^2$ embedded in $\mathbb{R}^3$ as a doughnut standing up as in Figure 1.4.2, and as before, $F(x,y,z) = -z$, the negative of the height function. The function F has four critical points on the torus, a maximum A, two saddles B and C, and a minimum D. All orbits of the gradient flow other than those fixed points and six special orbits described below approach the minimum D as $t \to +\infty$ and the maximum A as $t \to -\infty$. Two special orbits connect A with B, two more connect B with C, and the last two connect C with D.

Now tilt this torus slightly, that is, change the embedding but keep the function F the same. Equivalently, consider instead the function $F = -z + \epsilon x$ for small $\epsilon > 0$. Four critical points remain, as well as the special orbits connecting the maximum with the upper saddle and the lower saddle with the minimum. However, the orbits connecting the two saddles disappear. Instead of these two orbits we have four: two connecting the maximum with the lower saddle and two connecting the upper saddle with the minimum; see Figure 1.4.2.

Example 1.4.14 (Hot vinyl)**.** Orbits of a gradient flow need not be asymptotic to a single fixed point. Consider an old-fashioned vinyl record suspended flat from its rim but sagging towards the center. The music is encoded by a groove that spirals towards a circular groove around the center. Consider such a grooved “bowl” but with an infinite spiral towards a circle. The gradient flow then has the bottom of the spiral groove as an orbit that is asymptotic to that entire circle—with ever-diminishing speed.

Lyapunov functions impose a gradient-like structure on the dynamics, but without the requirement that critical sets consist of fixed points. We will see that this

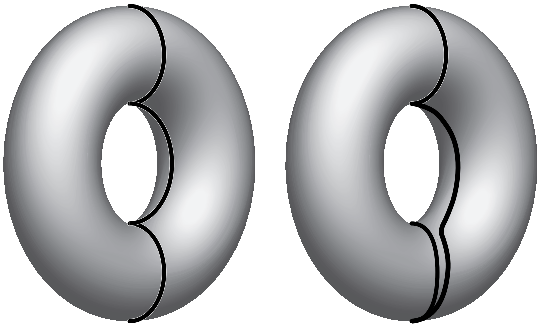


FIGURE 1.4.2. Gradient flows on the torus

makes them a universal tool for disentangling transient and recurrent dynamics. If, as in the case of gradient flows, a Lyapunov function is *strictly* decreasing along nonconstant orbits, then there is no nontrivial recurrence, and as in Example 1.4.8 this has long been a tool for establishing (asymptotic) stability in differential equations. Note that $L \equiv 0$ is always a Lyapunov function, so it is usually understood that a Lyapunov function is not meant to be constant. Functions that are merely nonincreasing along orbits can provide deep insights into the interplay between different invariant pieces of a dynamical system, as we now begin to demonstrate.

Definition 1.4.15. Let φ^t be a flow on a metric space X. A set $\varnothing \neq U \subsetneq X$ is a *trapping region* if $\varphi^t(U) \subset U$ for all $t \geq 0$ and there exists a $T > 0$ such that $\varphi^T(\bar{U}) \subset \operatorname{int}(U)$.[11]

Note that since a flow φ^t is a homeomorphism for each t that if U is a trapping region, then $X \smallsetminus \bar{U}$ is a trapping region for φ^{-t}. We need this fact in the next definition.

Definition 1.4.16. A set $A \subset X$ is an *attracting set* for the flow φ^t provided there exists a trapping region U such that $A = A_U := \bigcap_{t \geq 0} \varphi^t(U)$. We say that U is a trapping region for A. Similarly, the *repelling set* associated with U is $R_U := \bigcap_{t \leq 0} \varphi^t(X \smallsetminus \bar{U})$. For a given trapping region U the pair (A_U, R_U) of attracting and repelling sets for U is called an *attracting-repelling pair* for U. We denote the set of attracting-repelling pairs by $\mathscr{AR}(\Phi)$.

It is illuminating to explore these notions in the examples of flows that have appeared so far (Examples 1.1.5, 1.1.7, 1.1.8, 1.3.13, 1.3.5, 1.3.6, 1.3.9, 1.3.11, 1.3.12, 1.4.14), and a more subtle context is provided by Example 1.5.14 below.

[11] Note the improvement in Corollary 1.4.20.

The set of attracting-repelling pairs is not as large as one might suspect:

Lemma 1.4.17. *The set* $\mathscr{AR}(\Phi)$ *from Definition 1.4.16 is countable.*

Proof. Since X is a compact metric space, the topology has a countable base. For an attracting-repelling pair (A_U, R_U) there exists a neighborhood U of A_U, which is (without loss of generality) a finite union of elements of the countable base, such that $A_U \subset U$ and $R_U \subset X \smallsetminus \bar{U}$. Furthermore, A_U and R_U are the unique attracting-repelling pair associated with U, so we have a map $U \mapsto (A_U, R_U)$. Since there are at most countably many such U, the claim follows. □

Lemma 1.4.18. *If* φ^t *is a flow on a compact metric space, then any attracting or repelling set is nonempty, closed and invariant.*

Proof. If U is a trapping region and $T > 0$ is such that $\varphi^T(\bar{U}) \subset \operatorname{int}(U)$, then

$$\bigcap_{t\ge 0} \varphi^{t+T}(\bar{U}) \subset \bigcap_{t\ge 0} \varphi^t(U) = A_U \subset \bigcap_{t\ge 0} \varphi^t(\bar{U}) \subset \bigcap_{t\ge 0} \varphi^{t+T}(\bar{U}).$$

So A_U is an intersection of nonempty closed sets, hence nonempty and closed, and

$$\varphi^s(A_U) = \varphi^s\Big(\bigcap_{t\ge 0} \varphi^t(U))\Big) = \bigcap_{t\ge 0} \varphi^{s+t}(U) = \begin{cases} \bigcap_{t\ge s} \varphi^t(U) \subset \bigcap_{t\ge 0} \varphi^t(U) = A_U & \text{if } s \le 0, \\ \bigcap_{t\ge 0} \varphi^t(\underbrace{\varphi^s(U)}_{\subset U}) \subset \bigcap_{t\ge 0} \varphi^t(U) = A_U & \text{if } s \ge 0. \end{cases}$$

The proofs for repelling sets are similar. □

Attractor-repeller pairs are separated by Lyapunov functions:

Proposition 1.4.19. *Let* $(A, R) \in \mathscr{AR}$. *Then there is a Lyapunov function* $L\colon X \to [0,1]$ *such that* $L(A) = 0$, $L(R) = 1$, $L(X \smallsetminus (A \cup R)) \in (0,1)$, *and* L *is strictly decreasing along orbits of points outside* $A \cup R$.

Proof. A and R are disjoint compact sets, so

$$V(x) \coloneqq \frac{d(x, A)}{d(x, A) + d(x, R)}$$

is continuous with $V(A) = 0$ and $V(R) = 1$ and $V(X \smallsetminus (A \cup R)) \subset (0,1)$. From this we presently obtain a function that is strictly decreasing off $A \cup R$.

Since every orbit outside $A \cup R$ converges to A as $t \to \infty$ and to R as $t \to -\infty$, the supremum $\bar{V}(x) \coloneqq \sup V(\varphi^{[0,\infty)}(x)) = \max V(\varphi^{[0,t_x]}(x))$ is attained and hence continuous by compactness, continuity of V, and equicontinuity of the flow on $[0, t_x]$, where t_x is such that $V(\varphi^t(x)) < \bar{V}(x)/2$ for $t \ge t_x$. Also, $\bar{V}(\varphi^t(x)) \le \bar{V}(x)$ for all $x \in X$ and $t \ge 0$.

To make $\bar{V}$ strictly decreasing off $A \cup R$, let

$$L(x) := \int_0^\infty e^{-s} \bar{V}(\varphi^s(x))\,ds$$

be the weighted average of $\bar{V}$ along the forward orbit. Since $\bar{V}$ is continuous and nonincreasing, so is L: if $t \geq 0$, then

$$L(\varphi^t(x)) = \int_0^\infty e^{-s} \bar{V}(\varphi^{s+t}(x))ds \leq \int_0^\infty e^{-s} \bar{V}(\varphi^s(x))ds = L(x). \tag{1.4.1}$$

Now suppose $x \notin R$ is such that $L(\varphi^t(x)) = L(x)$ for some $t > 0$. Then on one hand $\varphi^t(x) \to A$ as $t \to \infty$ and on the other hand $\bar{V}(\varphi^{s+t}(x)) = \bar{V}(\varphi^s(x))$ for all $s > 0$ by (1.4.1), so $\bar{V}(x) = \bar{V}(\varphi^t(x)) \to 0$, and $x \in A$. □

This result allows us to slightly recast the terminology in Definition 1.4.16 by "improving" trapping regions as follows.

Corollary 1.4.20. *For any attracting set A there exists a trapping region U for A such that $\varphi^t(\bar{U}) \subset U$ for all $t > 0$.*

Proof. Let L be as in Proposition 1.4.19 and $U := L^{-1}([0,\epsilon))$ for $\epsilon > 0$ sufficiently small. Then $\bar{U} \subset L^{-1}([0,\epsilon])$ and $\varphi^t(\bar{U}) \subset L^{-1}([0,\epsilon))$ for all $t > 0$. □

The above results presage a remarkable general structural result: any flow admits a Lyapunov function, so the dynamics flows "downward" except for indecomposable dynamics on level sets of the Lyapunov function (Theorem 1.5.41). The next sections develop these indecomposability notions.

5. Recurrence properties and chain decomposition

Our study of dynamical behaviors has so far been limited to single orbits, and simple ones at that. We mainly considered fixed points, periodic orbits, and points that approach these orbits as time approaches infinity or minus infinity (asymptotic behavior). For example, orbits near an asymptotically stable fixed point have rather simple asymptotic behavior themselves; they converge to the fixed point. In particular, they are *transient* in the sense that there is a neighborhood to which they never return.

a. Recurrent points. We now develop terminology to describe more complicated asymptotic behavior.

Definition 1.5.1 (Limit set)**.** The *ω-limit set* of $x \in X$ for a flow Φ is the (closed) set

$$\omega(x) := \bigcap_{t \geq 0} \overline{\varphi^{[t,\infty)}(x)}$$

of accumulation points of the positive semiorbit. The α-limit set is defined similarly for negative time or as the ω-limit set for the inverse flow.

The closure $\mathscr{L}(\Phi)$ of the union of all ω-limit sets and all α-limit sets is the *limit set* of Φ.

Remark 1.5.2. For instance, the α-limit set and the ω-limit set of a periodic (or fixed) point both coincide with the orbit of that point. It is a good exercise here to determine these sets in the context of Examples 1.1.5, 1.1.7, 1.1.8, 1.3.13, 1.3.5, 1.3.6, 1.3.9, 1.3.11, 1.3.12, 1.4.14, 1.5.14, 1.6.2.

Note that $\omega(x)$ may be empty (but rarely so in this book, see Proposition 1.5.7). In the context of Definition 1.4.1, the fixed point is the ω-limit set for all orbits that ever come close enough. This motivates the following.

Proposition 1.5.3. *$q \in \omega(x) \Leftrightarrow$ there is a sequence $t_k \xrightarrow[k\to\infty]{} \infty$ with $\varphi^{t_k}(x) \xrightarrow[k\to\infty]{} q$.*

Proof. For $q \in \omega(x)$ and $k \in \mathbb{N}$ there exist $t_k \geq k$ such that $d(\varphi^{t_k}(x), q) < 1/k$. Conversely, $q = \lim_{k\to\infty} \varphi^{t_k}(x) \in \overline{\{\varphi^t(x) \mid t \geq m\}}$ for all $m \geq 0$. □

Starting earlier or later does not affect the asymptotics:

Proposition 1.5.4. *$\omega(x)$ is φ^t-invariant: If $s \in \mathbb{R}$, then $\varphi^s(\omega(x)) = \omega(x) = \omega(\varphi^s(x))$.*

Proof. φ^s is a homeomorphism, so

$$\bigcap_{T=0}^{\infty} \overbrace{\underbrace{\overline{\varphi^s(\varphi^{[T,\infty)}(x))}}_{=\overline{\varphi^{[T+s,\infty)}(x)}}}^{=\varphi^s(\overline{\varphi^{[T,\infty)}(x)})} = \begin{cases} \varphi^s\Big(\bigcap\limits_{T=0}^{\infty} \overline{\varphi^{[T,\infty)}(x)}\Big) = \varphi^s(\omega(x)) \\ \bigcap\limits_{T=0}^{\infty} \overline{\varphi^{[T,\infty)}(\varphi^s(x))} = \omega(\varphi^s(x)) \\ \bigcap\limits_{T=s}^{\infty} \overline{\varphi^{[T,\infty)}(x)} = \bigcap\limits_{T=0}^{\infty} \overline{\varphi^{[T,\infty)}(x)} = \omega(x). \end{cases} \qquad \square$$

Definition 1.5.5. If Λ is an invariant set for a flow Φ on X, define its *basin* of attraction or *stable set* and basin of repulsion or *unstable set* by

$$W^s(\Lambda) := \{x \in X \mid \varnothing \neq \omega(x) \subset \Lambda\},$$
$$W^u(\Lambda) := \{x \in X \mid \varnothing \neq \alpha(x) \subset \Lambda\}.$$

Remark 1.5.6. Compare with Definition 1.4.1. Examples 1.3.7 and 1.3.9 provide quite complementary simple instances of these sets. Figure 1.5.4 below shows a rather more interesting situation in this respect.

Proposition 1.5.7. *If $\mathscr{O}^+(x) := \varphi^{[0,\infty)}(x) \subset K$ with $K \subset X$ compact, then*

(1) $\varnothing \neq \omega(x) \subset K$,

(2) $\omega(x)$ is compact,

(3) *$d(\varphi^t(y),\omega(y)) \xrightarrow[t\to\infty]{} 0$: if $\omega(x)\subset O$ open $\Rightarrow \exists T\in\mathbb{R}$ with $\varphi^{[T,\infty)}(x)\subset O$.*

(4) *$\omega(x)$ is connected, so it is either a single point or infinite.*

Proof. $i\mapsto\varphi^i(x)$ has an accumulation point in K, so Proposition 1.5.3 gives (1).

(2): $\omega(x)\subset K$ is closed, hence compact.

(3): Otherwise there are $t_i\to+\infty$ with $\varphi^{t_i}(x)\in K\smallsetminus O$, and these points accumulate in the compact set $K\smallsetminus O$, contrary to $\omega(x)\subset O$.

(4): We show that if $p,q\in\omega(x)$ have disjoint neighborhoods O_p, O_q, then $\omega(x)\not\subset O_p\cup O_q$. Pick $\tau_n\to\infty$, $t_n\geq 0$ such that $p_n:=\varphi^{\tau_n}(x)\to p$ in O_p and $q_n:=\varphi^{t_n}(p_n)\to q$ in O_q, and let

$$s_n:=\max\{t\in[0,t_n] \mid \varphi^{[0,t)}(p_n)\subset O_p\}.$$

Then $\varphi^{s_n}(p_n)=\varphi^{\tau_n+s_n}(x)\in K\cap\partial O_p$ has an accumulation point which is in ∂O_p hence outside $O_p\cup O_q$ and on the other hand in $\omega(x)$ since $\tau_n\to\infty$. □

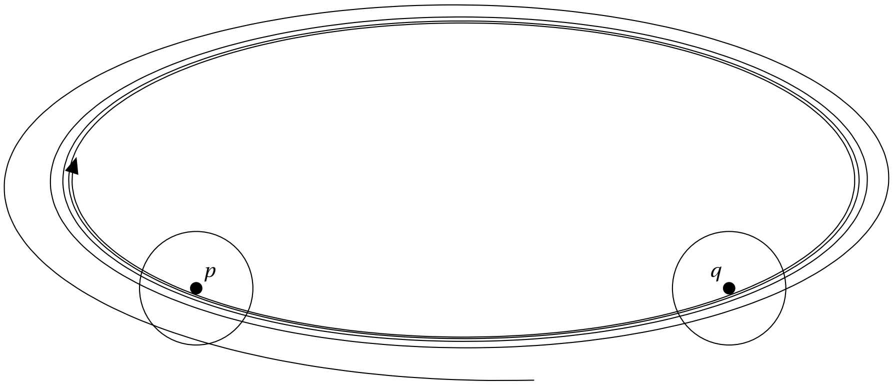


FIGURE 1.5.1. Proof of Proposition 1.5.7(4)

Corollary 1.5.8. *If h is a constant of motion (Definition 1.1.23) for a flow Φ on a* compact *space X, then $h(X)=h(\mathscr{L}(\Phi))$ (Definition 1.5.1).*

Proof. If $x\in X$, then $h(\{x\})=h(\overline{\varphi^{\mathbb{R}}(x)})=h(\omega(x))\subset h(\mathscr{L}(\Phi))$. □

Definition 1.5.9 (Recurrence)**.** A point x is *ω-recurrent* or *positively recurrent* if $x\in\omega(x)$, *α-recurrent* or *negatively recurrent* if $x\in\alpha(x)$, and *recurrent* (or *Poisson stable*) if $x\in\alpha(x)\cap\omega(x)$. We denote the closure of the set of recurrent points by $\mathscr{B}(\Phi)$—for "Birkhoff center" (Remark 1.5.37).

Remark 1.5.10. $\overline{\mathrm{Per}(\Phi)}\subset\mathscr{B}(\Phi)\subset\mathscr{L}(\Phi)$ (see Definition 1.1.10).

b. Nonwandering. The next generalization of recurrence that we study is that a point may not come back close to itself, but a different point arbitrarily close to a given point comes back close to the given point.

Definition 1.5.11 (Nonwandering)**.** $x \in X$ is *nonwandering* for a flow Φ on X if for any neighborhood U of x and $T_0 > 0$ there is a $t > T_0$ with $\varphi^t(U) \cap U \neq \varnothing$;[12] otherwise x is said to be *wandering*. The set of nonwandering points is denoted by $NW(\Phi)$. We say that Φ is *regionally recurrent* if $NW(\Phi) = X$.

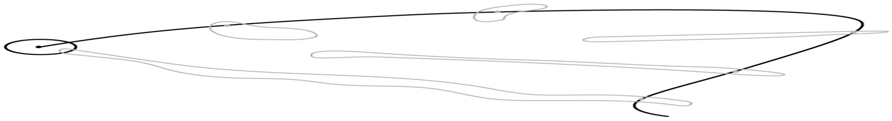


FIGURE 1.5.2. A nonwandering point

Remark 1.5.12 (Auslander)**.** Recurrence of x is defined in terms of the ω-limit set of x, and analogously, $x \in NW(\Phi)$ is equivalent to the existence of $x_i \xrightarrow[i\to\infty]{} x$, $t_i \xrightarrow[i\to\infty]{} +\infty$ such that $\lim_{i\to\infty} \varphi^{t_i}(x_i)$ and hence to

$$x \in PL(x) := \Big\{ \lim_{i\to\infty} \varphi^{t_i}(x_i) \;\Big|\; x_i \xrightarrow[i\to\infty]{} x,\ t_i \xrightarrow[i\to\infty]{} +\infty \Big\} = \bigcap_{t\in\mathbb{R}} \bigcap_{\epsilon>0} \overline{\varphi^{(t,\infty)}(B(x,\epsilon))},$$

the first *prolongational limit set* of x.

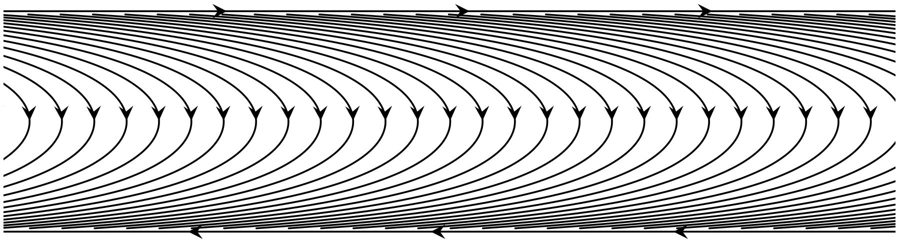


FIGURE 1.5.3. The first prolongational limit set of each point on the top line is the bottom line

[12] The following from [**159**, p. 22] may be helpful: "A better choice of words, suggested to us by K. Sigmund, is that a point is called *nostalgic* iff its neighborhoods U keep returning as in the definition of [nonwandering]. The point itself may or may not return near by, but its thoughts (nearby points) always do."

The definition of the nonwandering set looks asymmetric in time, but this is only apparent since $\varphi^t(U) \cap U \neq \varnothing \Leftrightarrow \varnothing \neq \varphi^{-t}(\varphi^t(U) \cap U) = \varphi^{-t}(U) \cap U$:

Proposition 1.5.13. *$NW(\varphi^t) = NW(\varphi^{-t})$, that is, a point $x \in X$ is nonwandering if and only if for all neighborhoods U of x and all $T_0 < 0$ there is a $t < T_0$ with $\varphi^t(U) \cap U \neq \varnothing$.*

Example 1.5.14. From the examples so far it is not apparent that being nonwandering is a strictly weaker notion than recurrence, and Figure 1.5.4 shows a planar flow[13] with nonwandering nonrecurrent points. Only the 3 fixed points are recurrent, but the nonwandering set includes the entire "∞" curve. Note as well that the flow *restricted* to this nonwandering set has only fixed nonwandering points.

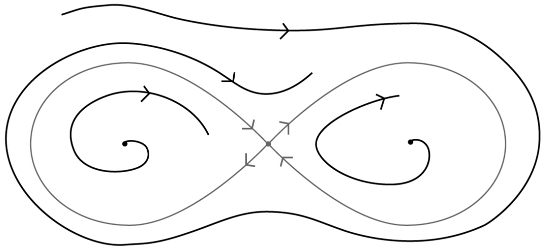


FIGURE 1.5.4. The Bowen–Katok "figure eight attractor" [**178**, p. 140]

Proposition 1.5.15. *$NW(\Phi)$ is closed and $\overline{\mathrm{Per}(\Phi)} \subset \mathscr{B}(\Phi) \subset \mathscr{L}(\Phi) \subset NW(\Phi)$.*

PROOF. A wandering x has a neighborhood U and a $T > 0$ with $\varphi^t(U) \cap U = \varnothing$ for all $t > T$. Then every point in U is wandering, so the set of wandering points is open. If $x \in X$, $y \in \omega(x)$ and $y \in O$ open, $T_0 > 0$, take $t_1 > 0$ and $t > T_0$ such that $\varphi^{t_1}(x) \in O$ and $\varphi^{t_1+t}(x) \in O$ (since $y \in \omega(x)$), hence $\varphi^{t_1+t}(x) \in \varphi^t(O) \cap O$. Thus, $\forall x\colon \omega(x) \subset NW(\Phi)$, so $\mathscr{L}(\Phi) \subset NW(\Phi)$ (Proposition 1.5.13). The rest follows from Remark 1.5.10 and Proposition 1.5.15. □

Remark 1.5.16. While examples show that each of the inclusions in Proposition 1.5.15 can be strict, a deep and important result, the proof of which is well beyond our scope, says that C^1-generically, they are not (Theorem 1.5.19).

Definition 1.5.17. For $k \geq 0$ the C^k-distance between two C^k flows on a C^k-manifold M is the usual (uniform) C^k-distance between their restrictions to $[0,1] \times$

[13] By including ∞ as a repelling fixed point, this becomes an example on the 2-sphere with 4 fixed points.

M. In a topological space, the intersection of a countable collection of open sets is called a G_δ-set. A property of elements of a topological space is said to be *generic* if it holds for each member of a dense G_δ-set.[14]

Theorem 1.5.18 (Pugh Closing Lemma [**13**, **155**, **252**–**254**])**.** *For a nonwandering point of a vector field there is an arbitrarily C^1-close vector field for which this point is periodic.*

PROOF STRATEGY. The basic task would seem to be rather obvious: consider a tube around an orbit segment that starts and ends near enough the nonwandering point p and make a perturbation of the vector field inside the tube that moves p onto this orbit at the start and of it at the end. The difficulty arises from the fact that we are aiming for the reverse of a usual perturbation result: those usually involve arbitrarily small modifications, but here we must, for a give length of nearby orbit, change the dynamics by a definite amount. For instance, a tube as described might well have to necessarily self-intersect because the nearby orbit is very long and tangled. Thinning the tube might avert the problem but make it difficult to perturb p enough and moreover, localizing perturbations more requires larger derivatives, which countervails the desired C^1-smallness of the perturbation. Instead, choosing many flow boxes along parts of that orbit that aren't too close to others is a better strategy... This balancing act makes for a formidable proof in which the gentlest possible deformations are just barely made to accumulate enough total change over the length of the orbit. Counterexamples to C^2 versions of this underscore the delicacy of what is required. (On the other hand, there are astonishing results in low dimensions [**16**, **166**].) □

Together with general genericity results and Theorem 6.1.6, this implies:

Theorem 1.5.19 (Pugh General Density Theorem [**253**])**.** $\overline{\mathrm{Per}(\Phi)} = NW(\Phi)$ *generically among C^1-flows.*

Recurrence other than periodicity is usually referred to a *nontrivial recurrence*. Since smooth curves locally separate the plane, flows on simply-connected surfaces have only trivial recurrence:

Theorem 1.5.20 (Poincaré–Bendixson Theorem)**.** *Let Φ be a C^1 flow on an open subset of the sphere S^2. Then all positively or negatively recurrent orbits are periodic. Furthermore, if the ω-limit set of a point contains no fixed points, then it consists of a single periodic orbit.*[15]

[14] While this notion can be defined in this generality, it is usually applied in complete metric spaces where the Baire Category Theorem can be used.

[15] The Poincaré–Bendixson Theorem won't be used in the sequel.

Proof. Suppose p is positively recurrent and neither fixed nor periodic. Take a short transversal γ at p and let t be the smallest positive number for which $\varphi^t(p) \in \gamma$. Then the union of the orbit segment $\{\varphi^s(p)\}_{0\le s\le t}$ and the piece of γ between p and $\varphi^t(p)$ is a simple closed curve $\mathscr{C}$ called a pretransversal. By the Jordan Curve Theorem the complement of $\mathscr{C}$ consists of two disjoint open sets A and B. We may label them such that near γ the flow goes from A to B. This implies that the positive semiorbit of $\varphi^t(p)$, hence the ω-limit set $\omega(p)$ of p, is in B. Since p is recurrent we have $A \ni \varphi^{-\epsilon}(p) \in \mathscr{O}(p) \subset \omega(p) \subset B$, a contradiction.

Now assume that $W := \omega(p)$ contains no fixed points. By Remark 1.6.29 below there are recurrent points in W. By the preceding, these are periodic. Thus let $q \in W$ be periodic. Consider a small transverse segment γ containing q. By continuity the return map to this segment is defined on a neighborhood of q in γ. Take a one-sided neighborhood I of q small enough so that the first point $\varphi^t(p)$ in γ is not in I, but infinitely many of these returns are. Parameterizing this neighborhood by $[0,\delta)$ gives a continuous map f from an interval $[0,\delta)$ to an interval $(0,\delta')$ that fixes 0. The orbit of p provides infinitely many $x \in (0,\delta)$ for which $f(x) < x$, so either $f(x) < x$ for all $x \in [0,\delta)$ or $[0,\delta)$ contains a fixed point y. The latter case is impossible, since the interval $[0,y]$ would be invariant under f and hence there would be an invariant annulus for the flow that separates the orbit of q from that of p, so $q \notin \omega(p)$. But if $f(x) < x$ then all $x \in (0,\delta)$ are positively and monotonically asymptotic to 0. Since the return times to I are bounded this means that the orbit segments of p between successive returns converge to the orbit of q, so $\omega(p)$ coincides with the orbit of q. □

By contrast, higher-dimensional flows can be rather more complex; it is a good exercise to also explore the various recurrence notions in the next examples. Example 1.5.21 and Example 1.5.23 will also serve as standard examples of hyperbolic flows that we describe later.

Example 1.5.21 (Smale horseshoe)**.** The flow we introduce here is a suspension or special flow over a map of R^2 or S^2 (or of any surface) which arises naturally in a Poincaré section. This map f squeezes a rectangle Δ vertically, stretches it

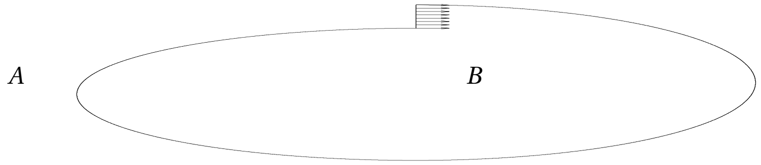


Figure 1.5.5. A pretransversal

horizontally and folds it over the original rectangle (Figure 1.5.6) in a horseshoe shape. Specifically, let us assume that the map is linear on the 2 halves, so the contraction factor is a constant $\lambda < 1/2$ and the expansion is by a factor $\mu > 2$ (to ensure that there are gaps between the branches) and that there are two complete strips that are folded back over Δ. The set $\Lambda := \bigcap_{n\in\mathbb{Z}} f^n(\Delta)$ of points whose orbits are in Δ is then a Cantor set with vertical contracting direction and horizontal expanding direction (Figure 1.5.7). A horseshoe flow is the time-1 suspension. Note that we have partially or implicitly defined a smooth flow but will focus on the continuous flow obtained by restricting to the suspension of Λ. Variants on this original construction allow for more crossings in Δ as well as nonlinearly expanding and contracting directions.

Example 1.5.22 (Linked horseshoes)**.** More generally, several rectangles might be mapped across each other in a like fashion, Figure 1.5.8 shows an instance that involves 2 rectangles with horizontal stretching; the black rectangle is mapped across both rectangles plus across itself a second time, while the other rectangle is mapped once across both rectangles.

Example 1.5.23 (Toral automorphism)**.** Consider the suspension (Definition 1.2.4) of the following map of the 2-torus. The linear map of $\mathbb{R}^2$ given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has integer entries and hence induces a well-defined map F_A of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Since it has unit determinant, the same goes for the inverse, which means that it defines a diffeomorphism (indeed, algebraic automorphism)

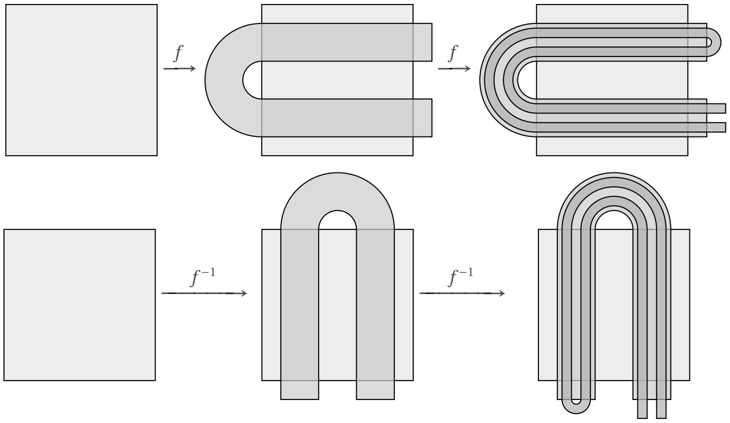


FIGURE 1.5.6. Horseshoe

of $\mathbb{T}^2$—which, furthermore, preserves area. The eigenvalues are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} > 1 \text{ and } \lambda_1^{-1} = \lambda_2 = \frac{3-\sqrt{5}}{2} < 1.$$

The eigenvectors for the first eigenvalue are on the line $y = \frac{\sqrt{5}-1}{2}x$. The family of lines parallel to it is invariant, and distances on those lines are expanded by a factor λ_1. Similarly, there is an invariant family of contracting lines $y = \frac{-\sqrt{5}-1}{2}x + \text{const.}$.

The tangent space spanned by the direction of expansion and contraction together with the flow direction define hyperbolicity (formally so in Definition 5.1.1). Note that the stable and unstable sets (Definition 1.3.24) for any point in the suspension flow are given by translations of the contracting and expanding eigendirections, respectively.

It is an interesting exercise to show that the collection of periodic points for the map F_A is exactly the set of points with rational coordinates, so the periodic orbits in the suspension flow are dense. But we will see later that there are also dense orbits for both F_A and the suspension, indeed, almost every orbit is dense.

Example 1.5.24. More generally, any $A \in \mathrm{GL}(m,\mathbb{Z})$ induces an automorphism F_A of $\mathbb{T}^m$ that preserves Lebesgue measure.[16] We say that it is hyperbolic if A has no

[16] Here, $\mathrm{GL}(m,\mathbb{Z})$ consists of the integer matrices that are invertible *among integer matrices*, which requires that they have determinant ±1.

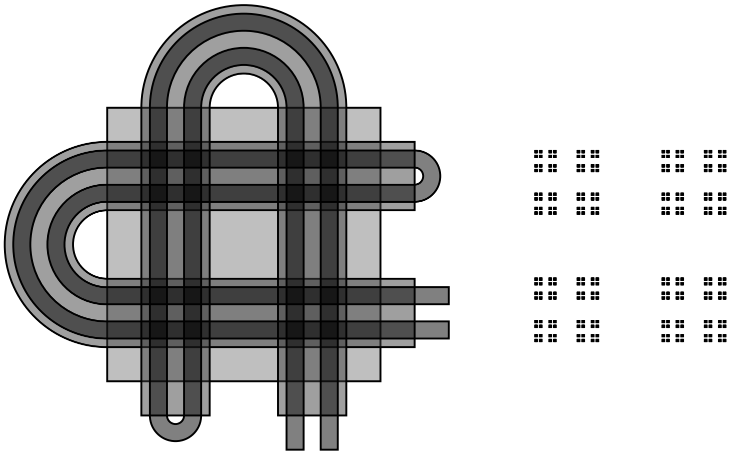


FIGURE 1.5.7. The invariant set of the Smale Horseshoe

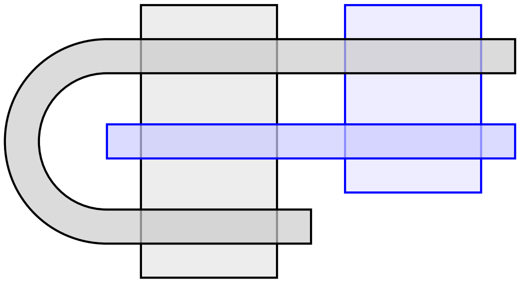


FIGURE 1.5.8. Linked horseshoes (Example 1.5.22)

eigenvalues on the unit circle. We will see that the suspension is then hyperbolic on the whole suspension manifold.

Remark 1.5.25. In contrast with a phenomenon we will see later (Theorem 9.5.1), the universal cover $\mathbb{R}^2 \times \mathbb{R}$ of the suspension manifold has a (dynamical) global product structure as follows: Each contracting plane $\left\{y = \frac{-\sqrt{5}-1}{2}x + \text{const.}\right\} \times \mathbb{R}$ meets each expanding plane $\left\{y = \frac{\sqrt{5}-1}{2}x + \text{const.}\right\} \times \mathbb{R}$ in an orbit (which is necessarily unique).

c. Chain recurrence. The notion of a nonwandering point involves nearby orbits; another variant of recurrence behavior is expressed in terms of objects that are nearly orbits.

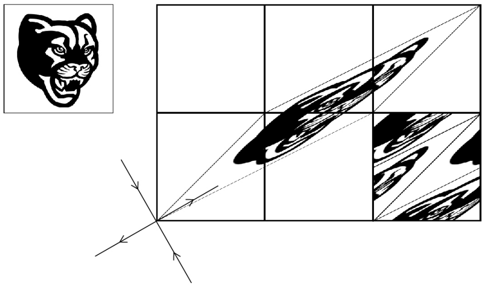


FIGURE 1.5.9. Example 1.5.23, Cat (cougar) map

Definition 1.5.26 (Pseudo-orbit, chain)**.** An *ϵ-pseudo-orbit* or *ϵ-chain* for a flow Φ on a space X is a map $g\colon I \to X$ on a nontrivial interval $I \subset \mathbb{R}$ such that

$$d(g(t+\tau), \varphi^\tau(g(t))) < \epsilon, \quad \text{for } t, t+\tau \in I \text{ and } |\tau| < 1.$$

It is a pseudo-orbit from x to y of length T if $0, T \in I$ and $g(0) = x$, $g(T) = y$.

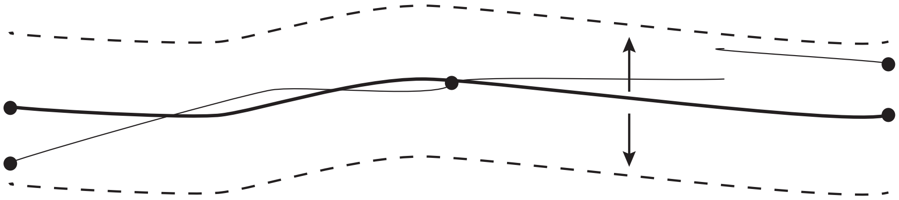


FIGURE 1.5.10. ϵ-pseudo-orbit

Note that g need not be continuous, see Figure 1.5.10.

This important notion is a little more involved for flows than for diffeomorphisms. Specifically, an alternate definition of an ϵ-pseudo-orbit is that there is a sequence of points $\{x = x_0, \dots, y = x_n\}$ and times $t_j \ge 1$ with $t_1 + \cdots + t_n = T$ and $d(\varphi^{t_j}(x_{j-1}), x_j) < \epsilon$ for all $1 \le j \le n$. These variants are related as follows.

Proposition 1.5.27. *Let X be a metric space, Φ a flow on X, $\epsilon > 0$, and $\delta > 0$ such that $d(x, y) < \delta \Rightarrow d(\varphi^t(x), \varphi^t(y)) < \epsilon$ for $0 \le t \le 2$. Then we have the following:*

1. *If there exist points $\{x = x_0, \dots, x_n = y\}$ and times $t_j \ge 1$ with $t_1 + \cdots + t_n = T$ and $d(\varphi^{t_j}(x_{j-1}), x_j) < \delta$ for all $1 \le j \le n$, then there is an ϵ-pseudo-orbit of length T from x to y.*
2. *If there is a δ-pseudo-orbit of length $T > 1$ from x to y, then there are points $\{x = x_0, \dots, y = x_n\}$ and times $t_j \ge 1$ with $t_1 + \cdots + t_n = T$ and $d(\varphi^{t_j}(x_{j-1}), x_j) < \epsilon$ for all $1 \le j \le n$.*

PROOF. (1): For $t \in [0, T]$ there is a unique $j \in \{1, \dots, n\}$ with $t_1 + \cdots + t_j < t \le t_1 + \cdots + t_j + t_{j+1}$. Define $g^t(x) = \varphi^{t-(t_1+\cdots+t_j)}(x_{j-1})$ and check that $g\colon [0, T] \to X$ is an ϵ-pseudo-orbit of length T from x to y.

(2): Let $g\colon [0, T] \to X$ be a δ-pseudo-orbit from x to y. Set $x_0 = x$, $x_n = y$, $n = \lceil T \rceil - 1 \in (T/2, T]$, $t_j = \frac{T}{n} \in [1, 2)$ for $j \in \{1, \dots, n\}$, and $x_j = g(j\frac{T}{n})$ for $j \in \{1, \dots, n-1\}$. Then the choice of δ gives $d(\varphi^{t_j}(x_{j-1}), x_j) < \epsilon$ when $1 \le j \le n$ □

By Proposition 1.5.27 the two ways of defining a pseudo-orbit can be used interchangeably, and so we will.

Remark 1.5.28. Pseudo-orbits arise in different ways. A pseudo-orbit might consist of orbit segments with jumps, that is, it is given by a sequence of points $x_k \in M$ and times $t_k \in \mathbb{R}^+$ such that $\inf t_k > 0$, $\sup t_k < \infty$, and $d(\varphi(t_k, x_k), x_{k+1}) < \delta$. The term "chain" seems particularly apt in this case. It might "drift" if it is the orbit of a perturbation of the given vector field; an orbit for the new vector field will be a pseudo-orbit for the old vector field. In this case there are no jumps (discontinuities) but there can be a "drift" from a true orbit. In full generality one may combine jumps and drift.

Moreover, the arguments in the proof of Proposition 1.5.27 combined with interpolation show that on a topological manifold one can without loss of generality take a pseudo-orbit to be continuous, and on a smooth manifold one can take it to be smooth. In that case one can with additional work furthermore arrange for the tangent vectors to the pseudo-orbit to be close to the vector field that generates Φ.

This notion lends itself to a surprising way of defining trapping regions that plays an important role in understanding the global structure of a flow.

Proposition 1.5.29. *The set $\mathscr{R}_\epsilon(x)$ of end-points of ϵ-pseudo-orbits that start at x*

(1) is open,
(2) satisfies $\varphi^{(0,\infty)}(x) \subset \mathscr{R}_\epsilon(x)$ (an orbit segment is an ϵ-chain),
(3) satisfies $y \in \mathscr{R}_\epsilon(x) \Rightarrow \mathscr{R}_\epsilon(y) \subset \mathscr{R}_\epsilon(x)$ (by concatenation of chains),
(4) satisfies $\varphi^t(\mathscr{R}_\epsilon(x)) \subset \mathscr{R}_\epsilon(x)$ for $t \geq 0$ (by concatenation of chains), and
(5) is a trapping region.

Proof. (1) If $y \in \mathscr{R}_\epsilon(x)$, then $B_\delta(y) \subset \mathscr{R}_\epsilon(x)$ for sufficiently small δ by modifying the connecting ϵ-orbit.

(5) If $y \in \overline{\mathscr{R}_\epsilon(x)}$ take $\mathscr{R}_\epsilon(x) \ni y_n \xrightarrow[n\to\infty]{} y$, so $\varphi^1(y_n) \xrightarrow[n\to\infty]{} \varphi^1(y)$ by continuity. Then there is an $N \in \mathbb{N}$ such that $d(\varphi^1(y_n), \varphi^1(y)) < \epsilon/2$ for all $n \geq N$, so $\varphi^1(y) \in \mathscr{R}_\epsilon(y_n) \subset \mathscr{R}_\epsilon(x)$. We have shown that $\varphi^1(\overline{\mathscr{R}_\epsilon(x)}) \subset \mathscr{R}_\epsilon(x)$. □

This helps understand the global structure of a flow via the following important recurrence notion.

Definition 1.5.30 (Chain recurrence, equivalence, components, decomposition)**.** Let Φ be a continuous flow on a metric space X. A point x is *chain recurrent* if $x \in \bigcap_{\epsilon>0} \mathscr{R}_\epsilon(x)$, that is, for all $\epsilon > 0$ there is an ϵ-pseudo-orbit from x to x. In other words, x lies on a closed ϵ-chain for any $\epsilon > 0$. The set $\mathscr{R}(\Phi)$ of chain recurrent points is the *chain recurrent set* of Φ.

For points $x, y \in \mathscr{R}(\Phi)$ we say $x \sim y$ or x, y are *chain-equivalent* or *chainable* if $x \in \bigcap_{\epsilon>0} \mathscr{R}_\epsilon(y)$ and $y \in \bigcap_{\epsilon>0} \mathscr{R}_\epsilon(x)$, that is, for all $\epsilon > 0$ there is an ϵ-pseudo-orbit from x to y and an ϵ-pseudo-orbit from y to x. In other words, x, y lie on a common closed ϵ-chain for any $\epsilon > 0$.

The equivalence classes of $\sim$ define the *chain decomposition* into the *chain (-transitive) components*, *chain recurrent classes*, *chain-equivalence classes*, or (in hyperbolic flows) *homoclinic classes* of $\mathscr{R}(\Phi)$.

Φ is said to be *chain-transitive* if $\mathscr{R}(\Phi) = X$ and there is only one chain component.

Remark 1.5.31. Again, while these notions are surprisingly effective for hyperbolic flows, they do not in themselves imply any complexity—see the Conley example (Figure 1.5.11) or, for that matter, the constant flow $(t, x) \mapsto x$ or (more naturally) Figure 1.1.4 (on the cylinder $S^1 \times \mathbb{R}$; this example also illustrates that a constant of motion need not be constant on $\mathscr{R}(\Phi)$) or the geodesic flow on the torus $\mathbb{T}^n$.

The following justifies the term “chain-equivalence”:

Proposition 1.5.32. *Chain-equivalence is an equivalence relation on* $\mathscr{R}(\Phi)$.

Proof. Symmetry is clear, and reflexivity follows by definition of $\mathscr{R}(\Phi)$. Transitivity: $x \sim y \sim z \Rightarrow x \in \mathscr{R}_\epsilon(y) \subset \mathscr{R}_\epsilon(z)$ because $y \in \mathscr{R}_\epsilon(z)$ (Proposition 1.5.29(3)). □

We note that “small” changes can make a big difference. The chain-recurrent set is very different for the south-south dynamics (Example 1.3.9) on the circle (chain-transitive) versus its interval counterpart (only the 2 fixed points are chain-recurrent). Likewise, the chain-recurrent set of the Akin flow on the interval (Example 1.3.12) is the ternary Cantor set, while the projection $A_\circ$ to the circle is chain-transitive.

Remark 1.5.33. $NW(\Phi) \subset \mathscr{R}(\Phi) = \overline{\mathscr{R}(\Phi)}$ (it is easy to check that the complement of $\mathscr{R}(\Phi)$ is open.) As before, it is good to examine this notion in the context of our examples so far, for instance by identifying the recurrent, nonwandering, and chain-recurrent sets in Figure 1.3.3, Figure 1.5.4 and Example 1.3.6 as well as in Conley’s example of a continuous vector field that is zero on the boundary of a rectangle and nonzero pointing downward inside (Figure 1.5.11). This is a somewhat “pathological” situation, which should induce scepticism about the notion of chain-recurrence, and Figure 1.1.4 shows a natural chain-transitive example where a meaningful analysis would produce much finer information than chain-transitivity alone. However, the value of the chain-decomposition in understanding the global structure of a continuous flow justifies the notion, particularly in the context of hyperbolicity, which precludes the occurrence of such pathology (Corollary 5.3.14(1)).

To summarize, Proposition 1.5.15 and Remark 1.5.33 give:

Proposition 1.5.34. $\overline{\operatorname{Per}(\Phi)} \subset \mathscr{B}(\Phi) \subset \mathscr{L}(\Phi) \subset NW(\Phi) = \overline{NW(\Phi)} \subset \mathscr{R}(\Phi) = \overline{\mathscr{R}(\Phi)}$.

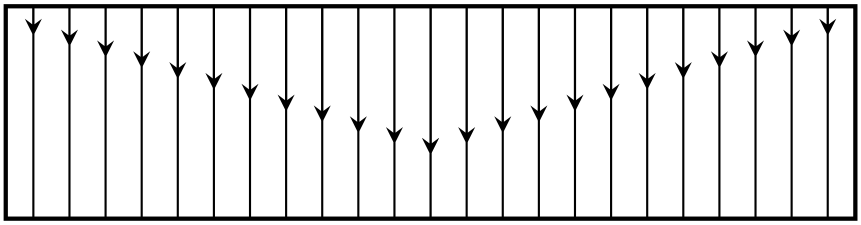


FIGURE 1.5.11. The Conley example

Remark 1.5.35. One should not expect a strengthening of Proposition 1.5.34—each of these inclusions can be strict (Exercise 1.21). This does not mean that we do not often have equality. When there is no nontrivial recurrence at all, then these levels of recurrence are all conflated, and they are also so C^1-generically (Theorem 1.5.19). This is the case in our simplest examples. More importantly for us, however, these sets tend to coincide in hyperbolic flows not despite but because of the complexity of the dynamics. This is the content of Proposition 5.3.31, and "semilocal" counterparts follow from Theorem 5.3.35.

Proposition 1.5.32 suggests studying a continuous flow through the strategy of restricting to chain components, so we pause to note that this is not a recursive process, that is, that chain components are themselves chain-recurrent:

Theorem 1.5.36 (Restriction property)**.** *Let Φ be a flow on a compact metric space X. Then $\mathscr{R}(\Phi\upharpoonright_{\mathscr{R}(\Phi)}) = \mathscr{R}(\Phi)$ and with the same chain-decomposition.*

PROOF. "$\subset$" is obvious: $\mathscr{R}(\Phi\upharpoonright_A) \subset A$ for any A, and if x, y lie on a common periodic ϵ-chain in $CR(\Phi)$ then these trivially are periodic ϵ-chains in X. Conversely, let $x \in \mathscr{R}(\Phi)$ and $g_n \colon \mathbb{R} \to X$ a periodic $1/n$-pseudo-orbit for $n \in \mathbb{N}$ with $g_n(0) = x$ (and $g_n(t) = y$ for some t to prove heredity of chain-equivalence). Note first that it suffices to show that for any neighborhood U of $\mathscr{R}(\Phi)$ there is an $N \in \mathbb{N}$ with $g_n(\mathbb{R}) \subset U$ for $n \geq N$. To show this, suppose (by compactness) to the contrary that there are a $z \notin \mathscr{R}(\Phi)$ and sequences $n_k \xrightarrow[k\to\infty]{} +\infty$ and t_k with $g_{n_k}(t_k) \xrightarrow[k\to\infty]{} z$. But then the periodic pseudo-orbits

$$\bar{g}_k(t) \coloneqq \begin{cases} g_{n_k}(t + t_k) & \text{if } i \notin p_k\mathbb{Z}, \\ z & \text{if } i \in p_k\mathbb{Z}, \end{cases}$$

where p_k is the period of g_{n_k}, show that $z \in \mathscr{R}(\Phi)$. □

Remark 1.5.37. In contrast to this heredity of chain-recurrence, $NW(\Phi\restriction_{NW(\Phi)}) \neq NW(\Phi)$ in general; see Example 1.5.14. The *Birkhoff center* of a flow is defined by recursively restricting to the nonwandering set, that is, from $NW(\Phi)$ pass to $NW(\Phi\restriction_{NW(\Phi)})$, etc.[17] The ultimate intersection is called the Birkhoff center and can be characterized as the maximal set C such that $C = NW(\Phi\restriction_C)$. It is a closed set that contains the recurrent points, and for flows on complete metric spaces it coincides with their closure. This explains the notation $\mathscr{B}$ in Definition 1.5.9.

Remark 1.5.38. The heredity of chain-recurrence gives it a somewhat intrinsic nature, and this makes it natural to note that ω-limit sets are characterized by being connected (Proposition 1.5.7) and chain-recurrent (Proposition 1.5.15): if Φ is a continuous flow on a connected space X and $X = \mathscr{R}(\Phi)$, then Φ is topologically conjugate to the restriction of some continuous flow to the ω-limit set of some point [**121**].

Lyapunov functions (Definition 1.4.9) and Proposition 1.5.29 make it possible to connect the notion of chain recurrence with stability as represented by attracting-repelling pairs (see Definition 1.4.16), incuding a characterization of chain-equivalence in terms of attracting-repelling pairs:

Theorem 1.5.39. *Let Φ be a flow on a compact metric space. Then*

$$\mathscr{R}(\Phi) = \bigcap_{(A,R)\in\mathscr{AR}} A \cup R.$$

If $x, y \in \mathscr{R}(\Phi)$, then $x \sim y$ if and only if for each $(A,R) \in \mathscr{AR}$ (see Definitions 1.4.16 and 1.5.30), x and y are either both in A or both in R.

Remark 1.5.40. In Example 1.3.11 one can describe attractor-repeller pairs explicitly. A connected component of a trapping region is an interval $[a,c)$ or $(c,b]$ with $f(c) \neq 0$ or (α,β) with $f(\alpha) > 0 > f(\beta)$. For example, if the trapping region is an interval $[a,c)$ or $(c,b]$ with $f(c) \neq 0$, then the corresponding attractor-repeller pair consists of $[a,c_1]$ and $[c_2,b]$ (which is the attractor and which is the repeller depends on the sign of $f(c)$), where $f(c_1) = 0 = f(c_2)$ and $f \neq 0$ on (c_1,c_2). An illustrative special case is $f^{-1}(\{0\}) = \{a,b,c\}$ with $f \geq 0$ (or $f \leq 0$), when each $A \cup R$ is either $[a,c] \cup \{b\}$ or $\{a\} \cup [c,b]$. In either case one member of the pair contains points that are not chain-recurrent, so the intersection over $\mathscr{AR}$ is essential in Theorem 1.5.39. (If f takes both positive and negative values, this is a little different.)

[17] More precisely, for each ordinal α set $N_\alpha = X$ if $\alpha = 0$, $N_\alpha = NW(\Phi\restriction_{N_\beta})$ when α is the successor of β, and $N_\alpha = \bigcap_{\beta<\alpha} N_\beta$ if α is a limit ordinal; this terminates after at most countably many steps because each closed set in this sequence is characterized by the elements in a countable base for the topology from which it is disjoint.

Proof. "$\supset$": If $x \notin \mathscr{R}(\Phi)$, then there is an $\epsilon > 0$ with $x \notin \mathscr{R}_\epsilon(x) \supset A_{\mathscr{R}_\epsilon(x)}$. On the other hand, $\varphi^t(x) \in \mathscr{R}_\epsilon(x)$ for $t > 0$ (Proposition 1.5.29), so $x \notin R_{\mathscr{R}_\epsilon(x)}$ since $R_{\mathscr{R}_\epsilon(x)}$ is invariant. Therefore, $x \notin A_{\mathscr{R}_\epsilon(x)} \cup R_{\mathscr{R}_\epsilon(x)}$, and we have shown that $x \notin \bigcap_{(A,R)\in\mathscr{AR}} A \cup R$.

Furthermore, if x and y are in different chain components, then there is no ϵ-chain from x to y for sufficiently small ϵ, so $y \notin \mathscr{R}_\epsilon(x)$ (see Proposition 1.5.29), and $x \in A_{\mathscr{R}_\epsilon(x)}$. Hence, $y \in R_{\mathscr{R}_\epsilon(x)}$.

"$\subset$": Let $x \notin A \cup R$ for some attracting-repelling pair. Proposition 1.4.19 yields a Lyapunov function L that is strictly decreasing off $A \cup R$. If $c_0 \coloneqq L(x)$ and $c_1 \coloneqq L(\varphi^1(x))$, then L is strictly decreasing on $L^{-1}([c_1, c_0])$. By compactness there is a $\delta \in (0, (c_0 - c_1)/2)$ such that if $y \in L^{-1}([c_1, c_1 + \delta])$, then $\varphi^1(y) \in L([0, c_1])$ and there is an $\epsilon > 0$ such that $L(y') < c_1 + \delta$ for all $y \in L^{-1}([0, c_1])$ and $y' \in B_\epsilon(y)$. Even with pseudo-orbits we "can't get back up", that is, ϵ-chains starting at x cannot be closed: To see this we use the sequence-definition of ϵ-chain (Proposition 1.5.27). Let $\{x = x_0, \dots, x_n; t_0, \dots, t_{n-1}\}$ be an ϵ-chain with $t_k \geq 1$ for each $0 \leq k \leq n-1$, then $L(\varphi^{t_0}(x_0)) \leq L(\varphi^1(x_0)) = c_1$. Also, $L(x_1) \leq c_1 + \delta$ by the choice of ϵ and $L(\varphi^{t_1}(x_1)) \leq c_1$. Inductively, we have $L(x_k) \leq c_1 + \delta$ for $1 \leq k \leq n$. Hence, there is no ϵ-chain from x to x, and $x \notin \mathscr{R}(\Phi)$. Thus, $\mathscr{R}(\Phi) \subset \bigcap_{(A,R)\in\mathscr{AR}} A \cup R$ by contraposition.

If $x \sim y$, then we see from this that x and y are in the same component of any attracting-repelling pair (A, R). □

Theorem 1.5.39 suggests that a continuous flow should be studied by analyzing the dynamics on the chain components, which can be done by restriction because they are compact invariant sets, and to then augment this analysis by determining the transient dynamics between them. We are indeed ready to give a complete global description of the dynamics: a Lyapunov function will disentangle transient and recurrent behavior systematically.

Theorem 1.5.41 (Conley's Fundamental Theorem of Dynamical Systems)**.** *Let Φ be a flow on a compact metric space X. Then there is a Lyapunov function $L\colon X \to [0,1]$ such that $L(\mathscr{R}(\Phi))$ is nowhere dense, $x \notin \mathscr{R}(\Phi) \Rightarrow L(\varphi^t(x)) < L(x)$ for all $t > 0$, and if $x, y \in \mathscr{R}(\Phi)$, then $L(x) = L(y) \Leftrightarrow x \sim y$ (see Proposition 1.5.32).*

Remark 1.5.42. A Lyapunov function with these properties is called a *complete Lyapunov function.* The proof also reveals that there are either finitely many or uncountably many chain-components. Example 1.3.12 is an instance of the latter.

Proof. By Lemma 1.4.17 write $\mathscr{AR}(\Phi) = \{(A_j, R_j)\}_{j=1}^M$ with $M \in \mathbb{N} \cup \{\infty\}$. Proposition 1.4.19 gives Lyapunov functions $L_j\colon X \to [0,1]$ that strictly decrease along orbits off $A_j \cup R_j$. The continuous function defined by the uniformly convergent series

$$L(x) := 2\sum_{j=1}^{M} 3^{-j} L_j(x) \in [0,1]$$

is nondecreasing along orbits since the summands are, and $L(\mathscr{R}(\Phi))$ is the ternary Cantor set or a finite subset.

If $x \notin \mathscr{R}(\Phi)$, then $x \notin A_j \cup R_j$ for some j, and $L_j(\varphi^t(x)) < L_j(x)$ for $t > 0$. Also, $L_k(\varphi^t(x)) \le L_k(x)$ for all k, so $L(\varphi^t(x)) < L(x)$ and L is strictly decreasing off $\mathscr{R}(\Phi)$.

Theorem 1.5.39 shows that $x, y \in \mathscr{R}(\Phi)$ are chain-equivalent if and only if for each attracting-repelling pair (A_j, R_j) they are in the same component. Hence, $L(x) = L(y)$. Conversely, if $x \not\sim y$, then there is a minimal $j < \infty$ with $x \in A_j$, $y \in R_j$ after possibly relabeling, so $L_j(x) = 0$ and $L_j(y) = 1$. Then

$$L(y) - L(x) \ge \frac{2}{3^j} - 2\sum_{k=j+1}^{M} \frac{1}{3^k} \ge \frac{2}{3^j} - \frac{2}{3^{j+1}}\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{1}{3^j} > 0. \qquad \square$$

With the chain-decomposition, the phase space or an essential part of it splits into a well-behaved union of closed invariant subsets, and the dynamics on these may be studied separately. This is highly effective, especially for hyperbolic flows. Therefore, our next agenda is to concentrate on such pieces, and we now investigate ways in which the recurrence on them can be stronger than just chain-recurrence.

Remark 1.5.43 (Generalized recurrent set)**.** As an aside we note that the finest decomposition of the space X by Lyapunov functions for the flow Φ is given by the (closed invariant) generalized recurrent set $GR(\Phi)$ of points along whose orbits *any* Lyapunov function for the flow is constant. Then $NW(\Phi) \subset GR(\Phi) \subset \mathscr{R}(\Phi)$. Each of these inclusions can be strict, see Exercise 1.27 and Remark 5.3.41, so in light of this and Proposition 1.5.34,

$$\overline{\mathrm{Per}(\Phi)} \subset \mathscr{B}(\Phi) \subset \mathscr{L}(\Phi) \subset NW(\Phi) \subset GR(\Phi) \subset \mathscr{R}(\Phi),$$

with set closed and each inclusion strict in some of our examples (Exercise 1.21).

Analogously to the proof of Conley's Theorem one shows:

Theorem 1.5.44 ([**17**, Theorem 2])**.** *There is a Lyapunov function f such that $x \in GR(\Phi)$ if and only if f is constant on $\mathscr{O}(x)$, and $x \notin GR(\Phi) \Rightarrow f(\varphi^t(x)) < f(x)$ for $t > 0$.*

Proof. The space $\mathscr{L}$ be the space of Lyapunov functions $f\colon X \to [-1,1]$ (with the topology of uniform convergence on compact sets) is separable, so there is a dense subset $\{f_k\}_{k\in\mathbb{N}}$, and $x \in GR(\Phi)$ if and only if $f_k(\varphi^t(x)) = f_k(x)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$. Then $F := \sum_{k\in\mathbb{N}} f_k/2^k \in \mathscr{L}$ and $f(x) := \int_0^\infty \frac{F(\varphi^s(x))}{s^2+1}\,ds$ is as desired: If $F(\varphi^t(x)) = F(x)$

for all $t > 0$, then $f_k(\varphi^t(x)) = f_k(x)$ for all $t > 0$ and $k \in \mathbb{N}$, so $x \in GR(\Phi)$. Conversely, if $x \notin GR(\Phi)$, there are $t_n \to +\infty$ with $F(\varphi^{t_{n+1}}(x)) < F(\varphi^{t_n}(x))$ for all $n \in \mathbb{N}$, hence the claim. □

The level sets of the Lyapunov function in Theorem 1.5.41 dynamically decompose the manifold in a way that is coherent with the chain components. The dynamics still can be (and for continuous systems in discrete time generically is [**5**]) rather complicated, but for hyperbolic flows (Definition 5.3.48) the chain components are open and hence finite in number (Corollary 5.3.34). Then this decomposition by level sets can even more effectively describe the overall dynamics.

Definition 1.5.45. Let Φ be a flow on a compact manifold M. A *filtration* $\mathbf{M}$[18] for Φ is a nested sequence $\varnothing = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M$ of compact sets such that $\varphi^t(M_i) \subset \operatorname{int}(M_i)$ for any $t > 0$ and any $i \in \{1, \dots, k\}$.

Remark 1.5.46. This notion is not obviously hereditary: a filtration for $\Phi\restriction_\Lambda$ does not imply the existence of a filtration for Φ.

So (the set of interiors of the members of) a filtration is a nested sequence of trapping regions. Note that $K_i^\Phi(\mathbf{M}) := \bigcap_{t\in\mathbb{R}} \varphi^t(M_i \smallsetminus M_{i-1})$ is compact and the maximally Φ-invariant subset in $M_i \smallsetminus M_{i-1}$ for $i \in \{1, \dots, k\}$. We let $K^\Phi(\mathbf{M}) := \bigcup_{i=1}^k K_i^\Phi(\mathbf{M})$.

Theorem 1.5.47 (Filtration)**.** *Let Φ be a continuous flow on X with finite chain-decomposition $\Lambda_1, \dots, \Lambda_k$. Then there is a filtration $\mathbf{M}$ of X composed of $M_0 \subset M_1 \subset \cdots \subset M_k$ such that $\Lambda_i = K_i^\Phi(\mathbf{M})$ for each $i \in \{1, \dots, k\}$.*

Proof. Theorem 1.5.41 gives a Lyapunov function $L\colon M \to \mathbb{R}$ for Φ with $L(\Lambda_k) > L(\Lambda_{k-1}) > \cdots > L(\Lambda_2) > L(\Lambda_1)$ after possibly relabeling. Fix $a_1, \dots, a_k \in \mathbb{R}$ such that

$$a_k > L(\Lambda_k) > a_{k-1} > L(\Lambda_{k-1}) > \cdots > a_2 > L(\Lambda_2) > a_1 > L(\Lambda_1).$$

The $M_i := L^{-1}(-\infty, a_i]$ define a filtration with $\Lambda_i \subset M_i \smallsetminus M_{i-1}$. If $x \in K_i^\Phi(\mathbf{M})$, then $\omega(x) \subset \mathscr{R}(\Phi) \cap K_i^\Phi(\mathbf{M}) \subset \Lambda_i$. Similarly, $\alpha(x) \subset \Lambda_i$, so $x \in \Lambda_i$. □

While Lyapunov functions and the chain-decomposition are effective in homing in on recurrent dynamics and organizing it to some extent, we saw that constants of motion can do so to some extent (Corollary 1.5.8) but also previously pointed to Figure 1.1.4 viewed on the cylinder $S^1 \times \mathbb{R}$ as an illustration that a constant of motion need not be constant on $\mathscr{R}(\Phi)$; indeed, in this chain-transitive example the level sets of energy provide a far better disaggregation of the dynamics: except for the energy level of the saddle, each level set here is an orbit. While this is

[18] More generally, a filtration is a decomposition into an indexed collection of sets where the index I is a totally ordered set such that if $i \le j$ in I then $M_i \subset M_j$.

untypically fine a decomposition, it motivates studying finer decompositions as well as stronger dynamical entanglements. This is the goal of the next section.

6. Transitivity, minimality, and topological mixing

As we mentioned after the proof of Theorem 1.5.41, the chain decomposition splits a flow into chain-transitive pieces. That chain-recurrence is the weakest recurrence notion in the previous section suggests to now describe ways in which orbits in a given chain-component might be more tightly entangled than chain-recurrence alone implies. This is our task for the present section.

Definition 1.6.1 (Topological transitivity)**.** We say that a flow on a metric space X is *topologically transitive* if there is a point $x \in X$ such that $\overline{\mathcal{O}^+(x)} = X$. A subset of X is said to be (topologically) transitive if it is an orbit closure.

This is a recurrence property in two ways: on one hand the point x is recurrent, and on the other hand, this property implies that every point is nonwandering.

Transitivity will also play a major role in studying hyperbolicity for flows. One of the fundamental notions in hyperbolicity is the idea of a basic set Definition 5.3.15 that is a transitive component of the flow.

Example 1.6.2. Taking $n = 1$ and $v \neq 0$ in Example 1.1.8 gives a trivial example of a topologically transitive system; it consists of a single periodic orbit. If $n = 2$ and $0 \neq v_1 = \alpha v_2$ with irrational α, then the corresponding linear flow is indeed topologically transitive (see also Remark 1.1.11). This can be shown by adapting the observation in Example 1.3.13 to reduce this to studying the rotation $x \mapsto x + \alpha$, whose orbits are $x_0 + \alpha\mathbb{Z}$ mod 1, hence dense. This shows that indeed *every* (semi-) orbit is dense(Definition 1.6.21). By contrast, all orbits are periodic if $\alpha \in \mathbb{Q}$ (Remark 1.1.11): If $pv_1 = qv_2$ and $t = \frac{q}{v_1} = \frac{p}{v_2}$, then $t(v_1, v_2) = (q, p)$, so $\varphi^{pq} = \text{Id}$.

Remark 1.6.3. A similarly homogeneous example arises below in a geometric context (Example 2.1.16), and it is profoundly different in terms of longitudinal behavior: while toral translations (Example 1.1.8) are suspensions (Example 1.3.13), those flows are not (Theorem 3.4.44).

The notion of transitivity proves useful immediately. For instance:

Proposition 1.6.4. *A topologically transitive flow has no constant of motion (Definition 1.1.23).*

Proof. A constant of motion is constant on the closure of the dense orbit. □

We can easily amplify this in the context of the chain-decomposition:

Proposition 1.6.5. *A flow on a connected space X whose chain-components are transitive and finite in number*[19] *has no constant of motion.*

Proof. A constant of motion h is constant on orbit closures, hence

- h is constant on each chain-component, and
- for any $x \in X$, $h(x) = h(\overline{\varphi^{\mathbb{R}}(x)}) = h(\omega(x))$, where $\omega(x) := \bigcap_{t\in\mathbb{R}} \overline{\varphi^{[t,\infty)}(x)}$ is contained in a chain-component.

Thus, $h(X)$ is finite and connected. □

Remark 1.6.6. This is a good moment to look back at Figure 1.3.3. None of those flows have a constant of motion, but only one has a dense orbit. None is topologically transitive, and the chain-recurrent set of the north-south dynamics consists of the fixed points, whereas it is the circle in the other 2 examples—which shows that Proposition 1.6.5 is not sharp.

Also, in all cases the nonwandering set consists of the fixed points, but when the south-north-south dynamics is included in Figure 1.1.4, then all its points are nonwandering.

Proposition 1.6.7. *A flow is transitive if and only if $\omega(x) = X$ for some $x \in X$.*

Proof. Since $\omega(x) \subset \overline{\mathcal{O}^+(x)}$, a flow is transitive if there exists a point $x \in X$ such that $\omega(x) = X$. Conversely, suppose $X = \overline{\mathcal{O}^+(x)}$. Unless x is periodic and hence $X = \mathcal{O}^+(x) = \mathcal{O}(x)$, we have $\varphi^{-1}(x) \in X \setminus \mathcal{O}^+(x) = \overline{\mathcal{O}^+(x)} \setminus \mathcal{O}^+(x) \subset \omega(x)$, so (since $\omega(x)$ is closed and invariant) $X = \overline{\mathcal{O}(x)} \subset \omega(x)$. □

It is common to define topological transitivity as the existence of a dense orbit, rather than a forward dense orbit. While there are flows that satisfy the first of these and not the latter (Example 1.3.6 or 1.3.9), this is a 1-dimensional phenomenon. This suggests a natural terminology in analogy to discrete-time dynamics, where the various definitions of topological transitivity agree on a *perfect set*, that is, a compact set without isolated points.

Definition 1.6.8. A compact set is said to be *flow-perfect* if it has no isolated segments, that is, no open subset is homeomorphic to an interval.[20]

Proposition 1.6.9 (Transitivity)**.** *For a continuous flow Φ on a flow-perfect metric space X, the following four conditions are equivalent:*

(1) Φ has a dense positive semiorbit (topological transitivity, Definition 1.6.1).
(2) Φ has a dense orbit.

[19] or, by Remark 1.5.42 equivalently, at most countable in number

[20] Isolated points are not a problem because if there is one, then all cases below are equivalent to it being the whole space.

(3) If $\varnothing \neq U, V \subset X$ are open, then there exists a $t \in \mathbb{R}$ such that $\varphi^t(U) \cap V \neq \varnothing$.
(4) If $\varnothing \neq U, V \subset X$ are open, then there exists a $t \geq 0$ such that $\varphi^t(U) \cap V \neq \varnothing$.

Remark 1.6.10. (4) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) are clear. We prove (2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(1). Note that (1)$\Rightarrow$(2)$\Rightarrow$(3) (and (4) $\Rightarrow$ (3)) use no assumptions on the topology of X. Considering Examples 1.3.6 or 1.3.9 in light of these 4 statements may help clarify Proposition 1.6.9 and its proof.

Remark 1.6.11. Item (3) can be strengthened. Since $\big\{B(x,\epsilon/2) \times B(y,\epsilon/2) \bigm| x, y \in X\big\}$ has a finite subcover by compactness of $X \times X$,

$$\forall \epsilon > 0\, \exists T \in \mathbb{R}\, \forall x, y \in X\, \exists t \in [0,T] \colon \varphi^t(B(x,\epsilon)) \cap B(y,\epsilon) \neq \varnothing.$$

PROOF OF PROPOSITION 1.6.9. (2)$\Rightarrow$(3). If $\varnothing \neq U, V \subset X$ are open and $\overline{\mathcal{O}(x)} = X$, then there are $t, s \in \mathbb{R}$ with $\varphi^t(x) \in U$ and $V \ni \varphi^s(x) = \varphi^{t-s}(\varphi^t(x)) \in \varphi^{t-s}(U)$, so $\varphi^{t-s}(U) \cap V \neq \varnothing$.

(3)$\Rightarrow$(4) is the "uphill" step. Here we "symmetrize time." To that end, first "symmetrize space" by considering the case $U = V =: W$ in (4). We show:

(1.6.1) Given $\varnothing \neq W \subset X$ open and $T > 0$ there is a $t \geq T$ with $\varphi^t(W) \cap W \neq \varnothing$.

It is important here that t can be taken arbitrarily large.

Claim 1.6.12. *For $\varnothing \neq W \subset X$ open there are $t \geq 1$ and $\varnothing \neq W' \subset W$ open with $\varphi^t(W') \subset W$.*

This implies (1.6.1) because applying it to W' recursively, we find that given $\varnothing \neq W \subset X$ and $T > 0$ there are an open $W' \subset W$ and $t \geq T$ such that $\varphi^t(W') \subset W$. We use the following notation several times.

Definition 1.6.13. For a topological space X and $x \in A \subset X$ we denote by $\mathbb{C}(A, x)$ the connected component of A containing x.

PROOF OF CLAIM 1.6.12. If W consists of fixed points, then so does its closure, and by (3) the closure is X, hence again by (3), X is a point, in which case (4) holds (trivially, as do the other 3 statements). Otherwise, pick a point $x \in W$ that is not fixed, and let $I := \mathbb{C}\big(\{t \in (-2,2) \bigm| \varphi^t(x) \in W\}, 0\big) \subset \mathbb{R}$. Then $\varphi^{[-1,1] \setminus I}(x)$ is compact, and we can replace W by $W \setminus \varphi^{[-1,1] \setminus I}(x)$. Since W is not homeomorphic to an interval, there is a $y \in W \setminus \varphi^I(x)$, and there are disjoint neighborhoods W_1 of $\varphi^I(x)$ and W_2 of y.[21] By (3) (and the choice of I) there is an $s \in \mathbb{R} \setminus [-1,1]$ with $\varphi^s(W_1) \cap W_2 \neq \varnothing$. Let $t := |s| \geq 1$.

- If $s < 0$ set $W' := \varphi^s(W_1) \cap W_2 \subset W_2 \subset W$ to get

$$\varphi^t(W') = \varphi^{-s}(f^s(Z_1) \cap Z_2) \subset Z_1 \subset W.$$

[21] This uses that X is a metric space, and "regular Hausdorff" would suffice.

- If $s > 0$ set $W' := W_1 \cap \varphi^{-s}(W_2) \subset W_1 \subset W$ to get
$$\varphi^t(W') = \varphi^s(W_1 \cap \varphi^{-s}(W_2)) \subset W_2 \subset W. \qquad \square$$

We now return to the proof of Proposition 1.6.9. (1.6.1) implies (3)⇒(4) in Proposition 1.6.9: If $\varnothing \neq U, V \subset X$ are open, then there exists an $s \in \mathbb{R}$ such that $W := \varphi^s(U) \cap V \neq \varnothing$. If $s > 0$ we are done. Otherwise, (1.6.1) gives a $t > -s$ with
$$\varnothing \neq \varphi^t(W) \cap W = \varphi^t(\varphi^s(U) \cap V) \cap \varphi^s(U) \cap V \subset \varphi^{t+s}(U) \cap V.$$
Since $t + s > 0$, this proves (4).

(4)⇒(1). Since X is second countable, let $U_1, U_2, \dots$ be a base for the topology. We inductively construct a semiorbit that intersects every U_n and is hence dense.

As the first step, take an open $W_1 \neq \varnothing$ with $\overline{W_1} \subset U_1 =: W_0$ compact and $t_1 = 0$.

Suppose for $1 \leq j \leq n$ there are $t_j \geq 0$ and open $\varnothing \neq W_j \subset \overline{W_j} \subset W_{j-1}$ with $\varphi^{t_j}(x) \in U_j$ for all $x \in W_j$. (4) then gives $t_{n+1} > 0$ with $\varphi^{t_{n+1}}(W_n) \cap U_{n+1} \neq \varnothing$. Since Φ is continuous, $W_n' := W_n \cap \varphi^{-t_{n+1}}(U_{n+1}) \neq \varnothing$ is open, and there is a nonempty open $W_{n+1} \subset \overline{W_{n+1}} \subset W_n'$.
Then $\varnothing \neq K := \bigcap_{j \in \mathbb{N}} \overline{W_j} \subset \bigcap_{j \in \mathbb{N}} W_{j-1}$ and $x \in K, j \in \mathbb{N} \Rightarrow \varphi^{t_j}(x) \in \varphi^{t_j}(W_j) \subset U_j$. $\square$

Remark 1.6.14. Examples 1.3.6 and 1.3.9 are not the only ones showing the need for the assumption on X in Proposition 1.6.9. More generally, if a point $x \in X$ has a dense positive semiorbit for a flow Φ, consider the cartesian product of Φ and the flow in Example 1.3.6 or 1.3.9. If y is a nonfixed point in the latter factor, then Φ restricted to the orbit closure of (x, y) has a dense orbit by definition, but no dense semiorbit.

Example 1.1.8 in dimension higher than 2 does not yield a transitive-versus-periodic dichotomy as in Example 1.6.2, but Proposition 1.6.9 gives a convenient criterion for transitivity.

Proposition 1.6.15. *A linear flow $x \mapsto x + tv$ on $\mathbb{T}^n$ is topologically transitive if and only if the components of v are rationally independent (that is, if $k \in \mathbb{Z}^n$ and $\langle k, v\rangle = 0$, then $k = 0$).*

We prove this via a converse to Proposition 1.6.4:

Lemma 1.6.16. *If Φ is a continuous flow on $\mathbb{T}^n$ and every bounded measurable Φ-invariant function is constant, then Φ is topologically transitive.*

Proof. If O is an open Φ-invariant set then χ_O is Φ-invariant, hence constant almost everywhere, so O has Lebesgue measure 0 or 1. Thus, there are no disjoint nonempty open Φ-invariant sets. If now $U, V \subset X$ are open then the Φ-invariant open sets $\varphi^{\mathbb{R}}(U)$ and $\varphi^{\mathbb{R}}(V)$ are therefore not disjoint, so $\varphi^t(U) \cap \varphi^s(V) \neq \varnothing$ for some $t, s \in \mathbb{R}$, and $\varphi^{t-s}(U) \cap V \neq \varnothing$. $\square$

PROOF OF PROPOSITION 1.6.15. We show both implications by contraposition. If there is a $k \in \mathbb{Z}^n \smallsetminus \{0\}$ with $\langle k, v\rangle = 0$, then $\sin\bigl(2\pi\langle k, x\rangle\bigr)$ is a nontrivial constant of motion, and Φ is not transitive by Proposition 1.6.4.

Conversely, suppose f is a nonconstant bounded measurable (hence L^2) invariant function and use the Fourier expansion:

$$\sum_{k\in\mathbb{Z}^n} f_k e^{2\pi i\langle k,x\rangle} = f(x) = f(x+tv) = \sum_{k\in\mathbb{Z}^n} f_k e^{2\pi i\langle k,x+tv\rangle} = \sum_{k\in\mathbb{Z}^n} f_k e^{2\pi it\langle k,v\rangle} e^{2\pi i\langle k,x\rangle}.$$

Since f is not constant, there is a $k \neq 0$ with $f_k \neq 0$, so the uniqueness of this expansion implies $e^{2\pi it\langle k,v\rangle} = 1$ for *all* $t \in \mathbb{R}$, hence $\langle k, v\rangle = 0$. □

The criteria in Lemmas 1.6.4 and 1.6.16 are not meant to be optimal, but they are well suited for the purpose at hand and also yield Proposition 3.3.6 below.

Remark 1.6.17. Proposition 1.6.15 gives a clean connection between a dynamical property and a parameter of the flow. This makes it natural to discuss this whole family of linear flows as such rather than viewing each in isolation. Among this family, flows with rather different kinds of orbit structures are tightly interspersed. A rational vector v gives rise to a flow all of whose orbits are closed, but arbitrarily near v there are rationally independent vectors, and they define flows with dense orbits; conversely each of these in turn is arbitrarily close to a rational vector and, on $\mathbb{T}^n$ with $n \geq 3$, also to “intermediate” flows with neither periodic nor dense orbits but orbit closures that form tori of smaller dimension. In particular, such distinct flows are definitely not orbit-equivalent. This indicates a great deal of structural “fragility” of these flows.

From this perspective we revisit some earlier examples. Example 1.4.13 has similarities but also a pronounced difference. We noted that the gradient flow on a “standup” torus undergoes a qualitative change when the torus is tilted slightly; this is akin to the “fragility” for toral flows. On the other hand, the description in Example 1.4.13 of the dynamics after this slight tilt did not depend on the amount of the tilt, so structurally all these perturbations of the initial gradient flow look rather the same. One might conjecture that they are pairwise orbit-equivalent. In a rather similar vein, Example 1.4.8 was obtained from the undamped pendulum and behaves quite differently—but as we change the amount of damping, Figure 1.4.1 changes geometrically (the spirals will approach the stable equilibrium more quickly) but not topologically, so here as well, we have a whole range of parameters with structurally “constant” behavior. The dynamics here is simple enough that one can try to slightly refine the ideas in the proof of Proposition 1.4.5 to show that any 2 of these damped pendulum flows are topologically conjugate.

Let us clarify as well that these features of the various families of flows are not an artefact of the parametrization; in a natural sense, these are continuous parametrizations in the following sense.

Definition 1.6.18 (C^r-closeness)**.** Two flows Φ and Ψ on M are said to be C^r-close if $\varphi\restriction_{[0,1]\times M}$ and $\psi\restriction_{[0,1]\times M}$ are uniformly C^r-close.

Remark 1.6.19. This is the compact-open C^r topology for maps on $\mathbb{R}\times M$. Since a flow Φ is determined by the mapping $\varphi\restriction_{[\alpha,\beta]\times M}$ for any $\alpha<\beta$ (Remark 1.1.4), this definition incorporates all information about the flows without unrealistically imposing bounds that are uniform in time.

For toral flows, changing v slightly produces slight changes in this C^r sense for any r. Similarly for the angle of tilting the torus with the gradient flow or for increasing the damping parameter.

Remark 1.6.20. Any 2 orbits of a linear flow on $\mathbb{T}^n$ are isometric (by a translation), so whenever such a linear flow is topologically transitive, every orbit is dense. The latter feature is a natural indecomposability condition for topological dynamical systems, a property stronger than topological transitivity and, after periodicity, the next case of strong and uniform recurrence.

Definition 1.6.21. A flow is said to be *minimal* if every orbit is dense or, equivalently, if every closed invariant set is empty or the whole space. A Φ-invariant set A is said to be minimal if $\Phi\restriction_A$ is minimal (or A has no proper closed invariant subset).

Remark 1.6.20 gives

Proposition 1.6.22. *A linear flow $x\mapsto x+tv$ on $\mathbb{T}^n$ is minimal iff the components of v are rationally independent (meaning: if $k\in\mathbb{Z}^n$ and $\langle k,v\rangle=0$, then $k=0$).*

Example 1.6.23. A topologically transitive flow that is not minimal is easy to construct from a minimal linear flow on a torus (which is generated by the constant vector field v) by considering the flow generated by the vector field fv with $f\colon\mathbb{T}^n\to\mathbb{R}$ such that $f^{-1}(0)$ (the set of fixed points) is nonempty and finite.

Theorem 1.6.24. *If a flow is minimal then so are its time-τ maps for all but countably many $\tau\in\mathbb{R}$.*

PROOF OUTLINE. If φ^τ is not minimal, then there is a proper minimal set A_τ for the map φ^τ. Note first that no orbit stays in A_τ for an interval $[0,\epsilon)$ of time because by minimality of $\varphi^\tau\restriction_{A_\tau}$, and by an approximation argument every point of A_τ would stay in A_τ for all $s\in[0,\epsilon)$ and hence forever, so A_τ is a proper invariant set for the flow φ^t, contrary to minimality. Thus every point of A_τ has a positive first-return time, which again by minimality of $\varphi^\tau\restriction_{A_\tau}$ and an approximation argument is a constant τ_1 on A_τ, and then $\tau \in \tau_1\mathbb{Z}$. We define a continuous nonconstant eigenfunction f_τ for φ^t by taking $f_\tau(x) = 1$ for $x \in A_\tau$ and imposing $f(\varphi^s(x)) = e^{2\pi is/\tau_1} f_\tau(x)$ (then $|f_\tau| = 1$). Note that the f_τ are distinct for different first-return times, so by separability of $C(M)$ there are hence only countably many τ for which φ^τ is not minimal. □

Proposition 1.6.25. *A continuous flow Φ on a compact metric space X is minimal if and only if for every $\epsilon > 0$ there is an $T > 0$ such that $\varphi^{[0,T]}(x)$ is ϵ-dense in X for each $x \in X$.*

Proof. The latter condition clearly implies minimality. On the other hand, if it fails then there is an $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there is an $x_n \in X$ for which $\varphi^{[-T,T]}(x_n)$ misses a ball $B(c_n,\epsilon)$. By compactness there are accumulation points x of $(x_n)_{n\in\mathbb{N}}$ and c of $(c_n)_{n\in\mathbb{N}}$, and we claim that the orbit of x misses $B(c,\epsilon/3)$. To that end, take $N \in \mathbb{N}$ and choose $n \geq N$ such that

- $c_n \in B(c,\epsilon/3)$,
- $\varphi^t(x_n) \in B(\varphi^t(x),\epsilon/3)$ for $|t| \leq N$.

Then for $|t| \leq N$ we have

$$d(\varphi^t(x),c) \geq d(\varphi^t(x_n),c_n) - d(\varphi^t(x_n),\varphi^t(x)) - d(c_n,c) \geq \epsilon - \epsilon/3 - \epsilon/3 = \epsilon/3.$$

Since N was arbitrary, this proves the claim. □

Remark 1.6.26. For the linear flows in Proposition 1.6.15 the exceptional values of τ are those of the form $\dfrac{l}{\langle k, v\rangle}$ with $l \in \mathbb{Z}$, $k \in \mathbb{Z}^n \smallsetminus \{0\}$ because for such τ we have $t\langle k, v\rangle = l$, so $\sin\big(2\pi\langle k, x\rangle\big)$, say, is φ^τ-invariant. This is illuminating even for $n = 1$.

The next result can be proved using Zorn's Lemma, but we will provide a different proof.

Proposition 1.6.27. *A continuous flow on a compact metric space has a nonempty minimal subset.*

Lemma 1.6.28. *The set of closed invariant sets of a flow Φ on a metric space is closed with respect to the Hausdorff metric.*

Proof. Φ acts homeomorphically on the collection of closed subsets with the Hausdorff metric, and invariant sets are the fixed points, so the set of these is closed. □

PROOF OF PROPOSITION 1.6.27. Let $m(B) = \max\{d(A,B) \mid A \subset B$ closed invariant$\}$ for B closed and invariant. Take M such that $m(M) = m_0 := \min m$. Then M has no proper closed invariant subsets: Otherwise $m_0 > 0$. Take a closed invariant $M_1 \subset M$ such that $d(M_1, M) = m_0$. By assumption M_1 is not minimal and contains M_2 such that $d(M_2, M_1) \geq m_0$ and hence $d(M_2, M) > m_0$. We continue this process to obtain a sequence M_i such that $d(M_i, M_j) \geq m_0$ contradicting compactness with respect to the Hausdorff metric. □

Remark 1.6.29. For a continuous flow Φ on a compact metric space denote the closure of the union of all invariant minimal sets by $M(\Phi)$. Then $\mathscr{B}(\Phi) \supset M(\Phi) \neq \varnothing$ since every point of a minimal set is recurrent.

An obvious and useful observation is:

Proposition 1.6.30. *Each of topological transitivity, minimality, and density of periodic orbits is invariant under time-changes and holds for a special flow if and only if its discrete-time counterpart (defined in the obvious way) holds for the base.*

While minimality is a strengthening of topological transitivity as defined by density of an orbit, a strengthening of transitivity as defined by open sets gives a criterion for much greater dynamical complexity: images of an open set persistently overlap with another given open set.

Definition 1.6.31 (Topological mixing)**.** A flow φ^t on a topological space X is said to be *topologically mixing* if for any two open sets U and V there exists a $T > 0$ such that $\varphi^t(U) \cap V \neq \varnothing$ for all $t \geq T$.

Remark 1.6.32. Figure 1.6.1 shows this in the context of Example 1.5.23 with a figure due to Grayson, Kitchens and Zettler on which some of those in their 1993 article [**135**] were based.

Analogously to Remark 1.6.11 this implies a uniform property if X is compact: $\forall \epsilon > 0\, \exists T \in \mathbb{R}\, \forall x, y \in X,\ t \geq T : \varphi^t(B(x,\epsilon)) \cap B(y,\epsilon) \neq \varnothing$.

This can be seen as an extreme form of unpredictability: if ϵ is taken to be the size of observational accuracy, then this statement says that after time T, an initial state can evolve to literally any state whatsoever, that is, no prediction at all is possible beyond this time T.

Proposition 1.6.9(4) immediately gives

Corollary 1.6.33. *Topologically mixing flows are topologically transitive.*

In contrast with Proposition 1.6.30, topological mixing depends on longitudinal effects, that is, the time-parametrization matters. The clearest illustration is given by suspensions:

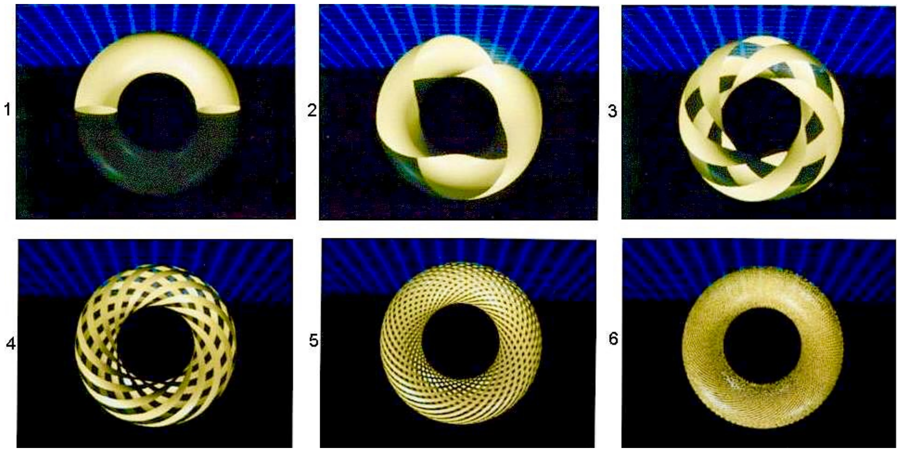


FIGURE 1.6.1. Mixing in Example 1.5.23

Example 1.6.34 (Suspensions are not mixing)**.** A suspension flow Φ over a homeomorphism of a space X is not mixing: if $U := X \times (0,1/2)$ and $V := X \times (1/2,1)$, then $\varphi^n(U) \cap V = \varnothing$ for all $n \in \mathbb{Z}$.

This is more generally true for special flows whose roof function is cohomologous to a constant: If Φ_f and $\Phi_{\hat{f}}$ are special flows over X, and f and $\hat{f}$ are cohomologous, that is,

$$\hat{f}(x) = f(x) + v(x) - v(\sigma x)$$

for some continuous function v, then by Proposition 1.3.17 the two flows are topologically conjugate via $\pi(x,t) = (x, t + v(x))$.

In particular, a flow under a function that is cohomologous to a constant is topologically conjugate to a suspension and hence not mixing.

For discrete-time dynamical systems we define topological mixing in the same way (but with integers t, T). Example 1.6.34 shows that unlike with topological transitivity, a special flow over a topologically mixing homeomorphism need not be topologically mixing. On the other hand, Example 1.6.35 below is a special flow with mixing base that is mixing. Thus, topological mixing (Definition 1.6.31) is sensitive to time-changes and hence to the choice of roof function for special flows.

Example 1.6.35. Φ_{r_c} is a mixing special flow over the map F_A (Example 1.5.23) when the roof function is $r_c\colon \mathbb{T}^2 \to \mathbb{R}^+$, $p \mapsto 1 + c\beta(d(p,0))$ with $c \notin \mathbb{Q}$ and $\beta\colon \mathbb{R} \to [0,1]$ smooth, even, decreasing on $[1/4,1/2]$, and such that $\beta(x) = 1$ for $|x| < 1/4$, $\beta(x) = 0$ for $|x| > 1/2$: r_c is irrational at the fixed point associated with the origin, but 1 at the period 3-point associated with the orbit $(1/2, 1/2)$, $(0, 1/2)$, and $(0, 1/2)$, so the periods of these two points for the special flow are incommensurate, and Φ_{r_c} is mixing by Proposition 6.2.19 below.

The details for the next example are given in the following chapter, but mention it now as another important example of a mixing flow. Furthermore, we will see that this is an important example of a hyperbolic flow.

Example 1.6.36. The geodesic flow on a compact factor of the hyperbolic plane (Section 2.4) is topologically mixing (Remark 8.1.14 and Corollary 9.1.4).

Remark 1.6.37. We pick up once more from Remark 1.6.20. Whether or not a toral translation is minimal, the orbit closures provide a natural decomposition of the torus: each orbit closure is a translate (or coset) of the closure of the orbit of 0, which is itself an embedded (sub)torus. This is, however, not an instance of a general phenomenon but rather a reflection of the homogeneity of toral translations, specifically the fact that any 2 orbits differ by a translation. (The orbit closure of 0 is a compact subgroup; other orbit closures are its cosets.) When this is not the case, a decomposition into orbit closures does not usually go well. The next section provides abundant examples of this. Recall, though, that while orbit closures do not usually partition the space neatly, Proposition 1.5.32 and Theorem 1.5.41 provide a natural and effective decomposition in great generality, which in also particularly well-suited to hyperbolic flows, where there is a finite partition by transitive pieces (Theorem 5.3.35).

7. Expansive flows

We now explore the concept of expansivity, a property that is central to hyperbolic flows and which, together with compactness of the space, provides a mechanism for complicated dynamical phenomena. Because special flows are our first examples of expansive flows and as a warm-up we first define expansivity for maps.

Definition 1.7.1 (Expansivity for maps)**.** A homeomorphism $f\colon X \to X$ is said to be *expansive* if there exists a constant $\delta > 0$ such that if $d(f^n(x), f^n(y)) < \delta$ for all $n \in \mathbb{Z}$ then $x = y$.

The adaptation of expansivity to flows is subtler because of the flow direction and the possibility of reparametrization. For any 2 orbits of a flow one expects to be able to reparametrize one of them in such a way that at some time the orbits are substantially separated. Expansivity says that this will happen for *any* reparametrization, or conversely, that *no* reparametrization can make 2 orbits stay close forever. This definition has proven to have the desired properties, and we now formalize it and then study some of its consequences as well as equivalent formulations.[22]

Definition 1.7.2 (Expansivity)**.** A flow Φ on a compact metric space X is *expansive* if for all $\epsilon > 0$ there is a $\delta > 0$, called an *expansivity constant* (for ϵ), such that:

if $x, y \in X$, $s\colon \mathbb{R} \to \mathbb{R}$ continuous, $s(0) = 0$, and $d(\varphi^t(x), \varphi^{s(t)}(y)) < \delta \ \forall\, t \in \mathbb{R}$,
then $y = \varphi^t(x)$ for some $|t| < \epsilon$.

Remark 1.7.3. By contraposition this says that any 2 orbits will separate by δ at some time, no matter how you reparametrize (one of) them.

For flows a few notable features of expansivity contrast with the discrete time.

- As in the discrete-time context, expansivity implies that points on different orbits separate by δ in the future or the past. In particular, no orbit is stable for both the flow and the reversed flow. (In discrete time, this characterizes expansivity.)
- Expansivity is independent of the metric and preserved by orbit equivalence (Theorem 1.7.7), time-changes (Corollary 1.7.8) and the forming of Cartesian products. (Likewise in discrete time for topological conjugacy and products.)
- A suspension is expansive if and only if the base is (Proposition 1.7.9).
- Expansivity implies that fixed points of Φ are isolated points of X, so one can omit these ($X \smallsetminus \{\text{fixed points}\}$ is compact) and thereby study flows without fixed points. Specifically, if x is fixed, $\epsilon > 0$, δ as in the definition, $d(x, y) < \delta$, $s \equiv 0$, then $d(\varphi^t(x), \varphi^{s(t)}(y)) = d(x, y) < \delta$ for all t, so $y = \varphi^t(x) = x$. (This also implies that there are only finitely many fixed points.) For flows without fixed points expansivity can be easier to check, see Theorem 1.7.5.
- We do not instead use the natural-looking simpler variant
$$\text{“}\exists\delta > 0 \ \big(d(\varphi^t(x), \varphi^t(y)) < \delta \ \forall\, t \in \mathbb{R}\big) \Rightarrow x = y\text{”}$$
because it does not hold for any nontrivial flow: $\forall\delta > 0\ \exists\eta > 0$ such that $\big(y = \varphi^s(x) \text{ with } |s| < \eta\big) \Rightarrow \big(d(\varphi^t(x), \varphi^t(y)) < \delta \ \forall\, t \in \mathbb{R}\big)$.
- On the other hand, it merely seems less natural to instead use
$$\text{“}\forall\epsilon > 0\ \exists\delta > 0\colon \big(d(\varphi^t(x), \varphi^t(y)) < \delta \ \forall\, t \in \mathbb{R}\big) \Rightarrow \big(y = \varphi^t(x) \text{ for some } |t| < \epsilon\big)\text{.”} \tag{1.7.1}$$
This notion turns out not to be invariant under orbit-equivalence(!) and holds for the “twist” flow $(x, y) \mapsto (x + ty \pmod 1), y)$ on $S^1 \times [1, 2]$ or equivalently, rotation of the annulus $1 \le r \le 2$ in $\mathbb{R}^2$ with constant *linear* speed, even though this can be time-changed to a rigid rotation and

[22] Our presentation follows that by Bowen and Walters [**59**].

no two orbits ($y =$ const.) separate. This example should also be out of bounds because it has a continuum of closed orbits. Hence the preference for allowing arbitrary continuous reparametrizations in the hypothesis.

In light of recent developments we point out that a nonuniform counterpart of (1.7.1) is weaker than Definition 1.7.2 but sufficient for existence and uniqueness of equilibrium states, which is a central motivation for the notion of expansivity.[23] That is to say, with respect to our principal purpose for this notion, this would serve. We prefer our choice because of the utility of Theorem 1.7.5 and because it reflects that no orbit can closely track another even if we are flexible with the timing, that is, the parametrization.

It is illuminating to show directly that the Smale horseshoe (Example 1.5.21) is an expansive flow; this also follows from Example 1.8.16 below. Likewise, Example 1.5.23 (the suspension of $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$) also provides an instance of an expansive flow. This is a consequence of Proposition 1.7.9 below but also not hard to see directly. Indeed, the orbits of 2 points x, y will separate (exponentially) for positive time unless y is in the local center-stable set of x, in which case such separation occurs in negative time. Hence, the only points that remain close are on the same orbit. It might be interesting to consider this argument in the case of a special flow over $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$, or one can reduce this to the suspension by invoking Proposition 1.3.28 and Theorem 1.7.7 or Proposition 1.7.10.

Compactness and contraposition give:

Proposition 1.7.4. *If Φ is a flow on a compact metric space X and δ an expansivity constant for $\epsilon > 0$ (Definition 1.7.2), then for any $\rho > 0$ there is a $T > 0$ with*

$$d(\varphi^t(x), \varphi^t(y)) < \delta \textit{ for all } t \in [-T, T] \Rightarrow d(y, \varphi^t(x)) < \rho \textit{ for some } t \in [-\epsilon, \epsilon].$$

PROOF. Otherwise, take $x_n, y_n \in X$ such that $d(y_n, \varphi^t(x_n)) > \rho$ for all $t \in [-\epsilon, \epsilon]$ and $d(\varphi^t(x_n), \varphi^t(y_n)) < \eta$ for all $t \in [-n, n]$, and (without loss of generality) $x_n \to x$ and $y_n \to y$. Then on one hand, $\varphi^t(x) \neq y$ when $|t| \le \epsilon$, while on the other hand for any $r \in \mathbb{R}$ we have $d(\varphi^r(x_n), \varphi^r(y_n)) < \eta$ for all $n \ge K := |r|$, so $d(\varphi^r(x), \varphi^r(y)) < \eta$, so, since r was arbitrary, $y = \varphi^t(x)$ for some $t \in [-\epsilon, \epsilon]$, a contradiction. □

Theorem 1.7.5. *Expansivity of a fixed-point-free flow Φ is equivalent to each of:*

(1) $\forall \epsilon > 0\ \exists \alpha > 0$ such that if $x, y \in X$, $h\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $h(0) = 0$, and $d(\varphi^t(x), \varphi^{h(t)}(y)) < \alpha\ \forall t \in \mathbb{R}$, then $y = \varphi^t(x)$ for some $|t| < \epsilon$.

[23] This can be found in [**87**, Section 2.5 (definition), Theorems A & 2.9 (application)] but that context is far outside our uniformly hyperbolic setting.

(2) $\forall \eta > 0\ \exists \delta > 0$ *such that if* $x, y \in X$, $s\colon \mathbb{R} \to \mathbb{R}$ *is continuous,* $s(0) = 0$, *and* $d(\varphi^t(x), \varphi^{s(t)}(y)) < \delta\ \forall t \in \mathbb{R}$, *then* $y \in \mathcal{O}(x)$ *and the orbit segment from* x *to* y *lies in the ball* $B_\eta(x)$.

(3) $\forall \epsilon > 0\ \exists \alpha > 0$ *as follows: if* $t_{\pm i} \xrightarrow[i\to+\infty]{} \pm\infty$, $0 < t_{i+1} - t_i \le \alpha$, $|u_{i+1} - u_i| \le \alpha$, $u_0 = t_0 = 0$, $d(\varphi^{t_i}(x), \varphi^{u_i}(y)) \le \alpha$ *for all* $i \in \mathbb{Z}$, *then* $\exists |t| < \epsilon\colon\ y = \varphi^t(x)$.[24]

Proof. That expansivity implies (1) and (2) is clear (for (2) take $\epsilon > 0$ such that $\varphi^t(x) \in B_\eta(x)$ for $|t| < \epsilon$). That (2) implies expansivity is also easy: For $\epsilon \in (0, T_0)$ take $\eta > 0$ such that $d(x, \varphi^\epsilon(x)) > \eta$ for all $x \in X$ by Proposition 1.1.12. Then the orbit segment from x to y lying in $B_\eta(x)$ implies $y = \varphi^t(x)$ with $|t| < \epsilon$.

Showing that (1) implies expansivity involves deforming a continuous $s(\cdot)$ in the definition of expansivity to a homeomorphism. As a first step we show that in a coarse way s is uniformly increasing.

Claim 1.7.6. *If* T_0 *is as in Proposition 1.1.12,* $T \in (0, T_0/3)$, *then there is a* τ_T *such that if* $x, y \in X$, $s\colon \mathbb{R} \to \mathbb{R}$ *continuous,* $s(0) = 0$, $d(\varphi^t(x), \varphi^{s(t)}(y)) < \delta_T := \gamma_T/3$ *(where* γ_T *is as in Proposition 1.1.12) for all* $t \in \mathbb{R}$, *then* $s(t + T) - s(t) \ge \tau_T$ *for all* $t \in \mathbb{R}$.

Proof. Proposition 1.1.12 gives

$$\begin{aligned} &d(\varphi^{s(t)}(y), \varphi^{s(t+T)}(y)) \\ &\quad \ge d(\varphi^t(x), \varphi^{t+T}(x)) - d(\varphi^t(x), \varphi^{s(t)}(y)) - d(\varphi^{s(t+T)}(y), \varphi^{t+T}(x)) \ge \gamma_T - 2\delta_T > 0, \end{aligned}$$

so continuity of Φ yields a $\tau_T > 0$ such that $|s(t + T) - s(T)| \ge \tau_T$ for all $t \in \mathbb{R}$.

We still need to "remove the absolute value", that is, to check that $s(t+T) \ge s(t)$ for all t, and it suffices to do so for $t = 0$. Suppose to the contrary that there is a $T \in (0, T_0/3)$ such that for all $n \in \mathbb{N}$ there are $x_n, y_n \in X$ and continuous $c_n\colon \mathbb{R} \to \mathbb{R}$ with $s_n(0) = 0$ for which $d(\varphi^t(x_n), \varphi^{s_n(t)}(y_n)) < 1/n$ for all $t \in \mathbb{R}$ but $s_n(T) < 0$ and (by passing to a subsequence) that $x_n \to x$ and hence $y_n \to x$. We will see that this produces a periodic orbit of period less than T_0, contrary to the choice of T_0.

If $s_n(T) \ge -T$ for infinitely many n, then $s_{n_i}(T) \to -L \in [-T, 0]$ for a subsequence, so $d(\varphi^T(x), \varphi^{-L}(x)) = 0$, and x is periodic with period $L + T < T_0$, a contradiction. Otherwise, $s_n(T) < -T$ for all large n, so $s_n(t_n) = -T$ for some $t_n \in [0, T]$ and $t_{n_i} \to t$, hence likewise $x = \varphi^{T+t}(x)$, a contradiction. □

We now return to the proof of the theorem. The claim above shows that if $d(\varphi^t(x), \varphi^{s(t)}(y)) < \delta_T$, then the desired increasing homeomorphism h_T of $\mathbb{R}$ is obtained from s by taking $h_T(nT) = s(nT)$ for $n \in \mathbb{Z}$ and linear in between. Moreover, for $t \in [nT, (n+1)T]$ there is a $t' \in [nT, (n+1)T]$ such that $h_T(t) = s(t')$

[24] This last characterization is particularly useful for Proposition 4.2.23 and hence Theorem 4.2.24.

and thus

$$d(\varphi^t(x), \varphi^{h_T(t)}(y)) = d(\varphi^t(x), \varphi^{s(t')}(y)) \le \underbrace{d(\varphi^t(x), \varphi^{t'}(x))}_{\le \sup_{x\in X\ u\in[0,T]} d(x,\varphi^u(x))} + d(\varphi^{t'}(x), \varphi^{s(t')}(y)).$$

Now we establish expansivity. For $\epsilon > 0$ and α as in (1) choose $T \in (0, T_0/3)$ such that $\sup_{x\in X\ u\in[0,T]} d(x, \varphi^u(x)) < \alpha/2$. Then

$$d(\varphi^t(x), \varphi^{s(t)}(y)) < \delta < \min(\delta_T, \alpha/2) \text{ for all } t \in \mathbb{R}$$

implies $d(\varphi^t(x), \varphi^{h_T(t)}(y)) < \alpha$ for all $t \in \mathbb{R}$, so $y = \varphi^t(x)$ for some $t \in (-\epsilon, \epsilon)$.

Finally, we prove that (3) is equivalent to expansivity. If Φ is expansive, $\epsilon > 0$, δ as in Definition 1.7.2, $\alpha > 0$ such that $\alpha + 2\sup\{d(z, \varphi^t(z)) \mid z \in X,\ |t| \le \alpha\} < \delta$, t_i, u_i, x, y as in (3), $s(t_i) \coloneqq u_i$, then interpolate linearly to $s(t)$ for $t \in [t_i, t_{i+1}]$ to get

$$d(\varphi^t(x), \varphi^{s(t)}(y)) \le \underbrace{d(\varphi^t(x), \varphi^{t_i}(x)) + d(\varphi^{t_i}(x), \varphi^{u_i}(y)) + d(\varphi^{u_i}(y), \varphi^{s(t)}(y))}_{\le \alpha + 2\sup\{d(z,\varphi^t(z)) \mid z\in X,\ |t|\le\alpha\}} < \delta,$$

so $y = \varphi^t(x)$ for some $|t| < \epsilon$ by choice of δ.

Conversely, choose $\epsilon > 0$ and α as in (3). If $d(\varphi^t(x), \varphi^{h(t)}(y)) < \alpha$ for all $t \in \mathbb{R}$ and an increasing homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ let $t_0 = 0$ and $t_{\pm i} \xrightarrow[i\to+\infty]{} \pm\infty$ such that $0 < t_{i+1} - t_i \le \alpha$ and $0 < h(t_{i+1}) - h(t_i) \le \alpha$. Then (3) with $u_i \coloneqq h(t_i)$ gives $y = \varphi^t(x)$ with $|t| < \epsilon$. □

Proposition 8.3.11 illustrates the utility of the characterization (3).

Theorem 1.7.7. *Expansivity is preserved by orbit-equivalence.*

Proof. If h is a homeomorphism that maps orbits of an expansive flow Φ on X to orbits of a flow Ψ on Y, then the fixed points of both flows are isolated and can hence be omitted (Remark 1.7.3). For $\eta' > 0$ choose $\eta > 0$ such that $h(B_\eta(x)) \subset B_{\eta'}(h(x))$ for all $x \in X$ and δ as in Theorem 1.7.5(2) as well as $\delta' > 0$ such that $d_Y(y_1, y_2) < \delta' \Rightarrow d_X(h^{-1}(y_1), h^{-1}(y_2)) < \delta$.

Suppose now that $x_1, x_2 \in X$ are such that there is a continuous $s\colon \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$ and $d_Y(\psi^t(h(x_1)), \psi^{s(t)}(h(x_2))) < \delta'$ for all $t \in \mathbb{R}$. Then Remark 1.3.23 and the choice of δ' give

$$d_X(\varphi^{\sigma_{x_1}(t)}(x_1), \varphi^{\sigma_{x_2}(t)}(x_2)) < \delta, \text{ that is, } d_X(\varphi^t(x_1), \varphi^{\sigma_{x_2}(s(\sigma_{x_1}^{-1}(u)))}(x_2)) < \delta$$

for all $t \in \mathbb{R}$. Thus, by Theorem 1.7.5(2), $x_2 \in \mathscr{O}(x_1)$, and the Φ-orbit segment from x_1 to x_2 is in $B_\eta(x_1)$, so $h(x_2) \in \mathscr{O}(h(h_1))$ and the Ψ-orbit segment from $h(x_1)$ to $h(x_2)$ is in $B_{\eta'}(h(x_1))$. □

Corollary 1.7.8. *A time-change of an expansive flow is expansive.*

Here is a counterpart of Proposition 1.6.30 (with Proposition 1.7.10 providing a broader one); Example 1.5.23 illustrates this.

Proposition 1.7.9. *A suspension is expansive if and only if the base is.*

Proof. With the conventions of Remark 1.2.6 suppose the suspension flow Φ is expansive. For $\epsilon \in (0,1/2)$ take $\delta > 0$ as in Definition 1.7.1 and suppose $y_1, y_2 \in M$ are such that $d(f^n(y_1), f^n(y_2)) < \delta$ for all $n \in \mathbb{Z}$. Then

$$\begin{aligned} d(\varphi^t((y_1,0),\varphi^t(y_2,0)) &\le \underbrace{\rho_{t-\lfloor t\rfloor}(\varphi^{\lfloor t\rfloor}(y_1),\varphi^{\lfloor t\rfloor}(y_1))}_{=(1-t+\lfloor t\rfloor)\rho(\varphi^{\lfloor t\rfloor}(y_1),\varphi^{\lfloor t\rfloor}(y_1))+(t-\lfloor t\rfloor)\rho(\varphi^{\lfloor t\rfloor+1}(y_1),\varphi^{\lfloor t\rfloor+1}(y_1))} \\ &< (1-t+\lfloor t\rfloor)\delta + (t-\lfloor t\rfloor)\delta = \delta. \end{aligned}$$

Thus, $(y_2,0) = \varphi^t(y_1,0)$ with $|t| < \epsilon < 1/2$, so $y_1 = y_2$, and f is expansive.

Conversely, if f is expansive and $\epsilon > 0$ take $\delta < \min(1/4,\epsilon)$ less than the expansivity constant of f with respect to ρ' and $x_1, x_2 \in M_f$ such that $d(\varphi^t(x_1),\varphi^{s(t)}(x_2)) < \delta$ for all $t \in \mathbb{R}$ and some continuous $s\colon \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$.

We will later reduce to the case where $x_1 \sim (y_1, 1/2) \in M \times [0,1]$, and with $x_2 \sim (y_2, t_2)$ we then get $\rho'(y_1,y_2) \le d(x_1,x_2) < \delta < 1/4$. Since $\varphi^1(x_1) \sim (f(y_1), 1/2)$ and $d(\varphi^t(x_1),\varphi^{s(t)}(x_2)) < \delta$ for all $t \in [0,1]$, we have $\varphi^{s(1)}(x_2) \sim (f(y_2), s)$, and therefore $\rho'(f(y_1), f(y_2) \le d(\varphi^1(x_1),\varphi^{s(1)}(x_2)) < \delta$. Continuing this gives $\rho'(f^n(y_1), f^n(y_2)) < \delta$ for all $n \in \mathbb{Z}$ and hence $y_1 = y_2$, which also gives $x_2 = \varphi^t(x_1)$ for some $|t| < \delta < \epsilon$.

For arbitrary x_1 find $r \in [-1/2, 1/2]$ with $x_1' := \varphi^r(x_1) \sim (y_1, 1/2)$. With $x_s' := \varphi^{s(r)}(x_2)$ this gives $d(\varphi^t(x_1'),\varphi^{s(t+r)-s(r)}(x_2')) < \delta$ for all $t \in \mathbb{R}$, so the foregoing implies $x_2' = \varphi^t(x_1')$ for some $|t| < \delta$, hence $x_2 = \varphi^{t+r-s(r)}(x_1)$ with $|t + r - s(r)| = d(x_1,x_2) < \delta < \epsilon$. Thus, Φ is expansive. □

With Proposition 1.3.28 and Theorem 1.7.7 this also implies:

Proposition 1.7.10. *A special flow is expansive if and only if the base is.*

In topological systems one often finds a weaker version of expansivity where some (but not necessarily all) nearby orbits separate in time, so that some microscopic deviation in initial conditions can lead to macroscopic differences in the orbits.

Definition 1.7.11 (Sensitive dependence)**.** Suppose Φ is a flow on a metric space X. A point $x \in X$ is said to exhibit *sensitive dependence on initial conditions* if there is an $\epsilon > 0$ as follows: for all $\delta > 0$ there is a $y \in X$ such that $d(y,x) < \delta$ and for any continuous map $s\colon \mathbb{R} \to \mathbb{R}$ there is a $t \in \mathbb{R}$ with $d(\varphi^t(x),\varphi^{s(t)}(y)) \ge \epsilon$. If this is the case for all $x \in X$, then we say that Φ has sensitive dependence on initial conditions.

Together with compactness of the metric space this can lead to chaotic dynamics. As hyperbolic flows are expansive (the stronger notion) we will not investigate sensitive dependence further.

8. Symbolic flows

We now describe a class of topological flows that provide the standard model for representing hyperbolic flows, in a way that we will later make explicit. A finite coding can help investigate the dynamics of deterministic systems that are so complex as to appear random. The flows that arise from coding a system are symbolic flows, and we will show that they are expansive. Chapter 2 will introduce the paradigmatic case of smooth flows for which these notions are pertinent.

Symbolic flows are particularly amenable to careful study of the orbit structure, as well as, later, statistical features. They are constructed as special flows over finite-state systems, that is, over a system that is described in terms of allowed sequences of symbols from a finite alphabet. Symbolic flows can also exhibit recurrence properties from among those listed in the previous sections and thus also provide new examples of systems with such features.

The symbolic examples with which we do this are central to the study of hyperbolic dynamical systems. In fact, we will show later that hyperbolic flows have a lift to a symbolic system that is uniformly finite-to-one and so will preserve many of the important properties of the hyperbolic flow (Section 6.4).

Definition 1.8.1. Let $\mathscr{A}_n$ be a finite set with the discrete topology (the "alphabet," whose members are called the symbols), where $n = \#\mathscr{A}_n$. Let $\Sigma_n = \{\mathscr{A}_n\}^{\mathbb{Z}}$. Then a point $\mathbf{t} = \{t_i\}_{i\in\mathbb{Z}} \in \Sigma_n$ is a bi-infinite sequence with each $t_i \in \mathscr{A}_n$. To give the set Σ_n the structure of a compact metric space we use the product topology.

For $a > 1$ (and usually $a = 2$) we define a metric on Σ_n by $d_a(\mathbf{s},\mathbf{t}) = a^{-N}$ where N is the largest nonnegative integer such that $s_i = t_i$ for all $|i| < N$.

The (left) *shift map* is the homeomorphism $\sigma\colon \Sigma_n \to \Sigma_n$ such that $\sigma(s)_i = s_{i+1}$. The space (Σ_n, σ) is the *full shift on n-symbols.* A set $\Lambda \subset \Sigma_n$ together with the shift map is a *subshift* if Λ is a closed shift invariant set.

If $A\colon \mathscr{A}_n \times \mathscr{A}_n \to \{0,1\}$ is a function (that is, an $n \times n$ matrix) such that for each $i \in \mathscr{A}_n$ there is a $j \in \mathscr{A}_n$ such that $A(i,j) = 1$ and for each $j \in \mathscr{A}_n$ there is a $i \in \mathscr{A}_n$ where $A(i,j) = 1$, then the subshift

$$\Lambda = \Sigma_A = \{\mathbf{s} \in \Sigma_n : A(s_i, s_{i+1}) = 1\ \forall\, i \in \mathbb{Z}\} \text{ with } \sigma_A := \sigma\restriction_{\Sigma_A}$$

is called a *subshift of finite type* or *topological Markov chain.*

The entries of the *transition matrix A* satisfy $a_{ij} = 1$ if and only if $A(i,j) = 1$, in which case we say that the transition from i to j is *allowed.* By assumption, each row and each column have a nonzero entry, and such a matrix is called an *adjacency matrix.*

Example 1.8.2. A subshift of infinite type is given by the sequences in $\{0,1\}^{\mathbb{Z}}$ that contain at most one occurrence of the symbol 1. It consists of a fixed point (the sequence of zeros) and an orbit whose α- and ω-limit sets are the fixed point. This is a discrete-time counterpart of Example 1.3.9. A discrete-time counterpart to Example 1.3.6 is given by the subshift defined by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Topological Markov chains provide the base for the special flows (Definition 1.2.7) that are the subject of this section.

A basis for the topology on Σ_n is given by the *cylinder sets*

$$C^{n_1,\dots,n_k}_{i_1,\dots,i_k} = \{\mathbf{s} \in \Sigma_n : s_{n_j} = i_j \text{ for all } 1 \le j \le k\}, \text{ where } n_j \in \mathbb{Z},\ i_j \in \mathscr{A}_n \tag{1.8.1}$$

consisting of the sequences with prescribed symbols in a finite set of locations. Since the complement of a cylinder is a union of cylinders, hence open, cylinders are both open and closed.

Definition 1.8.3. For a subshift Λ and a positive continuous function $f\colon \Lambda \to \mathbb{R}$, the *symbolic flow* φ^t_f is the flow over Λ under the function f. When Λ is a topological Markov chain and the roof function is Hölder-continuous, the symbolic flow φ^t_f is called a *hyperbolic symbolic flow*.

Here, and often, we use a regularity notion that is particularly natural for hyperbolic flows (Definition 12.1.1):

Definition 1.8.4. A map f between metric spaces is said to be Hölder continuous with exponent $\alpha \in (0,1]$ or α-Hölder if $d(f(x), f(y)) \le (d(x,y))^\alpha$ for nearby x and y.[25] A 1-Hölder map is said to be Lipschitz-continuous.

We use this assumption here because it naturally arises in hyperbolic dynamics and is at the same time needed in their study. The essential point is that hyperbolic behavior is connected with exponential growth or decay of distances between orbits, and Hölder continuity is well-adapted to this because exponentially small differences in inputs result in exponentially small differences in outputs. For symbolic flows, different natural choices of distance functions are related by Hölder regularity of the identity (and its inverse), and with a Hölder continuous roof function, the resulting flow has a natural Hölder structure.

Specifically, let

$$X = \{(\mathbf{s}, t) : t \in [0, f(\mathbf{s})], \mathbf{s} \in \Lambda\} \subset \Lambda \times \mathbb{R},$$

and identify the points $(\mathbf{s}, f(\mathbf{s}))$ and $(\sigma(\mathbf{s}), 0)$ for all $\mathbf{s} \in \Lambda$. On this identification space $\Lambda(f)$ the special flow over Λ with roof function f is described as follows (Definition 1.2.7). Let $\pi\colon X \to \Lambda(f)$ be the quotient map. Then $\varphi^t_f(\pi(\mathbf{s}, t_0)) = \pi(\sigma^k(\mathbf{s}, \tilde{t}))$

[25] See also Definition 7.1.1.

where $k \geq 0$ satisfies

$$\tilde{t} = t + t_0 - \sum_{j=0}^{k-1} f(\sigma^j(\mathbf{s}))$$

with $0 \leq \tilde{t} < f(\sigma^k(\mathbf{s}))$.

For these flows under functions it is of interest to connect dynamical properties of the base to those of the flow.

We are primarily interested in symbolic flows over subshifts of finite type. In this setting many of the dynamical properties of the subshift of finite type can be recovered from properties of the adjacency matrix A. For an adjacency matrix A there is an associated graph $\mathcal{G}_A$ on n vertices such that there is an edge from i to j if and only if $a_{ij} \neq 0$.[26] The reader is encouraged to draw the graphs for the matrices $A = \left(\begin{smallmatrix} 0&1&0&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&0&1&0 \end{smallmatrix}\right)$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Lemma 1.8.5. *Let A be an adjacency matrix and $\mathcal{G}_A$ be the associated graph on n vertices. If $i, j \in \mathcal{A}_n$, then the the number $\#_{ij}^m$ of distinct paths on $\mathcal{G}_A$ of length $m \in \mathbb{N}$ from i to j equals the i, j-th entry a_{ij}^m of A^m (the product of m copies of A).*

Proof. We use induction on m. The case $m = 0$ (or $m = 1$) is clear. The induction step is accomplished once we show that

$$\#_{ij}^{m+1} = \sum_{k \in \mathcal{A}_n} \#_{ik}^m a_{kj}. \tag{1.8.2}$$

For every $k \in \mathcal{A}_n$ every admissible path of length m connecting i and k produces exactly one admissible path of length $m+1$ connecting i and j by adding j to it, if and only if $a_{kj} = 1$. This proves (1.8.2). □

Corollary 1.8.6 (Periodic-orbit growth)**.** $\overline{\lim}_{n\to\infty} \frac{1}{n} \operatorname{card} \operatorname{Fix}(\sigma_A^n) = r(A)$, *where $r(A)$ is the spectral radius (Definition 12.3.1).*

Remark 1.8.7. If we let $\mathcal{W}$ be the set of finite length sequences that appear in Σ_A, then $w \in \mathcal{W}$ if and only if there is a corresponding allowed path on $\mathcal{G}_A$ following the prescribed vertices. We call such a finite sequence w an *allowed word* in Σ_A.

A matrix A with nonnegative integer entries is *irreducible* if for each $i, j \in \{1, \ldots, n\}$ there exists some $N = N(i, j)$ such that $a_{ij}^N \neq 0$.

Proposition 1.8.8. *A symbolic flow $(\Lambda(f), \sigma_f)$ over a subshift of finite type Σ_A has dense periodic points if A is irreducible. Furthermore, $\Lambda(f)$ is transitive if and only if A is irreducible.*

[26] The graphs we consider are directed, allow "loops", that is, an edge from a vertex to itself, and each vertex has at least one entering and one exiting edge (because otherwise it can't occur in a bi-infinite sequence).

Proof. By Proposition 1.6.30 it suffices to prove this for Σ_A.

To prove that the periodic points are dense in Σ_A, let $\mathbf{s} \in \Sigma_A$ and $\epsilon > 0$. Fix $N \in \mathbb{N}$ such that $a^{-N} < \epsilon$ where a is the constant in the metric. Let $w = \mathbf{s}_{-N} \cdots \mathbf{s}_N$. For the elements $\mathbf{s}_{-N}, \mathbf{s}_N \in \mathscr{A}_n$ there exists some $n \geq 2$ such that $a^n_{\mathbf{s}_N \mathbf{s}_{-N}} \neq 0$. So there exists an allowed word w' of length $n-2$ such that $ww'w$ is an allowed word, and we can define a periodic $\hat{\mathbf{s}} \in \Sigma_A$ with period $N+n-2$ by $\hat{s}_{-N} \cdots \hat{s}_{N+n-2} = ww'$. Then $d_a(\mathbf{s}, \hat{\mathbf{s}}) < \epsilon$ and periodic points are dense in Σ_A.

To show that Σ_A is transitive if A is irreducible, order $\mathscr{W}$ by first enumerating the words of length one (symbols in $\mathscr{A}_n$), then all the words of length 2, then all the words of length 3, etc. To prove there is a point with a dense forward orbit we connect the enumerated words. To do this let w_k and w_{k+1} be successive points in the enumerated words. Let $i \in \mathscr{A}_n$ be the final symbol of w_k and $j \in \mathscr{A}_n$ be the first symbol in w_{k+1}. Fix $n \in \mathbb{N}$ such that $a^n_{ij} \neq 0$ and let $w = s_1 \cdots s_n \in \mathscr{W}$ be a word of length n such that the first symbol is i and the last symbol is j. Fix $w' = s_2 \cdots s_{n-1}$ be the finite word obtained by removing the first and last symbols of w. Then $w_k w' w_{k+1}$ is an allowed word. Continuing by induction we then construct a forward infinite sequence containing all allowed words in Σ_A. Fix $\mathbf{s} \in \Sigma_A$ such that the forward sequence of terms in $\mathbf{s}$ agrees with the infinite sequence we constructed. It is not hard to see that under the shift map the forward orbit of $\mathbf{s}$ is dense in Σ_A.

The converse is much easier: Given $i, j \in \mathscr{A}_n$, transitivity implies that there is an $s \in \sigma_A$ that goes from the cylinder set $\{s_0 = i\}$ to the cylinder set $\{s_0 = j\}$, that is, that iwj is an allowed word for some word w. Thus $a_{ij} \neq 0$. □

Example 1.8.9. For a permutation matrix A (that is, a matrix with a single 1 in each row and each column), each symbol has a unique successor, so Σ_A consists of periodic orbits (one for each cycle of the permutation) and is hence transitive if and only if there is only one such orbit, that is, the permutation is cyclic and A is irreducible, such as $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. In fact, permutation matrices give the only cases of subshifts of finite type with finite cardinality.

Example 1.8.10. The matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not irreducible. For a roof function f the suspension flow Φ_f on $\Lambda(f)$ of Σ_A has a dense orbit, but does not have a dense forward orbit. This flow consists of two periodic orbits (coming from fixed points of Σ_A) and an orbit whose α-limit set is one of the periodic orbits and ω-limit set is the other periodic orbit. This flow is topologically conjugate to the cartesian product of the flow in Example 1.3.6 with that in Example 1.1.6.

Example 1.8.11. An irreducible matrix that appears similar to the previous example, but whose associated topological Markov chain has different dynamical properties, is given by $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ with $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. So not only is there an N for each i and j with $a_{ij}^N \neq 0$, but $N = 2$ works simultaneously for all ij-pairs.

Definition 1.8.12. An integer matrix A is *positive* if each entry is positive and *eventually positive* or *aperiodic* if there is an $N \in \mathbb{N}$ such that A^N is positive.

Then the proof of Proposition 1.8.8 gives:

Proposition 1.8.13. *If A is eventually positive, then Σ_A is topologically mixing.*

The next two results could have been proven earlier, and connect the results in this section to the results in the previous section.

Theorem 1.8.14. *Subshifts are expansive.*

Proof. Let Λ be a subshift, $\epsilon < 1$, and $\mathbf{s}, \hat{\mathbf{s}} \in \Lambda$. Then there is an $i \in \mathbb{Z}$ with $s_i \neq \hat{s}_i$, so $d_a(\sigma^i(\mathbf{s}), \sigma^i(\hat{\mathbf{s}})) = 1 > \epsilon$. □

As an immediate consequence of this and Theorem 1.8.14 we further have:

Proposition 1.8.15. *Symbolic flows are expansive.*

a. Symbolic codings. One of the main uses of symbolic flows for us will be in coding invariant sets for flows. In this case the coding is typically a semi-conjugacy and so does not preserve all of the properties of the original flow. However, the symbolic flow is usually easier to investigate and preserves sufficient properties to be useful. We now provide a few examples to show how this can be done. The more general theory on symbolic extensions will be given in Section 6.4.

Example 1.8.16. In Example 1.5.21, the dynamics on Λ is topologically conjugate to the full 2-shift by labeling the 2 image pieces overlapping with Δ as 0 and 1 and associating points and their itineraries. The flow is thus topologically conjugate to the symbolic flow over the full 2-shift with roof function equal to 1. Variants with more crossings in Δ are topologically conjugate to a full shift on more symbols. Therefore, the set Λ has a dense set of periodic points and is topologically transitive.[27]

Example 1.8.17. In Example 1.5.22, the dynamics on the natural invariant (Cantor) set is topologically conjugate to a shift on 5 symbols by proceeding analogously using the 5 overlap rectangles in the picture. In the rectangle to the right there are three rectangles that are preimages of the regions that overlap. In the rectangle

[27] And has positive topological entropy (Section 4.2).

to the right there are two rectangles that are the preimages of the regions that overlap. In the first two subrectangles of the rectangle that is to the left the image intersects all three of the preimages, and the third rectangle the image intersects the two preimages in the rectangle to the right. For the rectangle that is to the right the image of the first subrectangle intersects the three subrectangles in the rectangle to the left, while the image of the second subrectangle intersects the two subrectangles in the rectangle to the right. With suitable labeling, these allowed transitions are collected in the matrix

$$\mathbb{A} = \begin{pmatrix} 1&1&1&0&0\\ 1&1&1&0&0\\ 0&0&0&1&1\\ 1&1&1&0&0\\ 0&0&0&1&1 \end{pmatrix} \tag{1.8.3}$$

Thus, we have a "coding" by $\mathbb{A}$, that is, a homeomorphism h between the suspension of $\Sigma_{\mathbb{A}}$ and that of the invariant Cantor set in Figure 1.5.8 that intertwines the flows.

Example 1.8.18. Suspensions of hyperbolic toral automorphisms are factors of symbolic flows. Consider the suspension of the map

$$F_A(x, y) = (2x + y, x + y) \pmod 1 \tag{1.8.4}$$

of the two-torus from Example 1.5.23. Draw segments of the two eigenlines at the

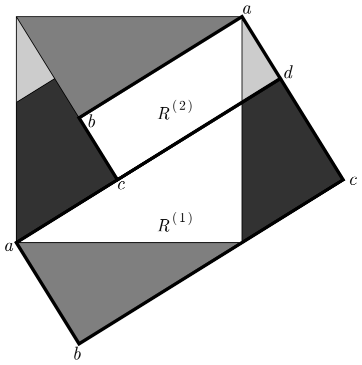


FIGURE 1.8.1. Partitioning the torus

origin until they cross sufficiently many times and separate the torus into disjoint rectangles. Although this prescription contains an ambiguity, direct inspection shows that it can be effected by taking a segment of the contracting line in the fourth quadrant until it intersects the segment of the expanding line twice in the first quadrant and once in the third quadrant (see Figure 1.8.1). The resulting configuration is a decomposition of the torus into two rectangles $R^{(1)}$ and $R^{(2)}$. Three pairs among the seven vertices of the plane configuration are identified, so there are only four different points on the torus which serve as vertices of the rectangles. This agrees with our description: those vertices are exactly the origin and three intersection points.

One can see even without explicit calculation that the image $F_A(R^{(i)})$ $(i = 1,2)$ consists of several "horizontal" rectangles of "full length". The union of the boundaries $\partial R^{(1)} \cup \partial R^{(2)}$ consists of the segments of the two eigenlines at the origin just described. The image of the contracting segment is a part of that segment. Thus, the images of $R^{(1)}$ and $R^{(2)}$ have to be "anchored" at parts of their "vertical" sides, that is, once one of the images "enters" either $R^{(1)}$ or $R^{(2)}$ it has to stretch all the way through it. Tracking where A sends integer points shows that $F_A(R^{(1)})$ consists of three components, two in $R^{(1)}$ and one in $R^{(2)}$. The image of $R^{(2)}$ has two components, one in each rectangle (see Figure 1.8.2). We can use these five components

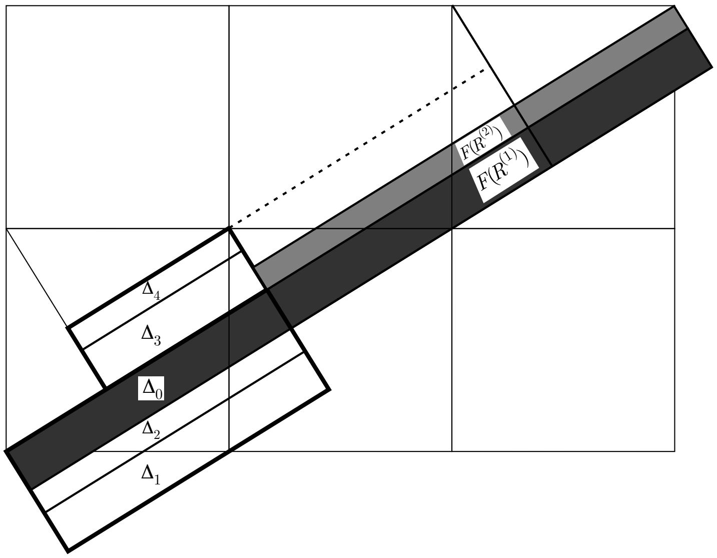


FIGURE 1.8.2. The image of the partition

$\Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4$ (or their preimages) as the pieces in our coding construction. Due to the contraction of F_A in the "vertical" direction and contraction of F_A^{-1} in the "horizontal" direction each intersection

$$\bigcap_{n\in\mathbb{Z}} F_A^{-n}(R^{(\omega_n)})$$

contains no more than one point. On the other hand, the "Markov" property, that is, the images going full length through rectangles, implies: If $\omega\in\Sigma_5$ and $F_A(\operatorname{Int}\Delta_{\omega_n})\cap\operatorname{Int}\Delta_{\omega_{n+1}}\neq\varnothing$ for all $n\in\mathbb{Z}$, then $\bigcap_{n\in\mathbb{Z}}\overline{F_A^n(\operatorname{Int}\Delta_{\omega_n})}\neq\varnothing$. In other words, we have a "coding," that is, a continuous map $h\colon\Sigma_{\mathbb{A}}\to\mathbb{T}^2$ with $\mathbb{A}$ from (1.8.3) such that

$$F_A\circ h = h\circ\sigma.$$

Thus, F_A is a (topological) factor of $\Sigma_{\mathbb{A}}$; in this case the term "semiconjugacy" for h is apt, because we will see that it is "mostly" bijective. Every point $q\in\mathbb{T}^2$ whose positive and negative iterates avoid the boundaries $\partial R^{(1)}$ and $\partial R^{(2)}$ has a unique preimage and vice versa. The points of Σ_A whose images are on those boundaries or their iterates under F_A fall into three categories corresponding to the three segments of stable and unstable sets through 0 which define parts of the boundary. Thus, sequences are identified in the following cases: They have a constant infinite right (future) tail consisting of 0's or 4's, and agree otherwise, or else an infinite left (past) tail (of 0's and 1's, or of 4's) and agree otherwise. We summarize some of the properties of the coding.

Proposition 1.8.19. *The induced factor map between the suspensions of $\sigma\restriction_{\Sigma_{\mathbb{A}}}$ and F_A is one-to-one on all periodic points (except for those coming from fixed points). The number of preimages of any point not negatively asymptotic to the suspension of the fixed point is bounded.*

Exercises

1.1. For a flow Φ on a space X and a point $x\in X$ prove that exactly one of the following hold:

(1) $t\mapsto\varphi^t(x)$ is one-to-one,
(2) there exists a smallest $t_0>0$ such that $\varphi^{t_0+t}(x)=\varphi^t(x)$ for all $t\in\mathbb{R}$, and
(3) $x=\varphi^t(x)$ for all $t\in\mathbb{R}$.

1.2. If $g:\mathbb{R}\to\mathbb{R}$ is continuous, then writing $v=\frac{dx}{dt}$ (velocity) converts the second-order differential equation $\dfrac{d^2x}{dt^2}+g(x)=0$, to the system

$$\begin{cases}\dfrac{dx}{dt}=v,\\[4pt] \dfrac{dv}{dt}=-g(x)\end{cases}$$

of first-order differential equations. Show that $H(x,v) = \frac{1}{2}v^2 + \int_0^x g(s)\,ds$ is a constant of motion.

1.3. Prove the converse of Theorem 1.4.3.

1.4. Carry out the "straightforward calculation" in the proof of Theorem 1.4.7.

1.5. Find all Lyapunov functions for the North-south flow Example 1.3.7 and the South-south flow Example 1.3.9.

1.6. In a compact metric space, show that $\{x\}$ is attracting (Definition 1.4.15) if and only if x is attracting (Definition 1.4.1).

1.7. Show that $W^s(x)$ (Definition 1.3.24) and $W^s(\{x\})$ (Definition 1.5.5) agree. (This is a tiny preview of Theorem 5.3.25.)

1.8. Prove Proposition 1.3.25.

1.9. Show that topological conjugacy (Definition 1.3.1) defines an equivalence relation among continuous flows.

1.10. Carry out the "illuminating" proof in Example 1.3.11.

1.11. Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are expanding maps with $|f'|$ bounded and $\|f-g\|_{C^1} < \infty$. Show that there is a unique $h\colon \mathbb{R} \to \mathbb{R}$ with $h - \mathrm{Id}$ bounded such that $f \circ g = g \circ f$ and that $hn := f^{-n} \circ g^n \xrightarrow[n\to\infty]{} h$ uniformly and $\|h_n - \mathrm{Id}\|_\infty \le K\|f-g\|_\infty \le K\|f-g\|_{C^1}$ for some $K > 0$.

1.12. Show that orbit-equivalence (Definition 1.3.21) defines an equivalence relation among continuous flows.

1.13. As suggested in Remark 1.6.17 show that any 2 versions of Figure 1.4.1 (for different damping parameters) are topologically conjugate by refining the ideas in the proof of Proposition 1.4.5.

1.14. Find the stable and unstable sets (Definition 1.3.24) of a fixed point of a topological Markov chain.

1.15. Find the stable and unstable sets (Definition 1.3.24) of a point in a topological Markov chain.

1.16. Find the stable and unstable sets (Definition 1.3.24) of a periodic point in a symbolic flow.

1.17. Find the stable and unstable sets (Definition 1.3.24) of a point in a symbolic flow.

1.18. Determine $\mathscr{L}$ (Definition 1.5.1), $\mathscr{B}$ (Definition 1.5.9), NW (Definition 1.5.11), $\mathscr{R}$ (Definition 1.5.30), $\mathscr{AR}$ (Definition 1.4.16) as well as the chain decomposition (Definition 1.5.30) in Examples 1.1.5, 1.1.7, 1.1.8, 1.3.13, 1.3.5, 1.3.6, 1.3.9, 1.3.11, 1.3.12, 1.4.14, 1.5.14, 1.5.23, 1.6.2 and Figures 1.1.4, 1.3.3, 1.4.1, 1.5.4, 1.5.11.

1.19. Find each basin of attraction and basin of repulsion (Definition 1.5.5) of any compact invariant sets that are apparent in Figures 1.1.4, 1.3.3, 1.4.1, 1.5.4, and 1.5.11.

1.20. Determine $NW(\Phi)$, $NW\big(\Phi\upharpoonright_{NW(\Phi)}\big)$, $NW\big(\Phi\upharpoonright_{NW(\Phi\upharpoonright_{NW(\Phi)})}\big)$ in Figures 1.1.4, 1.3.3, 1.4.1, 1.5.3, 1.5.4, and 1.5.11.

1.21. Find examples to show that each inclusion in Proposition 1.5.34 can be strict. (They can be found among examples presented in this chapter.)

1.22. In Figure 1.5.3 find the prolongational limit sets of any points not on the top line.

1.23. Prove that A_U and R_U in Definition 1.4.16 are nonempty, compact and Φ-invariant.

1.24. In the context of Remark 1.5.40 describe all possible trapping regions and attractor-repeller pairs.

1.25. In light of Proposition 1.6.7 prove or give a counterexample: If $\omega(x) \neq \varnothing$ then $\Phi\upharpoonright_{\omega(x)}$ is topologically transitive.

1.26. Show that the complement of $\mathscr{R}(\Phi)$ is open.

1.27. In Conley's example show that $GR(\Phi) \subsetneq \mathscr{R}(\Phi)$ (Remark 1.5.43).

1.28. Show that $\mathscr{R}(\Phi)/\sim$ (the space of chain-equivalence classes) is a Hausdorff topological space.

1.29. Show that $\mathscr{R}(\Phi)/\sim$ (the space of chain-equivalence classes) is either finite or a Cantor set.

1.30. Show that a continuous flow with infinite chain-decomposition has the Akin flow A (Example 1.3.12) as an orbit-factor.

CHAPTER 2

Hyperbolic geodesic flow*

Having built up more concepts for describing complicated flows we now pick up again from Subsection 1.1c to develop geodesic flows on hyperbolic surfaces. We will see later that these are the standard examples of hyperbolic flows. This chapter may be omitted, but provides details on the classical example that provided the impetus for studying hyperbolic flows.

This chapter assumes a basic knowledge of differential geometry. We will review some of the concepts, especially ones we will need for the dynamics of surfaces with negative curvature.

We begin with a description of the upper half-plane model of a hyperbolic metric with emphasis on the geometry and isometries of this model to have the tools we need for describing the dynamics of the geodesic flow, and we introduce the Poincaré disk as another standard model for hyperbolic geometry. We then describe the dynamics on the upper half-plane model and explain how we obtain compact factors of the Poincaré disk and hence flows on compact spaces with nontrivial recurrence. These compact factors are the classical examples of hyperbolic flows and illustrate many of the notions that we will develop in the second half of the book.

If one wants to only study the flows that have hyperbolic properties then one would study Subsections 2.1a, 2.1b, and 2.2a, together with Sections 2.3 and 2.4.

1. Isometries, geodesics, and horocycles of the hyperbolic plane and disk

The upper half-plane

$$\mathbb{H} := \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Im} z > 0 \right\} \subset \mathbb{C}$$

is an open subset of $\mathbb{C} \sim \mathbb{R}^2$, hence a smooth manifold, and

$$\langle u + iv, u' + iv' \rangle_z := \operatorname{Re} \frac{(u+iv)(u'-iv')}{(\operatorname{Im} z)^2}$$

for $z \in \mathbb{H}$, $u + iv$, $u' + iv' \in T_z\mathbb{H}$ is symmetric, $\mathbb{R}$-bilinear, and positive-definite, hence a Riemannian metric $\langle \cdot, \cdot \rangle$, called the *hyperbolic metric*. The half-plane $\mathbb{H}$

with this metric is called the *Poincaré upper half-plane* (or the Klein model or the *Lobachevsky plane*). The hyperbolic metric differs from the Euclidean metric $\operatorname{Re}(u+iv)(u'-iv')$ only by the scalar factor $(\operatorname{Im} z)^2$, so hyperbolic angles coincide with Euclidean angles.

Lemma 2.1.1. *The imaginary axis* $I := i\cdot(0,\infty)$ *is a geodesic with unit-speed parameterization* $t \mapsto ie^t$.

Proof. I minimizes length between any two of its points: The length of a curve $t \mapsto c(t) = x(t) + iy(t)$, $x(0) = x(1) = 0$, $y(0) = y_0$, $y(1) = y_1$ connecting iy_0 to iy_1 is

$$\ell(c) = \int_0^1 \sqrt{\langle \dot c(t), \dot c(t)\rangle_{c(t)}}\,dt = \int_0^1 \sqrt{\frac{(\dot x(t))^2 + (\dot y(t))^2}{(y(t))^2}}\,dt \geq \int_0^1 \frac{\frac{dy}{dt}}{y}\,dt = \ell(\gamma),$$

where γ is a parameterization of the segment $i[y_0, y_1] \subset I$. □

a. Isometries. The principal tool for understanding the geometry of $\mathbb{H}$ are its isometries. We begin with *linear fractional transformations.* Denote by $GL_+(2,\mathbb{R})$ the collection of real 2×2 matrices with positive determinant and associate to each $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_+(2,\mathbb{R})$ the map

$$T := \psi\begin{pmatrix} a & b \\ c & d\end{pmatrix} : \mathbb{H} \to \mathbb{H}, \qquad z \mapsto \frac{az+b}{cz+d}. \tag{2.1.1}$$

Then $T'(z) = \dfrac{ad-bc}{(cz+d)^2}$ and hence

$$\operatorname{Im} T(z) = \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\bar z+b}{c\bar z+d}\right) = \frac{(az+b)(c\bar z+d) - (a\bar z+b)(cz+d)}{2i(cz+d)(c\bar z+d)} = |T'(z)|\operatorname{Im}(z),$$

so T maps $\mathbb{H}$ to itself. $\mathcal{M} := \psi\big(GL_+(2,\mathbb{R})\big)$ is a group under composition and ψ is a homomorphism with kernel $\mathbb{R}\,\mathrm{Id}$. As a matrix group, this is $\mathrm{PSL}(2,\mathbb{R})$.

Lemma 2.1.2. *The maps* $T \in \mathcal{M}$ *are isometries of the hyperbolic metric.*

Proof. $$\underbrace{\operatorname{Re}\frac{T'(z)(u+iv)\overline{T'(z)(u'+iv')}}{(\operatorname{Im} T(z))^2}}_{=\langle T'(z)(u+iv),\,T'(z)(u'+iv')\rangle_{T(z)}} = \underbrace{\frac{T'(z)\overline{T'(z)}}{|T'(z)|^2}}_{=1}\underbrace{\operatorname{Re}\frac{(u+iv)(u'-iv')}{(\operatorname{Im}(z))^2}}_{=\langle u+iv,\,u'+iv'\rangle_z}.$$ □

Note that all $T \in \mathcal{M}$ extend naturally to $\mathbb{H}\cup\mathbb{R}\cup\{\infty\}$ by setting $T(-d/c) = \infty$ and $T(\infty) = a/c$ (or $T(\infty) = \infty$ if $c = 0$). Examples of linear fractional transformations are $z \mapsto -1/z$, $z \mapsto z+b$ ($b \in \mathbb{R}$), and $z \mapsto az$ ($a > 0$). They represent correspondingly three types of linear fractional transformation from the point of view of the intrinsic geometry of the Lobachevsky plane: *elliptic* (direct counterparts of Euclidean rotations), with a single fixed point inside the plane, *parabolic*, with no fixed points on the plane and no invariant geodesic, and *hyperbolic*, with no fixed points but a unique fixed geodesic (the axis). On $\mathbb{H}$ a parabolic map has a unique fixed point on $\mathbb{R}\cup\{\infty\}$ and a hyperbolic map has two fixed points on $\mathbb{R}\cup\{\infty\}$. Both parabolic and hyperbolic maps are counterparts of translations of the Euclidean plane.

There are also isometries other than linear fractional transformations. Clearly $z\mapsto -\bar{z}$ and $z\to 1/\bar{z}$ are examples. Geometrically the former is the reflection in the imaginary axis and the latter is the inversion with respect to the unit circle. We use linear fractional transformations now to study geodesics. Lemma 2.1.1 suggests to examine isometric images of the imaginary axis I (parameterized with unit speed by $t\mapsto ie^t$).

Lemma 2.1.3. *If C is a vertical line or a semicircle with center on the real line, then there exists a $T\in\mathcal{M}$ with $TI=C$. Furthermore, given any unit tangent vector v at a point of C one can take T such that it maps the upward vertical vector $\imath$ at $i\in I$ to v.*

Proof. If C is the vertical line $\{z \mid \operatorname{Re}(z)=b\}$ take $T(z)=z+b$. If C is a semicircle with end-points $x, x+r\in\mathbb{R}$ then note that $T_1\colon z\mapsto z/(z+1)$ maps I to the semicircle with end-points 0 and 1 (since $\left|\dfrac{it}{1+it}-\dfrac{1}{2}\right|=\left|\dfrac{2it-(1+it)}{2(1+it)}\right|=\dfrac{1}{2}$) and let $T_2(z)=rz$, $T_3(z)=z+x$, and $T=T_3\circ T_2\circ T_1$. To map tangent vectors as desired note that there is a linear fractional transformation T_0 such that $DT_0(\mathrm{i})=DT^{-1}(v)$, namely, either $T_0(z)=cz$ or $T_0(z)=-\dfrac{c}{z}$ for some $c\in\mathbb{R}_+$. Then $T\circ T_0$ is as desired. □

Corollary 2.1.4. *$\mathcal{M}$ acts transitively on the* unit tangent bundle $S\mathbb{H}$ *of* $\mathbb{H}$*: if $v\in T_z\mathbb{H}$, $w\in T_{z'}\mathbb{H}$, $\|v\|=1=\|w\|$, then there is a $T\in\mathcal{M}$ with $T(z)=z'$ and $T'(z)v=w$.*

Remark 2.1.5. Since any vertical line or semicircle with center on the real axis parameterized with unit speed is obtained via a linear fractional transformation from I parameterized by $t\mapsto ie^t$, they are all geodesics, and transitivity on $S\mathbb{H}$ implies that we have identified all geodesics. We note that the end-points of $\psi\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)(I)$ are $\frac{a\cdot 0+b}{c\cdot 0+d}=\frac{b}{d}$ and $\frac{a\cdot i\infty+b}{c\cdot i\infty+b+d}=\frac{a}{c}$.

b. Geodesics and geodesic flow. We are now able to describe the geodesic flow on the upper half-plane.

Theorem 2.1.6. *The geodesics of the Poincaré upper half-plane are precisely the vertical half-lines and the semicircles with center on the real axis.*

Remark 2.1.7. We also have a natural identification of $\mathrm{PSL}(2,\mathbb{R})$ and $S\mathbb{H}$ given by $\gamma\sim v:=\gamma\imath$, where $\imath$ is as in Lemma 2.1.3. Equivalently, set

$$\phi\colon S\mathbb{H}\to\mathrm{PSL}(2,\mathbb{R})\quad\text{by}\quad D\big(\psi(\phi(v))\big)(\imath)=v,$$

where Ψ is as in (2.1.1). With respect to this identification, the geodesic flow is given by $\gamma \mapsto \gamma \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

Since they are isometries and hence send geodesics to geodesics, we also have:

Proposition 2.1.8. *If C is a vertical line or a circle with center on the real axis and $\phi \in \mathcal{M}$ or $\phi(z) = -\bar{z}$ then $\phi(C)$ is a vertical line or a circle with center on the real axis.*

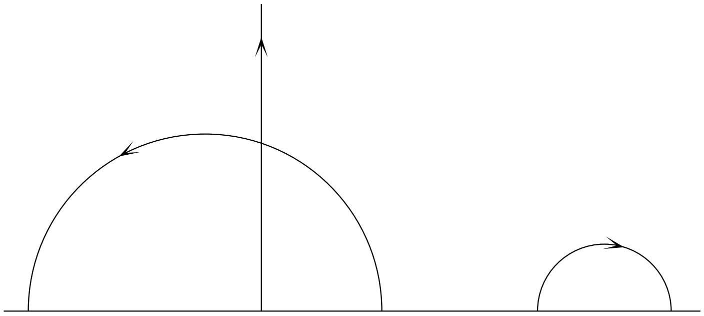


FIGURE 2.1.1. Geodesics on the Lobachevsky plane

The group Γ generated by the group $\mathcal{M}$ of linear fractional transformations and the transformation $S\colon x \mapsto -\bar{z}$ is the isometry group:

Proposition 2.1.9. *The group of isometries of $\mathbb{H}$ is generated by $\mathcal{M}$ and the symmetry $S\colon z \mapsto -\bar{z}$.*

PROOF. Let ϕ be any isometry of $\mathbb{H}$. Any isometry that preserves a geodesic and a tangent vector to it is the identity on that geodesic. Since $\phi(I)$ is a geodesic, Theorem 2.1.6 and Lemma 2.1.3 give a $T \in \mathcal{M}$ such that $T^{-1}\phi\restriction_I = \mathrm{Id}\restriction_I$. It suffices to show that $T^{-1}\phi$ is either the identity on $\mathbb{H}$ or coincides with the symmetry $S\colon z \mapsto -\bar{z}$. Consider the geodesic C with end-points $-r$ and r. It contains the point $ir \in I$ and hence so does $T^{-1}\phi(C)$ (since $T^{-1}\phi\restriction_I = \mathrm{Id}\restriction_I$). Since $T^{-1}\phi$ preserves angles, both these geodesics are orthogonal to I at ir. Hence they coincide up to orientation, that is, we either have $T^{-1}\phi(z) = z$ for $z \in C$ or $T^{-1}\phi(z) = -\bar{z}$ for $z \in C$, and hence the derivative of $T^{-1}\phi$ at ir is either the identity or the reflection in I. Since isometries are smooth, the same case occurs for all points on I; hence the same choice was made for all such geodesics, that is, $T^{-1}\phi = \mathrm{Id}$ or $T^{-1}\phi = S$ on $\mathbb{H}$. So $\phi \in \mathcal{M}$ or $\phi \circ S \in \mathcal{M}$. □

Proposition 2.1.10 (Stable manifolds)**.** *The orbits of upward vertical unit vectors at points $x+i \in \mathbb{R}+i$ are pairwise exponentially positively asymptotic under the geodesic flow $g^t\colon S\mathbb{H} \to S\mathbb{H}$.*

Proof. We use the canonical distance on $S\mathbb{H}$: If $z, z' \in \mathbb{H}$, $v \in S_z\mathbb{H}$, $w \in S_{z'}\mathbb{H}$, then there is a geodesic $\gamma\colon [0,1] \to \mathbb{H}$ (unique if $z \neq z'$) connecting z and z', and a unique continuous vector field X along γ such that $X(0) = v$ and $\measuredangle X(t), \dot\gamma(t) = \measuredangle v, \dot\gamma(0)$ for all $t \in [0,1]$. Then

$$d(v,w) \coloneqq \sqrt{(\measuredangle X(1), w)^2 + (d(z,z'))^2}.$$

Geometrically, this amounts to parallel-translating v along γ to $z' \in \mathbb{H}$ and measuring angles there.

In particular, if $v \in T_{x+iy}\mathbb{H}$, $w \in T_{x+d+iy}\mathbb{H}$ are vertical unit vectors then the angle term in this distance function is $2\tan^{-1}\dfrac{d/2}{y} \le \dfrac{d}{y}$, and an upper bound for the length of the connecting geodesic is given by the length of the connecting line segment, which is d/y. Thus,

$$d(v,w) < \sqrt{2}d/y. \tag{2.1.2}$$

The orbit of the upward vertical unit vector w at $x+i \in \mathbb{H}$ projects to the geodesic $t \mapsto x + ie^t$, and the distance between the corresponding upward unit vectors i_t at ie^t and w_t at $x+ie^t$ is bounded by $\sqrt{2}xe^{-t}$. □

Remark 2.1.11. By using the transformation $z \mapsto -1/z$ one also sees then that the orbits of the outward unit normals to the circle of radius $1/2$ centered at $i/2$ are negatively asymptotic to that of i. Together, we have thus identified the stable and unstable foliations explicitly, which we will much later produce in proper generality (Theorem 6.1.1).

Remark 2.1.12. We also note that in the proof of Proposition 2.1.10 one can let $y \to 0$ and conclude that 2 such vertical geodesics separate exponentially as $t \to -\infty$. In particular, geodesic arcs limiting on distinct boundary points diverge (exponentially) from each other. Contrariwise, if γ, η are geodesics such that $\{d(\gamma(t), \eta(t))\}_{t \ge 0}$ is bounded, then there is a $c \in \mathbb{R}$ such that $d(\gamma(t+c), \eta(t)) \xrightarrow[t\to+\infty]{} 0$. This also implies that if γ, η are geodesics such that $\{d(\gamma(t), \eta(t))\}_{t \in \mathbb{R}}$ is bounded, then there is a $c \in \mathbb{R}$ such that $\gamma(t+c) = \eta(t)$ for all $t \in \mathbb{R}$.

c. Horocycle flow. We are now able to define the horocycle flow for the upper half-plane model. Although this will not be a hyperbolic flow it will have some similar properties and is an important class for both dynamics and geometry.

Definition 2.1.13. Horizontal lines $\mathbb{R}+ir=\{t+ir \mid t\in\mathbb{R}\}$ are called *horocycles* centered at ∞. Circles tangent to $\mathbb{R}$ at $x\in\mathbb{R}$ are called *horocycles* centered at x. If $\gamma\colon\mathbb{R}\to\mathbb{H}$ is a geodesic then $\gamma(-\infty)$, $\gamma(\infty)\in\mathbb{R}\cup\{\infty\}$ are the limit points of γ as $t\to-\infty$ and $t\to+\infty$, respectively. If $v\in T_z\mathbb{H}$ then let $\pi(v):=z$.

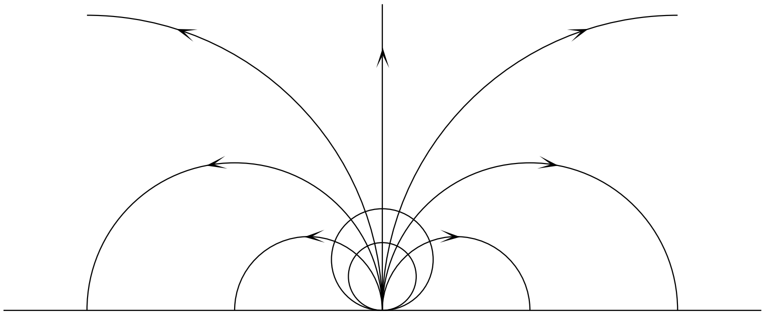


FIGURE 2.1.2. Geodesics and horocycles in the hyperbolic plane

Lemma 2.1.14. *For every horocycle H there is a $T\in\mathcal{M}$ with $T(\mathbb{R}+i)=H$.*

PROOF. If $H=\mathbb{R}+ir$ take $T(z)=rz$. If H is centered at $x\in\mathbb{R}$ and of Euclidean diameter r take $T_1(z)=-1/z$, $T_2(z)=rz$, $T_3(z)=z+x$, and $T=T_3\circ T_2\circ T_1$. □

Remark 2.1.15. With the identification from Remark 2.1.7, these horocycles are the orbits of the *horocycle flow* $h^s\colon\gamma\mapsto\gamma\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}$.

Example 2.1.16. The horocycle flow on a compact factor of the Poincaré disk (Section 2.3) is topologically transitive; indeed, the orbit of every g^t-periodic point is dense [**156**, Theorem 2.2] (see also Exercise 2.6, Exercise 6.7).

For some purposes it is useful to have an alternative model of the Lobachevsky plane (Figure 2.1.3).

Proposition 2.1.17 (Poincaré disk)**.** *The map $f\colon\mathbb{H}\to\mathbb{C}, z\mapsto\dfrac{z-i}{z+i}$ maps the Poincaré upper half-plane $\mathbb{H}$ onto the open unit disk $\mathbb{D}$ in $\mathbb{C}$ bounded by the unit circle $S^1=\{z\in\mathbb{C} \mid |z|=1\}$ since $|f(z)|=1$ when $z\in\mathbb{R}$ and $f(i)=0$. Pushing forward the hyperbolic Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathbb{H}$ to the metric given by*

$$\langle v,w\rangle:=\langle Df^{-1}v, Df^{-1}w\rangle$$

on the unit disk makes f an isometry. The unit disk with this metric is called the Poincaré disk. *Since f maps lines and circles into lines and circles and preserves*

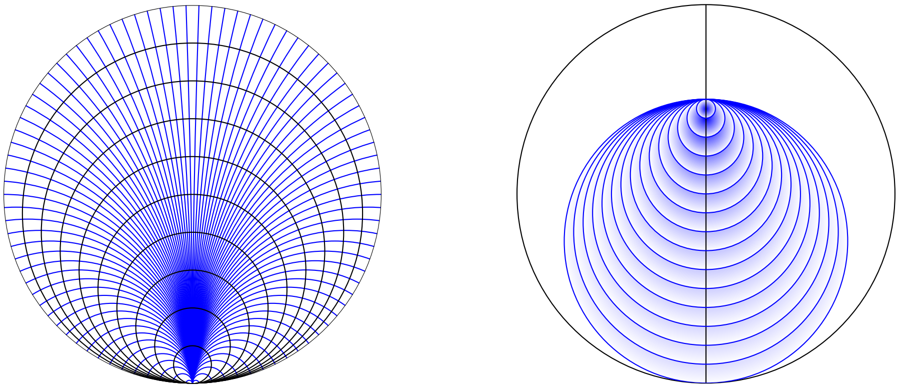


FIGURE 2.1.3. Geodesics and horocycles in the Poincaré disk with a common boundary point (Proposition 2.1.17), and a horocycle as a limit circle (Remark 2.1.18)

angles, the geodesics in the Poincaré disk are diameters of S^1 *and arcs of circles perpendicular to* S^1*, and the horocycles are circles tangent to* S^1 *(Figure 2.1.3).*

Remark 2.1.18 (Busemann function)**.** It is useful to note that the word "horocycle" (sometimes "oricycle") means "limit circle," which is due to the fact that these are limits of circles as follows: For a point ξ at infinity and $x \in \mathbb{D}$ consider the geodesic $\gamma = \gamma_{\xi,x}$ with $\gamma(0) = x$ and $\gamma(t) \xrightarrow[t\to+\infty]{} \xi$. The nested union $\bigcup_{t>0} B(\gamma(t), t)$ of disks is bounded by the horocycle through x determined by ξ (Figure 2.1.3). Alternatively it can be described as the set of points $y \in \mathbb{D}$ such that $d(\gamma(t), y) - t \xrightarrow[t\to+\infty]{} 0$. Indeed, more generally, the horocycles determined by ξ are the level sets of the *Busemann function*

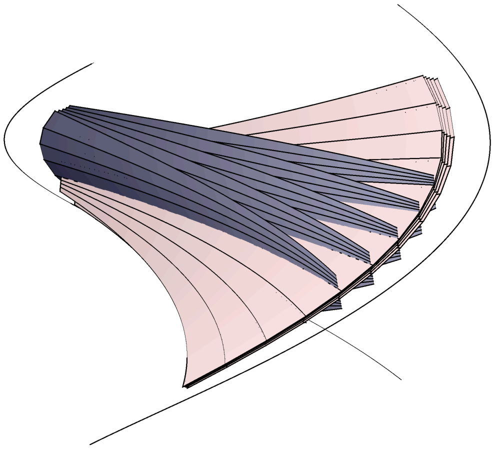
$$b_{\xi,x}(y) \coloneqq \lim_{t\to+\infty} d(\gamma_{\xi,x}(t), y) - t$$

illustrated by Figure 2.1.3. Busemann functions are Lipschitz continuous by the triangle inequality.[1] Furthermore, this description is altogether independent of having constant curvature.

Remark 2.1.19. Horocycles are *lines* because the point on the boundary of the Poincaré disk is not included. In fact, the dynamically natural objects are their normal vector fields (in $\mathrm{PSL}(2,\mathbb{R})$ or $S\mathbb{D}$ because they define the pairwise asymptotic geodesics—positively or negatively asymptotic according to whether one considers

[1] Thus, this pointwise limit is uniform on compact sets by Dini's Theorem: if a monotone sequence of continuous functions on a compact space converges pointwise to a continuous function, then the convergence is uniform.

the normal vector field pointing into or out of the horocycle. With this point of view, one can then moreover consolidate *all* unit vectors pointing to a common boundary point into a plane in PSL$(2,\mathbb{R})$, and likewise with vectors pointing away from a boundary point. Each of these 2 sets of planes is parametrized by the boundary circle, and Figure 2.1.4 shows them in a natural presentation in PSL$(2,\mathbb{R})$.



Animations at http://www.tsuboiweb.matrix.jp/showroom/public_html/animations/gif/T3image/T3image8.html, http://www.tsuboiweb.matrix.jp/showroom/public_html/animations/gif/geodflow/geodflowconft.html, and http://www.tsuboiweb.matrix.jp/showroom/public_html/animations/gif/geodflow/geodflowconftes.html

FIGURE 2.1.4. Horocycle foliations in PSL$(2,\mathbb{R})$ (after Tsuboi)

2. Dynamics of the natural flows

We now explore some of the dynamics for the geodesic flow and horocycle flow. We begin with the geodesic flow.

a. Dynamics of the geodesic flow. To further study the dynamics of the geodesic flow on $\mathbb{H}$ one can parameterize the set $S\mathbb{H}$ of unit vectors on $\mathbb{H}$ by $t, u, v \in \mathbb{R}$ as follows: Given a fixed reference vector $q \in S\mathbb{H}$ and $p \in S\mathbb{H}$ that does not point vertically downwards let H_p be the horocycle with p as inward (or upward) normal vector, γ the geodesic connecting the centers of H_q and H_p (that is, the points of tangency on the real axis), v the oriented hyperbolic length of the arc of H_p between $\gamma \cap H_p$ and the footpoint $\pi(p)$ of p, t the oriented arc length of the segment of γ between H_q and H_p, and u the oriented length of the arc of H_q between $\gamma \cap H_q$ and $\pi(q)$. It is easy to see that *locally* $\phi\colon (t,u,v) \mapsto p$ is a diffeomorphism between $\mathbb{R}^3$

and $S\mathbb{H}$. Note, however, that this does not parameterize any vertically downward vectors. A second chart starting from $-q$ would cover these.

If $W^s(p)$ denotes the collection of inward (or upward) unit normal vectors to H_p (the *stable manifold* of p), then the orbit of any $p' \in W^s(p)$ is positively asymptotic to that of p by Proposition 2.1.9, since the orbits of upward vertical unit vectors to $\mathbb{R} + i$ have pairwise asymptotic orbits. Note that $W^s(p)$ is a level set of (t, u). Indeed $W^s(q) = \phi(\{0\} \times \{0\} \times \mathbb{R})$. The set $W^{s0}(q) := \phi(\mathbb{R} \times \{0\} \times \mathbb{R})$ the *center-stable manifold* of q. Likewise the points of $W^u(p) := -W^s(-p)$ (the *unstable manifold* of p, outward unit vectors to H_{-p}) have negatively asymptotic orbits and $W^u(q) = \phi(\{0\} \times \mathbb{R} \times \{0\})$. The set $W^{u0}(q) := \phi(\mathbb{R} \times \mathbb{R} \times \{0\})$ is called the *center-unstable manifold* of of q. For vertically downward vectors we have to use the corresponding chart starting with $-q$ to make these definitions.

Proposition 2.1.10, particularly the estimate (2.1.2) of the decay of the distance between vertical tangent vectors combined with the fact that $t \mapsto x + ie^t$ is a geodesic, Definition 2.1.13, Lemma 2.1.14, and the preceding notions are summarized as follows:

Proposition 2.2.1. *The stable manifold of $v \in S\mathbb{H}$ with respect to the geodesic flow g^t is the unit normal vector field containing v to the horocycle centered at $\gamma_v(\infty)$. The unstable manifold of $v \in S\mathbb{H}$ is the unit normal vector field containing v to the horocycle centered at $\gamma_v(-\infty)$. In particular all stable and unstable manifolds are one-dimensional and the contraction and expansion rates are e^{-1} and e.*

Remark 2.2.2 (Hyperbolicity from the structure equations)**.** One can see the hyperbolic behavior of these geodesic flows directly from their algebraic structure. The unit tangent bundle has a framing by a vertical vector field V, a horizontal vector field H, and the vector field X that generates the geodesic flow. With respect to the representation in terms of $\mathrm{PSL}(2,\mathbb{R})$ they are given by elements of the Lie algebra (that is, traceless matrices) as follows. V is the initial derivative of the rotational flow (in unit tangent circles) given by the matrices $\begin{pmatrix} \cos {}^t\!/_2 & \sin {}^t\!/_2 \\ \sin {}^t\!/_2 & \cos {}^t\!/_2 \end{pmatrix}$,[2] so $V \sim \begin{pmatrix} 0 & -{}^1\!/_2 \\ {}^1\!/_2 & 0 \end{pmatrix}$, while $X \sim \begin{pmatrix} {}^1\!/_2 & 0 \\ 0 & -{}^1\!/_2 \end{pmatrix}$ is the initial derivative of $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, so taking

$$H := [V, X] \sim \begin{pmatrix} 0 & -{}^1\!/_2 \\ {}^1\!/_2 & 0 \end{pmatrix}\begin{pmatrix} {}^1\!/_2 & 0 \\ 0 & -{}^1\!/_2 \end{pmatrix} - \begin{pmatrix} {}^1\!/_2 & 0 \\ 0 & -{}^1\!/_2 \end{pmatrix}\begin{pmatrix} 0 & -{}^1\!/_2 \\ {}^1\!/_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^1\!/_2 \\ {}^1\!/_2 & 0 \end{pmatrix}$$

[2] We encountered this in Example 1.1.28 as an extreme magnetic flow; see also Remark 2.2.10 below.

gives the *canonical framing* X, H, V and the *structure equations*

$$[V, X] = H, \quad [H, X] = V, \quad [H, V] = X. \tag{2.2.1}$$

One can check (2.2.1) by using that in the PSL$(2,\mathbb{R})$-representation of $S\tilde{\Sigma}$, the vector fields of the canonical framing are given by

$$X \sim \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad H \sim \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad V \sim \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix},$$

A dynamically natural variant of this framing is the one by X and

$$H_\pm \coloneqq H \pm V, \quad \text{that is,} \quad H_+ \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_- \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with the corresponding bracket relations

$$[X, H_\pm] = [X, H] \pm [X, V] = \mp H_\pm \quad \text{and} \quad [H_+, H_-] = \underbrace{[H + V, H - V]}_{=-2[H,V]} = -2X. \tag{2.2.2}$$

A vector field $f H_\pm$ invariant under the geodesic flow satisfies

$$0 = [X, f H_\pm] = (\dot{f} \mp f) H_\pm,$$

which means that $\dot{f} = \pm f$, so $f = e^{\pm t}$. Thus, the differential of the geodesic flow expands and contracts, respectively, the directions $H_\pm$; this is the defining feature of hyperbolicity (Definition 5.1.1).

b. Dynamics of the horocycle flow.

Example 2.2.3 (The horocycle flow)**.** The vector fields X and $H_\pm$ each generate a flow we can describe explicitly (Remark 2.1.15):

$$X \rightsquigarrow \exp\left(\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} t\right) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \sim g^t,$$

$$H_+ \rightsquigarrow \exp\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} t\right) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \sim h_+^t,$$

$$H_- \rightsquigarrow \exp\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t\right) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \sim h_-^t.$$

The first is (again) the geodesic flow, and, as previewed in Remark 2.1.15, the latter flows are called the horocycle flows. Note that the matrix action is on the *right* (Remark 2.1.15).

Early on (Proposition 2.1.10, Proposition 2.2.1 and Remark 2.1.15) we noted that h_-^s parameterizes the stable manifold of Id, and we can now see by a matrix computation which gives the commutation relation

$$\text{(2.2.3)}\qquad \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} 1 & se^{-t} \\ 0 & 1 \end{pmatrix} \text{ or } g^t h_-^s g^{-t} = h_-^{se^{-t}},$$

which reflects the fact that geodesic flow contracts (or expands) orbits of the horocycle flow with the constant coefficient e^t. This plays important roles in the study of asymptotic behavior of both flows.[3] In addition to implying hyperbolicity of the geodesic flow, this also shows that the horocycle flow is *parabolic*, that is, characterized by polynomial behavior:

$$0 = [H_+, aX + bH_+ + cH_-] = (\dot a - 2c)X + (\dot b + a)H_+ + \dot c H_-$$

implies that as a function of t, c is constant ($\dot c = 0$), a is linear ($\dot a = 2c$), and b is quadratic ($\dot b = -a$).

We note that the bracket relation $[H_+, H_-] = -2X$ is also important because of its finitary counterpart, the *quadrilateral formula*:

$$\text{(2.2.4)}\qquad \begin{pmatrix} 1 & 0 \\ -\epsilon s & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s-1}{\epsilon} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\frac{1}{s}-1}{\epsilon} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix} \text{ or } h_-^{\frac{\frac{1}{s}-1}{\epsilon}} h_+^{\epsilon} h_-^{\frac{s-1}{\epsilon}} h_+^{-\epsilon s} = g^{2\ln s},$$

which is crucial below for mixing properties of the geodesic flow. Geometrically, this is a *quadrilateral* argument: For $s = 1 + \epsilon^2$ this says that a quadrilateral with $h_\pm$-sides about ϵ causes a $2\epsilon^2$ displacement along a geodesic: we approximately have $h_-^{-\epsilon} h_+^{\epsilon} h_-^{\epsilon} h_+^{-\epsilon} \approx g^{2\epsilon^2}$. However, for s further away from 1, this gives useful information by way of highly elongated quadrilaterals (Proposition 3.3.19).

In a different vein we note that h_+^s and h_-^s generate $\mathrm{PSL}(2,\mathbb{R})$ by (2.2.4).

Example 2.2.4 (The horizontal flow)**.** The structure equations (2.2.1) are invariant under the exchange of $X \leftrightarrow H$, $V \leftrightarrow -V$, so $\xi^\pm := V \pm X = \mp[H, \xi^\pm]$, which implies hyperbolicity of the flow generated by H. It is given by

$$e^{\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} t} = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix},$$

which sends I to the semicircle with end-points $\coth t/2$ and $\tanh t/2$ (these are reciprocals),[4] and the image of i ranges over the upper half of the unit circle as t ranges over $\mathbb{R}$—multiply

$$\frac{i\cosh\frac{t}{2} + \sinh\frac{t}{2}}{i\sinh\frac{t}{2} + \cosh\frac{t}{2}} = \frac{i\cosh\frac{t}{2} + \sinh\frac{t}{2}}{i\sinh\frac{t}{2} + \cosh\frac{t}{2}} \, \frac{\cosh\frac{t}{2} - i\sinh\frac{t}{2}}{\cosh\frac{t}{2} - i\sinh\frac{t}{2}} = \frac{2\sqrt{1+\sinh^2\frac{t}{2}}\,\sinh\frac{t}{2} + i}{1 + 2\sinh^2\frac{t}{2}}$$

[3] And well beyond this algebraic context Section 9.6.

[4] Thus, the dynamics induced on the boundary circle $\mathbb{R} \cup \{\infty\}$ is north-south dynamics (Example 1.3.7, Figure 2.3.2).

by its complex conjugate to check the absolute value; surjectivity is clear from $\frac{2\sqrt{1+\sinh^2\frac{t}{2}}\sinh\frac{t}{2}+i}{1+2\sinh^2\frac{t}{2}}\xrightarrow[t\to\pm\infty]{}\pm 1$.

Geometrically, this flow can be descibed as follows: Rotate a unit vector by $-\pi/2$, follow the corresponding geodesic for time t, then rotate the tangent vector back by $\pi/2$. Put differently, transport perpendicular vectors along geodesics. Presented this way, one sees that there is nothing special about "perpendicular" (Example 2.2.7).

c. Reeb flow. Let us describe a structure possessed by all geodesic flows that is in the present case particularly easy to discern because of its algebraic nature.

Definition 2.2.5 (Contact form, Reeb flow)**.** An (antisymmetric) n-form A on a smooth manifold M is a smooth map $A\colon TM^n \to \mathbb{R}$ that is linear in each fiber argument and antisymmetric. The exterior derivative dA of a 1-form A is the 2-form defined by

$$dA(X,Y) := \mathscr{L}_X A(Y) - \mathscr{L}_Y A(X) - A([X,Y]),$$

where $\mathscr{L}$ is the Lie derivative and $[X,Y]$ is the Lie bracket. The *contraction operator* inserts a vector field in the first slot of a differential form:

$$\iota_X A := A(X,\dots) =: A \lrcorner X.$$

A 1-form A on a 3-manifold M is called a *contact form* if

$$(A\wedge dA)(X,Y,Z) := A(X)dA(Y,Z) - A(Y)dA(X,Z) + A(Y)dA(Z,X)$$

defines a volume form, that is, is nonzero at every point. (See also Subsection 2.6d.) The associated plane field $\xi := \ker A$ is said to be a (cooriented) *contact structure*.

The *Reeb vector field* R_A associated to a contact form A is defined by $\iota_{R_A} A = A(R_A) = 1$ and $\iota_{R_A} dA = dA(R_A,\cdot) = 0$.[5] Its flow is called the *Reeb flow* (and it preserves the contact form because $\mathscr{L}_{R_A} A = \iota_{R_A} dA = 0$). Equivalently, R_A is the unique (up to a constant scalar factor) vector field that generates a flow which preserves the contact form. A *contact flow* is a flow that preserves a contact form.

In the case at hand, we can define a 1-form A uniquely by

$$A(X) = 1 \text{ and } A(V) = 0 = A(H). \tag{2.2.5}$$

[5] This is unique: the second condition determines R_A up to a scalar since dA is nondegenerate, and the first then fixes the scalar. Note that the Reeb vector field is associated to a contact form α rather than the contact structure: if $\alpha' = f\alpha$ with $f \in \mathscr{C}^\infty(M, \mathbb{R}\setminus\{0\})$, then $d\alpha' = df\wedge\alpha + f\,d\alpha$, and the condition $\iota_{R_{\alpha'}} d\alpha' = 0$ implies that R_α and $R_{\alpha'}$ are not collinear unless f is constant. A Reeb field on a contact manifold (M,ξ) is the Reeb field of a(ny) contact form α with $\xi = \ker\alpha$. These are exactly the nowhere-vanishing vector fields transverse to ξ whose flow preserves ξ.

For $Z \in \{V, H\}$ we then have $dA(X,Z) = \underbrace{\mathscr{L}_X \overbrace{A(Z)}^{\equiv 0}}_{=0} - \underbrace{\mathscr{L}_Z \overbrace{A(X)}^{\equiv 1}}_{=0} - \underbrace{A(\overbrace{[X,Z]}^{\in -\{V,H\}})}_{=0} = 0$, so $\iota_X dA := dA(X,\cdot) \equiv 0$, and $X = R_A$, while $A \wedge dA(X,V,H) = A(X)dA(V,H) = 1$ since

$$dA(V,H) = \underbrace{\mathscr{L}_V \overbrace{A(H)}^{\equiv 0}}_{=0} - \underbrace{\mathscr{L}_H \overbrace{A(V)}^{\equiv 0}}_{=0} - \underbrace{A(\overbrace{[V,H]}^{=-X})}_{=-1} = 1.$$

Thus, $A \wedge dA$ is indeed a volume; in fact a volume particularly well adapted to this canonical framing. We have shown that the geodesic flow on $\mathbb{H}$ is a contact flow with A the canonical contact form.

The aforementioned symmetry of the structure equations implies that the horizontal flow from Example 2.2.4 is also a contact flow: Set $B(H) = 1$, $B(V) = 0 = B(X)$ and either repeat the preceding calculations or observe that by symmetry they work out to the same effect, notably $B \wedge dB(H,X,V) = 1$.[6] (Compare Exercise 2.3 below.)

Example 2.2.6 (The vertical or fiber flow)**.** In the PSL(2,$\mathbb{R}$)-representation of $S\tilde{\Sigma}$, the 3 flows corresponding to the vector fields of the canonical framing are given by

$$X \rightsquigarrow \exp\left(\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} t\right) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

$$H \rightsquigarrow \exp\left(\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} t\right) = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix},$$

$$V \rightsquigarrow \exp\left(\begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} t\right) = \begin{pmatrix} \cos t/2 & -\sin t/2 \\ \sin t/2 & \cos t/2 \end{pmatrix}$$

We will explore the dynamics of X and H below (Remark 2.2.11). The last of these 3 flows is called the *vertical* or *fiber* flow. Unlike the other 2 it is not hyperbolic because of a sign change in the symmetry; this is reflected in the trigonometric functions in its representation: it is a periodic flow because it consists of "spinning" around the tangent fibers.[7] The arguments $t/2$ give the right period, by the way: $\left(\begin{smallmatrix} \cos 2\pi/2 & -\sin 2\pi/2 \\ \sin 2\pi/2 & \cos 2\pi/2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ mod ± 1. The horizontal has an easy geometric interpretation:

$$\begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} \cos -\pi/4 & -\sin -\pi/4 \\ \sin -\pi/4 & \cos -\pi/4 \end{pmatrix} = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix},$$

so "rotate $\pi/2$, follow the geodesic, rotate back $\pi/2$" or, in other words, translate a normal (rather than tangent) vector along a geodesic.

[6] In fact, $B = dA(V,\cdot)$.

[7] The counterpart to the earlier calculations is that $\xi^\pm := X \pm iH$ satisfies $[V,\xi^\pm] = \mp i\xi^\pm$, so a vector field $f\xi^\pm$ is V-invariant if and only if $0 = [V, f\xi^\pm] = \dot{f}\xi^\pm \mp if\xi^\pm$, that is, $\dot{f} = \pm if$ or $f = e^{\pm it}$.

As before, one can check that $C \coloneqq dA(H,\cdot)$ is a contact form invariant under the fiber flow generated by V but its Reeb field is $-V$, and $C \wedge dC(V,H,X) = -1$, so this volume has the opposite orientation from the ones defined by A and B and is hence not isotopic to either of them. The geodesic flow and the fiber flow are isotopic, however, via the magnetic flow construction from Example 1.1.28 (Remark 2.2.10), and Example 2.2.3 shows that the horocycle flow is the "midpoint" of a isotopy between the horizontal flow and the fiber flow (Remark 2.2.11) and the largest perturbation whose orbits reach the boundary.

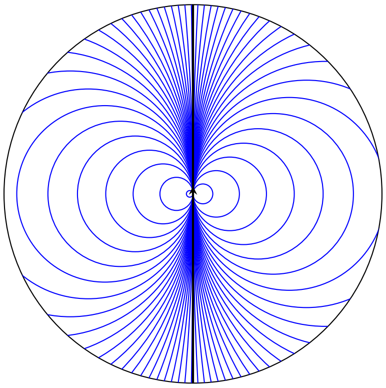


FIGURE 2.2.1. Magnetic perturbations of a geodesic diameter

Example 2.2.7 (A family of hyperbolic Reeb flows)**.** After (2.2.2) and in Example 2.2.4 we noted that X and H generate hyperbolic flows, and that they are the Reeb flows for A and B, respectively. More generally,

$$E \coloneqq E_\theta \coloneqq \cos\theta A + \sin\theta B$$

is a contact form with $R_E = P \coloneqq \cos\theta X + \sin\theta H$, and $\zeta^\pm \coloneqq \cos\theta H - \sin\theta X \pm V$ gives

$$[P,\zeta^\pm] = \mp\zeta^\pm \qquad \text{so} \qquad 0 = [P, f\zeta^\pm] = (\dot f \mp f)\zeta^\pm \Rightarrow f = \text{const.}\, e^{\pm t}.$$

as before. Thus R_E generates a family of hyperbolic flows parameterized by S^1. As suggested by our previous take on the horizontal flow, these consist of parallel translation along geodesics of vectors making an angle θ with the geodesic.

Remark 2.2.8. As a byproduct of Example 2.2.7 we note that the horocycle flows as well are each part of an S^1-family of natural flows generated by $\zeta^\pm$.

Remark 2.2.9 (Hopf coordinates)**.** We complement the infinitesimal version of hyperbolicity in Remark 2.2.2 by a description in *Hopf coordinates.* These are given by a homeomorphism

$$S\mathbb{D} \to \big(S^1 \times S^1 \smallsetminus \text{diagonal}\big) \times \mathbb{R}, \quad v \mapsto (v^-, v^+, \beta_{v^+}(0, \pi(v))),$$

where $v^\pm := \lim_{t\to\pm\infty} \gamma_v(t) \in \partial\mathbb{D} \sim S^1$, $\pi\colon S\mathbb{D} \to \mathbb{D}$ is the footpoint projection, and β is the *Busemann cocycle*

$$\mathbb{D} \times \mathbb{D} \times \partial\mathbb{D}, \quad x, y, \xi \mapsto \beta_\xi(x, y) := \lim_{t\to\infty} d(x, \xi_x(t)) - d(y, \xi_x(t)).$$

Here ξ_x is the geodesic with $\xi_x(0) = x$ and $\xi_x(t) \xrightarrow[t\to+\infty]{} \xi$. In these coordinates, the geodesic flow is given by $g^t(v^-, v^+, \tau) = (v^-, v^+, \tau + t)$, and it contracts the stable manifold (see Proposition 2.2.1 and Remark 2.1.7)

$$W^s(v^-, v^+, \tau) := \{(\xi, v^+, \tau) \mid \xi \in \partial\mathbb{D} \smallsetminus \{v^+\}\}.$$

Remark 2.2.10 (Magnetic flows)**.** As in Example 1.1.28 (and as promised in Example 2.2.6) one can interpolate between the geodesic and horocycle flows as follows. The geodesic flow takes a tangent vector along a curve with zero geodesic curvature with unit speed, and the horocycle flow does the same thing along curves with geodesic curvature 1. The interpolation is to choose a different (constant) geodesic curvature to obtain other defining curves for a flow. This does, in fact have a physical motivation in that while the geodesic flow models the motion of a force-free particle, constant nonzero geodesic curvature corresponds to the effect of a magnetic field perpendicular to the plane or disk on a charged particle, which is to produce constant acceleration perpendicular to the direction of motion and translates to constant geodesic curvature. These flows are called *magnetic flows.* (Note that depending on the orientation of the magnetic field one could drift right or left, which corresponds to making a consistent choice of horocycle, of which there are 2 through each tangent vector.) For a given initial tangent vector, increasing the intensity of the magnetic field (that is, geodesic curvature) produces ever smaller circles, which for curvature ± 1 just barely touch the boundary. These are the horocycles, and when one transports the normal rather than tangent unit vector, this is the horocycle flow (Example 2.2.3). A magnetic field that produces geodesic curvature greater than 1 produces motion along circles too small to reach the boundary, and therefore all orbits are periodic (as in Example 1.1.28), whereas none of the orbits are periodic for flows along curves of geodesic curvature between 1 and -1. To get periodic orbits for the geodesic flow requires passing to a compact factor. (We briefly return to magnetic flows on page 229.)

Let us briefly remark that in the spirit of Remark 1.6.17 we have here a continuous family (Definition 1.6.18) of flows and may be interested in how the dynamics changes as we make these deformations. Until the magnetic field, that is, the deviation from geodesic motion, becomes rather large, the flows look rather similar to each other. We will indeed see that for weak magnetic fields, any 2 of these flows are pairwise topologically orbit-equivalent (Theorem 5.4.5).

Remark 2.2.11. In summary, a linear combination of X, H, V can be written as a linear combination of E_θ (from Example 2.2.7) and V and generates a flow whose orbits project to curves on $\mathbb{H}$ with constant geodesic curvature given in terms of the coefficient of V with vectors transported along them that make an angle θ with the tangent vector of the curve in the surface (the size of which is determined by the coefficient of E)—unless the linear combination is just V, in which the curve in the surface is a point (zero speed since E has coefficient 0). The special cases we noted earlier are generated by X, V, $H_\pm$ and H. We also noted that X, H, V generate contact flows but that the contact form for V is not isotopic to either of the other 2.

3. Compact factors

Stable and unstable manifolds made their appearance earlier in Example 1.5.23 and Example 1.5.24, where they appeared as families of lines with irrational slope invariant under a hyperbolic automorphism of the two-torus and its suspension. The existence of families of stable and unstable manifolds is a hallmark of global hyperbolic behavior; flows on compact manifolds with such behavior are called *Anosov flows* (Definition 5.1.1).

Therefore it is natural to utilize our understanding of hyperbolic behavior of the geodesics in the hyperbolic disk in order to construct first examples of Anosov flows. All we need is to construct a compact factor of the hyperbolic disk and project the geodesic flow to that factor. We accomplish this by factoring out by a discrete group of isometries.

Draw a regular (hyperbolic) octagon $\mathcal{Q}$ in the Poincaré disk in $\mathbb{C}$ with vertices $v_k = de^{-k\pi i/4}$, $k = 0, \dots, 7$, joined by arcs of circles perpendicular to the unit circle (see Figure 2.3.1). Here $d \in (0,1)$ and as $d \to 1$, the sum of the internal angles converges to 0, and it goes to 6π, the value for the Euclidean octagon, as $d \to 0$. This becomes clear by keeping d fixed and increasing the size of the Poincaré disk indefinitely so that the arcs of circles approach line segments. Thus, we can fix d such that the internal angles add up to 2π. The identification space obtained from labeling and identifying the edges as in Figure 2.3.1 is a surface Σ of genus 2. Since the internal angles of $\mathcal{Q}$ add up to 2π, the identification map is smooth at the vertices (which are all identified to one point) and we can therefore push the metric on $\mathcal{Q}$ down to Σ. We obtain a compact manifold which is locally isometric to $\mathbb{H}$. Topologically this manifold is homeomorphic to the double torus or the sphere with two handles: the half with labels a, b is a torus with a hole, and so is the other

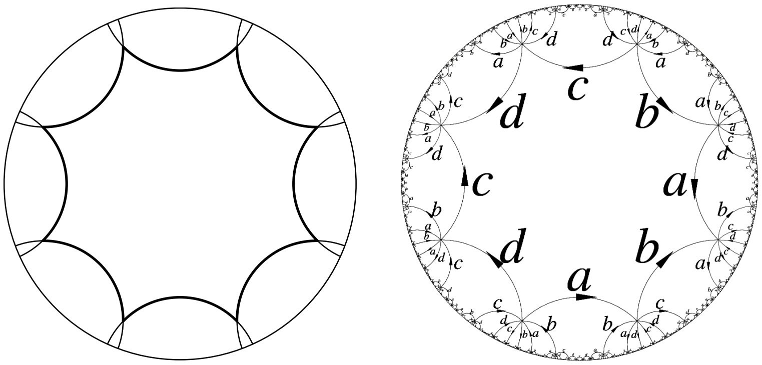


FIGURE 2.3.1. A hyperbolic octagon, identifications, and tiling by translates [Reproduced from [**181**] ©1995 Cambridge University Press. All rights reserved, and `http://topologygeometry.blogspot.ch/2010/06/notes-from-062310.html` with permission]

half; the hole is the common diameter along which these tori are glued together. One can also show that Σ is the space obtained by identifying orbits of the group Γ generated by the isometries mapping an edge to the one with which it is identified. In other words, the fundamental group of Σ can be identified with a discrete group Γ of hyperbolic linear fractional transformations.

Replacing eight arcs here by $4g \geq 8$ arcs gives a metric locally isometric to that of $\mathbb{H}$ on the orientable surface of genus g (sphere with g handles).

If a linear fractional transformation γ preserves a geodesic then such geodesic is unique and it is called the axis of γ. In fact, every $\gamma \in \Gamma$ has an axis. The projections of these geodesics to $M := \Gamma\backslash\mathbb{D}$ are precisely the closed geodesics of M. These are, of course, the projections of the closed orbits of the geodesic flow from the tangent bundle to M. The dynamics of any such γ (under iteration) restricts to a translation of the axis, and the action on the boundary circle is of north-south type much like in Example 1.3.7 as shown in Figure 2.3.2. The end-points of the axis are fixed points, one repelling (γ^-) and one attracting (γ^+). Indeed,

$$\lim_{n\to\pm\infty} \gamma^n(x) = \gamma^{\pm} \text{ for every } x \in \mathbb{D} \cup \partial\mathbb{D} \smallsetminus \{\gamma^{\mp}\}.$$

Associated to any C^2 Riemannian metric on a surface is the Gaussian curvature of the metric, an isometry-invariant real-valued function. Since the isometry group of $\mathbb{D}$ is transitive, the curvature of $\mathbb{D}$ is a constant k. Thus the induced metric on the compact factor Σ of genus 2 constructed from the octagonal fundamental domain

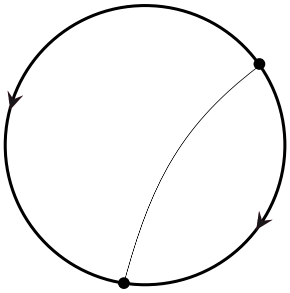


FIGURE 2.3.2. North-south dynamics on the boundary

has constant curvature k as well. The Gauss–Bonnet Theorem

$$k \cdot \operatorname{vol} M = 2\pi\chi$$

then shows that $k < 0$ because the Euler characteristic $\chi = 2 - 2g$ of Σ is negative. Conversely this then shows that any compact factor of $\mathbb{D}$ has negative Euler characteristic and hence genus at least 2. Thus the compact factors of $\mathbb{D}$ are homeomorphic to spheres with several handles attached. In fact, any compact orientable surface with a metric of constant negative curvature is isometric to a factor $\Gamma\backslash\mathbb{D}$ of $\mathbb{D}$ by a discrete group Γ of isometries of $\mathbb{D}$. To see how the picture developed for the octagonal fundamental domain looks in the general case, consider a discrete group of orientation-preserving isometries of the Poincaré disk $\mathbb{D}$ which produces a compact factor. One can choose a fundamental domain for Γ by considering the *Dirichlet domain*

$$D := D_p := \{x \in \mathbb{D} \mid d(x,p) \le d(x,\gamma p) \text{ for all } \gamma \in \Gamma\}$$

for any given point $p \in \mathbb{D}$. For any $\gamma \in \Gamma$ we evidently have $D_{\gamma p} = \gamma(D_p)$. The interiors of D_p and $D_{\gamma p}$ are disjoint when $\gamma \ne \mathrm{Id}$ and since Γ is discrete, there are only finitely many $\gamma \in \Gamma$ such that $D_p \cap D_{\gamma p} \ne \varnothing$. If $\gamma \in \Gamma$ is one of these elements, then $D_p \cap D_{\gamma p}$ consists of the points equidistant from p and γp, that is, is a geodesic segment. Thus D is a hyperbolic polygon, that is, bounded by finitely many geodesic arcs. Our assumption that $\Gamma\backslash\mathbb{D}$ is compact means that D is compact. By construction we also observe that the sets $D_{\gamma p}$ cover $\mathbb{D}$, so we have, in fact, tessellated $\mathbb{D}$ by the images of D under Γ.

Compact factors of the hyperbolic plane cannot be embedded isometrically in $\mathbb{R}^3$ because a compact embedded surface has positive curvature at the points of contact with a circumscribed sphere. An illustration of an isometrically embedded

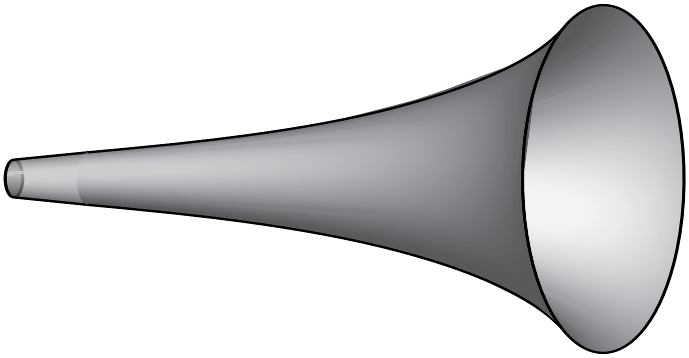


FIGURE 2.3.3. The pseudosphere [©Cambridge University Press, reprinted from [**181**] with permission]

surface of constant negative curvature is given by the pseudosphere in Figure 2.3.3.

4. The geodesic flow on compact hyperbolic surfaces

Unlike the geodesic flow on the round sphere and the flat torus considered in Subsection 1.1c, where the dynamics turned out to be rather simple, compact factors of the hyperbolic plane have geodesic flows of a complicated dynamical nature rather similar to hyperbolic symbolic flows. The full extent of this similarity will become clear as we develop the theory of hyperbolic dynamical systems, and indeed, these very geodesic flows were and still are among the primary motivations for studying hyperbolic dynamical systems. Therefore their study here is a precursor to the central object and Part 2 of this book. Thus we now establish for the geodesic flow on compact factors of the hyperbolic plane some of the properties that we tend to consider typical for complicated dynamical behavior, namely, density of closed orbits, and topological transitivity.

We first prove density of closed orbits:

Theorem 2.4.1 (Periodic orbits are dense)**.** *Let Γ be a discrete group of fixed-point-free isometries of $\mathbb{D}$ such that $M := \Gamma\backslash\mathbb{D}$ is compact. Then the periodic orbits of the geodesic flow on SM are dense in SM.*

PROOF. We use the model of the Poincaré disk $\mathbb{D}$. Let $v \in SM$, take a Dirichlet domain D for Γ, and let $w \in S\mathbb{D}$ be a lift of v with footpoint in D. Let c be the geodesic with $\dot{c}(0) = w$ in $\mathbb{D}$ and let x and y be the end-points of c on the boundary of the Poincaré disk. Our strategy is to find a hyperbolic element $\gamma \in \Gamma$ such that the end-points of its axis lie in given small δ-neighborhoods U and V, respectively, of the points $x =: c(-\infty)$ and $y =: c(\infty)$. Then among the tangent vectors to this axis one can find a vector that is close to w. The projection of the axis to M is the desired closed geodesic.

Minimality of the action of Γ on $\partial\mathbb{D}$ is the first step:

Lemma 2.4.2. *No proper closed subset of $\partial\mathbb{D}$ is invariant under the action of Γ.*

Proof. If $F \subset \partial\mathbb{D}$ is closed and Γ-invariant, then so is its convex hull E in $\mathbb{D}$, that is, the intersection of all hyperbolic half-spaces that contain F, and so also is the function $\delta(x) := d(x, E)$ on $\mathbb{D}$. Thus, δ is well-defined on the quotient and hence bounded—and identically zero (otherwise it is positive on a point of a geodesic orthogonal to the boundary of E and hence unbounded). Thus $F = \partial\mathbb{D}$. □

This implies that the set of end-points of axes in $\partial\mathbb{D}$ is dense, so we can find $\gamma, \eta \in \Gamma$ such that $\gamma^+ \in U$ and $\eta^- \in V$. If $\gamma = \eta$, we are done. Otherwise we may assume that $\gamma^\pm, \eta^\pm$ are 4 distinct points, and we will show that $\gamma^n\eta^n$ for large enough n is the desired isometry by using the north-south dynamics from Figure 2.3.2. If $W_\gamma \subset \partial\mathbb{D}$ is a neighborhood of γ^- and $W_\eta \subset \partial\mathbb{D}$ is a neighborhood of η^+ such that the closures of both of these and of U and V are pairwise disjoint, take $n \in \mathbb{N}$ such that $\eta^n(U) \subset W_\eta$ and $\gamma^n(\partial\mathbb{D} \smallsetminus W_\gamma) \subset U$, then $\gamma^n\eta^n(U) \subset U$, so $\gamma^n\eta^n$ has a (necessarily attracting) fixed point in U. Likewise $\eta^{-n}\gamma^{-n}$ (for possibly larger n) has an attracting fixed point in V. □

Remark 2.4.3. The interaction of γ and η is sometimes called "playing ping pong."

In the present context the multitude of closed orbits is "organized" rather neatly by the topology of the surface: Since these orbits are based on parameterized geodesics, they can be represented by those geodesics on the manifold itself, and it is a consequence of having negative curvature that there is at most one closed geodesic in each free homotopy class of loops, and there is indeed exactly one each by a curve-shortening argument in each such class. This means that likewise the periodic orbits in the unit tangent bundle are in pairwise different free homotopy classes except for the duplication of a geodesic with its reverse.

We emphasize that density of closed geodesics as orbits in the *phase space* is rather stronger than density of closed geodesics on the underlying surface—indeed, the latter is generic for any surface [**166**, Remark 1.6].

Theorem 2.4.4 (Transitivity)**.** *Let Γ be a discrete group of fixed-point-free isometries of $\mathbb{D}$ such that $M := \Gamma\backslash\mathbb{D}$ is compact. Then the geodesic flow on SM is topologically transitive (see Proposition 1.6.9).*

Remark 2.4.5. We emphasize that the dense orbit implied by topological transitivity is dense in the unit tangent bundle, which is stronger than the assertion that it traces a dense geodesic in the surface.

PROOF. By Theorem 2.4.1 and Proposition 1.6.9 it is sufficient to show that for any two periodic points $u, v \in SM$ (whose lifts to $\mathbb{D}$ we also denote by u and v) and neighborhoods U, V of u, v, respectively, there is $t \in \mathbb{R}$ such that $g^t(U) \cap V \neq \varnothing$. Take the geodesics c_u and c_v in $\mathbb{D}$ with $\dot c_u(0) = u$ and $\dot c_v(0) = v$. Replacing, if necessary, u by γu assume that $c_u(-\infty) \neq c_v(\infty)$, then denote by c the geodesic with end-points $c(-\infty) = c_u(-\infty)$ and $c(\infty) = c_v(\infty)$. By Proposition 2.2.1 we can find for each $t \in \mathbb{R}$ numbers $f(t), g(t) \in \mathbb{R}$ such that $d(\dot c_u(f(t)), c(t)) \xrightarrow[t\to-\infty]{\text{exponentially}} 0$ and $d(\dot c_v(g(t)), c(t)) \xrightarrow[t\to\infty]{\text{exponentially}} 0$. Since $\dot c_u$ and $\dot c_v$ project to closed orbits of the geodesic flow this shows that there exist t_1 and t_2 such that the projection of $\dot c(t_1)$ to SM is in U and the projection of $\dot c(t_2)$ to SM is in V. This then yields the claim. □

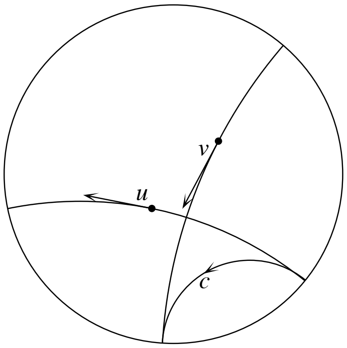


FIGURE 2.4.1. Transitivity of the geodesic flow

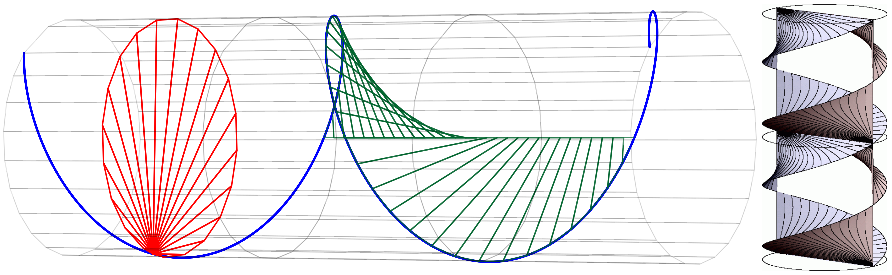
Remark 2.4.6 (Mixing)**.** As noted earlier, this geodesic flow is actually topologically mixing (Exercise 2.7, Remark 8.1.14, and Corollary 9.1.4) [**156**, Theorem 3.1].

Furthermore, Remark 2.1.12 implies:

Theorem 2.4.7 (Expansivity)**.** *The geodesic flow on* $\mathbb{H}$ *or* $\mathbb{D}$ *or any factor is* expansive *(Definition 1.7.2).*

Returning attention from compact factors to the universal cover, it is instructive to go further and consider the universal cover of the unit circle bundle $S\mathbb{D}$

(rather than the circle bundle of the universal cover $\mathbb{D}$). Unrolling the circle fibers shows that topologically this is $\mathbb{D} \times \mathbb{R}$, and Figure 2.4.2 shows a way to here visualize the sets of geodesics positively or negatively asymptotic to a given boundary point. The choice made here is that one can represent the set of geodesics positively asymptotic to a given boundary point as a $\mathbb{D}$-slice in the picture, shown here in red with those geodesics rendered as straight lines. In that case, the set of geodesics negatively asymptotic should be represented as in the green cross-section in the figure to show that the boundary point to which each geodesic is negatively asymptotic varies over the boundary circle in a way that corresponds to an interval's worth of red slices. An interesting consequence is that the red and green sets shown here do not intersect; each green slice meets a bounded interval of red slices, and, vice versa, each red slice meets a circle minus a point worth of green sections. This is in contrast with the global product structure of a suspension (Remark 1.5.25).



Animation at `http://www.tsuboiweb.matrix.jp/showroom/public_html/animations/gif/geodflow/geodflowconf.html`

FIGURE 2.4.2. The universal cover of $S\mathbb{D}$. Left (after Barthelmé): the (red) "flat" and (green) "spindle" fans do not intersect. Right (after Tsuboi): still picture from animation.

Remark 2.4.8. This geodesic flow is the original instance of an *Anosov flow* (Definition 5.1.1), which we study more carefully below. In that context, topological transitivity implies density of periodic orbits via a mechanism central to the study of their dynamics (shadowing, Section 5.3). Moreover, this geodesic flow is not only topologically transitive, but has strong ergodic properties (see for example, Theorem 8.1.13). These in turn imply that it is also topologically mixing (Definition 1.6.31).

In addition, we will further down describe surgeries that produce new (contact) flows from the 3 flows in Remark 2.2.2 and Example 2.2.4. Those turn out to have some profoundly different features from the ones we studied here.

5. Symmetric spaces

An important class of manifolds of negative curvature is obtained by an algebraic construction which generalizes the algebraic description of surfaces of constant negative curvature. This involves a substantial amount of differential geometry and Lie theory and is not required for other parts of this book.

The geometric property that enabled us to describe the geodesic flow on the sphere, the torus, and the hyperbolic plane was the presence of an isometry group that is transitive on unit tangent vectors. In general such spaces are called (globally) symmetric spaces. We begin with the traditional definition and then prove transitivity of the isometry group in the case of nonvanishing curvature.

Definition 2.5.1. A *Riemannian locally symmetric space* is a connected Riemannian manifold M such that for all $p \in M$ there is a neighborhood U on which $\exp_p \circ (-\mathrm{Id}) \circ \exp_p^{-1} \colon U \to M$ is an isometry. M is called a *globally symmetric space* if this local isometry extends to an isometry of M, that is, for every $p \in M$ there is an isometry σ_p of M with $\sigma_p(p) = p$ and $D\sigma_{p|_p} = -\mathrm{Id}$. σ_p is called the *(global) symmetry* at p. The space is said to have *rank one* if there is no isometrically embedded totally geodesic Euclidean plane.

Remark 2.5.2.
(1) An alternative definition is that the curvature tensor is parallel, that is, $\nabla R = 0$ and the space is simply connected.
(2) Since the end-points of any geodesic segment are exchanged by the symmetry at the midpoint and any two points are connected by a broken geodesic, the isometry group of a globally symmetric space or compact locally symmetric space is clearly transitive on points.
(3) Having rank 1 implies that all sectional curvatures are nonzero.
(4) S^n, $\mathbb{R}^n$, $\mathbb{H} = \mathbb{RH}^2$ are globally symmetric spaces; $\mathbb{T}^n$ is locally symmetric.[8]
(5) A complete simply connected locally symmetric space is globally symmetric.
(6) Thus the universal cover of a complete locally symmetric space is a globally symmetric space.

Proposition 2.5.3. *If M is a rank-one symmetric space then the isometry group is transitive on SM.*

Proof. Since transitivity on points is known we only need to show that the isometry group is transitive on any particular unit sphere S_pM. To that end it suffices to show that for every 2-plane $\Pi \subset T_pM$ the isometry group is transitive on $\Pi \cap S_pM$, which in turn follows once we see that there exists an $\epsilon > 0$ such that for $v \in \Pi \cap S_pM$

[8] $\mathbb{T}^n$ is a (globally) symmetric if one adopts the existence of a global symmetry as the definition, but not if simple connectedness is required.

there exists a family of isometries such that the images of v under their differentials cover an arc of length ϵ in $\Pi \cap S_pM$.

To that end consider a disk $D = \exp_p B(0,\delta)$ and a triangle in D with p as one vertex and interior angles α, β, γ. Consider the isometry I obtained by composing the three symmetries about the midpoints of the edges (in cyclic order). Since isometries preserve angles one easily sees by a picture that the angle between v and $DI(v)$ is $\alpha+\beta+\gamma$. Since Π has nonzero curvature the sum $\alpha+\beta+\gamma$ converges to π as the diameter of the triangle tends to 0 but it never equals π. Thus we obtain an arc of images whose size is independent of v. □

All symmetric spaces arise from an algebraic construction which generalizes the construction in the preceding subsections. To give an indication of how this comes about we begin with a direct generalization of a *geometric* construction of the hyperbolic space.

The Poincaré disk with the group of Möbius transformations can be obtained as follows. Consider the upper sheet $\mathscr{H}$ of the hyperboloid in $\mathbb{R}^3$ given by $Q(x) \coloneqq x_1^2 + x_2^2 - x_3^2 = -1$, $x_3 > 0$. The group $SO(2,1)$ of real 3×3 matrices preserving the indefinite quadratic form Q acts on the hyperboloid, and the index-two subgroup preserving $x_3 > 0$ therefore acts on $\mathscr{H}$. Since the action is linear in $\mathbb{R}^3$ it sends planes through 0 (that is, planes given by $ax_1 + bx_2 - cx_3 = 0$) to planes through 0 and hence the family $\mathscr{C}$ of curves given by the intersection of such planes with $\mathscr{H}$ is preserved.

If we change variables to $\eta_1 = x_1/x_3$, $\eta_2 = x_2/x_3$, $\eta_3 = 1/x_3$ the hyperboloid becomes the hemisphere $\eta_1^2 + \eta_2^2 + \eta_3^2 = 1$, $\eta_3 > 0$ and a plane $ax_1 + bx_2 - cx_3 = 0$ is mapped to the plane $a\eta_1 + b\eta_2 = c$ perpendicular to the $\eta_1\eta_2$-plane. Thus curves from $\mathscr{C}$ are mapped to circles orthogonal to the equator $\eta_3 = 0$. Finally apply the stereographic projection centered at $(0,0,-1)$ from the upper hemisphere to the disk $\eta_1^2 + \eta_2^2 < 1$. It is known to be conformal, so the curves from $\mathscr{C}$ now are (lines and) circles perpendicular to the boundary, that is, the geodesics of the Poincaré disk. One can show that the transformations that arise from $SO(2,1)$ in this process are exactly the Möbius transformations. In fact, the hyperboloid is an isometric embedding of the Poincaré disk into Minkowski space $(\mathbb{R}^3, q)$ with the pseudometric q induced by the form Q.

This geometric construction generalizes to give n-dimensional real hyperbolic spaces $\mathbb{RH}^n$. Consider the upper sheet of the hyperboloid $\mathscr{H}$ in $\mathbb{R}^{n+1}$ given by $Q(x) \coloneqq x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1$, $x_{n+1} > 0$. Again let $\mathscr{C}$ be the family of curves that lie on planes through 0, that is, on planes given by n simultaneous equations of the form $a_1x_1 + \cdots + a_nx_n - a_{n+1}x_{n+1} = 0$. The group $SO(n,1)$ of matrices preserving Q acts on $\mathscr{H}$. Change variables to $\eta_1 = x_1/x_{n+1}, \dots, \eta_n = x_n/x_{n+1}$, $\eta_{n+1} = 1/x_{n+1}$ and then apply the stereographic projection centered at $(0,\dots,0,-1)$ to map the resulting hemisphere to the open unit ball in $\mathbb{R}^n$. As before curves in $\mathscr{C}$ map to (lines and) circles perpendicular to the boundary of the unit ball $\mathbb{RH}^n$.

These spaces $\mathbb{RH}^n$ have (sectional) curvature -1 as well. This is clear for all tangent planes Π at $(0,\dots,0,1)$ since in the three-dimensional subspace of $\mathbb{R}^{n+1}$ containing Π the entire picture looks like the description of $\mathbb{RH}^2$.

For purposes of generalization it is more convenient to view $\mathbb{RH}^n$ as a subset of the n-dimensional real projective space $\mathbb{RP}^n$ of lines through 0 in $\mathbb{R}^{n+1}$ by identifying a point p on the upper hyperboloid with the line through 0 containing p. The Riemannian metric is, of course, not the induced one, but the tangent vectors to $\mathbb{RH}^n$ are tangent vectors of $\mathbb{RP}^n$. Hyperbolic distances are given as follows. Two points in this space correspond to two lines in $\mathbb{R}^{n+1}$. The plane defined by these intersects the cone $Q = 0$ in two more lines. The hyperbolic distance is given by the logarithm of the cross ratio of the four points in projective space determined by these four lines.

This latter description works over the complex field $\mathbb{C}$ as well. We obtain the n-dimensional complex hyperbolic space $\mathbb{CH}^n$ as a subset of complex projective space $\mathbb{CP}^n$, that is, the space of complex lines through the origin of $\mathbb{C}^{n+1}$, with a distance similarly defined by cross ratios. There is an important new phenomenon, however. Any tangent space can be viewed simultaneously as an n-dimensional complex linear space or a $2n$-dimensional real linear space. Thus a real vector v in a tangent space can be multiplied by $i = \sqrt{-1}$ to give a unique direction that is perpendicular to v with respect to the real structure but collinear to v with respect to the complex structure. One can check that this real 2-dimensional subspace has (sectional) curvature -4 and that multiplication by i is an isometry of the unit tangent bundle. Thus one has a natural real 1-dimensional subbundle on the unit tangent bundle $S\mathbb{CH}^n$ given by these directions. There is naturally a complementary subbundle defined by the vectors that are complex orthogonal to v and iv. Inside this subbundle all sectional curvatures are -1. This subbundle turns out to be nonintegrable.

For the geodesic flow these subbundles correspond to subbundles of vectors with expansion rates e^{2t} and e^t, respectively, and corresponding contraction rates.

For the quaternions $\mathbb{Q}$ one obtains hyperbolic spaces $\mathbb{QH}^n$ with a similar structure, but here one obtains a (real) 3-dimensional subbundle corresponding to planes of curvature -4. Even for the octonians $\mathbb{O}$ (Cayley numbers) one obtains a hyperbolic plane $\mathbb{OH}^2$, here with a corresponding 7-dimensional subbundle. The last construction, however, does not extend to higher dimension due to nonassociativity of the Cayley numbers. These examples in fact exhaust the list of Riemannian globally symmetric spaces of negative curvature. All of these spaces admit compact Riemannian factors obtained by the left action of a uniform lattice in the isometry group, so the geodesic flows on such factors provide examples of Anosov geodesic flows.

We now give, also without proof, an indication of the general *algebraic* description of globally symmetric spaces.

Proposition 2.5.4. *If M is a globally symmetric space then the identity component G of the isometry group of M acts transitively on M and the isotropy group K of any point is compact.*

Definition 2.5.5. A globally symmetric space M is said to be of *noncompact type* if G is semisimple with no compact factors and K is a maximal compact subgroup of G.

Remark 2.5.6. Unlike in the case of $\mathbb{RH}^2$ the group G for other globally symmetric spaces of rank 1 is substantially larger than the unit tangent bundle of the manifold we are considering.

Conversely for every connected semisimple Lie group with no compact factors and a maximal compact subgroup K (which is unique up to conjugacy by an inner automorphism of G) there is a natural globally symmetric structure on $M := G/K$, namely, every left-invariant Riemannian metric on G that is right-invariant under K then makes M a Riemannian manifold and the quotient of M under the left action of a lattice Γ in G is a compact Riemannian factor of M. This is the analog of the torus and compact factors of the hyperbolic plane $\mathbb{RH}^2$.

In this model geodesics through Id are given by one-parameter subgroups of G/K.

The general algebraic description of the geodesic flow on rank-one Riemannian symmetric spaces of noncompact type is as follows. Let G be a simple noncompact Lie group of real rank one. Such groups are $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $Sp(n,1)$, and F_4. Let K be a maximal compact subgroup of G. Then G/K is a globally symmetric space and its unit tangent bundle is of the form G/T, where T is a compact subgroup of K (namely, the isotropy subgroup of a tangent vector). The symmetric spaces are, correspondingly, n-dimensional real, complex, and quaternionic hyperbolic spaces and the 2-dimensional hyperbolic Cayley plane. The geodesic flow corresponds to the right action of a one-parameter subgroup that commutes with T. (Note that in the two-dimensional case $T = \{\mathrm{Id}\}$.)

The algebraic description of the geodesic flow on the hyperbolic plane and its factors allows another remarkable generalization to higher dimension. The idea is simply to replace $\mathrm{SL}(2,\mathbb{R})$ with $\mathrm{SL}(n,\mathbb{R})$ for larger n. For $n = 2$, as we have seen, the geodesic flow appears as the action of the positive diagonal subgroup; the natural generalization would be the following:

Definition 2.5.7. The right action of the positive diagonal subgroup

$$D_n^+ = \Big\{ \underbrace{\begin{pmatrix} \exp t_1 & & \\ & \ddots & \\ & & \exp t_n \end{pmatrix}}_{=:\mathrm{diag}(\exp t_1,\dots,\exp t_n)} \;\Big|\; (t_1,\dots,t_n) \in \mathbb{R}^n,\ \sum_{k=1}^{n} t_k = 0 \Big\} \cong \mathbb{R}^{n-1}$$

on $\mathrm{SL}(n,\mathbb{R})$ and its compact factors is called the *Weyl-chamber flow*.

This is our first example of an action of a higher-rank abelian group. Since it appears as a generalization of the geodesic flow on a surface $\Gamma/\mathbb{H}^2$ (an Anosov flow, Definition 5.1.1) it is natural to expect that its elements exhibit hyperbolic behavior. Note first that since all diagonal matrices commute, every element of the Weyl-chamber flow acts by isometries with respect to any left-invariant metric on $\mathrm{SL}(n,\mathbb{R})$ and hence to its projection to $\Gamma/\,\mathrm{SL}(n,\mathbb{R})$. Thus we should expect hyperbolicity *transverse* to the orbit direction.

Consider the one-parameter unipotent subgroup $u_{ij}(t) = \mathrm{Id} + t n_{ij}$ where the ij-entry of n_{ij} is 1 and all others are 0, and let W_{ij} be the foliation into left cosets of this subgroup. An explicit calculation gives

$$\underbrace{\mathrm{diag}(e^{t_1},\dots,e^{t_n})}_{=:G_{t_1,\dots,t_n}}\, u_{ij}(s)\mathrm{diag}(e^{-t_1},\dots,e^{-t_n}) = u_{ij}(se^{t_i-t_j}),$$

that is,

$$G_{t_1,\dots,t_n} H_s^{ij} G_{-t_1,\dots,-t_n} = H^{ij}_{s\exp(t_i-t_j)}\,, \tag{2.5.1}$$

if we denote by H_t^{ij} the right multiplication by $u_{ij}(t)$. The dynamical interpretation of (2.5.1) is that the element $G_{t_1,\dots,t_n}$ of the Weyl-chamber flow preserves the foliation W_{ij} and expands or contracts its leaves with coefficient $e^{t_i-t_j}$ depending on whether $i > j$ or $i < j$—much as the geodesic flow expands the horocycles from one family and contracts those from the other.[9]

Thus for all elements $G_{t_1,\dots,t_n}$ with $t_1 > t_2 > \cdots > t_n$ the stable and unstable foliations are the same; the set of such elements (or their indices) is called the *positive Weyl chamber*. When the sign of $t_i - t_j$ changes, the pair of foliations W_{ij} and W_{ji} switches roles. This is an essential higher-rank effect. A *Weyl chamber* is the subset of $D_n^+ \cong \mathbb{R}^{n-1}$ where all differences $t_i - t_j$ are nonzero and have the same sign.

The Weyl-chamber flow is a generalization of the geodesic flow on the symmetric space of the group $\mathrm{SL}(2,\mathbb{R})$, which can be described as the homogeneous space $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ provided with a Riemannian metric that is projected from a

[9] If the t_i for $i = 1,\dots n$ are pairwise different, then $G_{t_1,\dots,t_n}$ is partially hyperbolic in the sense of Definition 12.5.1 with the neutral direction being that of the orbits of Weyl-chamber flow.

left-invariant metric on $\mathrm{SL}(2,\mathbb{R})$ that is also $\mathrm{SO}(2)$ right-invariant. Hence one may naturally ask about connections between the Weyl-chamber flow and the geodesic flow on the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ provided with a Riemannian metric that is projected from a left-invariant metric on $\mathrm{SL}(n,\mathbb{R})$ that is also $\mathrm{SO}(n)$ right-invariant. It turns out that the latter geodesic flow has $n-1$ commuting (Definition 2.6.18) first integrals (Definition 1.1.23) and the restriction of the geodesic flow to any regular value of those integrals is smoothly conjugate to a Weyl-chamber flow with properly chosen generators.

Moreover, the Weyl-chamber flow provides the main instance of an algebraic $\mathbb{R}^k$-action, whose smooth rigidity is established in Theorem 10.1.24.

6. Hamiltonian systems

Both in this algebraic instance and when they appear in greater generality, it is useful to have a framework for decribing geodesic flows as mechanical or Hamiltonian systems rather than solely focusing on their geometric origin. This section gives a brief axiomatic introduction to the modern approach to Hamiltonian dynamics.

a. Symplectic geometry. The natural geometry for describing Hamiltonian systems is an antisymmetric counterpart to a Riemannian metric. Accordingly, we begin with nondegenerate antisymmetric 2-forms on linear spaces.

Definition 2.6.1. Let E be a linear space. A 2-tensor $\alpha\colon E\times E\to\mathbb{R}$ is said to be *nondegenerate* if $\alpha^\flat\colon v\mapsto\alpha(v,\cdot)$ is an isomorphism from E to its dual space E^*. It is said to be *antisymmetric* or *skew-symmetric* if $\alpha(v,w)=-\alpha(w,v)$. A nondegenerate antisymmetric 2-form is called a *symplectic* form. A linear space with a symplectic form is called a *symplectic* vector space. If (E,α), (F,β) are symplectic vector spaces then a linear map $T\colon E\to F$ is said to be *symplectic* if $T^*\beta=\alpha$.

Remark 2.6.2. If a scalar product $\langle\cdot,\cdot\rangle$ on E is fixed we can write $\alpha(\cdot,\cdot)=\langle\cdot,A\cdot\rangle$, so we identify the tensor with its matrix representation with respect to a given basis.

Proposition 2.6.3. *Let E be a linear space. If α is a symplectic form on E then $\dim E=2n$ for some $n\in\mathbb{N}$ and there is a basis $e_1,\dots,e_{2n}$ of E such that $\alpha(e_i,e_{n+i})=1$ if $i=1,\dots,n$ and $\alpha(e_i,e_j)=0$ if $|i-j|\neq n$. Hence, if one fixes a scalar product with respect to which $e_1,\dots,e_{2n}$ is an orthonormal basis, then $A=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$ with respect to this basis, where I is the $n\times n$ identity matrix.*

Proof. Since α is nondegenerate there exist e_1, e_{n+1} such that $\alpha(e_1,e_{n+1})\neq 0$, and we may take $\alpha(e_1,e_{n+1})=1$. By antisymmetry $\alpha(e_1,e_1)=\alpha(e_{n+1},e_{n+1})=0$ and $\alpha(e_{n+1},e_1)=-1$, so the matrix of $\alpha\restriction_{E_1}$, where $E_1=\operatorname{span}\{e_1,e_{n+1}\}$, with respect to

(e_1, e_{n+1}) is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The claim follows by induction on dimension: If $v \in E$, then

$$v - \alpha(v, e_{n+1})e_1 + \alpha(v, e_1)e_{n+1} \in E_2 := \{v \in E \mid \alpha(v, w) = 0 \text{ for all } w \in E_1\},$$

so $E_1 \oplus E_2 = E$ since $E_1 \cap E_2 = \{0\}$. □

Definition 2.6.4. A subspace V of a symplectic linear space (E, α) is said to be *isotropic* if $\alpha_{\restriction V} = 0$ and *Lagrangian* if furthermore $\dim V = \dim E/2$.

Remark 2.6.5. Thus the "adapted" basis of Proposition 2.6.3 gives a decomposition of E as a direct sum of two Lagrangian subspaces. Note that by nondegeneracy of α an isotropic subspace has dimension at most $\dim E/2$, so Lagrangian subspaces are maximal isotropic subspaces.

An interesting description of nondegeneracy is the following:

Proposition 2.6.6. *An antisymmetric 2-form α on a linear space E is nondegenerate if and only if $\dim E = 2n$ and the nth exterior power α^n of α is not zero.*

Proof. "⇐": If α is degenerate then $\alpha^\flat$ has nontrivial kernel, that is, there is a vector v such that $\alpha(v, w) = 0$ for all w, hence $\alpha^n(v, v_2, \dots, v_n) = 0$ for all $v_2, \dots, v_n$. "⇒": If α is nondegenerate write $\alpha = \sum_{i=1}^n dx_i \wedge dx_{i+n}$ by Proposition 2.6.3. Then

$$\alpha^n = \sum_{i_1,\dots,i_n=1}^{n} dx_{i_1} \wedge dx_{i_1+n} \wedge \cdots \wedge dx_{i_n} \wedge dx_{i_n+n} = n!(-1)^{[n/2]} dx_1 \wedge \cdots \wedge dx_{2n} \neq 0. \quad \square$$

An immediate observation from the preceding results is

Proposition 2.6.7. *If $T \colon (E, \alpha) \to (F, \beta)$ is a symplectic map, then T preserves volume and orientation. In particular T is invertible with Jacobian* 1.

Thus the set of symplectic maps $(E, \alpha) \to (E, \alpha)$ is a group which we call the *symplectic group* of (E, α). Assume a scalar product $\langle \cdot, \cdot \rangle$ is fixed and α is in standard form $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Here are some further simple properties of symplectic maps.

Proposition 2.6.8. *Suppose (E, α) is a symplectic vector space and $T \colon (E, \alpha) \to (E, \alpha)$ a symplectic map. If λ is an eigenvalue of T, then so are $\bar\lambda$, $1/\lambda$, $1/\bar\lambda$. If T has the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to a basis for which $\alpha(v, w) = \langle v, Jw \rangle$, then A^tC and B^tD are symmetric, and $A^tD - C^tB = I$.*

Proof. If T preserves α, and $\alpha(v, w) = \langle v, Jw \rangle$ then symplecticity means $T^tJT = J$. By calculation this implies that A^tC and B^tD are symmetric and $A^tD - C^tB = I$. If λ

is an eigenvalue then so is $\bar{\lambda}$ since the characteristic polynomial $P(\lambda) = \det(T - \lambda I)$ has real coefficients. Furthermore $JTJ^{-1} = (T^{-1})^t$, so

$$\begin{aligned} P(\lambda) = \det(T-\lambda I) = \det(J(T-\lambda I)J^{-1}) &= \det(T^{-1})^t(I - \lambda T^t) \\ &= \det((I-\lambda T)T^{-1}) = \det(\lambda(\lambda^{-1}I - T)) = \lambda^{2n}P(\lambda^{-1}); \end{aligned}$$

hence, since 0 is not an eigenvalue, $P(\lambda) = 0$ if and only if $P(1/\lambda) = 0$. □

Exercise 2.13 gives an appropriate version of a converse to this result.

Now we discuss symplectic forms on manifolds.

Definition 2.6.9. Let M be a smooth manifold. A differential 2-form ω is a smooth map from M to the space $\bigwedge^2 T^*M$ of antisymmetric 2-tensor fields, that is, it assigns to each $x \in M$ an antisymmetric 2-tensor on T_xM. A differential 2-form ω is said to be *nondegenerate* if it is nondegenerate at every point. A nondegenerate 2-form ω with $d\omega = 0$ is called a *symplectic form.* A pair (M, ω) of a smooth manifold and a symplectic form is called a *symplectic manifold.* If (M, ω) is a symplectic manifold then a subbundle of the tangent bundle TM of M is said to be *isotropic* if at every point $p \in M$ it defines an isotropic subspace of T_pM, and *Lagrangian* if at every point $p \in M$ it defines a Lagrangian subspace of T_pM. A smooth submanifold of a symplectic manifold is said to be *isotropic* if its tangent bundle is an isotropic subbundle, and *Lagrangian* if its tangent bundle is a *Lagrangian* subbundle of TM. A diffeomorphism $f\colon (M,\omega) \to (N,\eta)$ between symplectic manifolds such that $f^*\eta = \omega$ is said to be a *symplectic diffeomorphism* or *symplectomorphism.* If $(M,\omega) = (N,\eta)$ it is also called a *canonical transformation.*

Symplectic C^r diffeomorphisms of a symplectic manifold (M,ω) form a closed subset of $\mathrm{Diff}^r(M)$ with the C^r topology. Proposition 2.6.6 immediately yields:

Proposition 2.6.10. *If (M,ω) is a symplectic manifold then M is even-dimensional and ω^n is a volume form. In particular M is orientable.*

By Proposition 2.6.3 we can find coordinates around any given point x such that in T_xM the induced coordinates bring the symplectic form into standard form. This can be done by introducing any coordinate system and making an appropriate linear coordinate change in that system. Unlike in the case of a Riemannian metric, it is, however, possible to find a local chart such that the symplectic form is in standard form at *every* point of the chart. The proof uses an argument due to Moser sometimes called the "homotopy trick."

Theorem 2.6.11 (Darboux Theorem)**.** *Let (M,ω) be a symplectic manifold and $x \in M$. There is a neighborhood U of x and coordinates $\varphi\colon U \to \mathbb{R}^{2n}$ such that at every point $y \in U$ ω is in standard form with respect to the basis $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2n}}\right\}$.*

These coordinates are referred to as *Darboux* or *symplectic coordinates*.

PROOF (Moser homotopy trick). As noted, we may assume that we already have coordinates such that $M = \mathbb{R}^{2n}$ and ω is in standard form at $x = 0$ with respect to the basis $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2n}}\right\}$. Thus we need to find coordinates in which ω is constant. Denote by α the form with matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Let $\omega' \coloneqq \alpha - \omega$ and $\omega_t = \omega + t\omega'$ for $t \in [0,1]$. Then there is a ball around 0 on which all ω_t are nondegenerate (since there is such a ball for every t and it depends continuously on t). Thus $\omega' = d\theta$ for some one-form θ by the Poincaré Lemma, and without loss of generality $\theta(0) = 0$.

Since ω_t is nondegenerate, there is a unique (smooth) vector field X_t such that $\omega_t \lrcorner X_t \coloneqq X_t \lrcorner \omega_t \coloneqq \iota_{X_t}\omega_t \coloneqq \omega_t(X_t, \cdot) = -\theta$. Since $X_t(0) = 0$ one can integrate X_t on a small ball around 0 to get a 1-parameter family of diffeomorphisms $\{\varphi^t\}_{t\in[0,1]}$ such that $\dot{\varphi}^t = X_t$ and $\varphi^0 = \text{Id}$. Then

$$\frac{d}{dt}\varphi^{t*}\omega_t = \varphi^{t*}(\mathscr{L}_{X_t}\omega_t) + \varphi^{t*}\frac{d}{dt}\omega_t = \varphi^{t*}d(\omega_t \lrcorner X_t) + \varphi^{t*}\omega' = \varphi^{t*}(-d\theta + \omega') = 0,$$

so $\varphi^{1*}\omega_1 = \varphi^{0*}\omega_0 = \omega$, that is, φ^1 is the desired coordinate change. □

Remark 2.6.12. As mentioned before, this result is in contrast to the situation for Riemannian metrics, for which such charts exist only for flat metrics. An explanation is that the condition $d\omega = 0$ here may be considered an analog of flatness of a Riemannian metric.

b. Cotangent bundles. We now describe an important class of spaces with a canonical symplectic structure, the cotangent bundle of a smooth manifold. Not only does a cotangent bundle have a canonical symplectic structure, but furthermore the natural coordinates induced by coordinates on the underlying manifold are symplectic coordinates.

Let M be a smooth manifold and consider local coordinates $\{q_1, \dots, q_n\}$. On the cotangent bundle these induce coordinates $\{q_1, \dots, q_n, p_1, \dots, p_n\}$. Define a 1-form θ by setting

$$\theta = -\sum_{i=1}^{n} p_i \, dq_i. \tag{2.6.1}$$

Then its exterior derivative is $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$, that is, a symplectic form in Darboux coordinates. The next lemma shows that this definition does not depend on the choice of coordinates on the manifold. Alternatively it shows that diffeomorphisms of the manifold induce symplectomorphisms of the cotangent bundle:

Lemma 2.6.13. *Let M be a smooth manifold and $f\colon M \to M$ a diffeomorphism. Then the coderivative D^*f acting on the cotangent bundle T^*M preserves θ and ω.*

Proof. If we write $(Q_1,\dots,Q_n) = f(q_1,\dots,q_n)$ then

$$D^*f(q_1,\dots,q_n,p_1,\dots,p_n) = (Q_1,\dots,Q_n,P_1,\dots,P_n),$$

where $p_j = \sum_{i=1}^n \dfrac{\partial Q_i}{\partial q_j} P_i$. Thus

$$\sum_{i=1}^n P_i dQ_i = \sum_{i,j=1}^n P_i \frac{\partial Q_i}{\partial q_j} dq_j = \sum_{j=1}^n p_j dq_j$$

and θ, hence ω, is preserved. □

c. Hamiltonian vector fields and flows. Now we can begin to study the Hamiltonian equations.

Definition 2.6.14. Let (M,ω) be a symplectic manifold, and $H\colon M \to \mathbb{R}$ a smooth function. Then the vector field $X_H = dH^{\#}$ defined by $\omega \lrcorner X_H = dH$ is called the *Hamiltonian vector field* associated with H or the *symplectic gradient* of H. The flow Φ with $\dot{\varphi}^t = X_H$ is called the *Hamiltonian flow* of H.

A Hamiltonian vector field is C^r if and only if the Hamiltonian function is C^{r+1}. Thus one can identify the space of C^r Hamiltonian flows, which is a closed linear subspace of $\Gamma^r(TM)$, with the space $C^{r+1}(M,\mathbb{R})$ modulo additive constants.

This is indeed a formulation of usual Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}:$$

to see that $\dot{\varphi}^t = X_H$ gives these, we check that $X_H := \big(\dfrac{\partial H}{\partial p_i}, -\dfrac{\partial H}{\partial q_i}\big)$ satisfies $\omega \lrcorner X_H = dH$ in Darboux (symplectic) coordinates. But

$$\omega \lrcorner X_H = \sum_{i=1}^n (dq_i \wedge dp_i) \lrcorner X_H = \sum_{i=1}^n \underbrace{(dq_i \lrcorner X_H)}_{=\partial H/\partial p_i} \wedge dp_i - \sum_{i=1}^n dq_i \wedge \underbrace{(dp_i \lrcorner X_H)}_{=-\partial H/\partial q_i} = dH.$$

Remark 2.6.15. This can be restated as saying that a Hamiltonian flow is a skew-gradient flow in that X_H is orthogonal to the gradient of H (and has the same norm). This makes the next 2 propositions natural.

It is easy to see that Hamiltonian flows are instances of one-parameter groups of canonical transformations (Definition 2.6.9):

Proposition 2.6.16 (Liouville Theorem)**.** *Hamiltonian flows are symplectic and hence volume-preserving.*

PROOF. Let (M,ω) be a symplectic manifold, $H\colon M\to\mathbb{R}$ a smooth function, $\omega\lrcorner X_H = dH$, and $\dot\varphi^t = X_H$. Then

$$\frac{d}{dt}\varphi^{t*}\omega = \varphi^{t*}(\mathscr{L}_{X_H}\omega) = \varphi^{t*}(\overbrace{d(\underbrace{\omega\lrcorner X_H}_{=dH})}^{=ddH} + (\underbrace{d\omega}_{=0}\lrcorner X_H)) = \varphi^{t*}(ddH) = 0. \qquad \square$$

The converse is not true, that is, there are symplectic flows that are not Hamiltonian: A linear flow on the two-dimensional torus with the standard volume 2-form $dx\wedge dy$ preserves area and is hence symplectic. Its velocity vector field is constant $\neq 0$. Thus if it were a Hamiltonian flow the Hamiltonian would have to have constant nonzero gradient. On the other hand the Hamiltonian attains its maximum and thus has a critical point, a contradiction. Note, incidentally, that the lift of the linear flow to $\mathbb{R}^2$ is indeed Hamiltonian. If a vector field X generates a symplectic flow the calculation above shows that the 1-form $\omega\lrcorner X$ is closed. Thus the obstruction to being Hamiltonian is, in fact, of a topological nature (namely, vanishing of the cohomology class of the closed 1-form $\omega\lrcorner X$). See Exercise 2.14 for a discussion of a related phenomenon.

We note that the Hamiltonian is itself a constant of motion.

Proposition 2.6.17. *Let (M,ω) be a symplectic manifold, $H\colon M\to\mathbb{R}$ a smooth function, $\omega\lrcorner X_H = dH$, and $\dot\varphi^t = X_H$. Then $H(\varphi^t(x))$ does not depend on t.*

PROOF. $\displaystyle\frac{d}{dt}H_{|_{\varphi^t(x)}} = dH(\varphi^t(x))\dot\varphi^t(x) = \omega(X_H(\varphi^t(x)), \underbrace{\dot\varphi^t(x)}_{=X_H(\varphi^t(x))}) = 0.$ □

The Poisson bracket predates the symplectic approach to Hamiltonian mechanics and was traditionally used in coordinate calculations, but also illuminates the Lie algebraic structure underlying the geometry.

Definition 2.6.18. Let (M,ω) be a symplectic manifold and $f,g\colon M\to\mathbb{R}$ smooth functions. Then the *Poisson bracket* of f and g is defined by

$$\{f,g\} \coloneqq \omega(X_f,X_g) = df(X_g),$$

where $X_f = df^\#$ and $X_g = dg^\#$ (cf. Definition 2.6.14), that is, $\omega\lrcorner X_f = df$ and $\omega\lrcorner X_g = dg$. f and g are said to *commute* or be *in involution* if their Poisson bracket vanishes.

Proposition 2.6.19. *In symplectic coordinates $\{q_1,\dots,q_n,p_1,\dots,p_n\}$ we have*

$$\{f,g\} = \sum_{i=1}^{n}\Big(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\Big). \tag{2.6.2}$$

The Poisson bracket is antisymmetric and $\{\cdot,f\} = \mathscr{L}_{X_f}$. f is an integral of the Hamiltonian flow of H if and only if $\{f,H\}=0$.

PROOF. (2.6.2) follows by definition using $X_g = (\partial g/\partial p_i, -\partial g/\partial q_i)$. Antisymmetry follows from antisymmetry of ω. $\{\cdot, f\} = \mathscr{L}_{X_f}$ since

$$\mathscr{L}_{X_f} g = dg \lrcorner X_f = (\omega \lrcorner X_g) \lrcorner X_f = \omega(X_g, X_f) = \{g, f\}.$$

If φ^t is the Hamiltonian flow for H then $(d/dt) f \circ \varphi^t = \varphi^{t*} \mathscr{L}_{X_H} f = \varphi^{t*}\{f, H\}$ vanishes if and only if $\{f, H\}$ does. □

Remark 2.6.20. In particular we have reproved invariance of H since $\{H, H\} = 0$.

This gives a well-known result about Hamiltonian systems with symmetries:

Theorem 2.6.21 (Noether)**.** *Let (M,ω) be a symplectic manifold, $H\colon M \to \mathbb{R}$ smooth, $\omega \lrcorner X_H = dH$, and $\dot\varphi^t = X_H$. If H is invariant under the Hamiltonian flow for f, then f is a constant of motion of φ^t.*

PROOF. The hypothesis is that H is an integral for the flow of f, that is, $\{f, H\} = 0$, so conversely f is an integral for the flow of H. □

Remark 2.6.22. An interesting instance may arise when the phase space of the system is a cotangent bundle and the Hamiltonian is invariant under the action on the cotangent bundle of a one-parameter family of diffeomorphisms of the configuration space. Since such symmetries tend to be easy to detect, this result gives an easy way to find integrals of this sort.

Example 2.6.23. Consider the central-force or *Kepler problem* of two bodies moving freely, but subject to mutual gravitational attraction. In coordinates centered at the center of mass of the system the position of one body is $x \in \mathbb{R}^3 \smallsetminus \{0\}$ and its velocity is $v \in \mathbb{R}^3$. The potential energy of the gravitational field is given by $V(x) = -1/\|x\|$, so Newton's equation $F = ma$ becomes

$$\ddot x = \nabla \frac{1}{\|x\|} = -\frac{x}{\|x\|^3}$$

or

$$\dot x = v,$$
$$\dot v = -\frac{x}{\|x\|^3}.$$

The Hamiltonian $H(x, v) = \langle v, v\rangle/2 - 1/\|x\|$ (total energy) is invariant under rotations around the origin. In particular it is invariant under rotations in the xy-plane, which are generated by the Hamiltonian $q_1 p_2 - q_2 p_1$, if we choose to label the coordinates (q_1, q_2). Thus $q_1 p_2 - q_2 p_1$ is a first integral. It happens to be the z-component of angular momentum. The other two components are invariant by invariance under rotations in the other planes.

Definition 2.6.24. Let M be a smooth manifold. If X, Y are vector fields on M then the *Lie bracket* $[X,Y]$ is the unique vector field with $\mathscr{L}_{[X,Y]} = \mathscr{L}_Y\mathscr{L}_X - \mathscr{L}_X\mathscr{L}_Y$.

Remark 2.6.25. The Lie bracket measures to which extent the flows of two vector fields fail to commute. Indeed the Lie bracket of two vector fields vanishes identically if and only if the corresponding flows commute.

From the point of view of classical mechanics the most important (or at least the most traditional) symplectic manifolds are $\mathbb{R}^{2n}$ with the standard symplectic structure and the cotangent bundle of a differentiable manifold M (the *configuration space* of a mechanical system) with the symplectic form ω described in Subsection 2.6b, notably with the invariant 1-form (2.6.1). In both cases the symplectic manifold (phase space) itself is not compact, although in the second case the configuration space M may be compact; this is true in many important classical problems such as the motion of a rigid body. Of course $\mathbb{R}^{2n}$ can also be viewed as $T^*\mathbb{R}^n$, so the first case is a particular instance of the second.

In this book we primarily consider dynamical systems with compact phase space, and to apply our concepts and methods to a Hamiltonian system with Hamiltonian H one considers the restriction of the dynamics to the hypersurfaces $H = c$, which are compact in many situations, for example, for a geodesic flow on a compact Riemannian manifold, where those hypersurfaces are sphere bundles over the configuration space. Sometimes one can make a further reduction using the first integrals other than energy. If c is not a critical value of the Hamiltonian and the hypersurface $H_c := \{x \mid H(x) = c\}$ is compact then the Hamiltonian system preserves a nondegenerate $(2n-1)$-form ω_c[10].

d. Contact forms. There is an important situation when the invariant $(2n-1)$-forms can be described in a particularly natural way. In the case of both $\mathbb{R}^{2n}$ and T^*M the form ω is not only closed, but also *exact*. The 1-form θ defined by $\sum_{i=1}^n p_i dq_i$—globally in the first case, locally in the second—obviously satisfies $d\theta = \omega$. The calculation in the proof of Lemma 2.6.13 shows that θ is defined on T^*M independently of the choice of local coordinates. Of course in general a Hamiltonian system on T^*M does not preserve θ or any other 1-form whose exterior derivative is equal to ω. Let us see what conditions the invariance of θ

[10] This can be described as follows. One can locally decompose the $2n$-dimensional measure generated by ω into $(2n-1)$-dimensional measures on $H_{c+\delta}$ for all sufficiently small $|\delta|$ and consider the conditional measures, each of which is defined up to a multiplicative constant. Thus in this case due to Proposition 2.6.16 one can apply the Poincaré Recurrence Theorem 3.2.1, the Birkhoff Ergodic Theorem 3.2.16, and other facts from ergodic theory to the restriction of the Hamiltonian system to H_c.

imposes on the Hamiltonian:

$$\mathscr{L}_{X_H}\theta = d\theta \lrcorner X_H + d(\theta \lrcorner X_H) = dH + d(\theta \lrcorner X_H) = 0 \text{ if } \underbrace{\theta \lrcorner X_H}_{=-\sum p_i \frac{\partial H}{\partial p_i}} = -H.$$

Since the choice of Hamiltonian for a given vector field X_H is unique up to an additive constant, we have proved:

Proposition 2.6.26. *The Hamiltonian vector field X_H on T^*M preserves the 1-form θ if and only if the Hamiltonian can be chosen as positively homogeneous in p of degree one, that is, $H(q,\lambda p) = \lambda H(q,p)$ for $\lambda > 0$.*

The restriction of the form θ to the surface $H = c$ for a noncritical value of c of H is an example of a 1-form such that $\theta \wedge (d\theta)^{n-1}$ is nondegenerate. This motivates the following definition (see also Definition 2.2.5).

Definition 2.6.27. An alternating multilinear n-form ω on a smooth manifold is a map on n-tuples of vector fields X_i such that

$$\omega(X_{\sigma(1)},\dots,X_{\sigma(n)}) = \operatorname{sign}\sigma\,\omega(X_1,\dots,X_n),$$

where $\operatorname{sign}\sigma$ is the sign of the permutation σ, and ω is $C(\mathbb{R})$-linear in each entry. The *exterior product* or *wedge product* of a j-form α and a k-form β is defined by

$$\alpha \wedge \beta(X_1,\dots,X_{j+k}) \coloneqq \sum_{\substack{\sigma(1)<\cdots<\sigma(j)\\ \sigma(j+1)<\cdots<\sigma(j+k)}} \operatorname{sign}\sigma\,\alpha(X_{\sigma(1)},\dots,X_{\sigma(j)}) \cdot \beta(X_{\sigma(1)},\dots,X_{\sigma(k)}).$$

A 1-form θ on a $(2n-1)$-dimensional orientable manifold M is called a *contact form* if the $(2n-1)$-form $\theta \wedge (d\theta)^{n-1}$ (the power being with respect to the wedge product) is nondegenerate. Accordingly a pair (M,θ) of a smooth manifold with a contact form is said to be a *contact manifold*. A *contact flow* is a flow on M that preserves the contact form on M. The *Reeb flow* or *characteristic flow* of a contact form θ is the flow generated by the *Reeb vector field* R_θ defined by $\theta \lrcorner R_\theta = 1$ and $d\theta \lrcorner R_\theta = 0$. A diffeomorphism preserving the contact form is called a *contact diffeomorphism*.

Unlike a symplectic manifold, which admits a variety of Hamiltonian vector fields, a contact manifold (M,θ) comes with the canonical vector Reeb vector field R_θ, which is unique because the kernel of $d\theta$ is one-dimensional and disjoint from that of θ by the nondegeneracy assumption. Note that $\mathscr{L}_{R_\theta}\theta \equiv 0$ since $\theta \lrcorner R_\theta =$ const., so the *Reeb flow* of the contact form preserves θ and hence all structures defined in terms of θ, in particular the volume. Thus the Reeb flow provides a canonical example of a volume-preserving flow.

Suppose now that X is a vector field generating a flow preserving the contact form θ. Then it preserves $\ker d\theta$ as well and hence commutes with the Reeb flow of θ. Thus contact flows always arise as flows commuting with the Reeb flow of a contact form.[11] Combined with hyperbolicity this usually means that X is the Reeb field up to constant scaling (page 501).

Furthermore if the contact manifold is a level set for a homogeneous Hamiltonian then R_θ is exactly the Hamiltonian vector field. In fact, contact forms always arise in this manner from generalized homogeneous Hamiltonians (see Proposition 2.6.29). Conversely, we note:

Proposition 2.6.28. *Geodesic flows are Hamiltonian flows with a homogeneous Hamiltonian. Hamiltonian flows for homogeneous Hamiltonians, in particular geodesic flows, are Reeb flows, hence contact flows.*

Proposition 2.6.29. *Suppose (M,θ) is a contact manifold. Then M can be embedded into a symplectic manifold (N,ω) in such a way that the restriction of the ambient symplectic form to M is $d\theta$.*

Proof. If $N = M \times \mathbb{R}$ and $\omega_{x,t} = d(e^t\theta_x)$ then $\omega^n = e^{nt}(n\,dt \wedge \theta \wedge (d\theta)^{n-1})$ is a volume, so (N,ω) is a symplectic manifold and ω restricted to $M \times \{0\}$ is $d\theta$. □

Locally a contact form, similarly to a symplectic form, can be brought into a standard form. The following result is a simple consequence of the Darboux Theorem 2.6.11 for symplectic forms.

Theorem 2.6.30 (Darboux Theorem for contact forms)**.** *Let*

$$\theta_0 = x_1\,dy_1 + \cdots + x_n\,dy_n + dz$$

be the canonical contact form on $\mathbb{R}^{2n+1}$ and (M,θ) a contact $(2n+1)$-manifold. Then for $x \in M$ there exists a neighborhood U of x with coordinates in which $\theta = \theta_0$.

Proof. For $x \in M$ pick a neighborhood V_0 of 0 in $\ker\theta_x$ and let $V = V_0 \times (-\epsilon,\epsilon)$, $U' = \exp V$, $U'_t = \exp(V_0 \times \{t\}) \subset M$. $d\theta$ restricted to U'_t is a symplectic form so by the Darboux Theorem 2.6.11 each $y \in U'_t$ has a neighborhood $U_t \subset U'_t$ on which there are Darboux coordinates $x_1,\dots,x_n,y_1,\dots,y_n,z$, that is, $d\theta = \sum dx_i \wedge dy_i$. On $U := \bigcup_{-\epsilon<t<\epsilon} U_t$ we thus have $d(\theta - \sum dx_i \wedge dy_i) = 0$ whence $\theta = \sum dx_i \wedge dy_i + dz$ and $x_1,\dots,x_n,y_1,\dots,y_n,z$ are the desired coordinates. □

Exercises

2.1. Adapt the computations after (2.2.5) to B defined by $B(H) = 1$, $B(V) = 0 = B(X)$ to check that $B \wedge dB(H,X,V) = 1$ and $B = dA(V,\cdot)$.

2.2. Check the claims in Example 2.2.7.

[11] The same holds for a contact diffeomorphism.

2.3. Check the claims in Example 2.2.6: The discussion after Example 2.2.4 and Definition 2.2.5 showed that the geodesic flow and the horizontal flow each preserve a contact form (A and $B = dA(V,\cdot)$, respectively). Check that $C := dA(H,\cdot)$ is a contact form invariant under the fiber flow V in Remark 2.2.2 but that its Reeb field is $-V$ and that $C \wedge dC(V,H,X) = -1$, so this volume has the opposite orientation from the ones defined by A and B (and E in Example 2.2.7) and is hence not isotopic to either of them.

2.4. Since the generators $H_\pm$ for the horocycle flows are linear combinations of H and V, check whether a linear combination of the 1-forms B and C in Example 2.2.6 and before it is an invariant 1-form and if so, whether it is a contact form.

2.5. Prove the existence of the defining limit in Remark 2.1.18.

2.6. Show that the horocycle flow is minimal (by combining Example 2.1.16 with the use of homogeneity as in Example 1.6.2).

2.7. As in Theorem 2.4.4, let Γ be a discrete group of fixed-point-free isometries of $\mathbb{D}$ such that $M := \Gamma\backslash\mathbb{D}$ is compact. Prove that the geodesic flow is topologically mixing by combining Example 2.1.16 and the argument for (2)⇒(1) in Theorem 6.2.12 below.

A hypersurface M in $\mathbb{R}^n$ is said to be *star-shaped* if there exists a point c such that every half-line from c to ∞ intersects M in exactly one point.

2.8. Prove that any star-shaped hypersurface in $\mathbb{R}^{2n}$ provided with the standard symplectic structure is of contact type.

2.9. Describe the contact form and the Reeb vector field on $S^{2n-1} \subset \mathbb{R}^{2n}$ corresponding to the vector field $\xi = \frac{1}{2}\sum_{i=1}^{n}\Big(p_i\frac{\partial}{\partial p_i} + q_i\frac{\partial}{\partial q_i}\Big)$.

2.10. Consider a hypersurface M in the cotangent bundle T^*N of a smooth manifold that intersects each fiber in a star-shaped hypersurface. Prove that M is of contact type with respect to the standard symplectic structure.

2.11. Let L be a Lagrangian subspace of a symplectic vector space E. Prove that L has a *Lagrangian complement*, that is, a Lagrangian subspace M such that $L \cap M = \{0\}$.

2.12. Prove that in a Lagrangian subspace $L \subset E$ the basis $e_1,\dots,e_{2n}$ from Proposition 2.6.3 can be chosen in such a way that $e_1,\dots,e_n \in L$.

2.13. Let $\Lambda = (\lambda_1,\dots,\lambda_{2n})$ be a collection of nonzero complex numbers with the following properties:

1. Λ contains an even number of 1's and an even number of -1's.
2. If $\lambda \in \Lambda$ is real, $\lambda \neq \pm 1$, then $1/\lambda \in \Lambda$ (with the same multiplicity).
3. If $\lambda \in \Lambda$, $|\lambda| = 1$, and $\lambda \neq \pm 1$ then $\bar{\lambda} \in \Lambda$ (with the same multiplicity).
4. If $\lambda \in \Lambda$, $|\lambda| \neq 1$, $\lambda \notin \mathbb{R}$ then $\lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1} \in \Lambda$ (with the same multiplicities).

Prove that there exists a symplectic linear map $T\colon (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega)$, where ω is the standard symplectic form, such that Λ is the set of eigenvalues of T (with multiplicities).

2.14. * Prove that the 2-cohomology class of any nondegenerate closed 2-form on a $2n$-dimensional compact manifold M is nonzero.

2.15. Prove that there is no symplectic structure on the $2n$-sphere for $n \geq 2$, that is, there is no symplectic manifold (S^{2n}, ω).

2.16. Suppose $\{\omega_t\}_{0 \leq t \leq 1}$ is a family of nondegenerate closed differential 2-forms on a compact manifold M. Prove that there exists a family of diffeomorphisms $\varphi_t\colon M \to M$ such that $\varphi_t^* \omega_t = \omega_0$ if and only if the cohomology classes of the forms ω_t are the same.

2.17. Show that the geodesic flow on any surface of revolution has a first integral independent of the total energy. This integral is called the Clairaut integral.

2.18. Prove that any discrete subgroup of $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^k$ for some $k \leq n$ using the construction outlined in the proof of **??**.

2.19. * Let (M, ω) be a symplectic manifold and $\{\varphi^t\}$ a Hamiltonian flow all of whose orbits are periodic with the same minimal period T. Fix a value c of the Hamiltonian and consider the factor space N of the level surface M_c by the action of the flow. Show that the restriction of ω to M_c projects to a nondegenerate 2-form on N.

2.20. Show that the geodesic flow on the standard n-dimensional sphere satisfies the conditions of the previous exercise. Apply the procedure from that exercise to obtain a $(2n-2)$-dimensional symplectic manifold. Describe that manifold in detail for $n = 2$.

CHAPTER 3

Ergodic theory

Two important strands of 19th-century mathematics and physics led to the evolution of dynamical systems as we know it today. Celestial mechanics was a central motivation for Poincaré and his development of topological approaches to the study of dynamical systems (including the invention of topology itself). Statistical mechanics motivated a probabilistic approach to mechanical systems, where the preservation of volume provides a natural measure. This motivated von Neumann to formalize the foundations of ergodic theory [**218**, **219**], and this approach is broadly applicable, since there are usually many important invariant measures besides volume. Accordingly, we now study flows defined on measure spaces—in full generality. Later, in Chapter 8, we examine how these notions apply to hyperbolic flows.

This chapter introduces basic notions in measure theory, and then studies basic properties of measures invariant under a flow. We then examine the existence of time-averages of functions along orbits (Birkhoff's Ergodic Theorem). Next, we introduce ergodicity, which can be viewed as analogous to transitivity in topological dynamics, and a range of mixing properties much broader than topological mixing. Mixing is quite sensitive to time-changes, and this leads into a careful study of basic issues specific to continuous time: time-changes and special flows. This is significantly different from the theory of discrete systems. We conclude with spectral theory. This is an important subject in ergodic theory, but the section is optional since we will not use it in studying hyperbolic flows.

A number of the subjects in this chapter show significant differences between the discrete-time and continuous-time setting, such as invariant measures for time-changes and special flows, some aspects of ergodicity, properties of mixing, and measure-theoretic entropy. Especially the latter requires measure theory beyond what is usually done in an introductory graduate course, and we summarize those ideas in Section 11.1.

1. Flow-invariant measures and measure-preserving transformations

We now review basic notions of measures and measure spaces. Let X be a set and $\mathscr{T} \subset 2^X$ a σ-algebra (that is, $\varnothing, X \in \mathscr{T}$ and $\mathscr{T}$ is closed under complements and countable unions of sets). Then $(X, \mathscr{T})$ is called a *measurable space* and the elements of $\mathscr{T}$ are referred to as measurable sets. A *measure* is a function $\mu\colon \mathscr{T} \to [0,\infty]$ such that $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for pairwise disjoint $A_i \in \mathscr{T}$. A set is a *null set* if $\mu(A) = 0$. The σ-algebra $\mathscr{T}$ is *complete* if any subset of a μ-null set is in $\mathscr{T}$. The completion of a σ-algebra $\mathscr{T}$ is the σ-algebra $\overline{\mathscr{T}}$ generated by $\mathscr{T}$ and the μ-null sets. A complete σ-algebra $\mathscr{T}$ is said to be separable if it is the completion of a σ-algebra generated by a countable family of sets. A set A has *full measure* or is *conull* if $\mu(A^c) = 0$.[1] An assertion is said to be *essentially* true if it holds on a set of full measure. (Such as, an essentially constant function, an essential bound for a function, a flow with essentially no fixed points.)

A measure space $(X, \mathscr{T}, \mu)$ is *σ-finite* if X is a countable union of sets of finite measure. If $\mu(X) = 1$, then we call μ a *probability measure*. A point $x \in X$ is called an *atom* if $\mu(x) > 0$.

We usually assume that X is a probability measure and $\mathscr{T}$ is complete. We now define the basic notion of a measurable dynamical system.

Definition 3.1.1 (Measurability, measure-preservation, isomorphism)**.** A map between measurable spaces is *measurable* if the preimage of any measurable set is measurable. A measurable map between measure spaces is *non-singular* if the preimage of any null set is a null set. For a measurable map $T\colon (X, \mathscr{T}, \mu) \to (Y, \mathscr{A})$ we define the push-forward of μ by

$$T_* \mu(A) \coloneqq \mu(T^{-1}(A)).$$

A measurable map $T\colon (X, \mathscr{T}, \mu) \to (Y, \mathscr{A}, \nu)$ is *measure-preserving* if $T_*\mu = \nu$.

Two measure spaces are *isomorphic* if there exist sets $X' \subset X$ and $Y' \subset Y$ each of full measure and a measurable measure-preserving bijection $T : X' \to Y'$ with measurable inverse, the *isomorphism*. An isomorphism of a set to itself is called an *automorphism*.

A flow $\varphi^t : X \to X$ of a measure space (X, μ) is *measure-preserving* if each φ^t is measurable with $\mu(\varphi^t(A)) = \mu(A)$ for all measurable A.

A *function* $f\colon (X, \mathscr{T}) \to \mathbb{R}$ is said to be *measurable* if the preimage of any open set is measurable. Two measurable functions are equivalent if they coincide on a set of full measure. If $p \in [1,\infty)$, then

$$L^p(X) \coloneqq \Bigl\{f\colon X \to \mathbb{R} \text{ measurable} \Bigm| \int |f|^p < \infty\Bigr\}/\overset{\text{ae}}{=}$$

[1] A^c denotes the complement of A.

is the linear space of equivalence classes of functions whose pth power is integrable ($L^2(X,\mu)$ is a Hilbert space with inner product $\langle f,g\rangle := \int fg\,d\mu$), and

$$L^\infty(X) := \Big\{f\colon X\to\mathbb{R} \;\Big|\; \|f\|_\infty := \underbrace{\operatorname{essup}|f|}_{:=\inf\{M\,|\,\mu(|f|^{-1}((M,\infty]))=0\}} < \infty\Big\}\Big/\overset{\text{ae}}{=}$$

consists of the equivalence classes of essentially bounded measurable functions.

A function $f\colon X\to\mathbb{R}$ is said to be *essentially* Φ*-invariant* if there is a null set N off which $f\circ\varphi^t = f$ *for all* $t\in\mathbb{R}$. The salient point is that N does not depend on t.[2] A set is said to be *essentially* Φ*-invariant* if its characteristic function is. The σ-algebra of these sets is denoted by $\mathscr{I}$.

Remark 3.1.2. We remark that a.e. orbit is a measurable set; this is most easily seen from Theorem 3.6.2 below because orbits of measure-preserving suspensions are measurable.

Definition 3.1.3. A complete measure μ is said to be a *Borel measure* if it is defined on the Borel σ-algebra. It is a *Radon measure* if furthermore the measure of a compact set is finite, and a *Borel probability measure* if $\mu(X)=1$.

Example 3.1.4. We now illustrate these notions with previous examples.

- The flow in Example 1.1.5 (linear translations on $\mathbb{R}$) preserves Lebesgue measure on $\mathbb{R}$.
- Similarly, the translation flow on the circle from Example 1.1.6 preserves Lebesgue measure on the circle, and
- Example 1.6.2 is a flow on the torus that preserves Lebesgue measure.
- If a flow has a fixed point x then the *Dirac measure* δ_x on it is invariant, where $\delta_x(A) := \chi_A(x)$.
- For the flow on the circle with a single fixed point (Example 1.3.9), the Dirac measure at the fixed point is the only flow-invariant Borel probability measure: any interval that does not contain the fixed point in its closure has countably many disjoint images and preimages of pairwise equal measure by invariance, hence is a null set.
- The suspension of Example 1.8.2 has a single invariant Borel probability measure that is the Lebesgue measure on the single periodic orbit. Definition 3.6.1 and the discussion following will clarify invariant measures for suspension flows and how these relate to invariant measures on the base space.

[2] It is not required, but in the situation at hand, we can ultimately choose the exceptional set to be invariant, that is, f is measurable with respect to the completion of the σ-algebra of (properly) invariant sets.

- Orbits of suspension flows are clearly measurable sets. Up to measurable isomorphism, every measurable flow is of this type (Theorem 3.6.2), so almost every orbit of a measurable flow is a measurable set.

Theorem 3.1.5. *Every Borel probability measure μ on a metric space is regular: for each measurable set B and all $\epsilon > 0$ there exist an open set U_ϵ and a closed set C_ϵ such that $C_\epsilon \subset B \subset U_\epsilon$ and $\mu(U_\epsilon \smallsetminus C_\epsilon) < \epsilon$.*

Proof. We let $\mathscr{A}$ be the collection of $A \subset X$ such that for all all $\epsilon > 0$ there exist a closed set C_ϵ and an open set U_ϵ with $C_\epsilon \subset A \subset U_\epsilon$ and $\mu(U_\epsilon \smallsetminus C_\epsilon) < \epsilon$ (so $\mathscr{A}$ is the collection of regular sets). We will show that $\mathscr{A}$ is a σ-algebra and contains all open sets. It is then clear that $\mathscr{A}$ is complete, so it is the completion of the Borel σ-algebra.

First, $\varnothing, X \in \mathscr{A}$ since both are simultaneously open and closed. If $A \in \mathscr{A}$, $\epsilon > 0$, and C_ϵ and U_ϵ be as in the definition of regular, then $X \smallsetminus U_\epsilon \subset X \smallsetminus A$ is closed, and $X \smallsetminus C_\epsilon \supset X \smallsetminus A$ is open, and

$$\mu((X \smallsetminus C_\epsilon) \smallsetminus (X \smallsetminus U_\epsilon)) = \mu(U_\epsilon \smallsetminus C_\epsilon) < \epsilon.$$

Thus $X \smallsetminus A \in \mathscr{A}$.

Let $A_1, \dots \in \mathscr{A}$ and $A = \bigcup_{i=1}^\infty A_i$. Fix $\epsilon > 0$ and for each i let $U_{i,\epsilon}$ be an open set and $C_{i,\epsilon}$ be a closed set such that $C_{i,\epsilon} \subset A_i \subset U_{i,\epsilon}$ and $\mu(U_{i,\epsilon} \smallsetminus C_{i,\epsilon}) < \epsilon/3^i$. Let $\hat{C}_\epsilon := \bigcup_{i=1}^\infty C_{i,\epsilon}$ and $C_\epsilon := \bigcup_{i=1}^k C_{i,\epsilon}$, where k is such that $\mu(\hat{C}_\epsilon \smallsetminus C_\epsilon) < \epsilon/2$. Then $C_\epsilon \subset A$ is closed, $U_\epsilon := \bigcup_{i=1}^\infty U_{i,\epsilon} \supset A$ is open, and

$$\mu(U_\epsilon \smallsetminus C_\epsilon) \le \mu(U_\epsilon \smallsetminus \hat{C}_\epsilon) + \mu(\hat{C}_\epsilon \smallsetminus C_\epsilon) \le \underbrace{\sum_{i=1}^\infty \mu(U_{i,\epsilon} \smallsetminus C_{i,\epsilon})}_{<\sum_{i=1}^\infty \epsilon/3^i = \epsilon/2} + \underbrace{\mu(\hat{C}_\epsilon \smallsetminus C_\epsilon)}_{<\epsilon/2} < \epsilon.$$

So $A \in \mathscr{A}$, and $\mathscr{A}$ is closed under countable unions, hence a σ-algebra.

To finish the proof we show that $\mathscr{A}$ contains all closed $C \subset X$. Let $\epsilon > 0$. The $U_i := \{x \in X : d(x, C) < 1/i\} \subset C$ are open with $\varnothing = \bigcap_{i=1}^\infty U_i \smallsetminus C$, so $\mu(U_k \smallsetminus C) \to 0$, and there is a $k \in \mathbb{N}$ with $\mu(U_k \smallsetminus C) < \epsilon$. Taking $C = C_\epsilon$ and $U_\epsilon = U_k$ gives $C \in \mathscr{A}$. □

Corollary 3.1.6. *If μ is a Borel probability measure on a metric space X and B a measurable set, then*

$$\mu(B) = \sup_{C \subset B \text{ closed}} \mu(C) \quad (\mu \textit{ is } \text{inner regular}) \quad \textit{and} \quad \mu(B) = \inf_{U \supset B \text{ open}} \mu(U) \quad (\mu \textit{ is } \text{outer regular}).$$

Borel measures are identified by the integrals of continuous functions:

Theorem 3.1.7. *Let μ, ν be Borel probability measures on a metric space X. If $\int f\,d\mu = \int f\,d\nu$ for all continuous functions f, then $\mu = \nu$.*

Proof. This can be proved directly from the Riesz Representation Theorem for continuous functions (Theorem 3.1.10), but we provide a different independent proof. Let C be a closed subset of X and fix $\epsilon > 0$. Then there exists an open set U_ϵ such that $C \subset U_\epsilon$ and $\mu(U_\epsilon \smallsetminus C) < \epsilon$. Then $f : X \to [0,1]$ defined by

$$f(x) := \frac{d(x, X \smallsetminus U_\epsilon)}{d(x, X \smallsetminus U_\epsilon) + d(x, C)}$$

is a continuous function such that $f = 0$ on $X \smallsetminus U_\epsilon$, $f = 1$ on C. Hence,

$$\nu(C) \le \int f\,d\nu = \int f\,d\mu \le \mu(U_\epsilon) < \mu(C) + \epsilon.$$

thus, $\nu(C) \le \mu(C)$ since ϵ is arbitrary. Switching μ and ν gives the opposite inequality, so $\mu = \nu$ for closed, hence measurable, sets.. □

Proposition 3.1.8. *For a Borel measure μ on a separable metrizable space X:*

(1) The support $\operatorname{supp}\mu := \{x \in X \mid \mu(U) > 0 \text{ if } x \in U, U \text{ open}\}$ *of μ is closed.*
(2) $\mu(X \smallsetminus \operatorname{supp}\mu) = 0$.
(3) Any set of full measure is dense in $\operatorname{supp}\mu$.

Proof. (1) If $x \notin \operatorname{supp}\mu$ take $U_x \ni x$ open with $\mu(U_x) = 0$. Then $U_x \cap \operatorname{supp}\mu = \varnothing$.

(2) Since X is separable, $X \smallsetminus \operatorname{supp}\mu$ is covered by countably many U_x as above, so $\mu(X \smallsetminus \operatorname{supp}\mu) = 0$ by σ-additivity of μ.

(3) Contraposition: If $A \subset X$, $\varnothing \ne U := \operatorname{supp}\mu \smallsetminus \bar{A}$ then $\mu(X \smallsetminus A) \ge \mu(U) > 0$. □

Remark 3.1.9. If $\operatorname{supp}\mu = X$ then we say that μ has full support or is positive on open sets. In Example 3.1.4 we showed that the support of the sole invariant measure for Example 1.8.2 is the fixed point. We will see more interesting connections between (properties of) invariant measures and the topological dynamics on their support (Theorem 3.3.29, Exercise 3.2, Proposition 3.4.12).

Theorem 3.1.7 is related to the fact that measures define (positive) linear functionals. The converse, that positive linear functionals arise from measures is the content of the Riesz Representation Theorem from analysis, and this will give another way to obtain invariant measures.

Theorem 3.1.10 (Riesz Representation Theorem)**.** *Let X be a compact Hausdorff space. Then for each bounded linear functional F on $C^0(X)$ there exists a unique mutually singular pair μ, ν of finite Borel measures (Definition 3.1.3) such that $F(\varphi) = \int \varphi\,d\mu - \int \varphi\,d\nu$ for all $\varphi \in C^0(X)$.*

Remark 3.1.11. In particular, when F is positive (that is, nonnegative on positive functions) there is a unique finite Borel measure μ such that $F(\varphi) = \int \varphi\,d\mu$. This is an important class of functionals in this book. It is especially useful that the collection $\mathfrak{M}(X)$ of Borel probability measures on a compact metrizable space is a convex norm-bounded subset of the dual to $C(X)$. $\mathfrak{M}$ is closed with respect to the *weak* topology* (the product topology of setwise convergence) defined by $\mu_n \to \mu :\Leftrightarrow \int_X \varphi\, d\mu_n \to \int \varphi\, d\mu\ \forall \varphi \in C(X)$ (we say that μ_n *equidistributes* to μ), hence compact and sequentially compact by the Banach–Alaoglu Theorem.[3]

We continue our study of measure-preserving flows (Definition 3.1.1) by restating what it means to preserve a measure.

Theorem 3.1.12. *Let $\varphi^t : X \to X$ be a measurable flow of a measure space $(X, \mathscr{T}, \mu)$. Then $\int f\, d(\varphi^t_* \mu) = \int f \circ \varphi^t\, d\mu$ for all $f \in L^1(X, \mu)$ and all t.*

Proof. By definition, this holds for characteristic functions of Borel sets, hence for simple functions (linearity) and for nonnegative measurable functions (pointwise limits of increasing sequences of simple functions). Considering positive and negative parts gives the theorem. □

Corollary 3.1.13. *Let Φ be a measure-preserving flow of a measure space $(X, \mathscr{T}, \mu)$ and $f : X \to \mathbb{R}$ (or $\mathbb{C}$) integrable. Then $\int_X f(x)\, d\mu = \int_X f(\varphi^t x)\, d\mu$ for all $t \in \mathbb{R}$.*

Together with Theorem 3.1.7, this implies

Proposition 3.1.14. *$\mu \in \mathfrak{M}(X)$ is Φ-invariant iff $\int f \circ \varphi^t\, d\mu = \int f\, d\mu$ for all $f \in C(X)$.*

The next result can be proved in more generality, but this version will be sufficient for our needs.

Theorem 3.1.15 (Krylov–Bogolubov Theorem)**.** *Any continuous flow on a metrizable compact space has an invariant Borel probability measure.*

Proof. If $\varphi^t : X \to X$ continuous, $\mu \in \mathfrak{M}(X)$, then by Remark 3.1.11 there is a weak* accumulation point μ' of $\frac{1}{T}\int_0^T \varphi^t_* \mu \in \mathfrak{M}(X)$. μ' is φ^t_*-invariant. □

Theorem 3.1.16. *If Φ is a continuous flow of a compact metric space then the set $\mathfrak{M}(\Phi)$ of Φ-invariant Borel probability measures is a closed, hence compact, convex subset of $\mathfrak{M}(X)$.*

Proof. If $\{\mu_n\}_{n=1}^\infty \subset \mathfrak{M}(\Phi)$ and $\mu_n \to \mu$ in $\mathfrak{M}(X)$, then

$$\int f\, d(\varphi^t_* \mu) = \int f \circ \varphi^t\, d\mu = \lim_{n\to\infty} \int f \circ \varphi^t\, d\mu_n = \lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu$$

for all continuous functions $f : X \to \mathbb{R}$ and all $t > 0$. So $\mu \in \mathfrak{M}(\Phi)$. Convexity is clear since $\mathfrak{M}(X)$ is convex. □

[3] The (norm-) unit ball in the dual of a normed linear space B is weak*-compact (proved using the Tychonoff thm on compact products), and sequentially compact if B is separable (proved by a diagonal argument)—this implies that norm-bounded weak*-closed sets are compact/sequentially compact.

Definition 3.1.17. A continuous flow on a metrizable compact space is said to be *uniquely ergodic* if it has exactly one invariant Borel probability measure. It is said to be *strictly ergodic* if it is furthermore minimal.

Remark 3.1.18. If the measure μ used in the proof of Theorem 3.1.15 is invariant, then the process becomes trivial because the accumulating family is constant, yielding $\mu' = \mu$. Indeed, a number of invariant measures often arise in an obvious way. Dirac measures on fixed points (see Example 3.1.4) is the most self-evident. If p is periodic with period ℓ, then $\delta_{\mathcal{O}(p)} := \frac{1}{\ell}\int_0^\ell \delta_{\varphi^t(p)}\,dt$ is an invariant Borel probability measure, as are convex combinations of any number of invariant Borel probability measure.[4]

For a suspension flow over a μ-preserving transformation on X, the product of μ with Lebesgue measure on $[0,1]$ defines an invariant Borel probability measure. For a flow under a function r on X, likewise for continuous $F\colon \Lambda(r) \to \mathbb{R}$ the following equation

$$\int_{X_r} F\,d\mu_r = \frac{\int_X \left(\int_0^{r(x)} F(x,t)\,dt\right) d\mu(x)}{\int_X r(x)\,d\mu(x)} \tag{3.1.1}$$

defines an invariant Borel probability measure μ_r. We revisit this in Definition 3.6.1 below, where it turns out that any invariant Borel probability measure for a flow can be seen as arising in this way (Theorem 3.6.2).

The next theorem connects some of the notions on topological dynamics of Chapter 1 to the set of invariant measures for a flow.

Theorem 3.1.19 ([**211**])**.** *If Ψ is a time-change of a continuous flow Φ without fixed points, then there is an affine bijection between $\mathfrak{M}(\Phi)$ and $\mathfrak{M}(\Psi)$.*[5]

Definition 3.1.20. A flow $\varphi^t : X \to X$ of a measure space (X,μ) is *measure-theoretically isomorphic* to a flow $\psi^t : Y \to Y$ if there is an isomorphism $h\colon X \to Y$ such that $\psi^t \circ h \overset{\text{ae}}{=} h \circ \varphi^t$ for all $t \in \mathbb{R}$. These flows are *orbit-equivalent* if there is an isomorphism $h\colon X \to Y$ that sends orbits of Φ to orbits of Ψ. A flow Ψ on Y is a *factor* of Φ on X if there is a measure-preserving essentially surjective $h\colon X \to Y$ such that $\psi^t \circ h \overset{\text{ae}}{=} h \circ \varphi^t$ for all $t \in \mathbb{R}$.

Remark 3.1.21. For continuous flows the notion of orbit-equivalence proved natural, and the reader may have noted that we only introduced here the measurable counterpart of topological conjugacy. The reason for this is that the natural notion

[4] A convex combination is a linear combination with nonnegative coefficients that sum to 1.

[5] Thus, a time-change of a uniquely ergodic flow (Definition 3.1.17) without fixed points is uniquely ergodic; however, there are uniquely ergodic flows (with a fixed point) for which some time-change is not uniquely ergodic [**211**].

of measurable orbit-equivalence in the sense of “same orbits” is too weak to be interesting as the next result illustrates.

Theorem 3.1.22 (Dye’s Theorem [**110**, **111**])**.** *Between any two free ergodic measure-preserving flows there is a measurable isomorphism that sends orbits to orbits.*

What is missing is any control of time along orbits under this isomorphism. An important equivalence relation retains just enough control by requiring the isomorphism to be monotone along orbits.

Definition 3.1.23 (Monotone (or Kakutani) equivalence [**176**, **177**])**.** Two flows are *monotonically* or *Kakutani-equivalent* if one of them is measurably isomorphic to the other after a time-change which is smooth along orbits.[6]

Note that this does not only provide monotonicity in the orbit direction but average control of the speed-change as well.

We are motivated by continuous flows on compact metric spaces X. Here, the smallest σ-algebra containing all open sets is called the *Borel σ-algebra*. Although we will not need the following result, we mention that it is not very restrictive to focus on this context because there is the device of *continuous representation*:

Theorem 3.1.24 (Ambrose–Kakutani Theorem [**7**, Theorem 5])**.** *A measure-preserving flow Φ on a Lebesgue space* (Definition 11.1.1) *with essentially no fixed points is measure-theoretically isomorphic to a continuous special flow on a separable metric space with an invariant Borel probability measure.*

Remark 3.1.25. It is a natural and rather deeper question whether any probability-preserving flow can be realized as a *volume*-preserving flow, as conjectured by von Neumann in his foundational paper [**218**].

The next example is a very important class of flow, and we will refer back to the example a number of time in this chapter.

Example 3.1.26 (Bernoulli flow)**.** Consider the full shift (Definition 1.8.1) and endow the shift space σ_n with the Borel measure μ for which $\mu(C_i^0) = p_i$ with $\sum_i p_i = 1$ (see (1.8.1)). Together with shift-invariance, this uniquely defines a probability measure by Theorem 3.1.7, in fact, this is the product measure on $\mathscr{A}_n^{\mathbb{Z}}$, where $\nu(\{i\}) = p_i$ for $i \in \mathscr{A}_n$. The full shift with this measure is called a *Bernoulli shift*, and a flow is called a *Bernoulli flow* or said to have the *Bernoulli property* if every time-t map for $t \neq 0$ is measure-theoretically isomorphic to a Bernoulli shift (see Definition 3.4.3).

[6] Specifically, whose derivative along orbits is in $L^1(X,\mu)$. (If it is identically 1, then there is no time-change, and the flows are isomorphic.)

2. Ergodic Theorems

The purpose of studying invariant measures is to be able to meaningfully investigate probabilities in a statistical approach to long-term evolution. This necessitates knowing that such long-term statistics exist, and theorems to this effect are called ergodic theorems. The first of these was proved by von Neumann, and it served to crystallize the notion of ergodicity. Spurred by von Neumann's article, Birkhoff established a pointwise counterpart. We begin with a precursor to these.

In this section we prove results on ergodic theorems without defining ergodicity. The reason for this is that the theorems can be stated and proved in a more general setting, and often one needs the more general statement. Later we will explain the importance of the theorems in the context of ergodicity.

Poincaré viewed recurrence as a weaker form of stability, and he had the insight that this is ubiquitous in celestial mechanics, and indeed all mechanical systems, as a simple consequence of preserving a probability measure:

Theorem 3.2.1 (Poincaré Recurrence Theorem)**.** *Let Φ be a measure preserving flow of a probability space $(X,\mathscr{T},\mu)$. If A is measurable and $T \ge 0$, then for almost every $x \in A$ there exists $t > T$ such that $\varphi^t(x) \in A$ (that is, there are $t_i \to \infty$ with $\varphi^{t_i}(x) \in A$).*

Proof. $B := \{x \in A \mid \varphi^{iT}(x) \in A^c$ for all $i \in \mathbb{N}\} = A \smallsetminus (\bigcup_{i\in\mathbb{N}} \varphi^{-iT}(A))$ is measurable and the $\varphi^{-iT}(B)$ are pairwise disjoint and have the same measure as B. Therefore, $\mu(B) = 0$ since $\mu(X) = 1$. □

Corollary 3.2.2. *Let X be a separable metric space $\varphi^t : X \to X$ a continuous flow, μ a Φ-invariant Borel probability measure. Then $\mu(\mathscr{B}(\Phi)) = 1$ (hence $\mu(\mathscr{L}(\Phi)) = \mu(NW(\Phi)) = 1$ by Proposition 1.5.34).*

Proof. For a countable base $\{U_1, U_2, \dots\}$ of open subsets of X the set of all points $x \in U_m$ with $\varphi^{t_i}(x) \in U_m$ with $t_i \to \infty$ has full measure by the Poincaré Recurrence Theorem 3.2.1. □

Remark 3.2.3. This corollary is not in all cases as interesting as it seems. If μ is the Dirac measure on a fixed point, then essentially all points are fixed no matter how much orbit complexity there might be elsewhere.

While the Poincaré Recurrence Theorem establishes recurrence, a qualitative phenomenon, ergodic theorems are about using statistics. We present the von Neumann (convergence in the mean) and Birkhoff (pointwise convergence) ergodic theorems.

Theorem 3.2.4 (von Neumann Mean Ergodic Theorem)**.** *Let $\varphi^t\colon (X,\mu)\to(X,\mu)$ be a measure-preserving flow of a measure space, $f\in L^2(X,\mu)$. Then*

$$\frac{1}{T}\int_0^T f\circ\varphi^t\,dt \xrightarrow[T\to\infty]{L^2} P_\Phi(f),$$

where P_Φ is the orthogonal projection to the space $L^2(X,\mathscr{I},\mu\restriction_{\mathscr{I}})$ of φ^t-invariant functions.

Note that this theorem does not require the measure space to be a probability space. It follows from a Hilbert-space lemma, for which it is useful that one can associate with a measure-preserving map an isometric operator, and hence a 1-parameter family of such operators to a flow.

Definition 3.2.5 (Koopman operator)**.** For $p\ge 1$ one associates to a measure-preserving map $f\colon (X,\mu)\to(Y,\nu)$ an isometric operator

$$U_f\colon L^p(Y,\nu)\to L^p(X,\mu),\quad \varphi\mapsto\varphi\circ f$$

on complex-valued functions, the *Koopman operator*. For a measure-preserving flow Φ we have $U_\Phi^t := U_{\varphi^1}^t$, so we sometimes write $U_\Phi := U_{\varphi^1}$ and

$$U_{\varphi^t}(f) = f\circ\varphi^t.$$

Remark 3.2.6. The case $p=2$ is of particular interest. If $f\colon X\to X$ is invertible then so is U_f and in this case U_f defines a unitary operator on L^2. In particular, U_{φ^t} is a 1-parameter family of unitary operators on $L^2(X,\mu)$ if Φ is a μ-preserving flow on X.

Remark 3.2.7. When (X,μ) is a compact metric probability space and f is continuous, then $f\mapsto U_f h$ is continuous in the norm-topology—clearly for uniformly continuous h and the subspace of uniformly continuous functions is dense.

Theorem 3.2.8 (Alaoglu–Birkhoff Abstract Ergodic Theorem)**.** *Suppose H is a Hilbert space, G a group of unitary operators, and P_{H_G} the orthogonal projection to H_G, the space of its common fixed points. If $v\in H$, then $P_{H_G}(v)$ is the unique element of the closed convex hull $\overline{\text{co}}Gv$ of $Gv := \{gv \mid g\in G\}$ of minimal norm.*[7]

Proof. As a nonempty closed convex subset of a Hilbert space, $\overline{\text{co}}Gv$ contains a unique norm-minimizing element F. Since $\frac{1}{2}gF+\frac{1}{2}F\in\overline{\text{co}}Gv$ cannot have smaller norm, we have $gF=F$, that is, $F\in H_G$. To see that $F=P_{H_G}(v)$ we show that $v-F\perp H_G$. For $h\in H_G$ the set $\{w\in H \mid \langle w-F,h\rangle=\langle v-F,h\rangle\}\ni v$ is closed and convex and contains Gv (since each g is unitary) and hence F. Thus $\langle v-F,h\rangle=0$ (and indeed, $\{P_{H_G}(v)\}=H_G\cap\overline{\text{co}}Gv$). □

[7] We reproduce here a proof from Terence Tao's blog.

Proof of the von Neumann Ergodic Theorem. Take $v \in L^2$ and $\epsilon > 0$. By the Alaoglu–Birkhoff Abstract Ergodic Theorem there is a finite convex combination $v_\epsilon = \sum_{i=1}^n U_{\varphi^{t_i}} v$ with $\| v_\epsilon - P_\Phi v \| < \epsilon$, hence $\| \frac{1}{T} \int_0^T U_\Phi^t v_\epsilon \, dt - P_\Phi v \| < \epsilon$ for any $T > 0$, and $\overline{\lim}_{T \to \infty} \| \frac{1}{T} \int_0^T U_\Phi^t v \, dt - P_\Phi v \| < 2\epsilon$. □

The Birkhoff Ergodic Theorem addresses the question of the existence of the time averages in the sense of pointwise convergence. It applies on any probability space, and no topology is involved. Before stating it, we recall a standard result in measure theory in a slightly unconventional form.

Definition 3.2.9 (Absolute continuity)**.** If $(X, \mathscr{S}, \mu)$ and $(X, \mathscr{T}, v)$ are signed measure spaces then v is said to be *absolutely continuous* with respect to μ, written $v \ll \mu$, if every null set for μ is a null set for v.

Theorem 3.2.10 (Radon–Nikodym)**.** *If $(X, \mathscr{S}, \mu)$ and $(X, \mathscr{T}, v)$ are σ-finite signed measure spaces and $v \ll \mu$, then there is a μ-a.e. unique* density *or* Radon–Nikodym derivative $\left[\dfrac{dv}{d\mu} \right] \coloneqq \rho \colon X \to \mathbb{R}$ *of v with respect to μ that is measurable with respect to the completion $\overline{\mathscr{S}}$ of $\mathscr{S}$ and such that $v(A) = \int_A \rho \, d\bar{\mu}$, where $\bar{\mu}$ is the completion of μ, for every A in the completion of $\mathscr{T}$.*

In particular, $\overline{\mathscr{T}} \subset \overline{\mathscr{S}}$.

Corollary 3.2.11 (Conditional expectation)**.** *Suppose $(X, \mathscr{S}, \lambda)$ is a σ-finite measure space, $\mathscr{T} \subset \mathscr{S}$ a σ-algebra, $\varphi \in L^1(X, \mathscr{S}, \lambda)$. Denote by $\lambda_{\restriction_{\mathscr{T}}}$ the restriction, that is, $\lambda_{\restriction_{\mathscr{T}}}(A) = \lambda(A)$ for all $A \in \mathscr{T} \subset \mathscr{S}$. Then the* conditional expectation

$$E(\varphi \mid \mathscr{T}) \coloneqq \varphi_{\mathscr{T}} \coloneqq \left[\frac{d(\varphi\lambda)_{\restriction_{\mathscr{T}}}}{d\lambda_{\restriction_{\mathscr{T}}}} \right] \in L^1\left(X, \mathscr{T}, \lambda_{\restriction_{\mathscr{T}}}\right)$$

of φ on $\mathscr{T}$ is defined λ-a.e. uniquely by $\int_A \varphi_{\mathscr{T}} \, d\lambda = \int_A \varphi \, d\lambda$ for all $A \in \mathscr{T}$.

Proof. Apply Theorem 3.2.10 to $\lambda_{\restriction_{\mathscr{T}}} \gg v \coloneqq (\varphi\lambda)_{\restriction_{\mathscr{T}}}, A \mapsto \int \varphi \chi_A \, d\lambda$ for $A \in \mathscr{T}$. □

Proposition 3.2.12.

(1) *$E(\cdot \mid \mathscr{T}) \eqqcolon \pi_{\mathscr{T}} \colon L^1(\mu) \to L^1(\mu_{\restriction_{\mathscr{T}}}) \subset L^1(\mu)$ is a projection.*
(2) *$\pi_{\mathscr{T}}$ is linear and positive, that is, $f \geq 0 \Rightarrow f_{\mathscr{T}} \geq 0$.*
(3) *If g is $\mathscr{T}$-measurable and bounded, then $E(gf \mid \mathscr{T}) = gE(f \mid \mathscr{T})$.*
(4) *If $\mathscr{T}_2 \subset \mathscr{T}_1$ then $E(\cdot \mid \mathscr{T}_2) \circ E(\cdot \mid \mathscr{T}_1) = E(\cdot \mid \mathscr{T}_2)$.*

The proof is straightforward; we note that 1. follows from 4. but more directly from the obvious fact that $\pi_{\mathscr{S}} = \mathrm{Id}$.

We digress briefly to a contemplation of how this plays out in L^2.

Definition 3.2.13. Suppose H is a Hilbert space and $L \subset H$ is a closed subspace. Then each $v \in H$ uniquely[8] decomposes as $v = v_0 + v_\perp$, where $v_0 \in L$ and $v_\perp \perp L$, that is, $v_\perp \perp w$ for all $w \in L$, and the *orthogonal projection to* L is defined by

$$\pi_L \colon H \to L, \quad v_0 + v_\perp \mapsto v_0.$$

Proposition 3.2.14. *If $v \in H$, $w \in L$, then $\|v - \pi(v)\| \le \|v - w\|$ and $\langle v, w\rangle = \langle \pi_l(v), w\rangle$.*

Proof. $\|v - w\|^2 = \|v_0 + v_\perp - w\|^2 = \|v_0 - w\|^2$ is minimal iff $w = v_0 = \pi(v)$, and $\langle v, w\rangle = \langle v_0 + v_\perp, w\rangle = \langle v_0, w\rangle = \langle \pi_l(v), w\rangle$. □

Example 3.2.15. Suppose $(X, \mathscr{T}, \mu)$ is a probability space and $\mathscr{S} \subset \mathscr{T}$ is a σ-algebra in $\mathscr{T}$. Then $L := L^2(X, \mathscr{S}, \mu) \subset H := L^2(X, \mathscr{T}, \mu)$ is a closed subspace. For $f \in L^2(X, \mathscr{T}, \mu)$ and $A \in \mathscr{S}$ we then have $\chi_A \in L$ and hence by Proposition 3.2.14

$$\int_A f\, d\mu = \langle f, \chi_A\rangle = \langle \pi_L(f), \chi_A\rangle = \int_A \pi_L(f)\, d\mu.$$

In light of uniqueness in Corollary 3.2.11 we see that $\pi_{L^2(X,\mathscr{S},\mu)} = E(\cdot \mid \mathscr{S})\restriction_{L^2(X,\mathscr{T},\mu)}$, that is, the orthogonal projection to $L^2(X, \mathscr{S}, \mu)$ is given by conditional expectation.

We next prove the Birkhoff Ergodic Theorem for discrete time. The continuous-time counterpart (Theorem 3.2.19) then follows easily. If T is a measure-preserving transformation of a measure space $(\mathscr{B}, \mu)$ denote by $\mathscr{I} := \mathscr{I}_T := \{A \in \mathscr{B} \mid T^{-1}(A) = A\}$ the invariant σ-algebra.

Theorem 3.2.16 (Birkhoff Ergodic Theorem). *Let (X, μ) be a probability space, $T \colon X \to X$ μ-preserving, $f \in L^1(X, \mu)$. Then the time average exists:*

$$f_T := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k = f_{\mathscr{I}_T} \quad \mu\text{-a.e.}$$

In particular, f_T is measurable and T-invariant, and

$$\int f_T\, d\mu = \int f_{\mathscr{I}}\, d\mu = \int f\, d\mu. \tag{3.2.1}$$

Proof. If $g \in L^1(\mu)$, then $G_n := \max\limits_{k\le n} \sum\limits_{i=0}^{k-1} g \circ T^i \in L^1(\mu)$ is nondecreasing in n, and

$$\varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g \circ T^k \le \varlimsup_{n\to\infty} \frac{G_n}{n} \le 0 \quad \text{off} \quad A := \{x \mid G_n(x) \to \infty\} \in \mathscr{I}. \tag{3.2.2}$$

[8] $v_0 + v_\perp = w_0 + w_\perp \Rightarrow v_0 - w_0 = w^\perp - v^\perp \in L \cap L^\perp = \{0\}$.

$G_{n+1} = g + G_n \circ T \Leftrightarrow G_n \circ T \geq 0$, so $G_{n+1} - G_n \circ T = g - \min(0, G_n \circ T) \searrow g$ on A, and

$$0 \leq \int_A (G_{n+1} - G_n)\, d\mu = \int_A (G_{n+1} - G_n \circ T)\, d\mu \xrightarrow[\text{Theorem}]{\substack{\text{Monotone}\\ \text{Convergence}}} \int_A g\, d\mu = \int_A g_{\mathscr{I}}\, d\mu\restriction_{\mathscr{I}},$$

so $g_{\mathscr{I}} < 0 \Rightarrow \mu(A) = 0$. If $g := f - f_{\mathscr{I}} - \epsilon$, then $g_{\mathscr{I}} = -\epsilon < 0$, so (3.2.2) becomes

$$\varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (f \circ T^k) - f_{\mathscr{I}} - \epsilon \leq 0 \ \mu\text{-a.e.} \quad \text{with } \epsilon > 0 \text{ arbitrary.}$$

Replacing here f by $-f$ gives $\displaystyle\varliminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \geq f_{\mathscr{I}} - \epsilon$ μ-a.e.[9] □

Now consider a measurable map $(t, x) \mapsto \varphi^t(x)$ that To obtain the corresponding Birkhoff Ergodic Theorem for a flow Φ, we apply the Birkhoff Ergodic Theorem 3.2.16 to the measure-preserving transformation φ^1 and the function $f_1 := \int_0^1 f \circ \varphi^s\, ds$.

Proposition 3.2.17. *Let (X, μ) be a probability space, Φ a μ-preserving flow on X, $f \in L^1(X, \mu)$. Then* $\displaystyle\frac{1}{n} \int_0^1 f \circ \varphi^{n+s}\, ds \xrightarrow[n\to\infty]{\text{a.e.}} 0$.

Proof. The Birkhoff Ergodic Theorem 3.2.16 applied to the measure-preserving transformation φ^1 and the function $f_1 := \int_0^1 f \circ \varphi^s\, ds \in L^1(X, \mu)$ gives

$$\underbrace{\frac{1}{n} \int_0^1 f \circ \varphi^{n+s}\, ds}_{= f_1 \circ \varphi^n} = \frac{n+1}{n} \Big[\frac{1}{n+1} \sum_{k=0}^{n} f_1 \circ \varphi^k \Big] - \frac{1}{n} \sum_{k=0}^{n-1} f_1 \circ \varphi^k \xrightarrow[n\to\infty]{\text{a.e.}} 1 \cdot (f_1)_{\mathscr{I}} - (f_1)_{\mathscr{I}} \overset{\text{ae}}{=} 0. \ \square$$

Remark 3.2.18. Here and later, we use the Bachmann–Landau *"little O" notation*: $f(t) \in o(g(t)) :\Leftrightarrow \frac{f(t)}{g(t)} \xrightarrow[t\to a]{} 0$, where a is usually clear from context and most often equal to 0 or ∞. The corresponding *"big O" notation* is: $f(t) \in O(g(t)) :\Leftrightarrow \frac{f(t)}{g(t)}$ is bounded for t near a. We sometimes write $f(t) = o(g(t))$ and $f(t) = O(g(t))$.

Theorem 3.2.19 (Birkhoff Ergodic Theorem for flows)**.** *Let (X, μ) be a probability space, $\varphi^t \colon X \to X$ a μ-preserving flow, $f \in L^1(X, \mu)$. Then the time average exists:*

$$f_\Phi(x) := \lim_{t\to\infty} \frac{1}{t} \int_0^t f \circ \varphi^s ds = f_{\mathscr{I}} \quad \mu\text{-a.e.}$$

[9] This proof from [**181**] incorporates a shortcut by A. Fieldsteel and B. Bassler compared to the version originally communicated to us by Uwe Schmock who had first seen it in lecture notes by Erwin Bolthausen (a Managing Editor of this book series) with attribution to Jacques Neveu. (Neveu in turn explicitly told us that he was unaware of having given any such proof.)

Proof. We apply the Birkhoff Ergodic Theorem 3.2.16 to establish the existence of the limit and then show that it is $f_{\mathscr{I}}$. As a minor convenience we assume $f \geq 0$; the result follows from this by considering positive and negative parts.

First note that by Tonelli's Theorem

$$\infty > n\int f\,d\mu = \int_0^n \int_X f(\varphi^s(x))\,d\mu\,ds = \int_X \int_0^n f(\varphi^s(x))\,ds\,d\mu,$$

so $0 \leq f_n := \int_0^n f \circ \varphi^s ds$ is well-defined (and finite) off a null set E_n with $\int f_1 = n\int f$. The Birkhoff Ergodic Theorem 3.2.16 gives

$$\frac{1}{n}\int_0^n f \circ \varphi^s ds = \frac{1}{n}\sum_{k=0}^{n-1} f_1 \circ \varphi^k \xrightarrow[n\to\infty]{} E(f_1 \mid \mathscr{I}_{\varphi^1}) \text{ off a null set } F. \tag{3.2.3}$$

To pass from integer times to others, consider x outside the null set N defined as the union of the set F in (3.2.3), all the E_n above and the null set implicit in Proposition 3.2.17. Then Proposition 3.2.17 and $f \geq 0$ imply

$$0 \leq \int_0^{t-\lfloor t\rfloor} f(\varphi^s(\varphi^{\lfloor t\rfloor}(x))ds \leq f_1(\varphi^{\lfloor t\rfloor}(x)) \in o(t),$$

so (3.2.3) gives

$$\underbrace{f_\Phi(x)}_{=\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\varphi^s(x))ds} = \lim_{t\to\infty}\frac{\lfloor t\rfloor}{t}\frac{1}{\lfloor t\rfloor}\sum_{k=0}^{\lfloor t\rfloor-1} f_1(\varphi^k(x)) + \lim_{t\to\infty}\frac{1}{t}\int_0^{t-\lfloor t\rfloor} f(\varphi^s(\varphi^{\lfloor t\rfloor}(x))ds = (f_1)_{\mathscr{I}_{\varphi^1}} + 0.$$

Thus $\int f_\Phi = \int f$. Now apply what we proved so far to $g := f\chi_A$ for any $A \in \mathscr{I}$:

$$\int_A f_\Phi = \int f_\Phi\chi_A = \int f_\Phi(\chi_A)_\varphi = \int (f\chi_A)_\varphi = \int g_\varphi = \int g = \int f\chi_A = \int_A f,$$

and this, together with Φ-invariance, is the very definition of $f_\Phi = f_{\mathscr{I}}$.[10] □

The Birkhoff Ergodic Theorem also yields almost-everywhere convergence of negative and two-sided time averages:

Proposition 3.2.20. $\bar{f}_\Phi := \lim\limits_{t\to\infty}\frac{1}{t}\int_0^t f \circ \varphi^{-s}\,ds = f_{\mathscr{I}} \overset{\text{ae}}{=} f_\Phi$ *and* $\frac{1}{2t}\int_{-t}^t f \circ \varphi^s\,ds \xrightarrow{\text{a.e.}} f_{\mathscr{I}}$.

Remark 3.2.21. The Birkhoff Ergodic Theorem says that $f \mapsto f_\Phi$ is a projection to the Φ-invariant functions.

Remark 3.2.22. Another perspective on existence of time-averages is given by the *empirical measure* $\epsilon_{x,T} := \frac{1}{T}\int_0^T \delta_{\varphi^s(x)}\,ds$ for a given $x \in X$. If $f \in L^1(\mu)$, then $\epsilon_{x,T}(f)$ converges for μ-a.e. x by the Birkhoff Ergodic Theorem. Thus, if $L^1(\mu)$ is separable, then $\epsilon_{x,T}$ converges weakly for μ-a.e. x.

[10] This proof follows one in ***ETH*** lecture notes by Oscar Lanford.

The exceptional set where the positive or negative time averages do not exist may, of course, depend on the function f. However, it is negligible for any invariant measure.

Definition 3.2.23. Given a continuous flow Φ of a metric space X, we say that a subset $A \subset X$ has *total measure* if A has full measure with respect to *any* Φ-invariant Borel probability measure on X.

Corollary 3.2.24. *Let X be compact metrizable, Φ a continuous flow. Then*

$$\Big\{x \in X \;\Big|\; \lim_{t\to\infty} \frac{1}{t}\int_0^t f\circ\varphi^k(x)\,ds \text{ exists for all continuous functions } f\Big\}$$

has total measure, as does

$$\Big\{x \in X \;\Big|\; \lim_{t\to\infty} \frac{1}{t}\int_0^t f\circ\varphi^s(x)\,ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t f\circ\varphi^{-s}(x)\,ds \text{ for } f \in C(X)\Big\}.$$

Proof. For each f_j in a countable dense set of functions the averages converge on a set E_i of total measure. Lipschitz continuity of $f \mapsto \frac{1}{t}\int_0^t f\circ\varphi^s\,ds$ implies convergence on $\bigcap_i E_i$ for all continuous f, and having total measure is stable under countable intersection. □

By the Krylov–Bogolubov Theorem 3.1.15, a set of total measure is nonempty:

Corollary 3.2.25. *For any continuous flow Φ on a compact metric space X there is an $x \in X$ such that for every continuous function f on X the time averages $\frac{1}{t}\int_0^t f\circ\varphi^s(x)\,ds$ and $\frac{1}{t}\int_0^t f\circ\varphi^{-s}(x)\,ds$ both converge and have the same limit.*

Remark 3.2.26. We emphasize that while Corollary 3.2.24 produces an apparently large set of points whose Birkhoff averages exist, we have encountered instances of dynamical systems with a paucity of invariant measures; if these are moreover atomic, then the set promised by Corollary 3.2.24 may not look very large. For instance, in the south-south flow (Example 1.3.9) the fixed point is a set of total measure. Ruelle proposed to call points "historic" if they do not have a Birkhoff average, the idea being that the running average $\frac{1}{t}\int_0^t f\circ\varphi^s ds$ fluctuates significantly over time and thus in a vague way associates with a given average a time t or, rather, an "era" in the "history" of the orbit.[11] Among the questions one can raise when Lebesgue measure is defined on X but not invariant under a given flow, whether Lebesgue-almost all points have Birkhoff averages or whether instead there is a set of historic points with positive Lebesgue measure. While for hyperbolic flows this does not happen (Remark 8.4.8), a variant of Figure 1.5.4 shows

[11] As Ruelle put it, "This absence of limit is what we want to call historical behaviour. This means that, as the time... tends to ∞, the point... keeps having new ideas about what it wants to do." [**261**]

a situation where this is indeed the case: taking Figure 1.1.4 to represent a flow on $\mathbb{R}^2$, alter it, so orbits spiral out from the neutral fixed point to the homoclinic loop that connects adjacent saddles (Figure 3.2.1). Bowen observed that if f is a continuous function with different values at the adjacent saddles, then for any of those points whose orbits spiral towards the homoclinic loop, the Birkhoff averages do not exist—because the times spent near each of the saddles grows exponentially and therefore always moves the running averages back towards that value of f. We thus have an open set of historic points.[12]

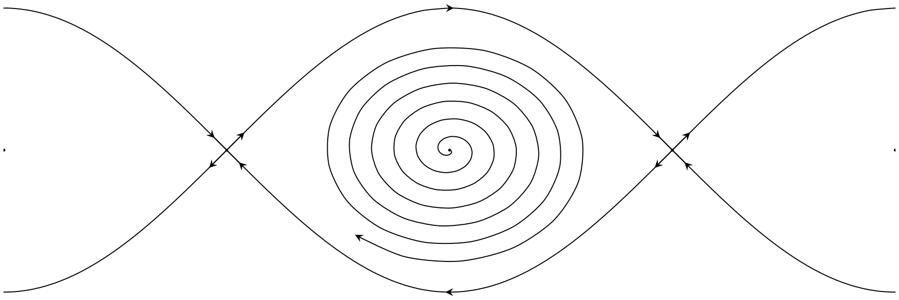


FIGURE 3.2.1. Spiraling towards a homoclinic loop

3. Ergodicity

We now introduce a central notion of this chapter. We discussed in Subsection 0.2c that ergodic theory arose from the desire for equality of time-averages, on whose existence we just elaborated, with the space average of an observable. Ergodicity of an invariant Borel probability measure is the very indecomposability notion which produces this circumstance. Despite their names, the ergodic theorems in the previous section do not presuppose the measure to be ergodic, and we will show how these general theorems specialize to ergodic systems to give in particular the equality of time- and space-averages (Corollary 3.3.11).

Definition 3.3.1. A measure μ is said to be *ergodic* with respect to Φ, or one says that Φ is ergodic with respect to μ, if for any measurable $A \subset X$ with $\varphi^{-t}(A) = A$ for all $t \in \mathbb{R}$ either $\mu(A) = 0$ or $\mu(X \smallsetminus A) = 0$.

[12] This example does not persist under typical perturbations because the homoclinic loop can disappear or become tangled instead; there are persistent examples, however, using homoclinic tangencies [**190**].

Remark 3.3.2. Φ-invariance of μ is not needed for this definition. Dirac measures are trivially ergodic, as is $\delta_{\mathscr{O}(p)}$ in Remark 3.1.18 ($\mathscr{O}(p)$ has no proper invariant subsets) and hence Lebesgue measure for the translation flow on the circle from Example 1.1.6. Proposition 3.3.6 and Proposition 3.3.7 below give the first nontrivial instances. It is clear from the definition (or from Proposition 3.3.12 below) that if μ is ergodic and $\nu \ll \mu \ll \nu$, then so is ν.

Ergodicity can be reformulated in functional language:

Proposition 3.3.3 (Characterization of ergodicity)**.** *The following are equivalent.*

(1) *Φ is ergodic with respect to μ.*
(2) *Any measurable Φ-invariant $f\colon X \to \mathbb{C}$ is constant μ-a.e.*
(3) *Any bounded measurable Φ-invariant $f\colon X \to \mathbb{R}$ is constant μ-a.e.*
(4) *Any Φ-invariant $f \in L^p(X,\mu)$ is constant μ-a.e.*
(5) *Any nonnegative measurable Φ-invariant $f\colon X \to \mathbb{C}$ is constant μ-a.e.*

Remark 3.3.4. The following also characterize ergodicity of a probability measure μ:

- $f \in C(X) \Rightarrow f_\Phi = \text{const.}$ μ-a.e. (Theorem 3.3.10).
- $\mu \gg \nu \in \mathfrak{M}(\Phi) \Rightarrow \mu = \nu$ (Proposition 3.3.12).
- μ is an extreme point of $\mathfrak{M}(\Phi)$ (Proposition 3.3.26).
- μ is ergodic for the time-τ map for all but countably many τ (Theorem 3.3.13, Proposition 3.3.14).

Proof. These (and other) characterizations arise from the following implications: Φ is not ergodic $\Rightarrow$ there is an invariant characteristic function (namely, of an invariant set of intermediate measure) that is not constant a.e. $\Rightarrow$ there is a nonnegative bounded invariant measurable function that is not constant a.e. $\Rightarrow$ there is a nonconstant invariant $f \in L^p$ $\Rightarrow$ there is an invariant measurable $\mathbb{C}$-valued function that is not constant a.e. $\Rightarrow$ Φ is not ergodic (because either the real or the imaginary part is a Φ-invariant measurable function $f\colon X \to \mathbb{R}$ and not constant almost everywhere, so there exists an $a \in \mathbb{R}$ such that $\mu(f^{-1}((a,\infty))) \notin \{0,1\}$, and this set is invariant). □

Remark 3.3.5. With any of these characterizations and keeping in mind that invariance of the measure is not needed to define ergodicity, it is easy to see that ergodicity is preserved by time-change, orbit-equivalence (for these, this holds both for the given measure or the one induced from it by the time-change or the orbit-equivalence), measure-theoretic isomorphism, and passing to factors or suspensions.[13] (To which *invariant* measures these various modifications lead is an altogether different and harder question.)

[13] A suspension is ergodic if and only if the base transformation is.

As we mentioned earlier, ergodicity can be thought of as the measurable analog to transitivity. Similarly to the above, transitivity is preserved by time-change, conjugacy, orbit-equivalence, and passing to factors or suspensions.

Ergodicity can be (and has been) viewed as having no *measurable* constant of motion. This is different from not having constants of motion, which follows from transitivity. Ergodicity does not follows from transitivity, even if the measure is a smooth volume, and there are even minimal nonergodic systems, though such examples are not easy to construct [**181**, Corollary 12.6.4].

Thanks to Proposition 3.3.3, the proof of Proposition 1.6.15 yields:

Proposition 3.3.6. *A linear flow $x \mapsto x + tv$ on $\mathbb{T}^n$ is ergodic with respect to Lebesgue measure if and only if the components of v are rationally independent.*

Proposition 3.3.7. *Consider $A \in \mathrm{GL}(m,\mathbb{Z})$, that is, an $m \times m$-matrix with integer entries and determinant ± 1, and assume that no eigenvalue of A is a root of unity. Then the suspension of the toral automorphism $F_A \colon \mathbb{T}^m \to \mathbb{T}^m$ induced by A is ergodic.*

Remark 3.3.8 (Walters)**.** Note that the hypotheses hold for $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and indeed any hyperbolic automorphism, but also for

$$W \coloneqq \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}).$$

Its characteristic polynomial $q(\lambda) \coloneqq \lambda^4 - 8\lambda^3 + 6\lambda^2 - 8\lambda + 1$ is irreducible over $\mathbb{Q}$ because so is $q(\lambda - 1) = \lambda^4 - 12\lambda^3 + 36\lambda^2 - 48\lambda + 24$ by Eisenstein's Criterion (the prime $p = 3$ divides all coefficients other than the leading one, but $p^2 = 9$ does not). The eigenvalues $2 - \sqrt{3} \pm i\sqrt{4\sqrt{3} - 6}$ lie on the unit circle, and the remaining two are real and off the unit circle. Therefore q is not a factor of $\lambda^n - 1$ for any n; since q is irreducible, the eigenvalues on the unit circle are thus not roots of unity.

Proof. A bounded measurable invariant function f does not depend on t, hence is naturally written as an F_A-invariant function on $\mathbb{T}^m$.[14] Fourier expansion gives

$$\sum_{k\in\mathbb{Z}^m} f_k \exp(2\pi i \langle k, x\rangle) = f(x) \overset{\text{ae}}{=} f(F_A(x)) = \sum_{k\in\mathbb{Z}^m} f_k \exp(2\pi i \underbrace{\langle k, Ax\rangle}_{=\langle A^t k, x\rangle})$$

Uniqueness of the Fourier expansion implies that $f_k = f_{(A^t)^n k}$ for $n \in \mathbb{N}$. Since no root of unity is an eigenvalue of A and hence of the transpose A^t, $(A^t)^l - \mathrm{Id}$ is

[14] Equivalently, we could invoke Remark 3.3.5.

invertible for every $l \in \mathbb{Z} \smallsetminus \{0\}$. So, for $k \in \mathbb{Z}^m \smallsetminus \{0\}$ the $(A^t)^n k$ (for $n \in \mathbb{Z}$) are pairwise distinct, that is, there are infinitely many $l \in \mathbb{Z}^m$ with $f_l = f_k$. But $f \in L^1$ implies $|f_k| = |f_l| \xrightarrow[|l|\to\infty]{} 0$, so $f_k = 0$. This means that $f \overset{\text{ae}}{=} f_0$, a constant. □

Remark 3.3.9. Proposition 3.3.3 simply states in various function spaces that the subspace of Φ-invariant functions is the space of constant functions. Remark 3.2.21 lets us determine the space of Φ-invariant functions as the range of the projection $f \mapsto f_\Phi$, and doing so for a dense set of functions gives the needed information. If X is a metric space, then density of $C(X)$ in L^p gives:

Theorem 3.3.10. *If $f_\Phi = \text{const.}$ μ-a.e. for every $f \in C(X)$, then μ is ergodic.*

The converse (that the time average equals the space average—to which we alluded at the start of this section) is an important corollary of the Birkhoff Ergodic Theorem 3.2.16.

Corollary 3.3.11 (Strong Law of Large Numbers)**.** *If $\mu(X) = 1$, Φ is an ergodic μ-preserving flow, and $f \in L^1(X,\mu)$, then*

$$f_\Phi(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\varphi^t(x))\,dt = \int_X f\,d\mu$$

for every x outside of a set of measure zero.

Proof. f_Φ is Φ-invariant, so constant a.e. By (3.2.1) the constant is $\int f\,d\mu$. □

Thus, an invariant measure determines the asymptotic distribution of μ-almost every point if it is ergodic. A nonergodic invariant measure μ may also determine the asymptotic distribution of *some* orbits, but such orbits are always a set of μ-measure zero.

Considering densities gives:

Proposition 3.3.12. *$\mu \in \mathfrak{M}(\Phi)$ is ergodic if and only if $\mu \gg \nu \in \mathfrak{M}(\Phi) \Rightarrow \mu = \nu$.*

Proof. $\mu \gg \nu \in \mathfrak{M}(\Phi) \Leftrightarrow \nu = \rho\cdot\mu$, where $\rho \in L^1(\nu)$ is the (unique hence Φ-invariant) Radon–Nikodym derivative. This is always constant ($\equiv 1$) iff ν is ergodic. □

The argument in the proof of Theorem 1.6.24 also establishes

Theorem 3.3.13. *If a probability measure is ergodic for a flow then for all but countably many τ it is ergodic for the time-τ map.*

Conversely, we clearly have

Proposition 3.3.14. *If the time-t map φ^t is ergodic for some t, then Φ is ergodic.*

Example 3.3.15. The time-t maps of the circle flow (Example 1.1.6) are ergodic (with respect to Lebesgue measure) exactly for irrational t. This can be seen via the Fourier decomposition of an invariant function f:

$$\sum_{i\in\mathbb{Z}} a_i e^{ix} = f(x) = f(x+t) = \sum_{i\in\mathbb{Z}} a_i e^{it} e^{ix} \Rightarrow \forall\, i \in \mathbb{Z} \quad a_i = a_i e^{it},$$

so either $a_i = 0$ for all $i \neq 0$ (so $f \equiv$ const.) or $it \in \mathbb{Z}$ for some $i \in \mathbb{Z}$, hence $t \in \mathbb{Q}$.

In light of Proposition 3.3.14, ergodicity of the geodesic flow in Section 2.3 with respect to the *Liouville measure* defined by the invariant contact form in (2.2.5) follows from:

Theorem 3.3.16. *For $t \neq 0$ the time-t-map of the geodesic flow on a finite-volume factor of the Poincaré disk (Section 2.3) is ergodic.*

Proof. If $f \circ g^t = f \in L^2$ (for fixed t), then $f \circ h_+^s - f$ (and likewise for h_-^s):

$$\|\underbrace{f\circ g^{nt}}_{=f}\circ h_+^s - \underbrace{f\circ g^{nt}}_{=f}\| \overset{(2.2.3)}{=\!=} \|f\circ h_+^{se^{nt}}\circ g^{nt} - f\circ g^{nt}\| = \|f\circ h_+^{se^{nt}} - f\| \xrightarrow[nt\to-\infty]{\text{Remark 3.2.7}} 0.$$

g^t, h_+^s and h_-^s generate SL$(2,\mathbb{R})$, so f is PSL$(2,\mathbb{R})$-invariant, that is, for all $g \in$ PSL$(2,\mathbb{R})$ $f\circ g \overset{\text{ae}}{=} f$, or, by the Fubini Theorem, for a.e. x we have $f(g(x)) = f(x)$ for a.e. $g \in$ PSL$(2,\mathbb{R})$. Thus, there is an x_0 with $f(g(x_0)) = f(x_0)$ for all $g \in$ PSL$(2,\mathbb{R})$, so $f \overset{\text{ae}}{=}$ const. □

Corollary 3.3.17. *The geodesic flow on a finite-volume factor of the Poincaré disk (Section 2.3) is ergodic with respect to the Liouville measure.*

Remark 3.3.18. Indeed, Theorem 3.3.16 implies more than ergodicity by Proposition 3.4.40 (Theorem 3.4.43). Yet stronger ergodic properties are obtained below with refined arguments (Theorem 3.4.32 and later on Theorem 8.1.13).

The horocycle flow from Theorem 3.3.16 is ergodic as well, and this is tightly connected with ergodicity of the geodesic flow.

Proposition 3.3.19. *An L^2 function invariant under a time-τ-map of the horocycle flow h_- (Example 2.2.3) on a finite-volume factor of the Poincaré disk (Section 2.3) is invariant under the time-$2\ln 2$-map of the geodesic flow. (Likewise for h_+.)*

Proof. If $f \in L^2$ is invariant under h_-^τ, then (2.2.4) with $\epsilon := 1/n\tau$ and $s = 2$ gives

$$\begin{aligned}\|f \circ g^{2\ln 2} - f\|_2 &= \|\underbrace{f \circ h_-^{-2n\tau}}_{=f} h_+^{1/n\tau} h_-^{n\tau} h_+^{-2/n\tau} - f\|_2 \\ &= \|f \circ h_+^{1/n\tau} h_-^{n\tau} h_+^{-2/n\tau} - \underbrace{f \circ h_-^{n\tau}}_{=f} h_+^{-2/n\tau} + f \circ h_+^{-2/n\tau} - f\|_2 \\ &\le \|f \circ h_+^{1/n\tau} - f\|_2 + \|f \circ h_+^{-2/n\tau} - f\|_2 \xrightarrow[n\tau \to +\infty]{\text{Remark 3.2.7}} 0 \qquad \square\end{aligned}$$

Corollary 3.3.20. *Each time-τ-map of the horocycle flow $h_\pm$ (Example 2.2.3) on a finite-volume factor of the Poincaré disk is ergodic.*

Remark 3.3.21. In fact, the horocycle flow is uniquely ergodic and hence strictly ergodic (Definition 3.1.17, Exercise 6.7, Corollary 3.4.35): Birkhoff averages converge uniformly (Theorem 3.3.32). (And more—Theorem 3.4.44 and Section 9.6.)

One can strengthen the statement that functions invariant under an ergodic flow are constant via the following simple observation:

Proposition 3.3.22. *If Φ is a μ-preserving flow and $f\colon X \to \mathbb{R}$ satisfies $f \circ \varphi^t \le f$ ("subinvariance"), then f is Φ-invariant.*

Proof. By assumption $A_r := \{x \in X \mid f(x) \le r\} \supset \{x \in X \mid f(\varphi^t(x)) \le r\} = \varphi^{-t}(A_r)$, while $\mu(\varphi^{-t}(A_r)) = \mu(A_r)$. Thus $\varphi^{-t}(A_r) \overset{\text{ae}}{=} A_r$ for all $r \in \mathbb{R}$. □

This and Proposition 3.3.3 yield

Corollary 3.3.23. *If μ is an ergodic Φ-invariant probability measure, $f\colon X \to \mathbb{R}$, and $f \circ \varphi^t \le f$, then f is constant μ-a.e.*

Proposition 3.3.24. *A probability-preserving flow Φ is ergodic iff*

$$\int_X \frac{1}{T}\int_0^T f \circ \varphi^t \, dt \, g \xrightarrow[T\to\infty]{} \int_X f \int_X g \tag{3.3.1}$$

for all $f, g \in L^2$, that is, if and only if $\dfrac{1}{T}\displaystyle\int_0^T f \circ \varphi^t \xrightarrow[T\to\infty]{\text{weakly}}$ const. *for all $f \in L^2$.*

Remark 3.3.25. For $f = \chi_A$ and $g = \chi_B$, (3.3.1) becomes

$$\frac{1}{T}\int_0^T \mu(\varphi^{-t}(A) \cap B) - \mu(A)\mu(B) \, dt \xrightarrow[T\to\infty]{} 0. \tag{3.3.2}$$

Proof. If $f = f \circ \varphi^t$, then $f = \frac{1}{T}\int_0^T f \circ \varphi^t \, dt \xrightarrow[T\to\infty]{\text{weakly}}$ const., so Φ is ergodic.

If Φ is ergodic, then Corollary 3.3.11 and the Vitali Convergence Theorem give (3.3.1) for all $f, g \in L^2$. □

Corollary 3.3.11 leads to the question of whether every continuous flow has an ergodic invariant measure. This becomes clear with an alternate characterization.

Proposition 3.3.26. *Ergodic measures are the extreme points of $\mathfrak{M}(\Phi)$: $\mu \in \mathfrak{M}(\Phi)$ is not ergodic iff there exist $\mu_1 \neq \mu_2 \in \mathfrak{M}(\Phi)$ and $0 < \lambda < 1$ such that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$.*

Proof. If $\varphi^{-t}(A) = A$ and $0 < \mu(A) < 1$, then $\mu = \mu(A)\mu_1 + (1-\mu(A))\mu_2$, where

$$\mu_1(B) \coloneqq \mu_A(B) \coloneqq \mu(B \mid A) \coloneqq \frac{\mu(B \cap A)}{\mu(A)} \tag{3.3.3}$$

is the *density of B in A*, and $\mu_2 \coloneqq \mu_{X \smallsetminus A} \perp \mu_A$.

$\mu_i \ll \mu$ for $i = 1,2$, so the Radon–Nikodym Theorem gives Φ-invariant $\rho_i \in L^1(\mu)$ with $\int f\,d\mu_i \equiv \int \rho_i f\,d\mu$. By assumption $\lambda\rho_1 + (1-\lambda)\rho_2 = 1 = \int \rho_1\,d\mu = \int \rho_2\,d\mu$, so $\mu_1 \neq \mu_2 \Rightarrow \rho_1 \neq \rho_2 \Rightarrow \rho_1 \not\equiv$ const., and μ is not ergodic. □

Theorem 3.3.27. *Every continuous flow on a metrizable compact space has an ergodic invariant Borel probability measure.*

Proof. By the Krein–Milman Theorem[15] $\mathfrak{M}(\Phi) \neq \varnothing$ has extreme points. □

Corollary 3.3.28. *A uniquely ergodic flow (Definition 3.1.17) is ergodic.*

Proof. $\mathfrak{M}(\Phi) = \{\mu\}$, so μ is extreme, hence ergodic. □

By the Krylov–Bogolubov Theorem 3.1.15 every minimal set is the support of an invariant measure, so:

Theorem 3.3.29. *A uniquely ergodic action has only one minimal set; in particular a topologically transitive uniquely ergodic action is minimal.*

Remark 3.3.30. Exercise 3.2 provides a related inference for ("plain") ergodicity.

Example 3.3.31. The flow in Example 1.3.5 is uniquely ergodic, so unique ergodicity is compatible with trivial recurrence—but only for Dirac measures.

The circle flow (Example 1.1.6) is uniquely ergodic. To see this note that the interval $[0, \frac{1}{n})$ has measure $1/n$ because all translates by multiples of $1/n$ have the same measure, and they sum to 1. By additivity, the measure of intervals with rational endpoints is their length; this defines Lebesgue measure.

A more generally useful criterion is:

[15] A compact convex set in a locally convex topological vector space is the closed convex hull of its extreme points, that is, $C = \overline{\text{co}}\,\text{ex}(C)$; less than this will do, of course, when only the existence of an extreme point is needed: Define a *face* F of a compact convex set K by $x + (0,1)(x-y) \subset F \Rightarrow x + [0,1](x-y) \subset F$; K itself has this property. The Hausdorff maximal principle gives a minimal face, and the Hahn–Banach Extension Theorem shows that it must be a point, hence an extreme point.

Theorem 3.3.32. *A continuous flow is uniquely ergodic if the time-averages of continuous functions converge uniformly to a constant.*

PROOF. If f is a continuous function, then $\frac{1}{T}\int_0^T f\circ\varphi^t \xrightarrow[T\to\infty]{\text{uniformly}} f_0 \in \mathbb{R}$. If μ is a Φ-invariant Borel probability measure, then $\int f\,d\mu = \int f_0\,d\mu = f_0$, so μ is uniquely defined on $C(X)$ and hence unique. □

Conversely:

Proposition 3.3.33. *If Φ is uniquely ergodic, then for every continuous function f the time averages $\frac{1}{T}\int_0^T f(\varphi^t(x))\,dt$ converge uniformly (to a constant).*

PROOF. If f is a continuous function for which this fails, then there are $a<b$, sequences of points $x_k, y_k \in X$, $k=1,2,\dots$, and a sequence $n_k\to\infty$ such that

$$\frac{1}{n_k}\int_0^{n_k} f(\varphi^t(x_k))\,dt < a, \qquad \frac{1}{n_k}\int_0^{n_k} f(\varphi^t(y_k)) > b.$$

A diagonal argument gives a subsequence n_{k_j} such that for every $g\in C(X)$ both

$$J_1(g) = \lim_{j\to\infty}\frac{1}{n_{k_j}}\int_0^{n_{k_j}} g(\varphi^t(x_{k_j});dt \text{ and } J_2(g) = \lim_{j\to\infty}\frac{1}{n_{k_j}}\int_0^{n_{k_j}} g(\varphi^t(y_{k_j});dt$$

exist. J_1 and J_2 are bounded linear positive Φ-invariant functionals; thus $J_1(g) = \int g\,d\mu_1$, $J_2(g) = \int g\,d\mu_2$ for Φ-invariant probability measures μ_1 and μ_2. Since $J_1(f)\le a<b\le J_2(f)$ we have $\mu_1\ne\mu_2$ so Φ is not uniquely ergodic. □

Theorem 3.3.34. *If a flow is uniquely ergodic then the time-τ maps for all but countably many τ are uniquely ergodic.*

PROOF (Veech). By Theorem 3.3.13, the unique Φ-invariant Borel probability measure μ is φ^τ-ergodic for all but countably many τ. To show that such φ^τ is uniquely ergodic, let ν be any φ^τ-invariant Borel probability measure. Then $\int_0^\tau \varphi^s(\nu)\,ds$ is Φ-invariant, so $\mu = \int_0^\tau \varphi^s(\nu)\,ds$ by unique ergodicity of Φ. However, μ is ergodic for φ^τ and hence an extreme point of the set of invariant measures, so the convex combination $\mu = \int_0^\tau \varphi^s(\nu)\,ds$ must be trivial, that is, $\varphi^s(\nu)=\mu$. Since μ is φ^s-invariant, this implies $\nu=\mu$, which establishes the claim. □

Remark 3.3.35. The examples of uniquely ergodic flows (as well as the majority of those one encounters in the early pertinent literature) suggest that unique ergodicity (and hence minimality) is closely tied to simple dynamics. This turns out to be wrong in the strongest possible way. Not only are there natural examples of uniquely ergodic weakly mixing flows (Definition 3.4.1, Theorem 3.4.44, Corollary 3.4.35), but by the Jewett–Krieger Theorem, *every* ergodic flow is measure-theoretically isomorphic to a uniquely ergodic one [**99**, **167**].

Proposition 3.3.26 connects decomposability of a measure (by convex combination) and decomposability of the space. One can sharpen that connection:

Proposition 3.3.36. *Different invariant ergodic probability measures for the same flow are mutually singular.*

Proof. Call them $\nu, \mu = \mu^{\ll} + \mu^{\perp}$ with $\mu^{\ll} \ll \nu \perp \mu^{\perp}$ (invariantly by uniqueness of Lebesgue decomposition); since μ is ergodic, hence extreme, we have either $\mu = \mu^{\perp}$ or $\mu = \mu^{\ll} = \nu$ by ergodicity of ν and Proposition 3.3.12. □

Proposition 3.3.36 means that any convex combination of finitely many ergodic measures produces a corresponding nontrivial finite partition of the space.

Moreover, every invariant measure for a measure-preserving transformation can be decomposed into—possibly uncountably many—*ergodic components.*

Theorem 3.3.37 (Ergodic Decomposition [**90**, Theorem 15, p. 152])**.** *Every invariant Borel probability measure for a continuous flow Φ of a metrizable compact space X decomposes into an integral of ergodic invariant Borel probability measures in the following sense: There is a partition (modulo null sets) of X into Φ-invariant subsets X_α, $\alpha \in A$,called the* ergodic components *of* (Φ, μ)*, with A a Lebesgue space, and each X_α carrying a Φ-invariant ergodic measure μ_α such that $\int f\,d\mu = \iint f\,d\mu_\alpha\,d\alpha$ for any function f.*

Remark 3.3.38. In metric spaces there is explicit description of the ergodic decomposition: For each ergodic measure consider the G_δ set of typical points with respect to all continuous functions, for example, points for which the Birkhoff averages for each continuous function converge to the integral of this function with respect to the measure in question (Theorem 3.2.16). This is a null set for all other ergodic measures and these sets are evidently pairwise disjoint. They are called *ergodic sets.* This essential uniqueness of the ergodic decomposition shows that $\mathfrak{M}(\Phi)$ is essentially a simplex.

4. Mixing

As the circle flow (Example 1.1.6) illustrates, ergodicity is compatible with fairly uncomplicated behavior. Notions of *mixing* provide stronger stochastic properties, and the relation to ergodicity is most apparent by comparison with (3.3.2).

Unlike in the topological setting there are various notions of mixing used in the measure theoretic setting. We first review the various definitions and list them in order of increasing strength.

Definition 3.4.1 (Mixing)**.** A measure-preserving flow $\varphi^t\colon (X,\mu) \to (X,\mu)$ is said to be *weakly mixing* or to have *continuous spectrum* (Remark 3.7.15) if for any two measurable sets A, B

$$\frac{1}{T}\int_0^T |\mu(A\cap\varphi^{-t}(B))-\mu(A)\mu(B)|\,dt \xrightarrow[T\to\infty]{} 0. \tag{3.4.1}$$

It is said to be *mixing* if for any two measurable sets A, B

$$\mu(A\cap\varphi^{-t}(B)) \xrightarrow[t\to\infty]{} \mu(A)\cdot\mu(B). \tag{3.4.2}$$

It is said to be *mixing of order N* if for any $N+1$ measurable sets A_i and with $t_0 := 0$

$$\mu\Big(\bigcap_{i=0}^{N}\varphi^{-t_i}(A_i)\Big) \xrightarrow[t_i-t_{i-1}\to\infty]{} \prod_{i=0}^{N}\mu(A_i). \tag{3.4.3}$$

It is said to be *multiply mixing* or *mixing of all orders* if it is mixing of order N for all $N\in\mathbb{N}$.

The next notion was introduced by Kolmogorov (under a different name) and is thus often referred to as the Kolmogorov property, or *K-property* for a flow.[16]

Definition 3.4.2 (K-mixing)**.** It is said to be *K-mixing* or to be a *K-flow* if for any measurable sets $A_0,\dots,A_m$ we have

$$\lim_{t\to\infty}\sup_{B\in\mathscr{A}_t(A_1,\dots,A_m)} |\mu(A_0\cap B)-\mu(A_0)\mu(B)| = 0,$$

where $\mathscr{A}_t(A_1,\dots,A_m)$ is the σ-algebra generated by the $\varphi^s(A_i)$ for $s\ge t$ and $1\le i\le m$. Equivalently (Definition 11.1.16),

$$\lim_{N\to\infty}\sup\Big\{\big|\mu(A\cap B)-\mu(A)\mu(B)\big| \;\Big|\; B\in\mathscr{A}\big(\bigvee_{i=n}^{N}T^i\xi\big)\Big\} \xrightarrow[n\to\infty]{} 0$$

for every measurable A and finite partition ξ.

Definition 3.4.3. A flow Φ is *Bernoulli* or said to have the *Bernoulli property* if for all $t\ne 0$ the time-t map is measure-theoretically isomorphic to a Bernoulli shift (Example 3.1.26).

Remark 3.4.4. A few comments on these notions and the relations between them:

- The circle flow (Example 1.1.6) is not weakly mixing: for $A=B=[0,1/2)$ and $T\in\mathbb{N}$, the integral in (3.4.1) is $1/8\ne 0$.
- Mixing is mixing of order 1.
- One can restate (3.4.2) as $\mu_B(\varphi^{-t}(A)) \xrightarrow[n\to\infty]{} \mu(A)$, that is, asymptotically $\varphi^{-t}(A)$ and B are independent sets.
- Clearly mixing implies weak mixing, so weak mixing is a weakened (average) version of the statement about asymptotic independence.

[16] Kolmogorov used "K" as an abbreviation for "quasiregular," which begins with a "K" in Russian, but it was quickly interpreted as the first letter of "Kolmogorov"

- By taking A invariant and $B \coloneqq X \smallsetminus A$ (or by comparing (3.4.1) and (3.3.1)) we find that weak mixing implies ergodicity. Thus, ergodicity is the weakest statement of this sort.
- One sharp distinction between ergodicity and these mixing notions is that ergodicity is a purely "transverse" property, whereas "longitudinal" issues (such as time-changes) affect mixing. This step up from ergodicity comes into sharp relief in Proposition 3.4.9: suspensions are never even weakly mixing.
- To clarify the intent of (3.4.3), we rewrite it for $N = 2$ as

$$\mu(\varphi^{-t}(A) \cap \varphi^{-s-t}(B) \cap C) \xrightarrow[s\to\infty \text{ and } t\to\infty]{} \mu(A)\mu(B)\mu(C).$$

- K-mixing means that the evolution of A_0 is eventually independent of anything involving the other A_i; this implies mixing of all orders but does not follow from it.
- The most effective criterion for the K-property is existence of a σ-algebra of measurable sets $\mathscr{A}$ such that $\mathscr{A} \subset \varphi^t \mathscr{A}$ for $t > 0$, $\bigcup_{t \ge 0} \varphi^t \mathscr{A}$ is dense in the σ-algebra $\mathscr{B}$ of all measurable sets, and $\bigcap_{t \ge 0} \varphi^{-t} \mathscr{A} = \mathscr{N}$, the trivial subalgebra of null sets and their complements. Equivalently, one can show the existence of a generator (Definition 11.3.6) with trivial tail.
- The K-property is also equivalent to triviality of the *Pinsker algebra* from entropy theory. We will have an opportunity to show how this is useful (Remark 8.3.19).
- The Kolmogorov zero-one law for independent random variables can be used to show that the Bernoulli property implies K-mixing. There are, however, K-mixing flows that are not Bernoulli [**223**].
- Weak mixing does not imply mixing, and there is a significant gap between these. If one uses the weak topology on the space of measure-preserving flows on a given probability space, then the weakly mixing ones form a set of second Baire category, while mixing ones form a set of first category, that is, in this sense most flows are weakly mixing and few are (strongly) mixing.
- However, for hyperbolic flows these mixing notions are usually conflated, that is, once a hyperbolic flow is known to be weakly mixing, the various stronger mixing properties hold as well (Remark 8.3.19, Theorem 8.4.17, Remark 8.4.18)—because, for instance, there is a generator with trivial tail. Accordingly readers focused on hyperbolic flows might choose to skip, for instance, the discussion of weak mixing on the following pages (for example, Propositions 3.4.5, 3.4.18, 3.4.19, 3.4.38, 3.4.40), save for statements that have implications for mixing.

- We nonetheless explore these notions with some care because there are occasions when we can explain how specifically a stronger mixing property can be proved directly (for example, in Theorem 8.1.13), and because at times weak mixing can be obtained with no additional effort over establishing ergodicity (such as in Proposition 8.3.16 or Theorem 3.4.44). This relies on some of the characterizations of weak mixing that we develop here (notably, Proposition 3.4.19, Proposition 3.4.40).
- It turns out that up to constant rescaling of time, any 2 Bernoulli flows are measure-theoretically isomorphic (Ornstein Isomorphism Theorem).

We mention the next result without proof as it provides a good interpretation of weak mixing as a mixing condition away from a "negligible" set of times:

Proposition 3.4.5. *A measure-preserving flow is weakly mixing if and only if for any two measurable sets A, B*

$$\text{there is an } E \subset \mathbb{R}^+ \text{ of density 0 such that } \lim_{E \not\ni t \to \infty} \mu(\varphi^{-t}(A) \cap B) = \mu(A) \cdot \mu(B). \tag{3.4.4}$$

Here we used the following notion and fact:

Definition 3.4.6. If $\lambda(E \cap [0,s]) - ds \xrightarrow[s\to\infty]{} 0$, then we say that $E \subset \mathbb{R}^+$ has *density d*. In particular, it has density 0 if $\lambda(E \cap [0,s]) \xrightarrow[s\to\infty]{} 0$.

For later use we note:

Proposition 3.4.7. *If f is bounded, then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |f| = 0 \quad \textit{if and only if} \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T f^2 = 0.$$

PROOF. Invoke Lemma 3.4.8: $\lim_{E \not\ni t \to \infty} f(t) = 0$ iff $\lim_{E \not\ni t \to \infty} f(t)^2 = 0$. □

Lemma 3.4.8. *If $f : \mathbb{R}^+ \to \mathbb{R}$ is bounded, then $\lim_{T\to\infty} \frac{1}{T} \int_0^T |f| = 0$ iff there is an $E \subset \mathbb{R}^+$ of density 0 such that $\lim_{E \not\ni t \to \infty} f(t) = 0$, that is, $0 = \lim_{t\to\infty} \begin{cases} f(t) & \text{if } t \notin E \\ 0 & \text{if } t \in E. \end{cases}$*

PROOF. "If": For $M := \|f\|_\infty$ and $\epsilon > 0$ there is an $S \in \mathbb{R}^+$ such that for $T \geq S$ we have

- $\displaystyle\int_{[0,T] \smallsetminus E} |f(t)|\, dt < \frac{\epsilon}{M+1}$ and
- $\displaystyle d_T(E) := \frac{1}{T} \lambda(E \cap [0,T]) < \frac{\epsilon}{M+1}$,

so $\displaystyle \frac{1}{T} \int_0^T |f|\, dt = \frac{1}{T} \Big(\int_{[0,T] \cap E} |f(t)|\, dt + \int_{[0,T] \smallsetminus E} |f(t)|\, dt \Big) < M d_T(E) + \frac{\epsilon}{M+1} < \epsilon.$

"Only if": Since $E_k := \big\{t \in \mathbb{R}^+ \,\big|\, |f(t)| \ge 1/k\big\} \subset E_{k+1}$ satisfies

$$d_T(E_k) = \frac{1}{T}\int_0^T \chi_{E_k}\,dt \le \frac{k}{T}\int_0^G |f(t)|\,dt \xrightarrow[T\to\infty]{} 0,$$

recursively take $l_k \ge l_{k-1}$ such that $d_T(E_k) < 1/k$ for $T \ge l_k$. Let $E := \bigcup_{k\in\mathbb{N}} E_k \cap [l_{k-1}, l_k)$ and $\epsilon > 0$. If $k > 1/\epsilon$ and $l_{k-1} < T \notin E$, then $T \notin E_k$, and $|f(T)| < 1/k < \epsilon$. To show $d_n(E) \to 0$ take $K > 2/\epsilon$, $T \ge l_K$ and $k \ge K$ such that $l_k \le T < l_{k+1}$. Since

$$E \cap [0,T) = (E \cap [0,l_k)) \cup (E \cap [l_k, T)) \subset (\underbrace{E_k \cap [0,l_k)}_{\subset E_k \cap [0,T)}) \cup (\underbrace{E_{k+1} \cap [l_k, T)}_{\subset E_{k+1} \cap [0,T)}),$$

we get $d_T(E) \le \dfrac{1}{T}\Big(T d_T(E_k) + T d_T(E_{k+1})\Big) < \dfrac{1}{k} + \dfrac{1}{k+1} < \dfrac{2}{k} < \epsilon$. □

Unlike ergodicity, mixing properties are sensitive to timing. This is starkly illustrated by the contrast between Remark 3.3.5 and the next result.

Proposition 3.4.9. *Suspensions are not weakly mixing.*

PROOF. For $A = B = X \times [0, 1/2)$ and $T \in \mathbb{N}$, the integral in (3.4.1) is $1/8 \neq 0$. □

Corollary 3.4.10. *Linear flows on tori are not weakly mixing.*

PROOF. They are suspensions of translations. □

Clearly, mixing and weak mixing are invariants of measure-theoretic isomorphism. The next result shows a stronger statement; this is an interesting result, but one we will not use later on and so state without proof.

Proposition 3.4.11. *If a flow is mixing (or weakly mixing), then so is any factor (Definition 3.1.1).*

This is another reason suspensions are not weakly mixing: they have circle flows (Example 1.1.6) as a factor (Proposition 1.3.3).

The next result relates the measure theoretic and topological notions of mixing.

Proposition 3.4.12. *If μ is a mixing invariant measure for a continuous flow Φ, then $\Phi\restriction_{\operatorname{supp}\mu}$ is topologically mixing.*

PROOF. If $A, B \subset \operatorname{supp}\mu$ are open, then $\mu(\varphi^{-t}(A) \cap B) > 0$ for all large t. □

Now we prove a criterion for mixing that allows us to use particularly convenient sets when checking mixing for specific dynamical systems.

Definition 3.4.13. A collection $\mathscr{C} \subset \mathscr{S}$ in a measure space $(X, \mathscr{S}, \mu)$ is said to be *sufficient* if finite disjoint unions of elements of $\mathscr{C}$ form a dense collection with respect to the symmetric-difference metric

$$d(A,B) := d_\mu(A,B) := \mu(A \triangle B) \in (0,\infty]. \tag{3.4.5}$$

Remark 3.4.14. This is closely related to the Rokhlin metric from (11.2.9), see Proposition 11.2.20 and Remark 11.2.21.

Proposition 3.4.15. *Suppose $\mathscr{C}$ is a sufficient collection of sets. Then*

(1) Φ is mixing if (3.4.2) *holds for any $A, B \in \mathscr{C}$,*
(2) Φ is weakly mixing if (3.4.1) *or* (3.4.4) *holds for any $A, B \in \mathscr{C}$,*
(3) Φ is ergodic if (3.3.1) *holds for any $A, B \in \mathscr{C}$,*
(4) Φ is mixing of order N if (3.4.3) *holds for any $A_i \in \mathscr{C}$.*

Proof. We prove (1) using Proposition 3.3.24, the other parts have like proofs. Let

$$A_1, \dots, A_k,\ B_1, \dots, B_l \in \mathscr{C},\ A_i \cap A_{i'} = \varnothing \text{ for } i \neq i',\ B_j \cap B_{j'} = \varnothing \text{ for } j \neq j'$$

and $A \coloneqq \bigcup_{i=1}^k A_i$, $B \coloneqq \bigcup_{j=1}^l B_j$. Then $\mu(A) = \sum_{i=1}^k \mu(A_i)$, $\mu(B) = \sum_{j=1}^l \mu(B_j)$, and by assumption

$$\mu(\varphi^{-t}(A) \cap B) = \sum_{i=1}^k \sum_{j=1}^l \mu(\varphi^{-t}(A_i) \cap B_j) \xrightarrow[t\to\infty]{} \sum_{i=1}^k \sum_{j=1}^l \mu(A_i) \cdot \mu(B_j) = \mu(A) \cdot \mu(B).$$

Thus (3.4.2) holds for any elements of the dense collection $\mathfrak{U}$ formed by finite disjoint unions of elements of $\mathscr{C}$. Now let A, B be arbitrary measurable sets. Find $A', B' \in \mathfrak{U}$ such that $\mu(A \triangle A') < \epsilon/4$, $\mu(B \triangle B') < \epsilon/4$. Then by the triangle inequality

$$\begin{aligned} |\mu(\varphi^{-t}(A) \cap B) - \mu(A)\mu(B)| &\le \mu(\varphi^{-t}(A \triangle A') \cap B) + \mu(\varphi^{-t}(A') \cap (B \triangle B')) \\ &\quad + |\mu(\varphi^{-t}(A') \cap B') - \mu(A')\mu(B')| \\ &\quad + \mu(A) \cdot \mu(B \triangle B') + \mu(B') \cdot \mu(A \triangle A') \\ &\le |\mu(\varphi^{-t}(A') \cap B') - \mu(A') \cdot \mu(B')| + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ can be chosen arbitrarily small, this implies (3.4.2). □

It is not only with respect to the sets in question, but also in the conclusion that a suitable approximation is good enough.

Proposition 3.4.16. *Let Φ be a continuous flow on a compact metric space X and μ a Φ-invariant Borel probability measure for which there exist constants $c, C > 0$ such that*

$$c\mu(P)\mu(Q) \le \varliminf_{t\to\infty} \mu(P \cap \varphi^{-t}(Q)) \le \varlimsup_{t\to\infty} \mu(P \cap \varphi^{-t}(Q)) \le C\mu(P)\mu(Q) \tag{3.4.6}$$

for all Borel sets $P, Q \subset X$. Then μ is mixing.

Proof. The left inequality in (3.4.6) implies that $\Phi \times \Phi$ is ergodic with respect to $\mu \times \mu$: If $A, B, C, D \subset X$ are Borel sets, then

$$\varliminf_{t\to\infty} \underbrace{(\mu \times \mu)((\varphi \times \varphi)^t(A \times C) \cap (B \times D))}_{=\mu(\varphi^t(A)\cap B)\cdot\mu(\varphi^t(C)\cap D)} \ge c^2 \underbrace{\mu(A) \cdot \mu(B)}_{=(\mu\times\mu)(A\times B)} \cdot \underbrace{\mu(C) \cdot \mu(D)}_{=(\mu\times\mu)(C\times D)}.$$

The same inequality holds if we replace $A\times C$ and $B\times D$ by finite disjoint unions of product sets, and such sets approximate every measurable $P,Q\subset X\times X$. Thus,

$$\varliminf_{t\to\infty}(\mu\times\mu)((\varphi\times\varphi)^t(P)\cap Q)\geq c^2(\mu\times\mu)(P)\cdot(\mu\times\mu)(Q),$$

and $\Phi\times\Phi$ is ergodic with respect to $\mu\times\mu$. (So Φ is weakly mixing by Proposition 3.4.19 below.)

Now let ν be the diagonal measure in $X\times X$ given by $\nu(E)=\mu(\pi_1(E\cap\Delta))$, where $\Delta=\{(x,x)\mid x\in X\}$ and $\pi_1\colon X\times X\to X$ is the projection to the first coordinate. The measure ν as well as its shift ν_t under the map $\varphi^t\times\mathrm{Id}$ are $(\Phi\times\Phi)$-invariant. Explicitly, $\nu_t(A\times B)=\mu(\varphi^t(A)\cap B)$. By the right inequality in (3.4.6) we have

$$\varlimsup_{t\to\infty}\nu_t(A\times B)=\varlimsup_{t\to\infty}\mu(\varphi^t(A)\cap B)<C\mu(A)\cdot\mu(B)=C(\mu\times\mu)(A\times B). \tag{3.4.7}$$

Let η be any weak limit point of the sequence ν_t. If $A,B\subset X$ are closed sets then $\eta(A\times B)\leq C(\mu\times\mu)(A\times B)$ by (3.4.7). Approximation by disjoint unions of products of closed sets gives $\eta(P)<C(\mu\times\mu)(P)$ for any Borel set $P\subset X\times X$, so $\eta\ll\mu\times\mu$, and $\eta=\mu\times\mu$ by Proposition 3.3.12 since η is $(\Phi\times\Phi)$-invariant and $\mu\times\mu$ is ergodic. For closed A,B with $\mu(\partial A)=\mu(\partial B)=0$ we have

$$\mu(\varphi^t(A)\cap B)=\nu_t(A\times B)\xrightarrow[t\to\infty]{}(\mu\times\mu)(A\times B)=\mu(A)\cdot\mu(B).$$

Since the collection of all such sets is sufficient, Φ is mixing with respect to μ by Proposition 3.4.15. □

The notions of mixing and weak mixing are remarkably well-behaved when passing to products:

Proposition 3.4.17. *A measure-preserving flow Φ on (X,μ) is mixing (weakly mixing) if and only if $\Phi\times\Phi$ is.*

Proof. If $\Phi\times\Phi$ is weakly mixing and $A,B\subset X$ then by Proposition 3.4.5 there is a set $E\subset\mathbb{N}$ of density 0 such that

$$\underbrace{\mu(\varphi^{-t}(A)\cap B)}_{=(\mu\times\mu)\left((\varphi\times\varphi)^{-t}(A\times X)\cap(B\times X)\right)}\xrightarrow[E\not\ni t\to\infty]{}(\mu\times\mu)(A\times X)\cdot(\mu\times\mu)(B\times X)=\mu(A)\mu(B),$$

so Φ is weakly mixing. Taking $E=\varnothing$ proves that $\Phi\times\Phi$ mixing $\Rightarrow$ Φ mixing.

Suppose now that Φ is weakly mixing. Then for measurable $A_1,A_2,B_1,B_2\subset X$ there exist sets $E_1,E_2\subset\mathbb{R}^+$ of density 0 such that

$$\lim_{E_i\not\ni t\to\infty}\mu(\varphi^{-t}(A_i)\cap B_i)=\mu(A_i)\cdot\mu(B_i)$$

for $i = 1,2$. Taking $E \coloneqq E_1 \cup E_2$ we find that

$$\underbrace{(\mu\times\mu)\Big(\overbrace{(\varphi\times\varphi)^{-t}(A_1\times A_2)\cap(B_1\times B_2)}^{=(\varphi^{-t}(A_1)\cap B_1)\times(\varphi^{-t}(A_2)\cap B_2)}\Big)}_{=\mu(\varphi^{-t}(A_1)\cap B_1)\mu(\varphi^{-t}(A_2)\cap B_2)} \xrightarrow[E\not\ni t\to\infty]{} \underbrace{\mu(A_1)\mu(B_1)\mu(A_2)\mu(B_2)}_{=(\mu\times\mu)(A_1\times A_2)(\mu\times\mu)(B_1\times B_2)}.$$

Since sets of the form $A\times B$ form a sufficient collection, Proposition 3.4.15.2 implies that $\Phi\times\Phi$ is weakly mixing. Taking $E_1 = E_2 = \varnothing$ gives Φ mixing $\Rightarrow \Phi\times\Phi$ mixing. □

One of the implications in Proposition 3.4.17 is easy to strengthen:

Proposition 3.4.18. *If $\Phi\colon X\to X$ is a measure-preserving flow and $\Phi\times\Phi$ is ergodic, then Φ is weakly mixing.*

Proof. Take A, B measurable and suppose $\Phi\times\Phi$ is ergodic. Then

$$\underbrace{\frac{1}{T}\int_0^T \underbrace{\mu(\varphi^{-t}(A)\cap B)}}_{=(\mu\times\mu)((\varphi\times\varphi)^{-t}(A\times X)\cap(B\times X))} \xrightarrow[n\to\infty]{} (\mu\times\mu)(A\times X)(\mu\times\mu)(B\times X) = \mu(A)\mu(B)$$

and

$$\underbrace{\frac{1}{T}\int_0^T \underbrace{\mu(\varphi^{-t}(A)\cap B)^2}}_{=(\mu\times\mu)((\varphi\times\varphi)^{-t}(A\times A)\cap(B\times B))} \xrightarrow[n\to\infty]{} (\mu\times\mu)(A\times A)(\mu\times\mu)(B\times B) = \mu(A)^2\mu(B)^2.$$

by Proposition 3.3.24. Thus,

$$\frac{1}{T}\int_0^T \underbrace{\Big(\mu(\varphi^{-t}(A)\cap B)-\mu(A)\mu(B)\Big)^2}_{=\mu(\varphi^{-t}(A)\cap B)^2-2\mu(\varphi^{-t}(A)\cap B)\mu(A)\mu(B)+\mu(A)^2\mu(B)^2} \xrightarrow[T\to\infty]{} 0.$$

This implies the claim by Proposition 3.3.24 and Proposition 3.4.7. □

In fact, we have:

Proposition 3.4.19. *The following are equivalent:*

(1) Φ is weakly mixing,
(2) $\Phi\times\Phi$ is weakly mixing,
(3) $\Phi\times\Phi$ is ergodic,
(4) $\Phi\times\Psi$ is ergodic whenever Ψ is.

Proof. The first 3 properties are equivalent by Propositions 3.4.17 and 3.4.18. (4)⇒(3): If $\Phi\times\Psi$ is ergodic whenever Ψ is, then for the constant flow Ψ on a single point, this implies ergodicity of Φ, so with $\Psi=\Phi$ we find that $\Phi\times\Phi$ is ergodic.

To show (1)⇒(4), we use Proposition 3.4.15.

$$\Big|\frac{1}{T}\int_0^T \overbrace{(\mu\times\nu)((\varphi\times\psi)^{-t}(A_1\times A_2)\cap B_1\times B_2)}^{=\overbrace{\mu(\varphi^{-t}(A_1)\cap B_1)}^{=:x_t}\overbrace{\nu(\psi^{-t}(A_2)\cap B_2)}^{=:y_t}} - \overbrace{(\mu\times\nu)(A_1\times A_2)(\mu\times\nu)(B_1\times B_2)}^{=\overbrace{\mu(A_1)\mu(B_1)}^{=:x}\overbrace{\nu(A_2)\nu(B_2)}^{=:y}}\Big|$$
$$=\frac{1}{T}\Big|\int_0^T x_t y_t - xy\,dt\Big| \le \underbrace{\frac{1}{T}\int_0^T |x_t - x|\cdot y_t\,dt}_{\le(\sup_t y_t)\frac{1}{T}\int_0^T |x_t-x|\,dt\to 0\ (\Phi\text{ weakly mixing})} + x\cdot\underbrace{\Big|\frac{1}{T}\int_0^T y_t - y\,dt\Big|}_{\to 0\ (\Psi\text{ ergodic})} \xrightarrow[T\to\infty]{} 0. \qquad\square$$

Remark 3.4.20. (4) motivates saying that a flow Φ is *mildly mixing* if $\Phi\times\Psi$ is ergodic whenever Ψ has a *possibly infinite* ergodic invariant measure.

Corollary 3.4.21. *If Φ is weakly mixing then so is $\Phi\times\Phi\times\cdots\times\Phi$ for any finite number of products.*

Proof. Recursively taking $\Psi=\Phi\times\Phi$, $\Psi=\Phi\times\Phi\times\Phi$ and so on in Proposition 3.4.19(4) shows that if Φ is mixing, then $\Phi\times\Phi\times\cdots\times\Phi$ is ergodic. Using (3)⇒(1) with $2n$ copies of Φ then shows that the product of n copies is weakly mixing. □

It may be interesting to give an evidently equivalent formulation:

Corollary 3.4.22. *If $\Phi\times\Phi$ is ergodic then so is $\Phi\times\Phi\times\cdots\times\Phi$ for any finite number of products.*

Just as ergodicity can be expressed in terms of functions rather than sets, so can the various notions of mixing. In probabilistic terms, sets are events and functions are random variables. The preceding notions of ergodicity and mixing involve various forms of eventual independence of events, and they can be recast in terms of eventual independence of random variables using the *covariance* of L^2-functions.

Definition 3.4.23. The *covariance* of $f,g\in L^2$ is defined as

$$\operatorname{cov}(f,g) := \underbrace{\langle f-\langle f,1\rangle, g-\langle g,1\rangle\rangle}_{=\int\left(f-\int f\right)\overline{\left(g-\int g\right)}} = \underbrace{\langle f,g\rangle-\langle f,1\rangle\langle 1,g\rangle}_{\int f\bar g-\int f\int\bar g}.$$

That is, we project both functions to the orthocomplement $1^\perp\subset L^2$ of the constant functions by subtracting their average (to focus on their variation) and then take the inner product.

Remark 3.4.24. Like the inner product, the covariance is sesquilinear (linear in the first entry and and antilinear in the second) and invariant under isometric operators (that is, $\langle U\cdot, U\cdot\rangle = \langle\cdot,\cdot\rangle \Rightarrow \operatorname{cov}(U\cdot, U\cdot)=\operatorname{cov}$). If either of the functions is constant, then the covariance is zero, so it is unaffected by the addition of constants to either function. For many statements about covariance, this allows us to assume without loss of generality that the functions in question have zero average, that is, are in $1^\perp$. Indeed, "*polarization*"[17] allows us to consider the same function in both entries:

$$\operatorname{cov}(f,g) = \frac{1}{4}[\operatorname{cov}(f+g, f+g) - \operatorname{cov}(f-g, f-g)].$$

The covariance also satisfies the Cauchy–Schwarz inequality: $|\operatorname{cov}(f,g)| \le \|f\|\|g\|$.

Proposition 3.4.25. *If $\Xi \subset L^2$ is a complete system, that is, $\overline{\operatorname{span}(\Xi)} = L^2$, then*

- Φ *is ergodic if and only if* $\frac{1}{T}\int_0^T \operatorname{cov}(U_\Phi^t(f), g) \xrightarrow[T\to\infty]{} 0$ *for all $f, g \in \Xi$,*
- Φ *is weakly mixing if and only if*

$$\frac{1}{T}\int_0^T \left|\operatorname{cov}(U_\Phi^t(f), g)\right| \xrightarrow[T\to\infty]{} 0 \tag{3.4.8}$$

  for all $f, g \in \Xi$,
- Φ *is weakly mixing if and only if for all $f, g \in \Xi$, there exists an $E \subset \mathbb{R}^+$ of density 0 (Definition 3.4.6) such that*

$$\operatorname{cov}(U_\Phi^t(f), g) \xrightarrow[E\not\ni t\to\infty]{} 0,$$

- Φ *is mixing if and only if*

$$\operatorname{cov}(U_\Phi^t(f), g) \xrightarrow[t\to\infty]{} 0 \tag{3.4.9}$$

  for all $f, g \in \Xi$.
- Φ *is* mixing of order N *if*

$$\int \prod_{i=0}^N f_i \circ \varphi^{t_i}\, d\mu \xrightarrow[t_i - t_{i-1}\to\infty]{} \prod_{i=0}^N \int f_i\, d\mu$$

  for any $\{f_0, \dots, f_N\} \subset \Xi$.

Proof. To see how to pass from a complete system to L^2 note first that sesquilinearity of covariance means that checking any of these statements for all $f, g \in \Xi$ implies the same for all $f, g \in \operatorname{span}(\Xi)$. Now take arbitrary $f, g \in L^2$ and $f', g' \in \operatorname{span}(\Phi)$ such that $\|g - g'\| < \epsilon/2\|f\|$ and $\|f - f'\| < \epsilon/2\|g'\|$. Then

$$\begin{aligned}|\operatorname{cov}(U_\Phi^t(f), g)| &= |\operatorname{cov}(U_\Phi^t(f), g - g') + \operatorname{cov}(U_\Phi^t(f) - U_\Phi^t(f'), g') + \operatorname{cov}(U_\Phi^t(f'), g')| \\ &\le |\operatorname{cov}(U_\Phi^t(f'), g')| + \epsilon.\end{aligned}$$

Now, for each of these statements, knowing it for all $f, g \in L^2$ implies the corresponding mixing property by taking $f = \chi_A$ and $g = \chi_B$ for measurable sets A, B.

[17] $\|u+v\|^2 - \|u-v\|^2 = 4\langle u, v\rangle$

To see the converse note that characteristic functions of measurable sets (or linear combinations of those of a sufficient collection) form a complete system in L^2 for which the statement about covariance boils down to the respective mixing property. □

Remark 3.4.26. Note that we have in particular reproved Proposition 3.4.15.

Remark 3.4.27. The characterization of mixing in (3.4.9) invites the question of how fast the covariance goes to 0 with t. This depends on the 2 functions involved, and the convergence can be arbitrarily slow. However, for a smooth dynamical system and functions selected from a suitable class—smooth, Lipschitz or Hölder continuous (Definition 1.8.4), for instance—hyperbolicity can produce exponential convergence to 0. This is known as exponential *decay of correlations*. Since the classes of functions needed for this are not preserved by measure-theoretic isomorphism, neither is this property, so it is a meaningful property of a smooth dynamical system rather than of its measure-theoretic isomorphism class. Note that this is sensitive to time-changes; for instance, suspensions are not mixing and hence have no correlation decay at all. We elaborate on this in Section 8.5.

Parsing Definition 3.4.23, this characterization of mixing can be restated thus:

Proposition 3.4.28. *If $\Xi \subset L^2$ is a complete system, that is,* span(Ξ) *is dense in L^2, then Φ is mixing if and only if $U^t_\Phi(f) \xrightarrow[t\to\infty]{\text{weakly}} \int f$ for all $f \in \Xi$.*

Likewise we have:

Proposition 3.4.29. *An Φ-invariant probability measure μ is N-mixing if and only if given any $f_i \in L^2(\mu)$, any weak accumulation point $\psi_n \xrightarrow{\text{weakly}} \psi$ of $\prod_{i=1}^N f_i \circ \varphi^{t_i}$ (with $t_i - t_{i-1} \xrightarrow[t\to\infty]{} \infty$) is constant.*

Proof. "Only if" is clear. To get "if", we recursively verify that the constant is correct.

First, take $f_i \equiv 1$ for $i \neq 1$, including taking the test function $f_0 \equiv 1$. Then the weak-accumulation statement becomes

$$\int f_1 = \int f_1 \circ \varphi^t \cdot 1 \to \text{const.} \int 1 = \text{const.},$$

so the constant is $\int f_1$ for each such subsequence, and thus $f_1 \circ \varphi^t \xrightarrow{\text{weakly}} \int f_1$. By symmetry, $f_i \circ \varphi^t \xrightarrow{\text{weakly}} \int f_i$ for all i. In particular, $f_2 \circ \varphi^{t_2 - t_1} \xrightarrow[t_2 - t_1 \to\infty]{\text{weakly}} \int f_2$. Supposing next that $f_i \equiv 1$ for $i \notin \{1, 2\}$, this implies

$$\int f_1 \circ \varphi^{t_1}\, f_2 \circ \varphi^{t_2} \cdot 1 = \int f_2 \circ \varphi^{t_2 - t_1}\, f_1 \xrightarrow[t_2 - t_1 \to\infty]{} \int \Big(\int f_2\Big) f_1 = \int f_1 \int f_2,$$

so $f_2 \circ \varphi^{t_2}\, f_1 \circ \varphi^{t_1} \xrightarrow[t_2 - t_1 \to\infty]{\text{weakly}} \int f_1 \int f_2$ with like statements for any pair of the f_i. This can be continued. □

Remark 3.4.24 suggests

Proposition 3.4.30. *In each of the statements in Proposition 3.4.25 one can replace* $\operatorname{cov}(U_\Phi^t(f),g)$ *by* $\operatorname{cov}(U_\Phi^t(f),f)$ *or by* $\langle U_\Phi^t(f),f\rangle$ *if* $f\perp 1$. *For instance,* Φ *is mixing if and only if*

$$\operatorname{cov}(U_\Phi^t(f),f)\xrightarrow[t\to\infty]{}0$$

for all f *in a complete set* L^2, *which happens if and only if*

$$\langle U_\Phi^t(f),f\rangle\xrightarrow[t\to\infty]{}0$$

for all f *in a complete set for* $1^\perp$.

Proof. While Remark 3.4.24 applies if the hypothesis is known for all $f\in L^2$, the step from a complete system to L^2 requires attention because $f\mapsto\operatorname{cov}(U_\Phi^t(f),f)$ is not linear. The following lemma covers the mixing case, and the others are analogous. The last statement follows directly from Remark 3.4.24. □

Lemma 3.4.31. *If* $\operatorname{cov}(U_\Phi^t(f),f)\xrightarrow[t\to\infty]{}0$, *then* $\operatorname{cov}(U_\Phi^t(f),g)\xrightarrow[t\to\infty]{}0$ *for all* $g\in L^2$.

Proof. $M_f:=\{g\in L^2\ \mid\ \operatorname{cov}(U_\Phi^t(f),g)\xrightarrow[t\to\infty]{}0\}$ is a closed subspace of L^2 that contains 1 and f, and $U_\Phi M_f\subset M_f$: If $g\in M_f$ and $t\in\mathbb{R}^+$, then, since U_Φ is an isometry, $\langle U_\Phi^t(f),U_\Phi(g)\rangle=\langle U_\Phi(U_\Phi^{t-1}(f)),U_\Phi(g)\rangle=\langle U_\Phi^{t-1}(f),g\rangle\to 0$. Thus,

$$M_f\supset m_f:=\bigcap\big\{E\subset L^2\text{ closed}\ \mid\ 1,f\in E,\ U_\Phi(E)\subset E\big\}\supset U_\Phi(m_f).$$

If $g\in m_f^\perp$, then $\langle 1,g\rangle=0$ and $\langle U_\Phi^t(f),g\rangle=0$ for all t since $U_\Phi^t(f)\in U_\Phi^t(m_f)\subset m_f$, so $g\in M_f$. Thus, $L^2=m_f\oplus m_f^\perp\subset M_f\oplus M_f=M_f$. □

As an application of Proposition 3.4.28 and (2.2.3) we show:

Theorem 3.4.32. *The geodesic flow on a finite-volume factor of the Poincaré disk (Section 2.3) is mixing.*

Lemma 3.4.33 (Mautner phenomenon)**.** $f\circ g^{t_i}\xrightarrow[t_i\to\infty]{\text{weakly}}f_0\in L^2\Rightarrow f_0\circ h_-^s=f_0$.

Proof. $\|\underbrace{f\circ g^{t_i}h_-^s-fg^{t_i}}_{\xrightarrow[i\to\infty]{\text{weakly}}f_0\circ h_-^s-f_0}\|\overset{(2.2.3)}{=\!=}\|f\circ h_-^{se^{-t}}g^{t_i}-f\circ g^{t_i}\|=\|f\circ h_-^{se^{-t}}-f\|\xrightarrow[t\to\infty]{\text{Remark 3.2.7}}0.$ □

Proof of Theorem 3.4.32. If $f\circ g^{t_i}\xrightarrow[t_i\to\infty]{\text{weakly}}f_0\in L^2$, then f_0 is h_--invariant, hence by Corollary 3.3.20 constant, so $f_0=\int f$. Thus, $f\circ g^t\xrightarrow[t\to\infty]{\text{weakly}}\int f$, which gives the claim by Proposition 3.4.28. □

The mixing property has applications in this case. First, we obtain:

Proposition 3.4.34. *If $f\colon S\Sigma \to \mathbb{R}$ is continuous on the unit tangent bundle of a compact factor Σ of the Poincaré disk (Section 2.3), then for $x \in S\Sigma$*

$$\frac{1}{2}\int_{-1}^{1} f \circ g^t(h_-^s(x))\,ds \xrightarrow[t\to+\infty]{\textit{uniformly}} \int_{S\Sigma} f.$$

Proof outline. To apply mixing thicken the arc $h_-^{[-1,1]}(x)$ to $U := B^{cu} \times h_-^{[-1,1]}(x)$ of positive volume (using local product coordinates); here $B^{cu} \sim h_+^{[-\epsilon,\epsilon]}(x) \times g^{[-\epsilon,\epsilon]}(x)$. Then $\operatorname{area}(B^{cu}) \cdot \dfrac{1}{2}\displaystyle\int_{-1}^{1} f \circ g^t(h_-^s(x))\,ds \approx \int f \circ g^{-t}(y)\chi_U(y) \xrightarrow[t\to+\infty]{\text{mixing}} \operatorname{vol}(U)\int f$. It is essential here that g^{-t} does not expand in the B^{cu}-direction; this ensures uniformity of the approximation in t and equicontinuity of $x \mapsto \frac{1}{2}\int_{-1}^{1} f \circ g^t(h_-^s(x))\,ds$ for $t \ge 0$. Since $\operatorname{vol}(U) \approx \operatorname{area}(B^{cu})$, the claim follows by letting $\epsilon \to 0$. □

Corollary 3.4.35. *The horocycle flow on a compact factor of the Poincaré disk is uniquely ergodic.*

Remark 3.4.36. Compactness is not needed [**265**].

Proof outline. By Theorem 3.3.32 we check uniform convergence of Birkhoff averages to a constant using (2.2.3) and Proposition 3.4.34:

$$\frac{1}{2e^t}\int_{-e^t}^{e^t} f \circ h_-^s(g^{-t}(x))\,ds = \frac{1}{2}\int_{-1}^{1} f \circ g^t(h_-^s(x))\,ds \xrightarrow[t\to+\infty]{\text{uniformly}} \int_{S\Sigma} f. \qquad \square$$

We now take a spectral point of view.

Definition 3.4.37. A complex $f \in L^2(\mu) \smallsetminus \{0\}$ is said to be an *eigenfunction* of a measure-preserving flow $\varphi^t\colon X \to X$ on a probability space (X,μ) if $f \circ \varphi^t = e^{2\pi i\omega t} f$ for all $t \in \mathbb{R}$; in this case ω is called the corresponding eigenfrequency and $e^{2\pi i\omega}$ the corresponding eigenvalue.

Thus, a flow is ergodic if 1 is a simple eigenvalue, and *weakly mixing* if every eigenfunction is constant almost everywhere:

Proposition 3.4.38. *Eigenfunctions of a weakly mixing measure-preserving flow are constant, that is, if $f \in L^2$ and $f \circ \varphi^t = e^{it\omega} f$ for some $\omega \in \mathbb{R}$, then $f =$ const.—and hence $\omega = 0$, so Φ has only one eigenvalue.*

Proof. If $f \in L^2$ and $f \circ \varphi^t = e^{it\omega} f$, then either $e^{it\omega} \equiv 1$ and $f =$ const. by ergodicity (which follows from weak mixing), or $\omega \ne 0$, in which case

$$\langle f, 1\rangle = \int f = \int f \circ \varphi^t = \int e^{it\omega} f = e^{it\omega}\langle f, 1\rangle$$

implies $\langle f, 1\rangle = 0$ and hence

$$\int |f|^2 = \frac{1}{T}\int_0^T e^{it\omega}|\int f\bar{f}\,dt = \frac{1}{T}\int_0^T \left|\int e^{it\omega} f\bar{f}\right| dt = \frac{1}{T}\int_0^T \left|\int (f \circ \varphi^t)\bar{f}\right| dt \xrightarrow[T\to\infty]{} 0$$

by (3.4.8) since Φ is weakly mixing. □

Remark 3.4.39. This provides yet another reason linear flows on tori are not weakly mixing: the coordinate projections are nonconstant eigenfunctions.

Proposition 3.4.40. *A flow is weakly mixing iff* every *time-t map for $t \neq 0$ is ergodic.*

Remark 3.4.41. Compare this with Theorem 3.3.13.

Proof. "$\Leftarrow$": If $f \circ \varphi^t = e^{2\pi i\omega t} f$ for all $t \in \mathbb{R}$ then either $\omega = 0$ and f is an invariant function for φ^1 (say), or $\omega \neq 0$ and f is an invariant function for $\varphi^{1/\omega}$. Either way, $f = \text{const.}$.

"$\Rightarrow$": If $f \circ \varphi^\tau = f \perp 1$, then $g_k := \int_0^\tau e^{2\pi ikt/\tau} f \circ \varphi^t\, dt$ is φ^t-invariant, hence equals $\int g_k = \int \int_0^\tau e^{2\pi ikt/\tau} f \circ \varphi^t\, dt = \int_0^\tau e^{2\pi ikt/\tau} (\int f \circ \varphi^t)\, dt = 0$, so $\mathfrak{f}_x(t) := f(\varphi^t(x))$ satisfies $0 \overset{\text{ae}}{=} g_k(x) = \int_0^\tau e^{2\pi ikt/\tau} \mathfrak{f}_x(t)\, dt$ for all $k \in \mathbb{Z}$, so $\mathfrak{f}_x \overset{\text{ae}}{=} 0$, and $f \overset{\text{ae}}{=} 0$. □

Remark 3.4.42. This is another reason why suspensions are not weakly mixing.

Together with Theorem 3.3.16, this has the following corollary:

Theorem 3.4.43. *The geodesic flow on a compact factor of the Poincaré disk is weakly mixing with respect to the Liouville measure (hence not a suspension).*

Likewise, Proposition 3.4.40 and Proposition 3.3.19 give weak mixing of the horocycle flow.

Theorem 3.4.44. *The horocycle flow from Remark 2.1.15 is weakly mixing with respect to Lebesgue measure (hence not a suspension).*

Of course, we already saw that stronger mixing holds for the geodesic flow (Theorem 3.4.32). Theorem 8.1.13 and even more so Theorem 8.4.17 go further, and this is a good time to lay some of the ground work. We will do more with the horocycle flow later on to find that it is indeed mixing of all orders with respect to Lebesgue measure, and that this is not tied to the algebraic nature of this flow but to the "commutation" relation with the geodesic flow (Section 9.6).

The Bernoulli property differs from the other mixing notions in that verifying it appears to require finding a symbolic flow as well as a measure-theoretic isomorphism to it. It is easy to believe that this can be challenging, so it is important to have criteria for the Bernoulli property that can be verified in ways more in line with the other mixing properties. We here provide an important one without proof.

To define this new notion, we first weaken the notion of "almost everywhere" to an approximate one: we say that a property holds for ϵ-a.e. point of a set E in a measure space (X,μ) if the set B where it fails satisfies $\mu_E(B) < \epsilon$, using the conditional measure from (3.3.3). Likewise, an invertible map $f\colon (X,\mu) \to (Y,\nu)$

of measure spaces (not necessarily probability spaces) is said to be ϵ-measure-preserving if there is a $B \subset X$ with $\mu_X(B) < \epsilon$ and $|\nu(f(A))/\mu(A) - 1| < \epsilon$ for all $A \subset X \smallsetminus B$. Using the notions and notations from Definition 11.1.6 we then make the definition.

Definition 3.4.45. A measure-preserving flow Φ is said to be *very weak Bernoulli* if $f := \varphi^t$ is very weak Bernoulli for every $t \neq 0$ which in turn means that f admits arbitrarily fine partitions (or a generating partition) $\xi = \{C_1, \dots, C_k\}$ that are very weak Bernoulli as follows: Define $\alpha\colon X \to \{1, \dots, k\}$ by $\xi(x) = C_{\alpha(x)}$ and suppose that for $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and ϵ-a.e. atom E of $\bigvee_{j=N}^{n} f^j(\xi)$ there is an ϵ-measure-preserving map $\theta\colon (E \times [0,1], \mu_E \times m) \to (X, \mu)$ with

$$\overline{\lim_{k\to\infty}} \sum_{j=1}^{k} |\alpha(f^j(x)) - \alpha(f^j(\theta(x,u)))| < \epsilon$$

for ϵ-a.e. $(x,u) \in E \times [0,1]$.

This says that x and $\theta(x,u)$ have on average almost exactly the same future as described by itineraries with respect to ξ. With respect to θ, the extra flexibility from considering $E \times [0,1]$ is helpful.

Remark 3.4.46. This notion was introduced as a broader condition under which two systems with the same entropy are measurably isomorphic [**222**]; previously Ornstein had shown this for Bernoulli shifts. Specifically, the new development was that if a generating partition is very weak Bernoulli, then it is "finitely determined," another notion original to that paper, and that if two measure-preserving transformations acting on Lebesgue spaces have finitely determined generating partitions and the same entropy, then they are isomorphic. 4 years later, Ornstein proved the following result, which reveals this property to provide an easier way of establishing the Bernoulli property.

Theorem 3.4.47 ([**223**])**.** *A measure is Bernoulli if it has the* very weak Bernoulli *property (Definition 3.4.45).*

5. Invariant measures under time-change

Time-changes of flows were first described in Proposition 1.2.2. Definition 1.2.1 explained that this is equivalent to scaling the generating vector field, that is, passing from a generating vector field V to the generating vector field $W = \rho V$ for some $\rho\colon X \to (0,\infty)$. While this is straightforward for smooth flows, we now make this explicit for measurable flows and furthermore examine how an invariant measure for a flow corresponds to an (absolutely continuous) invariant measure for a time-change of that flow.

Let us first do the straightforward calculation for smooth flows. A volume form ω is preserved by a flow generated by X if $\mathscr{L}_X\omega = 0$, and we have

Proposition 3.5.1. *If V preserves the volume ω, then $W = \rho V$ preserves the volume ω/ρ: $\mathscr{L}_{\rho V}(\omega/\rho) = 0$.*

Proof. For scalar functions α, the Cartan formula

$$\mathscr{L}_X(\alpha\omega) = \iota_X \underbrace{d(\alpha\omega)}_{=0\text{ (maximum rank)}} + d(\iota_X\alpha\omega) = d(\alpha\iota_X\omega) = d\alpha\wedge\iota_X\omega$$

implies that

$$\mathscr{L}_{\rho V}(\alpha\omega) = \rho\underbrace{\mathscr{L}_V(\alpha\omega)}_{=d\alpha\wedge\iota_V\omega} + \underbrace{d\rho\wedge\iota_X(\alpha\omega)}_{=\alpha d\rho\wedge\iota_V\omega} = (\underbrace{\rho d\alpha + \alpha d\rho}_{=d(\alpha\rho)})\wedge\iota_X\omega = 0$$

when $\alpha\rho = \text{const}$. □

Remark 3.5.2. The last line reflects the fact that constant rescaling of a vector field does not affect whether it preserves a given volume, and constant rescalings of a volume do not affect invariance under a given vector field.

In the measurable context, we first note that such "scaling of the generating vector field" gives a cocycle α as in Proposition 1.2.2 in the context of a μ-preserving flow Φ on X. Since Proposition 1.2.2 produces a cocycle over the time-changed flow, we here effectively study a "backwards" time-change, which explains the apparent mismatch between Theorem 3.5.4 and Proposition 3.5.1. It is easiest to read Theorem 3.5.4 as saying that Proposition 3.5.1 holds in the measurable context.

Proposition 3.5.3. *If $0 < \rho \in L^1(X,\mu)$, then for $t \ge 0$ and a.e. $x \in X$ the equation*

$$\int_0^\alpha \rho(\varphi^\tau(x))\,d\tau = t$$

has a unique solution $\alpha = \alpha(t,x) \ge 0$. So then does $-\int_\alpha^0 \rho(\varphi^\tau(x))\,d\tau = t$ for $t < 0$, here with $\alpha < 0$, and clearly $t \mapsto \alpha(t,x)$ is strictly increasing, $\alpha(0,x) \equiv 0$, and $\lim_{t\to\pm\infty}\alpha(t,x) = \pm\infty$.

Proof. Since $\alpha \mapsto \int_0^\alpha \rho(\varphi^\tau(x))\,d\tau = t$ is continuous and strictly increasing, the conclusion holds for all x such that $\lim_{\alpha\to\infty}\int_0^\alpha \rho(\varphi^\tau(x))\,d\tau = \infty$, and we show that this is a set of full measure by showing that $\rho_\Phi := \lim_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha \rho(\varphi^\tau(x))\,d\tau > 0$ a.e.

By the Birkhoff Ergodic Theorem, this latter limit exists on a Φ-invariant conull set X', and we claim that the (Φ-invariant) set $E := \{x \in X' \mid \rho_\Phi(x) = 0\}$ is a null set: We have $\int_{X'}\rho_\Phi\,d\mu = \int_{X'}\rho\,d\mu$, and by the Birkhoff Ergodic Theorem

$$\int_{X'\smallsetminus E}\rho_\Phi\,d\mu = \int_X(\rho\chi_{X'\smallsetminus E})_\Phi\,d\mu = \int_X \rho\chi X'\smallsetminus E\,d\mu = \int_{X'\smallsetminus E}\rho\,d\mu,$$

so $\int_E \rho\, d\mu = \int_E \rho_\Phi\, d\mu = 0$. This implies $\mu(E) = 0$ since $\rho > 0$. □

Note that $x \mapsto \alpha(t,x)$ is measurable since

$$\{x \in X \mid \alpha(t,x) < \alpha\} = \{x \in X \mid \int_0^\alpha \rho(\varphi^\tau(x))\, d\tau > t\} \text{ for } \alpha > 0.$$

Then the "backwards" time change, $\varphi_\rho^t(x) \coloneqq \varphi^{\alpha(t,x)}(x)$, defines a measurable flow because $\alpha(t_1,x) + \alpha(t_2,\varphi^{t_1}(x)) = \alpha(t_1+t_2,x)$ by uniqueness:[18]

$$\underbrace{\int_0^{\alpha(t_1,x)+\alpha(t_2,\varphi^{t_1}(x))} \rho(\varphi^\tau(x))\, d\tau}_{=\int_0^{\alpha(t_1,x)} \rho(\varphi^\tau(x))\,d\tau + \int_0^{\alpha(t_2,\varphi^{t_1}(x))} \rho(\varphi^\tau(\varphi^{t_1}(x)))\,d\tau} = t_1 + t_2 = \int_0^{\alpha(t_1+t_2,x)} \rho(\varphi^\tau(x))\, d\tau.$$

Theorem 3.5.4 (Measurable Proposition 3.5.1)**.** *If Φ preserves μ and $0 < \rho \in L^1(\mu)$, then the time-change Φ_ρ preserves the probability measure $d\mu_\rho = \rho\, d\mu / \int \rho\, d\mu$.*

Corollary 3.5.5. *A continuous time change of a uniquely ergodic continuous flow on a compact metric space is itself uniquely ergodic (Definition 3.1.17).*

Proof. We show that $\int f \circ \varphi_\rho^\tau\, d\mu_\rho = \int f\, d\mu_\rho$ for $f \in L^\infty(\mu_\rho)$.

$$f_t(x) \coloneqq \frac{1}{t}\int_0^t \underbrace{f(\varphi_\rho^\tau(x))}_{f(\varphi^{\alpha(\tau,x)}(x))}\, d\tau = \frac{1}{t}\underbrace{\int_0^{\alpha(t,x)} f(\varphi^v(x))\rho(\varphi^v(x))\, dv}_{v=\alpha(\tau,x),\quad dv=\frac{d\alpha(\tau,x)}{d\tau}d\tau=\frac{d\tau}{\rho(\varphi^v(x))}}$$

$$= \overbrace{\frac{1}{t}\alpha(t,x)}^{\xrightarrow[t\to\infty]{} \lim\limits_{\alpha\to\infty} \frac{\alpha}{\int_0^\alpha \rho(\varphi^\tau(x))\,d\tau} = \frac{1}{\rho_\Phi(x)} \text{ (time average)}} \underbrace{\frac{1}{\alpha(t,x)}\int_0^{\alpha(t,x)} (\rho f)(\varphi^v(x))\, dv}_{\xrightarrow[t\to\infty]{} (\rho f)_\Phi(x) \text{ (time average)}},$$

so $\rho_\Phi f_t \xrightarrow[t\to\infty]{\text{a.e.}} (\rho f)_\Phi$ and hence

$$\int_X \rho_\Phi f_t\, d\mu \xrightarrow[t\to\infty]{} \int_X (\rho f)_\Phi\, d\mu = \int_X \rho f\, d\mu = \int_X \rho\, d\mu \int_X f\, d\mu_\rho.$$

Applying this to $f \circ \varphi_\rho^\tau$ instead of f gives

$$\int_X \rho_\Phi f_t \circ \varphi_\rho^\tau\, d\mu \xrightarrow[t\to\infty]{} \int_X \rho\, d\mu \int_X f \circ \varphi_\rho^\tau\, d\mu_\rho.$$

The right-hand sides agree because the two left-hand sides have the same limit:

$$\left|\int_X \rho_\Phi f_t \circ \varphi_\rho^\tau\, d\mu - \int_X \rho_\Phi f_t\, d\mu\right| \le \frac{1}{t}\int_X \rho_\Phi \underbrace{\left|\int_0^t f \circ \varphi_\rho^{\tau+s} - f \circ \varphi_\rho^s\, ds\right|}_{\le 2\tau\|f\|_\infty} d\mu \xrightarrow[t\to\infty]{} 0. \quad \square$$

[18] And because $(x,t) \mapsto (x, \alpha(t,x))$ is measurable on $X \times \mathbb{R}$.

6. Flows under a function

We introduce special flows (Definition 1.2.7) in the measurable context.

Definition 3.6.1. Let F be an invertible measure-preserving transformation of a finite measure space (Y,μ) and $0 < r \in L^1(Y)$ a "roof" function. On $X := \{(y,s) \in Y \times (0,\infty) \mid s < r(y)\}$ with the measure ν induced by $\mu \times m$ recursively define the *flow under r* by

$$\varphi^t(y,s) := \begin{cases} (y, s+t) & \text{if } 0 \le s+t < r(y), \\ \varphi^{s+t-r(y)}(F(y),0) & \text{if } s+t > r(y), \\ \varphi^{s+t+r(F^{-1}(y))}(F^{-1}(y),0) & \text{if } s+t < 0. \end{cases}$$

This is a ν-preserving flow on X. Note that ν may not be a probability measure even if μ is, but ν is finite. The measure ν is given by the formula

$$\int_X f\,d\nu = \int_Y \left(\int_0^{r(x)} f(x,t)\,dt \right) d\mu(x) \tag{3.6.1}$$

where f is any bounded measurable function.

Also, such a flow has no fixed points, or at least the set of these has measure 0. It turns out that this property of having essentially no fixed point is sufficient for being of this form. This next theorem has no counterpart for special flows in the topological setting.

Theorem 3.6.2 (Ambrose–Rokhlin Special-Flow Representation [**6**])**.** *Let Φ be a measure-preserving flow on a Lebesgue space (Definition 11.1.1) with essentially no fixed points. Then Φ is measure-theoretically isomorphic to a special flow (that is, represented as a special flow).*

Remark 3.6.3. For an aperiodic flow (that is, it has essentially no closed orbits) one can choose this special representation in such a way that the roof function is arbitrarily close to a given constant in the uniform topology.

This can be viewed as a global counterpart to the local construction of flow boxes in Proposition 1.1.14, but even locally, this is a nontrivial insight into the structure of a flow. Notably, it implies that the time-dependence is quite regular, which is not apparent from the definition of a measurable flow. In particular, this implies that the orbit of a.e. point is a measurable set. We note, however, that being global, this is very different from topological dynamics, where being a special flow constrains the topology of the underlying manifold and important fixed-point free flows are not of this type, for example, geodesic flows.

We mention without proof (see [**89**] for one) that it is possible to strengthen this result as follows.

Theorem 3.6.4 (Rudolph)**.** *If Φ is ergodic and $p, q, s > 0$ with $p/q \notin \mathbb{Q}$ and $s < 1$, then the roof function in the representation as a special flow can be chosen such that $r(Y) = \{p, q\}$ and $\mu(r^{-1}(\{p\})) = s$.*

The proof of the Ambrose–Rokhlin Special-Flow Representation Theorem proceeds in two main steps. Proposition 3.6.6 produces the "geometry" of a special flow, that is, the partition of the space by the orbit segments which (a posteriori) run from the base to the roof. Proposition 3.6.7 then builds the dynamics accordingly. The needed properties of the partition are as follows.

Definition 3.6.5. A partition ξ of X is said to be an *orbit-segment* partition for Φ if

(1) each partition element is an orbit segment $C = \{\varphi^\tau(x) \mid 0 \le \tau < l\}$ in such a way that the representation of any $y \in C$ as $y = \varphi^\tau(x)$ with $0 \le \tau < l$ is unique (we call x the *bottom endpoint* and l the *length* of the orbit segment), and

(2) the function $C \ni \varphi^\tau(x) = y \mapsto (L, T)(y) := (l, \tau)$ is measurable.

Proposition 3.6.6. *A measure-preserving flow Φ on a Lebesgue space (X, μ) with essentially no fixed points admits an invariant set of positive measure with an orbit-segment partition for which $L \ge c$ for some $c > 0$.*

Proposition 3.6.7. *If a Lebesgue space (X, μ) with a μ-preserving flow Φ has an orbit-segment partition, then Φ is measure-theoretically isomorphic to a special flow.*

Proof of Theorem 3.6.2. Proposition 3.6.6 gives an invariant set E of positive measure with an orbit-segment partition, on which the flow then is measure-theoretically isomorphic to a special flow by Proposition 3.6.7. We now apply this recursively.

Let $C_0 = \varnothing$, and for $i \ge 1$ there either is a set $E \subset X \smallsetminus \bigcup_{j<i} C_j$ of measure at least $1/i$ on which Φ is measure-theoretically isomorphic to a special flow, and we let C_i be the (without loss of generality countable) union of such E, or else we set $C_i = \varnothing$. Then the restriction of Φ to $C := \bigcup_{i\in\mathbb{N}} C_i$ is measure-theoretically isomorphic to a special flow, and C has full measure: Otherwise, Proposition 3.6.6 and Proposition 3.6.7 yield an $E \subset X \smallsetminus C$ with $\mu(E) > 1/i$ for some $i \in \mathbb{N}$, hence $E \cap C \ne \varnothing$ by construction of C, a contradiction. □

Proof of Proposition 3.6.6. We will find two disjoint sets both of which orbits revisit indefinitely; the "exit points" from one of these on the way to the other form a good candidate for the base of a special flow. We will use averaging in the flow direction:

$$\tau \mapsto \operatorname{avg}_A^\alpha(\tau, x) := \frac{1}{\alpha}\int_0^\alpha \chi_A \circ \varphi^{t+\tau}(x)\, dt \quad \text{is} \quad \frac{2}{\alpha}\text{-Lipschitz for } x \in X \tag{3.6.2}$$

because $\frac{1}{\alpha}\Big|\underbrace{\int_0^\alpha \chi_A\circ\varphi^{t+\tau_1}\,dt-\int_0^\alpha \chi_A\circ\varphi^{t+\tau_2}\,dt}_{=\int_{\tau_1}^{\tau_1+\alpha}\chi_A\circ\varphi^t\,dt-\int_{\tau_2}^{\tau_2+\alpha}\chi_A\circ\varphi^t\,dt}\Big|\le\frac{2}{\alpha}|\tau_1-\tau_2|$. Since Φ has essentially no fixed points, there is a measurable A and a $t_0\in\mathbb{R}$ with $\delta\coloneqq\mu((X\setminus A)\cap\varphi^{t_0}(A))>0$. Let

$E_1{:=}\{x\in X\mid \operatorname{avg}_A^\alpha(0,x)<1/4\},\quad E_2{:=}\{x\in X\mid \operatorname{avg}_A^\alpha(0,x)>3/4\},\quad E{:=}E_1\cap\varphi^{t_0}(E_2),$

with (by Lemma 3.6.8) $\alpha>0$ small enough that

$$\mu((X\setminus A)\triangle E_1)<\delta/2\quad\text{and}\quad\mu(A\triangle E_2)<\delta/2.$$

Lemma 3.6.8 (Wiener)**.** *If $f\in L^\infty(X,\mu)$, then $\frac{1}{t}\int_0^t f\circ\varphi^s\,ds\xrightarrow[t\to0]{L^2\ \&\ \text{in measure}}f$.*

Proof. It suffices to prove convergence in L^2, and to that end we use the *spectral measure* σ with $\langle U_\Phi^t f,f\rangle=\int_X f\circ\varphi^t\cdot f=\int_{\mathbb{R}}e^{i\lambda t}\,d\sigma(\lambda)$ (Definition 3.7.8) to get

$$\Big\|\frac{1}{t}\int_0^t f\circ\varphi^s\,ds-f\Big\|^2=\int_X\underbrace{\Big(\frac{1}{t}\int_0^t f\circ\varphi^s\,ds-f\Big)\overline{\Big(\frac{1}{t}\int_0^t f\circ\varphi^s\,ds-f\Big)}}_{=\frac{1}{t^2}\int_0^t f\circ\varphi^s\,ds\overline{\int_0^t f\circ\varphi^r\,dr}-\frac{1}{t}\int_0^t\bar f f\circ\varphi^s\,ds-\frac{1}{t}\int_0^t f\overline{f\circ\varphi^r}\,dr+f\bar f}$$

$$\overset{\text{Fubini}}{=\!=}\frac{1}{t^2}\int_0^t\int_0^t\overbrace{\int_{\mathbb{R}}e^{i\lambda(s-r)}\,d\sigma(\lambda)}^{=\langle U_\Phi^s f,U_\Phi^r f\rangle}\,ds\,dr$$

$$-\frac{1}{t}\int_0^t\overbrace{\int_{\mathbb{R}}\overline{e^{i\lambda s}}\,d\sigma(\lambda)}^{=\overline{\langle U_\Phi^s f,f\rangle}}\,ds-\frac{1}{t}\int_0^t\int_{\mathbb{R}}e^{i\lambda r}\,d\sigma(\lambda)\,dr+\overbrace{\int_{\mathbb{R}}d\sigma(\lambda)}^{=\langle U_\Phi^0 f,f\rangle}$$

$$\overset{\text{Fubini}}{=\!=}\int_{\mathbb{R}}\Big|\frac{e^{i\lambda t}-1}{i\lambda t}-1\Big|^2d\sigma(\lambda)\xrightarrow[t\to0]{\text{Lebesgue Dominated-Convergence Theorem}}0$$

because $\frac{1}{t^2}\int_0^t\int_0^t e^{i\lambda s}\overline{e^{i\lambda r}}\,ds\,dr=\underbrace{\frac{1}{t}\int_0^t e^{i\lambda s}\,ds}_{=\frac{1}{t}\frac{e^{i\lambda s}}{i\lambda}\Big|_0^t=\frac{e^{i\lambda t}-1}{i\lambda t}\xrightarrow[t\to0]{}1}\overline{\frac{1}{t}\int_0^t e^{i\lambda r}\,dr}$. □

As a consequence,

$$\mu(E)\ge\mu((\underbrace{X\setminus A}_{\sim E_1}\cap\underbrace{\varphi^{t_0}(A)}_{\sim\varphi^{t_0}(E_2)})-\mu((X\setminus A)\triangle E_1)-\mu(A\triangle E_2)>0,$$

so there is an invariant set E' with positive measure of points that visit E for arbitrarily large positive and negative times. Also,

$$E_1'\coloneqq\varphi^{[0,\alpha/8]}(E_1)\quad\text{and}\quad E_2'\coloneqq\varphi^{[0,\alpha/8]}(E_2)$$

are disjoint because if $x \in E_1' \cap E_2'$, then $x_i \coloneqq \varphi^{\tau_i}(x) \in E_i$ for suitable $\tau_i \in [0, \alpha/8]$, so

$$\frac{1}{2} < |\underbrace{\operatorname{avg}_A^\alpha(0, x_1) - \operatorname{avg}_A^\alpha(0, x_2)}_{=\operatorname{avg}_A^\alpha(-\tau_1, x) - \operatorname{avg}_A^\alpha(-\tau_2, x)}| \le \frac{2}{\alpha}|\tau_1 - \tau_2| \le \frac{2}{\alpha}\frac{\alpha}{8} = \frac{1}{4}$$

by (3.6.2), a contradiction.[19] The "points of exit from E_1' on the way to E_2'" now form the base Y of a special-flow representation as follows.

For any orbit in E', the set of times when it hits E_1' is open and without upper or lower bound, and its connected components ("E_1'-intervals") have length at least $\alpha/8$. Likewise for E_2'-intervals, and no E_1'-interval overlaps with any E_2'-interval, so for every point in E' there is a well-defined nearest such interval "below" and likewise "above." We then take $Y \subset E'$ to be the set of top endpoints of E_1'-intervals with the additional property that the nearest interval above is an E_2'-interval. This ensures that it takes more time than $\alpha/8$ to return to Y, that is, $f(x) \coloneqq \min\{t > 0 \mid \varphi^t(x) \in Y\} \ge \alpha/8$ for $x \in Y$.

The desired partition elements are now given by $\varphi^{[0, f(x)]}(x)$ for $x \in Y$, and $(L, T)(\varphi^s(x)) = (f(x), s)$, so it only remains to show measurability of L and T. For T this follows from measurability of

$$B_k \coloneqq \{x \in X \mid \frac{k}{n} \le T(x) < \frac{k+1}{n}\} \quad \text{for} \quad n \ge 8/\alpha,\ k \in \mathbb{N},$$

which reduces to that of B_0 since $B_{k+1} = \varphi^{1/n}(B_k) \smallsetminus B_0$. B_0 is measurable because $x \in B_0$ means that x just exited E_1' on the way to E_2', that is, $x \notin E_1' \ni \varphi^{-1/n}(x)$ and there is an $i \in \mathbb{N}$ with $\varphi^{1/n}(x), \varphi^{2/n}(x), \dots, \varphi^{i/n}(x) \notin E_1' \cup E_2'$ but $\varphi^{i+1/n}(x) \in E_2'$. Finally, L is measurable because $L^{-1}((c,\infty)) = \varphi^{\mathbb{Q} \cap [0,c]}(T^{-1}((c,\infty))) \cup \varphi^{-c}(T^{-1}([c,\infty)))$. This completes the proof of Proposition 3.6.6 $\square$

Remark 3.6.9. We could arrange for L to be bounded by refining the partition elements $\varphi^{[0, L(x))}(x)$ into the $\varphi^{[ic,(i+1)c)}(x)$ and $\varphi^{[kc, L(x))}(x)$, where $c \coloneqq \inf L < L(x) - kc < 2c$. The new roof function L' then satisfies $c \le L' < 2c$.

Proof of Proposition 3.6.7. The essence of the argument is that the bottom endpoints of the partition elements form the base. For checking the details of why this works it is convenient to reparametrize the flow in a piecewise linear way so all partition elements are traversed in unit time, that is, by multiplying the "speed" by $L(x)$ on the partition element containing x. Another way of putting this is to write

[19] The E_i are measurable: continuity of $\tau \mapsto \operatorname{avg}_A^\alpha(0, \varphi^\tau(x)) = \operatorname{avg}_A^\alpha(-\tau, x)$ implies $E_i' = \varphi^{\mathbb{Q} \cap [0, \alpha/8]}(E_i)$.

(for $y := \varphi^{L(x)-T(x)}(x)$, and $t'(x) = \frac{L(x)-T(x)}{L(x)}$)

$$\bar{\varphi}^t(x) = \begin{cases} \varphi^{tL(x)}(x) & \text{for } 0 \le t < t'(x) \\ \varphi^{(t-t'(x))L(y)}(y) & \text{for } 1 > t \ge t'(x). \end{cases}$$

This defines a measurable flow $\bar{\Phi}$ on $(X, \mathscr{A})$ with the same orbit-segment partition, and by Theorem 3.5.4 it preserves the measure $\bar{\mu} = L\mu / \int L\,d\mu$. It suffices to show that this flow is measure-theoretically isomorphic to a special flow with $\bar{T} = T/L$ and roof function $\bar{L} \equiv 1$, that is, a suspension.

To that end note that the map $\pi\colon X \to X_1$ that sends each point to the bottom endpoint of its partition element makes $(X_1, \mathscr{A}_1 := \pi_* \mathscr{A} := \{A \subset X_1 \mid \pi^{-1}(A) \in \mathscr{A}\}, \mu_1 := \pi_*(\bar{\mu}))$ a measure space, where $\mu_1(A) = \bar{\mu}(\pi^{-1}(A))$ is preserved by the base transformation $F := \bar{\varphi}^1$. This is represented as a suspension flow $\psi^t = h \circ \bar{\varphi}^t \circ h^{-1}$ via the bijection $h\colon X \to X' := X_1 \times [0,1)$, $\bar{\varphi}^t(x) \mapsto (x, t)$.

Lemma 3.6.10. *$\mu' := h(\mu) = \mu_1 \times$ Lebesgue$=: \bar{\nu}$ on $\mathscr{A}' := h(\mathscr{A}) = \mathscr{A}_1 \times \mathscr{B}$, the product σ-algebra in $X' = X_1 \times [0,1)$.*

Up to reversing the above time-change, this proves Proposition 3.6.7 □

The proof of Lemma 3.6.10 involves careful applications of the basic notions of measure theory more than dynamical ideas, and the main effort is to show that $\mathscr{A}_1 \times \mathscr{B} = \mathscr{A}'$. The inclusion $\mathscr{A}_1 \times \mathscr{B} \subset \mathscr{A}'$ is clear: If $\mathscr{A}_1 \times \mathscr{B} \ni A = A_1 \times [t_1, t_2)$ with $A_1 \in \mathscr{A}_1$, then $h^{-1}(A) = \{y \mid \pi(y) \in A_1,\ t_1 \le \bar{T}(y) < t_2\} \in \mathscr{A}$. Therefore the main effort is the reverse inclusion, and here it is central that we are dealing with Lebesgue spaces. This is put to use via notions of “open” and “boundary” in the absence of any topology by using the flow parameter as follows: If $E \subset \mathbb{R}$ and $x \in X$, then the (flow-)closure of $C := \varphi^E(x)$ is $\bar{C} := \varphi^{\bar{E}}(x)$ and $\partial C := \varphi^{\partial E}(x)$ is the (flow-)boundary of C. More generally, then, the flow-closure and flow-boundary of $A \subset X$ are defined by

$$\varphi^{\mathbb{R}}(x) \cap \bar{A} = \overline{\varphi^{\mathbb{R}}(x) \cap A} \quad \text{and} \quad \varphi^{\mathbb{R}}(x) \cap \partial A = \partial(\varphi^{\mathbb{R}}(x) \cap A)$$

for all $x \in X$. $A \subset X$ is said to be (flow-)open if $\{t \mid \varphi^t(x) \in A\}$ is open for all $x \in X$. Then we can approximate measurable sets by flow-open ones as follows.

Lemma 3.6.11. *For $A \in \mathscr{A}$ and $\epsilon > 0$ there is an $A_\epsilon \in \mathscr{A}$ such that*

(1) A_ϵ is flow-open,
(2) $\mu(\partial A_\epsilon) = 0$, and
(3) $\mu(A \triangle A_\epsilon) < \epsilon$.

Proof. We take $A_\epsilon := A_{n,\beta} := \{x \in X \mid \operatorname{avg}_A^{1/n}(0, x) > \beta\}$ for $n \in \mathbb{N}$ and $\beta \in (0,1)$ to be determined. Note that $\{t \mid \varphi^t(x) \in A_{n,\beta}\}$ is open for all $x \in X$ by (3.6.2), and that

$\partial A_{n,\beta} \subset \{x \in X \mid \operatorname{avg}_A^{1/n}(0,x) = \beta\}$, so there is a $\beta \in (0,1)$ with $\mu(\partial A_{n,\beta}) = 0$ for all $n \in \mathbb{N}$. By Lemma 3.6.8 we can choose n such that $\mu(A \triangle A_{n,\beta}) < \epsilon$. □

Lemma 3.6.12. *If $A \in \mathscr{A}$ is flow-open, then $\pi(A) \in \mathscr{A}_1$, that is, $\bar{A} := \pi^{-1}(\pi(A)) \in \mathscr{A}$.*

PROOF. This, and $\mu_1(\pi(A)) = \bar{\mu}(\bar{A})$, follows from

$$\bar{A} = \Big(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n+1}-2} \bar{A}_{n,k}\Big) \cup \Big(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^n-1} \bar{A}_{k/2^n}\Big),$$

where $A_{n,k} := \mathscr{A} \cap \bar{T}^{-1}((\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}))$,

$$\begin{aligned}\bar{A}_{n,k} &:= \pi^{-1}(\pi(A_{n,k}))\\ &= \Big[\bar{T}^{-1}((\frac{1}{2^{n+1}},\infty)) \cap \bar{\varphi}^{\mathbb{Q}\cap[0,1-\frac{k}{2^{n+1}})}(A_{n,k})\Big] \cup \Big[\bar{T}^{-1}([0,1-\frac{1}{2^{n+1}})) \cap \bar{\varphi}^{\mathbb{Q}\cap[-\frac{k+1}{2^{n+1}},0]}(A_{n,k})\Big],\end{aligned}$$

and $A_{k/2^n} := \mathscr{A} \cap \bar{T}^{-1}(\{\frac{k}{2^n}\})$, $\bar{A}_{k/2^{n+1}} := \pi^{-1}(\pi(A_{k/2^{n+1}})) = \bar{\varphi}^{[-\frac{k}{2^{n+1}},1-\frac{k}{2^{n+1}}]}(A_{k/2^{n+1}})$.

□

We note also that

$$A \subset D := \Big[\bigcap_{l\in\mathbb{N}} \bigcup_{n\geq l} \bigcup_{k=1}^{2^{n+1}-2} \pi(A_{n,k}) \times \Big(\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\Big)\Big] \cup \Big[\bigcup_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} A_{k/2^n}\Big] \subset \bar{A} \quad \text{(flow-closure)}.$$

The point is that $h(D) \in \mathscr{A}_1 \times \mathscr{B}$ since the $\pi(A_{n,k}) \in \mathscr{A}_1$ by Lemma 3.6.12 since the $A_{n,k}$ are flow-open.

PROOF OF LEMMA 3.6.10. $\bar{\mu}(A) = \bar{\nu}(h(A))$ for every $A \in \mathscr{M} := \{h^{-1}(A) \mid A \in \mathscr{A}_1 \times \mathscr{B}\} \subset 2^X$ because the σ-algebra $\mathscr{M}$ is generated by sets A for which $h(A) = A_1 \times I$, where $A_1 \in \mathscr{A}_1$ and $I \subset [0,1)$ is an interval, and for such sets this is clear. Thus, $\mathscr{M}$ is complete with respect to μ. The preceding observation and Lemma 3.6.12 imply that for any $A \in \mathscr{A}$ and $\epsilon > 0$ there is an $A_\epsilon \in \mathscr{M}$ with $\mu(A \triangle A_\epsilon) < \epsilon$. Since X is a Lebesgue space and $\mathscr{M}$ is complete, this implies $\mathscr{M} = \mathscr{A}$—and we also have $\bar{\nu} = h(\bar{\mu})$. □

Even though the proof of Theorem 3.6.2 is not entirely constructive and hence does not give a straightforward explicit representation as a special flow, the mere existence of such a representation is useful, notably with respect to studying the interplay between entropy and time-changes (Theorem 4.1.8 and Theorem 4.1.9).

As mentioned at the beginning of the section the measure ν given by (3.6.1) is not necessarily a probability measure. Therefore, an invariant probability measure for the special flow can be defined by

$$\int_X f\,d\mu_r = \frac{\displaystyle\int_Y \Big(\int_0^{r(x)} f(x,t)\,dt\Big)d\mu(x)}{\displaystyle\int_Y r(x)\,d\mu(x)} \tag{3.6.3}$$

for any bounded measurable function f.

In the context of special flows it is possible to produce a flow-counterpart to the Kac Lemma 11.3.35, a basic result in discrete-time ergodic theory, which involves the *return time*. For measure-preserving flows this issue is far trickier than in discrete time because for a set that is far from open, closed or convex, even defining "return time" is challenging. For returns to the base of a special flow, however, there is a simple analog of the Kac Lemma.

Proposition 3.6.13 (Flow Kac Lemma [**284**, Corollary 1])**.** *If F is a μ-preserving map on a topological probability space (X,μ), $0 < r \in L^1(\mu)$, $\mu(A) > 0$, then*

$$\int_A T_A(x)\,d\mu_A = \frac{1}{\mu(A)}\int_X r\,d\mu,$$

with μ_A the conditional measure from (3.3.3) *and* $T_A(x) := \min\{t > 0 \mid \varphi_r^t(x,0) \in A\}$.

7. Spectral theory*

Although we will hardly use it in our study of hyperbolic flows, we describe here some elements of the spectral approach to ergodic theory. The central idea is to connect properties of the Koopman operator (Definition 3.2.5) for a flow with dynamical properties of the flow. or to use them for the classification problem.

Note first that for a μ-preserving flow Φ on X the operators $U_\Phi^t = U_{\varphi^t}$ associated with a flow form a 1-parameter group of unitary operators on $L^2(X,\mu)$—and here it is useful to consider complex-valued functions[20]. 1 is always an eigenvalue, because constant functions are invariant. Therefore it is often natural to restrict attention to $1^\perp \subset L^2$, the space of functions with integral 0. We will usually assume that μ is a Borel probability measure, in which case $L^2(X,\mu)$ is separable. This turns out to imply that U_Φ^t is a *continuous* group of unitary operators. One useful simple property of these operators that makes them special beyond linearity is that $U_\Phi(fg) = U_\Phi(f)U_\Phi(g)$.

An easy connection to the classification problem is that if a flow Φ on (X,μ) and a flow Ψ on (Y,ν) are measure-theoretically isomorphic, then their Koopman

[20] The results we obtain in this context can be used for real linear spaces E by passing to the complexification $E_{\mathbb{C}}$ (that is, the space $E \otimes \mathbb{C}$ obtained by allowing complex scalars) and then suitably restricting attention to the real part.

operators are conjugate (or, as one says in this context, unitarily equivalent): let $h\colon X \to Y$ be an isomorphism such that $h\circ\varphi^t = \psi^t \circ h$, then $U_{\varphi^t}\circ U_h = U_{h\circ\varphi^t} = U_{\psi^t\circ h} = U_h\circ U_{\psi^t}$. It is interesting when one can go the other way around: If one can show that the unitary operators for Φ and Ψ are conjugate, then one may hope to utilize this somehow to show that Φ and Ψ are measure-theoretically isomorphic. This is, of course, not always so.

Thus spectral invariants of U_f, for example, eigenvalues with their multiplicities or the spectrum, are invariants of measure-theoretic isomorphism of f.

Definition 3.7.1. Two measure-preserving transformations are said to be *spectrally isomorphic* if their Koopman operators are unitarily equivalent. An invariant of spectral isomorphism is called a *spectral invariant.*

Let us illustrate how dynamical properties might be expressible in terms of the spectrum of the Koopman operator.

Proposition 3.7.2. *A μ-preserving flow Φ on X is ergodic if and only if 1 is a simple eigenvalue of the associated Koopman operator.*

Proof. We noted that 1 is always an eigenvalue, and simplicity of this eigenvalue is equivalent to saying that U_{φ^t}-invariant functions are constant, which is equivalent to ergodicity. □

From this, we conclude

Proposition 3.7.3. *Ergodicity is a spectral invariant.*

Definition 12.3.1 formally introduces the spectrum in this context.

Proposition 3.7.4. *If Φ is a probability-preserving flow, then*

1. *The eigenvalues of U_Φ lie on the unit circle.*
2. *The spectrum of U_Φ lies on the unit circle.*
3. *The eigenvalues of U_Φ form a subgroup of the unit circle.*
4. *The eigenspaces of U_Φ are pairwise orthogonal.*

Proof. 1.: If A is an isometry and $Av = \lambda v$, then $\|v\| = \|Av\| = \|\lambda v\| = |\lambda|\|v\|$.

2.: If A is unitary then $r(A^{\pm1}) \le \|A^{\pm1}\| = 1$, so $\sigma(A^{\pm1}) \subset \{\lambda \mid |\lambda| \le 1\}$. $A \in \mathrm{Aut}(V)$ implies $0 \notin \sigma(A)$ and hence $\sigma(A^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma(A)\}$ because $(1/\lambda)I - A^{-1}$ is invertible if and only if $-\lambda A[(1/\lambda)I - A^{-1}] = \lambda I - A$ is.

3.: If $U_\Phi(f) = \lambda f$ and $U_\Phi(g) = \mu g$, then $\mu\lambda^{-1}$ is also an eigenvalue:

$$U_\Phi(f\cdot\bar g) = U_\Phi(f)\overline{U_\Phi(g)} = \mu\bar\lambda\cdot f\cdot\bar g = \mu\lambda^{-1}\cdot f\cdot\bar g,$$

This shows closure under inverses (take $\mu = 1 = g$) and then under multiplication.

4.: If $U_\Phi(f) = \lambda f$ and $U_\Phi(g) = \mu g$, then

$$\langle f, g\rangle = \langle U_\Phi(f), U_\Phi(g)\rangle = \langle \lambda f, \mu g\rangle = \lambda\bar{\mu}\langle f, g\rangle = \lambda\mu^{-1}\langle f, g\rangle,$$

so $\lambda\mu^{-1} = 1$ or $\langle f, g\rangle = 0$. □

Remark 3.7.5. We emphasize that we are here considering eigenvalues of $U_\Phi = U_{\varphi^1}$. If $e^{i\alpha}$ is an eigenvalue of U_Φ, then there is an eigenfunction f with $U_{\varphi^t}(f) = e^{i\alpha t} f$ for all $t \in \mathbb{R}$. This itself produces a multiplicative subgroup, so for 1-parameter roups of unitary operators it is conventional to call $\alpha \in \mathbb{R}$ an eigenvalue of $(U^t)_{t\in\mathbb{R}}$ if $e^{i\alpha}$ is an eigenvalue of U^1. Then Proposition 3.7.4 tells us that the eigenvalues of a 1-parameter group of unitary operators are an additive subgroup of $\mathbb{R}$ (with pairwise orthogonal eigenspaces). See also Definition 3.4.37.

Ergodicity easily provides information about other eigenspaces.

Proposition 3.7.6. *A probability-preserving flow Φ is ergodic iff*

(1) All eigenfunctions have constant absolute value.
(2) All eigenspaces are 1-dimensional.

Proof. 1.: $U_\Phi(|f|) = |U_\Phi(f)| = |\lambda||f| = |f|$, so $|f|$ is invariant.

2.: If f, g are nonzero eigenfunctions for λ, then they are nonzero a.e. by 1., so f/g is a well-defined invariant function. □

It is also easy to see the following.

Proposition 3.7.7. *Mixing is a spectral invariant (Definition 3.7.1).*

Proof. Suppose Φ on (X, μ) is mixing, Ψ ν-preserving on Y, $W \circ U_\Phi = U_\Psi \circ W$, W unitary, and $f_i = W(g_i) \in L^2(Y, \nu)$ for $i = 1, 2$. Then $W1 = 1$ by Proposition 3.7.3, and

$$\begin{aligned}\langle U_\Psi^t(f_1), f_2\rangle = \langle U_\Psi^t(W(g_1)), W(g_2)\rangle = \langle W(U_\Phi^t(g_1)), W(g_2)\rangle = \langle U_\Phi^t(g_1), g_2\rangle \\ \xrightarrow[t\to\infty]{} \langle g_1, 1\rangle\langle g_2, 1\rangle = \langle Wg_1, W1\rangle\langle Wg_2, W1\rangle = \langle f_1, 1\rangle\langle f_2, 1\rangle.\end{aligned}$$ □

Both because this was used in the proof of Lemma 3.6.8 and because it is an important aspect of studying spectral properties, we now introduce spectral measures, which are defined by something much like a Fourier transform.

Definition 3.7.8. If Φ is a measure-preserving flow on a Lebesgue space (X, μ) and $f \in L^2(X, \mu)$, the *spectral measure* σ_f of f on $\mathbb{R}$ is defined by

$$\langle U_\Phi^t f, f\rangle = \int_{\mathbb{R}} e^{it\lambda} d\sigma_f(\lambda).$$

By taking $t = 0$ we find that $\sigma_f(\mathbb{R}) = \|f\|_2^2$. That such a measure exists is due to a theorem of Bochner.[21]

Example 3.7.9. If f is an eigenfunction of U_Φ with eigenvalue $\lambda = e^{i\alpha}$, then

$$e^{it\alpha}\|f\|_2^2 = \langle U_{\varphi^t}(f), f\rangle = \int_{\mathbb{R}} e^{it\lambda}\, d\sigma_f(\lambda), \tag{3.7.1}$$

which is equivalent to $\sigma_f = \|f\|_2^2 \delta_\alpha$, a Dirac measure at α. Conversely, (3.7.1) implies $|\langle U_{\varphi^t}(f), f\rangle| = \|f\|_2^2 = \|U_{\varphi^t}(f)\|\|f\|$, so $U_{\varphi^t}(f)$ and f are proportional by the equality case of the Cauchy–Schwarz inequality, so f is an eigenfunction—with the eigenvalue given by (3.7.1). Thus, weak mixing is equivalent to the following: Every $f \in L^2(X,\mu)$ whose spectral measure is a point mass is constant. Likewise, ergodicity is equivalent to the following: Every $f \in L^2(X,\mu)$ whose spectral measure is δ_0 is constant.

The following notion is natural for describing a situation in which a measure-preserving transformation is "spectrally rigid".

Definition 3.7.10. We say that Φ has *pure point spectrum* or *discrete spectrum* if Φ is ergodic and there is a basis of eigenfunctions of U_Φ.

Remark 3.7.11. The terminology goes back to that in Definition 12.3.1 in that the spectrum consists entirely of eigenvalues. Note also that by Proposition 3.7.6 these λ are pairwise distinct; this produces enough information for spectral isomorphism.

Proposition 3.7.12. *Ergodic measure-preserving flows with discrete spectrum and with the same eigenvalues are spectrally isomorphic.*

Proof. For each eigenvalue map the corresponding eigenfunction for one transformation to that for the other (see Proposition 3.7.6); extend by linearity and continuity. □

Remark 3.7.13. In this case the dynamics of U_Φ consists of a product of rotations of the eigenspaces; the essential information is contained in what happens to normalized eigenfunctions. This can be exploited to show that, in fact, here the eigenvalues determine Φ up to a measure-theoretic isomorphism.

Although we omit the proof (other than Proposition 3.4.38), one can characterize weakly mixing flows analogously to the way we previously characterized ergodicity.

[21] …and that $t \mapsto \langle U_\Phi^t f, f\rangle$ is "positive definite": if $(z_1,\dots,z_m) \in \mathbb{C}^m$ and $(t_1,\dots,t_m) \in \mathbb{R}^m$, then $0 \le \left\|\sum_{k=1}^m z_k U_\Phi^{t_k}(f)\right\|^2 = \left\langle \sum_{i=1}^m z_i U_\Phi^{t_i}(f), \sum_{j=1}^m z_j U_\Phi^{t_j}(f)\right\rangle = \sum_{i,j=1}^m \langle U_\Phi^{t_i - t_j} f, f\rangle z_i z_j$.

Proposition 3.7.14. *For a measure-preserving flow Φ the following are equivalent:*

- *Φ is weakly mixing,*
- *all eigenfunctions are constant,*
- *σ_f is nonatomic ("continuous") for every $f \perp 1$.*

Remark 3.7.15. The third of these versions is the reason one also describes this property as having continuous spectrum.

Exercises

3.1. Determine $\mathfrak{M}(\Phi)$ in Examples 1.1.5, 1.1.7, 1.3.5, 1.3.6, 1.3.9, 1.3.11, 1.4.14 and Figures 1.4.1, 1.5.4, 1.5.11, 1.1.4.

3.2. Prove: If μ is an ergodic invariant measure for a continuous flow Φ, then $\Phi\restriction_{\operatorname{supp}\mu}$ is topologically transitive. (Compare Proposition 3.4.12.)

3.3. Show that measurable isomorphism (Definition 3.1.1) and monotone equivalence (Definition 3.1.23) of flows are equivalence relations.

3.4. Prove: If μ is an ergodic invariant measure for a continuous flow Φ, then the orbit of μ-a.e. x is dense in $\operatorname{supp}\mu$.

3.5. Theorem 3.3.13 and Proposition 3.4.40 combine to imply that for a suspension of an ergodic probability-preserving transformation there is countable set of $\tau \in \mathbb{R}$ for which the time-τ-map is not ergodic. Describe this exceptional set.

3.6. Show that K-mixing implies mixing (Definition 3.4.1).

3.7. Show that K-mixing implies multiple mixing (Definition 3.4.1).

3.8. Show that the Bernoulli property implies mixing (Definition 3.4.1).

3.9. Show that the Bernoulli property implies multiple mixing (Definition 3.4.1).

3.10. Show that the Bernoulli property implies K-mixing (Definition 3.4.1).

CHAPTER 4

Entropy, pressure, and equilibrium states

The preceding chapters developed important notions for the study of qualitative features of dynamical systems in topological and probabilistic ways. We now introduce quantitative notions for describing the complexity of a dynamical system. The principal notion is entropy. Its probabilistic version measures complexity on an exponential scale by an approach modeled on information theory. The topological version was developed in analogy to measure-theoretic entropy and turns out to bo closely connected to other measures of orbit complexity, such as growth of periodic orbits. Inspired by the study of thermodynamics, a notion of pressure builds on these notions, and connecting these various notions in turn provides new ways of constructing measures of particular dynamical interest.

1. Measure-theoretic entropy

The measure-theoretic entropy of a flow is usually defined in terms of the action of its time-1 map.[1]

Definition 4.1.1 (Measure-theoretic entropy)**.** If Φ is a μ-preserving flow, then the measure-theoretic (or Kolmogorov–Sinai) entropy of Φ is defined by $h_\mu(\Phi) := h_\mu(\varphi^1)$ (see Definition 11.2.27).

While it would be desirable to have a definition for flows that avoids passing to the time-1 map, the definition of entropy in terms of the action on measurable partitions of a measure space $(X, \mathscr{S}, \mu)$ does not translate naturally to continuous time. We outline this approach to illustrate this. The entropy of a partition is defined by

$$H_\mu(\xi) = -\sum_{C \in \xi} \mu(C) \log \mu(C),$$

where $0 \log 0 := 0$. We denote by $\mathscr{P}_H$ the collection of measurable partitions (mod 0) with finite entropy, and we refer to these as *finite-entropy partitions.*

For two measurable partitions ξ and η of X the joint partition is

$$\xi \vee \eta := \{C \cap D \mid C \in \xi, D \in \eta\} \tag{4.1.1}$$

[1] Or as $\sup_{t>0} \frac{1}{t} h_\mu(\varphi^t)$.

For a measurable partition ξ and a measure-preserving (not necessarily invertible) transformation f we define the *joint partition for f of ξ* as follows. For $I \subset \mathbb{R}$ set

$$\xi^f_I := \bigvee_{i \in I \cap \mathbb{Z}} f^i(\xi), \qquad \xi^f_{-n} := \xi^f_{[-n,0)}, \qquad \xi^f_- := \xi^f_{(-\infty,0)}.$$

The measure-theoretic entropy of a measure preserving transformation $f : X \to X$ relative to the partition ξ is $h(f,\xi) := h_\mu(f,\xi) := \lim_{n\to\infty} H(\xi^f_{-n})/n$. The *entropy* of f with respect to μ (or the entropy of μ) is

$$h(f) := h_\mu(f) := \sup\left\{h_\mu(f,\xi) \;\middle|\; \xi \in \mathscr{P}_H\right\}.$$

The difficulty with continuous-time systems is that the join of a partition over an interval in $\mathbb{R}$ does not lend itself to defining a natural notion of complexity. Accordingly, we outline the definitions and properties of entropy for maps in Chapter 11. Readers unfamiliar with measure-theoretic entropy for maps will want to refer to the concepts, definitions, and results there.

The focus of this book is on continuous fixed-point free flows on compact metric spaces, and for these we can take a different approach to define measure-theoretic entropy directly rather than via time-one maps.

Definition 4.1.2 (Measure-theoretic entropy of a flow [**280**])**.** For a continuous fixed-point free flow Φ on a compact metric space X and $t \in \mathbb{R}$ define the (t,ϵ,Φ)-ball around $x \in M$ as

$$B(x,t,\epsilon,\Phi) := \{y \in X \mid \exists \alpha \in \operatorname{Rep}([0,1]),\, d(\varphi^{\alpha(s)t}x, \varphi^{st}y) < \epsilon \text{ for } 0 \le s \le 1\},$$

where

$$\operatorname{Rep}([0,a]) := \{\alpha : [0,a] \to \mathbb{R} \text{ strictly increasing continuous with } \alpha(0) = 0\}$$

is the set of all reparametrizations.

For an ergodic Φ-invariant Borel measure μ and $\delta \in (0,1)$ let $N(\delta,t,\epsilon,\Phi)$ be the minimum number of (t,ϵ,Φ)-balls whose union has measure at least $1-\delta$ and define

$$\bar{h}_\mu(\Phi) := \lim_{\epsilon\to 0} \overline{\lim_{t\to\infty}} \frac{1}{t} \log N(\delta,t,\epsilon,\Phi).$$

(This is indeed independent of δ.)

This formulation of measure-theoretic entropy for flows does not require using the time-1 map, and for a continuous flow on a compact metric space without fixed points coincides with $h_\mu(\Phi)$, see [**280**].

Theorem 4.1.3. *Let $\Phi : X \to X$ be a continuous flow on a compact metric space. If $\mu, \nu \in \mathfrak{M}(\Phi)$ and $p \in [0,1]$, then*

$$h_{p\mu+(1-p)\nu}(\Phi) = p h_\mu(\Phi) + (1-p) h_\nu(\Phi).$$

Proof. If ξ is a finite partition, then Lemma 11.2.15 gives

$$\begin{aligned}0 &\le H_{p\mu+(1-p)\nu}(\xi) - pH_\mu(\xi) - (1-p)H_\nu(\xi)\\ &\le -(p\log p + (1-p)\log(1-p))\\ &\le \log 2.\end{aligned}$$

When η is a finite partition and $\xi \coloneqq \bigvee_{i=0}^{n-1}\varphi^{-i}\eta$, this implies that

$$h_{p\mu+(1-p)\nu}(\Phi,\eta) = ph_\mu(\Phi,\eta) + (1-p)h_\nu(\Phi,\eta).$$

One one hand, taking the supremum over η gives.

$$h_{p\mu+(1-p)\nu}(\Phi) \le ph_\mu(\Phi) + (1-p)h_\nu(\Phi).$$

For the reverse inequality, take $c_\mu < h_\mu(\Phi)$, $c_\nu < h_\nu(\Phi)$ and partitions ξ_μ, ξ_ν such that $h_\mu(\Phi,\xi_1) > c_\mu$ and $h_\nu(\Phi,\xi_1) > c_\nu$. Then $\xi \coloneqq \xi_\mu \vee \xi_\nu$ satisfies

$$\begin{aligned}h_{p\mu+(1-p)\nu}(\Phi,\xi) &= ph_\mu(\Phi,\xi) + (1-p)h_\nu(\Phi,\xi)\\ &\ge ph_\mu(\Phi,\xi_m u) + (1-p)h_\nu(\Phi,\xi_n u) > pc_\mu + (1-p)c_\nu.\end{aligned}$$

Thus, $h_{p\mu+(1-p)\nu}(\Phi,\xi) \ge ph_\mu(\Phi) + (1-p)h_\nu(\Phi)$ since c_μ, c_ν were arbitrary. □

We now describe how to obtain the entropy of a flow under a function (Definition 3.6.1) from the entropy of the base map and relevant information about the function. (If the invariant measure is not normalized, then the entropy will be computed using the associated normalized measure (3.6.3).)

Theorem 4.1.4 (Abramov)**.** *With the notations from Definition 3.6.1, consider a μ-preserving transformation $F\colon (Y,\mu) \to (Y,\mu)$, where μ is a probability measure, and let $\Phi = \Phi_{F,r}$ be the special flow under the roof function r. Suppose there is an $r_0 > 0$ such that $r(y) \ge r_0$ for all $y \in Y$. Then*

$$h_{\mu_r}(\Phi) = h_\mu(F)\Big/\int r\,d\mu. \tag{4.1.2}$$

Proof. Proposition 11.3.15(4) lets us scale t by any rational number, so assume without loss of generality that $0 < t < r_0$ and set $X_t \coloneqq Y \times [0,t) \subset X$. The map of X_t induced by Φ is of the form $\Phi_{X_t}(y,s) = (F(y), s + r(y) - t\lfloor r(y)/t\rfloor)$. So Theorem 11.3.36 and Proposition 11.3.32 give $h_{\mu_r}(\varphi^t) = h_{(\mu_r)_{X_t}}(\Phi_{X_t})\mu_r(X_t) = h_\mu(F)\cdot\frac{t}{\int r\,d\mu}$. □

For the special case of suspensions (that is, $r \equiv 1$) we have

Corollary 4.1.5. $h_{\mu\times m}(F_\circ) = h_\mu(F)$.

Example 4.1.6. Consider F acting on two copies $A = B = (Y,\mu)$, and write $h := h_\mu(F)$. Set $r \equiv a$ on A, $r \equiv b$ on B, and $\Phi_A := \Phi_{\upharpoonright_A}$, $\Phi_B := \Phi_{\upharpoonright_B}$. Then, with self-explanatory notation, $h_{\mu_A}(\Phi_A) = h/a$, $h_{\mu_B}(\Phi_B) = h/b$, and Proposition 11.3.15(2) gives

$$h_\mu(\Phi) = \frac{\nu_A(X)h_\mu(\Phi_A) + \nu_B(X)h_\mu(\Phi_A)}{\nu(X)} = \frac{2}{a+b}\Big(\frac{a}{2}\frac{h}{a} + \frac{b}{2}\frac{h}{b}\Big) = \frac{2h}{a+b} = \frac{h}{\int r} = h_\mu(\Phi)$$

by Abramov's formula (4.1.2).

The context of the Kac Lemma (Proposition 3.6.13) provides an application of Abramov's formula to special flows..

Proposition 4.1.7 ([**284**, Corollary 1])**.** *If F is a μ-preserving map on a topological probability space (X,μ), $0 < r \in L^1(\mu)$, $\mu(A) > 0$, then*

$$\int_A T_A(x)\,d\mu_A = \frac{h_{\mu_A}(F_A)}{h_{\mu_r}(\Phi_r)},$$

where μ_A is the conditional measure from (3.3.3)*, $T_A(x) := \min\{t > 0 \mid \varphi_r^t(x,0) \in A\}$, and F_A the return map from* (11.3.6)*.*

Abramov's formula also provides insights into the effect of time-changes.

Theorem 4.1.8 (Abramov)**.** *If $0 < \rho \in L^1(X,\nu)$, then the time-change Φ_ρ (Theorem 3.5.4) of the special flow $\Phi = \Phi_{F,r}$ satisfies*

$$h_{\nu_\rho}(\Phi_\rho) = h_\nu(\Phi)\int \rho\,d\nu.$$

Proof. Φ_ρ is measure-theoretically isomorphic to a special flow over F with a roof function r_ρ that satisfies

$$r(y) = \int_0^{r_\rho(y)} \rho,$$

which is the "distance" traveled by Φ_ρ in time $r_\rho(y)$. By Fubini's Theorem we have

$$\int r\,d\mu = \int_Y \int_0^{r_\rho(y)} \rho\,d\mu = \int \rho\,d\nu.$$

Now apply Abramov's formula (4.1.2). □

By the Ambrose–Kakutani–Rokhlin Special-Flow Representation Theorem 3.6.2, every measurable flow with essentially no fixed points and with an invariant Borel probability measure is measure-theoretically isomorphic to a special flow, so the preceding result implies one for time-changes in full generality.

Theorem 4.1.9 (Abramov)**.** *If Φ is a measurable flow on (X,μ) with essentially no fixed points and $0 < \rho \in L^1(\mu)$, then $h_{\mu_\rho}(\Phi_\rho) = h_\mu(\Phi)\int \rho\,d\mu$.*

Corollary 4.1.10. *If Φ is a measurable flow on (X,μ), then $h_\mu(\varphi^T) = |T| h_\mu(\varphi^1)$.*

Proof. φ^T is the time-1 map of the flow $\Psi\colon t \mapsto \psi^t := \varphi^{tT}$, so if $T > 0$, then

$$h_\mu(\varphi^T) = h_\mu(\Psi) \overset{\text{Theorem 4.1.9}}{=\!=\!=} T h_\mu(\Phi) = T h_\mu(\varphi^1). \qquad \square$$

Corollary 4.1.5 puts us in a position to study the entropy in a familiar example.

Example 4.1.11. Consider the toral automorphism from Example 1.5.23 with Lebesgue measure m as the invariant Borel probability measure. To simplify notation write F for this hyperbolic toral automorphism and denote the maximal eigenvalue by λ. Let ξ be a finite partition of $\mathbb{T}^2$ into elements of diameter less than $1/10$. We estimate $H(\bigvee_{k=-n}^{n} F^k(\xi)) = H(\xi_{-2n-1}^F)$ from below by estimating from above the diameter and hence the Lebesgue measure of the elements of ξ_{-2n-1}^F. Let $C \in \bigvee_{k=-n}^{n} F^k(\xi)$ and $x, y \in C$. Consider the line parallel to the eigenvector with eigenvalue $\lambda > 1$ passing through the point x and the line parallel to the second eigenvector passing through y. Define z as the first point of intersection of these lines. Then $d(F^k(x), F^k(y)) \le d(F^k(x), F^k(z)) + d(F^k(z), F^k(y))$. First, let $k > 0$. Then $d(F^k(z), F^k(y)) = \lambda^{-k} d(z, y) \le \lambda^{-k} d(x, y) < \lambda^{-k}/10$. Since for $k = 1, \dots, n$ the points $F^k(x)$, $F^k(y)$ belong to the same element of the partition ξ we have $d(F^k(x), F^k(y)) < 1/10$ and hence $d(F^k(x), F^k(z)) < 1/10 + \lambda^{-k}/10 < 1/5$. This implies by induction that the length of the line segment connecting $F^k(x)$ and $F^k(z)$ is also less than $1/5$. Hence $d(x,z) = \lambda^{-n} d(F^n(x), F^n(z)) < \lambda^{-n}/5$. A similar argument for negative k shows that $d(y,z) < \lambda^{-n}/5$ and hence we have $d(x,y) < 2\lambda^{-n}/5$. Thus the diameter of any element of $\bigvee_{-n}^{n} F^{-k}(\xi)$ is at most $2\lambda^{-n}/5$ and hence by the isoperimetric inequality its Lebesgue measure is at most $2\pi\lambda^{-2n}/25$. Thus the left inequality in Proposition 11.3.1(1) gives $h(F,\xi) \ge \log\lambda$ and hence

$$h_m(F) \ge \log\lambda$$

for Lebesgue measure m. This is also the entropy of the suspension. Furthermore, comparison with Proposition 4.2.17 and Corollary 4.3.8 implies that we actually have equality.

2. Topological entropy

We now return to topological dynamics to introduce a counterpart of entropy in this setting, topological entropy. One way of looking at it is to ask naively "how many orbits are there?" Since we have already studied periodic points as anchors for nearby dynamical behavior, counting periodic orbits is a way to seek additional information. This plays out in slightly different ways for flows than for discrete-time dynamical systems because in the latter case the lengths of periodic orbits are integers, so one can simply note how many periodic points there are for a given integer.

Furthermore, for a flow it makes no sense to try to count periodic *points* because there are either none or uncountably many of them, so we count periodic *orbits* instead. This can be done in two different ways. It would be closest to the discrete-time case to count periodic orbits weighted by their length, which is what counting of periodic points amounts to in that case (Corollary 1.8.6). On the other hand one can count just the number of periodic orbits without weighting by their lengths. If, however, the number of periodic orbits grows exponentially then the distinction is immaterial because most orbits of length up to T have length close to T, so the growth rate is the same.

Definition 4.2.1 (Periodic orbit growth)**.** Let $P_T(\Phi)$ be the number of all periodic orbits of period up to T and

$$p(\Phi) \coloneqq \varlimsup_{T\to\infty} \frac{1}{T}\log(\max(P_T(\Phi),1))$$

the exponential growth rate of the number of periodic orbits for a flow.

Going beyond periodic points, topological entropy is the most important numerical invariant related to the orbit growth and represents the exponential growth rate of the number of orbit segments distinguishable with arbitrarily fine but finite precision. In a sense, the topological entropy describes in a crude but suggestive way the total (rather than average) exponential complexity of the orbit structure with a single number.

Let $\Phi = \{\varphi^t\}$ be a continuous flow on a compact metric space (X,d). The family of metrics d_t^Φ defined by

$$d_t^\Phi(x,y) \coloneqq \max_{0\le\tau\le t} d(\varphi^\tau(x),\varphi^\tau(y))$$

measures the distance between the orbit segments $\varphi^{[0,t]}(x)$ and $\varphi^{[0,t]}(y)$ and defines the *Bowen balls*

$$B_\Phi(x,\epsilon,t) \coloneqq \{y\in X \mid d_t^\Phi(x,y)<\epsilon\}. \tag{4.2.1}$$

A set $E\subset X$ is said to be (t,ϵ)*-spanning* or (t,ϵ)*-dense* if $X\subset\bigcup_{x\in E} B_\Phi(x,\epsilon,t)$. Let $S_d(\Phi,\epsilon,t)$ be the minimal cardinality of a (t,ϵ)-spanning set, or equivalently the cardinality of a *minimal* (t,ϵ)*-spanning set.* This is the minimal number of initial conditions whose behavior up to time t approximates the behavior of *any* initial condition up to ϵ. Consider its exponential growth rate

$$h_d(\Phi,\epsilon) \coloneqq \varlimsup_{t\to\infty}\frac{1}{t}\log S_d(\Phi,\epsilon,t). \tag{4.2.2}$$

Obviously $h_d(\Phi,\epsilon)$ does not decrease with ϵ.

Definition 4.2.2. The *topological entropy* of Φ is

$$h_{\text{top}}(\Phi) := h(\Phi) := h_d(\Phi) = \lim_{\epsilon \to 0} h_d(\Phi, \epsilon).$$

Remark 4.2.3. For future reference we note that for compact $K \subset X$ we can likewise define the entropy $h_{\text{top}}(\Phi, K)$ of Φ on K: replace $S_d(\Phi, \epsilon, t)$ by the minimal cardinality of an $E \subset K$ that is (t, ϵ)-dense in K, then take the exponential growth rate and let $\epsilon \to 0$.

Topological entropy is defined in terms of the metric d but does not depend on it:

Proposition 4.2.4. *If d' is another metric on X which defines the same topology as d, then $h_{d'}(\Phi) = h_d(\Phi)$.*

Proof. Consider the set D_ϵ of all pairs $(x_1, x_2) \in X \times X$ for which $d(x_1, x_2) \geq \epsilon$. This is a compact subset of $X \times X$ with the product topology. The function d' is continuous on $X \times X$ in that topology and consequently it reaches its minimum $\delta(\epsilon)$ on D_ϵ. This minimum is positive; otherwise there would be points $x_1 \neq x_2$ such that $d'(x_1, x_2) = 0$. Thus, if $d'(x_1, x_2) < \delta(\epsilon)$, then $d(x_1, x_2) < \epsilon$, that is, any $\delta(\epsilon)$-ball in the metric d' is contained in an ϵ-ball in the metric d. This argument extends immediately to the metrics d'^{Φ}_t and d^{Φ}_t. Thus for every t we have $S_{d'}(\Phi, \delta(\epsilon), t) \geq S_d(\Phi, \epsilon, t)$ so $h_{d'}(\Phi, \delta(\epsilon)) \geq h_d(\Phi, \epsilon)$ and $h_{d'}(\Phi) \geq \lim_{\epsilon \to 0} h_{d'}(\Phi, \delta(\epsilon)) \geq \lim_{\epsilon \to 0} h_d(\Phi, \epsilon) = h_d(\Phi)$. Interchanging the metrics d and d' one obtains $h_d(\Phi) \geq h_{d'}(\Phi)$. □

Corollary 4.2.5. *The topological entropy is an invariant of topological conjugacy.*

Proof. Let $\Phi \colon X \to X$, $\Psi \colon Y \to Y$ be topologically conjugate via a homeomorphism $h \colon X \to Y$. Fix a metric d on X and define d' on Y as the pullback of d, that is, $d'(y_1, y_2) = d(h^{-1}(y_1), h^{-1}(y_2))$. Then h is an isometry so $h_d(\Phi) = h_{d'}(\Psi)$. □

There are several quantities similar to $S_d(\Phi, \epsilon, t)$ that can be used to define topological entropy. For example, let $D_d(\Phi, \epsilon, t)$ be the minimal number of sets whose d^{Φ}_t-diameter is at most ϵ and whose union covers X. The diameter of an ϵ-ball is at most 2ϵ so every covering by ϵ-balls is a covering by sets of diameter $\leq 2\epsilon$, that is,

$$D_d(\Phi, 2\epsilon, t) \leq S_d(\Phi, \epsilon, t). \tag{4.2.3}$$

On the other hand, any set of diameter $\leq \epsilon$ is contained in the ϵ-ball around each of its points so

$$S_d(\Phi, \epsilon, t) \leq D_d(\Phi, \epsilon, t). \tag{4.2.4}$$

Lemma 4.2.6. *For any $\epsilon > 0$ the limit $\lim_{t \to \infty}(1/t) \log D_d(\Phi, \epsilon, t) < \infty$ exists.*

Proof. We show that the sequence $a_n := \log D_d(\Phi,\epsilon,n)$ is subadditive: $a_{m+n} \le a_n + a_m$. Then $\lim_{n\to\infty} a_n/n$ exists by Lemma 4.2.7. This implies the claim by monotonicity of $t \mapsto D_d(\Phi,\epsilon,t)$.

To prove that $D_d(\Phi,\epsilon,s+t) \le D_d(\Phi,\epsilon,t)\cdot D_d(\Phi,\epsilon,s)$ for all s,t, note that if A has d_t^Φ-diameter less than ϵ and B has d_s^Φ-diameter less than ϵ, then $A\cap\varphi^{-t}(B)$ has $d_{s+t}^{\varphi^t}$-diameter less than ϵ. Thus if $\mathfrak{A}$ is a cover of X by $D_d(\Phi,\epsilon,t)$ sets of d_t^Φ-diameter less than ϵ and $\mathfrak{B}$ is a cover of X by $D_d(\Phi,\epsilon,s)$ sets of d_s^Φ-diameter less than ϵ, then the cover by all sets $A\cap\varphi^{-t}(B)$, where $A\in\mathfrak{A}$, $B\in\mathfrak{B}$, which contains at most $D_d(\Phi,\epsilon,t)\cdot D_d(\Phi,\epsilon,s)$ sets, is a cover by sets of $d_{s+t}^{\varphi^t}$-diameter less than ϵ. □

Lemma 4.2.7 (Bowen–Fekete Lemma, subadditivity)**.** *If there are k,L such that $a_{m+n} \le a_m + a_{n+k} + L$ for all $m,n\in\mathbb{N}$ then* $\frac{a_n}{n} \xrightarrow[n\to\infty]{} \inf_{n\in\mathbb{N}} \frac{a_{n+k}+L}{n} \in [-\infty,\infty)$.

Proof. $l = r + in$ with $0\le r<n$ gives $\frac{a_l}{l} \le \frac{a_r + i(a_{n+k}+L)}{r+in}$. If $l\to\infty$ (with n fixed, so $i\to\infty$), then $\overline{\lim}_{l\to\infty}\frac{a_l}{l} \le \inf_n \frac{a_{n+k}+L}{n} \le \underline{\lim}_{n\to\infty}\frac{a_{n+k}+L}{n} = \underline{\lim}_{l\to\infty}\frac{a_l}{l}$[2]. □

From (4.2.3) and (4.2.4) we see that

$$\tilde{h}_d(\Phi,\epsilon) := \lim_{n\to\infty}(1/t)\log D_d(\Phi,\epsilon,t) \ge h_d(\Phi,\epsilon) \ge \tilde{h}_d(\Phi,2\epsilon),$$

and similarly for $\underline{h}_d(\Phi,\epsilon) := \underline{\lim}_{t\to\infty}(1/t)\log S_d(\Phi,\epsilon,t)$ instead of $h_d(\Phi,\epsilon)$. Thus,

$$\lim_{\epsilon\to 0}\tilde{h}_d(\Phi,\epsilon) = \lim_{\epsilon\to 0}\underline{h}_d(\Phi,\epsilon) = h(\Phi),$$

and $\lim_{\epsilon\to 0}\big(h_d(\Phi,\epsilon) - \underline{h}_d(\Phi,\epsilon)\big) = 0$. So the topological entropy can be defined in terms of the number of open sets whose d_t^Φ-diameter is at most ϵ and whose union covers X.

One more way to define topological entropy is via the numbers $N_d(\Phi,\epsilon,t)$, the maximal number of points in X with pairwise d_n^Φ-distances at least ϵ. We call such a set of points (t,ϵ)-*separated.* Such points generate the maximal number of orbit segments of length t that are distinguishable with precision ϵ. A maximal (t,ϵ)-separated set is a (t,ϵ)-spanning set, that is, for any such set of points the ϵ-balls around them cover X, because otherwise it would be possible to increase the set by adding any point not covered. Thus

$$N_d(\Phi,\epsilon,t) \ge S_d(\Phi,\epsilon,t). \tag{4.2.5}$$

On the other hand, no ϵ-ball can contain two points 2ϵ apart. Thus

$$S_d(\Phi,\epsilon,t) \ge N_d(\Phi,2\epsilon,t). \tag{4.2.6}$$

[2] This extends to $k,L\ge 0$ depending on n so long as both are $o(n)$.

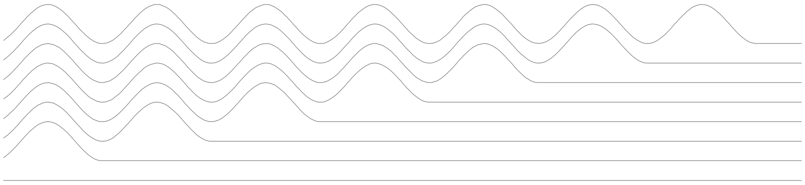


FIGURE 4.2.1. A separated set [©Cambridge University Press, reprinted from [**149**] with permission]

Using (4.2.5) and (4.2.6) we obtain

$$\underline{h}_d(\Phi,\epsilon) \le \varliminf_{t\to\infty} \frac{1}{t} \log N_d(\Phi,2\epsilon,t) \le \varlimsup_{t\to\infty} \frac{1}{t} \log N_d(\Phi,2\epsilon,t) \le h_d(\Phi,\epsilon)$$

and hence

$$h_{\text{top}}(\Phi) = \lim_{\epsilon\to0} \varlimsup_{t\to\infty} \frac{1}{t} \log N_d(\Phi,\epsilon,t) = \lim_{\epsilon\to0} \varliminf_{t\to\infty} \frac{1}{t} \log N_d(\Phi,\epsilon,t),$$

justifying the description as “the exponential growth rate of the number of orbit segments distinguishable with arbitrarily fine but finite precision.”

For a map $f : X \to X$ the family of metrics d_n^f is defined, similar to flows, by

$$d_n^f(x,y) \coloneqq \max_{0\le i\le n} d(f^i(x), f^i(y)) \tag{4.2.7}$$

Then the topological entropy is similarly defined as

$$h_{\text{top}}(f) \coloneqq h_d(f) = \lim_{\epsilon\to0} h_d(f,\epsilon) = \lim_{\epsilon\to0} \varlimsup_{n\to\infty} \frac{1}{n} \log S_d(\Phi,\epsilon,n).$$

Equicontinuity of $\{\varphi^s \mid |s| \le 1\}$ implies

Proposition 4.2.8. $h_{\text{top}}(\Phi) = h_{\text{top}}(\varphi^1)$.

Remark 4.2.9. See also Proposition 4.3.6.

PROOF. Let $\epsilon > 0$. Fix $\delta > 0$ such that $d(\varphi^t x, \varphi^t y) < \epsilon$ for all $0 < t \le 1$ when $d(x,y) < \delta$. Let E be an (n,δ)-spanning set for φ^1, then E is (t,ϵ)-spanning for Φ where $t \le n$. Then $S_d(\Phi,\epsilon,t) \le S_d(\varphi^1,\delta,n)$ for $t \le n$. Hence,

$$\varlimsup_{t\to\infty} \frac{1}{t} \log S_d(\Phi,\epsilon,t) \le \varlimsup_{n\to\infty} \frac{1}{n} \log S_d(\varphi^1,\delta,n).$$

From this we see that $h_{\text{top}}(\Phi) \le h_{\text{top}}(\varphi^1)$.

The other direction follows directly from the definitions. Indeed,

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log S_d(\Phi,\epsilon,t) \geq \overline{\lim_{n\to\infty}} \frac{1}{n}\log S_d(\Phi,\epsilon,n) \geq \overline{\lim_{n\to\infty}} \frac{1}{n}\log S_d(\varphi^1,\epsilon,n),$$

so $h_d(\Phi,\epsilon) \geq h_d(\varphi^1,\epsilon)$ for each $\epsilon > 0$, and $h_{\text{top}}(\Phi) \geq h_{\text{top}}(\varphi^1)$. □

Corollary 4.2.10. *For $t \in \mathbb{R}$ we have $h_{\text{top}}(\varphi^t) = |t| h_{\text{top}}(\varphi^1) = |t| h_{\text{top}}(\Phi)$.*

Proof. For $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $d(x,y) < \delta(\epsilon) \Rightarrow d(\varphi^r(x),\varphi^r(y)) < \epsilon$ for $0 \leq r \leq s$. If E is (n,δ)-spanning for φ^s, then E is (m,ϵ)-spanning for φ^t so long as $mt \leq ns$. So $S_d(\varphi^t, m,\epsilon) \leq S_d(\varphi^1,\delta,\lfloor\frac{mt}{s}\rfloor + 1)$, hence

$$\begin{aligned}\overline{\lim_{m\to\infty}} \frac{1}{m}\log S_d(\varphi^t,\epsilon,m) &\leq \overline{\lim_{m\to\infty}} \frac{1}{m}\log S_d(\varphi^s,\delta,\lfloor\frac{mt}{s}\rfloor + 1) \\ &= \overline{\lim_{m\to\infty}} \left(\frac{1}{m}\big(\lfloor\frac{mt}{s}\rfloor + 1\big)\right) h_d(\varphi^s,\delta) = \frac{t}{s} h_d(\varphi^s,\delta).\end{aligned}$$

So $s h_{\text{top}}(\varphi^t) \leq t h_{\text{top}}(\varphi^s)$. By symmetry we have equality, and setting $s = 1$ gives the claim for $t \geq 0$.

Finally, the image of a (t,ϵ)-separated set for Φ is (t,ϵ)-separated for the inverse flow and vice versa. So $h_{\text{top}}(\varphi^{-1}) = h_{\text{top}}(\varphi^1)$. The result now follows. □

Proposition 4.2.11. *If Ψ is a factor (Definition 1.3.1) of Φ, then $h_{\text{top}}(\Psi) \leq h_{\text{top}}(\Phi)$.*

Proof. Let $\Phi\colon X \to X$, $\Psi\colon Y \to Y$, $\pi\colon X \to Y$, $\pi\circ\Phi = \Psi\circ\pi$, $\pi(X) = Y$, and d_X, d_Y be the distance functions in X and Y, correspondingly.

π is uniformly continuous, so for any $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that if $d_X(x_1,x_2) < \delta(\epsilon)$, then $d_Y(\pi(x_1),\pi(x_2)) < \epsilon$. Thus the image of any $(d_X)_t^\Phi$ ball of radius $\delta(\epsilon)$ lies inside a $(d_Y)_t^\Psi$ ball of radius ϵ, that is,

$$S_{d_X}(\Phi,\delta(\epsilon),t) \geq S_{d_Y}(\Psi,\epsilon,t).$$

Taking logarithms and limits, we obtain the result. □

We amplify this with Theorem 4.2.13 below.

The following proposition contains an incomplete list of standard elementary properties of topological entropy. The proofs demonstrate the usefulness of switching back and forth from one of the three definitions to another.

Proposition 4.2.12. *(1) If Λ is closed and Φ-invariant, then $h_{\text{top}}(\Phi\restriction_\Lambda) \leq h_{\text{top}}(\Phi)$.*

(2) If $X = \bigcup_{i=1}^m \Lambda_i$, where Λ_i, $(i = 1,\dots,m)$ are closed Φ-invariant sets, then $h_{\text{top}}(\Phi) = \max_{1\leq i\leq m} h_{\text{top}}(\Phi\restriction_{\Lambda_i})$.

(3) $h_{\text{top}}(\varphi^{mt}) = |m| h_{\text{top}}(\varphi^t)$.

(4) $h_{\text{top}}(\Phi \times \Psi) = h_{\text{top}}(\Phi) + h_{\text{top}}(\Psi)$.

Here if $\Phi\colon X \to X$, $\Psi\colon Y \to Y$, *then* $\Phi \times \Psi\colon X \times Y \to X \times Y$ *is defined by* $(\varphi^t \times \psi^t)(x, y) = (\varphi^t(x), \psi^t(y))$.

We note that (3) is the best we can do: there is no Abramov-like theorem for topological entropy as in the measurable case.

Proof. (1): every cover of X by sets of d_t^{Φ}-diameter less than ϵ is a cover of Λ.

(2): the union of covers of $\Lambda_1, \dots, \Lambda_m$ by sets of diameter less than ϵ is a cover of X, so

$$D_d(\Phi, \epsilon, t) \le \sum_{i=1}^{m} D_d(\Phi\restriction_{\Lambda_i}, \epsilon, t),$$

that is, for at least one i

$$D_d(\Phi\restriction_{\Lambda_i}, \epsilon, t) \ge \frac{1}{m} D_d(\Phi, \epsilon, t).$$

Since there are only finitely many i, at least one i works for infinitely many t, so

$$\varlimsup_{t\to\infty} \frac{\log D_d(\Phi\restriction_{\Lambda_i}, \epsilon, t)}{t} \ge \varlimsup_{t\to\infty} \frac{\log D_d(\Phi, \epsilon, t) - \log m}{t} = \tilde{h}_d(\Phi, \epsilon).$$

This proves (2).

For positive m (3) follows from $d_t^{\varphi^{mt}} = d_{mt}^{\varphi^t}$. If $m = -1$ note that $B_{\varphi^t}(x, \epsilon, t) = B_{\varphi^{-t}}(\varphi^t(x), \epsilon, t)$ and $S_d(\varphi^t, \epsilon, t) = S_d(\varphi^{-t}, \epsilon, t)$, so $h_{\text{top}}(\varphi^t) = h_{\text{top}}(\varphi^{-t})$.

For negative m (3) follows from the statement for $m > 0$ and $m = -1$.

(4): balls in the product metric

$$d((x_1, y_1), (x_2, y_2)) := \max(d_X(x_1, x_2), d_Y(y_1, y_2))$$

on $X \times Y$ are products of balls on X and Y. The same is true for balls in $d_t^{\varphi^t \times \psi^t}$. Thus

$$S_d(\Phi \times \Psi, \epsilon, t) \le S_{d_X}(\Phi, \epsilon, t) S_{d_Y}(\Psi, \epsilon, t)$$

and $h_{\text{top}}(\Phi \times \Psi) \le h_{\text{top}}(\Phi) + h_{\text{top}}(\Psi)$. On the other hand, the product of any (t, ϵ)-separated set in X for Φ and any (t, ϵ)-separated set in Y for Ψ is a (t, ϵ)-separated set for $\Phi \times \Psi$. Thus

$$N_d(\Phi \times \Psi, \epsilon, t) \ge N_{d_X}(\Phi, \epsilon, t) \times N_{d_Y}(\Psi, \epsilon, t)$$

and hence $h_{\text{top}}(\Phi \times \Psi) \ge h_{\text{top}}(\Phi) + h_{\text{top}}(\Psi)$. □

We will see later that in the case of hyperbolic flows one of the standard methods to compute entropy is to find an extension that is uniformly finite-to-one and whose entropy is easier to compute. This works because the entropies of the two systems are equal:

Theorem 4.2.13. *If $\Phi : X \to X$ and $\Psi : Y \to Y$ are continuous flows on compact metric spaces and $\pi : X \to Y$ is a semiconjugacy from Φ to Ψ that is uniformly finite-to-one, then $h_{\text{top}}(\Phi) = h_{\text{top}}(\Psi)$.*

Proof. Proposition 4.2.11 gives $h(\Phi) \ge h(\Psi)$. Proposition 4.2.8 reduces showing that $h(\Phi) \le h(\Psi)$ to proving that $h(\varphi^1) \ge h(\psi^1)$, and using time-1 maps lets us set up a combinatorial argument in discrete time for effective control of the number of orbits. X, Y come with metrics d, d', respectively.

For $\epsilon > 0$, $C \ge \max_{y\in Y} \#\pi^{-1}(y)$, $m \in \mathbb{N}$, $y \in Y$ let

$$U_y = U_{y,n,\epsilon} = \{x \in X \mid d_n^{\varphi^1}(x,z) < \epsilon \text{ for some } z \in \pi^{-1}(y)\} \supset \pi^{-1}(y).$$

Since φ is continuous there is an open neighborhood W_y of y such that $\pi^{-1}(W_y) \subset U_y$.

Since Y is compact there is a finite cover $\{W_{y_1}, \dots, W_{y_p}\}$. Let $\beta > 0$ be a Lebesgue number of this cover (that is, if $y \in Y$ there exists W_{y_j} such that $\overline{B_\beta(y)} \subset W_{y_j}$). For sufficiently small $\epsilon > 0$ we will show that

$$\frac{1}{n}\log(N_d(\varphi^1, 2\epsilon, n)) \le \frac{1}{n}\log(S_{d'}(\psi^1, \beta, n)) + \frac{1}{m}\log C + \frac{1}{n}\log C. \tag{4.2.8}$$

This completes the proof because then

$$h_d(\varphi^1, 2\epsilon) \le h_{d'}(\psi^1, \beta) + \frac{1}{m}\log C, \text{ hence } h_d(\varphi^1, 2\epsilon) \le h_{d'}(\psi^1, \beta)$$

since m is arbitrary. If $\epsilon \to 0$, then $\beta \to 0$, so indeed $h(\varphi^1) \le h(\psi^1)$.

So, for $n \in \mathbb{N}$ let $\ell \in \mathbb{N}$ such that $(\ell - 1)m < n \le \ell m$. Let $A \subset X$ be a maximal $(n, 2\epsilon)$-separated set for φ^1 and $B \subset Y$ be a minimal (n, β)-spanning set for ψ^1.

For $y \in B$ let $q(j, y) \in \{y_1, \dots, y_p\}$ such that $\overline{B_\beta(\varphi^j(y))} \subset W_{q(j,y)}$. Now define

$$\pi_\ell : A \to B \times X^\ell \text{ by } \pi_\ell(x) = (y, x_0, \dots, x_{\ell-1})$$

where $d'_n(y, \pi(x)) \le \beta$, $y \in B$, and $x_s \in \pi^{-1}(q(sm, y))$ such that $d_m(\varphi^{sm}(x), x_s) < \epsilon$ for all $0 \le s < \ell$; this is possible since

$$\pi \circ \varphi^{sm}(x) = \psi^{sm} \circ \pi(x) \in \overline{B_\beta(\varphi^{sm}(y))} \subset W_{q(sm,y)}$$

implies

$$\varphi^{sm}(x) \in \pi^{-1}(W_{q(sm,y)}) \subset U_{q(sm,y),m,\epsilon}.$$

Claim 4.2.14. *π_ℓ is* 1-1.

Proof. If $\pi_\ell(x) = \pi_\ell(x')$, $0 \le t < m$, and $0 \le s \le \ell$, then

$$d(\varphi^{sm+t}(x), \varphi^{sm+t}(x')) \le d_m(\varphi^{sm}(x), x_s)) + d_m(x_s, \varphi^{sm}(x')) \le \epsilon + \epsilon = 2\epsilon.$$

Since $m\ell \ge n$ we get $d_n(x, x') \le 2\epsilon$, hence $x = x'$ since A is 2ϵ-separating. □

This gives (4.2.8): If $y \in B$, then

$$\underbrace{\#\Big(\pi_\ell(A) \cap \Big(\{y\} \times X^\ell\Big)\Big)}_{\le \#(\pi_\ell(A)) = N_d(\varphi^1, 2\epsilon, n)} \le \prod_{s=0}^{\ell-1} \#\big(\pi^{-1}(q(sm, y)\big) \le C^\ell.$$

There are $\#(B) = S_{d'}(\psi^1, \beta, n)$ choices of y, so $N_d(\varphi^1, 2\epsilon, n) \le S_{d'}(\psi^1, \beta, n)C^\ell$, and

$$\frac{1}{n}\log(N_d(\varphi^1, 2\epsilon, n)) \le \frac{1}{n}\log(S_{d'}(\psi^1, \beta, n)) + \underbrace{\frac{1}{n}\log C^\ell}_{=\frac{\ell m}{nm}\log C \le \frac{n+m}{nm}\log C}. \qquad \square$$

From Theorem 4.2.13 and Proposition 1.8.19 we see that the suspension of the symbolic flow constructed for the toral automorphism F_A in (1.8.4) and the suspension of F_A itself have the same entropy. We will see later that this is a more general result for codings of hyperbolic flows. In the more general setting it can be difficult to compute the entropy for the hyperbolic flow, but easier to compute the entropy for the symbolic coding.

Furthermore, from Proposition 11.3.15(4) we have a nice connection between measure theoretic entropy of a map and the special flow with a roof function. For topological entropy there is not such a nice connection between the topological entropy of the special flow and the topological entropy of the base. However, for (constant-time) suspensions there is a direct correspondence between flow and base.

Proposition 4.2.15. *The topological entropy of a suspension flow equals that of the base.*

Proof. By Proposition 4.2.8 we want to show that the entropy of the time-1 map is that of the base. The time-1 map is the cartesian product of the base and the identity; the latter has zero entropy because there is no dependence on n in (4.2.7), so the discrete-time counterpart of Proposition 4.2.12(4) yields the claim. □

This helps compute the topological entropy for several suspensions. Notably, we can apply the following to the corresponding suspensions (compare Corollary 1.8.6):

Proposition 4.2.16. $h_{\text{top}}(\sigma\restriction_{\Sigma_A}) = \log|\lambda_A^{\max}|$ *for any topological Markov chain* Σ_A.

Proof. We endow the space Σ_N with the metric $d = d_{10N}$ given by

$$d_{10N}(\omega, \omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega'_n|}{(10N)^{|n|}}.$$

Then for $\alpha = (\alpha_{-m}, \dots, \alpha_m)$ the symmetric cylinder $C^m_\alpha = \{\omega \in \Sigma_N \mid \omega_i = \alpha_i \text{ for } |i| \le m\}$ is at the same time the ball of radius $\epsilon_m = (10N)^{-m}/2$ around each of its points. Similarly if we fix numbers $\alpha_{-m}, \dots, \alpha_{m+n}$, the cylinder

$$C^{-m,\dots,n+m}_{\alpha_{-m},\dots,\alpha_{n+m}} = \{\omega \in \Sigma_N \mid \omega_i = \alpha_i \text{ for } -m \le i \le m+n\} \tag{4.2.9}$$

is at the same time the ball of radius ϵ_m around each of its points with respect to the metric d_n associated with the shift σ. Thus, any two d_n balls of radius ϵ_m are either identical or disjoint and there are exactly N^{n+2m+1} different ones of the form (4.2.9). The covering of Σ_N by those balls is obviously minimal, so $S_{d_{10N}}(\sigma, \epsilon_m, n) = N^{n+2m+1}$ and

$$h_{\text{top}}(\sigma\upharpoonright_{\Sigma_N}) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{n} \log N^{n+2m+1} = \log N.$$

Similarly, for the topological Markov chain Σ_A, we have $S_d(\sigma, \epsilon_m, n)$ equals the number of those cylinders (4.2.9) that have nonempty intersection with the set Σ_A. Assume each row of the matrix A contains at least one 1. Since the number of admissible paths of length n that begin with the symbol i and end with the symbol j is equal to the entry a^n_{ij} of the matrix A^n, the number of nonempty cylinders of rank $n+1$ in Σ_A is equal to $\sum_{i,j=0}^{N-1} a^n_{ij} < C \cdot \|A^n\|$ for some constant C. On the other hand, since all numbers a^n_{ij} are nonnegative, $\sum_{i,j=0}^{N-1} a^n_{ij} > c\|A^n\|$ for another constant $c > 0$. Thus, we have

$$S_{d_{10N}}(\sigma, \epsilon_m, n) = \sum_{i,j=0}^{N-1} a^{n+2m}_{ij}$$

and

$$\begin{aligned} \log|\lambda^{\max}_A| = \log r(A) &= \lim_{n\to\infty} \frac{1}{n} \log\|A^n\| = \lim_{n\to\infty} \frac{1}{n} \log\|A^{n+2m}\| \\ &= \lim_{n\to\infty} \frac{1}{n} \log S_{d_{10N}}(\sigma, \epsilon_m, n) = h_{\text{top}}(\sigma\upharpoonright_{\Sigma_A}), \end{aligned}$$

where $r(A)$ is the spectral radius of the matrix A (Definition 12.3.1). □

As we noted before, Theorem 4.2.13 and Proposition 1.8.19 imply that the suspension of the symbolic flow constructed for the toral automorphism F_A in (1.8.4) and the suspension of F_A itself have the same entropy, so Proposition 4.2.16 now enables us to compute the entropy of the latter. We can at the same time determine the growth of the number of periodic orbits (Definition 4.2.1).

Proposition 4.2.17. *If $F = F_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ is given by $F(x, y) = (2x + y, x + y) \pmod 1$, then its suspension $F_\circ$ satisfies*

$$h_{\text{top}}(F_\circ) = p(F_\circ) = \log\frac{3+\sqrt{5}}{2},$$

the larger eigenvalue of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ from Example 1.5.23.

Proof. To show that $h_{\text{top}}(F_\circ) = \frac{3+\sqrt{5}}{2}$ we show that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1&1&0&1&0 \\ 1&1&0&1&0 \\ 1&1&0&1&0 \\ 0&0&1&0&1 \\ 0&0&1&0&1 \end{pmatrix}$ from (1.8.3) have the same maximal eigenvalue (in fact, the same set of nonzero eigenvalues): subtract column 4 of $A - \lambda\,\mathrm{Id}$ from the first two columns and column 5 from the third, then add rows 1 and 2 to row 4 and row 3 to row 5:

$$\begin{pmatrix} 1-\lambda & 1 & 0 & 1 & 0 \\ 1 & 1-\lambda & 0 & 1 & 0 \\ 1 & 1 & -\lambda & 1 & 0 \\ 0 & 0 & 1 & -\lambda & 1 \\ 0 & 0 & 1 & 0 & 1-\lambda \end{pmatrix} \to \begin{pmatrix} -\lambda & 0 & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & 0 \\ \lambda & \lambda & 0 & -\lambda & 1 \\ 0 & 0 & \lambda & 0 & 1-\lambda \end{pmatrix} \to \begin{pmatrix} -\lambda & 0 & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 1 & 1-\lambda \end{pmatrix}.$$

To determine the growth of periodic orbits, let $\lambda = \frac{3+\sqrt{5}}{2}$ and $G = F^n - \mathrm{Id}$. Then $\mathrm{Fix}(F_A^n) = G^{-1}(0,0)$ is parametrized by $\mathbb{Z}^2 \cap (A^n - \mathrm{Id})([0,1) \times [0,1))$. The cardinality is

$$\mathrm{area}\big((A^n - \mathrm{Id})([0,1] \times [0,1])\big) = |\det(A^n - \mathrm{Id})| = |(\lambda^n - 1)(\lambda^{-n} - 1)| = \lambda^n + \lambda^{-n} - 2,$$

which has exponential growth rate $\log\lambda$. □

The coincidence of topological entropy with the growth rate of periodic orbits (Definition 4.2.1) is not accidental but due to expansivity (Remark 4.2.25).

Proposition 4.2.18. *Let Φ be a continuous expansive flow on a compact metric space X, 2η an expansivity constant. Then for $\epsilon \in (0,\eta)$, $\delta > 0$ there is a $C_{\delta,\epsilon}$ such that $N_d(\Phi,\delta,t) \le C_{\delta,\epsilon} N_d(\Phi,\epsilon,t)$ for all $t > 0$.*

Proof. Proposition 1.7.4 and equicontinuity of $\Phi\restriction_{[-T,T]\times X}$ give $T, \alpha > 0$ with

$$d^\Phi_{2T}(\varphi^{-T}(x), \varphi^{-T}(y)) \le 2\epsilon \Rightarrow d(x,y) < \delta \text{ and } d(x,y) < \alpha \Rightarrow d^\Phi_{2T}(\varphi^{-T}(x), \varphi^{-T}(y)) \le \delta.$$

Let E be a maximal (t,δ)-separated set and F a maximal (t,ϵ)-separated set. For $x \in E$ there is a $z = S(x) \in F$ with $d^\Phi_t(x,z) < \epsilon$, so $\operatorname{card} E \le \sum_{z\in F} \operatorname{card} S^{-1}(\{z\})$, and we bound $\operatorname{card} S^{-1}(\{z\})$ as follows.

If $x \neq y \in S^{-1}(\{z\})$, then $d_t^{\Phi}(x,y) \le 2\epsilon$ by definition of S, so $d(\varphi^s(x), \varphi^s(y)) \le \delta$ for $s \in [T, t-T]$ by choice of T, and the choice of α implies that either $d(x,y) > \alpha$ or $d(\varphi^t(x), \varphi^t(y)) > \alpha$. Thus,

$$\begin{aligned}\operatorname{card} S^{-1}(\{z\}) &= \operatorname{card}\{(x, \varphi^t(x)) \mid S(x) = z\} \\ &\le \max\left\{\operatorname{card} A \mid A \subset X \times X \text{ and } d(a,b) > \alpha \text{ for } a, b \in A\right\} =: C_{\delta,\epsilon},\end{aligned}$$

since the $(x, \varphi^t(x))$ form just such a separated set. □

With (4.2.2), (4.2.5), (4.2.6), and Definition 4.2.2 this implies:

Theorem 4.2.19. *If Φ is a continuous expansive flow on a compact metric space and 4δ is an expansivity constant, then $h_{\text{top}}(\Phi) = h_d(\Phi, \delta)$.*

Remark 4.2.20 (Entropy-expansiveness)**.** Although we do not prove it, expansivity can be replaced in these applications to entropy (and pressure) by a broader notion called entropy-expansiveness (or h-expansiveness) defined as follows. Φ is *entropy-expansive* if

$$h_{\text{top}}^*(\Phi, \epsilon) := \sup_{x \in X} h_{\text{top}}(\Phi, \bigcap_{t \in \mathbb{R}} \varphi^{-t}(B_\epsilon(\varphi^t(x))) = 0$$

(Remark 4.2.3) for some $\epsilon > 0$, which is then called an h-expansivity constant. (Expansivity is a special case in which the intersection is a short orbit segment of x.) In particular, Theorem 4.2.19 has the following counterpart.

Theorem 4.2.21 ([**52**, Corollary 2.5])**.** $h_{\text{top}}(\Phi) \le h_d(\Phi, \epsilon) + h_{\text{top}}^*(\Phi, \epsilon)$*, so if ϵ is an h-expansivity constant, then $h_{\text{top}}(\Phi) = h_d(\Phi, \epsilon)$.*

It is useful to augment our notation beyond Definition 4.2.1:

Definition 4.2.22. Denote by $\mathbb{O}_t(T)$ the set of periodic orbits γ of Φ for which a period $\pi(\gamma)$ is in $[T-t, T+t]$ (this is finite by expansivity, and π is well defined on $\mathbb{O}_t(T)$), and let $\mathbb{P}_t(T) := \bigcup_{\gamma \in \mathbb{O}_t(T)} \gamma$ be the set of points with these periods. For a periodic orbit γ denote by $\pi'(\gamma)$ its shortest or prime period and $\mathbb{O}_t'(T) := (\pi')^{-1}([T-t, T+t])$.

Proposition 4.2.23 (Periodic points are separated)**.** *With α as in Theorem 1.7.5(3), taking one point from each $\gamma \in \mathbb{P}_{\alpha/2}(t)$ gives a (t,α)-separated set.*

Proof. If $x, y \in \mathbb{P}_{\alpha/2}(t)$ with periods a, b, respectively, are not (t,α)-separated, set $t_{pm+q} = pa + q\alpha$ and $u_{pm+q} = pb + q\alpha$ with $0 \le q < m := 1 + \lfloor \frac{t - \alpha/2}{\alpha} \rfloor$ to get

$$d(\varphi^{t_{pm+q}}(x), \varphi^{u_{pm+q}}(y)) = d(\varphi^{q\alpha}(x), \varphi^{q\alpha}(y)) \le \alpha,$$

so x, y are on the same orbit by Theorem 1.7.5(3). □

Theorem 4.2.24. *If Φ is an expansive flow, then $p(\Phi) \le h_{\text{top}}(\Phi)$.*[3]

PROOF. If α is as in Theorem 1.7.5(3), then $\operatorname{card}\mathbb{O}_{\alpha/2}(t) \le N_d(\Phi, \alpha/2, t)$ by Proposition 4.2.23, so with the notation of Definition 4.2.1,

$$P_t(\Phi) \le \sum_{n=1}^{\lfloor t/\alpha \rfloor} \operatorname{card}\mathbb{O}_{\alpha/2}(n\alpha) \le \frac{t}{\alpha} N_d(\Phi, \alpha/2, t),$$

since $t \mapsto N_d(\Phi, \alpha/2, t)$ is nondecreasing. As $t \to \infty$ invoke Theorem 4.2.19. □

Remark 4.2.25. Remark 8.3.13 gives a sufficient condition for equality in Theorem 4.2.24, the specification property. This means that for hyperbolic flows, topological entropy is the exponential growth rate of periodic orbits. Theorem 8.7.9 refines this substantially.

Theorem 4.2.26. *Let Φ be a continuous flow on a compact metric space, $NW(\Phi)$ its nonwandering set. Then $h_{\text{top}}(\Phi) = h_{\text{top}}(\Phi\restriction_{NW(\Phi)})$.*

Remark 4.2.27. Looking ahead, this is a corollary of the Variational Principle (Theorem 4.3.7) because Φ-invariant probability measures are supported on $NW(\Phi)$.

PROOF. $h_{\text{top}}(\Phi\restriction_{NW(\Phi)}) \le h_{\text{top}}(\Phi)$ since $NW(\Phi) \subset X$. To show the other inequality we use a combinatorial argument, so we switch to the time-1 map as in the proof of Theorem 4.2.13.

Fix $n \ge 1$ and $\epsilon > 0$. Let A be an (n, ϵ)-spanning set of minimum cardinality for $\varphi^1\restriction_{NW(\Phi)}$. Let

$$U = \{x \in X \mid d_n(x, y) < \epsilon \text{ for some } y \in A\}.$$

So U is an open neighborhood of $NW(\Phi)$. Since $U^c = X \setminus U$ is compact and all its points are wandering, there exists a uniform $\beta > 0$ such that $0 < \beta \le \epsilon$ and for all $y \in U^c$ we have

$$\varphi^m(B_\beta(y)) \cap B_\beta(y) = \varnothing \text{ for all } m \ge 1.$$

Now take a minimal (n, β)-spanning set B for U^c. Then $C := A \cup B$ is an (n, ϵ)-spanning set for X. Let $l \in \mathbb{N}$ and define $\pi_l : X \to C^l$ by $\pi_l(x) = (y_0, \dots, y_{l-1})$ with

$$d_n(\varphi^{in}(x), y_i) < \begin{cases} \epsilon \text{ and } y_i \in A & \text{if } \varphi^{in}(x) \in U, \\ \beta \text{ and } y_i \in B & \text{if } \varphi^{in}(x) \in U^c \end{cases}.$$

Claim 4.2.28. *If $\pi_l(x) = (y_0, \dots, y_{l-1})$, then $y_i \in B$ does not repeat in the l-tuple.*

PROOF. Since $B_\beta(y_i)$ is wandering the result follows. □

Claim 4.2.29. *If $ln \ge m$, then π_l is 1-1 for $(m, 2\epsilon)$-separated points.*

[3] In particular, Φ has only countably many closed orbits.

Proof. If $\pi_l(x) = \pi_l(x')$, then for $0 \le j \le n$ and $0 \le i < l$ we have

$$d(\varphi^{in+j}(x), \varphi^{in+j}(x')) \le d_n(\varphi^{in}(x), y_i) + d_n(y_i, \varphi^{in}(x')) < \epsilon + \epsilon = 2\epsilon,$$

so $d_m(x, x') \le d_{ln}(x, x') < 2\epsilon$. □

Claim 4.2.30. *Let q be the minimum cardinality of an (n, β)-spanning set of U^c for φ^1 and p be the minimum cardinality of an (n,ϵ)-spanning set of $NW(\Phi)$ for φ^1. Then $\#(\pi_l(E)) \le (q+1)!l^q p^l$ for an $(n,2\epsilon)$-separated set E.*

Proof. Let I_j be the subset of l-tuples in $\pi_l(E)$ with exactly j of the $y_i \in B$. Since y_i cannot be repeated in $\pi_l(x)$ we have $j \le q$, and there are $\binom{q}{j}$ ways of picking the j points $y_i \in B$, $l!/(l-j)!$ ways of arranging the choices of positions, and at most $p^{l-j} \le p^l$ ways of picking the remaining terms. So

$$\#(I_j) \le \binom{q}{j} \frac{l!}{(l-j)!} p^l \quad \text{and} \quad \#(\pi_l(E)) = \sum_{j=0}^{q} I_j \le \sum_{j=0}^{q} \underbrace{\binom{q}{j}}_{\le q!} \underbrace{\frac{l!}{(l-j)!}}_{\le l_j \le l^q} p^l \le (q+1)!l^q p^l. \ \square$$

We now return to the proof of the theorem. Since $(q+1)!l^q$ grows at most polynomially in l, wandering points do not contribute to the entropy:

$$\begin{aligned} h_{2\epsilon}(\varphi^1) &= \overline{\lim_{n\to\infty}} \frac{1}{n} \log(N_d(\varphi^1, 2\epsilon, n)) \\ &\le \overline{\lim_{l\to\infty}} \frac{\log((q+1)!) + q\log(l) + l\log(p))}{(l-1)n} \\ &= \frac{\log(p)}{n} = \frac{\log(S_d(\varphi^1\restriction_{NW(\Phi)}, \epsilon, n))}{n} \xrightarrow[n\to\infty]{} h_\epsilon(\varphi^1\restriction_{NW(\Phi)}), \end{aligned}$$

so $h_{\text{top}}(\Phi) = h_{\text{top}}(\varphi^1) \le h_{\text{top}}(\varphi^1\restriction_{NW(\Phi)}) = h_{\text{top}}(\Phi\restriction_{NW(\Phi)})$ by letting $\epsilon \to 0$. □

Example 4.2.31. Example 1.1.8 is a flow of isometries, so $d_t = d$ for all $t \ge 0$, and $S_d(\Phi, \epsilon, t) = S_d(\Phi, \epsilon, 0)$ for all $t \ge 0$. Hence, the topological entropy is zero.

Example 1.3.6 and Example 1.3.9 are flows for which the nonwandering set is finite. By Theorem 4.2.26, the entropy is zero.

We now show that if the flow is sufficiently regular and the dimension of the space is finite that the topological entropy is finite and bounded by a product of the dimension of the space and the Lipschitz constant of the map.

Definition 4.2.32. Let (X, d) be a metric space. A flow $\Phi : X \to X$ is *Lipschitz continuous* if

$$L(\Phi) := \exp\left\{ \sup_{0<t\le 1} \sup_{x\ne y} \frac{1}{t} \log\left(\frac{d(\varphi^t(x), \varphi^t(y))}{d(x,y)} \right) \right\} < \infty.$$

The constant $L(\Phi)$ is the *Lipschitz constant* of Φ.

This definition implies $d(\varphi^t x, \varphi^t y) \le L(\Phi)^t d(x,y)$ for $x, y \in X$ and $t \in [0,1]$.

Remark 4.2.33. If V is a Lipschitz continuous vector field on a compact Riemannian manifold, then it generates a flow Φ via (1.1.1) for which

$$\frac{d(\varphi'(x), \varphi'(y))}{d(x,y)} \le L,$$

where L is a Lipschitz constant of V. Then a straightforward computation shows that the flow Φ is Lipschitz continuous. Here, orbits are solutions of (1.1.1) and hence C^1 curves, which makes the Lipschitz assumption on V a more stringent condition than Definition 4.2.32.

Definition 4.2.34. Let (X,d) be a compact metric space and $\#B_d(\epsilon)$ be the minimum cardinality of a covering of X by ϵ-balls. The *box dimension*[4] of X is

$$BD(X) := \overline{\lim_{\epsilon \to 0}} \frac{\log \#B_d(\epsilon)}{|\log \epsilon|} \in [0,\infty].$$

Remark 4.2.35. It is easy to see that this is invariant under bi-Lipschitz maps and that $BD(\bigcup_{i=1}^n X_i) = \max_i BD(X_i)$. Thus, $BD([0,1]^n) = n$, so the box dimension of a Riemannian manifold is the topological dimension.

Theorem 4.2.36. *Let (X,d) be a compact metric space with finite box dimension $BD(X)$ and $\Phi\colon X \to X$ a Lipschitz continuous flow on X. Then*

$$h_{\text{top}}(\Phi) \le BD(X) \log \max\{1, L(\Phi)\}.$$

Proof. Let $L = \max\{1, L(\Phi)\}$. Then $B_d(x, L^{-t}\epsilon) \subset B_{d_t^\Phi}(x,\epsilon)$ for all $x \in X$, $t \ge 0$, $\epsilon > 0$. This implies $S_d(\Phi,\epsilon,t) \le \#B_d(L^{-t}\epsilon)$. Now $|\log(L^{-t}\epsilon)| = t \log L + \log \epsilon$, so

$$t = \frac{|\log L^{-t}\epsilon| - \log\epsilon}{\log L} = \frac{|\log L^{-t}\epsilon|}{\log L}\left(1 - \frac{\log \epsilon}{|\log L^{-t}\epsilon|}\right) = \frac{|\log L^{-t}\epsilon|}{\log L}\left(1 + O\left(\frac{1}{t}\right)\right),$$

and

$$\overline{\lim_{t\to\infty}} \frac{\log S_d(\Phi,\epsilon,t)}{t} \le \overline{\lim_{t\to\infty}} \log \frac{\#B_d(L^{-t}\epsilon)}{t} = (\log L)\, \overline{\lim_{t\to\infty}} \frac{\log \#B_d(L^{-t}\epsilon)}{|\log L^{-t}\epsilon|} = BD(X) \log L. \quad \square$$

Corollary 4.2.37. *If Φ is a Lipschitz continuous flow on a compact Riemannian manifold M, then $h_{\text{top}}(\Phi) < \infty$.*

[4] Or box-counting dimension, Minkowski dimension, upper box dimension, entropy dimension, Kolmogorov dimension, Kolmogorov capacity, limit capacity, upper Minkowski dimension

The topological entropy for a flow is obviously invariant under flow equivalence. It changes under time change and hence under orbit equivalence in a rather complicated way. However, arguing similarly to the proof of Proposition 8.7.15 one can show that if a continuous flow without fixed points has zero (or finite) topological entropy, then so does any time-change (Theorem 4.3.14). Let us comment on the much harder question of how topological entropy changes under perturbation of a flow. This dependence need not even be continuous[5] and even in discrete time the picture is quite subtle [**171**, p. 584]. For hyperbolic flows, the subject of this book, this plays out exceptionally well, however (Theorem 5.4.26).

As a preview of theory to be presented further on, we present its historic precursor here. In 1964 William Parry proved:

Theorem 4.2.38 (Parry)**.** *The topological entropy of a topologically mixing (Proposition 1.8.13) shift Σ_A (Definition 1.8.1) is the maximal eigenvalue λ of A (Proposition 4.2.16). If v is a corresponding positive right eigenvector, P is defined by $\lambda_i v_i P_{ij} = A_{ij} v_j$ and p is the probability vector with $pP = p$, then the σ_A-invariant Markov measure m_P defined on cylinders* (1.8.1) *by*

$$m_P\big(C^{0,\dots,k}_{i_1,\dots,i_k}\big) = p_{i_0} P_{i_0 i_1} \dots P_{i_{k-1} i_k}$$

has entropy λ, and all other σ_A-invariant Borel probability measure have smaller entropy. It is called the Parry measure.

Thus, in this case the entropy of each measure is at most the topological entropy, this upper bound is attained, and by a unique invariant Borel probability measure.

3. Topological pressure and equilibrium states

To extend the notion of entropy recall that it is calculated by counting the elements of a maximal (t,ϵ)-separated set, that is, by summing 1 over the elements of the set. It is natural to instead allow weighted sums over separating or spanning sets. This leads to the notion of pressure, a term motivated by statistical mechanics.

One of the fundamental laws of thermodynamics is that the entropy of an isolated system can never decrease in time. An isolated system then approaches a state where the entropy cannot increase and so therefore remains constant. This relates to the notion of a measure of maximal entropy and the states in the support of the measure are the points in the state space where the energy has been maximized.

[5] The easiest example is in discrete time and noninvertible: $z \mapsto \lambda z^2$ on the unit disk in $\mathbb{C}$ has entropy $\log 2$ for $\lambda = 1$ and 0 for $\lambda < 1$.

If a system is not isolated, for instance, if the system is placed inside a heat bath, then it will tend towards an equilibrium that is called a thermodynamic equilibrium. The free energy of a system is the amount of work that a thermodynamic system can perform, so the free energy is the internal energy of a system minus the amount of energy that cannot perform work. In thermodynamics the entropy is just the unusable energy multiplied by the temperature. So in an isolated system maximizing the entropy is equivalent to minimizing the free energy. The system will then evolve to a state where the free energy cannot decrease and so remains constant. The measures associated with this equilibrium are a generalization of the measures of maximal entropy and are called equilibrium states.

In our context, this notion is defined using a continuous $f : X \to \mathbb{R}$, called a *potential* or *observable*. The term "observable" reflects the fact that an observation of a system usually yields a real number (the measurement) that depends on the state of the system, that is, a point in phase space. We use these functions as weights in sums over spanning sets:

Definition 4.3.1. Let X be a compact metric space and $\Phi\colon X \to X$ be a continuous flow. For $f \in C^0(X)$, and $t \geq 0$ set $S_t f := \int_0^t f \circ \varphi^\tau \, d\tau$ and

$$N_d(\Phi, f, \epsilon, t) := \sup\Big\{ \sum_{x\in E} e^{S_t f(x)} \Bigm| E \subset X \text{ is } (t,\epsilon)\text{-separated}\Big\},$$

$$S_d(\Phi, f, \epsilon, t) := \inf\Big\{ \sum_{x\in E} e^{S_t f(x)} \Bigm| X = \bigcup_{x\in E} B_\Phi(x,\epsilon,t)\Big\},$$

$$D_d(\Phi, f, \epsilon, t) := \inf\Big\{ \sum_{C\in E} \inf_{x\in C} e^{S_t f(x)} \Bigm| X \subset \bigcup_{C\in E} C \text{ and } \operatorname{diam}_{d_t^\Phi}(C) \leq \epsilon \text{ for } C \in E \subset 2^X\Big\}.$$

The expressions $\sum_{x\in E} e^{S_t f(x)}$ are sometimes called *statistical sums*.[6] Then

$$P(f) := P(\Phi, f) := \lim_{\epsilon\to 0} \overline{\lim_{t\to\infty}} \frac{1}{t} \log S_d(\Phi, f, \epsilon, t)$$

is called the *topological pressure* of Φ with respect to f.

Remark 4.3.2. The definition implies that $P(f + c) = P(f) + c$ for any $c \in \mathbb{R}$, and if f and g are cohomologous (Definition 1.3.20), then $P(f) = P(g)$.

Analogously to (4.2.5), (4.2.6), (4.2.3) and (4.2.4) we have

$$\begin{aligned} N_d(\Phi, f, 2\epsilon, t) &\leq S_d(\Phi, f, \epsilon, t) \leq N_d(\Phi, f, \epsilon, t), \\ D_d(\Phi, f, 2\epsilon, t) &\leq S_d(\Phi, f, \epsilon, t) \leq D_d(\Phi, f, \epsilon, t), \end{aligned} \tag{4.3.1}$$

which shows that

$$P(\Phi, f) = \lim_{\epsilon\to 0} \overline{\lim_{t\to\infty}} \frac{1}{t} \log N_d(\Phi, f, \epsilon, t) = \lim_{\epsilon\to 0} \underline{\lim_{t\to\infty}} \frac{1}{t} \log N_d(\Phi, f, \epsilon, t), \tag{4.3.2}$$

[6] Or *partition sums.*

and

$$P(\Phi,f)=\lim_{\epsilon\to 0}\lim_{t\to\infty}\frac{1}{t}\log D_d(\Phi,f,\epsilon,t)$$

by an argument similar to that following Lemma 4.2.6, since $D_d(\Phi,f,\epsilon,t)$ is submultiplicative similarly to Lemma 4.2.6.

Remark 4.3.3. When $f=0$, Definition 4.3.1 gives topological entropy: $P(\Phi,0)=h_{\text{top}}(\Phi)$. If $c\in\mathbb{R}$, then $S_t(f+c)=tc+S_tf$, so $S_d(\Phi,f+c,\epsilon,t)=e^{tc}S_d(\Phi,0,\epsilon,t)$ and $P(f+c)=P(f)+c$. We also have $S_d(\Phi,f,\epsilon,t)\le\|e^{S_tf}\|_{C^0}\cdot S_d(\Phi,\epsilon,t)$ and thus

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log S_d(\Phi,f,\epsilon,t)\le\|f\|_{C^0}+\overline{\lim_{t\to\infty}}\frac{1}{t}\log S_d(\Phi^t,\epsilon,t).$$

Thus, if Φ is a smooth flow on a compact manifold and f is continuous, then

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log S_d(\Phi,f,\epsilon,t)\le\|f\|_{C^0}+\overline{\lim_{t\to\infty}}\frac{1}{t}\log S_d(\Phi^t,\epsilon,t)<\infty.$$

Finally, $t\mapsto N_d(\Phi,f,\epsilon,t)$ and $t\mapsto S_d(\Phi,f,\epsilon,t)$ are nondecreasing, so $S_d(\Phi,f,\epsilon,\lfloor t\rfloor)\le S_d(\Phi,f,\epsilon,t)\le S_d(\Phi,f,\epsilon,\lceil t\rceil)$ and hence

$$P(\Phi,f)=\lim_{\epsilon\to 0}\overline{\lim_{t\to\infty}}\frac{1}{t}\log S_d(\Phi,f,\epsilon,t).$$

Remark 4.3.4. The proof of Proposition 4.2.4 extends to show that pressure is independent of the metric (inducing a given topology) used to define it, thus justifying some of our notation. This implies that pressure is invariant under topological conjugacy, that is, if $\varphi^t=\pi^{-1}\circ\psi^t\circ\pi$ and $g=f\circ\pi$ then $P(\varphi^t,f)=P(\psi^t,g)$.

For what follows it is convenient to work with the time-1 map for a while.

Definition 4.3.5. $P(\varphi^1,f):=\lim_{\epsilon\to0}\overline{\lim}_{n\to\infty}\frac{1}{n}\log S_d(\varphi^1,f,\epsilon,n)$, where[7]

$$S_d(\varphi^1,f,\epsilon,n):=\inf\Big\{\sum_{x\in E}e^{S_nf(x)}\Big|X=\bigcup_{x\in E}\underbrace{B_{\varphi^1}(x,\epsilon,n)}_{:=\{y\in X\ \mid\ d_t^{\varphi^1}(x,y):=\max_{0\le k\le n}d(\varphi^k(x),\varphi^k(y))<\epsilon\}}\Big\}.$$

Proposition 4.3.6. $P(\Phi,f)=P(\Phi,S_1f)=P(\varphi^1,f)$.

Proof. Equicontinuity of $\{\varphi^t\mid |t|\le s\}$ implies $P(\Phi,f)=P(\Phi,f\circ\varphi^s)$ for any $s\in\mathbb{R}$, and a computation shows that $\|S_tS_1f-S_{t-2}f\circ\varphi^1\|_\infty\le\|f\|_\infty$ for all $t\ge2$. □

[7] Although we did not adapt the notation accordingly, this is a definition of pressure for a homeomorphism.

Theorem 4.3.7 (Variational Principle)**.** *If Φ is a continuous flow on a compact metric space X and $f : X \to \mathbb{R}$ is continuous, then*

$$P(\Phi, f) = \sup_{\mu\in\mathfrak{M}(\Phi)} \left[h_\mu(\Phi) + \int f\, d\mu \right]. \tag{4.3.3}$$

Corollary 4.3.8. *If Φ is a continuous flow on a compact metric space X, then*

$$h_{\text{top}}(\Phi) = \sup_{\mu\in\mathfrak{M}(\Phi)} h_\mu(\Phi).$$

Before we prove this theorem we explore some related results. The quantity $h_\mu(\Phi) + \int f\, d\mu$ is called the *free energy*. The topological pressure then tries to maximize the free energy. Up to a change in sign this is the same as the free energy as we described previously. Due to the change in sign thermodynamics tries to minimize the free energy. When $f \equiv 0$ this gives the Variational Principle for entropy.

Definition 4.3.9. A measure $\mu \in \mathfrak{M}(\Phi)$ such that $P(\Phi, f) = h_\mu(\Phi) + \int f\, d\mu$ is an *equilibrium measure* or *equilibrium state* for Φ associated with f. A measure $\mu \in \mathfrak{M}(\Phi)$ such that $h_{\text{top}}(\Phi) = h_\mu(\Phi)$ is a *measure of maximal entropy* for Φ.

Example 4.3.10. Example 4.1.11 and Proposition 4.2.17 show that Lebesgue measure is a measure of maximal entropy for the suspension of the toral automorphism from Example 1.5.23; in particular, the supremum in Corollary 4.3.8 is attained in this case.

We briefly describe the collection of equilibrium states.

Theorem 4.3.11. *Let $\Phi : X \to X$ be a continuous flow of a compact metric space and $f : X \to \mathbb{R}$ be continuous. Then*

(1) *the set $\mathfrak{M}_f(\Phi)$ of equilibrium states for f is convex,*
(2) *if $f - g$ is cohomologous to $c \in \mathbb{R}$ (Definition 1.3.20), then $\mathfrak{M}_f(\Phi) = \mathfrak{M}_g(\Phi)$,*[8]
(3) *if $h_{\text{top}}(\Phi) < \infty$ and $\mathfrak{M}_f(\Phi) \neq \varnothing$, then then the extreme points of $\mathfrak{M}_f(\Phi)$ are exactly the ergodic equilibrium states, and*
(4) *If the entropy map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous,*[9] *then $\mathfrak{M}_f(\Phi)$ is nonempty and compact.*

Proof. (1) follows from the affine property of entropy (see Theorem 4.1.3). Indeed, let $\mu, \nu \in \mathfrak{M}_f(\Phi)$ and $p \in [0, 1]$. Then

$$\underbrace{h_{p\mu+(1-p)\nu}(\Phi)}_{=ph_\mu(\Phi)+(1-p)h_\nu(\Phi)} + \underbrace{\int f\, d(p\mu + (1-p)\nu)}_{=p\int f\,d\mu+(1-p)\int f\,d\nu} = pP(\Phi, f) + (1-p)P(\Phi, f) = P(\Phi, f).$$

[8] The converse is Theorem 8.3.21.

[9] This is the case if Φ is expansive (Corollary 11.3.14).

(2): The assumption is that f is cohomologous to $g+c$, so on both sides of (4.3.3) replacing f by $g+c$ amounts to adding c.

(3): If $\mu\in\mathfrak{M}_f(\Phi)$ is ergodic, then it is an extreme point of $\mathfrak{M}(\Phi)$ and hence of $\mathfrak{M}_f(\Phi)$. Conversely, if $\mu=p\mu_1+(1-p)\mu_2\in\mathfrak{M}_f(\Phi)$ is an extreme point of $\mathfrak{M}_f(\Phi)$ with $\mu_1,\mu_2\in\mathfrak{M}(\Phi)$, then

$$P(\Phi,f)=h_\mu(\Phi)+\int f\,d\mu=ph_{\mu_1}(\Phi)+(1-p)h_{\mu_2}(\Phi)+p\int f\,d\mu_1+(1-p)\int f\,d\mu_2\le P(\Phi,f),$$

so $\mu_1,\mu_2\in\mathfrak{M}_f(\Phi)$. Thus, $\mu=\mu_1=\mu_2$, and μ is an extreme point of $\mathfrak{M}(\Phi)$, hence ergodic by Theorem 3.1.16.

(4): If $\mu\mapsto h_\mu(\Phi)$ is upper semicontinuous, then so is $\mu\mapsto h_\mu(\Phi)+\int f\,d\mu$. An upper semicontinuous function on a compact space has a maximum, so $\mathfrak{M}_f(\Phi)\ne\varnothing$, and upper semicontinuity further implies that $\mathfrak{M}_f(\Phi)$ is compact. □

Proposition 4.3.12 (High-pressure measure). *Let (X,d) be a compact metric space, Φ a continuous flow on X, $f\in C^0(X)$, $E_n\subset X$ an (n,ϵ)-separated set,*

$$\nu_n:=\Big(\sum_{x\in E_n}e^{S_nf(x)}\Big)^{-1}\sum_{x\in E_n}e^{S_nf(x)}\delta_x,\quad\textit{and}\quad \mu_n:=\frac{1}{n}\int_0^n\varphi_*^s\nu_n\,ds.$$

Then there exists a weak-accumulation point $\mu\in\mathfrak{M}(\Phi)$ of $\{\mu_n\}_{n\in\mathbb{N}}$ that satisfies*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log\sum_{x\in E_n}e^{S_nf(x)}\le h_\mu(\Phi)+\int f\,d\mu.$$

Corollary 4.3.13. $P(\Phi,f)\le\sup_{\mu\in\mathfrak{M}(\Phi)}h_\mu(\Phi)+\int f\,d\mu$.

Proof. Fix $\delta>0$ and let $\{E_n\}_{n\in\mathbb{N}}$ be (n,ϵ)-separated sets in X such that

$$\sum_{x\in E_n}e^{S_nf(x)}\ge N_d(\Phi,f,\epsilon,n)-\delta.$$

Proposition 4.3.12 then gives

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log N_d(\Phi,f,\epsilon,n)\le h_\mu(\Phi)+\int f\,d\mu$$

for an accumulation point μ of μ_n. Taking the supremum over μ and letting $\epsilon\to0$ gives the claim. □

Corollary 4.3.13 is half of the Variational Principle.

Proof of Proposition 4.3.12. Let n_k be a subsequence such that

$$\lim_{k\to\infty}\log\sum_{x\in E_{n_k}}e^{S_{n_k}f(x)}=\overline{\lim_{n\to\infty}}\log\sum_{x\in E_n}e^{S_nf(x)}.$$

Let μ be an accumulation point of μ_{n_k}. Notice that although we are allowing $n\in\mathbb{N}$ we choose $\mu_n = \frac{1}{n}\int_0^n \varphi_*^s \nu_n\,ds$, and so as in the proof of Theorem 3.1.15 the weak*-accumulation point is Φ invariant.

Let ξ be a partition whose elements have diameter less than ϵ and $\mu(\partial\xi)=0$. Let $E_n=\{x_1,\dots,x_m\}$ be an (n,ϵ)-separated set. Then (see (11.1.1) and Definition 11.2.1)

$$H_{\nu_n}(\xi^{\varphi^1}_{-n}) + \underbrace{n\int f\,d\mu_n}_{=\int S_n f\,d\nu_n} = \sum_{x\in E_n}[-\nu_n(\{x\})\log(\nu_n(\{x\}))+\nu_n(\{x\})S_nf(x)] = \log\sum_{x\in E_n}e^{S_nf(x)}.$$

Here the last equality is a simple computation or an application of (the easy "=" part of) Lemma 11.2.14 below. If $a(k)=\lfloor (n-k)/q\rfloor$ for $0\le k<q<n$, then this gives

$$\frac{q}{n}\log\sum_{x\in E_n}e^{S_nf(x)} = \underbrace{\frac{q}{n}H_{\nu_n}(\xi^{\varphi^1}_{-n})}_{=\frac{1}{n}\sum_{k=0}^{q-1}H_{\nu_n}(\xi^{\varphi^1}_{-n})}+q\int f\,d\mu_n$$

$$[\text{Proposition 11.2.6(4)}\Rightarrow]\le\sum_{k=0}^{q-1}\Big(\sum_{r=0}^{a(k)-1}\frac{1}{n}H_{\varphi_*^{rq+k}\nu_n}(\xi^{\varphi^1}_{-q})+\frac{2q}{n}\log\#(\xi)\Big)+q\int f\,d\mu_n$$

$$[\text{Proposition 11.2.6(6)}\Rightarrow]\le H_{\mu_n}(\xi^{\varphi^1}_{-q})+\frac{2q^2}{n}\log\#(\xi)+q\int f\,d\mu_n.$$

Hence, $\displaystyle\lim_{k\to\infty}\frac{1}{n_k}\log\sum_{x\in E_{n_k}}e^{S_{n_k}f(x)}\le\frac{1}{q}\lim_{k\to\infty}H_{\mu_{n_k}}(\xi^{\varphi^1}_{-q})+\int f\,d\mu_{n_k}=\frac{1}{q}H_\mu(\xi^{\varphi^1}_{-q})+\int f\,d\mu$

and $\displaystyle\overline{\lim_{n\to\infty}}\frac{1}{n}\log\sum_{x\in E_n}e^{S_nf(x)}\le h_\mu(\varphi^1,\xi)+\int f\,d\mu\le h_\mu(\varphi^1)+\int f\,d\mu=P_\mu(f,\Phi).$ □

PROOF OF THEOREM 4.3.7. In light of Corollary 4.3.13, it remains to show that

$$h_\mu(\Phi)+\int f\,d\mu\le P(\Phi,f)\quad\text{for every }\mu\in\mathfrak{M}(\Phi). \tag{4.3.4}$$

Let $\xi=\{C_1,\dots,C_k\}$ be a measurable partition of X. Then $\mu(C_i)=\sup\{\mu(B)\mid B\subset C_i\text{ is closed}\}$, so there are compact $B_i\subset C_i$ (think of these as the "islands") such that $H_\mu(\xi|\mathscr{B})\le 1$ for $\mathscr{B}=\{B_0,\dots,B_k\}$ with $B_0=X\smallsetminus(\bigcup_{j=1}^k B_i)$ (think of this as "the sea"). Then

$$h_\mu(\Phi,\xi)\le h_\mu(\Phi,\mathscr{B})+H_\mu(\xi|\mathscr{B})\le h_\mu(\Phi,\mathscr{B})+1.$$

Now let $d=\min\{d(B_i,B_j)\mid i,j\in\{1,\dots,k\},i\neq j\}>0$ and $\delta\in(0,d/2)$ such that $|f(x)-f(y)|<1$ whenever $d(x,y)<\delta$. Let $E\subset X$ be an (n,δ)-spanning set. For $C\in\mathscr{B}^{\varphi^1}_{-n}$ there is an $x_C\in\overline{C}$ such that $(S_nf)(x_C)=\sup\{S_nf(x)\mid x\in C\}$ and a $y_C\in E$

such that $d_n^\Phi(x_C, y_C) \le \delta$, so $S_n f(x_C) \le S_n f(y_C) + n$. Then

$$H_\mu(\mathscr{B}^{\varphi^1}_{-n}) + \int S_n f\,d\mu \le \underbrace{\sum_{C\in\mathscr{B}^{\varphi^1}_{-n}} \mu(C)(-\log\mu(C) + \underbrace{S_n f(x_C)}_{\le S_n f(y_C)+n})}_{\le\log\sum_{C\in\mathscr{B}^{\varphi^1}_{-n}} e^{S_n f(y_C)+n} \text{ by Lemma 11.2.14}} \le n + \log(2^n \underbrace{\sum_{x\in E} e^{S_n f(x)}}_{\delta<d/2,\ y\in E\Rightarrow\#\{C\in\mathscr{B}^{\varphi^1}_{-n} \mid y_C=y\}\le 2^n})$$

and

$$\frac{1}{n}H_\mu(\mathscr{B}^{\varphi^1}_{-n}) + \underbrace{\int f\,d\mu}_{=\frac{1}{n}\int S_n f\,d\mu} \le 1 + \log 2 + \frac{1}{n}\log\sum_{x\in E} e^{S_n f(x)}.$$

Therefore, $h_\mu(\Phi,\xi) + \int f\,d\mu \le h_\mu(\Phi,\mathscr{B}) + 1 + \int f\,d\mu \le 2 + \log 2 + P(\Phi, f)$, and hence $h_\mu(\varphi^1) + \int f\,d\mu \le 2 + \log 2 + P(\Phi, f)$. Applying this to φ^n and $S_n f$ gives

$$h_\mu(\Phi) + \int f\,d\mu \le P(\Phi, f) + (2 + \log 2)/n \xrightarrow[n\to\infty]{} P(\Phi, f). \qquad \square$$

If Φ_r is a special flow, then the Variational Principle (Theorem 4.3.7) and the Abramov Theorem 4.1.4 imply

$$h_{\mathrm{top}}(\Phi_r) = \sup_{\mu_r\in\mathfrak{M}(\Phi_r)} h_{\mu_r}(\Phi_r) = \sup_{\mu\in\mathfrak{M}(\sigma)} \frac{h_\mu(\sigma)}{\int r\,d\mu}. \tag{4.3.5}$$

This is useful with respect to an earlier question.

Theorem 4.3.14. *If a continuous flow without fixed points has zero (or finite) topological entropy, then so does any time-change.*

Proof outline. Because there are no fixed points assume without loss of generality that the flows Φ, Ψ are special flows (Theorem 3.6.2) over the same base transformation σ under roof functions r_Φ, r_Ψ with (by compactness) bounded logarithms. Then the right-hand of (4.3.5) is the same for both Φ and Ψ. If $h_{\mathrm{top}}(\Phi) < \infty$, then this supremum is finite, and hence $h_{\mathrm{top}}(\Psi) < \infty$; if $h_{\mathrm{top}}(\Phi) = 0$ then likewise, $h_{\mathrm{top}}(\Psi) = 0$. □

We next relate the pressure of a special flow to the pressure of the base dynamics, provided there is a unique equilibrium state for the base dynamics (for which the discrete-time counterpart of Theorem 8.3.6 gives sufficient conditions).

Proposition 4.3.15. *Let X be a compact metric space, F a homeomorphism on X with $h_{\mathrm{top}}(F) < \infty$, $r: X \to (0,\infty)$ continuous, Φ_r the special flow on X_r, $G \in C(X_r)$, and $g(x) := \int_0^{r(x)} G(x,t)\,dt \in C(X)$.*

- *There is a unique $c \in \mathbb{R}$ with $P(F, g - cr) = 0$.*

- *If F has a unique equilibrium state m for $g-cr$, then m_r (from* (3.6.3)*) is the unique equilibrium state of G for Φ_r, and $c = P(\Phi_r, G)$.*
- *If F has a unique equilibrium state m for $-h_{\text{top}}(\Phi_r)r$, then m_r is the unique measure of maximal entropy for Φ_r on X_r.*

PROOF. We first show that the continuous map $c \mapsto P(F, g-cr)$ is strictly decreasing. If μ is an F-invariant measure and $c_1 < c_2$, then continuity of r together with $r > 0$ and $h_{\text{top}}(F) < \infty$ imply

$$\begin{aligned} h_\mu(F) + \int (g - c_1 r)d\mu &= h_\mu(F) + \int g d\mu - c_1 \int r d\mu \\ &> h_\mu(F) + \int g d\mu - c_2 \int r d\mu = h_\mu(F) + \int (g - c_2 r) d\mu. \end{aligned}$$

Let μ_n be a sequence of F-invariant probability measures with $\lim_{n\to\infty} h_{\mu_n}(F) + \int g - c_2 r d\mu_n = P(F, g - c_2 r)$. By taking a subsequence we can assume that $\mu_n \xrightarrow[n\to\infty]{\text{weak}^*} \mu$. Then

$$\begin{aligned} P(F, g - c_1 r) &\geq \lim_{n\to\infty} h_{\mu_n}(F) + \int (g - c_1 r) d\mu_n \\ &= \lim_{n\to\infty} h_{\mu_n}(F) + \int (g - c_2 r) d\mu_n + (c_2 - c_1) \int r d\mu_n \\ &= P(F, g - c_2 r) + (c_2 - c_1) \int r d\mu > P(F, g - c_2 r), \end{aligned}$$

so $c \mapsto P(F, g - cr)$ is strictly decreasing. Furthermore, $\lim_{c\to\pm\infty} P(F, g - cr) = \mp\infty$ since $h_{\text{top}}(F) < \infty$, so there is a unique $c \in \mathbb{R}$ with $P(F, g - cr) = 0$.

If m is an equilibrium state of F for $g - cr$, then $h_m(F) + \int (g - cr)\, dm = P(F, g - cr) = 0$, hence

$$c = \frac{h_m(F)}{\int r dm} + \frac{\int g dm}{\int r dm} \overset{\text{(Theorem 4.1.4)}}{=\!=\!=\!=} h_{m_r}(\Phi_r) + \int G dm_r.$$

If m is the unique equilibrium state of F for $g - cr$, then m_r is the unique equilibrium state of Φ_r for G because for any F-invariant probability measure $\mu \neq m$ we have $0 > h_\mu(F) + \int (g - cr) d\mu$, so

$$h_{m_r}(\Phi_r) + \int G dm_r = c > \frac{h_\mu(F) + \int g d\mu}{\int r d\mu} \overset{\text{(Theorem 4.1.4)}}{=\!=\!=\!=} h_{\mu_r}(\Phi_r) + \int G d\mu_r.$$

This also implies that $c = h_{m_r}(\Phi_r) + \int G dm_r = P(\Phi_r, G)$. Furthermore, from Theorem 3.6.2 we know that each measure μ_r arises from a measure μ concluding the proof of the second part of the theorem.

If $G = 0$, then $g = 0$, so $P(F, 0 - cr) = 0$ when $c = P(\Phi_r, 0) = h_{\text{top}}(\Phi_r)$. So, if there is a unique equilibrium state m for $-h_{\text{top}}(F)r$, then m_r is a unique measure of maximal entropy. □

Remark 4.3.16. Under the assumptions of Theorem 4.3.11 and Theorem 4.3.7, $h_\mu(\Phi)$ and hence $h_\mu(\Phi)+\int f\,d\mu$ on the right-hand side of (4.3.3) is upper semicontinuous if Φ is expansive (Corollary 11.3.14)—thus Theorem 4.3.11(4) gives existence of an equilibrium measure. This is notable, but the importance of equilibrium states rests in great part on our ability to study them carefully, and a nonconstructive existence result is of limited use in this respect. Therefore, we now give more restrictive sufficient conditions for the existence of equilibrium states because they allow us to construct them explicitly (Theorem 4.3.21). These involve controlling the "dynamical distortion" of the potential (Definition 4.3.17). We will much later see that in the principal context of this book, equilibrium states are unique (Theorem 8.3.6).

Definition 4.3.17. Let X be a metric space, Φ a flow. With the notation from Definition 4.3.1, the set $V(\Phi)$ of *Bowen-bounded* functions [**55**, p. 193] for Φ is

$$\big\{f\in C^0(X)\ \big|\ \exists K,\epsilon>0\,\forall t>0\colon d_t^\Phi(x,y)<\epsilon\Rightarrow|S_tf(x)-S_tf(y)|<K\big\},\tag{4.3.6}$$

and the set $V_0(\Phi)$ of *Walters-continuous* functions [**286**, p. 125] for Φ is

$$\big\{f\in C^0(X)\ \big|\ \forall\epsilon>0\,\exists\delta>0\,\forall t>0\colon d_t^\Phi(x,y)<\delta\Rightarrow|S_tf(x)-S_tf(y)|<\epsilon\big\}.\tag{4.3.7}$$

These regularity conditions may look technical but arise naturally in hyperbolic flows: Hölder continuous functions (Definition 1.8.4) are Walters-continuous (and hence Bowen-bounded) for a hyperbolic flow (Proposition 8.3.1) due to a quantitative (exponential) version of Proposition 1.7.4 (Proposition 6.2.4). Periodic data determine a Walters-continuous function, or rather its cohomology (Theorem 5.3.23). The utility of Bowen-boundedness lies in the following, which makes Proposition 4.3.12 the main step in the construction of equilibrium states.

Lemma 4.3.18. *Let Φ be an expansive flow on a compact metric space X with expansivity constant δ_0 (cf. Definition 1.7.1). Then for $f\in V(\Phi)$, $\epsilon\in(0,\delta_0/2)$, and $\delta>0$ there exists $C_{\delta,\epsilon}$ such that (for all $t>0$)*

$$N_d(\Phi,f,\delta,t)\le C_{\delta,\epsilon}N_d(\Phi,f,\epsilon,t).$$

Remark 4.3.19. If $\delta>\epsilon$ we can take $C_{\delta,\epsilon}=1$.

Proof. For $0<\epsilon<\delta_0/2$ expansivity gives a $T>0$ such that

$$d_{2T}^\Phi(\varphi^{-T}(x),\varphi^{-T}(y))\le2\epsilon\Rightarrow d(x,y)<\delta,$$

and equicontinuity gives $\alpha>0$ such that $d(x,y)\le\alpha\Rightarrow d_{2T}^\Phi(\varphi^{-T}(x),\varphi^{-T}(y))\le\delta$. If E is a maximal (t,δ)-separated set and F a maximal (t,ϵ)-separated set, then for $x\in E$ there is a $z(x)\in F$ such that $d_t^\Phi(x,z(x))<\epsilon$. The cardinality of $E_z:=\{x\in E\ |\ z(x)=z\}$ is bounded uniformly in t: If $x,y\in E_z$ then $d_t^\Phi(x,y)\le2\epsilon$ by definition of E_z, hence $d(\varphi^s(x),\varphi^s(y)) \le \delta$ for $s \in [T, t-T)$ by choice of T, and thus, by choice of α and since $\{x,y\}$ is (t,δ)-separated, $d(x,y) > \alpha$ or $d(\varphi^t(x),\varphi^t(y)) > \alpha$. Therefore

$$\begin{aligned}\operatorname{card}(E_z) &= \operatorname{card}\{(x,\varphi^t(x)) \mid x \in E_z\}\\ &\le \max\{\operatorname{card} A \mid A \subset X \times X \text{ and } (a,b) \in A,\ a \ne b \Rightarrow d(a,b) > \alpha\} =: M\end{aligned}$$

since the $(x,\varphi^t(x))$ form just such an α-separated set. Now take ϵ, K as in (4.3.6) so that $|S_t f(x) - S_t f(z)| \le K$ for $x \in E_z$ and

$$\sum_{x\in E} e^{S_t f(x)} \le \sum_{z\in F} \underbrace{\operatorname{card} E_z}_{\le M} e^K e^{S_t f(z)} \le \underbrace{Me^K}_{=:C_{\delta,\epsilon}} N_d(\Phi, f, \epsilon, t). \qquad \square$$

Together with (4.3.1) and (4.3.2) this gives

Proposition 4.3.20. *If Φ is expansive, 3ϵ an expansivity constant, $f \in V(\Phi)$, then*

$$P(\Phi, f) = \lim_{t\to\infty} \frac{1}{t} \log S_d(\Phi, f, \epsilon, t).$$

With Proposition 4.3.12, this in turn gives existence of equilibrium states:

Theorem 4.3.21. *If Φ be an expansive flow on a compact metric space, $f \in V(\Phi)$, and $P(\Phi, f) < \infty$ (Definition 4.3.1), then every weak*-accumulation point of the μ_n in Proposition 4.3.12 is an equilibrium state for Φ associated with f.*

While we will be able to establish uniqueness of equilibrium states for hyperbolic flows later, this may not hold for systems beyond this context, even though much progress has recently been made for nonuniformly hyperbolic dynamical systems. We note that systems with more than 1 equilibrium state are said to be in a *phase transition.*

We will revisit Theorem 4.3.21 in a context where equilibrium states are unique (Theorem 8.3.6). For that work and elsewhere, another way of singling out an invariant Borel probability measure is important, and we now define this property and connect it to equilibrium states.

Definition 4.3.22 (Gibbs measure)**.** For a continuous flow Φ on a compact metric space X and a potential function $f\colon X \to \mathbb{R}$, a measure $\mu \in \mathfrak{M}(\Phi)$ is a *Gibbs measure* for f with constant P if for $\delta > 0$ there is a constant $C > 0$ such that for $x \in X$ and $t > 0$ we have

$$\frac{1}{C} \le \frac{\mu(B_\Phi(x,t,\delta))}{\exp(S_t f(x) - tP)} \le C.$$

For hyperbolic Φ, Proposition 8.3.14 says that for each $f \in V(\Phi)$ there is a Gibbs measure with constant $P = P(\Phi, f)$ (Definition 4.3.1). Our present object is to show that this is an alternate way of producing equilibrium states.

Theorem 4.3.23. *If Φ is a continuous flow on a compact metric space X and $f \in V(\Phi)$, then a Gibbs measure with constant $P = P(\Phi, f)$ is an equilibrium state for f.*

Proof. Fix $t > 0$ and $\epsilon > 0$. Let $\beta > 0$ such that $d(x, y) \le \beta$ implies that $d_t(x, y) \le \epsilon$. Let $\xi = \{B_1, ..., B_m\}$ be a measurable partition of X such that $\operatorname{diam}(B_i) \le \beta$ for all $1 \le i \le m$ and hence $\operatorname{diam}\varphi^s(A) \le \epsilon$ for all $A \in \xi_{-n}^{\varphi^t}$, $n \in \mathbb{N}$ and $s \in [0, nt]$. By the Gibbs property we have $\mu(A) \le C \exp(S_{nt} f(x) - ntP)$.

We also have

$$h_\mu(\varphi^t) + \int S_t f(x)\, d\mu \ge h_\mu(\varphi^t, \xi) + \int S_t f(x)\, d\mu = \lim_{n\to\infty} \frac{1}{n} H(\xi_{-n}^{\varphi^t}) + \int S_t f(x)\, d\mu.$$

$H(\xi_{-n}^{\varphi^t}) \ge -\log C + Pnt - \int S_{nt} f(x)\, d\mu$ by Proposition 11.2.6(1), and $\int S_t f(x)\, d\mu = \frac{1}{n} \int S_{nt} f(x)\, d\mu$, so $h_\mu(\varphi^t) + \int S_t f(x)\, d\mu \ge Pt$. Hence, $h_\mu(\varphi) + \int f\, d\mu = P(\varphi, f)$ since $h_\mu(\varphi^t) = t h_\mu(\varphi^1)$ and $\int S_t f(x)\, d\mu = t \int f\, d\mu$. □

Additional assumptions on a flow imply uniqueness of equilibrium states (Theorem 8.3.6) for Bowen-bounded potentials. Instead of presenting this strengthening here, we defer it to the context of hyperbolic flows, where Bowen-boundedness is particularly natural—and invariant under topological equivalence, unlike in the present context.

We close by remarking that while we gave a motivation for the study of equilibrium states in terms of thermodynamical concepts that can be transferred to dynamical systems, the principal motivation of dynamicists in studying them is that they provide a collection of measures (rather than just the measure of maximal entropy) that are deeply connected to the dynamics of a flow and have strong stochastic properties (Remark 8.3.19). Furthermore, among these is the equilibrium state for a special potential, the *geometric potential*, which is of exceptional interest in its own right: This Sinai–Ruelle–Bowen measure is central to the description of hyperbolic attractors (Theorem 8.4.7) and stands in for volume when this is not invariant; as a corollary, volume is an equilibrium state if invariant, and enjoys the stochastic properties we derive in full generality—and sometimes even more (Theorem 8.4.17). Lastly, this Sinai–Ruelle–Bowen measure can be a tool for establishing results about smooth dynamics whose statements make no reference to probabilistic aspects (Theorem 10.2.7), and the question of when it coincides with the measure of maximal entropy leads to interesting rigidity results (Section 10.4). This theory has recently been developed also for geodesic flows on noncompact negatively curved manifolds [**231**].

4. Equilibrium states for time-t maps*

We digress to connect equilibrium states for a flow and for its time-t maps. The problem is that the set of invariant measures for the time-t map of a flow may be larger than the set of invariant measures for the flow. We begin with measures of maximal entropy.

In Proposition 4.2.8 we showed that the topological entropy of a flow is equal to the topological entropy of the time-1 map of the flow, and Corollary 4.2.10 gives $|t|h_{\text{top}}(\Phi) = h_{\text{top}}(\varphi^t)$ for any t. Therefore, any measure of maximal entropy for Φ is a measure of maximal entropy φ^t. However, there may be measures of maximal entropy for the time-t map that are not measures of maximal entropy for the flow. For instance, if we start with a map $f : X \to X$ with a measure of maximal entropy and a constant-time suspension with roof function 1, then the time-1 map will have an invariant measure supported on each $X \times \{c\}$ for $0 \le c < 1$, but these are not flow-invariant and hence not measures of maximal entropy for the flow. Weak mixing avoids this problem:

Theorem 4.4.1. *If Φ has a unique measure μ of maximal entropy and μ is weakly mixing, and if $t > 0$, then μ is the unique measure of maximal entropy for the time-t map of Φ.*

Proof (Communicated by Federico Rodriguez Hertz). Let ν be a measure of maximal entropy for the time-t map φ^t. Then the measure $\int_0^t \varphi_*^s \nu\, ds$ is Φ-invariant and a measure of maximal entropy since $\varphi_*^s \nu$ is invariant under the time-t map and has the same entropy as ν and μ. So

$$\int_0^t \varphi_*^s \nu\, ds = \mu,$$

and μ is a linear combination (by the integral) of φ^t–invariant measures. But by Proposition 3.4.40, φ^t is μ-ergodic, so $\varphi_*^s \nu = \mu$ for every s, in particular $\nu = \mu$. □

For an equilibrium state associated with a potential function $f : X \to \mathbb{R}$ the Variational Principle implies that for the time-1 map we have

$$\begin{aligned} P(\Phi, f) &= \sup_{\mu \in \mathfrak{M}(\Phi)} h_\mu(\Phi) + \int f\, d\mu \\ &= \sup_{\mu \in \mathfrak{M}(\Phi)} h_\mu(\varphi^1) + \int f\, d\mu \\ &\le \sup_{\mu \in \mathfrak{M}(\varphi^1)} h_\mu(\varphi^1) + \int f\, d\mu \\ &= P(\varphi^1, f). \end{aligned}$$

More generally, for the time-t map one usually replaces the potential function f by $f_t = \int_0^t f(\varphi^s x)ds$ and considers $P(\varphi^t, f_t) = \sup_{\mu\in\mathfrak{M}(\varphi^t)} h_\mu(\varphi^t) + \int f_t d\mu$. The pressure above does not readily relate to $P(\Phi, f)$ for general f.

Part 2

Hyperbolic flows

We now come to the principal subject matter of this book, hyperbolic dynamics in continuous time. Chapter 5 defines hyperbolicity and develops its essential features as well as a range of new examples. This leads to a definition not just of what hyperbolic behavior is but of a hyperbolic flow. Chapter 6 refines the toolkit and our understanding of hyperbolic dynamics by utilizing the manifold structure of stable and unstable sets. Related regularity issues are refined in an optional chapter (Chapter 7), at which point we are prepared to study the statistical aspects of hyperbolic flows (Chapter 8). Hyperbolic dynamics is deterministic but of such complexity that a probabilistic approach is natural. We finally pursue 2 topics further. A study of Anosov flows (Chapter 9) explores dynamical and structural features of these that have often proved interesting also to topologists. Chapter 10 explores a range of situations in which the generally rare circumstance of smooth conjugacy (or orbit-equivalence) arises in natural contexts from the coincidence of some dynamical features with those of a algebraic counterpart. Particularly these latter chapters contain rather new mathematics, but so do several other ones in this part; even among the first examples there are quite recent ones.

CHAPTER 5

Hyperbolicity

This chapter begins to home in on the main subject of the book with the definition of a hyperbolic set, that is, a definition of what we mean by hyperbolic behavior. We almost immediately (with the Alekseev cone criterion) observe that this is a robust property, which persists under perturbation. This criterion then also proves effective in checking hyperbolicity in a collection of "physical" examples (geodesic flows, billiards, gases, and linkages). We then implement the Anosov–Katok program to establish much of the qualitative dynamical features of a hyperbolic flow from the *shadowing property* [**175**, § 2], also known as *pseudo-orbit tracing property* (Section 5.3). A core result is that in the hyperbolic context the chain decomposition, which in Section 1.5 seemed somewhat theoretical, comes into its own as exactly the right tool for describing the overall orbit structure of a hyperbolic set: There are finitely many chain components, and each contains a dense orbit as well as a dense collection of periodic orbits. This also leads us to a natural global definition of hyperbolicity of a flow, which was at best implicit in earlier literature (Definition 5.3.48). We are also able to dig much more deeply into the issue of persistence of hyperbolicity: Shadowing implies that under perturbation of a flow, not only does the presence of hyperbolic behavior persist, but the entire orbit structure arising from hyperbolicity is indestructible even under C^0 perturbations. More: C^1 perturbations can also not create additional hyperbolicity, and the full dynamics of a hyperbolic set remains present in the perturbation (structural stability), and even more, the recurrent part of the orbit structure is in its entirety rigid under C^1 perturbations—the perturbation has exactly the same recurrent dynamics, no more, and no less (Ω-stability). We note that only hyperbolic flows have this remarkable feature.

We emphasize that no part of this chapter uses (explicitly or implicitly) the existence of stable and unstable manifolds. These are central to the hyperbolic theory, but we chose to emphasize how much of the core dynamics can be obtained from shadowing alone. The invariant foliations will be introduced and immediately put to use in Chapter 6.

1. Hyperbolic sets and basic properties

The geodesic flow on compact factors of the hyperbolic plane (Remark 2.2.2) and its horizontal twin (Example 2.2.4) are iconic examples of the kind of flow in which we are interested (Remark 5.1.3). This helps give context for the definition and provides intuition for the defining properties of these sets.

Definition 5.1.1 (Hyperbolic set)**.** Let M be a smooth manifold[1] and Φ a smooth flow on M. A compact Φ-invariant set Λ is a *hyperbolic set* for Φ if there exist a finite number of hyperbolic fixed points $\{p_1,\dots,p_k\}$ and a closed set Λ' such that $\Lambda = \Lambda' \cup \{p_1,\dots,p_k\}$ and there exist a splitting $T_{\Lambda'}M = E^s \oplus E^c \oplus E^u$ and constants $C \ge 1$, $\lambda \in (0,1)$, $\mu > 1$ such that

- $E^c(x) := \mathbb{R}V(x) \neq \{0\}$ for all $x \in \Lambda'$, where $V := \dot{\varphi}$ as in (1.1.2)
- $\|D\varphi^t\restriction_{E^s_x}\| \le C\lambda^t$ for all $t > 0$ and all $x \in \Lambda'$, and
- $\|D\varphi^{-t}\restriction_{E^u_x}\| \le C\mu^{-t}$ for all $t > 0$ and all $x \in \Lambda'$.

A smooth flow Φ on a closed[2] connected manifold M is said to be an *Anosov flow* (or *hyperbolic flow*) if M is hyperbolic for Φ. If $\dim M = 3$, then such Φ is called an Anosov 3-flow.

Remark 5.1.2. This definition of a hyperbolic set allows the existence of (isolated!) fixed points, in contrast to what is often done elsewhere. Allowing fixed points gives greater generality, and we will find that the main results are no different. The inclusion of fixed points is also a natural adaptation for the study of stability properties later.

Remark 5.1.3 (Examples)**.** Numerous prior examples are of this kind:

- The suspensions in Examples 1.5.23 and 1.5.24 are Anosov flows.
- By Theorem 5.1.16 below, so are time-changes of these, hence all special flows over these automorphisms.
- So are geodesic flow on compact factors of the hyperbolic plane—the discussion in Remark 2.2.2 establishes the requirements of Definition 5.1.1, and
- Example 2.2.4 does so for the horizontal flow generated by H, which therefore gives yet another example of an Anosov flow.
- Other examples of hyperbolic flows will appear in Remark 5.1.12 and Sections 6.3, 6.5, 9.3, 9.2, and 5.2b.

Anosov flows were conceived as a codification of the salient features of geodesic flows of compact manifolds with negative curvature. These in turn were studied

[1] Implicitly assumed connected throughout this book.

[2] that is, compact and without boundary

as the first examples of ergodic, indeed chaotic, mechanical systems and hence remain the primary continuous-time example in this theory. Theorem 5.2.4 below establishes that these are indeed Anosov flows, whether or not the curvature is constant as in Chapter 2. Its proof addresses the fact that we usually cannot identify the contracting and expanding subspaces as readily as in the algebraic case (for example, as in Remark 2.2.2). This idea likewise underlies the other "physical" examples in Section 5.2.

Proposition 5.1.4. *Let Λ be a hyperbolic set for a flow Φ, $\tau \in \{u, s, c, cs, cu\}$. Then*

- *$x \mapsto E^\tau_x$ is Φ-invariant and continuous,*
- *$\dim E^\tau_x$ is locally constant,*
- *E^τ_x are pairwise uniformly transverse for $\tau = u, s, c$: there is $\alpha_0 > 0$ such that for any $x \in \Lambda$, the angle between $\xi \in E^\tau_x$ and $\eta \in E^{\tau'}_x$ is at least α_0 when $\tau \neq \tau'$.*

Proof. This holds trivially at any fixed point in the hyperbolic set. Elsewhere, the inequalities $\|D\varphi^t \xi\| \le C\lambda^t \|\xi\|$ invariantly characterize E^s_x, and similarly for $\tau \in \{u, cs, cu\}$. By continuity of $D\varphi^t$ the set of (x, ξ) on which they hold is closed, so $\lim_{x \to x_0} E^\tau_x \subset E^\tau_{x_0}$. Then $\dim E^u_{x_0} + \dim E^s_{x_0} = \dim M - 1 = E^u_x + E^s_x$ implies that neither inclusion is strict, so $E^\tau_{x_0} = \lim_{x \to x_0} E^\tau_x$.

Since the angle between $\xi \in E^\tau_x$ and $\eta \in E^{\tau'}_x$ is continuous and positive ($E^\tau_x \cap E^{\tau'}_x = \{0\}$ it has a positive minimum. □

We note that one can do better than continuity: Theorem 7.4.1 establishes Hölder continuity (Definition 7.1.1). The next lemma produces a metric such that we can take $C = 1$ in Definition 5.1.1. Such a metric is called an *adapted metric* or *Lyapunov metric.*

Proposition 5.1.5 (Adapted metric)**.** *Let Λ be a hyperbolic set for a flow Φ with λ, μ, C as in Definition 5.1.1 and $\underline{\lambda} \in (\lambda, 1)$, $\underline{\mu} \in (1, \mu)$. Then there is a continuous Riemannian metric such that for the induced norm $\|\cdot\|^*$, for $t \ge 0$ and for $x \in \Lambda$ we have*

$$\|D\varphi^t\restriction_{E^s_x}\|^* \le \underline{\lambda}^t \quad \textit{and} \quad \|D\varphi^{-t}\restriction_{E^u_x}\|^* \le \underline{\mu}^{-t}.$$

Proof. We adapt the norm on each of the spaces E^s and E^u. For $v \in E^s_x$ define

$$\left(\|v\|^s_x\right)^2 := \underbrace{\int_0^\infty \underline{\lambda}^{-2s} \left(\|D\varphi^s v\|_{\varphi^s(x)}\right)^2 ds}_{\le \int_0^\infty \underline{\lambda}^{-2s} C\lambda^{2s} \|v\|_x ds < \infty}.$$

As an integral of quadratic forms, this is a quadratic form and hence the norm arises from an inner product. This is the desired norm on E^s_x because if $v \in E^s_x$ and

$t > 0$, then

$$\begin{aligned}\left(\|D\varphi^t v\|^s_{\varphi^t x}\right)^2 &= \underbrace{\int_0^\infty \underline{\lambda}^{-2s}\left(\|D\varphi^{t+s}v\|_{\varphi^{t+s}x}\right)^2 ds}_{=\underline{\lambda}^{2t}\int_0^\infty \underline{\lambda}^{-2(t+s)}(\|D\varphi^{t+s}v\|_{\varphi^{t+s}x})^2 ds}\\ &= \underline{\lambda}^{2t}\underbrace{\int_t^\infty \underline{\lambda}^{-2s}\left(\|D\varphi^s v\|_{\varphi^s(x)}\right)^2 ds}_{\le\int_0^\infty \underline{\lambda}^{-2s}(\|D\varphi^s v\|_{\varphi^s(x)})^2 ds} \le \underline{\lambda}^{2t}\left(\|v\|^s_x\right)^2.\end{aligned}$$

Similarly, the desired metric on E^u is

$$\left(\|v\|^u_x\right)^2 = \int_0^\infty \underline{\mu}^{-2s}\left(\|D\varphi^{-s}v\|_{\varphi^{-s}(x)}\right)^2 ds.$$

For $v = v_s + v_u \in E^s_x \oplus E^u_x$, where $v_\tau \in E^\tau_x$, let $(\|v\|^*_x)^2 := (\|v_s\|^s_x)^2 + (\|v_u\|^u_x)^2$; this is a metric on $E^s_x \oplus E^u_x$ with $E^s \perp E^u$. For the nonfixed points in Λ one can extend this to a metric in the center direction. For $v = v_c + v_s + v_u \in T_xM$ where $v_s \in E^s_x$, $v_u \in E^u_x$, and $v_c \in E^c_x$, let $(\|v\|^*_x)^2 := (\|v_s\|^s_x)^2 + (\|v_u\|^u_x)^2 + (\|v_c\|_x)^2$. This induces a metric on T_xM, which is continuous since the components are continuous.

Furthermore, this metric can be extended to a continuous metric on all of M and changed into a smooth Riemannian metric on all of M by a perturbation so small as to preserve the defining inequalities. □

Checking that a given set is hyperbolic for a flow involves the challenge of finding the invariant subbundles E^u and E^s. Outside of algebraic situations, it is not clear how to go about that. Fortunately, there it turns out that approximate knowledge of these suffices, and in practice, one can establish that a set is hyperbolic by using cone fields.

Definition 5.1.6. For a set $X \subset M$ with a splitting $T_xM = E_x \oplus F_x$ for each $x \in X$, and for $\beta \in (0,1)$ the *β-cone field* consists of the *β-cone*

$$C_\beta(E,F) = \{v + w \mid v \in E, w \in F, \|w\| < \beta\|v\|\}.$$

of E_x and F_x at each $x \in X$.

Proposition 5.1.7 (Alekseev Cone Field Criterion)**.** *A compact Φ-invariant set Λ is hyperbolic if and only if there exist constants $\lambda, \beta \in (0,1)$, $C \ge 1$, and a decomposition $T_x\Lambda = S_x \oplus E^c \oplus U_x$ for each $x \in \Lambda$ such that for all $x \in \Lambda$ and all $t > 0$ we have*

- *$E^c_x = \mathbb{R}X(x) \ne \{0\}$ for all nonfixed points $x \in \Lambda$ and $E^c_x = \{0\}$ for the fixed points, where X is the generating vector field,*
- *$D\varphi^t\big(\overline{C_\beta(U_x, E^c_x \oplus S_x)}\big) \subset C_\beta(U_{\varphi^t(x)}, E^c_{\varphi^t(x)} \oplus S_{\varphi^t(x)})$,*
- *$D\varphi^{-t}\big(\overline{C_\beta(S_x, E^c_x \oplus U_x)}\big) \subset C_\beta(S_{\varphi^t(x)}, E^c_{\varphi^t(x)} \oplus U_{\varphi^t(x)})$,*

- $\|D\varphi^t\xi\| \le C\lambda^t\|\xi\|$ *for* $\xi \in C_\beta(S_x, E^c_x \oplus U_x)$*, and*
- $\|D\varphi^{-t}\xi\| \le C\lambda^t\|\xi\|$ *for* $\xi \in C_\beta(U_x, E^c_x \oplus S_x)$*.*

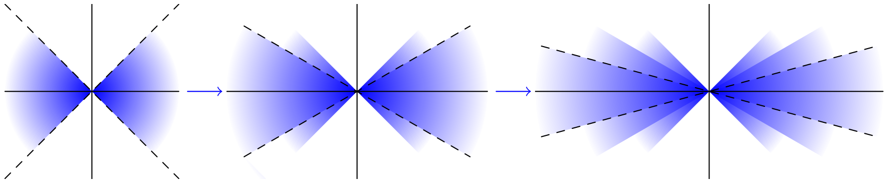


FIGURE 5.1.1. Invariant cones

PROOF. "Only if" is an easy consequence of the definitions. "If": We show that

$$E^u_x \coloneqq \bigcap_{t>0} D\varphi^t\big(\overline{C_\beta(U_{\varphi^{-t}(x)}, E^c_{\varphi^{-t}(x)} \oplus S_{\varphi^{-t}(x)})}\big)$$

and

$$E^s_x \coloneqq \bigcap_{t>0} D\varphi^{-t}\big(\overline{C_\beta(S_{\varphi^{t}(x)}, E^c_{\varphi^{t}(x)} \oplus U_{\varphi^{t}(x)})}\big).$$

are as in Definition 5.1.1. By construction, they are expanded and contracted, respectively, and equivariant, so we need only show that these are linear subspaces of the right dimension. To that end, let S^∞_x be an accumulation point of $(D\varphi^t(S_{\varphi^{-t}(x)}))_{t>0}$ in the following sense: By compactness of the unit sphere, orthonormal bases in $D\varphi^t(S_{\varphi^{-t}(x)})$ accumulate to a frame, and we denote its linear hull by $S^\infty_x \subset E^u_x$. Then $\dim S^\infty_x = \dim S_x$. Defining $T^\infty_x \subset E^s_x$ in like manner, we now show that with the definitions above, $E^u_x = S^\infty_x$, and a like argument then gives $E^s_x = T^\infty_x$.

If $v \in E^u_x$, then $v = v^u + v^{cs}$, where $v^u \in S^\infty_x$ and $v^{cs} \in E^c_x \oplus E^s$, and there is a $K \in \mathbb{R}$ such that

$$\|v^{cs}\| = \|D\varphi^t(D\varphi^{-t}(v - v^u))\| \le K \underbrace{\|D\varphi^{-t}(v - v^u)\|}_{=\|D\varphi^{-t}(v) - D\varphi^{-t}(v^u)\| \le C\lambda^t(\|v\| + \|v^u\|)} \xrightarrow[t\to\infty]{} 0. \qquad \square$$

One can also express the Alekseev cone criterion in terms of Lorentz metrics that behave analogously to Lyapunov functions or metrics.

Definition 5.1.8. A Lorentz metric is a nondegenerate bilinear form g with signature $(n-1, 1)$, that is, of the n values of the quadratic form $Q(x) \coloneqq g(x, x)$ on an orthogonal basis, one is negative and all others are positive.[3]

[3] By Sylvester's law of inertia, this is independent of the choice of such basis. A Riemannian metric is a like form of signature $(n, 0)$, that is, positive-definite.

Proposition 5.1.9. *A smooth flow $\varphi^t\colon M \to M$ of a 3-manifold M is an Anosov flow if and only if there are two continuous Lorentz metrics Q^+ and Q^- on M and constants $a, b, c, T > 0$ such that*

(1) for all $v \in T_xM$, $t > T$, if $Q^\pm(v) > 0$ then $Q^\pm(D_x\varphi^{\pm t}(v)) > ae^{bt}Q^\pm(v)$,
(2) $C^+ \cap C^- = \varnothing$, where $C^\pm$ is the $Q^\pm$-positive cone,
(3) $Q^\pm(X) = -c$ where X is the generating vector field,
(4) $D_x\varphi^{\pm T}(\overline{C^\pm(x)}) \smallsetminus \{0\} \subset C^\pm(\varphi^{\pm T}(x))$.

Proof. If φ^t is an Anosov flow we can choose disjoint cones around the strong stable and unstable directions, neither of which contains X. These define (up to a factor) the Lorentz metrics, and choosing $c = 1$ fixes the metrics; we omit the details.

Assume now the above conditions for two continuous Lorentz metrics $Q^\pm$ and constants $a, b, c, T > 0$. The cone fields $C^\pm$ induce fields $\mathscr{E}^\pm$ of ellipses in the projectivization PTM of TM, and φ^t acts on fields of ellipses by $(\varphi^t_*\mathscr{E})(x) := PD_{\varphi^{-t}(x)}\varphi^t(\mathscr{E}(\varphi^{-t}(x)))$. Then

- condition (2) implies that $\mathscr{E}^+_t(x) \cap \mathscr{E}^-_t(x) = \varnothing$,
- condition (4) implies that $\overline{\mathscr{E}^\pm_T(x)} \subset \operatorname{int}\mathscr{E}^\pm(x)$.

If we endow each $\mathscr{E}^\pm(x)$ with the Hilbert metric then this last property (strict nesting) implies that $D\varphi^{\pm T}$ induces contractions $\mathscr{E}^\pm(x) \to \mathscr{E}^\pm(\varphi^{\pm T}(x))$ of the Hilbert metrics with a factor that can be chosen uniformly by compactness of M. Thus, the diameter of $\mathscr{E}^\pm_t(x) \subset \mathscr{E}^\pm(x)$ as measured by the Hilbert metric on $\mathscr{E}^\pm(x)$ goes to 0 exponentially, so $\Delta^\pm(x) := \bigcap_{t>T}\mathscr{E}^\pm_t(x)$ are points, and $\Delta^+(x) \neq \Delta^-(x)$ for all $x \in M$ since $\mathscr{E}^+_t(x) \cap \mathscr{E}^-_t(x) = \varnothing$.

Clearly $\Delta^\pm$ define φ^t-invariant line fields $E^\pm$, and since $X \notin C^\pm$ by condition (3), $\Delta^+(x) \neq X(x) \neq \Delta^-(x)$.

Now choose a continuous Riemannian metric on M whose unit spheres intersect $E^\pm$ in points for which $Q^\pm = 1$. Then condition (1) implies that $E^\pm$ are exponentially expanding and contracting, respectively, as required. □

Note that the subbundles S and T in Proposition 5.1.7 need not be invariant. They simply need to be close to an invariant subbundle by a factor of β. This flexibility makes them easily extendible with the same defining properties, so while it follows directly from the definitions that every closed invariant subset of a hyperbolic set for Φ is also a hyperbolic set, the cone field criterion allows us to conclude more interestingly, that one can sometimes envelop a given hyperbolic set by a larger one.

Proposition 5.1.10 (Persistence of hyperbolicity)**.** *A compact hyperbolic set $\Lambda \subset X$ for a flow Φ has a neighborhood $U \subset X$ such that $\Lambda_\varphi^U := \bigcap_{t\in\mathbb{R}} \varphi^t(\bar{U})$ is a hyperbolic set, and moreover, so is $\Lambda_\Psi^U := \bigcap_{t\in\mathbb{R}} \psi_t(\bar{U})$ when Ψ is sufficiently C^1-close to Φ.*

Proof. Let Λ be a hyperbolic set for a flow Φ. First assume we have an adapted metric on Λ with hyperbolic constants $\lambda \in (0,1)$ and $\mu > 1$, and fix $\underline{\lambda} \in (\lambda, 1)$ and $\underline{\mu} \in (1,\mu)$. Extend the splitting on Λ to a continuous splitting (not necessarily invariant) in a sufficiently small neighborhood V of Λ, and fix $\beta > 0$ sufficiently small and cones $C_\beta(E_x^s, E_x^c \oplus E_x^u)$ and $C_\beta(E_x^u, E_x^c \oplus E_x^s)$. If $x \in \Lambda$ and $t > 0$, then

$$D\varphi^{-t} C_\beta(E_x^s, E_x^c \oplus E_x^u) \subset C_{\lambda^t\beta}(E_x^s, E_x^c \oplus E_x^u)$$

and

$$D\varphi^{t} C_\beta(E_x^u, E_x^c \oplus E_x^s) \subset C_{\mu^{-t}\beta}(E_x^u, E_x^c \oplus E_x^s).$$

Also, we can choose V and β such that

$\|D\varphi^{-t}\xi\| \le \underline{\mu}^{-t}\|\xi\|$ for $\xi \in C_\beta(E_x^u, E_x^c \oplus E_x^s)$ & $\|D\varphi^t\eta\| \le \underline{\lambda}^t\|\eta\|$ for $\eta \in C_\beta(E_x^s, E_x^c \oplus E_x^u)$.

For a possibly smaller neighborhood U of Λ and $x \in \overline{U}$ the conditions in Proposition 5.1.7 hold not only for Φ, but also any flow Ψ that is C^1 close to Φ. □

Although Proposition 5.1.10 does not assert that $\Lambda_\Psi^U \neq \varnothing$(!), the next result is a direct consequence.

Corollary 5.1.11. *Any sufficiently small C^1-perturbation of an Anosov flow is an Anosov flow.*

Remark 5.1.12. Thus, the magnetic flows from Remark 2.2.10 are Anosov flows when the magnetic field is weak enough.

Unfortunately, in this observation and in Proposition 5.1.10 itself, there is no control over how large a perturbation one can allow. We are a little more fortunate in the context of magnetic flows, so let us elaborate on this observation. We first define magnetic flows in more satisfying generality.

Definition 5.1.13 (Magnetic flow)**.** On a Riemannian manifold M suppose $\mathfrak{m}\colon TM \to TM$ is antisymmetric tensor, that is, $\langle \mathfrak{m}v_1, v_2\rangle + \langle v_1, \mathfrak{m}v_2\rangle = 0$ for all $v_1, v_2 \in TxM$ and all $x \in M$;[4] and consider the flows defined on SM by the following counterpart to the geodesic equation (5.2.1):

$$\nabla_{\dot\gamma}\dot\gamma = \mathfrak{m}\dot\gamma.$$

[4] In Remark 2.2.10 this was a 90° rotation combined with a constant scaling.

The size of $\mathfrak{m}$ is a natural measure of the size of the magnetic perturbation to the geodesic flow, and it is natural to ask how large this perturbation can be without losing hyperbolicity. It turns out that here we have the rare case of explicit control of the hyperbolicity domain.

Theorem 5.1.14 ([**138**, Théorème 4.1])**.** *If the sectional curvatures K of a closed Riemannian manifold M satisfy $-k_2^2 \le K \le -k_1^2 < 0$, then magnetic flows with $\frac{5}{4}\|\mathfrak{m}\|_\infty^2 + \|\nabla \frac{m}{\|}_{\infty} < k_1^2$ are Anosov flows.*

In a variety of contexts it is useful that for symplectic systems the cone criterion can be established merely by producing strictly invariant cone families; uniform expansion and contraction is then a consequence. Geometrically, this is intuitive: "squeezing" a cone should push points outward. We establish this in dimension 2 using convenient local coordinates.

Theorem 5.1.15 (Wojtkowski cones)**.** *Suppose $A_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ are matrices such that for some $\epsilon > 0$, all $k \in \mathbb{Z}$ and all $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with $xy > 0$ we have $|\det A_k| \ge 1$ and*

$$A_k v \in C_\epsilon := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \epsilon y \le x \le y/\epsilon \right\}.$$

Then there are $c > 0$ and $\lambda > 1$ such that

$$\|A_{k-1} \dots A_{k-i} v\| \ge c\lambda^i \|v\|$$

for all $k \in \mathbb{Z}$, $i \in \mathbb{N}$ and $v \in C_\epsilon$.

Proof (Wojtkowski, Kourganoff)**.** Since $\frac{\epsilon}{2}(x^2+y^2) \le P\Big(\begin{pmatrix} x \\ y \end{pmatrix}\Big) := xy \le \frac{2}{\epsilon}(x^2+y^2)$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in C_\epsilon$, we check the conclusion for $\sqrt{P}$ instead of $\|\cdot\|$. Specifically, we show $P(A_k v) \ge \frac{1}{1-\epsilon^2} P(v)$ for $v \in C_\epsilon$ and $k \in \mathbb{Z}$.

Without loss of generality $\det A_k > 0$ (otherwise left-multiply A_k by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) and all entries of A_k are positive (otherwise multiply by $-\mathrm{Id}$), so

$$1 \le a_k d_k - b_k c_k \le \frac{1}{\epsilon} b_k \frac{1}{\epsilon} c_k - b_k c_k = \big(\frac{1}{\epsilon^2} - 1\big) b_k c_k$$

since $\begin{pmatrix}a_k\\c_k\end{pmatrix} = A_k\begin{pmatrix}1\\0\end{pmatrix} \in C_\epsilon$ and $\begin{pmatrix}b_k\\d_k\end{pmatrix} = A_k\begin{pmatrix}0\\1\end{pmatrix} \in C_\epsilon$ by continuity. This implies that $b_k c_k \ge \dfrac{1}{\frac{1}{\epsilon^2}-1} = \dfrac{\epsilon^2}{1-\epsilon^2} = \dfrac{1}{1-\epsilon^2} - 1 > \dfrac{1/2}{1-\epsilon^2} - \dfrac{1}{2}$. For $v = \begin{pmatrix}x\\y\end{pmatrix} \in C_\epsilon$ we thus have

$$\begin{aligned} P(A_k v) &= (a_k x + b_k y)(c_k x + d_k y) \ge (a_k d_k + b_k c_k)xy \\ &= \underbrace{(a_k d_k - b_k c_k)}_{\ge 1} xy + 2b_k c_k xy \ge (1 + 2b_k c_k)P(v) \ge \frac{1}{1-\epsilon^2} P(v). \quad \square \end{aligned}$$

The last "generic" application of the Alekseev cone criterion is rather basic:

Theorem 5.1.16 (Hyperbolicity of time-changes)**.** *Let Λ be a hyperbolic set for a flow Φ. If Ψ is a smooth time-change of Φ, then Λ is a hyperbolic set for Ψ.*

Proof. Write $\psi^t(x) = \varphi^{\alpha(t,x)}(x)$ as in Proposition 1.2.2 with $\alpha(0,\cdot) = 0$. Choose for each $x \in \Lambda$ local coordinates $x = (x^0, x^u, x^s)$ centered at x and adapted to the splitting $T_x M = E_x^0 \oplus E_x^+ \oplus E_x^-$ so that with respect to these coordinates

$$D\varphi^t(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_t & 0 \\ 0 & 0 & B_t \end{pmatrix}$$

with $\|B_t\| \le \lambda^t < 1$ and $\|A_t^{-1}\| \le \mu^{-t} < 1$. In these coordinates

$$D\psi^t(0) = \begin{pmatrix} 1 & \alpha_{x^u}(t,x) & \alpha_{x^s}(t,x) \\ 0 & A_{\alpha(t,x)} & 0 \\ 0 & 0 & B_{\alpha(t,x)} \end{pmatrix},$$

where $\alpha_{x^u}(t,x)$ and $\alpha_{x^s}(t,x)$ are the partial derivatives of α with respect to x^u and x^s, respectively. By compactness of Λ we may take $Kt > 0$ as an upper bound for their size when $t > 0$. To prove hyperbolicity of ψ^t we use the cone criterion. We write vectors in $T_x\Lambda = E_x^0 \oplus E_x^+ \oplus E_x^-$ as (u, v, w) with $u \in E_x^0$, $v \in E_x^+$, $w \in E_x^-$ and let

$$\|u, v, w\|^2 := \epsilon^2 \|u\|^2 + \|v\|^2 + \|w\|^2,$$

where a sufficiently small $\epsilon > 0$ will be specified later. For $\gamma < \sqrt{\mu^2 - 1}$ we now check whether the γ-cone given by

$$\epsilon^2 \|u\|^2 + \|w\|^2 \le \gamma^2 \|v\|^2$$

is $D\psi^t$-invariant for $t \in [0,1]$. Take ϵ such that

$$K^2 t^2 \epsilon^2 + \lambda^{2\alpha(t,x)} \le 1 \text{ for } t \in [0,1].$$

If $(u', v', w') = D\psi^t(u, v, w)$ then

$$\begin{aligned}
\epsilon^2\|u'\|^2 + \|w'\|^2 &= \epsilon^2\|u + \alpha_{x^u} v + \alpha_{x^s} w\|^2 + \|B_{\alpha(t,x)} w\|^2\\
&\le \epsilon^2(\|u\| + Kt\|v\| + Kt\|w\|)^2 + \lambda^{2\alpha(t,x)}\|w\|^2\\
&= \epsilon^2\|u\|^2 + (K^2t^2\epsilon^2 + \lambda^{2\alpha(t,x)})\|w\|^2\\
&\quad + \epsilon^2Kt(Kt\|v\|^2 + 2\|u\|\|v\| + 2\|u\|\|w\| + 2Kt\|v\|\|w\|)\\
&\le \gamma^2\|v\|^2 + \epsilon^2Kt\Big(Kt\|v\|^2 + \frac{2\gamma}{\epsilon}\|v\|^2 + \frac{2\gamma^2}{\epsilon}\|v\|^2 + 2\gamma Kt\|v\|^2\Big)\\
&= \gamma^2\Big(1 + \frac{\epsilon Kt}{\gamma^2}(\epsilon Kt(1+2\gamma) + 2\gamma(1+\gamma))\Big)\|v\|^2\\
&< \gamma^2\mu^{2\alpha(t,x)}\|v\|^2 \le \gamma^2\|v'\|
\end{aligned}$$

for sufficiently small $\epsilon > 0$ and $t \in (0,1]$. Thus γ-cones are ψ^t-invariant. To check that vectors in γ-cones are expanded note that $\epsilon^2\|u'\|^2 + \|w'\|^2 \ge \delta^{\alpha(t,x)}(\epsilon^2\|u\|^2 + \|w\|^2)$ for some $\delta > 0$ and take $\gamma > 0$ small enough so that

$$\frac{\mu^{2\beta} + \delta^\beta\gamma^2}{1+\gamma^2} \ge \eta^\beta \tag{5.1.1}$$

for some $\eta > 1$ and all $\beta > 0$. Then if $\epsilon^2\|u\|^2 + \|w\|^2 \le \gamma^2\|v\|^2$ we have

$$\begin{aligned}
\epsilon^2\|u'\|^2 + \|v'\|^2 + \|w'\|^2 &\ge \delta^{\alpha(t,x)}(\epsilon^2\|u\|^2 + \|w\|^2) + \|A_{\alpha(t,x)}v\|^2\\
&\ge \eta^{\alpha(t,x)}(\epsilon^2\|u\|^2 + \|w\|^2)\\
&\quad + (\delta^{\alpha(t,x)} - \eta^{\alpha(t,x)})(\epsilon^2\|u\|^2 + \|w\|^2)\\
&\quad + \mu^{2\alpha(t,x)}\|v\|^2\\
&\ge \eta^{\alpha(t,x)}(\epsilon^2\|u\|^2 + \|w\|^2)\\
&\quad + [(\delta^{\alpha(t,x)} - \eta^{\alpha(t,x)})\gamma^2 + \mu^{2\alpha(t,x)}]\|v\|^2\\
&\ge \eta^{\alpha(t,x)}(\epsilon^2\|u\|^2 + \|v\|^2 + \|w\|^2),
\end{aligned}$$

where the last inequality follows from (5.1.1).

Since ψ^{-t} is a time change of φ^{-t} there is a corresponding cone family for ψ^{-t}. □

Remark 5.1.17. We give another proof later; see page 294.

2. Physical flows: geodesic flows, billiards, gases, and linkages

While in the examples of Remark 5.1.3 the definition of hyperbolicity was easily checkable directly, the cone criterion provides a convenient way to establish hyperbolicity in particular of various classes of "mechanical" flows (beyond Remark 2.2.10), and these are explored in this section. Specifically, we show that geodesic flows of negatively curved manifolds are Anosov flows (Theorem 5.2.4) and substantially weaken the needed hypotheses in the case of surfaces (Theorem 5.2.8); the same approach then establishes hyperbolicity of dispersing billiards (Theorem 5.2.18), and these are in turn connected to the gas models that motivated Maxwell and Boltzmann (Theorem 5.2.31). Finally, we describe Anosov systems that are mechanical in a way that could be made into an actual desktop model (Theorem 5.2.36).

a. Geodesic flows. We begin with geodesic flows beyond the geodesic flow on the hyperbolic plane and its compact factors in Chapter 2. This requires a little differential geometry (which is less important for our purposes than the results). Geodesic flows of negatively curved manifolds are an important example both historically and mathematically. Indeed, as mentioned in Chapter 0, the concept of an Anosov flow arose as Anosov axiomatized the arguments used in working with geodesic flow on manifolds of negative sectional curvature.

To formally introduce the geodesic flow in full generality, let M be a compact Riemannian manifold. The *geodesic equation* is a suitable way to write $\ddot{\gamma} = 0$, that is, zero acceleration, which corresponds to free-particle motion. $\dot{\gamma}$ is the tangent vector to a curve $t \mapsto \gamma(t)$, and the second derivative can be expressed using the *Levi-Civita connection* ∇ or *Riemannian covariant derivative* as follows:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0. \tag{5.2.1}$$

This, then, defines a flow on the unit tangent bundle of M as before.

To introduce curvature, which has an essential effect on the dynamics, let R be the curvature tensor defined by

$$R(u,v)w \coloneqq \nabla_v\nabla_u w - \nabla_u\nabla_v w + \nabla_{[u,v]}w.$$

Then $\langle R(u,v)w,x\rangle = \langle R(w,x)u,v\rangle$ and $R(u,u) = 0$ for $u,v,w,x \in T_pM$. If $u,v \in T_pM$ are linearly independent, then the *sectional curvature*

$$K(S) \coloneqq \frac{\langle R(u,v)u,v\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}$$

depends only on the 2-plane $S \subset T_pM$ spanned by the vectors[5] u and v and is the Gaussian curvature at p of the 2-manifold $\exp_p S$ with respect to the Riemannian metric induced from M, where exp is the Riemannian exponential map. We usually

[5] Because changing to a base (u',v') of S can be accomplished by repeated application of the steps $(u,v) \mapsto (v,u)$, $(u,v) \mapsto (au,v)$ and $(u,v) \mapsto (u+av,v)$, none of which change K.

assume that this is always negative and hence, by compactness, bounded from above by $-k^2 < 0$.

Jacobi fields help discern the effect of curvature on the dynamics of the geodesic flow. For a geodesic $\gamma\colon \mathbb{R} \to M$, a Jacobi field $Y\colon\ t \mapsto Y(t) = \frac{\partial V}{\partial s}$ is an "*infinitesimal variation*" of a *geodesic variation* $V\colon \mathbb{R} \times (a,b) \to M$, that is, each $\gamma_s \coloneqq V(\cdot, s)$ is a geodesic with $\gamma_0 = \gamma$

Proposition 5.2.1. *An infinitesimal variation is a solution of the Jacobi equation*

$$\ddot{Y}(t) + \underbrace{K(t)}_{\coloneqq R(\dot\gamma(t),\cdot)\dot\gamma(t)} Y(t) = 0, \tag{5.2.2}$$

where dots denote differentiation with respect to t.

Proof. Since $[\frac{\partial V}{\partial t}, \frac{\partial V}{\partial s}] = 0$ we have $\nabla_{\frac{\partial V}{\partial t}} \frac{\partial V}{\partial s} = \nabla_{\frac{\partial V}{\partial s}} \frac{\partial V}{\partial t}$. Thus, for $s = 0$ we have

$$\ddot{Y} = \nabla_{\frac{\partial V}{\partial t}} \nabla_{\frac{\partial V}{\partial t}} \frac{\partial V}{\partial s} = \nabla_{\frac{\partial V}{\partial t}} \nabla_{\frac{\partial V}{\partial s}} \frac{\partial V}{\partial t} = -\Big(\nabla_{\frac{\partial V}{\partial s}} \underbrace{\nabla_{\frac{\partial V}{\partial t}} \frac{\partial V}{\partial t}}_{=0\text{ (geodesic equation)}} - \nabla_{\frac{\partial V}{\partial t}} \nabla_{\frac{\partial V}{\partial s}} \frac{\partial V}{\partial t} - \nabla_{[\frac{\partial V}{\partial s}, \frac{\partial V}{\partial t}]} \frac{\partial V}{\partial t}\Big)$$

$$= -R\Big(\frac{\partial V}{\partial s}, \frac{\partial V}{\partial t}\Big) \frac{\partial V}{\partial t} = -R(\dot\gamma, Y)\dot\gamma. \qquad \square$$

Conversely,

Proposition 5.2.2. *Solutions of the Jacobi equation are infinitesimal variations.*

Proof. If Y is a solution of the Jacobi equation along γ let $h_i(s)$ for $|s| < \epsilon$ be curves with $(h_i(0), h_i'(0)) = (\gamma(t_i), Y(t_i))$ for $i = 1, 2$. If ϵ and $t_1 - t_2$ are small enough then for all s there is a unique shortest geodesic $V(\cdot, s)$ from $h_1(s)$ to $h_2(s)$. Y and the vector field $X = \frac{\partial V}{\partial s}$ along γ are solutions of the Jacobi equation that agree at t_1 and t_2, hence everywhere because they solve the same second-order differential equation. □

Remark 5.2.3 (Orthogonal Jacobi fields)**.** A tangential Jacobi field (sometimes referred to as a parallel Jacobi field) is of the form $Y(t) = f(t)\dot\gamma(t)$ with $\ddot f(t) = 0$ (since $\ddot\gamma(t) = 0 = K(t)\dot\gamma(t)$) and hence linear in time. On the other hand the projection Y_T onto $\mathbb{R}\dot\gamma$ of any Jacobi field Y is of the same form with $f(t) = \langle Y(t), \dot\gamma(t)\rangle$. But $\ddot f = \langle \ddot Y, \dot\gamma\rangle = -\langle K\dot\gamma, Y\rangle = 0$ and thus the tangential projection Y^T of Y is a Jacobi field. By linearity of the Jacobi equation the same holds for $Y^\perp \coloneqq Y - Y^T$, which is orthogonal to $\dot\gamma$. Another way to represent *orthogonal Jacobi fields* is to note that if $Y(t)$ is a Jacobi field along a geodesic γ and both $Y(t_0)$ and $\dot Y(t_0)$ are orthogonal to $\dot\gamma(t_0)$ for some t_0, then $Y(t)$ and $\dot Y(t)$ are orthogonal to $\dot\gamma(t)$ for all t. We denote the set of orthogonal Jacobi fields by $\mathscr{J}(\gamma)$.

If $\dim(M) = n$ and γ is a geodesic in M, then the dimension of the space of Jacobi fields along γ is $2n$. The space of orthogonal Jacobi fields is then $2n-2$-dimensional since the space of tangential Jacobi fields is 2-dimensional.[6]

We now make more precise how the behavior of Jacobi fields reflects the dynamics of the geodesic flow g^t. For $p \in M$, $v \in T_pM$ denote by γ_v the geodesic with $\gamma_v(0) = p$, $\dot\gamma_v(0) = v$. Then there are isomorphisms

$$\psi_v \colon T_vTM \to T_pM \oplus T_pM, \quad \xi \mapsto (x, x') \quad \text{with} \quad \psi_{g^tv}(Dg^t\xi) = \big(Y(t), \dot Y(t)\big),$$

where Y is the Jacobi field along γ_v with $Y(0) = x$ and $\dot Y(0) = x'$.

Theorem 5.2.4. *The geodesic flow of a compact Riemannian manifold with negative sectional curvature is an Anosov flow.*

Proof. We establish the cone conditions for Proposition 5.1.7 by connecting curvature and the Jacobi equation (5.2.2), with Lemma 5.2.5 as the key step.

Let M be a compact Riemannian manifold with tangent bundle TM, unit tangent bundle $SM := \{v \in TM \mid \|v\| = 1\}$, and geodesic flow $g^t : SM \to SM$. Its dynamics can be described in terms of the evolution of Jacobi fields, that is, we can describe an action of g^t (or Dg^t, rather) on Jacobi fields. Two linearly independent tangential Jacobi fields with linear growth correspond to affine reparameterizations of the geodesic, that is, shifts of the initial point and uniform changes of speed. The first variation corresponds to the flow direction for the geodesic flow in the unit tangent bundle SM; the second is transverse to SM. Thus, in order to establish that the geodesic flow in SM is an Anosov flow it is sufficient to show that the space of orthogonal Jacobi fields admits a splitting into exponentially contracting and exponentially expanding invariant subspaces.

To study orthogonal Jacobi fields it suffices to know that they are solutions of the Jacobi equation (5.2.2) and that the operator K in that equation is negative-definite and symmetric: the curvature assumption together with compactness implies the existence of $k, \kappa > 0$ such that $-k^2$ is an upper bound for the sectional curvature and

$$\langle KY, Y\rangle \le -k^2\langle Y, Y\rangle \text{ when } Y \perp \dot\gamma, \quad \text{and} \quad \langle KY, KY\rangle < \frac{1}{\kappa^2} \text{ for } Y \in SM.$$

To show hyperbolicity of the geodesic flow define a new norm on $T_pM \oplus T_pM$ by $\|u, v\| := \sqrt{\langle u, u\rangle + \epsilon\langle v, v\rangle}$ for $u, v \in T_pM$ and for some fixed $\epsilon < 1/\kappa$, and note that $0 \le \langle u - v, u - v\rangle = \langle u, u\rangle - 2\langle u, v\rangle + \langle v, v\rangle$ and hence

$$2\epsilon\langle u, v\rangle \le \epsilon\langle u, u\rangle + \epsilon\langle v, v\rangle \le \|u, v\|^2.$$

[6] If we restrict to the unit tangent bundle, that is, to unit-speed geodesics, then the dimension of the space of Jacobi fields is $2n-1$ and the space of tangential Jacobi fields is 1-dimensional, so the space of orthogonal Jacobi fields is $2n-2$-dimensional in either case.

Then $\langle Y, \dot{Y}\rangle / \| Y, \dot{Y}\|^2 \geq \delta$ defines a cone in the sense of Definition 12.5.6, and from the discussion above we see that the cone families $C^+ = \{Y \in \mathscr{J}(\gamma) \mid \langle Y, \dot{Y}\rangle \geq 0\}$ and $C^- = \{Y \in \mathscr{J}(\gamma) \mid \langle Y, \dot{Y}\rangle \leq 0\}$ can be equivalently defined by

$$C^{\pm}_{(p,v)} = \{(x, x') \in T_pM \oplus T_pM \mid \langle x, \dot{\gamma}_v(0)\rangle = 0, \langle x', \dot{\gamma}_v(0)\rangle = 0, \pm\langle x, x'\rangle \geq 0\}.$$

Lemma 5.2.5. *The family of cones* C^+_δ *given by* $\langle Y, \dot{Y}\rangle \geq 0$ *is strictly invariant, and vectors in it grow exponentially in time.*

Proof. $\dfrac{d}{dt}\langle Y, \dot{Y}\rangle = \langle \dot{Y}, \dot{Y}\rangle + \underbrace{\langle Y, \ddot{Y}\rangle}_{=-\langle R(Y,\dot{\gamma})\dot{\gamma},Y\rangle=-\langle K(t)Y,Y\rangle\geq k^2\langle Y,Y\rangle} \geq \overbrace{\langle \dot{Y}, \dot{Y}\rangle + k^2\langle Y, Y\rangle}^{>0 \text{ unless } Y=0=\dot{Y}} \underbrace{\geq 2k\langle Y, \dot{Y}\rangle}_{0\leq\langle\dot{Y}-kY,\dot{Y}-kY\rangle=\langle\dot{Y},\dot{Y}\rangle-2k\langle Y,\dot{Y}\rangle+k^2\langle Y,Y\rangle} \geq 0$, therefore $(d\varphi^t)(C^+(x)) \subset \operatorname{int}\big(C^+_{\varphi^t x}\big)$ and[7]

$$\| Y(t), \dot{Y}(t)\|^2 \geq \frac{1}{2\epsilon}\langle Y(t), \dot{Y}(t)\rangle \geq \frac{1}{2\epsilon}e^{2kt}\langle Y(0), \dot{Y}(0)\rangle \geq \frac{\delta}{2\epsilon}e^{2kt}\| Y(0), \dot{Y}(0)\|. \qquad \square$$

One could likewise show that C^- is strictly invariant and expanding in negative time, but this follows from reversibility of the geodesic flow (Remark 1.1.29): by definition

$$g^{-t}(v) = -g^t(-v). \tag{5.2.3}$$

We thus obtain a splitting $T_vSM = S_v \oplus E^c_v \oplus U_v$ and cones satisfying the conditions of Proposition 5.1.7 to obtain Theorem 5.2.4. □

Remark 5.2.6. Jacobi fields not only determine cone fields as above but also the stable and unstable subbundles. To that end consider the orthogonal Jacobi vector field determined (uniquely) by the boundary-value problem $Y_s(0) = v$, $Y_s(s) = 0$ for any $v \perp \dot{\gamma}(0)$. Then $Y := \lim_{s\to+\infty} Y_s$ (pointwise) is a *stable Jacobi field*, that is, with $Y(t) \xrightarrow[t\to+\infty]{} 0$. Stable Jacobi fields define infinitesimal variations of pairwise forward-asymptotic geodesics, that is, stable vectors. A like construction gives unstable Jacobi fields.

Later on (Section 6.2) we likewise obtain stable and unstable manifolds (sets of positively or negatively asymptotic geodesics), whereas Proposition 2.1.10 and Proposition 2.2.1 did so by using the algebraic structure in an essential way.

It is plausible that having negative curvature everywhere is not strictly needed for Theorem 5.2.4, and in his seminal papers on ergodicity of geodesic flows of negatively curved surfaces Hopf recognized the essential features of hyperbolicity

[7] Note that k appears below, when k^2 arose as a curvature bound; the dynamical growth and contraction rates are indeed related to curvature data via square roots.

and commented on the possibility of even allowing some positive curvature.[8] We instead explore how much flatness can be allowed for surfaces by developing more carefully the mechanism that gives hyperbolicity (Theorem 5.2.8).

The technical ingredient is to "projectivize" the action on Jacobi fields. In the 2-dimensional case orthogonal Jacobi fields are represented by the scalars $y = \langle Y, n\rangle$, where n is a unit normal vector field to the geodesic, and the Jacobi equation becomes $\ddot{y} + Ky = 0$. Where $y \neq 0$ we can projectivize this to $u := \dot{y}/y$, which then satisfies the *Riccati equation*

$$\dot{u} = \frac{d}{dt}\frac{\dot{y}}{y} = \frac{\ddot{y}y - (\dot{y})^2}{y^2} = -K - u^2,$$

with $K(t)$ the Gauss curvature at $\gamma(t)$, as before.[9]

Proposition 5.2.7. *The geodesic flow g^t of a closed surface M is an Anosov flow if there is an $m > 0$ such that for any solution u of the Riccati equation along any geodesic $\gamma\colon [0,1] \to M$ with $u(0) = 0$, we have $u(1) \geq m$ (and u is defined on $[0,1]$).*

Proof. For $v \in S_xM$ let $\gamma = \gamma_v$ be the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$ and choose a smooth orthogonal basis $(\dot{\gamma}, e_1, e_2)$ at each $\gamma(t)$. It suffices to check that

$$A_k = D_{(\gamma(k),\dot{\gamma}(k))}g^1 \text{ on } \dot{\gamma}(k)^{\perp}$$

with respect to the basis (e_1, e_2) is as in Theorem 5.1.15, and since $|\det A_k| = 1$ (g^t is volume-preserving), it suffices to show that with $\epsilon := \min(1/4, K_{\max}, m) > 0$ all solutions u of the Riccati equation along a geodesic $\gamma\colon [0,1] \to M$ with $u(0) > 0$ are defined on $[0,1]$ and satisfy $\epsilon \leq u(1) \leq 1/\epsilon$. Here $-K_{\max}$ is the minimum of the Gauss curvature. (Theorem 5.1.15 gives expanding cones, and (5.2.3) then gives the contracting ones.)

The easy direction is that $u(1) \geq \epsilon$: If u_0 is the solution with $u_0(0) = 0$, then $u(1) \geq u_0(1) \geq m \geq \epsilon$ by assumption.

The other inequality follows by contradiction: Suppose $u(1) > 1/\epsilon$. Then $u(t) > 1/\epsilon$ for $t \in [0,1]$ because $u(t) > 1/\epsilon \Rightarrow \dot{u}(t) \leq \frac{1}{\epsilon} - u(t)^2 < 0$. Thus, $\dot{u} \leq \frac{1}{\epsilon} - u^2$ and hence

$$-\frac{d}{dt}\frac{1}{u} = \frac{\dot{u}}{u^2} \leq \frac{1}{\epsilon u^2} - 1 \leq -\frac{1}{2}, \quad \text{so} \quad \frac{1}{u(1)} > \frac{1}{u(1)} - \frac{1}{u(0)} \geq \frac{1}{2} > \epsilon,$$

contrary to our assumption. □

[8] He specifically illustrated this by giving explicit finitary geometric criteria to control the effects of positive curvature [**163**, p. 593f].

[9] This generalizes to higher dimension by considering a symmetric operator U defined by $\dot{Y} = UY$ and obtaining a Riccati equation for it.

We now give a curvature condition that implies the hypotheses of Proposition 5.2.7 and hence hyperbolicity. If the Gauss curvature is zero at each point of a geodesic, then the Jacobi equation shows that this geodesic is not hyperbolic. Remarkably, the existence of such a geodesic is the only obstruction to hyperbolicity of the geodesic flow:

Theorem 5.2.8 ([**189**])**.** *The geodesic flow of a closed nonpositively curved Riemannian surface is Anosov if every geodesic contains a point where the curvature is negative.*

Lemma 5.2.9. *With this assumption there are $M, T > 0$ such that every unit-speed geodesic γ satisfies $\int_0^t K(\gamma(s))\,ds \le -M$ for $t \ge T$.*

Proof. Otherwise there are geodesics γ_n on $[-n, n]$ with $\int_{-n}^n K(\gamma(t))\,dt \ge -\frac{1}{n}$. By the Arzela-Ascoli Theorem a subsequence converges uniformly on each $[-n, n]$ to a geodesic γ on $\mathbb{R}$ with $\int_{\mathbb{R}} K(\gamma(t))\,dt = 0$ (Dominated-Convergence Theorem). □

Proof of Theorem 5.2.8 (Kourganoff)**.** Take $M, T < 1$ as in Lemma 5.2.9 ($T < 1$ by possibly scaling the metric). To check the hypotheses of Proposition 5.2.7 let u be the solution of the Riccati equation along a geodesic γ for which $u(0) = 0$. Showing that u is defined on (at least) $[0, 1]$ is the main effort and yields a uniform lower bound for $u(1)$ as a byproduct.

If u is defined on $[0, 1]$, then let $t_1 = 1$; otherwise there is a $t_1 \in (0, 1]$ such that $[0, t_1)$ is the maximal interval on which u is defined.

Let $t_2 := \sup\{t \in [0, t_1] \mid u(t) \ge M\} \in [0, t_1]$ and $t \in [t_2, t_1)$. Then

$$u(t) = u(t_2) + \int_{t_2}^t \dot u(s)\,ds = u(t_2) - \int_{t_2}^t K(s)\,ds - \int_{t_2}^t u^2(s)\,ds.$$

If $t_2 = 0$, this gives

$$u(t) = 0 - \int_{t_2}^t \underbrace{K(s)}_{\le 0}\,ds - \int_{t_2}^t u^2(s)\,ds \ge \begin{cases} -M^2 \\ M - M^2 & \text{if } t = 1 \text{ (Lemma 5.2.9).} \end{cases}$$

Otherwise,

$$u(t) = \underbrace{u(t_2)}_{\ge M} - \int_{t_2}^t \underbrace{K(s)}_{\le 0}\,ds - \int_{t_2}^t \underbrace{u^2(s)}_{\le M^2}\,ds \ge M - M^2;$$

that is, $u(t) \ge -M^2$ in either case, which means that u is defined on an open interval around t of uniform size, so $t_1 = 1$, and u is defined on $[0, 1]$.

With this in hand, the preceding shows that $u(1) \ge M - M^2 > 0$, as needed. □

One reason one might in the case of surfaces be interested in the extent to which *positive* curvature is allowed is that a compact surface isometrically embedded in $\mathbb{R}^3$ has points of positive curvature (for instance, any point touching a smallest sphere that contains the embedded surface). Michael Herman asked: Are there compact surfaces in $\mathbb{R}^3$ with Anosov geodesic flow? The answer turns out to be affirmative [**108**]: If one takes a sufficiently large, sufficiently thin spherical shell with sufficiently hyperboloid-like holes drilled through from outside to inside in a dense-enough pattern, the hyperbolicity produced from the negative curvature in the holes outweighs the small positive curvature between encounters with such holes. This raises a new question, of course: can this be done with surfaces of low genus? Can it be done with genus 2?[10]

The following does not address the question as posed but provides a visually appealing counterpart *in* S^3. The spherical billiard shown on the left of Figure 5.2.1 is uniformly hyperbolic by Theorem 5.2.18 below, and Theorem 5.2.38 below then implies that a sufficiently "thin" surface as shown on the right of Figure 5.2.1 (embedded isometrically in S^3 and presented here in stereographic projection to $\mathbb{R}^3$) has Anosov geodesic flow [**188**]. Regrettably, one cannot make this construction work in $\mathbb{R}^3$; there are necessarily conjugate points.

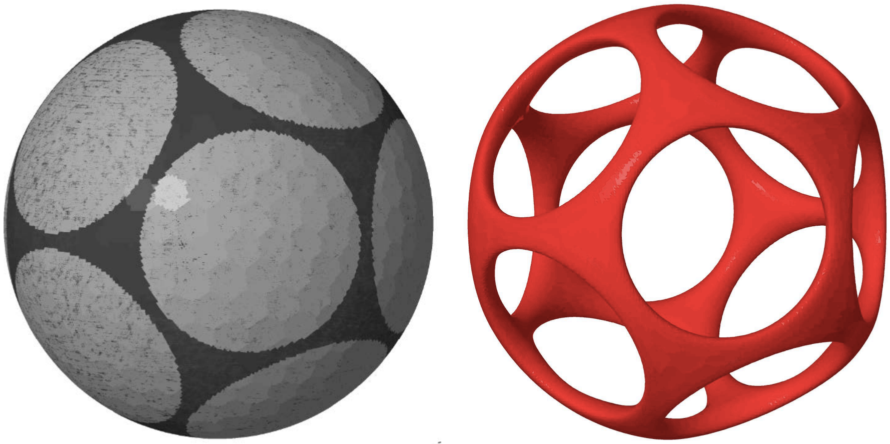


FIGURE 5.2.1. A genus-11 Anosov surface projected stereographically from S^3 [**188**]

[10] Progress towards bounding the needed genus has been made only just now [**107**].

b. Benoist–Hilbert geodesic flows. We now introduce geodesic flows that differ significantly from those we previously encountered—although Chapter 2 provides good preparation. They do not arise from Riemannian metrics and manifest the distinction in remarkable ways while at the same time being amenable to rather pedestrian explicit computations. We will introduce them in the proper context and establish that they are indeed Anosov flows. Without entering their study more deeply, we point to some of their particularly interesting features.

Definition 5.2.10 (Projective convexity, divisibility)**.** Let $\mathrm{PGL}(\mathbb{R}^m)$ be the group of projective transformations of the projective space $\mathbb{P}(\mathbb{R}^m)$,[11] that is, $\mathrm{GL}(\mathbb{R}^m)$ modulo homotheties. An open set $\Omega \subset \mathbb{P}(\mathbb{R}^m)$ is said to be *convex* if it intersects each projective line in a connected set, *projectively (or properly) convex*, if there is moreover a projective hyperplane that does not intersect the closure of Ω, and *strictly convex* if every projective line intersects the boundary $\partial\Omega$ in at most 2 points. A projectively convex Ω is said to be *divisible* if there is a discrete torsion-free[12] subgroup $\Gamma < \mathrm{PGL}(\mathbb{R}^m)$ that preserves Ω and with compact quotient $M = \Gamma\backslash\Omega$ (following Furstenberg and Benoist we say that Γ *divides* Ω.) One can prove and we will use that $\partial\Omega$ is C^1.

Example 5.2.11. The *ellipsoid* $\Omega_0 := \{[v] \in \mathbb{P}(\mathbb{R}^m) \mid q(v) > 0\}$, where q is a quadratic form on $\mathbb{R}^m$ with signature $(1, m-1)$ is strictly convex and divided by any cocompact lattice in its isometry group $\mathrm{SO}(q)$.

Definition 5.2.12 (Hilbert distance and geodesic flow)**.** The *Hilbert distance* d_Ω on a projectively convex $\Omega \subset \mathbb{P}(\mathbb{R}^m)$ is defined by $d_\Omega(x,y) := |\log((a,b;x,y))$, where

$$(a,b;x,y) := \frac{x-a}{x-b}\Big/\frac{y-a}{y-b} = \frac{(x-a)(y-b)}{(x-b)(y-a)}$$

is the *cross ratio* with $a, b \in \partial\Omega$ such that a, b, x, y lie on the line $\langle x, y\rangle$ through $x \neq y$. (This distance is invariant under all Ω-preserving projective transformations.) This implies that the shortest curve between any 2 points of Ω is a line segment. The *geodesic flow* on Ω is defined by $\tilde{g}^t(x,\xi)$ being the unit tangent vector in the direction of ξ at the point x_t at distance t from x on the line through x defined by ξ. Its projection g^t is called the geodesic flow on $M = \Gamma\backslash\Omega$.

As promised, we will prove that these geodesic flows are in scope for us:

Theorem 5.2.13. *The geodesic flow of a compact factor of a divisible strictly convex subset of* $\mathbb{P}(\mathbb{R}^m)$ *is an Anosov flow.*

[11] This can be viewed as the space of lines through $0 \in \mathbb{R}^m$, the points on the unit sphere in $\mathbb{R}^m$ or the equivalence classes of $\mathbb{R}^m \smallsetminus \{0\{$ modulo collinearity.

[12] that is, only the identity has finite order

We will prove this result later, but the point of this section is an exploration of the dynamical features of these flows, analogously to Chapter 2. Theorem 5.2.13 will then be a rather easy consequence.

We begin by making the notions from Definition 5.2.12 more explicit and amenable to computation. First we take a global affine chart, that is, we may assume that $\overline{\Omega} \subset \mathbb{R}^{m-1} \subset \mathbb{P}(\mathbb{R}^m)$, a suitable affine hyperplane, so Ω is a bounded convex subset of $\mathbb{R}^{m-1}$, and the tangent bundle is $\Omega \times \mathbb{R}^{m-1}$. Define C^1 maps

$$\left.\begin{aligned} p&\colon T\Omega \to \Omega,\\ p^{\pm}&\colon T\Omega \to \partial\Omega,\\ \sigma^{\pm}&\colon T\Omega \to (0,\infty) \end{aligned}\right\} \text{ by } \sigma^+(w)(p^+(w)-x) = \xi = \sigma^-(w)(x-p^-(w)) \text{ and } p(\underbrace{w}_{=(x,\xi)\in T\Omega\smallsetminus\{0\}}) = x.$$

This allows us to define the *Hilbert norm* $\|w\|_\Omega \coloneqq \sigma^+(w)+\sigma^-(w)$ of vectors $w = (x,\xi) \in T\Omega \smallsetminus \{0\}$. $p, p^{\pm}, \|\cdot\|_\Omega$ are independent of the affine chart, and the *unit tangent bundle* is $SM = \{w \in T\Omega \mid \|w\|_\Omega = 1\}$. One can check that the geodesic flow (unit-speed motion along lines) on $S\Omega$ is thus given by

$$\tilde{g}^t(w) \eqqcolon w_t = (x_t, \xi_t) = \big(x + \frac{e^t-1}{\sigma^+(w)e^t+\sigma^-(w)}\xi, \frac{e^t}{(\sigma^+(w)e^t+\sigma^-(w))^2}\xi\big).$$

Remark 5.2.14. This shows in particular, that this is a C^1 flow—and no more regular than that unless the boundary is more smooth. One can improve this by way of reparametrization as follows. For a smooth Γ-invariant Riemannian metric g on Ω follow Hilbert-geodesics with constant g-speed.

The following will turn out to be the stable foliation for the geodesic flow:

$$\widetilde{W}^{cs}(w) \coloneqq (p^+\restriction_{S\Omega})^{-1}(p^+(w)) \quad \text{for} \quad w \in S\Omega,$$

the collection of unit vectors pointing to the same boundary point. Isolating strong stable leaves geometrically requires a little extra work.

Claim 5.2.15.

$$\begin{aligned}\widetilde{W}^{ss}(w) \coloneqq &\big\{v_1 = (x_1,\xi_1) \in \widetilde{W}^{cs}(w) \mid \ w_1 = w \textit{ or } \langle x, x_1\rangle \cap \langle p^-(w), p^-(w_2)\rangle \subset \underbrace{\mathcal{T}_{p^+(w)}}_{\coloneqq\text{tangent space to the boundary}}\big\}\\ =&\big\{v_1 = (x_1,\xi_1) \in S\Omega \mid \ d_\Omega(\underbrace{p(\tilde{g}^t(w))}_{\eqqcolon x_t}, \underbrace{p(\tilde{g}^t(w_1))}_{\eqqcolon x_{1,t}}) \xrightarrow[t\to\infty]{} 0\big\}.\end{aligned}$$

Proof. This is a nice application of the fact that a cross ratio is naturally defined for a set of lines in the following sense: If points a, b, x, y are collinear and 4 lines are drawn from a distinct point {q} to these, then for any other line not through q with corresponding intersection points A, B, X, Y, the cross-ratios agree, that is,

$(a,b;x,y)=(A,B;X,Y)$, and conversely, their agreement implies that the 4 lines through a and A etc. are concurrent.

Note first that $p^+(w)\neq p^+(w_1)\Rightarrow d_\Omega(x_t,x_{1,t})\xrightarrow[t\to\infty]{}\infty$, so we may assume $p^+(w)=p^+(w_1)=:p_1^+$. Then

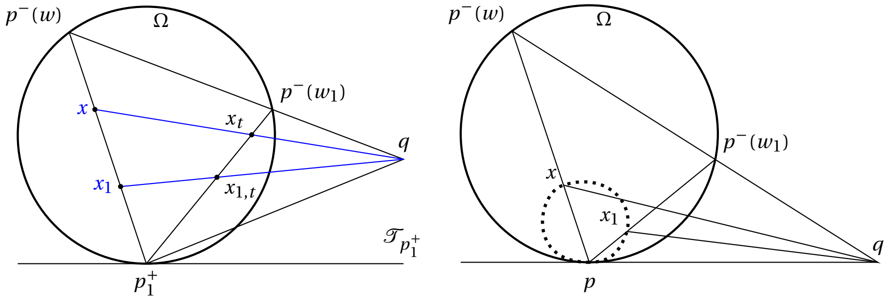


FIGURE 5.2.2. Strong stable leaves and Busemann functions

$$(p_1^+,p^-(w);x,x_t)=e^t=(p_1^+,p^-(w_1);x_1,x_{1,t})$$

implies (see Figure 5.2.2) that

$$\langle x_t,x_{1,t}\rangle\ni q:=\langle p^-(w),p^-(w_1)\rangle\cap\langle x,x_1\rangle.$$

Since the line $\langle x_t,x_{1,t}\rangle$ converges to $\langle p_1^+,q\rangle$, this implies the claim:

$$d_\Omega(x_t,x_{1,t})\xrightarrow[t\to\infty]{}0\Leftrightarrow\langle p_1^+,q\rangle\text{ is tangent to }\partial\Omega\Leftrightarrow q\in\mathcal{T}_{p_1^+}.\qquad\square$$

The (footpoint) projection in Ω of a strong stable leaf is a horocycle, and we now give alternate descriptions of this. One is as the limit of $d-\Omega$-spheres through $x\in\Omega$ as their centers tend to $p\in\partial\Omega$. Another is as a 0-level set of a *Busemann function*:

$$\mathcal{H}_{(x,\xi)}=\{x_1\in\Omega\mid b_x(x_1,p^+(x,\xi))=0\},$$

where the Busemann function b is defined on $\Omega\times\Omega\times\partial\Omega$ by

$$b_{x_1}(x_2,p):=\lim_{x\to p}\big(d_\Omega(x_1,x)-d_\Omega(x_2,x)\big)$$

or equivalently (see Figure 5.2.2) as the logarithm of the cross ratio of the 4 lines through $q:=\mathcal{T}_p\cap\langle p_1^-,p_2^-\rangle$ and p,p_1^-,x_2,x_1, respectively, where p_i^- is the other boundary point on $\langle x_i,p\rangle$.

While the *definition* of the stable subbundle E^s as the tangent bundle of the stable foliation is universal, the explicit formulas in this context give an equally explicit representation of this subbundle:

$$E^s_w = \{(y, -\sigma^+(w)y) \in T_w S\Omega \mid y \in T_x\mathscr{H}_w\}, \quad E^u_w = \{(y, \sigma^-(w)y) \in T_w S\Omega \mid y \in T_x\mathscr{H}_w\}.$$

By construction, these are Γ- and g^t-invariant, and $TS\Omega = E^s \oplus \mathbb{R}X \oplus E^u$.

Proof of Theorem 5.2.13. The flip map $v \mapsto -v$ conjugates the geodesic to its reverse, so it suffices to check that vectors in E^s contract exponentially. To that end we reduce to considerations in $T\Omega$ by observing that the existence of a compact factor implies that for any Riemannian norm $\|\cdot\|$ on $S\Omega$ there is a $C \geq 1$ such that

$$\|(y, -\sigma^+(w)y)\|/C \leq \|y\|_\Omega \leq C\|(y, -\sigma^+(w)y)\|.$$

Thus, it suffices to show that for $\lambda \in (0,1)$ there is a $T > 0$ as follows: If $(y, -\sigma^+(w)y) \in E^s_w$, hence

$$D\tilde{g}^t((y, -\sigma^+(w)y)) = (y_t, -\sigma^+(\tilde{g}^t(w))y_t),$$

then $\|y_T\|_\Omega \leq \lambda\|y\|_\Omega$.

To that end an explicit description of y_t suggested by Figure 5.2.2 helps: Writing $w = (x, \xi)$ and $w_t := \tilde{g}^t(w) = (x_t, \xi_t)$, we find that y_t is the unique vector tangent to the horosphere $\mathscr{H}_{w_t}$ such that $p^+(w)$, $x + y$ and $x_t + y_t$ lie on a line.

Since $\partial\Omega$ is strictly convex, the map $t \mapsto \|y_t\|_\Omega = \sigma^+(y_t) + \sigma^-(y_t)$ is strictly decreasing, and indeed to 0 as $t \to \infty$ since $\partial\Omega$ is C^1. Thus, writing $E^s_1 := \{v \in E^s \mid \|v\| = 1\}$, the function $F\colon E^s_1 \times \mathbb{R} \to \mathbb{R}$, $(v,t) \mapsto \|y_t\|_\Omega / \|y\|_\Omega$ is continuous, decreasing in t with $F(\cdot, 0) \equiv 1$ and $F(v,t) \xrightarrow[t\to\infty]{} 0$, so there is a unique (and continuous and Γ-invariant) $\tau\colon E^s_1 \to (0,\infty)$ such that $F(v, \tau(v)) = \lambda$. Take $T := \max\tau$. □

Remark 5.2.16. It is nontrivial to show that there are examples of this type beyond Example 5.2.11; this is a substantial part of the work of Benoist [**36**–**42**]. Another is the investigation of what can occur if one does not require strict convexity. Benoist showed that there are nontrivial instances of this and studied their features. From the dynamical point of view, including their smooth ergodic theory, the definitive study at this time is [**60**].

c. Billiards. Billiard flows provided our first example of a flow that is naturally represented as a flow under a function (Example 1.2.9) because in the cases that then came to mind, the boundary of the billiard table is a global section (Figure 0.1.2). To discuss billiards with hyperbolic behavior we begin with a formal definition of a billiard.

Definition 5.2.17. A smooth *billiard* table B in $R = \mathbb{T}^2$ (which is a good model for motion in a periodic crystal) or $R = \mathbb{R}^2$ is the closure of an open set B° of R whose boundary is a finite disjoint union of smoothly embedded circles called the walls of B. A billiard is said to be *dispersing* if every wall γ has negative curvature (that is, if T is the tangent vector of γ and N the normal vector pointing into the table, then $\langle\frac{\partial T}{\partial s}, N\rangle < 0$, where s is the arc-length parameter; Figure 0.1.2 instead has a boundary with positive curvature). The phase space of the billiard is the unit tangent bundle SB° with the billiard flow φ^t defined as in Example 1.2.9 (straight-line motion with optical reflection) except for

- t such that φ^t is at the boundary (while one could adjust the definition in such a way as to make the flow continuous at such points, it cannot be differentiable),
- $t \geq T$ if φ^T is tangent to the boundary (this is a removable discontinuity but necessarily a failure of differentiability).

We define the *regular set* Ω to be those points in SB° for which the second possibility (grazing collisions) occurs for no positive or negative time; this is a residual conull flow-invariant set, and φ^t is smooth on it. B has *finite horizon* if the boundary is a global section, that is, every orbit meets the boundary.

The *Sinai billiard* is $\mathbb{T}^2$ minus a disk; it is dispersing with infinite horizon. Removing instead disks of diameter $3/5$ around $(0,0)$ and $(1/2, 1/2)$ gives finite horizon.

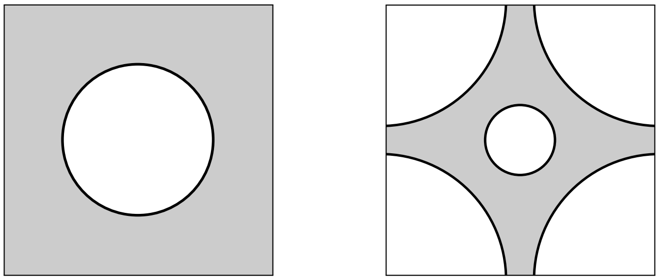


FIGURE 5.2.3. Dispersing billiards on $\mathbb{T}^2$ with infinite and finite horizon (the particle moves in the shaded region)

Similarly to Theorem 5.2.8, one can show that finite-horizon dispersing billiards are uniformly hyperbolic away from the collision singularities:

Theorem 5.2.18 ([**189**])**.** *The regular set of a finite-horizon dispersing billiard is uniformly hyperbolic, that is, it has all the required properties from Definition 5.1.1 except for compactness.*[13]

[13] And there are no fixed points.

This result is obtained by introducing Jacobi fields and a Riccati equation, but in this case hyperbolicity comes from the collisions.

Let $V\colon (a,b)\times(c,d)\to M$ be a variation of a billiard orbit $\gamma = V(\cdot,0)$, that is, $V(\cdot,s)$ is a unit-speed billiard orbit for each s with collision times $t_i(s)$. Then $Y := \frac{\partial V}{\partial s}$ is called a Jacobi field where defined. $\ddot{Y}=0$ away from collisions, and we now investigate the jump discontinuities at collisions. The reflection of a billiard orbit rotates the tangent vector by 2θ, where θ is the angle of incidence; we write $R_{2\theta}$ for rotation by 2θ and now show the counterpart for Jacobi fields.

Lemma 5.2.19. *If Y^- and Y^+ are the values of the Jacobi field before and after collision with incidence angle θ, then $Y^+ = -R_{2\theta}Y^-$.*

Corollary 5.2.20. *Orthogonal Jacobi fields remain orthogonal after a collision.*

Proof. Denote by $\tau(s)$ the time of collision of $s\mapsto V(\cdot,s)$ with a point $\Gamma(r(s))$ of a boundary piece Γ parametrized by arc length. Denote by $\omega^\pm(s)$ the angle between the horizontal axis and $\frac{\partial}{\partial t}\big|_{t=t^\pm} V(t,s)$ (before and after collision). Then $\theta = \frac{1}{2}(\omega^+ - \omega^-)$, and $\psi := \frac{1}{2}(\omega^+ + \omega^-)$ is the angle between the horizontal axis and the tangent to Γ at the collision point. For small s and $t^\pm$ near τ, we then have

$$V(t^\pm,s) = \Gamma(r(s)) + R_{\omega^\pm}(t^\pm - \tau(s)).$$

Differentiating with respect to s at $s=0$ then gives

$$\begin{aligned} Y(t^\pm) = \frac{\partial r}{\partial s} R_{\psi(s)}\begin{pmatrix}1\\0\end{pmatrix} - \frac{\partial \tau}{\partial s} R_{\omega^\pm(s)}\begin{pmatrix}1\\0\end{pmatrix} + R_{\pi/2}R_{\omega^\pm}(t^\pm - \tau(s))\frac{\partial \omega^\pm(s)}{\partial s}& \\ \xrightarrow[t^\pm\to\tau]{} \frac{\partial r}{\partial s} R_{\psi}\begin{pmatrix}1\\0\end{pmatrix} - \frac{\partial \tau}{\partial s} R_{\omega^\pm}\begin{pmatrix}1\\0\end{pmatrix},& \end{aligned}$$

so $Y^+ + R_{2\theta}Y^- = \frac{\partial r}{\partial s}\underbrace{R_\psi(\mathrm{Id}+R_{2\theta})}_{=2\cos\theta R_{\omega^+}}\begin{pmatrix}1\\0\end{pmatrix} - 2\underbrace{\frac{\partial \tau}{\partial s}}_{=\frac{\partial\tau}{\partial r}\frac{\partial r}{\partial s}} R_{\omega^+}\begin{pmatrix}1\\0\end{pmatrix} = 0.$ □

As with geodesic flows, orthogonal Jacobi fields are described by a scalar y using a unit vector field orthogonal to the orbit, and writing $u = \dot{y}/y$, we obtain the Riccati equation $\dot{u} = -u^2$ between collisions, and:

Lemma 5.2.21. *At a collision, $y^+ = -y^-$, $\quad \dot{y}^+ = -\dot{y}^- + \dfrac{2\kappa y^-}{\sin\theta}$, and $u^+ = u^- - \dfrac{2\kappa}{\sin\theta}$, where κ is the curvature of Γ at the collision point (negative for dispersing billiards).*

Proof. $y^+ = -y^-$ follows from the previous lemma. Next,

$$\dot{y}^+ + \dot{y}^- = \frac{\partial(\omega^+ + \omega^-)}{\partial s} = 2\frac{\partial\psi}{\partial s} = 2\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial s} = 2\kappa\frac{y^-}{\sin\theta},$$

and $u^+ - u^- = \dfrac{\dot y^+}{y^+} - \dfrac{\dot y^-}{y^-} = -\dfrac{\dot y^+ + \dot y^-}{y^-} = -\dfrac{2\kappa}{\sin\theta}$. □

Corollary 5.2.22. *In a dispersing billiard,* $u(0) \ge 0 \Rightarrow u(t) \ge 0$ *for all* $t \ge 0$.

Proof. If there is no collision between time 0 and t_1, then $\dot u = -u^2$, so either $u(0 = 0$ and hence $u \equiv 0$ on $[0, t_1]$ or $u(0) > 0$ and hence

$$\frac{d}{dt}\frac{1}{u} = -\frac{\dot u}{u^2} = 1,$$

so $\frac{1}{u}$ is increasing, hence positive on $[0, t_1]$. And the previous lemma shows that collisions increase u. □

Analogously to Lemma 5.2.9 we here have

Lemma 5.2.23. *For a finite-horizon billiard there is a* $T >$ *such that every unit-speed billiard orbit has a collision in* $[0, T]$,

Proof. Otherwise, there are billiard orbits γ_n without collision on $[-n, n]$, and by compactness (and suitable choice of parametrizations) a subsequence of $(x_n, v_n) := (\gamma_n(0), \dot\gamma_n(0))$ converges to $(x, v) \in SB^\circ$, which then defines a limit geodesic γ, necessarily periodic, and with period τ, say (because it is contained in the billiard table B, hence not dense in $\mathbb{T}^2$). If γ has no collision, we are done. Otherwise, there is a ball $B_0 \subset \mathbb{T}^2 \smallsetminus B$ tangent to γ and also a ball $B_1 \subset \mathbb{T}^2 \smallsetminus B$ tangent to γ *on the other side* because if not, then a geodesic with initial vector (x', v) close to $(x, v) := (\gamma(0), \dot\gamma(0))$ for x' close enough to x on that other side, is collision-free. If $v_n = v$ for any $n \ge \tau$, then γ_n, being τ-periodic, is collision-free, so $v_n \ne v$ for all $n \in \mathbb{N}$. This, however implies that for large enough n, γ_n intersects B_0 or B_1 on $[-2\tau, 2\tau]$, contrary to their construction. □

Proof of Theorem 5.2.18. For $(x, v) \in SB^\circ$ write $W_{(x,v)} := v^\perp \subset T_{(x,v)}B^\circ$. For an orbit γ and $(w, w') \in W_{(\gamma(0),\dot\gamma(0))}$ there is an orthogonal Jacobi field Y along γ with $(Y, \dot Y) = (w, w')$. Denoting by t_k the collision times and $\tilde t_k := \frac{1}{2}(t_k + t_{k+1})$, the linear maps

$$A_k := D_{(\gamma(\tilde t_k),\dot\gamma(\tilde t_k))}\varphi^{\tilde t_{k+1} - \tilde t_k} \colon W_{(\gamma(\tilde t_k),\dot\gamma(\tilde t_k))} \to W_{(\gamma(\tilde t_{k+1}),\dot\gamma(\tilde t_{k+1}))}$$

have determinant ±1 since the billiard flow φ^t preserves volume, so uniform hyperbolicity follows from Theorem 5.1.15 once we show that $u := \dot y / y$ satisfies

$$T_1 \le \frac{1}{u(\tilde t_{k+1})} \le T_2 - \frac{1}{\kappa_{\max}},$$

where $0 < T_1 \le \frac{1}{2}(t_{k+1} - t_k) \le T_2 < \infty$ for all k, and $\kappa_{\max} < 0$ is the minimum curvature of the boundary. To see this note that $\dot{u} = -u^2$ on $(t_k, \tilde{t}_k)$, so $\frac{d}{dt}\frac{1}{u} \equiv 1$, and

$$\frac{1}{u(\tilde{t}_{k+1})} = \overbrace{\frac{1}{u(t^+_{k+1})}}^{\le -1/\kappa_{\max}} + \overbrace{\tilde{t}_{k+1} - t_{k+1}}^{\in[T_1,T_2]}. \qquad \square$$

d. Gases of particles. So far billiards have appeared solely as "toy models," and while this is sufficient motivation for studying them, their role in dynamical systems is much larger because, as we now show following [**92**], the natural microscopic model of a gas of hard particles is itself a billiard problem. We first illustrate this in the simplest nontrivial case.

Example 5.2.24 (2-disks billiard)**.** Consider 2 disks of unit mass and with radius r moving freely on $\mathbb{T}^2$ and colliding elastically with each other. With respect to a frame with origin at their joint center of mass, their positions are opposites, so one of them describes a configuration completely. The possible configurations are those in which the disks do not overlap, that is, the centers are at least $2r$ apart. In our choice of coordinates, this system is modeled by free motion of a point mass in $\mathbb{T}^2$ with a disk of radius $2r$ removed. This is the configuration space of a dispersing billiard, though it remains to check that the direction changes at collisions correspond to reflection in this model.

To address the latter point in this example, let us formally define billiards in arbitrary dimension.

Definition 5.2.25. A billiard table is a compact Riemannian manifold B with boundary. A billiard orbit is a unit-speed geodesic with reflection in $T_p\partial B \subset T_pB$ at points $p \in \partial B$. Here in an inner-product space E we define reflection in a codimension-one subspace V by $x \mapsto x - 2\langle x, u\rangle u$ for a unit vector $u \perp V$.

Our object in what follows is to verify that particles moving freely and with elastic collisions are indeed billiard systems as in Definition 5.2.25. To that end it is helpful to clealy describe what we mean by mechanical systems with collisions.

First, the configuration space for a mechanical system with collisions consists of a subset $B = \bigcap_i D_i^{-1}([0,\infty))$ of an n-manifold M, where the $D_i\colon M \to \mathbb{R}$ are piecewise C^1-functions with nonzero differential.

Collisions occur on $\partial B \subset \bigcap_i D_i^{-1}(0)$, which has a well-defined tangent space away from the intersections of 2 such level sets and away from those points where a D_i is not C^1, the singular points. A collision at such a point is said to be singular, and orbits are not defined beyond such times. other collisions are regular, and the regular set consists of those orbits that never have a singular collision.

In the case of a multi-particle systshoudle D_i are the signed pairwise distances between particles. Singularities correspond to multiple (simultaneous) particle collisions or to collisions between 2 particles that involve more than one point of contact.

Definition 5.2.26. Free motion in a mechanical system is geodesic motion with respect to the Riemannian metric whose norm is the total kinetic energy of the system, called the *kinetic-energy metric*.

Thus, conservation of energy corresponds to constant speed, which is essential for optical reflection.

Collisions at regular points $p \in \partial B$ are described in terms of the D_i with $D_i(p) = 0$ by a map $R_p\colon V_- \to V_+$, where $V_\pm := \{v \in T_pM \mid \pm dD(v) \le 0\}$; then $R_p(v) = V \Rightarrow v \in V_- \cap V_+ = \ker dD$, and D_i is decreasing before the collision and increasing thereafter. We extend R_p to T_pM by imposing $R_p(-v) = -R(v)$ and now describe further properties of R that determine it explicitly,

Definition 5.2.27 (Elastic collision)**.** A linear map $R\colon E \to E$ of an n-dimensional inner product space E is an *elastic collision* if

(1) R preserves the norm,
(2) there is a vector N such that $R(V_\pm) = V_\mp := \{v \in E \mid \mp\langle v, n\rangle \ge 0\}$,
(3) there is a full-rank linear map $L\colon E \to \mathbb{R}^{n-1}$ with $LR = L$ (called a *sufficient set of linear invariants*).

As we indicated previously, the first property reflects conservation of (kinetic) energy, and the second one says that $N^\perp$ is not crossed. We will obtain the linear invariants from conservation of momentum. For now we note that elastic collisions are reflections in $N^\perp$.

Proposition 5.2.28. *An elastic collision is a reflection in $N^\perp$ (Definition 5.2.27).*

Proof. Fix a unit vector $u \in \ker L$. For $v \in E$ we have $LRv = Lv$, hence $Rv - v = \tau(v)u \in \ker L$ defines $\tau \in E^*$, so

$$\langle v, v\rangle = \langle Rv, Rv\rangle = \langle v + \tau(v)u, v + \tau(v)u\rangle = \langle v, v\rangle + 2\tau(v)\langle u, v\rangle + (\tau(v))^2, \tag{5.2.4}$$

that is, $\tau(v) = 0$ or $\tau(v) = -2\langle u, v\rangle$. In light of Definition 5.2.25 we show that the latter possibility holds for all $v \in E$ and that $\ker L = \mathbb{R}N$. The reason is that

$$\tau(v) = 0 \Rightarrow Rv = v \Rightarrow v \in V_+ \cap V_- = N^\perp \tag{5.2.5}$$

by Definition 5.2.27(2). Thus $v \perp \ker L \Rightarrow \langle u, v\rangle = 0 \xRightarrow{(5.2.4)} \tau(v) = 0 \Rightarrow v \perp N$, so $(\ker L)^\perp \subset N^\perp$ hence $(\ker L)^\perp = N^\perp$ since both are 1-dimensional. Thus, $\ker L = \mathbb{R}N$.

(5.2.5) also implies $\tau(v) = 0 \Rightarrow v \in N^\perp = (\ker L)^\perp \Rightarrow \langle u, v\rangle = 0 = -\frac{1}{2}\tau(v) \Rightarrow \tau(v) = -2\langle u, v\rangle$. □

Corollary 5.2.29. *A mechanical system with collisions whose regular collisions are elastic can be modeled as a billiard system.*

PROOF. By definition of "mechanical" the motion is along geodesics, and by Proposition 5.2.28 the boundary collisions are reflections in $(\ker dD_i(p))^{\perp} = (T_p(\partial B))^{\perp}$, where p is the collision point and D_i is such that $D_i(p) = 0$. □

This has further implications. R preserves velocity components tangent to ∂B, that is, if π is the projection to $\ker dD_p$, then $\pi \circ R_p = \pi$. And $F\colon T_pM \to \mathbb{R}$ is collision-invariant if and only if $FRv \equiv Rv$ if and only if $\ker F \ni Rv - v = -2\langle u, v\rangle u$ for all v if and only if $(\ker dD_p)^{\perp} \subset \ker F$.

Example 5.2.30 (2 point masses on the interval)**.** We illustrate this formalism in the particularly simple example of 2 point masses m_1, m_2 at $x_1, x_2 \in [0,1]$, respectively, that collide elastically with each other and with the end-points. We describe the configuration space as $M = \mathbb{R}^2$ with $D_1 = 1 - x_1$ (that is, $x_1 \le 1$), $D_2 = x_2$ (that is, $x_2 \ge 0$), and $D_3 = x_1 - x_2$ (that is, x_1 is to the right of x_2). Thus, $B = \bigcap_{i=1}^{3} D_i^{-1}([0,\infty))$ with inner product $\langle (v_1, v_2), (w_1, w_2)\rangle = m_1 v_1 w_1 + m_2 v_2 w_2$.

A collision between the 2 masses is described by $D_3 = 0$, and the linear-invariants map is the linear momentum $L(v) = m_1v_1 + m_2v_2$, so $\ker L = \mathbb{R}(m_2, -m_1)$, and with $u := \frac{1}{\sqrt{m_1^2+m_2^2}}(m_2, -m_1)$ we get

$$R(v)=v-2\langle u,v\rangle u=(v_1,v_2)-2\underbrace{\frac{m_2v_1-m_1v_2}{m_1^2+m_2^2}}_{=:C(v)}(m_2,-m_1)=(v_1-C(v)m_2,v_2+C(v)m_1).$$

We note that the more common approach to this example is to use the standard inner product rather than the one giving kinetic energy and to accordingly change the configuration space to a triangle with sides $\sqrt{m_1}$ and $\sqrt{m_2}$.

The main result from these endeavors is:

Theorem 5.2.31 (Cowan [**92**])**.** *The gas of hard particles is a point billiard.*

Remark 5.2.32. The particles are not assumed to be spherical, so angular momentum and its transfer between particles is in scope.

PROOF. The gas of hard particles is modeled by N piecewise smooth rigid bodies B_i moving freely in $\mathbb{R}^3$ and having nonsingular intertial tensor. The "position" of B_i is a point in $M_i := F_i \times G_i$, where $F_i \sim \mathbb{R}^3$ parametrizes possible locations of the center of mass and $G_i \sim \mathrm{SO}(3)$ decribes the orientation of B_i. Thus, the configuration space is $B := \{p \in M := M_1 \times \cdots \times M_N \mid D_{ij}(p) \ge 0 \text{ for } 1 \le i, j \le N\}$ using the signed distances $D_{ij}\colon M \to \mathbb{R}$ between ∂B_i and ∂B_j chosen to be positive away from overlaps. If, furthermore, each B_i is constrained to remain in $A \subset \mathbb{R}^3$ with piecewise smooth boundary $\partial A = \bigcup_{k=1}^m A_k$ we instead have

$$B = \{p \in M \mid D_{ij}(p) \geq 0, D'_{ik} \geq 0 \text{ for } 1 \leq i, j \leq N,\ 1 \leq k \leq m\},$$

where the D'_{ik} are the signed distances between ∂B_i and A_k. Regular boundary points are those where only one inequality among these fails to be strict; other (singular) boundary points represent double collisions or configurations where more than 1 point of a particle is in contact with another particle or with ∂A.

The total kinetic energy combines translational and rotational energy, and for the latter, the intertial tensors I_i of the B_i are the counterparts of mass:

$$\langle(v,\omega),(v',\omega')\rangle = \sum_{i=1}^{M} \langle m_i v_i v'_i + I_i \omega_i \omega'_i.$$

We note that this defines free motion in a noneuclidean space, so the motion is not along lines.

We finally show that the collisions between these particles are elastic. For 2-particle collisions consider B_1 and B_2 to fix ideas. We need to find the linear-invariants map $L\colon T_pM \to \mathbb{R}^{6N-1}$. $6(N-2)$ obvious linear invariants are given by the velocity components of the B_i for $i > 2$. The needed 11 additional linear invariants are

- 3 components L_1, L_2, L_3 of total linear momentum,
- 3 components each $(L_4, \dots, L_9)$ of angular momentum of B_1 and B_2 with respect to the collision point because relative to that point the torque from the collision is zero,
- 2 velocity components L_{10}, L_{11} of B_1 projected to the collision plane.

To check that the L built from these has maximal rank, we adduce the velocity L_{12} of B_1 normal to the collision plane, and show that the resulting extension L' has trivial kernel: the trivial invariants tell us that the B_i for $i > 2$ have zero velocities. If $L_{10} = L_{11} = L_{12} = 0$, then the translational velocity of B_1 is zero, and $L_1 = L_2 = L_3 = 0$ implies the same for B_2, so the angular momenta come from angular velocities, which are therefore 0.

To conclude we note that collisions with ∂A play out the same way because the 5 nontrivial invariants are $L_4, L_5, L_6, L_{10}, L_{11}$ from the previous arguments. Thus, Corollary 5.2.29 gives Theorem 5.2.31. □

We conclude with brief remarks on what are called no-slip billiards. This particle-collision model also brings in exchanges of angular momentum, even between spherical particles [**67**]. While its governing rules are not quite as principled as the one in the previous subsection, they have a sound physical justification and they have been studied significantly more, with some surprising results.[14] The central definition is a little less abstract than Definition 5.2.27:

Definition 5.2.33. Let M be the configuration manifold of 2 rigid bodies with smooth boundaries in $\mathbb{R}^n$ endowed with the kinetic-energy metric. The collision map $C\colon T_qM \to T_qM$ is said to be *strict* if

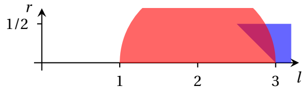
1. Kinetic energy is preserved,
2. linear and angular momentum are conserved,
3. C is an involution (time-reversibility),
4. collision forces act only at the point of impact.

In $\mathbb{R}^2$, these requirements imply that collisions are either elastic or of the no-slip type [**94**, Theorem 1.1];[15] this is a central motivation for the definition of no-slip billiards, as is the fact that this system preserves the usual Liouville measure [**94**, Theorem 1.2]. These systems have been studied to great effect by Feres and collaborators, and we recomnend the richly illustrated introduction [**94**].

e. Linkages. We illustrate our next class of systems with a particularly salient example. Linkages consist of rods connected by joints at each of which there may or may not be a mass. Instead of formalizing that definition, we present the instance of interest.

Definition 5.2.34 (Kourganoff linkage)**.** The *Kourganoff linkage* consists of points $(a,0),(b,0),(0,c),(d,e),(-2,f),(2,g)\in\mathbb{R}^2$ and connected by massless rods of lengths 1, l and r (Figure 5.2.4) subject to

$$(l-2)^2+r^2<1, \text{ and } 3-l<r<1/2 \tag{5.2.6}$$



(for instance, $l = 11/4$, $r = 1/3$) with mass ϵ^2 at $(0,c)$, no mass at (d,e), and unit masses at the other joints. (Thus, the massive joints are constrained to motion along lines.) Its configuration space $\mathscr{C}$ is the set of $(a,b,c,d,e,f,g)\in\mathbb{R}^7$ with

$$\begin{aligned}(a+2)^2+f^2 &= 1 \ = (b-2)^2+g^2,\\ (a-d)^2+e^2 &= l^2 = (b-d)^2+e^2,\\ (c-e)^2+d^2 &= r^2\end{aligned}$$

[14] For instance, the motion of a "sticky" disk bouncing between parallel lines is bounded, and the "stadium" billiard is not ergodic.

[15] In $\mathbb{R}^n$, one obtains the disjoint union of orthogonal Grassmannian manifolds $\mathrm{Gr}(k,n-1)$, $k=0,\dots,n-1$, of all k-dimensional planes in $\mathbb{R}^{n-1}$.

and endowed with the kinetic-energy metric (Definition 5.2.26), whose geodesic flow represents free motion of the linkage.

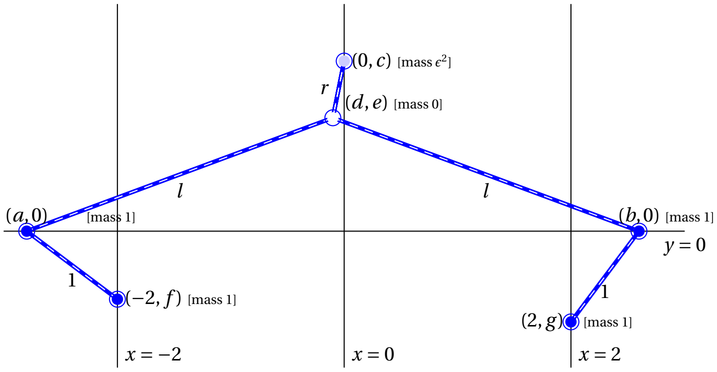


FIGURE 5.2.4. The Kourganoff linkage

Remark 5.2.35. A look at Figure 5.2.4 shows that the configuration space is described by the 2 circles that describe the orientation of the unit-length rods plus a real parameter for the shortests rod. That is to say, this linkage has a 2-dimensional configuration space: a 2-torus parametrizes the orientation of the 2 lower pairs of vertices (the ends of the unit-length rods), and once these are fixed, (d,e) is fixed modulo the sign of e (completely fixed if $e=0$), and once that choice is made, there are 2 possibilities for c. This naturally immerses the configuration space in $\mathbb{T}^2\times\mathbb{R}$ and defines a natural projection to $\mathbb{T}^2$ with at most 4 preimages per point (Figure 5.2.6). More formally, since $(a+2,f)$ and $(b-2,g)$ lie on the unit circle (Figure 5.2.4), the configuration space $\mathscr{C}$ in Definition 5.2.34 is contained in $\mathbb{T}^2\times\mathbb{R}^3$ parametrized by

$$(\theta,\phi,c,d,e)\mapsto(a,b,c,d,e,f,g)=(-\cos\theta-2,\cos\phi+2,c,d,e,\sin\theta,\sin\phi).$$

One can imagine the physical construction of such a linkage using rotational joints and with 5 vertices attached to *prismatic joints*, frictionless sleeves that slide along the respective constraint lines (Figure 5.2.7). (Those sleeve joints are a mere convenience, and linkage-purists can replace each by a massless *Peaucellier* or *Hart linkage* (Figure 5.2.5), which produces straight-line motion using only rotational joints, or they can instead be approximated by arcs of vast circles traced by the ends of additional very long rods.)

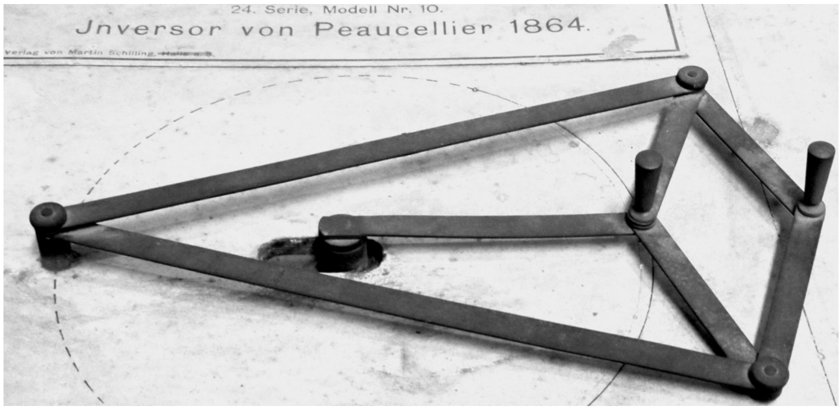

24. Serie, Modell Nr. 10.
Jnversor von Peaucellier 1864.
Verlag von Martin Schilling, Halle a.S.

FIGURE 5.2.5. The Peaucellier linkage
at the University of Tokyo http://www.ms.u-tokyo.ac.jp/models/models/invertors.pdf

Theorem 5.2.36. *For sufficiently small ϵ the free motion of the Kourganoff linkage is an Anosov flow.*

Remark 5.2.37. This, finally, is a realistic physical system whose dynamics is Anosov. Specifically, since the Anosov property is persistent, a Kourganoff linkage with rods of sufficiently small (rather than zero) mass is Anosov, and if constructed with sufficiently small friction will exhibit corresponding dynamics. It should be noted that ϵ itself arises from a like use of stability and is therefore not explicit. Nonetheless, unlike any previously known Anosov linkages, the geometry of the Kourganoff linkage is completely explicit [**164**, **187**].

We note as well that the point of this is not merely the existence of an Anosov linkage—a universality theorem asserts that any compact Riemannian manifold is the configuration space of a linkage, and this includes negatively curved ones. However, the linkages obtained from the application of that theorem are astronomically more complicated than the one here. This is a "realistic" linkage.

The proof strategy is to establish that for small enough ϵ the configuration space with the kinetic-energy metric is so close to a hyperbolic billiard that the geodesic flow is necessarily Anosov. This involves a result to the effect that "compressing" a surface in $\mathbb{T}^2 \times \mathbb{R}$ along the z-direction asymptotically gives a billiard in $\mathbb{T}^2$ in the sense that the geodesic flow uniformly converges to the billiard dynamics under this procedure. If the limiting billiard is hyperbolic (such as by Theorem 5.2.18), then stability of hyperbolicity implies that the geodesic flow of the surface is Anosov once it is sufficiently compressed towards $\mathbb{T}^2$.

This "flattening idea" goes some time back. In the 1920s Birkhoff noted that if one of the principal axes of an ellipsoid tends to 0, then the geodesic flow of this ellipsoid appears to tend to the billiard flow of the limiting ellipse.[16] Arnold suggested a reverse idea in the 1960s in hopes that it would establish hyperbolicity of dispersing billiards: that a dispersing billiard in $\mathbb{T}^2$ can be approximated by the geodesic flow of a surface of negative curvature made by gluing together two copies of the billiard (ergodicity of that billiard was later proved by Sinai using a different approach).[17] This makes these ideas explicit:

Theorem 5.2.38 ([**187**, Theorem 5])**.** *Let $B \subset \mathbb{T}^2$ be a finite-horizon dispersing billiard with smooth scatterers and $\Sigma \subset E := \mathbb{T}^2 \times \mathbb{R}$ an immersed surface such that $B = \pi(\Sigma)$, where π is the natural projection to $\mathbb{T}^2$. Then the Euclidean metric on E induces a Riemannian metric h_ϵ on $\Sigma_\epsilon := f_\epsilon(\Sigma)$, where*

$$f_\epsilon \colon E \to E, \quad (x, y, z) \mapsto (x, y, \epsilon z),$$

which in turn induces the Riemannian metric $g_\epsilon := f_\epsilon^*(h_\epsilon)$ on Σ. Suppose

(1) the surface $\pi^{-1}(\operatorname{Int} B) \cap \Sigma$ is transverse to the fibers of π (no vertical tangent planes) and

(2) the curvature of $\Sigma \cap V$ is nonzero at $q \in \pi^{-1}(\partial B) \cap \Sigma$ (nondegenerate boundary projection),[18]

where V is a neighborhood of q in the vertical affine plane through q that is perpendicular to $T_q\Sigma$. Then for small-enough ϵ, the geodesic flow of (Σ, g_ϵ) is Anosov.

[16] "In order to see how the theorem of Poincaré and its generalization can be applied, we will consider first a special but highly typical system of this sort, namely that afforded by the motion of a billiard ball upon a convex billiard table. This system is very illuminating for the following reason: Any Lagrangian system with two degrees of freedom is isomorphic with the motion of a particle on a smooth surface rotating uniformly about a fixed axis and carrying a conservative field of force with it. In particular if the surface is not rotating and if the field of force is lacking, the paths of the particle will be geodesics. If the surface is now flattened to the form of a plane convex curve C, the 'billiard ball problem' results. But in this problem the formal side, usually so formidable in dynamics, almost completely disappears, and only the interesting qualitative questions need to be considered. If C is an ellipse an integrable problem results, namely the limiting ease of an ellipsoid treated by Jacobi." [**47**, p. 169f]

[17] "In precisely the same way a torus billiard table can be be regarded as a two-sided torus with a hole on which the point moves along a geodesic. But if the two-sided ellipse is an oblate ellipsoid, the two-sided torus with a hole will be an oblate "Kringel" [this is the northern German term for "pretzel"] (of genus 2). Thus, motion on our torus billiard table is a limiting case of motion along a geodesic on the knot-shaped surface. . . Thus, a two-sided torus billiard table can be regarded as an oblate surface with negative curvature everywhere: on flattening, all the curvature is accumulated along the circumference." [**?**, Chapter VI, §4].

[18] This ensures that any geodesic in that preimage is unstable, that is, has sensitive dependence on initial conditions.

Proof of Theorem 5.2.36. Figure 5.2.6 makes it plausible that the hypotheses of Theorem 5.2.38 hold. The proof consists of verifying this explicitly.

We first check that the configuration space $\mathscr{C}$ is a smooth submanifold of $\mathbb{T}^2\times\mathbb{R}$ as follows.

- Near $p\in\mathscr{C}$ with $c\neq e\neq 0$, $\mathscr{C}$ is the graph over $\mathbb{T}^2$ (that is, over θ,ϕ) of

$$\begin{aligned} d &= \frac{-\cos\theta+\cos\phi}{2}\\ e &= \pm\sqrt{l^2-\Big(\frac{\cos\theta+\cos\phi}{2}+2\Big)^2}\\ c &= e\pm\sqrt{r^2-\Big(\frac{\cos\theta-\cos\phi}{2}\Big)^2}, \end{aligned}\tag{5.2.7}$$

  with "±" depending on p.
- Near $p\in\mathscr{C}$ where $\phi\neq 0 \bmod \pi$ and $(-\cos\theta-2,0)$, (d,e) and $(0,c)$ are not aligned, $\mathscr{C}$ is a graph over θ and c: d and e are simple roots of a second-order polynomial, hence depend smoothly on θ and c, and $\phi=\pm\cos^{-1}(2d+\cos\theta)$.
- Likewise, near $p\in\mathscr{C}$ where $\theta\neq 0 \bmod \pi$ and $(\cos\theta+2,0)$, (d,e) and $(0,c)$ are not aligned, $\mathscr{C}$ is a graph over ϕ and c.

For each $p\in\mathscr{C}$ at least one of these scenarios applies: if the latter 2 scenarios do not apply, suppose $\theta=0 \bmod \pi$ and $\phi=0 \bmod \pi$, so $\theta=\phi \bmod 2\pi$ since $r<1/2$, hence $\theta=\phi=\pi \bmod 2\pi$ since $l<3$, so $c\neq e\neq 0$ contrary to our assumption. Thus (by symmetry without loss of generality) instead $(-\cos\theta-2,0)$, (d,e) and $(0,c)$ are

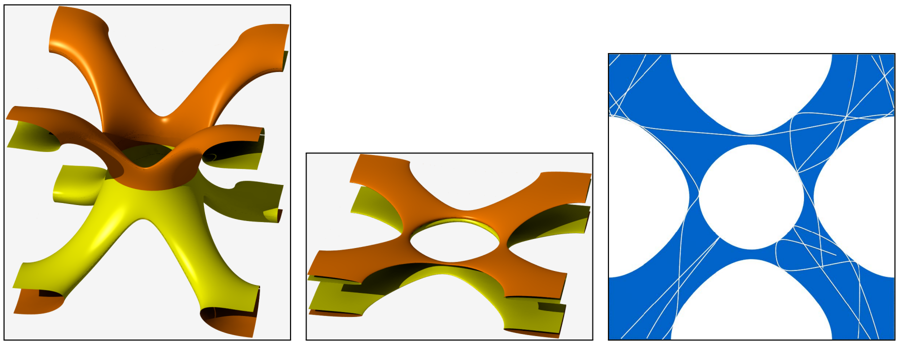


Figure 5.2.6. The configuration space with ϵ large, small, and zero

aligned, and failure of the first scenario implies $e \in \{0, c\}$, so $(-\cos\theta - 2, 0)$, (d, e) and $(0, c)$ are all on the x-axis, contrary to $l + r > 3$.

The kinetic-energy metric on $\mathscr{C}$ is given by

$$g_\epsilon = da^2 + df^2 + db^2 + dg^2 + \epsilon^2 dc^2 = d\theta^2 + d\phi^2 + \epsilon^2 dc^2,$$

and it is nondegenerate on $\mathscr{C}$ because of the local embedding as a graph. As mentioned before, its geodesic flow is the free motion of the Kourganoff linkage.

The nature or the embedding further implies that the projection

$$p\colon \mathbb{T}^2 \times \mathbb{R}^3 \to \mathbb{T}^2 \times \mathbb{R}, \quad (\theta, \phi, c, d, e) \mapsto (\theta, \phi, c)$$

restricts to an isometric immersion of $\mathscr{C}$ to a surface Σ in $\mathbb{T}^2 \times \mathbb{R}$ with the metric $g_\epsilon = d\theta^2 + d\phi^2 + \epsilon^2 dc^2$. The projection $\pi\colon \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2$ maps Σ to

$$B = \pi(\mathscr{C}) = \{(\theta, \phi) \in \mathbb{T}^2 \;\big|\; |\cos\theta - \cos\phi| \le 2r,\ \cos\theta + \cos\phi \le 2l - 4\} \tag{5.2.8}$$

with boundary

$$\big\{\cos\theta - \cos\phi = 2r\big\} \cup \big\{\{\cos\phi - \cos\theta = 2r\big\} \cup \big\{\cos\theta + \cos\phi = 2l - 4\big\}.$$

We later show that this is a finite-horizon dispersing billiard but first establish Theorem 5.2.38(1) (this is clear) and Theorem 5.2.38(2). Consider one of the boundary components and suppose $\pi(q) \in \big\{(\theta, \phi) \in \mathbb{T}^2 \;\big|\; \cos\theta + \cos\phi = 2l - 4\big\}$.

Denoting by N the normal vector at q and by subscripts θ and ϕ the corresponding projections, parametrize the (θ, ϕ)-projection of a normal line by $\theta(t) := q_\theta + tN_\theta$, $\phi(t) := q_\phi + tN_\phi$, $F(\theta, \phi) := \frac{\cos\theta + \cos\phi}{2} + 2$, and

$$c(t) = \pm\sqrt{l^2 - (F(\theta(t)\phi(t)))^2} \pm \sqrt{r^2 - \Big(\frac{\cos\theta(t) - \cos\phi(t)}{2}\Big)^2},$$

so that $(\theta(t), \phi(t), c(t)) \in \Sigma$ according to (5.2.7) and $(\theta(0), \phi(0), c(0)) = q$ by choosing "±" appropriately. For t near 0 we then have

$$c(t) = \pm\sqrt{t \frac{d}{dt}\Big|_{t=0} F(\theta(t), \phi(t)) + O(t^2)} \pm \sqrt{r^2 - \Big(\frac{\cos\theta(0) - \cos\phi(0)}{2}\Big)^2} + O(t).$$

It suffices to show that $t \mapsto c(t)$ is invertible near $c(0)$ and that the inverse has nonzero second derivative. To that end note that

$$(c(t) - c(0))^2 = \pm t \underbrace{\frac{d}{dt}\Big|_{t=0} F(\theta(t), \phi(t))}_{=\begin{pmatrix}\theta'(0)\\ \phi'(0)\end{pmatrix} \nabla F(\theta(0), \phi(0)) \ne 0 \text{ since } 2<l<3} + o(t)$$

and hence $t = \pm \frac{1}{\frac{d}{dt}\big|_{t=0} F(\theta(t), \phi(t))} (c(t) - c(0))^2 + o(c(t) - c(0))^2$, as required.

Finally, we show that B is a finite-horizon dispersing billiard—which is easy to believe from Figure 5.2.6. As to dispersion, consider the boundary component $F(\theta,\phi) := \cos\theta + \cos\phi) = 2l - 4 \in (0,2)$ (since $2 < l < 3$). Its curvature is

$$\nabla \cdot \frac{\nabla F}{\|\nabla F\|} = -\nabla \cdot \frac{1}{\sqrt{\sin^2\theta + \sin^2\phi}} \begin{pmatrix} \sin\theta \\ \sin\phi \end{pmatrix} = -\frac{\cos\theta \sin^2\phi + \sin^2\theta\cos\phi}{\sqrt{\sin^2\theta + \sin^2\phi}^{\,3}}.$$

As required, the numerator is positive because it equals

$$\begin{aligned} \cos\theta(1-\cos^2\phi) + \cos\phi(1-\cos^2\theta) &= (\cos\theta + \cos\phi)\cos^2\theta \\ &\quad - (\cos^2\theta + 2\cos\theta\cos\phi + \cos^2\phi)\cos\theta \\ &\quad + (\cos\theta + \cos\phi) \\ &= \underbrace{(2l-4)}_{>0}\big[\cos^2\theta - \underbrace{(2l-4)}_{<2}\cos\theta + 1\big] > 0. \end{aligned}$$

For the boundary component $\{\cos\phi - \cos\theta = 2r\}$ the corresponding numerator is

$$\sin^2\phi\cos\theta - \sin^2\theta\cos\phi = 2r\big[\cos^2\theta + \underbrace{2r}_{<2}\cos\theta + 1\big] > 0,$$

and similarly for $\{\cos\theta - \cos\phi = 2r\}$.

We conclude the proof by showing that if $(l-2)^2 + r^2 < 1$ and $r < 1/2$, then B has finite horizon.

Otherwise there is a bi-infinite geodesic $(\theta(t), \phi(t))$, and we first show that it has slope ± 1. Up to exchanging θ and ϕ we may assume the slope is in $[-1,1]$, so there is a t_0 with $\theta(t_0) = 0 \bmod 2\pi$, so $G := \{\phi(t) - \phi(t_0) \bmod 2\pi \;\big|\; t \in \mathbb{R},\ \theta(t) = 0 \bmod 2\pi\}$ is a subgroup of $\mathbb{R}/2\pi\mathbb{Z}$, and for $t \in G$ we have $|\cos\theta(t) - \cos\phi(t)| \le 2r$ by (5.2.8), so $\cos\phi(t) \ge 2 - 2r > 0$ and hence $G \subset (-\frac{\pi}{2}, \frac{\pi}{2}) \bmod 2\pi$, which means that G is a point and the slope is in $\{0, \pm 1\}$. The slope cannot be 0 because in that case taking t such that $\cos(t) = -1$ and (5.2.8) give $-1 + 2r \ge \cos\phi(t) \ge 1 - 2r > 0$, contrary to $r < 1/2$.

Thus, the slope is 1 (up to replacing θ by $-\theta$). Therefore, there are t_1, t_2 with

$$\begin{aligned} \phi(t_1) + \theta(t_1) &= \pi \bmod 2\pi, \\ \phi(t_2) + \theta(t_2) &= 0 \bmod 2\pi. \end{aligned}$$

Averaging these 2 equations and using $\theta(t_2) - \theta(t_1) = \phi(t_2) - \phi(t_1) \bmod 2\pi$ (slope 1) gives $\phi(t_2) - \phi(t_1) = \frac{\pi}{2} \bmod \pi$, and hence $\cos\phi(t_2)\cos\phi(t_1) = -\sin\phi(t_2)\sin\phi(t_1)$. Squaring both sides here gives

$$\begin{aligned} \cos^2\phi(t_2)\cos^2\phi(t_1) &= (1 - \cos^2\phi(t_2))(1 - \cos^2\phi(t_1)) \\ &= 1 - \cos^2\phi(t_2) - \cos^2\phi(t_1) + \cos^2\phi(t_2)\cos^2\phi(t_1), \end{aligned}$$

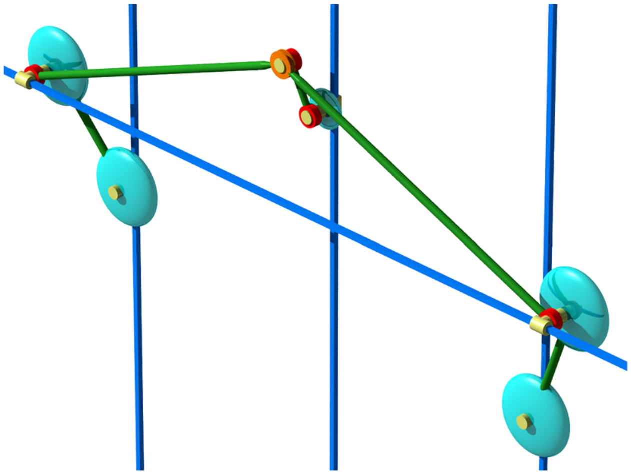


FIGURE 5.2.7. The Kourganoff linkage, a mechanical Anosov system animated by Jos Leys at http://mickael.kourganoff.fr/videos/anosov-linkage.mov; see also https://icerm.brown.edu/video_archive/?play=1138

so $\cos^2\phi(t_2)+\cos^2\phi(t_1)=1$. The choice of t_1 implies that

$$\underbrace{\cos\phi(t_1)}_{=-\cos\theta(t_1)}=\frac{1}{2}(\cos\phi(t_1)-\cos\theta(t_1))\le\frac{1}{2}2r=r$$

by (5.2.8), and the choice of t_2 and (5.2.8) imply

$$\underbrace{\cos\phi(t_2)}_{=\cos\theta(t_2)}=\frac{1}{2}(\cos\phi(t_2)+\cos\theta(t_2))\le\frac{1}{2}(2l-4)=l-2.$$

Thus, $1=\cos^2\phi(t_2)+\cos^2\phi(t_1)\le r^2+(l-2)^2<1$ (by (5.2.6)), a contradiction. □

3. Shadowing, expansivity, closing, specification, and Axiom A

The orbit structure of hyperbolic dynamical systems has a distinctive and iconic richness and complexity, and these features can be derived from what thereby appears as the very core feature of hyperbolic dynamics: The shadowing of orbits. This feature is that in a hyperbolic system anything one can imagine approximately happening is, to good approximation, actually happening in the system. This section shows that the Shadowing Lemma (Theorem 5.3.2) produces the essential richness of the orbit structure of a hyperbolic dynamical system: expansivity (Corollary 5.3.4), the Anosov Closing Lemma (Theorem 5.3.10), specification

(Theorem 5.3.59), spectral decomposition (Theorem 5.3.35), and a natural definition of hyperbolicity (Theorem 5.3.45) as well as topological stability (Theorem 5.3.6, Theorem 5.3.7).

The stronger Anosov Shadowing Theorem 5.4.1 further implies structural stability (Theorem 5.4.5).[19] We reserve this for the next section and emphasize that Section 5.4 (through Theorem 5.4.10) is independent of this one, that is, a reader can learn about structural stability directly without working through the present section first.

Definition 5.3.1. Let Φ be a flow on a metric space M and g be an ϵ-pseudo-orbit for Φ (Definition 1.5.26). Then g is said to be δ-*shadowed* if there exists a point $p \in M$ and a homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) - t$ has Lipschitz constant δ and $d(g(t), \varphi^{\alpha(t)}(p)) \le \delta$ for all $t \in \mathbb{R}$. A set $Y \subset M$ has the *shadowing property* if for any $\delta > 0$ there is an $\epsilon > 0$ such that any ϵ-pseudo-orbit in Y is δ-shadowed by a point $p \in M$. We say that Φ has the shadowing property if this holds for $Y = M$. A set $Y \subset M$ has L-*Lipschitz shadowing for* $\epsilon_0 > 0$ if any ϵ-pseudo-orbit in Y with $\epsilon \le \epsilon_0$ is $L\epsilon$-shadowed by a point $p \in M$.

Theorem 5.4.1 below implies that hyperbolic sets have this property:

Theorem 5.3.2 (Shadowing Lemma)**.** *A hyperbolic set for a flow has* a neighborhood *with L-Lipschitz shadowing for some $\epsilon_0 > 0$ and for some $L > 0$. The shadowing* point *need not be unique because neither is the choice of the parameterization. But the shadowing* orbit *is unique and any 2 parameterizations differ by a constant that is at most $L\epsilon / \min \|X\|$ in absolute value, where X is the generating vector field.*

Remark 5.3.3. Implicitly Theorem 5.3.2, or, rather, Definition 5.3.1, controls the timing of the shadowing orbit to within a percentage error, where the percentage is small for small ϵ.

The uniqueness assertion of Theorem 5.3.2 implies that no two orbits can shadow each other:

Corollary 5.3.4. *The restriction of a flow to a (sufficiently small neighborhood of a) hyperbolic set is expansive (Definition 1.7.2).*

Proof. If $\mathscr{O}(x)$ and $\mathscr{O}(y)$ $L\epsilon_0$-shadow each other, then both $L\epsilon_0$-shadow the pseudo-orbit $\mathscr{O}(x)$ and hence agree by uniqueness. □

Remark 5.3.5. We continue to derive consequences of the Shadowing Lemma, but the reader is encouraged to verify that these consequences can equivalently be obtained by combining the Shadowing Property (without the uniqueness assertion) with expansivity.

[19] . . . and symbolic descriptions, which we do not include here [**181**, Theorem 18.2.5].

As a preview of coming attractions, we note that the Shadowing Lemma implies stability:

Theorem 5.3.6 (Topological stability)**.** *Anosov flows are topologically stable, that is, any sufficiently C^0-close flow is an extension (Definition 1.3.1).*

The proof *idea* is straightforward: The orbits of the perturbation are pseudo-orbits for the given flow and hence shadowed by genuine orbits of the flow; this correspondence between orbits of the perturbation and those of the given flow gives the factor map—but one needs to check that it is continuous [**287**]. While this is possible, we will instead step up from the Shadowing Lemma to the Shadowing Theorem 5.4.1, where such continuous dependence is built into the conclusion. In passing, we note that topological stability implies a nontrivial variant of structural stability (Theorem 5.4.5) for C^0-perturbations:

Theorem 5.3.7. *Any 2 sufficiently C^0-close Anosov flows are orbit equivalent.*

Proof. The factor map in Theorem 5.3.6 is injective because the orbit of the perturbation that shadows a given one is unique by expansivity (from the Anosov property) of the perturbation. □

Remark 5.3.8. The argument actually shows, of course, that Anosov flows are C^0 structurally stable (Definition 5.4.4) among expansive flows.

When the Anosov flows are geodesic flows, this last observation has a remarkable refinement.[20]

Theorem 5.3.9 ([**126**, Théorème B])**.** *Any Anosov geodesic flows of a closed manifold that supports a Riemannian metric with constant negative curvature are pairwise topologically orbit-equivalent.*

We now apply the Shadowing Lemma to study the structure of hyperbolic sets. The uniqueness assertion of Theorem 5.3.2 implies not only expansivity but also that the shadowing orbit is periodic when one starts with a periodic pseudo-orbit:

Theorem 5.3.10 (Anosov Closing Lemma)**.** *If Λ is a hyperbolic set for a flow Φ then there are a neighborhood U of Λ and numbers $\epsilon_0, L > 0$ such that for $\epsilon \le \epsilon_0$ any periodic ϵ-pseudo-orbit in U is $L\epsilon$-shadowed by a unique periodic orbit for Φ.*

Remark 5.3.11. The definition of hyperbolicity allows isolated hyperbolic fixed points (Definition 5.1.1), and these are pseudo-orbits shadowed only by themselves, so in the Closing Lemma *and henceforth* "periodic point" is meant to include fixed points.

[20] Towards which the "averaging" idea underlying Proposition 1.3.27 was developed.

Proof. Except for “periodic,” this is just Lipschitz shadowing. Uniqueness forces the shadowing orbit to close up: If the pseudo-orbit has period T and $\mathscr{O}(x)$ is a shadowing orbit, then so is $\mathscr{O}(\varphi^{nT}(x))$ for $n \in \mathbb{Z}$. If $nT > L\epsilon / \min \|X\|$, then uniqueness gives $\varphi^{T'}(x) = x$ for T' near nT. □

This can also be proved directly rather than as a corollary of Theorem 5.3.2.

Remark 5.3.12. Except for particularly short pseudo-orbits one can take $n = 1$ in the proof of Theorem 5.3.10, so the shadowing orbit has a length comparable with that of the pseudo-orbit; the accuracy of shadowing controls a percentage difference in orbit length beyond the “misparamatrization” of the pseudo-orbit itself. The latter effect is apparent for a pseudo-orbit $t \mapsto \varphi^{1.1t}(x)$ of φ^t for which the shadowing orbit is $t \mapsto \varphi^t(x)$, which has a 10% difference in speed. In important applications, the pseudo-orbit is an almost periodic orbit segment, for which it may be desirable to control the timing more finely, and Proposition 6.2.4 below (a quantitative version of Proposition 1.7.4) does so by bounding an *absolute* error instead (Remark 6.2.5), which is critical for several of those applications.

We have actually proved:

Proposition 5.3.13. $\overline{\mathrm{Per}(\Phi)} = \mathscr{R}(\Phi)$*, the chain recurrent set, if* Φ *is expansive with shadowing.*

Corollary 5.3.14. *Let* Φ *be a smooth flow on a compact manifold* M*. Then:*

(1) *If* $\mathscr{R}(\Phi)$ *is hyperbolic, then* $\overline{\mathrm{Per}(\Phi)} = \mathscr{B}(\Phi) = \mathscr{L}(\Phi) = NW(\Phi) = \mathscr{R}(\Phi)$.
(2) *If* $NW(\Phi)$ *is hyperbolic, then* $\overline{\mathrm{Per}(\Phi)} = NW(\Phi|_{NW(\Phi)})$.
(3) *If the limit set* $\mathscr{L}(\Phi)$ *is hyperbolic, then* $\overline{\mathrm{Per}(\Phi)} = \mathscr{L}(\Phi)$.
(4) *If* Λ *is a hyperbolic set for* Φ *and* V *a neighborhood of* Λ *such that* Λ_Φ^V *(Proposition 5.1.10) is hyperbolic, then* $\overline{\mathrm{Per}(\Phi\restriction_{\Lambda_\Phi^V})} = NW(\Phi\restriction_{\Lambda_\Phi^V})$.

Proof. (1): $\forall\delta$ each $x \in \mathscr{R}(\Phi)$ is in a periodic δ-chain in $\mathscr{R}(\Phi)$ (Theorem 1.5.36), which is $L\delta$-shadowed by a periodic p (Theorem 5.3.10), so $x \in \overline{\mathrm{Per}(\Phi)}$, and $\mathscr{R}(\Phi) \subset \overline{\mathrm{Per}(\Phi)} \subset \mathscr{B}(\Phi) \subset \mathscr{L}(\Phi) \subset NW(\Phi) \subset \mathscr{R}(\Phi)$ (Proposition 1.5.34).

(2): “$\subset$” is clear. “$\supset$”: $x \in NW(\Phi|_{NW(\Phi)})$ implies that x is arbitrarily near periodic pseudo-orbits in $NW(\Phi)$, hence in $\overline{\mathrm{Per}(\Phi)}$ by Theorem 5.3.10 applied to the hyperbolic set $NW(\Phi)$.

(3): “$\subset$” is Remark 1.5.10. “$\supset$”: it suffices to show that $x \in \omega(y) \Rightarrow x \in \overline{\mathrm{Per}(\Phi)}$ (Definition 1.5.1). $d(\varphi^t(y), \omega(y)) \xrightarrow[t\to\infty]{} 0$ by Proposition 1.5.7(3). Given $\delta > 0$ there exist $t_0, t_1 > 0$ with $d(\varphi^{t_0}(y), \varphi^{t_0+t_1}(y)) < \delta$, $d(\varphi^{t_0}(y), x) < \delta$, and $d(\varphi^t(y), \omega(y)) < \delta$ for $t_0 \le t \le t_0 + t_1$. The periodic δ-chain $\varphi^{[t_0, t_0+t_1]}(y)$ is within δ of $\omega(y)$ and shadowed by a periodic orbit $\mathscr{O}$ with $d(x, \mathscr{O}) < \delta + L\delta$. Thus, $x \in \overline{\mathrm{Per}(\Phi)}$.

(4). For $\epsilon > 0$ sufficiently small denote by U_ϵ the $\epsilon/(2L+1)$-neighborhood of $x \in NW(\varphi\restriction_{\Lambda_\Phi^V})$ in Λ_Φ^V, where L is as in the Closing Lemma. For some $T > 1$ there exists a $y \in \varphi^T(U_\epsilon) \cap U_\epsilon$, and then $d(\varphi^T(y), y) < 2\epsilon/(2L+1)$, so the Closing Lemma gives a periodic $z \in \Lambda_\Phi^V$ with $d(\varphi^t(z), \varphi^t(y)) < 2L\epsilon/(2L+1)$ for $0 \le t < T$. Then $d(x,z) \le d(x,y) + d(y,z) \le \dfrac{(2L+1)\epsilon}{2L+1} = \epsilon$. □

Λ and Λ_Φ^V coincide in our examples, and this is useful.

Definition 5.3.15 (Local maximality, basic set)**.** A hyperbolic set Λ for Φ is said to be *locally maximal* or *isolated* if there is a neighborhood V of Λ (an *isolating neighborhood*) such that $\Lambda = \Lambda_\varphi^V$ (Proposition 5.1.10). If furthermore $\varphi^t\restriction_\Lambda$ has a positive semiorbit that is dense in Λ, then Λ is said to be a *basic set*.[21]

Remark 5.3.16 (Basic sets are regionally recurrent)**.** $NW(\varphi^t\restriction_\Lambda) = \Lambda$ if Λ is a basic set (Corollary 5.3.14(4)).

Example 5.3.17. A natural example of a closed invariant hyperbolic set that is not locally maximal is given by a hyperbolic periodic orbit together with the orbit of a transverse homoclinic point (see Figure 6.3.1; dynamically this is similar to Example 1.3.9 with a periodic orbit rather than a fixed point at the center of attention).

This situation appears in the horseshoe (Figure 1.5.6), for example, coded by the set Λ_0 of sequences of 0's and 1's that have no more than one 1. This set is not locally maximal since for every $N \in \mathbb{N}$ it is contained in the closed set Λ_0^N consisting of all sequences such that any two 1's are separated by at least N 0's and for any open neighborhood V of Λ_0 we have $\Lambda_0^N \subset V$ for sufficiently large N.

It is not hard to see that Λ_0^N is indeed locally maximal, so for any neighborhood V of Λ_0 there is an invariant locally maximal hyperbolic set $\widetilde{\Lambda}$ such that $\Lambda_0 \subset \widetilde{\Lambda} \subset V$.

Indeed, although any closed invariant subset of the horseshoe is hyperbolic and may have an extremely complicated structure, it can always be enveloped by a locally maximal one (such as, Λ_V^f for an appropriate open neighborhood V as in Proposition 5.1.10).

In general however, if Λ is a hyperbolic set and V an open neighborhood of Λ, there may not exist a locally maximal hyperbolic invariant set $\widetilde{\Lambda}$ such that $\Lambda \subset \widetilde{\Lambda} \subset V$ (Theorem 6.5.1).

Remark 5.3.18 (Reader beware!)**.** The literature quite frequently assumes implicitly that a hyperbolic set is either locally maximal or included in a locally maximal set.

[21] This notion appears to go back to Anosov [**11**].

As we noted, this does not always hold, so when these assumptions are not stated, readers may want to check whether they are actually needed or not.

We do note that Theorem 5.3.35, one of the central results of this chapter, implies local maximality.

Proposition 5.3.19. *If Λ is a locally maximal hyperbolic set, then for $\delta > 0$ sufficiently small there exist $\gamma > 0$ and $\epsilon \in (0, \gamma)$ such that any ϵ-pseudo orbit that stays within γ of Λ is δ-shadowed by a point* in Λ.

PROOF. Let U be an isolating neighborhood of Λ, and $\eta > 0$ such that $\bigcup_{x\in\Lambda} B_\eta(x) \subset U$. Let $\delta = \eta/2$ and fix $\epsilon_1 > 0$ such that any ϵ_1-pseudo orbit in Λ is $\delta/2$-shadowed. By uniform continuity of Φ there exists $\gamma \in (0, \delta/4)$ and $\epsilon \in (0, \gamma)$ such that any ϵ-pseudo orbit $g : I \to X$ that stays within γ of Λ has an ϵ_1-pseudo orbit $g' : I \to X$ such that $d(g(t), g'(t)) < \delta/2$ for all $t \in I$. Then g' is $\delta/2$-shadowed by a point in Λ, and this implies that the pseudo orbit g is δ-shadowed by a point in Λ. □

We now have the following immediate consequence.

Corollary 5.3.20. *The restriction of a flow to a locally maximal hyperbolic set has the shadowing property.*

This shows that if V is sufficiently small and Λ is locally maximal then the shadowing orbits in all prior results are in Λ, so Λ has many periodic orbits.

Corollary 5.3.21. *If Λ is a locally maximal hyperbolic set for Φ, then periodic points are dense in $NW(\Phi\restriction_\Lambda)$. In particular, periodic points are dense in basic sets.*

Arguing as in the proof of Theorem 5.3.25 shows

Proposition 5.3.22. *A hyperbolic set is locally maximal if and only if the restriction to it has the shadowing property.*

To give another expression of the abundance of closed orbits, we show a precursor of a result that periodic data determine a function[22] (Theorem 7.2.1). Suppose f is null-cohomologous (Definition 1.3.20). Then $\varphi^T(x) - x \Rightarrow \int_0^T f(\varphi^t(x)\,dt = F(\varphi^T(x)) - F(x) = 0$. For Walters-continuous functions, this obvious necessary condition for being null-cohomologous is sufficient:

Theorem 5.3.23 (Topological Livshitz Theorem)**.** *Let Λ be a basic set for a flow Φ generated by a vector field X, f Walters-continuous for Φ (Definition 4.3.17). If $\varphi^T(x) = x \Rightarrow \int_0^T f(\varphi^t(x))\,dt = 0$, then f is null-cohomologous (Definition 1.3.20), that is, there is a continuous $F \colon \Lambda \to \mathbb{R}$ with $f = XF$, the derivative in the flow direction. F is unique up to an additive constant.*

[22] or, rather, a cocycle.

Proof. Uniqueness is clear: If $XF = XF'$, then $X(F - F') \equiv 0$, so $F - F'$ is constant on the dense orbit, hence constant. If $\Lambda = \overline{\mathscr{O}(x_0)}$, set $F(\varphi^t(x_0)) := \int_0^t f(\varphi^s(x_0))\,ds$. We next show that F is uniformly continuous on $\mathscr{O}(x_0)$. This implies that F has a unique continuous extension to $\Lambda = \overline{\mathscr{O}(x_0)}$, and since f and XF are continuous and agree on a dense set, they coincide, concluding the proof.

Given $\epsilon > 0$ take $\delta < \epsilon/2\|f\|_\infty$ as in Bowen-boundedness (4.3.7) for $\epsilon/2$ and $\eta = \delta/L$ with L as in the Anosov Closing Lemma (Theorem 5.3.10). If $t_1 < t_2$ and $d(\varphi^{t_1}(x_0), \varphi^{t_2}(x_0)) < \eta$, then $\varphi^{[t_1,t_2]}(x_0)$ is δ-shadowed by a T-periodic point y with $|T - t_2 + t_1| < \delta$, so $d^\Phi_{t_2-t_1}(x_0, y) < \delta$. Then

$$|\underbrace{S_{t_2-t_1} f(x_0)}_{=F(\varphi^{t_2}(x_0))-F(\varphi^{t_1}(x_0))} - \underbrace{S_{t_2-t_1} f(y)}_{=S_{T-t_2+t_1} f(y)}| < \epsilon/2,$$

and $|F(\varphi^{t_2}(x_0)) - F(\varphi^{t_1}(x_0))| < \epsilon/2 + |S_{T-t_2+t_1} f(y)| < \epsilon/2 + \delta\|f\|_\infty < \epsilon$. □

Remark 5.3.24. While interesting, this result does not have obvious applications because we do not have a ready supply of Walters-continuous functions.

The next consequence of shadowing is that being asymptotic to a compact locally maximal hyperbolic set implies being asymptotic to a specific point in that set. To formalize this, the *local* counterparts of the stable and unstable sets of a point (Definition 1.3.24) are defined by

$$\begin{aligned} W^s_\epsilon(x) &:= \{y \in W^s(x) \mid d(\varphi^t(x), \varphi^t(y)) \le \epsilon \text{ for } t \ge 0\}, \\ W^u_\epsilon(x) &:= \{y \in W^u(x) \mid d(\varphi^t(x), \varphi^t(y)) \le \epsilon \text{ for } t \le 0\}. \end{aligned} \tag{5.3.1}$$

Theorem 5.3.25 (In-Phase Theorem). *If Λ is a compact locally maximal hyperbolic set for Φ on M, then with the terminology of Definition 1.5.5 and* (1.3.1)

$$W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x) \quad \textit{and} \quad W^u(\Lambda) = \bigcup_{x\in\Lambda} W^u(x),$$

and for each $\epsilon > 0$, Λ has a neighborhood U with $\bigcap_{t\ge 0} \varphi^{-t}(U) \subset W^s_\epsilon(\Lambda) := \bigcup_{x\in\Lambda} W^s_\epsilon(x)$ (and analogously for W^u).

Remark 5.3.26. Here "$\supset$" follows from the definition, and "$\subset$" says that a point asymptotic to Λ approaches Λ in a way that is "in phase" with an orbit of Λ.

Proof. If $y \in W^s(\Lambda)$ and $\eta > 0$, then there is a $T > 0$ such that for all $t \ge T$ we have an $x_t \in \Lambda$ with $d(\varphi^t(y), x_t) < \eta$ (Proposition 1.5.7(4)). If $\epsilon > 0$ and δ is as in the Shadowing Lemma (Theorem 5.3.2), then by uniform continuity of φ^1 we can choose η such that

$$d(\varphi^1(x_t), x_{t+1}) \le d(\varphi^1(x_t), \varphi^1(\varphi^t(y))) + d(\varphi^{t+1}(y), x_{t+1}) < \delta,$$

so $(x_t)_{t\ge T}$ is ϵ-shadowed by some $x\in\Lambda$. Then $y\in W^s(x)$ because

$$t\ge T\Rightarrow d(\varphi^t(y),\varphi^t(x))\le d(\varphi^t(y),x_t)+d(x_t,\varphi^t(x))\le\delta+\epsilon. \qquad \square$$

We note from this that attractors cannot have unstable sets "sticking out":

Theorem 5.3.27. *If* Λ *is a hyperbolic attractor for a flow* Φ*, then* $W^u(\Lambda)\subset\Lambda$.

Proof. There are a trapping region U for Λ and $\epsilon>0$ such that $W^u_\epsilon(\varphi^t(x))\subset U$ for each $x\in\Lambda$ and $t\in\mathbb{R}$. Then $W^u(x)\subset\bigcap_{t\ge0}\varphi^t(U)=\Lambda$ since

$$t\ge0\Rightarrow W^u(x)=\varphi^t(\underbrace{W^u(\varphi^{-t}(x))}_{=\bigcup_{s\ge0}\varphi^s(W^u_\epsilon(\varphi^{-s-t}(x)))\subset U})\subset\varphi^t(U). \qquad \square$$

In his seminal paper, Smale introduced the following property to focus on dynamical systems for which hyperbolicity is the dominant feature:

Definition 5.3.28 (Axiom A)**.** A flow Φ satisfies *Axiom A* if $NW(\Phi)$ is hyperbolic and the closure of the periodic orbits.

Remark 5.3.29. Analogously to Remark 5.1.2, our definition of Axiom A allows for hyperbolic fixed points, whereas Smale's original definition of Axiom A excluded singularities (he used "Axiom A′ " as the name for our Axiom A). Our choice follows Bowen's terminology.

The second feature in this axiom is slightly stronger than what the Anosov Closing Lemma would imply from the first one; Smale thought it possible that it is a consequence of the hyperbolicity of $NW(\Phi)$, and he was "generically right": although any manifold of dimension at least 4 supports a flow whose nonwandering set is hyperbolic, but which is not Axiom A [**96**], for C^1-generic flows the nonwandering set is the closure of the periodic points (Theorem 1.5.19), so if the nonwandering set is hyperbolic, then it generically satisfies Axiom A. Corollary 5.3.14(2) implies:

Proposition 5.3.30. *If a flow* Φ *satisfies* Axiom A*, then* $NW\big(\Phi\restriction_{NW(\Phi)}\big)=NW(\Phi)$.

Corollary 5.3.14(1) implies (see Definitions 1.5.30, 1.5.1, and 1.5.9):

Proposition 5.3.31. *If* $\mathscr{R}(\Phi)$ *is hyperbolic, then* Φ *satisfies Axiom A and* $\overline{\mathrm{Per}(\Phi)}=\mathscr{B}(\Phi)=\mathscr{L}(\Phi)=NW(\Phi|_{NW(\Phi)})=NW(\Phi)=\mathscr{R}(\Phi)$ *(and more; see Theorem 5.3.42).*

Transitive Anosov flows satisfy Axiom A by the Anosov Closing Lemma. The suspension of an Axiom A diffeomorphism (defined analogously) is an Axiom A flow.

The chain decomposition (Proposition 1.5.32) is particularly effective here because of the following observation.

Proposition 5.3.32. *If* $\overline{\mathrm{Per}(\Phi)}$ *is hyperbolic,*[23] *then there is an* $\epsilon > 0$ *such that any periodic points* p, q *with* $d(p,q) < \epsilon$ *are chain-equivalent. In particular, the chain-decomposition of* $\overline{\mathrm{Per}(\Phi)}$ *is finite.*

Proof. If ϵ is small enough for Theorem 5.3.2, then the concatenation of $\varphi^{(-\infty,0)}(p)$ and $\varphi^{[0,\infty)}(q)$ is $L\epsilon$-shadowed by a ("heteroclinic") $\mathcal{O}(z)$. Uniqueness and Proposition 1.7.4 give $\alpha(z) \cap \mathcal{O}(p) \neq \varnothing \neq \omega(z) \cap \mathcal{O}(q)$. For any desired $\rho > 0$, concatenation of $\varphi^{(-\infty,T_1)}(p)$, $\varphi^{[T_1,T_2)}(z)$ and $\varphi^{[T_2,\infty)}(q)$ for suitable T_1, T_2 then includes a ρ-chain from p to q. □

Remark 5.3.33. A pertinent variant of chain-equivalence (Definition 1.5.30) would be $x \sim_\epsilon y :\Leftrightarrow x \in \mathcal{R}_\epsilon(y)$ & $y \in \mathcal{R}_\epsilon(x)$, and in this case the equivalence classes are obviously open. Proposition 5.3.32 shows that this stabilizes in the present context, that is, "$\sim$"="$\sim_\epsilon$" for small ϵ.

Proposition 5.3.13, Theorem 1.5.36, and Proposition 5.3.32 imply

Corollary 5.3.34. *If either* Φ *or* $\Phi\restriction_{\mathcal{R}(\Phi)}$ *is expansive with shadowing, then the chain components of* Φ *are open in* $\mathcal{R}(\Phi)$*, so by compactness they are finite in number and admit a filtration* (Theorem 1.5.47).

We now show that the chain-components are basic sets:

Theorem 5.3.35 (Spectral Decomposition, Smale [**279**])**.** *In each of the following situations* Λ *is a finite disjoint union of basic sets* Λ_i *(hence locally maximal).*

(1) $\Lambda = NW(\Phi\restriction_K)$ *for some compact locally maximal hyperbolic set* K*.*
(2) $\Lambda = NW(\Phi)$ *and* Φ *satisfies Axiom A.*
(3) $\Lambda = \mathcal{R}(\Phi)$ *is hyperbolic.*
(4) $\Lambda = \mathcal{L}(\Phi)$ *is hyperbolic.*

Proof. (1): The Λ_i are the intersections of Λ with the chain components of $\Phi\restriction_K$ (which is expansive with shadowing, so Corollary 5.3.34 applies). To see that they are transitive suppose $U, V \subset \Lambda_i$ are open and $\epsilon > 0$. There is a periodic ϵ-chain in K that meets both, and for small-enough ϵ, so does the shadowing periodic orbit $\mathcal{O}$ from Theorem 5.3.10, which lies in an isolating neighborhood, hence in K by local maximality, then in $\Lambda = NW(\Phi\restriction_K)$ by periodicity. Thus, Proposition 1.6.9(4) holds.

Local maximality of Λ_i: If the orbit of x is in a sufficiently small neighborhood of Λ_i, which is also disjoint from the other Λ_j, then $x \in K$ by local maximality of K, so $\varnothing \neq \omega(x) \subset K$, that is, $\mathcal{O}^+(x)$ accumulates on a $y^+ \in \Lambda = NW(\Phi\restriction_K)$; likewise with a y^- in the α-limit set, so with a segment of a dense orbit in Λ_i from near y^- to near

[23] or Φ is expansive with shadowing

y^+ we get a closed chain, and by Theorem 5.3.10, $x \in \overline{\mathrm{Per}(\Phi{\restriction_K})} \subset NW(\Phi{\restriction_K}) = \Lambda$, hence $x \in \Lambda_i$.

In the remaining cases $\Lambda = \overline{\mathrm{Per}(\Phi)}$ (Corollary 5.3.14), so the chain-components Λ_i of $\Phi{\restriction_\Lambda}$ are open (Proposition 5.3.32), hence finite in number. Λ_i is topologically transitive because if $U, V \subset \Lambda_i$ are open and $\epsilon > 0$, there is a periodic ϵ-chain in Λ that meets both, and for small-enough ϵ, so does the shadowing periodic orbit $\mathscr{O} \subset \mathrm{Per}(\Phi) \subset \overline{\mathrm{Per}(\Phi)} = \Lambda$ from Theorem 5.3.10. Local maximality follows as in (1) by obtaining $\omega(x) \cup \alpha(x) \subset \Lambda_i$ from $\omega(x) \cup \alpha(x) \subset \mathscr{L}(\Phi) \subset \Lambda$. □

Remark 5.3.36. The Λ_i can also be described by an equivalence relation defined in terms other than chain-equivalence (Section 6.2). A remarkable variant of the spectral decomposition appears in Theorem 9.3.3 and the constructions described thereafter.

Remark 5.3.37. This is a good moment to emphasize a distinction with the corresponding decomposition for discrete-time dynamics. In that context, the basic sets are topologically transitive, but can be further decomposed into topologically mixing components. For flows this is in general not possible, as illustrated by suspensions.

Proposition 5.3.38. *Let Φ be a flow such that either $\mathscr{L}(\Phi)$ or $\mathscr{R}(\Phi)$ is hyperbolic or Φ is Axiom A (Definition 5.3.28). Then $M = \bigcup_{i=1}^m W^s(\Lambda_i) = \bigcup_{i=1}^m W^u(\Lambda_i)$ with each union disjoint, where $\{\Lambda_i\}_{i=1}^k$ is the Spectral Decomposition (Theorem 5.3.35).*

Proof. There are pairwise disjoint open $U_i \supset \Lambda_i$ for $i \in \{1, \dots, k\}$. If $x \in M$, then (Proposition 1.5.15) $\omega(x) \subset \Lambda = \bigcup_{i=1}^k \Lambda_i \subset \bigcup_{i=1}^k U_i$ is connected (Proposition 1.5.7(4)), so there is a unique i with $\omega(x) \subset U_i$ (and hence $x \in W^s(\Lambda_i)$). Reversing the flow shows the same for W^u. □

Remark 5.3.39. Proposition 5.3.38 and Theorem 5.3.25 imply that if we have a spectral decomposition of Λ (which is the case if $\mathscr{L}(\Phi)$ or $\mathscr{R}(\Phi)$ is hyperbolic or Φ is Axiom A) and $x \in M$, then $x \in W^s(y)$ and $x \in W^u(z)$ for some $y, z \in \Lambda$. So there are nontrivial stable and unstable sets for any point of M even if these points may not be contained in Λ.

The last few results played out quite similarly when $\mathscr{L}(\Phi)$ or $\mathscr{R}(\Phi)$ is hyperbolic or Φ is Axiom A, even though these are not equivalent. It turns out that there is a common underlying notion—which then makes a natural definition of hyperbolicity—obtained by adding an extra condition under which these 3 scenarios become equivalent. Specifically, we show that hyperbolicity of the chain recurrent set is equivalent to the flow satisfying Axiom A (Definition 5.3.28) and having no cycles among the basic sets as defined below.

Definition 5.3.40 (Cycles)**.** Suppose Φ is a flow on a compact manifold M satisfying Axiom A or such that either $\mathscr{L}(\Phi)$ or $\mathscr{R}(\Phi)$ is hyperbolic. Define a partial ordering $\gg$ on the basic sets $\Lambda_1, \dots, \Lambda_n$ from the Spectral Decomposition Theorem 5.3.35 by

$$\Lambda_i \gg \Lambda_j \text{ if } \big(W^u(\Lambda_i) \smallsetminus \Lambda_i\big) \cap \big(W^s(\Lambda_j) \smallsetminus \Lambda_j\big) \neq \varnothing.$$

A *k-cycle* consists of a sequence $\Lambda_{i_1} \gg \Lambda_{i_2} \gg \cdots \gg \Lambda_{i_k} \gg \Lambda_{i_1}$ of basic sets. The flow Φ *has no cycles* if this happens for no k.

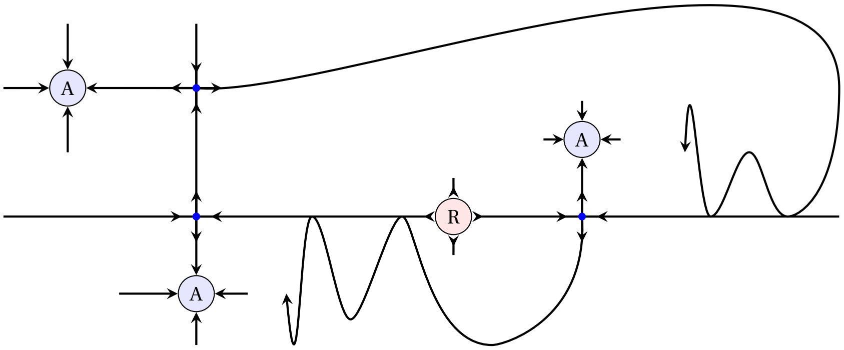


FIGURE 5.3.1. Axiom A with a cycle

Remark 5.3.41. To see how the presence of a cycle is compatible with Axiom A, note that for a flow with a section as shown in Figure 5.3.1 (suggested to us by Hayashi) the nonwandering set is finite and includes a 3-cycle of saddles; the intersections of stable and unstable manifolds of succesive saddles are either an interval or a tangency; the attractors and repeller (including a repeller at ∞) are strategically placed to keep the nonwandering set finite.

Cycles are precluded by having a filtration, so Corollary 5.3.34 gives

Theorem 5.3.42. *If Φ is a flow on a compact manifold M and $\mathscr{R}(\Phi)$ is hyperbolic, then Φ has no cycles.*

Theorem 5.3.43. *If $\mathscr{L}(\Phi)$ is hyperbolic and Φ has no cycles, then $\mathscr{L}(\Phi) = \mathscr{R}(\Phi)$.*

Lemma 5.3.44. *If $p \in \mathscr{R}(\Phi) \smallsetminus \mathscr{L}(\Phi)$, then $p \in W^s(\Lambda_i)$ for some $i \in \{1, \dots, k\}$ (Remark 5.3.39), and there is a $q \in W^u(\Lambda_i) \cap \big(\mathscr{R}(\Phi) \smallsetminus \mathscr{L}(\Phi)\big)$.*

PROOF OF THEOREM 5.3.43. We proceed by contraposition: if there is an $x_1 \in \mathscr{R}(\Phi) \smallsetminus \mathscr{L}(\Phi)$, then $x_1 \in W^s(\Lambda_{i_1})$ for some $i_1 \in \{1, \dots, k\}$. By Lemma 5.3.44 there is an $x_2 \in W^u(\Lambda_{i_1}) \cap \big(\mathscr{R}(\Phi) \smallsetminus \mathscr{L}(\Phi)\big)$ and hence an i_2 such that $x_2 \in W^s(\Lambda_{i_2})$. By

Lemma 5.3.44, there is an $x_3 \in W^u(\Lambda_{i_2}) \cap \big(\mathscr{R}(\Phi) \smallsetminus \mathscr{L}(\Phi)\big)$—and so on. Since there are only finitely many Λ_i, this sequence contains a cycle. □

Proof of Lemma 5.3.44. We first pick a compact neighborhood U of Λ_i such that $p \notin U$ and $\varphi^t(U) \cap \Lambda_j = \varnothing$ for $j \neq i$ and $0 \le t \le 1$.

For $n \in \mathbb{N}$ we fix a $1/n$-chain $g_n : I_n \to M$ that starts and ends at p, and a point p_n in g_n that is closest to Λ_i. Since $p \in W^s(\Lambda_i)$, we know by possibly taking a subsequence that $d(p_n, \Lambda_i) \to 0$ as $n \to \infty$. For each n let $t_n \in I_n$ such that $g_n(t_n) = p_n$. Then for n large there exists some $s_n \ge 1$ such that $g_n(t_n + s) \in \operatorname{int}(U)$ for $0 \le s < s_n$, but $q_n := g_n(t_n + s_n) \notin \operatorname{int}(U)$.

Since $p_n \xrightarrow[n\to\infty]{} \Lambda_i$ we have $s_n \xrightarrow[n\to\infty]{} \infty$. Let q be a limit point of q_n. Then $q \in \mathscr{R}(\Phi) \smallsetminus \mathscr{L}(\Phi)$. By construction, $\varphi^t(q) \in U$ for $t < 0$. So $\alpha(q) \subset \Lambda_i$, and $q \in W^u(\Lambda_i)$. □

Theorem 5.3.45 (Axiom A, no cycles)**.** *For a C^1 flow Φ, the following are equivalent:*

(1) Φ satisfies Axiom A and has no cycles.
(2) $\mathscr{L}(\Phi)$ is hyperbolic and has no cycles.
(3) $\mathscr{R}(\Phi)$ is hyperbolic.

Remark 5.3.46. By Proposition 5.3.31, the pertinent hyperbolic set is the same in these 3 equivalent cases: $\overline{\operatorname{Per}(\Phi)} = \mathscr{B}(\Phi) = \mathscr{L}(\Phi) = NW(\Phi|_{NW(\Phi)}) = NW(\Phi) = \mathscr{R}(\Phi)$.

Proof. (1)⇒(2) from the definition (Definition 5.3.28 and Proposition 1.5.34).

(2) implies that $\mathscr{L}(\Phi)$ is the closure of the periodic points, has a spectral decomposition, and no cycles, so Theorem 5.3.43 gives $\mathscr{L}(\Phi) = \mathscr{R}(\Phi)$, hence (3).

(3)⇒(1) because $\mathscr{R}(\Phi) = \overline{\operatorname{Per}(\Phi)}$ (Corollary 5.3.14) and has a spectral decomposition (Theorem 5.3.35) without cycles (Theorem 5.3.42). □

Remark 5.3.47. In the literature one variously finds the assumption of Axiom A with no cycles, or of hyperbolic chain recurrent set, or of hyperbolic limit set with no cycles. By Theorem 5.3.45 these are equivalent. The variety of such usage also underscores the importance of this concept, so we make it our definition of hyperbolicity.

Definition 5.3.48 (Hyperbolic flow)**.** A flow Φ is said to be hyperbolic if one of the following equivalent conditions holds:

- Φ satisfies Axiom A and has no cycles.
- $\mathscr{L}(\Phi)$ is hyperbolic and has no cycles.
- $\mathscr{R}(\Phi)$ is hyperbolic.

Following Bowen, we write

$$\mathscr{A} := \big\{\Phi \;\big|\; \Phi \text{ is hyperbolic}\big\}. \tag{5.3.2}$$

Remark 5.3.49. This notion is of a global nature compared to Definition 5.1.1. Therefore, it is less apparent that this is an open condition. Our other results about persistence of hyperbolicity are either potentially vacuous (Proposition 5.1.10), highly specialized (Corollary 5.1.11), or only imply that the *presence* of hyperbolicity is an open condition (Theorems 5.3.6, 5.4.5). However, a global counterpart (Theorem 5.4.13) to Theorem 5.4.5 does control the entire chain-recurrent set and thus finally establishes that C^1-perturbations of hyperbolic flows are themselves hyperbolic (Corollary 5.4.14).

We note an obvious consequence of spectral decomposition for Anosov flows:

Theorem 5.3.50. *For an Anosov flow Φ on a manifold M the following are equivalent.*

1. *The spectral decomposition of Φ is $\{M\}$.*
2. *Φ is* regionally recurrent *(Definition 1.5.11).*
3. *Φ is topologically transitive.*
4. *Periodic points are dense in M.*

We develop this further in Theorem 6.2.11 but mention a related observation.

Theorem 5.3.51. *The interior of the nonwandering set of an Anosov flow is either empty or the whole manifold.*

Proof [**240**, Lemma 4.2]. If Λ_0 is a basic set with nonempty interior, then it contains a periodic point p and a neighborhood U of p. The weak stable and unstable leaf of p are dense in Λ_0, and $W^{cu}(p) \subset \bigcup_{t\geq 0} \varphi^t(U)$, $W^{cs}(p) \subset \bigcup_{t\leq 0} \varphi^t(U)$, so $W^{cu}(p) \cup W^{cs}(p) \subset \Lambda_0$, and $\overline{W^{cu}(p)} \cup \overline{W^{cs}(p)} \subset \Lambda_0$, so Λ_0 is W^{cu}- and W^{cs}-saturated. For any $x \in \Lambda_0$ we thus have $W^{cu}(x) \cup W^{cs}(x) \subset \Lambda_0$, so density of $W^{cu}(x)$ in Λ_0, hence in $W^{cs}(x)$ implies that Λ_0 contains a product neighborhood of x, so Λ_0 is open and closed in M, hence $\Lambda_0 = M$. □

Meanwhile, we formalize an observation from Corollary 5.3.34:

Theorem 5.3.52. *Let $\Phi \in \mathscr{A}$ and $\Lambda_1, \dots, \Lambda_k$ the spectral decomposition. Then there is a filtration $\mathbf{M}$ of M composed of $M_0 \subset M_1 \subset \cdots \subset M_k$ such that $\Lambda_i = K_i^{\Phi}(\mathbf{M})$ for each $i \in \{1, \dots, k\}$.*

Thus, $\gg$ is a total and linear order. In particular:

Theorem 5.3.53. *If $\Lambda_1, \dots, \Lambda_k$ is the spectral decomposition of a hyperbolic flow Φ, then Λ_i is an attractor if there is no j with $\Lambda_i \gg \Lambda_j$.*

Remark 5.3.54. The constructions in Theorem 9.3.11 virtually reverse-engineer this by gluing together filtrating neighborhoods of hyperbolic sets in order to produce examples of hyperbolic flows.

Volume-preserving hyperbolic flows have neither an attractor nor any cycles:

Corollary 5.3.55. *The spectral decomposition of a volume-preserving hyperbolic flow has only one piece.*

We have come a long way, and we repeat that the preceding are all consequences of the Shadowing Lemma.[24] Of course, we also have yet to prove the Shadowing Lemma. We will do so presently. First we combine shadowing with transitivity.

Bowen introduced *specification* as a notion that formalizes how shadowing and transitivity make it possible to prescribe the evolution of an orbit to the extent of specifying a finite collection of arbitrarily long orbit segments and any fixed precision: as long as one allows for enough time between the specified segments one can find a single (periodic) orbit approximating this entire itinerary and the time between the segments depends only on the quality of the approximation and not on the length of the specified segments.

Definition 5.3.56 (Specification)**.** Let X be a compact metric space and Φ be a flow on X. Then Φ satisfies *specification* if for any $\epsilon > 0$ there exists some T_ϵ such that given any finite collection of points $x_0, \dots, x_n \in X$ and times $t_0, \dots, t_n \in [0, \infty)$ there exists a point $y \in X$, and $s_0, \dots, s_n \in [0, T_\epsilon]$, and for each $i \in \{0, \dots, n\}$ we have

$$d(\varphi^t y, \varphi^{t-\sum_{j=0}^{i-1} t_j + s_j} x_i) < \epsilon \quad \text{for} \quad t \in [0, t_i] + \sum_{j=0}^{i-1} t_j + s_j, \quad \text{and} \quad \varphi^{\sum_{i=0}^{n} t_i + s_i}(y) = y.$$

Remark 5.3.57. The "transition" times s_j here are controlled only to the extent that they need not be very long—depending on the desired accuracy of the approximation. With both more tools and stronger assumptions, we will later be able to prescribe the transition times exactly (Theorem 8.3.4).

The idea is to associate the orbit segment $\varphi^{[0,t_i]}(x_i)$ with the pair $(x_i, t_i) \in X \times [0, \infty)$. Such a collection of orbit segments has specification if given $\epsilon > 0$ there is a closed orbit that stays within ϵ of each orbit segment in turn provided we allow a transition time between the orbit segments, which can be chosen to be no more than T_ϵ.

There are variants of this definition in the literature. The orbit y might not be required to be a closed orbit, or it is required that the transition time is equal between each of the orbit segments. For the hyperbolic case any of the various versions hold (possibly subject to assuming topological mixing), but in other situations one may need to choose one specific variant. A stronger counterpart requires that the transition time (rather than bounds on it) is prescribed (Definition 8.3.2).

[24] Or, alternatively, shadowing and expansivity.

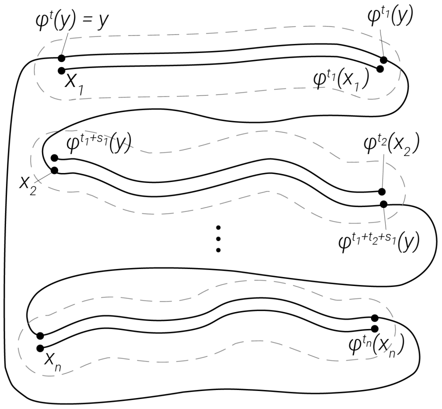


FIGURE 5.3.2. Specification of orbit segments

Combined with expansivity, specification forces exponential orbit complexity:

Proposition 5.3.58. *An expansive continuous flow Φ with specification on a compact space X with more than 1 orbit has exponential growth of periodic orbits (Definition 4.2.1) and hence positive topological entropy (Theorem 4.2.24).*

PROOF. Since $X \neq \varnothing$ there is an $x_0 \in X$, and by the specification property there is a closed orbit p that starts near x_0. Denote by T_0 its least period. Suppose $\epsilon > 0$ is an expansivity constant. With the notations of Definition 4.2.22 it suffices to show that $\#\mathbb{O}'_{T_0+T_\epsilon}(T+2T_0+2T_\epsilon) \geq 2\#\mathbb{O}'_{T_0+T_\epsilon}(T)$ for all $T > T_0 + T_\epsilon$. To see this, consider $q \in \mathbb{O}'_{T_0+T_\epsilon}(T)$ and apply the specification property with the specifications q, p and q, p, p to get 2 (by expansivity distinct) elements of $\mathbb{O}'_{T_0+T_\epsilon}(T+2T_0+2T_\epsilon)$. □

Theorem 5.3.59 (Specification Theorem). *Let Λ be a basic set for a flow φ^t. Then $\varphi^t\restriction_\Lambda$ has the specification property.*

PROOF. By Remark 1.6.11 the orbit segments of the specification can be interpolated to a closed $\epsilon/2L$-orbit by orbit segments whose length is bounded in terms of ϵ as follows. The first interpolating segment begins within $\epsilon/2L$ of $\varphi^{t_1}(x_1)$ and ends after time t_1' within $\epsilon/2L$ of x_2. The next one begins within $\epsilon/2L$ of $\varphi^{t_1+t_1'+t_2}(x_2)$ and ends within $\epsilon/2L$ of x_3. and so on; the last one ends within $\epsilon/2L$ of x_1. By Theorem 5.3.2, this pseudo-orbit is $\epsilon/2$-shadowed by an orbit, which is then as desired. □

Remark 5.3.60. The proof reveals that shadowing and transitivity together imply specification (and that shadowing and specification both hold for basic sets), though more is needed for the stronger specification property in Theorem 8.3.4: on one hand mixing rather than just transitivity is needed, and on the other hand, finer control using the invariant foliations is essential. Conversely, however, specification implies transitivity but not shadowing because of the required transition times (strong shadowing implies topological mixing, but also not shadowing).

Bowen's Specification Theorem, suitably strengthened, is a useful tool for the study of statistical properties of orbits within hyperbolic sets (Theorem 8.3.6). Proposition 5.3.58 gives a much simpler application:

Proposition 5.3.61. *Unless it is an orbit or empty, a basic set has exponential growth of periodic orbits and hence positive topological entropy.*[25]

4. The Anosov Shadowing Theorem, Structural and Ω-stability

We finally present the shadowing result, which makes the proof of Theorem 5.3.6 easier, implies the Shadowing Lemma (Theorem 5.3.2), and leads to structural stability (Corollary 5.4.7).

Theorem 5.4.1 (Anosov Shadowing Theorem)**.** *If M is a Riemannian manifold, φ^t a C^1 flow, then any compact hyperbolic set $\Lambda \subset M$ for Φ has a neighborhood V and $\epsilon_0, \delta_0, C > 0$ such that if*

- *$\psi^t\colon V \to M$ is generated by a vector field X,*
- *$d_{C^1}(\varphi^t, \psi^t) < \epsilon_0$ for $|t| \le 1$,*
- *N is a topological space,*
- *$\sigma^t\colon N \to N$ a continuous flow and*
- *$\alpha \in C^0(N, V)$ such that $Y := \frac{d}{dt}|_0 \alpha \circ \sigma^t$ exists and*
- *$d_{C^0}(Y, X \circ \alpha) < \epsilon < \epsilon_0$,*

then there are $\beta \in C^0(N, \Lambda_V^{\psi})$ and $\tau\colon N \times \mathbb{R} \to \mathbb{R}$ with

- *$\beta \circ \sigma^t = \psi^{\tau(\cdot,t)} \circ \beta$,*
- *$d_{C^0}(\alpha, \beta) < C\epsilon$, and*
- *$\mathrm{Lip}(\tau(x,\cdot) - \mathrm{Id}) < C\epsilon$.*

Moreover, β is locally transversely unique: $\bar{\beta} \circ \sigma^t = \psi^{\bar{\tau}(\cdot,t)} \circ \bar{\beta}$ and $d_{C^0}(\alpha, \bar{\beta}) < \delta_0 \Rightarrow \bar{\beta}(x) = \psi^{\theta(x)}(\beta(x))$ for some continuous $\theta\colon N \to [-C\delta_0, C\delta_0]$.

Remark 5.4.2. The Shadowing Lemma (Theorem 5.3.2) follows by taking $\Psi = \Phi$, $N = \mathbb{R}$, $\sigma^t(x) = x + t$, $\alpha(t) = g(t)$ (differentiable with derivative near $X \circ \alpha$, see Remark 1.5.28).

[25] See also Remark 4.2.25

Proof. By the Whitney embedding theorem $M \subset \mathbb{R}^n$ for suitable n, so we take $M = \mathbb{R}^n$ without loss of generality: If the result is known for $\mathbb{R}^n$, embed M and augment M to a tubular neighborhood $U' \subset \mathbb{R}^n$ while extending Φ and a C^1-close Ψ to U' by the same contraction normal to M and apply the result. It gives a β consisting of full orbits of the extension of Ψ, so $\beta(N) \subset M$ because Ψ contracts normally to M and indeed, $\beta(N) \subset V$, hence $\beta(N) \subset \Lambda_V^\Psi$ because it consists of orbits.

Hyperbolicity is the central ingredient in the proof, and it will be used in a standard way to set up a contraction, whose fixed point is the desired object. However, hyperbolicity plays out transversely to the flow direction, so we isolate this transverse behavior with the following device. Let $X^\perp$ be the normal bundle, that is, $X_p^\perp + p = X^\perp(p)$ is the hyperplane through $p \in V$ orthogonal to X, and $X_\epsilon^\perp$ denotes the ϵ-ball around p in $X(p)$. For small-enough ϵ_0 this gives well-defined projections $\pi_p \colon \psi^{[-1,1]}(X_{\epsilon_0}^\perp) \to X^\perp$, $\psi^t(x) \mapsto x$ for $|t| \le 1$, $x \in X_{\epsilon_0}^\perp$. We also denote by $C_\alpha^\perp(N, V)$ the space of continuous $\beta \colon N \to V$ such that $\beta(x) \in X^\perp(\alpha(x))$ for $x \in N$.

We then seek a fixed point of

$$F \colon C_\alpha^\perp(N, V) \to C_\alpha^\perp(N, \mathbb{R}^n),\ \beta \mapsto \pi_\alpha \circ \psi^1 \circ \beta \circ \sigma^{-1} \colon x \mapsto \pi_{\alpha(x)}(\psi^1(\beta(\sigma^{-1}(x)))) \in X^\perp(\alpha(x)).$$

Represent $\beta \in C^0(Y, \mathbb{R}^n)$ by the vector field $v_\beta := \beta - \alpha \in X_{\alpha(\cdot)}^\perp$ (a section of the bundle $\{(y, X_{\alpha(y)}^\perp) \mid y \in N\}$ over N). In these terms F is represented by

$$F_\alpha \colon v \mapsto \pi_\alpha \circ \psi^1(\alpha \circ \sigma^{-1} + v \circ \sigma^{-1}) - \alpha =: (\underbrace{DF_{\alpha|_0}}_{\text{linear part}} + \underbrace{H}_{\text{higher-order terms}})(v).$$

Then $v = F_\alpha(v) = (DF_{\alpha|_0} + H)(v) \quad \Leftrightarrow \quad v = -((DF_\alpha)_0 - \mathrm{Id})^{-1} H(v) =: T(v).$

Lemma 5.4.3. *There are a neighborhood $V \supset \Lambda$, $\epsilon_0, \epsilon > 0$, and $R > 0$ independent of N, Ψ, α with $\|((DF_\alpha)_0 - \mathrm{Id})^{-1}\| < R$ when $d_{C^1}(\Phi, \Psi) < \epsilon_0$, $d_{C^0}(Y, X \circ \alpha) < \epsilon$.*

Proof. For $\delta > 0$ there are $\epsilon_0 > 0$, $\mu < 1$ and a neighborhood $V \supset \Lambda$ to which the splitting $T_\Lambda M = E^u \oplus E^s \oplus X$ extends (maybe not invariantly) for Ψ such that $d_{C^1}(\Phi, \Psi) < \epsilon_0$. Then with respect to $E^u \oplus E^s \oplus X$, we have

$$D\psi^1 = \begin{pmatrix} a_{uu} & a_{uu} & * \\ a_{us} & a_{ss} & * \\ * & * & b \end{pmatrix},$$

where $\|a_{uu}\|^{-1} < \mu$, $\|a_{ss}\| < \mu$, $\|a_{uu}\| < \delta^2\mu$, $\|a_{us}\| < \delta^2\mu$, and $\| * \| < \delta^2\mu$. With respect to the decomposition into unstable and stable vector fields in $X^\perp$

$$((DF_\alpha)_0 \xi)(y) = d\pi_\alpha D\psi^1|_{\alpha(\sigma^{-1}(y))} \xi(\sigma^{-1}(y)) \quad \text{splits into} \quad (DF_\alpha)_0 = \begin{pmatrix} A_{uu} & A_{uu} \\ A_{us} & A_{ss} \end{pmatrix},$$

where $d_{C^0}(\alpha,\psi^1\alpha\sigma^{-1})<\epsilon$ and $d_{C^1}(\Phi,\Psi)<\epsilon_0$ imply

$$\|A_{uu}\|^{-1}<\frac{1+\mu}{2},\quad \|A_{uu}\|<\delta\mu,\quad \|A_{us}\|<\delta\mu,\quad \|A_{ss}\|<\frac{1+\mu}{2}. \qquad \square$$

To show that T contracts, we control H. If $k_i(t):=H_i(v+th)$ (components with respect to the canonical basis in $\mathbb{R}^n$) then $k_i(1)-k_i(0)=\int_0^1 k_i'(t)\,dt$ gives

$$H(v+h)-H(v)=h\int_0^1 DH(v+th)\,dt=h\int_0^1 DF_\alpha|_{v+th}-DF_\alpha|_0\,dt.$$

ψ^1, hence F_α, is C^1, so there is a δ_0 with $\|DF_\alpha|_{v+th}-DF_\alpha|_0\|\le\frac{1}{2R}$ if $\|v\|$, $\|v+h\|<\delta_0$, $t\in[0,1]$.[26] Thus

$$\|v_1\|,\|v_2\|<\delta_0\Rightarrow\|T(v_1)-T(v_2)\|<\frac{1}{2}\|v_1-v_2\|.$$

With $\theta=\min(\delta,\delta_0)$ and $\epsilon<\theta/(2R)$ as in Lemma 5.4.3, $d_{C^0}(\alpha\sigma^1,\psi^1\alpha)<\epsilon$, which follows from $d_{C^0}(Y,X\circ\alpha)<\epsilon$, gives

$$\|T(0)\|<R\|H(0)\|=R\|\psi^1\circ\alpha\circ\sigma^{-1}-\alpha\|=Rd_{C^0}(\alpha,\psi^1\alpha\sigma^{-1})=Rd_{C^0}(\alpha\sigma^1,\psi^1\alpha)\le\frac{\theta}{2},$$

so T is a $1/2$-contraction on the closed ball of $X^\perp$-vector fields of norm up to δ_0. By the Contraction Mapping Principle (Proposition 12.1.3), T has a unique fixed point, which yields the desired β. Uniqueness of the fixed point implies transverse uniqueness; θ in the statement of the theorem is continuous and small because β, $\bar\beta$ and Ψ are continuous, Ψ has positive speed, and β, $\bar\beta$ are close. □

As promised, we first use this to prove topological stability.

Proof of Theorem 5.3.6. $h:=\beta$ from Theorem 5.4.1 with $\Lambda=M=N$, $\Psi=\Phi$, $\sigma=\Phi'$ sufficiently C^0-close to Φ, and $\alpha=\mathrm{Id}$ is the desired factor map (surjective since it is a continuous perturbation of Id). □

If we could apply the same reasoning to Φ' to get a factor map the other way around, we would expect it to be the inverse of the h in Theorem 5.3.6, which would then be a homeomorphism. This can indeed be done if Φ' is hyperbolic but that requires C^1-closeness, rather than C^0-closeness. Accordingly, while the previous application focused on approximate orbits by taking $\Phi=\Psi$ in Theorem 5.4.1, we now obtain a profound strengthening of Proposition 5.1.10 by C^1-perturbing Φ. This is interesting in no small part because where applications motivate the study of a dynamical system, those parts of its behavior are of particular interest that are *robust*, that is, which persist under small perturbations of the system and hence are

[26] δ_0 is determined by R alone, which is a measure of hyperbolicity. "More hyperbolicity" means smaller R and hence less constraint on δ_0.

not sensitive to the parameters of the model at hand. In this respect it is especially interesting if the entire orbit structure is topologically unchanged under small perturbations.

Definition 5.4.4 (Structural stability)**.** A flow Φ is said to be *structurally stable* if there is a C^1 neighborhood (Definition 1.6.18) U of Φ in the class of C^1 flows such that any flow in U is orbit-equivalent to Φ (see Definition 1.3.21).

We note that we have already encountered flows with this feature. This is explicit in restricted form in Proposition 1.4.5 and suggested as an exercise (about the damped pendulum) in Remark 1.6.17. When expressed globally rather than locally, the Hartman–Grobman Theorem (Corollary 12.4.11 for continuous time or Theorem 5.6.3 for perturbations of linear maps obtained from localization (Theorem 12.4.12)) is a similar instance. In the present context, the dynamics is incomparably more complicated, however.

Interest in structural stability first arose from Smale's agenda of classifying dynamical systems up to topological equivalence because this gives open (hence manageable) equivalence classes. Theorem 5.4.1 implies that hyperbolic systems are structurally stable, though not in the exact same sense, except in the case of Anosov flows:

Theorem 5.4.5 (Strong structural stability of hyperbolic sets)**.** *Suppose Λ is a compact hyperbolic set for a C^1 flow Φ on M. Then there are*

- *a C^1-neighborhood U of Φ,*
- *a C^0-neighborhood V of the inclusion ι of Λ in M (which can be viewed as the identity) and*
- *a continuous map $h\colon U \to C(\Lambda, M)$, $\Psi \mapsto h_\Psi$*

such that $h_\Phi = \iota$ and for each $\Psi \in U$

1. *h_Ψ is a continuous embedding,*
2. *h_Ψ is the transversely unique map in V for which $\psi^{\tau(t)} \circ h_\Psi = h_\Psi \circ \varphi^t \restriction_\Lambda$, where τ is as in Theorem 5.4.1,*
3. *$\Lambda_\Psi := h_\Psi(\Lambda)$ is a hyperbolic set for Ψ.*

Definition 5.4.6. The map $\Psi \mapsto \Lambda_\Psi$ is called the *continuation* of Λ.

Proof. We use symmetry and uniqueness by applying the Shadowing Theorem 5.4.1 three times.

With $0 < \epsilon < \delta_0/2$ as in the Shadowing Theorem, $Y = \Lambda$, $\alpha = \mathrm{Id}\restriction_\Lambda$, $\sigma = \Phi$, we get a transversely unique $\beta\colon \Lambda \to V$ and a monotone τ with $\beta \circ \varphi^t = \psi^{\tau(t)} \circ \beta$. By Proposition 5.1.10 $\Lambda' := \beta(\Lambda)$ is hyperbolic.

To show that β is injective apply the Shadowing Theorem the other way around: With ϵ as before, $Y = \Lambda'$, $\alpha' = \mathrm{Id}\restriction_{\Lambda'}$, interchange Φ and Ψ (which we can do if ϵ is small enough) to obtain β' with $\beta' \circ \psi^t \restriction_{\Lambda} = \varphi^{\tau'(t)} \circ \beta'$.

To see that β is a homeomorphism note that $h := \beta' \circ \beta$ satisfies

$$\beta' \circ \beta \circ \varphi^t = \beta' \circ \psi^{\tau(t)} \circ \beta = \varphi^{\tau'(\tau(t))} \circ \beta' \circ \beta,$$

where $\tau'(\tau(t))$ is increasing and close to t. Since Id does the same ($\mathrm{Id} \circ \varphi^t = \psi^t \circ \mathrm{Id}$), transverse uniqueness in Theorem 5.4.1 implies that $\beta'(\beta(x)) = \psi^{\theta(x)}\,\mathrm{Id}(x)$, where θ can be taken increasing by the last argument in the proof of Theorem 1.7.5, and then $\beta' \circ \beta$ is surjective by continuity of θ. □

Corollary 5.4.7. *Anosov flows are structurally stable. The orbit-equivalence is unique up to small time-shifts when chosen near the identity.*

Remark 5.4.8. This proof of structural stability ultimately relies on the Contraction Mapping Principle because this was the main device used in the proof of the Shadowing Theorem. The fixed point of a contraction depends smoothly on the contraction when this is meaningful in a given application, and accordingly, it turns out that the conjugacy given by structural stability of a hyperbolic C^{k+1} embedding depends C^k on the perturbation (in the C^0 topology for conjugacies) [**207**].[27]

Remark 5.4.9. It is natural to ask whether for small-enough perturbations the conjugacy is more regular. A suggestive result by Palis–Viana [**225**] and de la Llave (unpublished) says that in dimension 2 (and discrete time) the Hölder exponent of the conjugacy is close to 1, that is, for each $\alpha \in (0,1)$ there is a neighborhood of the dynamical system such that the conjugacy between the respective hyperbolic sets has Hölder exponent α for each perturbation in this neighborhood. de la Llave further showed, however, that this fails in higher dimension.

Another question about structural stability is how large a perturbation can be without ceasing to be orbit-equivalent to the given flow. Numerous subjects in dynamics would benefit from general results along these lines, but there are few. Among them is a criterion for magnetic flows (Definition 5.1.13) that is easy to apply.

Theorem 5.4.10 ([**138**, Théorème 3.2])**.** *If M is a Riemannian manifold whose sectional curvatures K satisfy $-k2^2 \le K \le -k_1^2 < 0$, then all magnetic flows with $\|\mathfrak{m}\|_\infty < k_1$ are pairwise orbit-equivalent.*

[27] Here the loss of one derivative is related to the fact that an operator such as $\beta \mapsto g \circ \beta \circ \sigma^{-1}$ in the proof of the Shadowing Theorem that involves a composition is C^k if the maps in question are C^{k+1}.

We later address the question of how regular (beyond continuity) the orbit-equivalence is (that is, the homeomorphism in its definition). Note that this is not a well-defined question because it is not "the" homeomorphism—we only have transverse uniqueness. Therefore the question is how regular this h can be chosen. Proposition 1.3.27 describes the extent to which altering a given choice of orbit-equivalence can improve the regularity of the dependence on time. Transverse regularity is fixed by transverse uniqueness, and we will study that later (Theorem 7.3.3).

One of the crowning achievements in dynamical systems is well beyond the scope of this text but we state it here to complete the picture: Hyperbolicity is indeed *equivalent* to structural stability as follows:

Theorem 5.4.11 (Hayashi [**153**, **154**, **288**])**.** *A C^1 flow Φ is structurally stable iff Φ satisfies Axiom A and* strong transversality:[28] $W^s(\gamma_1) \cap W^u(\gamma_2) = W^s(\gamma_1) \pitchfork W^u(\gamma_2)$ *for any orbits γ_1 and γ_2 in $NW(\Phi)$.*

We next produce a corresponding "global" stability result (Theorem 5.4.13), albeit only the "if" part. (The major achievement of Hayashi is the much harder "only if" direction.) This is a proper "global" counterpart to the Structural Stability Theorem 5.4.5. Not only does a compact locally maximal hyperbolic set persist under perturbation, and with topologically identical dynamics, but for an Axiom A flow with the no-cycles property the nonwandering set of a perturbation is topologically and dynamically the same as that for the original flow. This is called Ω-stability because the original notation for the nonwandering set was Ω. Our preceding work easily produces a nominally stronger version of this classical result (stability of $\mathscr{R}(\Phi)$ rather than just of $NW(\Phi)$), but we refer to it by the original name.

Definition 5.4.12. For $r \geq 1$ a C^r flow Φ is said to be (C^r-)Ω*-stable*, and $\mathscr{R}(\Phi)$ is said to be (C^r-)*stable* if for any flow Ψ that is sufficiently C^r close to Φ (see Definition 1.6.18) there is an orbit-equivalence (in a given neighborhood of the identity) between $\Phi\restriction_{\mathscr{R}(\Phi)}$ and $\Psi\restriction_{\mathscr{R}(\Psi)}$.

Theorem 5.4.13 (Ω-Stability Theorem)**.** *C^1 hyperbolic flows*[29] *are C^1-Ω-stable.*

Corollary 5.4.14. *Hyperbolicity (Definition 5.3.48) is a C^1-open condition.*[30]

Proof. By the Spectral Decomposition Theorem 5.3.35 there are disjoint basic sets $\Lambda_1, \ldots, \Lambda_m$ for Φ such that $\mathscr{R}(\Phi) = \bigcup_{i=1}^m \Lambda_i$, and without cycles. Furthermore, $\Lambda_i = W^u(\Lambda_i) \cap W^s(\Lambda_i)$ for each i, and we choose isolating open sets U_i for Λ_i with pairwise disjoint closures.

[28] This notion uses stable and unstable *manifolds*; see Theorem 6.1.1.

[29] as in Definition 5.3.48

[30] See also Theorem 6.1.6.

If Ψ is a flow sufficiently C^1-close to Φ, then by the Structural Stability Theorem 5.4.5 there are disjoint hyperbolic basic sets $\tilde{\Lambda}_i \subset U_i$ for Ψ that are orbit equivalent to Λ_i for each $i \in \{1, \dots, m\}$, and for which the U_i are isolating neighborhoods. Therefore, it suffices to show that $\mathscr{R}(\Psi) \subset \bigcup_{i=1}^m U_i$.

We show $NW(\Psi) \subset \bigcup_{i=1}^m U_i$ (with a proof different from [**247**]), so Ψ satisfies Axiom A with no cycles, and $\mathscr{R}(\Psi) = NW(\Psi) \subset \bigcup_{i=1}^m U_i$ (Remark 5.3.46).

Suppose to the contrary that there is a sequence of flows $\Phi_n \xrightarrow[n\to\infty]{C^r} \Phi$ such that for each n there is a nonwandering point of Φ_n in $M \smallsetminus \bigcup_{i=1}^m U_i$. Since $M \smallsetminus \bigcup_{i=1}^m U_i$ is compact, there exists a point $p \in M \smallsetminus \bigcup_{i=1}^m U_i$ and a sequence of points $p_n, q_n \in M$ converging to p (by possibly replacing Φ_n with a subsequence) and a sequence of times $t_n \xrightarrow[n\to\infty]{} \infty$ with $\varphi_n^{t_n}(p_n) = q_n$. Then there exist some $i_0 \neq i_1 \in \{1, \dots, m\}$ with $p \in W^u(\Lambda_{i_0}) \cap W^s(\Lambda_{i_1})$ by Proposition 5.3.38.

By continuity of the flow and passing to a subsequence there is a sequence of times $t_n^1 \in (0, t_n)$ such that $\varphi_n^{t_n^1}(p_n) \xrightarrow[n\to\infty]{} \Lambda_{i_1}$. Since $\varphi_n^{t_n}(p_n) = q_n \to p$, there is a sequence $T_n^1 \in (t_n^1, t_n)$ such that $\varphi_n^{T_n^1}(p_n) \in \partial U_{i_1}$ and $\varphi_n^{[t_n^1, T_n^1)}(p_n) \subset U_i$. Since Λ_{i_1} is Φ-invariant and $\varphi_n^{T_n^1}(p_n) \notin U_{i_1}$, we have $T_n^1 - t_n^1 \xrightarrow[n\to\infty]{} \infty$.

By compactness of ∂U_{i_1} there is a subsequence with $\varphi_n^{T_n^1}(p_n) \to x_1 \in \partial U_{i_1}$. Since $T_n^1 - t_n^1 \to \infty$ we know that $\mathscr{O}^-(x_1) \subset U_{i_1}$ and so $x_1 \in W^u(\Lambda_{i_1})$. Since $x_1 \notin \mathscr{R}(\Phi)$ there is an i_2 with $x_1 \in W^s(\Lambda_{i_2})$ and $i_2 \notin \{i_0, i_1\}$ by construction: $i_2 \neq i_1$ because $x_1 \in W^u(\Lambda_{i_1}) \smallsetminus \Lambda_{i_1} \Rightarrow x_1 \notin W^s(\Lambda_{i_1})$, and $i_2 \neq i_0$ because otherwise we have a cycle and are done.

Arguing likewise with x_1 gives a sequence of times $t_n^2 \xrightarrow[n\to\infty]{} \infty$ such that $t_n^2 \in (T_n^1, t_n^1)$ and $\varphi_n^{t_n^2}(p_n) \xrightarrow[n\to\infty]{} \Lambda_{i_2}$, and a sequence of times $T_n^2 \in (t_n^2, t_n)$ such that $\varphi_n^{T_n^2}(p_n) \in \partial U_{i_2}$ and $\varphi_n^t(p_n) \in U_{i_2}$ for $t \in [t_n^2, T_n^2)$. Furthermore, as before we have $T_n^2 - t_n^2 \to \infty$ as $n \to \infty$ by possibly taking a subsequence.

As before by taking a subsequence as necessary we obtain $\varphi_n^{T_n^2}(p_n) \xrightarrow[n\to\infty]{} x_2 \in \partial U_{i_2}$, and $x_2 \in W^u(\Lambda_{i_2}) \cap W^s(\Lambda_{i_3})$ for some $i_3 \notin \{i_0, i_1, i_2\}$: $i_3 \neq i_2$ because $W^u(\Lambda_{i_2}) \cap W^s(\Lambda_{i_2}) = \Lambda_{i_2}$ and $i_3 \notin \{i_0, i_1, i_2\}$ because otherwise we have a cycle and are done.

Continuing in this manner we obtain a sequence of points $\{x_n\}$, which for $n \geq m$ contradicts the no-cycles assumption. Hence, $\mathscr{R}(\Psi) = \bigcup_{i=1}^m \tilde{\Lambda}_i$ for Ψ sufficiently C^r close to Φ. □

We previously mentioned the Structural-Stability Theorem 5.4.11 as a high point in dynamics, and it comes with a counterpart to Theorem 5.4.13—C^1-Ω-stability characterizes hyperbolicity:

Theorem 5.4.15 (Hayashi [**153**, **154**, **288**]). *C^1-Ω-stable flows are hyperbolic.*[31]

These results (together with the discrete-time counterparts) were long known as the (Palis–)Smale Stability conjectures. While a proof is far outside our scope, we briefly discuss some of the ingredients.

PROOF INGREDIENTS. The principal contribution was Mañé's proof of the discrete-time counterpart to Theorem 5.4.11, on which Palis quickly built his proof of the discrete-time counterpart of Theorem 5.4.15. Hayashi overcame the formidable additional difficulties for flows.

A promising approach to Theorem 5.4.15 is to use contraposition: use any failure of hyperbolicity to make a change in the orbit structure that disproves Ω-stability. This is still a formidable task, and Axiom A (for instance) also requires density of periodic points $NW(\Phi)$.

To see this, a simple general principle is helpful: Any C^1-generic property of $NW(\Phi)$ that is invariant under orbit-equivalence holds for Ω-stable flows—because invariance means that it either holds for all flows in a C^1-neighborhood (and hence for the Ω-stable flow itself) or it fails for every flow in a neighborhood—contrary to its genericity. Applying this to the Pugh's General Density Theorem 1.5.19 tells us that one part of Axiom A automatically holds for Ω-stable flows: $\overline{\mathrm{Per}(\Phi)} = NW(\Phi)$.

With Theorem 6.1.6, this principle further tells us that fixed and periodic points of Ω-stable flows are hyperbolic. With this, it looks like we are close. Yet lots of hard work awaits—eased, however by having much hyperbolicity at our disposal.

The fundamental hurdle in the case of flows, which has no discrete-time counterpart, is the need to rule out fixed points in the closure of the periodic ones (Definition 5.1.1). Thus, in order to otherwise follow the strategy implemented by his predecessors in the discrete-time context, Hayashi needed to prove:

Theorem 5.4.16 ([**154**]). $\overline{\mathrm{Per}(\Phi)}$ *contains no fixed points.*

It is here where his most prominent contribution enters.

Theorem 5.4.17 (Hayashi Connecting Lemma [**154**]). *Consider a flow Φ with an isolated hyperbolic set Λ which has an almost homoclinic point, that is to say, $\big(\overline{W^s(\Lambda)} \cap W^u(\Lambda)\big) \cup \big(\overline{W^u(\Lambda)} \cap W^s(\Lambda)\big) \smallsetminus \Lambda \neq \varnothing$.*

Then for every C^1 neighborhood U of Φ there is a $\Psi \in U$ that agrees with Φ on a neighborhood of Λ and has a homoclinic point, that is, $\big(W^u(\Lambda) \cap W^s(\Lambda)\big) \smallsetminus \Lambda \neq \varnothing$.

The proof uses delicate perturbations along the lines of the Pugh Closing Lemma (Theorem 1.5.18). This helps prove Theorem 5.4.16 as follows. If periodic points accumulate on a fixed point, then that (hyperbolic) fixed point comes with an almost-homoclinic point (Figure 5.4.1); the Connecting Lemma perturbs this

[31] as in Definition 5.3.48

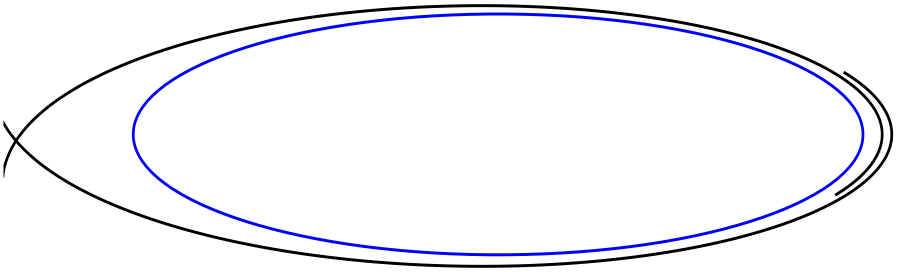


FIGURE 5.4.1. A periodic orbit close to a hyperbolic fixed point

to a homoclinic point, which is nonwandering but not transverse, an unstable phenomenon that thereby rules out Ω-stability. □

Remark 5.4.18. We emphasized the shadowing property as an essential mechanism for a number of core features of hyperbolic dynamics, and while we clarified that what was mainly being used was the combination of shadowing and expansivity, we used them in the full strength provided by hyperbolicity in the first place. (We note that Lipschitz Shadowing implies structural stability [**237**], but the proof is on a completely different level from anything we have done here, in that it uses Theorem 5.4.11.)

We recall that the exposition in this chapter was guided by wanting to show the power of the Anosov–Katok approach in which shadowing (with uniqueness or expansivity) are used to develop the topological dynamics and stability of hyperbolic sets [**175**]. Another approach to doing so without first introducing the invariant foliations directly uses the contraction principle to produce the desired structures and traces back to Moser and Mather; it is well-presented in the concise introduction by Yoccoz [**291**]. Finally, we have on occasion brushed up against the limitations of this approach, and these are averted by building on the invariant foliations early on.

We conclude with brief remarks (without proof) on the implications of these properties per se rather than with the additional strength in which we saw them, in some cases with the additional strong requirement that these properties hold robustly. That is, we examine the implications of expansivity, sometimes together with shadowing without Lipschitz shadowing or uniqueness, and sometimes in restriction to the interior of the set of systems possessing these features.

We first illustrate the potential power of expansivity alone:

Theorem 5.4.19 (Mañé [**209**])**.** *For a C^1-residual set of flows on a given compact manifold (that is, C^1-generically)*

- *expansivity implies hyperbolicity [**269**] (in fact, a like conclusion holds assuming only "measure-expansivity" [**199**], that is, one allows a null set of exceptions to expansivity in a suitable way);*
- *if the homoclinic class of a hyperbolic periodic orbit is expansive and isolated, then it is hyperbolic [**200**];*
- *a measure-expansive locally maximal homoclinic class is hyperbolic [**199**].*

Since we have not discussed genericity results much, we should note that genericity in the C^1-topology is appealing in that it can produce interesting phenomena as well as results like this one, but that genericity in finer topologies is much harder, so there is often a desire to go from C^1-genericity results either to analogous results in a finer topology or a more explicit description of the exceptions to the generic circumstance.

Results about all systems (rather than generic ones) can be obtained, for instance, if instead of expansivity one assumes stable (or robust) expansivity. Put differently, these would be results not about the collection of expansive systems but about the interior of this collection, usually in the C^1-topology. The first such result is in the volume-preserving category:

Theorem 5.4.20 ([**3**, **4**])**.** *C^1-robustly expansive volume-preserving flows are Anosov.*

Absent volume-preservation, there are a few characterizations of robust expansivity with or without shadowing:

Theorem 5.4.21 ([**216**])**.** *Consider a flow Φ on a compact manifold.*
The following are equivalent:

- *Φ is C^1-robustly expansive and has the shadowing property,*
- *Φ is C^1-robustly expansive and structurally stable,*
- *Φ is Anosov.*

The following are equivalent:

- *Φ is C^1-robustly expansive,*
- *Φ is quasi-Anosov,*
- *Φ has no fixed points and is Axiom A and quasi-transverse.*

Here, we used:

Definition 5.4.22. An invariant set Λ of a flow Φ is said to be *quasi-hyperbolic* if $\|D\varphi^t(v)\|_{t\in\mathbb{R}}$ is unbounded for $T_\Lambda M \ni v \perp \frac{d\varphi^t}{dt}|_{t=0}$ and *quasi-transverse* if $T_xW^u(x)\cap T_xW^s(x)=\{0\}$ for $x\in\Lambda$. If $\Lambda = M$, then Φ is quasi-Anosov and quasi-transverse, respectively.

It should be noted that robustness of any dynamical property is a severe strengthening over just assuming that property by itself, and this, rather than merely the strength of expansivity, is manifest in the preceding results. We illustrate this with a 3-dimensional counterpart that involves topological transitivity:

Theorem 5.4.23 ([**102**]). *Robustly transitive 3-flows are Anosov.*

We now remark on our definition of hyperbolicity of a flow as hyperbolicity of the chain-recurrent set (Definition 5.3.48). It turns out that there is a sufficient (and obviously necessary) criterion for this that may be easier to verify.

Theorem 5.4.24 ([**86**, **264**]). *A compact Φ-invariant set Λ is hyperbolic if it is quasi-hyperbolic and $\Phi\restriction_{\Lambda}$ is chain-recurrent.*

Indeed, this gives an alternative proof that the geodesic flow of a negatively curved Riemannian manifold is Anosov [**86**, Theorem 4.1].

At the end of Section 4.2 we raised the question of how topological entropy depends on a flow, promising that for hyperbolic flows this plays out better than it does in general. Structural stability is the key ingredient.

Theorem 5.4.25 (Continuity of entropy). *The topological entropy of a hyperbolic flow changes continuously under C^1-perturbations.*

We should say that this is an aspect of dynamical systems that plays out rather differently in discrete time: Structural stability gives a conjugacy in that case, so topological entropy is locally constant. We are now investigating a subject that is quite specific to flows.

For Anosov flows, one can go well beyond continuous dependence, and we will describe some pertinent results now. Structural stability is a central ingredient, though some pertinent results can be obtained without it. In this context, however, our approach to structural stability shows a weakness: Our proof gives limited information on how the orbit-equivalence depends on the perturbation, other than continuously. Another proof does; it is due to Moser and obtains the orbit-equivalence by applying the Contraction Principle/Implicit Function Theorem directly to the problem rather than taking a detour through shadowing (see page 288). Corollary 12.4.11 is another exemplar of this approach, and the detailed information on how the fixed point of a contraction varies with the contraction (Proposition 12.1.3) indicates that with this approach one expects the orbit-equivalence to depend smoothly on an Anosov flow in a suitable sense—and when chosen properly, of course; as we noted, there is flexibility in the orbit direction. The implications for topological entropy are astonishing:

Theorem 5.4.26 (Smoothness of entropy [**171**, **172**, **174**]). *Suppose $s \mapsto \Phi_s$ is a 1-parameter family of Anosov flows for $s \in (-\epsilon, \epsilon)$.*

- *If $s \mapsto \Phi_s$ and Φ_0 are C^{k+1} and $1 \le k \le \infty$, then $s \mapsto h_{\text{top}}(\Phi_s)$ is C^k.*
- *If $s \mapsto \Phi_s$ and Φ_0 are C^1, then so is $s \mapsto h_{\text{top}}(\Phi_s)$.*
- *If $s \mapsto \Phi_s$ and Φ_0 are analytic, then so is $s \mapsto h_{\text{top}}(\Phi_s)$ is C^k.*

That geodesic flows of negatively curved manifolds are Anosov flows gives immediate applications in that context. However, for these, one can strengthen the results.

Theorem 5.4.27 ([**174**, Theorem 1])**.** *Suppose (M, g) is a C^2 closed Riemannian manifold without conjugate points, and $s \mapsto g_s$ is a C^1 (!) perturbation by metrics without conjugate points. Then $s \mapsto h_{\text{top}}(g_s)$ is Lipschitz continuous.*[32]

Theorem 5.4.28 ([**174**, Theorem 3 & Remark c) + [**185**]])**.** *If (M, g_0) is a C^2 closed Riemannian manifold of nonpositive sectional curvature and $s \mapsto g_s$ is a C^2 perturbation, then $s \mapsto h_{\text{top}}(g_s)$ is C^1.*

Once a function is differentiable, one can aim to compute and use the derivative. Indeed, [**174**] obtains derivative formulas, and criteria for vanishing of the derivative give interesting precursors to rigidity results in Section 10.4.

5. The Mather–Moser method*

Sections 5.3 and 5.4 developed the topological orbit structure of hyperbolic sets, including the notion of a hyperbolic flow itself, plus structural (and omega-) stability, and an underlying agenda was to do so using shadowing and expansivity as the source of all these phenomena. To give a fuller picture of the methods used in hyperbolic dynamics we briefly show an alternate route to structural stability. The most focused presentation of this approach due in large part to Moser and Mather develops the core theory in a self-contained way and establishes the Hartman–Grobman Theorem, expansivity, structural stability, the Shadowing Lemma, stable/unstable manifolds, local product structure, spectral decomposition in that order (and in discrete time) [**291**] (see also [**270**]). In this section we limit ourselves to structural stability for illustrative purposes, but we use the same approach to establish Corollary 12.4.11, which we also use here to establish expansivity, and which underlies the next section.

The beginning of this section introduces notation, terminology and basic facts that will be useful elsewhere as well; the core starts with Remark 5.5.9 on page 287.

It is not essential but helpful to define partial hyperbolicity here (for the discrete-time context). It will be convenient to use the following notation.

[32] Here we used shorthand: $h_{\text{top}}(g_s)$ is the topological entropy of the geodesic flow of g_s.

Definition 5.5.1 (Conorm)**.** We define the *conorm* $\lfloor\!\lfloor A \rfloor\!\rfloor$ of a linear map A by

$$\lfloor\!\lfloor A \rfloor\!\rfloor := \inf\big\{\|Av\|/\|v\| \;\big|\; \|v\| = 1\big\}.$$

This is complementary to the usual norm $\|A\| := \sup\big\{\|Av\|/\|v\| \;\big|\; \|v\| = 1\big\}$.

Definition 5.5.2. An embedding f is said to be *partially hyperbolic* on Λ (in the narrow sense) if there exists a Riemannian metric called a *Lyapunov metric* in an open neighborhood U of Λ for which there are numbers[33]

$$0 < \lambda < \zeta \le \xi < \mu \text{ with } \lambda < 1 < \mu \tag{5.5.1}$$

and a pairwise orthogonal invariant splitting into stable, center and unstable directions

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x), \quad d_xfE^\tau(x) = E^\tau(f(x)),\ \tau = s, c, u \tag{5.5.2}$$

such that

$$\|d_xf \restriction E^s(x)\| \le \lambda < \zeta \le \lfloor\!\lfloor d_xf \restriction E^c(x) \rfloor\!\rfloor \le \|d_xf \restriction E^c(x)\| \le \xi < \mu \le \lfloor\!\lfloor d_xf \restriction E^u(x) \rfloor\!\rfloor.$$

In this case we set $E^{cs} := E^c \oplus E^s$ and $E^{cu} := E^c \oplus E^u$.

Remark 5.5.3. This is equivalent to requiring that for any Riemannian metric there is a constant C for which there are numbers λ_i, μ_i, $i = 1, 2, 3$ as in (5.5.1) and an invariant splitting (5.5.2) such that

$$\|d_xf^n \restriction E^s(x)\| \le C\lambda^n,$$
$$C^{-1}\zeta^n \le \lfloor\!\lfloor d_xf \restriction E^c(x) \rfloor\!\rfloor \le \|d_xf \restriction E^c(x)\| \le C\xi^n,$$
$$C^{-1}\mu^n \le \lfloor\!\lfloor d_xf \restriction E^u(x) \rfloor\!\rfloor.$$

Example 5.5.4. The time-1 map of an Anosov flow is partially hyperbolic.

It is useful to have a characterization of (partial) hyperbolicity in terms of the action of the differential on vector fields.

Theorem 5.5.5 (Mather)**.** *Let M be a smooth manifold, $U \subset M$ an open subset, $f\colon U \to M$ a C^1 embedding, and $\Lambda \subset U$ a compact f-invariant set. Denote by Γ_b the set of bounded vector fields on Λ and by $\Gamma_c \subset \Gamma_b$ the set of continuous vector fields on Λ (these are sections of the bundle $T_\Lambda M := TM_{\restriction\Lambda}$), and for a vector field X on Λ define $\mathscr{F}(X)$ by*

$$\mathscr{F}(X)(f(x)) := Df_x(X(x)).$$

Then for $\ell^- < \ell^+$ the following are equivalent:

(1) *There exist $\lambda < \ell^-$ and $\mu > \ell^+$ such that Λ is (partially)* hyperbolic *with λ, μ as in Definition 5.5.2.*

[33] We chose the letters ζ and ξ for the middle numbers because ξ looks “just a little bigger” than ζ.

(2) $\operatorname{sp}(\mathscr{F}\upharpoonright_{\Gamma_b}) \cap \{z \in \mathbb{C} \mid \ell^- \le |z| \le \ell^+\} = \varnothing$.

(3) $\operatorname{sp}(\mathscr{F}\upharpoonright_{\Gamma_c}) \cap \{z \in \mathbb{C} \mid \ell^- \le |z| \le \ell^+\} = \varnothing$.

Proof. (1)⇒(2): Check that the splitting $\Gamma_b(T_\Lambda M) = \Gamma_b(E^\lambda) \oplus \Gamma_b(E^\mu)$ has the desired properties.

(2)⇒(3): Since $\Gamma_c \subset \Gamma_b$ is an invariant Banach subspace, $\operatorname{sp}(\mathscr{F}\upharpoonright_{\Gamma_b}) \subset \operatorname{sp}(\mathscr{F}\upharpoonright_{\Gamma_b})$.

(3)⇒(1): This involves 2 simple steps.

Lemma 5.5.6. *The projections* $\pi^\pm$ *that define the splitting* $\Gamma_c = \mathscr{E}^\lambda \oplus \mathscr{E}^\mu$ *are* $C^0(\Lambda)$*-linear.*

A map $L\colon \Gamma_c \to \Gamma_c$ is said to be $C^0(\Lambda)$-linear if $L(\varphi X) = \varphi \cdot L(X)$ for all $\varphi \in C^0(\Lambda)$. This lets us apply a general fact about continuous maps of bundles.

Lemma 5.5.7. *A* $C^0(\Lambda)$*-linear map* $L\colon \Gamma_c \to \Gamma_c$ *is pointwise defined, that is, there is a continuous family* $(L_x\colon T_xM \to T_xM)_{x\in\Lambda}$ *of linear maps such that* $L(X)(x) = L_x(X(x))$ *for all* $x \in \Lambda$.

Now, Lemma 5.5.6 provides the hypotheses for Lemma 5.5.7 applied to $\pi^\pm$, so we obtain fiberwise linear maps $\pi_x^\pm$, and these are complementary projections since $\pi^\pm$ are (check that $(\pi^\pm)^2 = \pi^\pm$ and $\pi^- + \pi^+ = \mathrm{Id}$ imply the same for $\pi_x^\pm$). This gives continuous subbundles $E_x^\lambda \coloneqq \pi_x^+(T_xM)$ and $E_x^\mu \coloneqq \pi_x^-(T_xM)$ with the desired properties. □

Proof of Lemma 5.5.6. The main point is that the subspaces $\mathscr{E}^\lambda$ and $\mathscr{E}^\mu$ are $C^0(\Lambda)$-closed: If $X \in \mathscr{E}^\lambda$ and $\varphi\colon \Lambda \to \mathbb{R}$ is continuous (hence bounded), then $\varphi X \in \mathscr{E}^\lambda$ because $\mathscr{F}^n(\varphi X) = \varphi \circ f^{-n} \cdot \mathscr{F}^n(X)$. Thus $\Gamma_c = \mathscr{E}^\lambda \oplus \mathscr{E}^\mu$ as $C^0(\Lambda)$-modules; since $\pi^\pm$ is $C^0(\Lambda)$-linear on $\mathscr{E}^\lambda$ and $\mathscr{E}^\mu$ (it is 0 or Id), the claim follows. □

Proof of Lemma 5.5.7. If $X \equiv 0$ on an open set U then $\pi^\pm(X) = 0$ on U: For $x \in U$ take $\varphi \in C^0(\Lambda)$ such that $\varphi(x) = 1$ and $\varphi X \equiv 0$ to get

$$\pi^\pm(X)(x) = 1 \cdot \pi^\pm(X)(x) = \varphi(x) \cdot \pi^\pm(X)(x) = \pi^\pm(\varphi X)(x) = \pi^\pm(0)(x) = 0.$$

If $X \in \Gamma_c$ and $X(x) = 0$ take $X_n \to X$ with $X_n = 0$ on $B(x, 1/n)$ and hence $\pi^\pm(X)(x) = \lim \pi^\pm(X_n)(x) = 0$.

If $(x, v) \in T_\Lambda M$, $X \in \Gamma_c$ and $X(x) = v$, then $\pi_x^\pm(v) \coloneqq \pi^\pm(X)(v)$ is thus independent of such X. □

Definition 5.5.8 (Fibered linear automorphisms)**.** Suppose K is a compact mertic space, $\pi\colon E \to K$ a finite-dimensional vector bundle, $f\colon K \to K$ a homeomorphism. Then $F\colon E \to E$ is called a linear automorphism of E fibered over f if for every $x \in K$ the restriction F_x of F to $E_x \coloneqq \pi^{-1}(x)$ is a linear isomorphism onto $E_{f(x)}$ depending continuously on x. We denote by $\Gamma_b(E)$ and $\Gamma_c(E)$ the (Banach) space of bounded, respectively continuous, sections of π. The action $\mathscr{F}$ of F on sections X of π is given by

$$F_x(X(x)) = \mathscr{F}(X)(f(x)).$$

($\mathscr{F}$ is linear and preserves Γ_b and Γ_c.)

Remark 5.5.9 (Transverse bundle)**.** The setting of Theorem 5.5.5 is an instance of this situation, with E being the tangent bundle and $F = Df$. For invariant sets $\Lambda \subset M$ of flows, a related useful bundle is the *transverse bundle* $T^\Phi_\Lambda M$ defined by $T^\Phi_x M = T_x M / \dot{\varphi}$, the linear space $T_x M$) modulo the flow direction, which inherits a norm or inner product from $T_x M$. Theorem 5.5.5 tells us that time-t maps of hyperbolic flows induce a *hyperbolic* action F on the transverse bundle E, that is, there is an invariant splitting

$$E(x) = E^s(x) \oplus E^u(x), \quad FE^\tau(x) = E^\tau(f(x)),\ \tau = s, u$$

such that

$$\|F\restriction_{E^s(x)}\| \le \lambda < 1 < \mu \le \lfloor\!\lfloor F\restriction_{E^u(x)} \rfloor\!\rfloor.$$

Theorem 12.4.8 applied to $\mathscr{G}$ defined by $G_x(X(x)) = \mathscr{G}(X)(f(x))$ gives

Theorem 5.5.10 (Invariant section)**.** *If $F\colon E \to E$ is a hyperbolic linear automorphism fibered over a homeomorphism f of K and $G\colon E \to E$ is fibered over f such that $\ell := L(G - F) < \epsilon := \min(1 - \lambda, 1 - \mu^{-1})$ (see Definition 12.1.1), then there is a unique bounded section X of E such that $\mathscr{G}(X) = X$, and X is continuous with $\|X\| \le (\epsilon - \ell)^{-1} \sup_{x \in K} \|G_x(0)\|$.*

Localization (Theorem 12.4.12) provides applications of results like this to a compact hyperbolic set of diffeomorphisms, with G being the localization of Df—variously on the tangent bundle or the transverse bundle. For instance, the Hartman–Grobman implies expansivity without first establishing shadowing with uniqueness: With $K := \Lambda^U_\varphi$ (as in Proposition 5.1.10) hyperbolic, the localization G on the transverse bundle of $D\varphi^t$ from Theorem 12.4.12 is Lipschitz-close to $D\varphi^t$ on the transverse bundle, so we can apply the Hartman–Grobman Theorem (Corollary 12.4.11) to conclude that for $x \in K$, $v \in E_x \smallsetminus \{0\}$ the $\mathscr{G}$-orbit of the section X with $X(x) = v$, $X(y) = 0$ for $y \ne x$ is unbounded, so the orbit of $\exp_x v$ does not stay in localization neighborhoods around the orbit of x.

Structural stability is a like application of Theorem 5.5.10.

MATHER–MOSER PROOF OF THEOREM 5.4.5. By assumption, we can localize Ψ to a fibered action on the transverse bundle which is Lipschitz-close to that of

$D\Phi$. The unique bounded (and then continuous) section X from Theorem 5.5.10 is continuous in Ψ and gives the orbit-equivalence h by

$$(h(x) = \exp_x(X(x)). \tag{5.5.3}$$

As in Theorem 5.3.7, h is injective by expansivity □

6. The Hartman–Grobman Theorem

Returning to a much more modest (and local) context, we now closely explore how well the dynamics near a hyperbolic fixed point (Definition 1.1.24) is described by the dynamics of the linearization. This can be viewed as a local counterpart to structural stability, but there are interesting contrasts to point out. One is that there is an extension that is not at all perturbative, and the other is that, being local, this result can produce a conjugacy rather than an orbit-equivalence.

Specifically, we now show how the Hartman–Grobman Theorem 12.4.14 for discrete-time systems translates to a corresponding result for flows. One such translation is too straightforward to state as a separate result: the return map to a transversal through a hyperbolic periodic point of a flow is a map with a hyperbolic fixed point, so the Hartman–Grobman Theorem 12.4.14 applies directly to the return map. We here develop the other application, to hyperbolic fixed points of flows. The purpose is twofold: It provides insight into the dynamics of a flow near a hyperbolic fixed point, but it also illustrates a mechanism by which conjugacies between time-1 maps are conjugacies between flows in some generality.

Theorem 5.6.1 (Hartman–Grobman)**.** *Let M be a smooth manifold, Φ a continuously differentiable flow on M and $p \in M$ a hyperbolic fixed point of Φ. Then for each $T > 0$ there exist neighborhoods U of p and V of $0 \in T_pM$ as well as a homeomorphism $h\colon U \to V$ such that $\varphi^t = h^{-1} \circ D\varphi_p^t \circ h$ on U for all $t \in [-T, T]$.*

Proof. Without loss of generality (Theorem 12.4.12) assume that $M = \mathbb{R}^n$, $p = 0$, $L^t := D_0\varphi^t$ is a hyperbolic linear map for $t \neq 0$, with $\Delta := \varphi^1 - L^1$ bounded, ϵ as in (12.4.1), and $\ell := L(\Delta) < \epsilon$. Corollary 12.4.11 gives a unique homeomorphism $h\colon E \to E$ with $h - \mathrm{Id}$ bounded and $h \circ L^1 = \varphi^1 \circ h$. It suffices to check that

$$\varphi^t \circ h \circ L^{-t} = h \quad \text{for all } t \in \mathbb{R} \tag{5.6.1}$$

because this establishes that h is a conjugacy between any of the time-t maps and its linearization; localization then translates this into the conclusion of Theorem 5.6.1. To check (5.6.1) for a given $t \in \mathbb{R}$ note that $\varphi^t \circ h \circ L^{-t}$ conjugates φ^1 and L^1:

$$\varphi^1 \circ [\varphi^t \circ h \circ L^{-t}] \circ L^{-1} = \varphi^t \circ \underbrace{[\varphi^1 \circ h \circ L^{-1}]}_{=h} \circ L^{-t} = \varphi^t \circ h \circ L^{-t}.$$

Uniqueness of such conjugacy implies $\varphi^t \circ h \circ L^{-t} = h$ by boundedness of

$$\varphi^t \circ h \circ L^{-t} - \mathrm{Id} = \underbrace{(\varphi^t - L^t)}_{=0 \text{ outside a compact set}} \circ h \circ L^{-t} + L^t \circ \underbrace{(h - \mathrm{Id})}_{\text{bounded}} \circ L^{-t}. \qquad \square$$

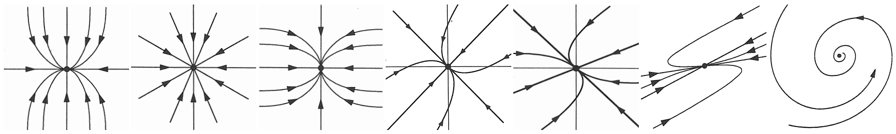


FIGURE 5.6.1. Pairwise locally conjugate attracting points

Remark 5.6.2. Figure 5.6.1 shows that topological linearization does not necessarily convey geometric information about the dynamics near an attracting fixed point. This raises the question of whether the homeomorphism in Theorem 5.6.1 has enough regularity to do so.

The Hartman–Grobman conjugacy is Hölder continuous (Definition 1.8.4) [**29**, **257**], a fact we will later use. Indeed, the conjugacy can be taken to have Hölder exponent arbitrarily close to 1 [**293**] when the flow is C^∞.

Furthermore, Hartman proved that the conjugacy is C^1 if the manifold is a surface (and it is clear in that case that the derivative at the fixed point is the identity). This is of interest because it tells us more than the topological dynamics near the fixed point: Figure 5.6.1 illustrates that the phase portrait of an attracting fixed point can look quite different depending on whether there are 2 distinct eigenvalues of the linear part, and if not, on whether there are 2 linearly independent eigenvectors or not; the eigenvectors themselves affect the phase portrait. Topologically, none of these distinctions can be detected, but having a differentiable conjugacy whose linear part is the identity tells us that the phase portrait for the flow looks very much like that of the linearization in every respect, save for gentle bending of the orbits as we move away from the fixed point. This means that often a fairly good phase portrait of a flow on a surface can be obtained by starting with thumbnails of linearized phase portraits at each hyperbolic fixed point. Example 7.7.4 shows how useful this is.

Regrettably, this convenient fact fails in higher dimension; Hartman gave examples in dimension 3 whose linearizing homeomorphism is not C^1. In Theorem 10.1.10 and Corollary 10.1.11 we see that it is possible for a stable or unstable hyperbolic fixed point to have a smooth linearization if the eigenvalues satisfy certain conditions. Only much more recently was it proved that the conjugacy can be taken differentiable *at the fixed point* (and with derivative equal to the identity, Theorem 7.7.1), which has the desired consequence for drawing phase portraits: a thumbnail of the phase portrait of the linearization pasted in at the fixed point gives a geometric accurate representation of the actual phase portrait in a neighborhood as in Example 7.7.4.

By contrast with the question of smooth linearization, the Hartman–Grobman Theorem 5.6.1 can be "globalized" as a purely topological statement by going beyond the need to match linear parts. More precisely, the index of a hyperbolic fixed point classifies local conjugacy classes nearby.

Theorem 5.6.3. *Let M, N be smooth manifolds, Φ, Ψ continuously differentiable flows on M and N, respectively, and $p \in M$, $q \in N$ hyperbolic fixed points of Φ, Ψ, respectively, with the same indices. Then for each $T > 0$ there exist neighborhoods U of p and V of $0 \in T_pM$ as well as a homeomorphism $h\colon U \to V$ such that $\varphi^t = h^{-1} \circ D\varphi_p^t \circ h$ on U for all $t \in [-T, T]$.*

Proof. By the Hartman–Grobman Theorem 5.6.1, both flows are locally conjugate to their linear parts, and by Theorem 1.4.7, these are conjugate. □

Exercises

5.1. Show that in the proof of Proposition 5.1.5 one directly obtains a smooth adapted metric by taking $\left(\|v\|_x^s\right)^2 := \int_0^S \underline{\lambda}^{-2s}\left(\|D\varphi^s v\|_{\varphi^s(x)}\right)^2 ds$ for large enough S.

5.2. Show that the Smale horseshoe presented in Example 1.5.21 is locally maximal (Definition 5.3.15).

5.3. Show that hyperbolic attractors are locally maximal.

5.4. Show that the stable and unstable sets of a hyperbolic fixed point (Definition 1.3.24) are topological manifolds.

5.5. Show that h in (5.5.3) is as claimed in Theorem 5.4.5.

5.6. Let M be an m-dimensional Riemannian manifold with sectional curvatures in $[-K^2, -k^2]$. Prove that

$$k(m-1) \le v(M) := \lim_{r\to\infty} \frac{1}{r} \log \operatorname{vol}(B(x,r)) \le K(m-1)$$

(volume growth).

5.7. Prove that the fundamental group $\pi_1(M)$ of a compact manifold that admits a metric of negative sectional curvature has exponential growth, that is, for any given system Γ of generators of $\pi_1(M)$ the number of elements $\gamma \in \pi_1(M)$ that can be represented by words of length at most n grows exponentially with n.

5.8. Prove that the universal cover of a manifold of negative sectional curvature is diffeomorphic to Euclidean space.

5.9. Prove that all geodesics c on a manifold of negative sectional curvature are minimal, that is, for any two points on the lift of c the segment of the lift between these points is the shortest curve between its ends.

CHAPTER 6

Invariant foliations

A key objective of Chapter 5 was to show how much of the orbit structure in hyperbolic dynamics can be discerned solely from shadowing and expansivity. One can go further yet, notably, of course, by applying the results from that chapter to other ends. Restricting to properties that result from shadowing and expansivity is on the other hand inherently limiting because these are topological properties, while significant parts of the hyperbolic theory are built on more subtle geometric properties. The central such property is that stable and unstable sets of points and orbits form (flow-invariant) foliations that are tangent to the stable and unstable subbundles. Put differently, the stable sets from (1.3.1) and (5.3.1) are submanifolds, and this additional structure is essential to, for instance, major parts of the ergodic theory of hyperbolic flows (Chapter 8). On the other hand, as part of the structure that comes with a hyperbolic flow, these invariant foliations are of interest in their own right and with respect to refined and new questions one can ask about hyperbolic dynamical systems. For instance, there are characterizations of topological transitivity and mixing in terms of the invariant foliations (Theorem 6.2.12) and of whether a flow is of algebraic type or not (Section 10.3).

1. Stable and unstable foliations

We begin this chapter by showing that the stable and unstable sets of points in a hyperbolic flow are smooth manifolds.

Theorem 6.1.1 (Stable- and Unstable-Manifold Theorem)**.** *Let* Λ *be a hyperbolic set for a* C^r *flow* Φ *on* M*,* $r \in \mathbb{N}$*,* C, λ, μ *as in Definition 5.1.1, and* $t_0 > 0$*. Then for each* $x \in \Lambda$ *there is a pair of embedded* C^r*-discs* $W^{ss}_{\text{loc}}(x)$*,* $W^{uu}_{\text{loc}}(x)$*, depending continuously on* x *in the* C^1*-topology and called the* local strong stable manifold *and the* local strong unstable manifold *of* x*, respectively, such that*

(1) $T_x W^{ss}_{\text{loc}}(x) = E^s_x$*,* $T_x W^{uu}_{\text{loc}}(x) = E^u_x$*;*
(2) $\varphi^t(W^{ss}_{\text{loc}}(x)) \subset W^{ss}_{\text{loc}}(\varphi^t(x))$ *and* $\varphi^{-t}(W^{uu}_{\text{loc}}(x)) \subset W^{uu}_{\text{loc}}(\varphi^{-t}(x))$ *for* $t \geq t_0$*;*

(3) for every $\delta > 0$ there exists $C(\delta)$ such that

$$d(\varphi^t(x),\varphi^t(y)) < C(\delta)(\lambda+\delta)^t d(x,y) \qquad \text{for } y \in W^{ss}_{\text{loc}}(x),\ t>0,$$
$$d(\varphi^{-t}(x),\varphi^{-t}(y)) < C(\delta)(\mu-\delta)^{-t} d(x,y) \qquad \text{for } y \in W^{uu}_{\text{loc}}(x),\ t>0;$$

(4) there exists a continuous family U_x of neighborhoods of $x \in \Lambda$ such that

$$W^{ss}_{\text{loc}}(x) = \{y \mid \varphi^t(y) \in U_{\varphi^t(x)} \text{ for } t>0, \qquad d(\varphi^t(x),\varphi^t(y)) \xrightarrow[t\to+\infty]{} 0\},$$
$$W^{uu}_{\text{loc}}(x) = \{y \mid \varphi^{-t}(y) \in U_{\varphi^{-t}(x)} \text{ for } t>0, \qquad d(\varphi^{-t}(x),\varphi^{-t}(y)) \xrightarrow[t\to+\infty]{} 0\}.$$

Proof. The Hadamard–Perron Theorem 12.5.2 applied to the time-t_0 map φ^{t_0} with $T_xM = E^s \oplus (E^c_x \oplus E^u_x)$ yields the existence of $W^{ss}_{\text{loc}}(x) \in C^r$ satisfying (1)–(4) for $t \in \mathbb{N}t_0$. The same with $T_xM = (E^s_x \oplus E^c_x) \oplus E^u_x$ yields $W^{uu}_{\text{loc}}(x) \in C^r$ satisfying (1)–(4) with $-t \in \mathbb{N}t_0$.

Observe now that (4) holds for positive multiples of t_0 if and only if it holds for real t. Once (3) holds for $t \in \mathbb{N}t_0$ it trivially holds for $t>0$ by adjusting the constant $C(\delta)$ since $\{\varphi^t\}_{t\in[0,t_0]}$ is equicontinuous and M is compact. □

Remark 6.1.2. With a little care one can replace the condition $t \ge t_0$ in (2) by $t>0$.

The sets

$$W^{ss}(x) := W^s(x) := \bigcup_{t>0} \varphi^{-t}(W^{ss}_{\text{loc}}(\varphi^t(x))) = \{y \in M \mid d(\varphi^t(x),\varphi^t(y)) \xrightarrow[t\to\infty]{} 0\},$$
$$W^{uu}(x) := W^u(x) := \bigcup_{t>0} \varphi^{t}(W^{u}_{\text{loc}}(\varphi^{-t}(x))) = \{y \in M \mid d(\varphi^{-t}(x),\varphi^{-t}(y)) \xrightarrow[t\to\infty]{} 0\}$$

are defined independently of a particular choice of local stable and unstable manifolds, and are smooth injectively immersed manifolds called the global *strong stable* and *strong unstable* manifolds. The manifolds

$$W^{cs}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^{ss}(x)) \text{ and } W^{cu}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^{uu}(x))$$

are called the *weak stable* and *weak unstable* manifolds (or *center-stable* and *center-unstable* manifolds) of x. Note that $T_xW^{cs} = E^s_x \oplus E^c_x$, $T_xW^{cu} = E^c_x \oplus E^u_x$.

Locally, then we have a picture like Figure 6.1.1 for each nonfixed point $p \in \Lambda$.

Proposition 5.1.4 and compactness imply:

Proposition 6.1.3. *Let Λ be a hyperbolic set for a flow Φ. Then there are a neighborhood U of Λ and $\alpha > 0$ such that if $x, y \in \Lambda$ and $z \in W^{ss}(x) \cap W^{uu}(y) \cap U$, then for any $\xi \in T_zW^{ss}(x)$ and $\eta \in T_zW^{uu}(y)$ the angle between ξ and η is at least α.*

Remark 6.1.4. We proved hyperbolicity of time-changes (Theorem 5.1.16) with just the Alekseev cone criterion much earlier, but this is an opportunity to show

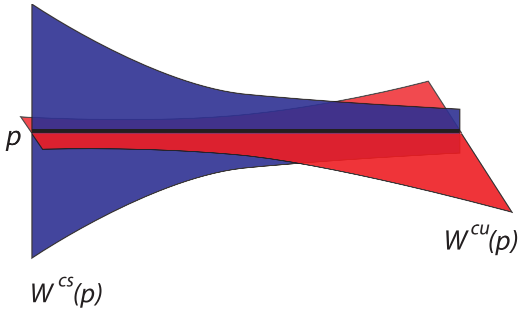


FIGURE 6.1.1. Local center-stable and center-unstable leaves

an alternative argument that tracks what happens to strong stable leaves under a time-change.

SECOND PROOF OF THEOREM 5.1.16. Since Ψ and Φ have the same orbits, Λ is Ψ-invariant, as are the center-stable and center-unstable manifolds for Φ. We use this to determine stable (and unstable) manifolds for Ψ. For $x \in \Lambda$ and $y \in W^{ss}_{\Phi}(x)$ we have

$$d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t\to\infty]{\text{exponentially}} 0 \text{ hence } d(\underbrace{\varphi^{\alpha(t,x)}(x)}_{=\psi^t(x)}, \varphi^{\alpha(t,x)}(y)) \xrightarrow[t\to\infty]{\text{exponentially}} 0,$$

where α is as in (1.2.1). Here, $\varphi^{\alpha(t,x)}(y)$ is a parametrization of the orbit of y which asymptotically lags (or leads) the corresponding Ψ-orbit by a constant:

$$\alpha(t,x) - \alpha(t,y) = \int_0^t \underbrace{\rho(\varphi^s(x)) - \rho(\varphi^s(y))}_{\text{exponentially small}}\, ds \xrightarrow[t\to\infty]{\text{exponentially}} a(y) \in \mathbb{R},$$

so the triangle inequality gives $d(\psi^t(x), \psi^{t-a(y)}(y)) \xrightarrow[t\to\infty]{\text{exponentially}} 0$, and by the theorem about differentiation inside an integral, $a(y)$ depends smoothly on $y \in W^{ss}_{\Phi}(x)$. Thus,

$$W^{ss}_{\Psi}(x) = \{\psi^{-a(y)}(y) \mid y \in W^{ss}_{\Phi}(x)\}$$

is the Ψ-stable manifold of x for Ψ. Due to the exponential contraction, its tangent space at x is the Ψ-stable subspace at x. A like argument for the unstable direction establishes hyperbolicity of Λ for Ψ. □

Since we now have the needed terminology we digress to point out that hyperbolic flows are generic in the following sense.

Definition 6.1.5. A fixed point p of a local flow is said to be *transverse* if the differential at p of any time-t map for $t \neq 0$ does not have 1 as an eigenvalue. Equivalently, the linear part of the vector field at p does not have 0 as an eigenvalue.

A periodic point p of period $t > 0$ for a flow is said to be *transverse* if 1 is a simple eigenvalue of the differential at p of the time-t map of the flow. Equivalently, p is a transverse fixed point for the Poincaré map on a transversal to the flow near p.

A smooth flow is said to be a *Kupka–Smale flow to order t* if all fixed points and all periodic orbits of period less than t are hyperbolic and the t-balls in their stable and unstable manifolds are pairwise transverse. It is called a *Kupka–Smale flow* if it is a Kupka–Smale flow to order t for all $t > 0$.

Theorem 6.1.6 (Kupka–Smale Theorem)**.** *Let $0 < r \le k \le \infty$ and M a compact C^k manifold. Then for any $t > 0$, Kupka–Smale flows of order t are a C^r-dense C^1-open set and hence Kupka–Smale flows are a C^r-dense C^1-G_δ set in the space of C^r flows.*

The *Inclination Lemma* illuminates the local geometry of the stable and unstable manifolds near a hyperbolic periodic point and can be useful in proving a number of results on the structure of hyperbolic sets. We note, however, that in those arguments it often suffices to instead invoke only smoothness and continuous dependence of the invariant manifolds. Thus, the remainder of this section is optional reading.

We here obtain consequences for flows of the Inclination Lemma (or λ-Lemma) for diffeomorphisms (Theorem 12.6.1) or of its proof. The first application is the most direct one: applying Theorem 12.6.1 to the diffeomorphism $f = \varphi^1$ gives

Proposition 6.1.7 (Inclination Lemma for Fixed Points)**.** *Suppose p is a hyperbolic fixed point of a smooth flow Φ and $\mathscr{D}$ is a disk that transversely intersects $W^{ss}(p)$. Then the $\varphi^t(\mathscr{D})$ accumulate on $W^{uu}(p)$ in the C^1-topology as $t \to +\infty$. Specifically, for any disk Δ in $W^{uu}(p)$ and any $\epsilon > 0$ there is an $t > 0$ and a $\mathscr{D}' \subset \mathscr{D}$ such that $d_{C^1}(\varphi^t(\mathscr{D}'), \Delta) < \epsilon$.*

Since our interest in flows centers primarily on those without fixed points, it is more interesting to have analogous statements for hyperbolic periodic points. The first restricts attention to a section.

Proposition 6.1.8 (Inclination Lemma in a section)**.** *Consider a hyperbolic periodic point p for a C^r flow Φ on a manifold M, and a C^r Poincaré section S transverse to the flow containing p. Then the Poincaré return map is a local diffeomorphism f, and if $\mathscr{D} \subset S$ is a disk that transversely intersects $W_f^{ss}(p) \subset S$, then the $f^n(\mathscr{D})$ accumulate on $W_f^{uu}(p) \subset S$ in the C^1-topology as $n \to +\infty$, that is, for any disk Δ in $W_f^{uu}(p)$ and any $\epsilon > 0$ there is an $n \in \mathbb{N}$ and a $\mathscr{D}' \subset \mathscr{D}$ such that $d_{C^1}(f^n(\mathscr{D}'), \Delta) < \epsilon$.*

Proof. This is a direct application of Theorem 12.6.1, except that here f is a local diffeomorphism. So one can either check that the proof of Theorem 12.6.1 works in this context or extend f to a diffeomorphism of Euclidean space by Theorem 12.4.12. □

Remark 6.1.9. It is not really needed that the point of transverse intersection lie in the local section because one can always achieve that by first applying φ^t for sufficiently large t because Φ preserves transversality.

Finally, a version of the Inclination Lemma that treats periodic orbits of flows directly rather than through sections uses not the Inclination Lemma for diffeomorphisms but its proof.

Proposition 6.1.10 (Inclination Lemma for Flows)**.** *Let p be a hyperbolic periodic orbit of least period T for a flow Φ of a manifold M with a splitting $T_pM = E^s \oplus E^c \oplus E^u$. Let D be an embedded disk intersecting $W^{ss}(p)$ transversely at some point $q \in W^{ss}(p)$ such that* $\dim(D) = \dim(E^u) + 1$. *Then for any $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for each $n \geq N$ there is an embedded disk $D_n \subset D$ containing q such that $\varphi^{Tn}(D_n)$ is C^1 ϵ close to $W^{cu}(p)$.*

Proof. Although the diffeomorphism φ^T does not quite satisfy the hypotheses of the Inclination Lemma (Theorem 12.6.1), the proof produces this result nonetheless (Remark 12.6.3) because no expansion is needed in order to establish that iterates of $\mathscr{D}$ are C^1-close to the center-unstable manifold of p. It is used solely to assert that large disks in $W^{cu}(p)$ are approximated by images of $\mathscr{D}$, and Proposition 6.1.10 makes no such claim. □

Remark 6.1.11. This argument illuminates what one should expect: That any disk in $W^{uu}(p)$ has a neighborhood Δ in $W^{cu}(p)$ such that for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ and a $\mathscr{D}_n \subset \mathscr{D}$ with $d_{C^1}(\varphi^{Tn}(\mathscr{D}_n), \Delta) < \epsilon$.

Remark 6.1.12. We repeat that many invocations of the Inclination Lemma can be averted by instead using that center-stable and center-unstable manifolds are C^r and depend continuously on the base point.

2. Global foliations and local maximality

The presence of the invariant foliations for a hyperbolic flow allows us to complement the previous dynamical insights for hyperbolic sets by a geometric understanding, which in turn augments our description of the dynamics on these sets. This section and those that follow give a panorama of ways in which this can be done.

Remark 5.3.39 indicates that the invariant foliations are meaningful well beyond a neighborhood of a hyperbolic set.

In the case of geodesic flows, the global foliations can be described geometrically. As we do this it may be instructive to revisit the discussion of surfaces of constant negative curvature at the end of Subsection 2.11 and Theorem 5.2.4 as well as Remark 5.2.6. The universal cover $\widetilde{M}$ of a negatively curved Riemannian manifold M is diffeomorphic to $\mathbb{R}^n$ (Exercise 5.8). We begin with unstable manifolds. Fix $v \in S\widetilde{M}$ and let

$$B_T := \{\gamma(0) \mid \gamma \text{ geodesic}, \gamma(-T) = \gamma_v(-T)\},$$
$$W_T := \{\dot\gamma(0) \mid \gamma \text{ geodesic}, \gamma(-T) = \gamma_v(-T)\},$$

the outside unit normal vectors to B_T. W_T is a smooth submanifold of $S\widetilde{M}$ of dimension $n-1$, where $n = \dim M$.

Consider any curve in W_T. Associated with the corresponding geodesic variation is a Jacobi field Y with $Y(-T) = 0$ (see the proof of Theorem 5.2.4). Unless $Y = 0$ we have $\langle Y(t), \dot Y(t)\rangle > 0$ for $t > -T$ since $\dot Y(-T) \neq 0$ and $Y(t-T) = t\dot Y(-T) + o(t)$ whence $\langle Y(t-T), \dot Y(t-T)\rangle > 0$ for small positive values of t. But we showed that this must then hold for all $t > 0$.

Thus, every tangent vector to W_T is contained in a cone from the invariant family. As in the Hadamard–Perron Theorem this implies that $W_T \xrightarrow[T\to\infty]{} W^u(v)$, a smooth $(n-1)$-dimensional submanifold of $S\widetilde{M}$. Since the projection $\pi\colon S\widetilde{M} \to \widetilde{M}$ is smooth, the spheres B_T converge to a smooth submanifold B_∞ called a *horosphere* (which means limit sphere).

$W^u(v)$ in fact consists of the outward unit normals to B_∞, which itself can be described as $\{\gamma(0) \mid \gamma \text{ geodesic}, d(\gamma(t), \gamma_v(t)) \xrightarrow[t\to-\infty]{} 0\}$.

Remark 6.2.1. If M is oriented then one can orient unstable leaves consistently by taking for each $v \in SM$ a positive orthonormal frame whose first vector is v; this orients a horosphere, and this orientation lifts to the corresponding unstable leaf.

Another example in which we can explicitly see the properties of the stable and unstable manifolds is the suspension of a hyperbolic toral automorphism (Example 1.5.23): for any point the stable and unstable manifolds are lines obtained as the projections of the contracting and expanding eigendirections translated to the base point.

We now characterize local maximality through local stable and stable manifolds (local product structure), and then revisit the spectral decomposition in this light.

Proposition 6.2.2 (Bowen bracket)**.** *For a hyperbolic set Λ for a flow Φ and $\epsilon > 0$ sufficiently small there exists a $\delta > 0$ such that if $x, y \in \Lambda$ such that $d(x, y) < \epsilon$, then there exists some $t = t(x, y) \in (-\epsilon, \epsilon)$ such that*

$$W^{ss}_\epsilon(\varphi^t(x)) \cap W^{uu}_\epsilon(y)$$

consists of a single point $[x, y]$, *called the* Bowen bracket[1] *of* x *and* y, *and there exists* $C_0 = C_0(\delta) > 0$ *such that if* $x, y \in \Lambda$ *and* $d(x, y) < \delta$, *then* $d_s(\varphi^{t(x,y)}(x), [x, y]) < C_0 d(x, y)$ *and* $d_u(y, [x, y]) < C_0 d(x, y)$ *where* d_s *and* d_u *denote the distances along the stable and unstable manifolds.*

Remark 6.2.3. Thus, the Bowen bracket is defined on the ϵ-neighborhood of the diagonal in $\Lambda \times \Lambda$ and maps to an ϵ-neighborhood of Λ in the manifold. (2.2.4) is a particularly concrete context for this concept.

Proof. Proposition 6.1.3 implies uniform transversality of W^{uu} and W^{cs} and that there is exactly one point z of intersection of $W^{cs}_\epsilon(x)$ and $W^{uu}_\epsilon(y)$, which, as we now show, depends continuously on x and y. By continuous dependence of $W^{cs}_\epsilon(x)$ and $W^{uu}_\epsilon(y)$, we can choose a chart to $R^l \oplus \mathbb{R}^k$ near z in which leaves of $W^{cs}_\epsilon(x')$ are graphs of Lipschitz maps $F_{x'}(\cdot)$ over $\mathbb{R}^l$ and leaves of $W^{uu}_\epsilon(y')$ are graphs of Lipschitz maps $G_{y'}(\cdot)$ over $\mathbb{R}^k$, in both cases with Lipschitz constants less than 1. Then $W^{cs}_\epsilon(x') \cap W^{uu}_\epsilon(y') = \{(x_0, y_0)\} = \{(G_{x'}(F_{y'}(x_0)), G_{x'}(F_{y'}(y_0)))\}$, so both coordinates are fixed points of contractions depending continuously on (x', y'). Since Λ is compact we obtain uniform ϵ and δ. □

We note that this lets us improve on shadowing (and on Proposition 1.7.4) as suggested by Figure 6.1.1:

Proposition 6.2.4 (Exponential expansivity)**.** *Let* Λ *be a hyperbolic set for a flow* Φ, *and* λ, μ *as in Definition 5.1.1. Then for any* $\eta \ge \max(\lambda, \mu^{-1})$ *there exist* $\delta > 0$ *and* $C > 0$ *such that if* $x \in \Lambda$, $d(\varphi^t(y), \varphi^t(x)) < \delta$ *for* $t \in [0, T]$, *then*

$$\begin{aligned} d(\varphi^t(\varphi^\tau(x)), \varphi^t(y)) &< C'\big(\eta^t d(x, y) + \eta^{T-t} d(\varphi^T(x), \varphi^T(y))\big) \\ &\le C\,\eta^{\min(t,T-t)} \cdot \big(d(x, y) + d(\varphi^T(x), \varphi^T(y))\big). \end{aligned}$$

(We use $\tau = t(x, y) \in (-\delta, \delta)$ *and* $[\cdot, \cdot]$ *from Proposition 6.2.2.)*

Proof. $d(\varphi^{t+\tau}(x), \varphi^t(y)) \le d(\varphi^{t+\tau}(x), \varphi^t([x, y])) + d(\varphi^t([x, y]), \varphi^t(y))$. □

Remark 6.2.5. This result improves the Anosov Closing Lemma (Theorem 5.3.10) in situations where the closed pseudo-orbit consists of a small number of orbit segments, such as a single orbit segment that almost closes up: the percentage error in the Closing Lemma is improved to a small *absolute* difference in periods. Proposition 6.2.19 below is an instance where this is critical (Remark 6.2.20).

Definition 6.2.6. A hyperbolic set Λ has a *local product structure* if $[x, y] \in \Lambda$ for x, y and (sufficiently small) ϵ as in Proposition 6.2.2.

[1] Or Smale bracket, see [**245**].

Theorem 6.2.7. *A hyperbolic set for a flow is locally maximal if and only if it has a local product structure.*

Proof. If Λ is locally maximal fix an adapted metric in a neighborhood V of Λ, an isolating neighborhood $U \subset V$ of Λ, and $\epsilon > 0$ such that $\bigcup_{x\in\Lambda} B_\epsilon(x) \subset U$ and such that there is a δ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $z := [x, y]$ as in Proposition 6.2.2 is well-defined—and by definition both forward and backward asymptotic to Λ. Since we use an adapted metric, $\mathcal{O}(z) \subset U$. But U is an isolating neighborhood, so $z \in \Lambda$, and Λ has a local product structure.

If Λ has a local product structure, fix $\epsilon > 0$, $\delta > 0$, and $C_0 > 0$ as in Proposition 6.2.2. To show that Λ is locally maximal, we establish that if an orbit is sufficiently close to Λ then it must be in Λ.

For V a sufficiently small neighborhood of Λ we know that Λ_V is hyperbolic and satisfies the same constants $\epsilon, \delta, and C_0$ as above. Meaning that if $x, y \in \Lambda_V$ and $d(x, y) < \delta$, then $d_s(\varphi^{t(x,y)}(x), [x, y]) < C_0 d(x, y)$ and $d_u(y, [x, y]) < C_0 d(x, y)$.

Fix $\alpha < \frac{\delta}{3}\min(1, 2C_0)$ such that $d(\varphi^t(x), \varphi^t(y)) < \frac{\delta}{3}$ for $0 \le t \le 1$ whenever $x \in \Lambda$, $y \in W^{uu}_\alpha(x)$. If furthermore $d(\mathcal{O}^+(y), \Lambda) < \alpha/C_0$, then for each $n \in \mathbb{N}$ there is a $y_n \in \Lambda$ with $d(y_n, \varphi^n(y)) < \alpha/C_0$. Since $\varphi^1(x), y_1 \in \Lambda$ and $d(\varphi^1(x), y_1) \le d(\varphi^1(x), \varphi^1(y)) + d(\varphi^1(y), y_1) < \delta/3 + \alpha/C_0$ we see that $x_1 = [y_1, \varphi^1(x)] \in \Lambda$ and $\varphi^1 y \in W^{uu}_\alpha(x_1)$. Continuing, we construct a sequence of points $x_n = [y_n, \varphi^1(x_{n-1})] \in \Lambda$ with $\varphi^n(y) \in W^{uu}_\alpha(x_n)$. Let $z_n = \varphi^{-n}(x_n)$. Then $z_n \xrightarrow[n\to\infty]{} y$ by the uniform contraction in the local unstable manifolds. Thus $y \in \Lambda$ since Λ is closed. Similarly, if $y \in W^{ss}_\alpha(x)$ and α is analogously defined for $-1 \le t \le 0$, and $\mathcal{O}^-(y)$ stays within α/C_0 of Λ, then $y \in \Lambda$.

If $\alpha_1 \in (0, \alpha)$ is sufficiently small and $d(\mathcal{O}(y), \Lambda) < \alpha_1$ then $\Lambda \cup \mathcal{O}(y)$ is a hyperbolic set by Proposition 5.1.10. Furthermore, for α_1 possibly smaller, if we let $x \in \Lambda$ such that $d(x, y) < \alpha_1$, then we can define $z_1 = [x, y]$ and $z_2 = [y, x]$, the above argument shows that $z_1, z_2 \in \Lambda$. Hence, $y \in \Lambda$ by the product structure of Λ. □

Remark 6.2.8. One can define a topological equivalent to a locally maximal hyperbolic set. This is the notion of a Smale flow or topological Anosov flow , which is defined as an expansive flow on a compact metric space whose local stable and unstable sets have a unique point of intersection (giving a well-defined Bowen bracket). Almost all of the theory we introduce about hyperbolic sets for flows can be shown to hold for Smale flows [**245**].

We now connect the spectral decomposition to the invariant foliations; this will have relevance later when we investigate Markov partitions. This also explains the name of the components of the decomposition.

Second proof of Theorem 5.3.35. We first define a relation on the periodic points contained in a locally maximal hyperbolic set Λ. Let $p, q \in \mathrm{Per}(\Phi\restriction_\Lambda)$. Then

$p \sim q :\Leftrightarrow$ "$W^{cu}(p) \cap W^{cs}(q) \cap \Lambda \neq \varnothing \neq W^{cs}(p) \cap W^{cu}(q) \cap \Lambda$ with both intersections transverse in at least one point." This relation is clearly reflexive and symmetric, and we presently show that it is also transitive. We will then see that this equivalence relation defines the set Λ_i as the closure of an equivalence class.

Transitivity: If $x, y, z \in \operatorname{Per}(\Phi\upharpoonright_{\Lambda})$ and $p \in W^{cu}(x) \cap W^{cs}(y) \cap \Lambda$, $q \in W^{cu}(y) \cap W^{cs}(z) \cap \Lambda$ are transverse intersection points, then by continuity of unstable leaves the images of a ball around p in $W^{cu}(p) = W^{cu}(x)$ accumulate on $W^{cu}(y)$ so $W^{cu}(x)$ and $W^{cs}(z)$ have a transverse intersection in Λ.

By Theorem 6.2.7 the equivalence classes are open, so by compactness there are finitely many of them, and we denote by $\Lambda_1, \dots, \Lambda_m$ their (pairwise disjoint) closures. By Corollary 5.3.14(4) $NW(f\upharpoonright_{\Lambda}) \subset \overline{\operatorname{Per}(\Phi\upharpoonright_{\Lambda})}$ since Λ is locally maximal, so $\bigcup_{i=1}^{m} \Lambda_i = NW(\Phi\upharpoonright_{\Lambda})$.

To show that $\varphi^t\upharpoonright_{\Lambda_i}$ is topologically transitive note first that if $p \in \Lambda_i$ is periodic and $p \sim q$ with q periodic, then there is by definition a $z \in W^{cu}(p) \cap W^{cs}(q) \cap \Lambda$ that is a point of transverse intersection, so $W^{cu}(p)$ accumulates on the orbit of q. Thus, $W^{cu}(p) \cap \Lambda$ is dense in $\Lambda_i \cap \operatorname{Per}(\varphi^t\upharpoonright_{\Lambda})$, hence in its closure Λ_i for every periodic $p \in \Lambda_i$.

We conclude by showing that a hyperbolic set $\Lambda = NW(\varphi^t\upharpoonright_{\Lambda})$ is topologically transitive if $W^{cu}(p) \cap \Lambda$ is dense in Λ for every periodic $p \in \Lambda$. We need to check that for any two open sets V and W in Λ there exists an $T \in \mathbb{R}^+$ such that $\varphi^t(V) \cap W \neq \varnothing$ for all $t \geq T$ (Definition 1.6.31). For open $V, W \subset \Lambda$ density of periodic points implies the existence of a periodic point $p \in V \cap \Lambda$. Since V is open there exists $\delta > 0$ such that $W^u_\delta(p) \subset V$. Since $W^{cu}(p) \cap \Lambda$ is dense there exists $T_0 \in \mathbb{R}$ such that $W \cap \varphi^{T_0}(W^u_\delta(p)) \neq \varnothing$. Let T_1 be the period of p. Then for $n \in \mathbb{N}$ and $t = T_0 + nT_1$ we have $W \cap \varphi^t(V) \neq \varnothing$. □

This proof yields a new corollary of Theorem 5.3.35:

Proposition 6.2.9. *Let Λ be a basic set for Φ and $p \in \Lambda$ be a periodic point. Then*

- *the center-stable manifold of p is dense in $W^s(\Lambda)$ and*
- *the center-unstable manifold of p is dense in $W^u(\Lambda)$.*

Proof. Let $p, q \in \Lambda$ be periodic points. Then $W^{cu}(p) \pitchfork W^{cs}(q) \neq \varnothing$ and $W^{cs}(p) \pitchfork W^{cu}(q) \neq \varnothing$. Furthermore, the local stable manifold of the orbit of p accumulates on the local stable manifold for q. By iteration this implies that the stable manifold of the orbit of p accumulates on the stable manifold of q. Since the periodic points q are dense, Theorem 5.3.25 says that the stable manifold of p is dense in $W^s(\Lambda)$. Reversing the flow proves the same for unstable manifolds. □

Corollary 6.2.10. *A compact locally maximal hyperbolic set Λ is topologically transitive if and only if periodic points are dense in Λ and the center-unstable manifold of every periodic point is dense in Λ.*

Proof. The spectral decomposition is trivial. $\square$

We can refine this (and Theorem 5.3.50) in the case of Anosov flows.

Theorem 6.2.11. *For Anosov flows the following are equivalent.*

(1) The spectral decomposition has one piece (the whole manifold).
(2) The flow is regionally recurrent *(Definition 1.5.11).*
(3) The flow is topologically transitive.
(4) Periodic points are dense.
(5) All center-unstable leaves are dense.
(6) All center-stable leaves are dense.

Proof. Equivalence of (1), (2), (3), (4) is Theorem 5.3.50. We show (4)$\Rightarrow$(6)$\Rightarrow$(2), and this for the reversed flow establishes (4)$\Rightarrow$(5)$\Rightarrow$(2).
(4)$\Rightarrow$(6): If $x \in M$, then $M = X := \overline{W^{cu}(x)}$ because X is open as follows. By (4), it suffices to check that $z \in X$, $\varphi^T(p) = p \in R_\epsilon(z) := \{[a,b] \mid a \in W^u_\epsilon(z),\ b \in W^{cs}_\epsilon(z)\} \Rightarrow p \in X$, and this is clear because $X \ni \varphi^{-iT}([p,z]) \xrightarrow[i\to\infty]{} p$, so $p \in \bar{X} = X$.
(6)$\Rightarrow$(2) follows if $W^{cu}(p)$ is dense for some periodic point p because this implies (2) since $W^{cs}(p)$ is dense by (6), so homoclinic points are dense, and they are nonwandering (see for example, Corollary 6.3.3 below).

Suppose to the contrary that $M \neq X := \overline{W^{cu}(p)}$. X is a union of unstable leaves and Φ-invariant. Take $\epsilon > 0$ such that

$$U := \bigcup_{x\in X} W^{ss}_\epsilon(x) \neq M.$$

Then $\varphi^t(U) \subset \bigcup_{x\in X} W^{uu}_{\lambda^t\epsilon}(x)$ for $t \geq 0$ and a suitable $\lambda \in (0,1)$, and $X = \bigcap_{t\geq 0}\varphi^t(U)$ (nested intersection) and $Y := \bigcap_{t\geq 0}\varphi^{-t}(M \smallsetminus U)$ are disjoint, and Y is closed and Φ-invariant, so there is a $\delta > 0$ with $d(x,y) \geq \delta$ for all $x \in X$ and $y \in Y$. Thus Y is a union of stable leaves because if $y \in Y$ and $z \in W^{ss}(y)$ then $z \in Y$ because otherwise $Y \ni \varphi^t(z) \xrightarrow[t\to\infty]{} X$ contrary to (6) since $\varphi^t(x) \in Y$ and $d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t\to\infty]{} 0$. $\square$

The following is a counterpart for the mixing case (and note the pertinent dichotomy in Theorem 9.1.1).

Theorem 6.2.12. *For a locally maximal hyperbolic set Λ the following are equivalent:*

(1) $\Phi\restriction_\Lambda$ is topologically mixing.
(2) The periodic points of Λ are dense in Λ and for each periodic point $p \in \Lambda$ we have $\overline{W^{ss}(p) \cap \Lambda} = \Lambda$ and $\overline{W^{uu}(p) \cap \Lambda} = \Lambda$.

(3) *$\Phi_{\restriction_\Lambda}$ is transitive and each open set contains periodic points with incommensurate periods.*[2]

Definition 6.2.13. Here we say that the periods of a set of periodic points are *commensurate* if they are all in $p\mathbb{Z}$ for some $p > 0$; incommensurate otherwise (that is, they generate a dense subgroup of $\mathbb{R}$).

Proof. We prove (3)⇔(1)⇔(2).

- (1)⇒(2): This strengthening of the contrapositive is of independent interest:

Proposition 6.2.14. *If $p \in \Lambda$ is a periodic point of Φ and $W^{uu}(p)$ is not dense, then $\Phi_{\restriction_\Lambda}$ is the suspension of a homeomorphism f of $K := \Lambda \cap \overline{W^{uu}(p)}$.*

Proof. Let T be the period of p. Analogously to Proposition 1.6.27 there is a minimal nonempty $L \subset K$ such that

$$L \text{ is closed, } W^u\text{-saturated and } \varphi^T\text{-invariant.} \tag{6.2.1}$$

The compact, hence closed, set $\varphi^{[0,T]}(L)$ is W^{cu}-saturated, hence equal to Λ by density of weak-unstable manifolds.

Claim 6.2.15. *If $\varphi^t(L) \cap L \neq \varnothing$ then $\varphi^t(L) = L$.*

Proof. $L \cap \varphi^t(L)$ satisfies (6.2.1), so $L \cap \varphi^t(L) = L$ by minimality, that is, $L \subset \varphi^t(L)$, so $\varphi^{-t}(L) \subset L$. However, $\varphi^{-t}(L)$ also satisfies (6.2.1), so $\varphi^{-t}(L) = L$ by minimality. Apply φ^t to get the claim. □

Claim 6.2.16. *There is a smallest $s > 0$ such that $\varphi^s(L) \cap L \neq \varnothing$ (and hence $\varphi^s(L) = L$).*

Proof. Since $\mathscr{S} := \{s \geq 0 \mid \varphi^s(L) \cap L \neq \varnothing\}$ is closed, we just need to find a positive lower bound. We show that if there is no positive lower bound, then $L = \Lambda$.

Note that recursively, if $t \in \mathscr{S}$ and hence $\varphi^t(L) = L$, then $\varphi^{nt}(L) = L$ for all $n \in \mathbb{Z}$. If there are arbitrarily small such t, then $\mathscr{S}$ is therefore dense in $\mathbb{R}$ and hence equal to $\mathbb{R}$, which implies that L is W^{cu}-saturated, hence dense, and hence equal to Λ. □

This choice of s gives $\Lambda = \varphi^{[0,s)}(L)$, and this is a disjoint union. This allows us to recognize the suspension as follows. Λ is a bundle over S^1 using the projection

$$\pi\colon \Lambda \to S^1, \quad \varphi^t(x) \mapsto t \pmod{s},$$

where $x \in L$, and $f = \varphi^s{\restriction_L}$ is the base map.

Since L is W^u-saturated, $W^{uu}(p)$ must lie in some $\varphi^t(L)$, so $L = K$. □

- (2)⇒(1): The main step is to establish uniformity of density.

[2] By Proposition 6.2.19 we can replace "each open set contains" with "has".

Lemma 6.2.17. *For $\epsilon > 0$ and each periodic point p there is an $R > 0$ such that the R-disk $W^u_R(p)$ in $W^{uu}(p)$ is ϵ-dense in Λ.*

Proof. Otherwise there are ϵ and p such that for each $N \in \mathbb{N}$ there is an $x_n \in \Lambda$ such that $B(x_n, \epsilon)$ and the N-ball in $\overline{W^{uu}(p)}$ are disjoint. Passing to a subsequence, let $x = \lim_{n\to\infty} x_n$ and note that $x \notin \overline{W^{uu}(p)}$. □

The same conclusion holds uniformly for all $q \in \mathscr{O}(p)$: Let T be a period of p, $\epsilon > 0$, and $\epsilon' > 0$ such that if $t \in [0, T]$, then the preimage under φ^t of an ϵ-ball contains an ϵ'-ball. Let $R' > 0$ be as in Lemma 6.2.17 for ϵ', and $R > 0$ such that $W^u_{R'}(p) \subset \varphi^{-t}(W^u_R(\varphi^t(p)))$ when $0 \le t \le T$. Now, if $q = \varphi^{t_q}(p) \in \mathscr{O}(P)$ with $0 \le t_q \le T$, and B is an ϵ-ball, then $B \cap W^u_R(q) \ne \varnothing$.

Now suppose $U, V \subset \Lambda$ are (relatively) open and choose $\epsilon > 0$ such that V contains an ϵ-ball and U contains an ϵ-neighborhood of a periodic point p. Take $T > 0$ such that if $t \ge T$, then $\varphi^t(W^{uu}_\epsilon(p))$ contains an R-disk in $W^{uu}(\varphi^t(p))$. Then $\varphi^t(U) \cap V \ne \varnothing$ for $t \ge T$.

- (1)⇒(3): Contraposition. Let $O \subset \Lambda$ be an open set such that the period of each periodic point in O is in $\pi\mathbb{Z}$. Consider a flow box (see Proposition 1.1.14) $U = \varphi^{(-\pi/4,\pi/4)}(D_x) \subset O$ of diameter less than $\delta < \pi v/2L$, where D_x is a local transversal through $x \in O$, $v := \min|X| > 0$ is the minimum speed, and L is as in Theorem 5.3.10. Suppose $\varphi^T(U) \cap U \ne \varnothing$, that is, there is a $p \in U$ such that $\varphi^T(p) \in U$. Then $\varphi^{0,T]}(p)$ defines a periodic δ-pseudo-orbit and is hence $L\delta$ shadowed by a periodic orbit, hence $|T - n\pi| \le L\delta/v < \pi/2$ for some $n \in \mathbb{N}$, so Φ is not mixing.
- (3)⇒(1): Suppose $U, V \subset \Lambda$ are (relatively) open and $p, q \in U$ are periodic with incommensurate periods π_p and π_q. By Proposition 6.2.9 there is, analogously to Lemma 6.2.17, an R such that $W^{cu}_R(p) \cap V \ne \varnothing$, that is, there is an open neighborhood W of some $p' \in \mathscr{O}(p)$ such that $W^u_R(x) \cap V \ne \varnothing$ for all $x \in W$. Also, there is a $q' \in \mathscr{O}(q)$ such that $W^s_{\text{loc}}(p') \cap W^u_{\text{loc}}(q') \ne \varnothing$. Thus, there are $\epsilon > 0$ and $n \in \mathbb{N}$ such that $\varphi^{[n\pi_q - \epsilon, n\pi_q + \epsilon]}(q') \subset W$ and $\varphi^{[n\pi_p - \epsilon, n\pi_p + \epsilon]}(p') \subset W$ for all $n \ge N$. Since π_p and π_q are incommensurate, this shows that for all sufficiently large t there are $x \in W \subset U$ with $\varphi^t(x) \in V$. □

Proposition 6.2.18. *If Λ is a compact locally maximal hyperbolic set, periodic points are dense in Λ, and $W^{uu}(p)$ or $W^{ss}(p)$ is dense for some periodic p, then $W^{uu}(z)$ and $W^{uu}(z)$ are dense for all $z \in \Lambda$.*

Proof. The contrapositive of Proposition 6.2.14 and its counterpart for W^{ss} show that $W^{uu}(p)$ and $W^{ss}(p)$ are dense for all periodic p. To show that $W^{uu}(z)$ (and hence likewise $W^{ss}(z)$) is dense for all $z \in \Lambda$, introduce convenient local neighborhoods $O_r(x) := \bigcup_{y \in W^{uu}_r(x)} W^{cs}_r(y)$ for $r > 0$ and $x \in \Lambda$. We can choose a $\delta(r)$ independently of x such that $B(x, \delta(r)) \subset O_r(x)$ and moreover, every W^u-leaf that meets

$B(x,\delta(r))$ "goes across $O_r(x)$": $W^{uu}(z)\cap B(x,\delta(r))\neq\varnothing \Rightarrow W^{uu}(z)\cap W^{cs}_r(y)\neq\varnothing$ for all $y\in W^{uu}_r(x)$.[3]

For $x\in\Lambda$ and $\epsilon>0$ we will find a point of $W:=W^{uu}(z)$ within ϵ of x. Take periodic points $p_1,\dots,p_k$ such that the balls $B(p_i,\delta(\epsilon/2))$ cover Λ and note that if $T>0$ is large enough, then $\varphi^T(W^{uu}_{\epsilon/2}(p_i))\cap B(x,\epsilon/2)\neq\varnothing$ for $1\le i\le k$, and that there is an i such that $\varphi^{-T}(W)\cap B(p_i,\delta(\epsilon/2))\neq\varnothing$. By choice of T there is a $q\in W^{uu}_{\epsilon/2}(p_i)\cap\varphi^{-T}(B(x,\epsilon/2))$. Since these points are in $O_{\epsilon/2}(p_i)$, there is also a $y\in\varphi^{-T}(W)\cap W^{cs}_{\epsilon/2}(q)$.

Then $d(q,y)<\epsilon/2$, $d(\varphi^T(q),\varphi^T(y))<\epsilon/2$, and

$$d(x,\varphi^T(y))\le d(x,\varphi^T(q))+d(\varphi^T(q),\varphi^T(y))<\frac{\epsilon}{2}+\frac{\epsilon}{2},$$

so $\varphi^T(y)\in W$ is the desired point. □

The following answers a natural question arising from Theorem 6.2.12: its third characterization is equivalent to the set of all periods being incommensurate.

Proposition 6.2.19. *If the periods of a basic set are incommensurate, then so are those of periodic points that intersect a given open set.*

Proof. We need to show that the subgroup P_O of $\mathbb{R}$ generated by periods of periodic points in O is dense in $\mathbb{R}$ given that the subgroup P generated by all periods is. To that end we show that if O is open, $\epsilon>0$, and p is ρ-periodic, then there is a $\tau\in\mathbb{R}$ such that for all $n\in\mathbb{N}$ there are periodic points $p_n\in O$ whose period is within $\epsilon/2$ of $\tau+n\rho$. This implies the claim because it shows that P_O contains elements ϵ-close to any element of P, so density of P implies density of P_O.

By transitivity, there is an orbit segment from within $\epsilon/2L$ of p (where L is as in the Anosov Closing Lemma, Theorem 5.3.10) to the center of an $\epsilon/2$-ball in O and an orbit segment from within $\epsilon/2L$ of that point to within $\epsilon/2L$ of p. Denote by τ the sum of their lengths. For each $n\in\mathbb{N}$ the $\epsilon/2L$-pseudo-orbit consisting of these orbit segments and n periods of p is $\epsilon/2$-shadowed by a periodic orbit that contains a point $p_n\in O$ and has the desired period. □

Example 1.6.35 used the above result to establish topological mixing for a special flow over the hyperbolic toral automorphism (Example 1.5.23).

Remark 6.2.20. Theorem 5.3.10 is less explicit about the timing of the shadowing orbit than needed for this proof, and Proposition 6.2.4 implies that the periods we obtain are as asserted.

Remark 6.2.21. If all periods are a multiple of $k\in\mathbb{R}$, then the ζ-function from (8.7.9) is a product of rational functions of $z=e^{-ks}$.

[3] We only use the existence of such a y.

From the preceding one can extract the following (compare Theorem 6.2.12):

Theorem 6.2.22 ([**56**, p. 77],[**54**])**.** *For a basic set Λ there are 3 mutually exclusive possibilities:*

- *Λ is a point,*
- *the restriction of the flow to Λ is a (constant-time) suspension,*
  - *$\Leftrightarrow$ the restriction of the flow to Λ is not topologically mixing,*
  - *$\Leftrightarrow$ the periods of the restriction of the flow to Λ are commensurate,*
- *every strong stable or unstable manifold is dense in Λ.*

The equivalence of the formulations of the second possibility and the Ω-Stability Theorem imply genericity of mixing:

Corollary 6.2.23. *The subset of $\mathscr{A}$ (see* (5.3.2)*) for which an infinite basic set is not topologically mixing is of the first category in the C^r topology for $r \geq 1$.*

Proof. By Theorem 6.2.22 it suffices to show that having commensurate periods is a first-category phenomenon in $\mathscr{A}$. Specifically, denote by $V_n \subset \mathscr{A}$ the flows such that for some (infinite) element X of the spectral decomposition there is a $\tau \geq 1/n$ for which $\varphi^t(x) = x \in X \Rightarrow t \in \mathbb{Z}\tau$. Then it suffices to show that V_n is closed and nowhere dense (that is, closed with dense complement) in $\mathscr{A}$, since $\bigcup_n V_n$ is the set of flows in $\mathscr{A}$ with commensurate periods.

To see that V_n is closed, suppose $V_n \ni \Phi_k \xrightarrow[C^1]{} \Psi \in \mathscr{A}$. The Ω-Stability Theorem 5.4.13 implies that for large enough k there is an orbit-equivalence between $\Phi_k\upharpoonright_{NW(\Phi_k)}$ and $\Psi\upharpoonright_{NW(\Psi)}$, and this orbit-equivalence identifies the spectral decompositions $NW(\Phi_k) = \bigcup_i \Lambda_i^k$ and $NW(\Psi) = \bigcup_i \Lambda_i^\Psi$, so we can pass to a subsequence we can find a fixed i such that there is a $\tau_k \geq 1/n$ for which $\varphi^t(x) = x \in X_k := \Lambda_i^k \Rightarrow t \in \mathbb{Z}\tau_k$ (and each X_k is identified with $X := \Lambda_i^\Psi$).

Ω-stability further implies that each Ψ-periodic orbit γ in X corresponds to periodic orbits γ_k in X_k whose (least) periods converge to that of γ as $k \to \infty$. Since these are each upper bounds for τ_k, we can take $\tau_k \in [1/n, M]$ for some $M \in \mathbb{R}$ and hence $\tau_k \to \tau \in [1/n, M]$ after passing to a subsequence. That done, we now find for arbitrary closed orbits γ in X^Ψ that By the assumption on X_k the corresponding periodic orbits γ^k in X_k satisfy $n_k\tau_k = \text{period}(\gamma_k) \xrightarrow[k\to\infty]{} \text{period}(\gamma)$, so $\gamma \in \mathbb{Z}\tau$, hence $\Psi \in V_n$. Thus, V_n is closed.

That the complement of V_n is dense is the easy part; indeed the complement of $\bigcup_n V_n$ is dense: take 2 distinct periodic orbits γ_1, γ_2 in a basic set and disjoint neighborhoods of them. Perturb the flow in one of them (only) in an arbitrarily C^r-small way so that $\frac{\text{period}(\gamma_1)}{\text{period}(\gamma_2)} \notin \mathbb{Q}$. □

These arguments can be varied to show that often no 2 periods are commensurate.

Proposition 6.2.24. *For a C^r-generic ($r \in \mathbb{N}$) element of $\mathscr{A}$ (see* (5.3.2)*) the periods of closed orbits are pairwise incommensurate.*

PROOF. We show that the set of such flows is a C^r-dense intersection of C^r-open subsets of $\mathscr{A}$.

To show density enumerate the periodic orbits of a $\Phi_0 \in \mathscr{A}$ as $\{p_n \mid n \in \mathbb{N}\}$, this enumeration carries to any Φ sufficiently near Φ_0. For $\epsilon > 0$ such that $\mathscr{A}$ contains the C^r-ϵ-ball around Φ_0 and $i \in \mathbb{N}$ recursively define a time-change Φ_i of Φ_{i-1} localized in the complement of $\bigcup_{j\le i} p_j$ and C^r-$\epsilon 2^{-i-1}$-close to Φ_{i-1} for which the periods of the p_j for $j \le i+1$ are pairwise incommensurate. The limit is within ϵ of Φ_0 in the C^r topology and as desired.

Consider the set V_n of $\Phi \in \mathscr{A}$ for which a pair of distinct closed orbits has periods at most n and a period ratio p/q with $p, q \in \mathbb{N}$ and $q \le n$. This is closed, and their union over n is the set of Φ with a commensurate pair of closed orbits. Thus, the complement is C^r-generic. □

The next result (by Mañé) provides a criterion for a flow to be Anosov. It can be used in Section 9.3 to provide examples of nontransitive Anosov flows.

Theorem 6.2.25 (Mañé criterion)**.** *A smooth flow on a compact manifold is an Anosov flow if and only if the chain-recurrent set is hyperbolic, the (weak) stable and unstable manifolds intersect transversely at one (hence every) point of each orbit, and their dimension is constant.*

PROOF. That Anosov flows satisfy the criterion is clear. To show the other direction we use that if the chain recurrent set is hyperbolic then the flow is Axiom A with no cycles. If the (weak) stable and unstable manifolds intersect transversely at every point of each orbit, then we have strong transversality. Now assume that the dimension of the stable and unstable splitting is constant on the manifold. By strong transversality, each point $p \in M$ is contained in a strong stable manifold and strong unstable manifold, and the sum of the dimensions of these is $n-1$, where n is the dimension of M. Therefore, we have a splitting $T_pM = E^s \oplus E^c \oplus E^u$ that is continuous and flow-invariant. For the basic sets $\Lambda_1, \dots, \Lambda_n$ there are adapted metrics and constants of hyperbolicity. Furthermore, the constants can be extended to neighborhoods of the basic sets $\mathscr{O}_1, \dots, \mathscr{O}_n$ as we have done previously. Let $\mathscr{O} = \bigcup_{i=1}^n \mathscr{O}_i$. Since $X := M \setminus \mathscr{O}$ is compact, there is a uniform bound $T > 0$ such that no orbit will spend more than the amount of time T in X. Therefore there exists a constant $C > 0$ that compensates for the time spent in X using the constants of hyperbolicity for the basic sets. Hence, M is a hyperbolic set, and the flow is Anosov. □

3. Horseshoes and attractors

As discussed in Chapter 0, in studying the 3-body problem, Poincaré observed complicated orbit structures. Specifically, he noticed that if p is a hyperbolic periodic orbit whose center-stable and center-unstable manifolds intersect transversely off of the orbit for p that this intersection point creates complicated orbit structures as shown by the Poincaré section in Figure 6.3.1. Later Birkhoff noticed that the transverse intersection point off of the orbit of p was in the closure of periodic orbits. Later, Smale introduced the notion of the horseshoe as a dynamical object that encapsulates the complexity seen near the transverse intersection.

Nonlinear versions of Example 1.5.21 have the same qualitative features (both in the base and the roof function)—this is the content of the Structural Stability Theorem 5.4.5. Thus, we now define a horseshoe with this in mind as a model and show how horseshoes naturally arise for transverse homoclinic points..

Definition 6.3.1. Let Φ be a flow on a manifold M. A hyperbolic set for Φ is a *horseshoe* if it is orbit equivalent to a hyperbolic symbolic flow (Definition 1.8.3).

If we take a Poincaré section for a periodic orbit we obtain a fixed point in the section. If we further suppose that the stable and unstable manifolds for the periodic orbit intersect transversely in the section at a point off of the periodic orbit we obtain a transverse homoclinic point. From this we can obtain a horseshoe as described below.

This situation arises for suspensions of maps with transverse homoclinic points as in Figure 6.3.1. When the stable and unstable curve of a hyperbolic fixed point intersect transversely, they produce tangles (Figure 6.3.1), and these

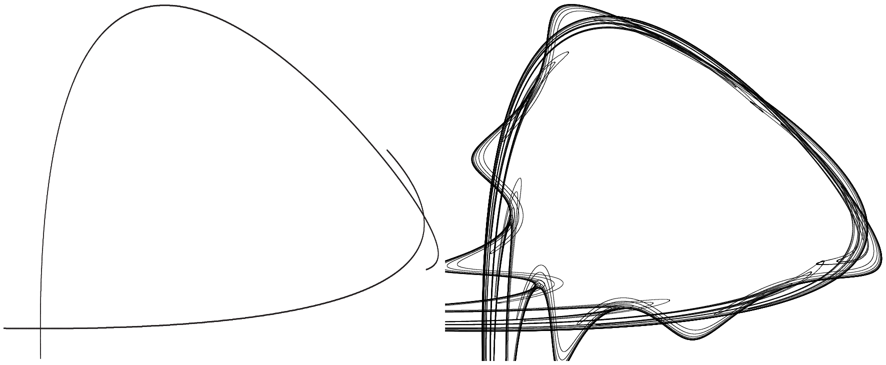


FIGURE 6.3.1. Transverse homoclinic point and homoclinic tangles [©Cambridge University Press, reprinted from [**181**] with permission]

in turn produce horseshoes for an iterate of the map; this is the Birkhoff–Smale Theorem, illustrated in Figure 6.3.2.

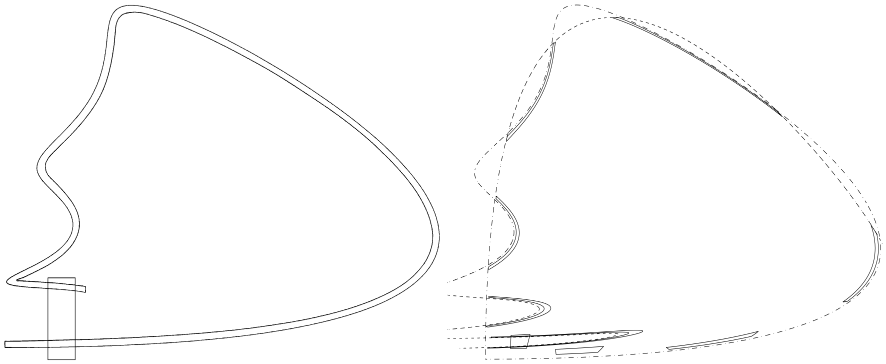


FIGURE 6.3.2. Horseshoes from tangles [©Cambridge University Press, reprinted from [**149**, **181**] with permission]

Theorem 6.3.2 (Birkhoff–Smale Theorem)**.** *Let M be a compact Riemannian manifold and Φ be a smooth flow on M. If p is a hyperbolic periodic orbit for Φ, q is a transverse homoclinic point for p in a smooth section for the flow near p with return map f, and U is a neighborhood of $\{p,q\}$, then there is an $n \in \mathbb{N}$ such that f^n contains a hyperbolic invariant set $\Lambda \subset U$ topologically conjugate to the full 2-shift. Furthermore, the suspension of Λ is orbit equivalent to a hyperbolic flow over the 2-shift.*

Corollary 6.3.3. *Every transverse homoclinic point for a hyperbolic periodic orbit of smooth flow is in the closure of the periodic points and is hence a nonwandering point for the flow.*

Theorem 5.3.53 indicates that hyperbolic attractors arise somewhat naturally, but it is useful to exhibit nontrivial examples explicitly. To that end, we now produce the derived-from-Anosov or *DA flow*, which is in its own right a prominent example, but also plays a pivotal role in a later construction of an anomalous Anosov flow (Theorem 9.3.1), and we derive from it the *Plykin attractor*, on which in turn rests the construction of hyperbolic sets that are not enveloped by a locally maximal one (Definition 6.3.8, Section 6.5).

Let F be the Anosov diffeomorphism of $\mathbb{T}^2$ induced by $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$. We are going to modify it by deforming it near the origin to make that a repelling fixed point. Then the complement of a small closed disk around the origin is a trapping region, and we will describe the associated attractor (for the suspension).

Denote by v^u and v^s the normalized eigenvectors corresponding to the eigenvalues $\lambda_1 = (3+\sqrt{5})/2$ and $\lambda_2 = \lambda_1^{-1} = (3-\sqrt{5})/2$, respectively, and let e^u and e^s be the stable and unstable vector fields obtained from v^u and v^s by parallel translation. Then $E^u(p) = \mathbb{R}e^u(p)$ and $E^s(p) = \mathbb{R}e^s(p)$ and $DF_p e^u(p) = \lambda_1 e^u(F(p))$ and $DF_p e^s(p) = \lambda_2 e^s(F(p))$ for all $p \in \mathbb{T}^2$.

On a disk U centered at 0 introduce coordinates (x_1, x_2) diagonalizing A, that is, such that $F(x_1, x_2) = (\lambda_1 x_1, \lambda_2 x_2)$ on U.

Definition 6.3.4. The derived-from-Anosov flow or DA flow is the suspension of the diffeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by (6.3.1) below.

Remark 6.3.5. We next establish that f has a hyperbolic attractor Λ whose complement W is dense in $\mathbb{T}^2$ and is the basin of the attracting fixed point 0 for f^{-1}. Thus, the DA flow has a closed hyperbolic attractor with empty interior.

To construct f let $\phi\colon \mathbb{R} \to [0,1]$ be an even C^∞ function such that

$$\phi(t) = \begin{cases} 1 & \text{if } |t| \le 1/8, \\ 0 & \text{if } |t| \ge 1/4, \end{cases}$$
$$\phi'(t) < 0 \quad \text{if } 1/8 < t < 1/4,$$

and note that

$$g(x) := \frac{x\phi(x)}{1+bx^2} \text{ with } b > \frac{\min\phi'}{\frac{1}{4} - \lambda_2} \text{ sufficiently large}$$

is odd with $g'(x) = \underbrace{\frac{x\phi'(x)}{1+bx^2}}_{\frac{|x|}{1+bx^2} \le \frac{1}{2b}} + \phi(x) \underbrace{\frac{1-bx^2}{(1+bx^2)^2}}_{\frac{1-z}{(1+z)^2} \ge -\frac{1}{8}} \ge \frac{\min\phi'}{2b} - \frac{1}{8} > -\frac{\lambda_2}{2}$. Let

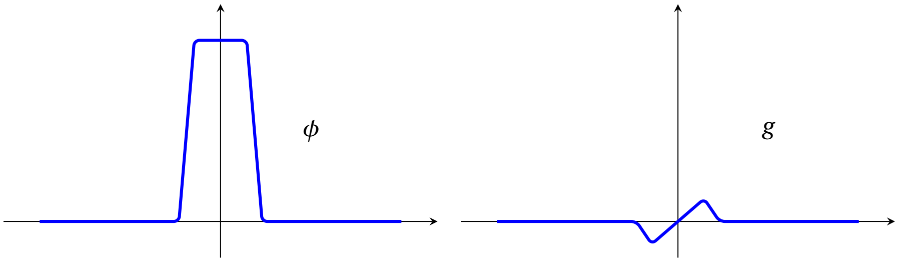


FIGURE 6.3.3. Bump functions

$$f\colon \mathbb{T}^2 \to \mathbb{T}^2, \quad x \mapsto \begin{cases} F(x) & \text{if } x \notin U, \\ F(x_1,x_2) + (0, \phi(x_1)g(x_2)) & \text{if } x = (x_1,x_2) \in U. \end{cases} \tag{6.3.1}$$

The fixed points of f are in U, where $(x_1,x_2) = f(x_1,x_2)$ is equivalent to

$$\begin{aligned} x_1 &= \lambda_1 x_1, \\ x_2 &= \lambda_2 x_2 + \phi(x_1)g(x_2). \end{aligned}$$

The first equation implies $x_1 = 0$ so the second equation reduces to

$$\left(\lambda_2 - 1 + \frac{\phi(x_2)}{1+bx_2^2}\right)x_2 = 0$$

with solutions $x_2 = 0, \pm\bar{x}$, where $\phi(\bar{x}) = (1-\lambda_2)(1+b\bar{x}^2)$.

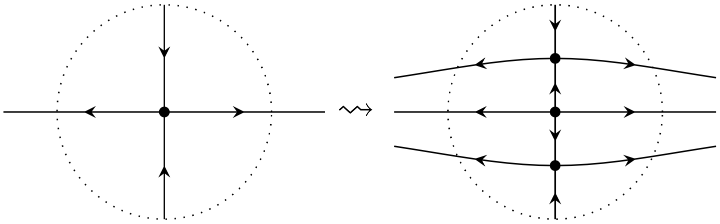


FIGURE 6.3.4. The DA-bifurcation of the fixed point

In order to determine the nature of these fixed points we note that

$$Df_{(x_1,x_2)} = \begin{pmatrix} \lambda_1 & 0 \\ \phi'(x_1)g(x_2) & h(x_1,x_2) \end{pmatrix} = \begin{cases} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 + 1 \end{pmatrix} & \text{when } (x_1,x_2) = (0,0), \\ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 + g'(\pm\bar{x}) \end{pmatrix} & \text{when } (x_1,x_2) = (0,\pm\bar{x}), \end{cases}$$

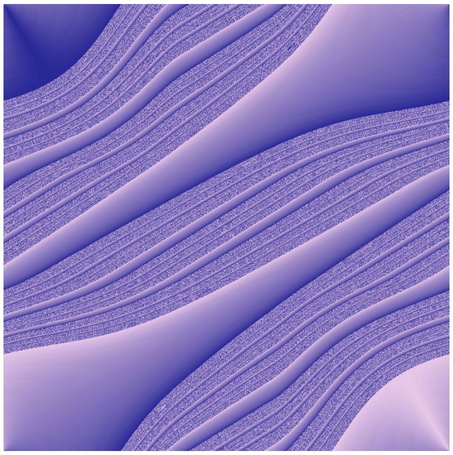
with $h(x_1,x_2) := \lambda_2 + \phi(x_1)g'(x_2)$ and $g'(\pm\bar{x}) = \frac{\pm\bar{x}\phi'(\pm\bar{x})}{1+b\bar{x}^2} + (1-\lambda_2)\frac{1-b\bar{x}^2}{1+b\bar{x}^2} < 0 + 1 - \lambda_2$. Thus, $(0,0)$ is a repelling fixed point and $(0,\pm\bar{x})$ are hyperbolic fixed points.

The stable manifold of 0 is f-invariant, and Df preserves the stable subbundle E^s of F although it may not contract vectors in E^s everywhere, and in fact permutes the stable manifolds for F in the same way as F does. The unstable manifold

$$W = W^{uu}(0) = \{p \in \mathbb{T}^2 \mid \alpha(p) = \{0\}\} = \bigcup_{n\in\mathbb{N}} f^n(U_0)$$

of 0, where U_0 is a sufficiently small open neighborhood of 0, is open and we show later that it is dense.

Proposition 6.3.6. *If b is sufficiently large, then $\Lambda := \mathbb{T}^2 \smallsetminus W$ is a hyperbolic attractor.*



Picture after Yves Coudène, from `https://www.lpsm.paris/pageperso/coudene/dyn1.html`

FIGURE 6.3.5. The unstable manifold of 0

Lemma 6.3.7. *There exists $\lambda' < 1$ such that $h(x_1, x_2) < \lambda'$ on Λ.*

PROOF. By compactness it suffices to show that $h < 1$ on Λ. Let

$$V := \{(x_1, x_2) \in U \mid h(x_1, x_2) \geq 1\} = \bigcup_{(x_1,x_2)\in U} \big(\{x_1\} \times V_{x_1}\big)$$

where $V_{x_1} = \{x_2 \mid h(x_1, x_2) \geq 1\}$. Note that $h(x_1, t) = \lambda_2 + \phi(x_1) g'(t) \geq 1$ if and only if $g'(t) \geq \dfrac{1-\lambda_2}{\phi(x_1)}$. Since g' is an even function it follows that V_{x_1} is a symmetric interval for all x_1 and furthermore if $x > y \geq 0$ then $\phi(x) \leq \phi(y)$ and $V_x \subset V_y$. On the other hand if we write $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) = (x_1', x_2')$ then $h(x_1, x_2) = \dfrac{\partial f_2}{\partial x_2}(x_1, x_2)$ and $f^{-1}(\{x_1'\} \times V_{x_1'}) = \{x_1\} \times V'_{x_1}$ (since f permutes unstable leaves of F) with V'_{x_1} symmetric and of length no more than that of V_{x_1} (since $h(x_1', x_2') \geq 1$). Thus $V'_{x_1} \subset V_{x_1}$ and since $f(x_1, 0) = F(x_1, 0)$ we conclude that $f^{-1}(V) \subset V$. Moreover, since $\{0\} \times V_0 \subset \mathbb{T}^2 \smallsetminus \Lambda$ we find $\{x_1\} \times V_{x_1} \subset \mathbb{T}^2 \smallsetminus \Lambda$ for all x_1 sufficiently close to 0. Consequently $f^{-n}(v) \subset \mathbb{T}^2 \smallsetminus \Lambda$ for some $n \in \mathbb{N}$ and hence $V \subset \mathbb{T}^2 \smallsetminus \Lambda$. □

PROOF OF PROPOSITION 6.3.6. $\Lambda := \mathbb{T}^2 \smallsetminus W$ is an attractor by definition (and $\mathbb{T}^2 \smallsetminus \bar{U}_0$ is a trapping region for it). The lemma shows that on Λ the diagonal elements of Df are λ_1 and a function bounded from above by $\lambda' < 1$. For large b the off-diagonal element is close to zero since $\phi'(x_1)$ is bounded and $g(x_2) \leq \dfrac{x}{1+bx^2} \leq \dfrac{1}{2b}$.

One obtains hyperbolicity of Λ as follows: If we write $Df_x = \begin{pmatrix} \lambda_1 & 0 \\ G(x) & H(x) \end{pmatrix}$ and take $\epsilon \in (0, \sqrt{\lambda_1^2 - 2})$ such that $\dfrac{1}{\lambda_1 - \lambda'} < \sqrt{1 + 1/\epsilon^2} - 1$, then $|G(x)| < \epsilon$ for all $x \in \Lambda$ if b is sufficiently large. Consider now horizontal cones of the form $|v| < \gamma|u|$ with $\dfrac{\epsilon}{\lambda_1 - \lambda'} < \gamma < \sqrt{\epsilon^2 + 1} - \epsilon < 1$. Note that these are invariant under Df since if we let $(u', v') \coloneqq Df_x(u, v)$ then

$$|v'| = |G(x)u + H(x)v| < \epsilon|u| + \lambda'|v| < (\lambda'\gamma + \epsilon)|u| < \gamma\lambda_1|u| = \gamma|u'|$$

since $\epsilon < (\lambda_1 - \lambda')\gamma$. To see that vectors in γ-cones expand, note that

$$\begin{aligned}
|(u', v')|^2 &= u'^2 + v'^2 = \lambda_1^2 u^2 + (G(x)u + H(x)v)^2 \\
&\geq \lambda_1^2 u^2 + G^2(x)u^2 - 2\underbrace{|G(x)|\,H(x)\,|u||v|}_{\leq \gamma |G(x)|\,H(x)\,u^2} + \underbrace{H^2(x)v^2}_{= -[1-H^2(x)]v^2 + v^2 \geq -\gamma^2[1-H^2(x)]u^2 + v^2} \\
&\geq [\lambda_1^2 + \underbrace{G^2(x)}_{>-\epsilon^2} - 2\gamma\underbrace{|G(x)|}_{<\epsilon}\,\underbrace{H(x)}_{<1} - \gamma^2\underbrace{(1 - H^2(x))}_{<1}]u^2 + v^2 \\
&> (\lambda_1^2 - \underbrace{(\epsilon^2 + 2\gamma\epsilon + \gamma^2)}_{=(\epsilon+\gamma)^2 < \epsilon^2 + 1 < \lambda_1^2 - 2 + 1 = \lambda_1^2 - 1})u^2 + v^2 > u^2 + v^2.
\end{aligned}$$

Since the stable manifolds are given, hyperbolicity follows from Proposition 5.1.7 without studying vertical cones. □

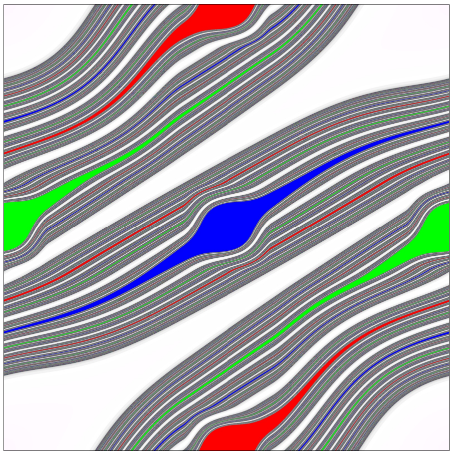
To show that $W = \mathbb{T}^2 \smallsetminus \Lambda$ is dense in $\mathbb{T}^2$, consider $p \in \Lambda$ and any open neighborhood U_p of p. Then there is a point $q \in U_p$ that is periodic for F, with period n, say. The stable manifold L of q (under F) is thus f^n-invariant and dense. Density of W follows if we can find $N \in \mathbb{N}$ such that $f^{-Nn}(L^1) \cap W \neq \varnothing$, where $L^1 = L \cap U_p$. But this is necessarily the case, since otherwise $L_f \coloneqq \bigcup_{n \in \mathbb{N}} f^{-Nn}(L^1) \subset \Lambda$ and by hyperbolicity f^{-n} expands L_f so $L_f = L$. But L is dense in $\mathbb{T}^2$, so we would have $\Lambda = \mathbb{T}^2$, a contradiction. Thus Λ is the complement of an open dense set.

We have thus produced a hyperbolic attractor on $\mathbb{T}^2$.

We conclude this section with the presentation of a closely related attractor on a sphere or a disk.

Definition 6.3.8. The *Plykin attractor* is suspension of the attractor on S^2 or a disk obtained from the DA attractor by identification of x and $-x$ as shown in Figure 6.3.7 and described below.

The DA map f is invariant under $J : \mathbb{T}^2 \to \mathbb{T}^2$, $J(x) = -x \pmod 1$,that is, $f \circ J = J \circ f$, and $(1/2, 1/2)$ is a periodic point of f since $f(1/2, 1/2) = F(1/2, 1/2) = (1/2, 0)$, $f(1/2, 0) = (0, 1/2)$, and $f(0, 1/2) = (1/2, 1/2)$. Thus, undertaking a like construction on those 3 additional points gives a map $f_P : \mathbb{T}^2 \to \mathbb{T}^2$, which commutes with J, has a single repelling fixed point, a repelling period 3-orbit, and a hyperbolic attractor Λ (Figure 6.3.6).



Picture after Yves Coudène, from `https://www.lpsm.paris/pageperso/coudene/dyn1.html`

FIGURE 6.3.6. The DA construction repeated on a period-3 orbit

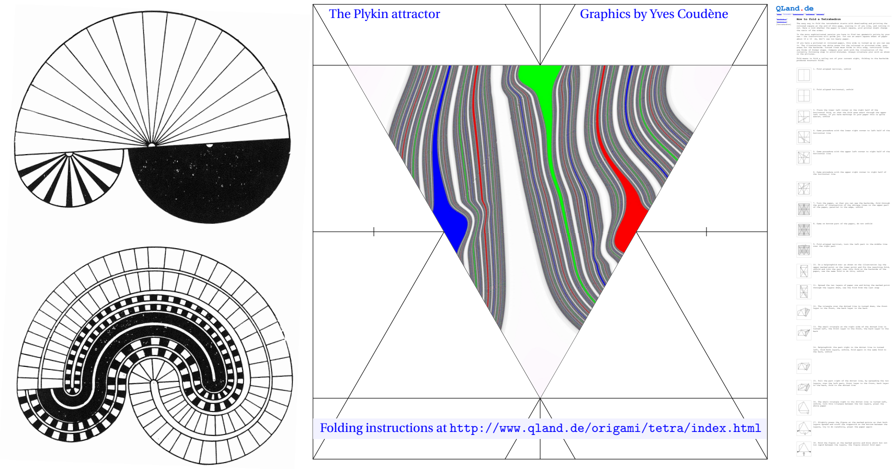
Note that J fixes these 4 repelling points:

$$-\left(\frac{1}{2},\frac{1}{2}\right)=\left(\frac{1}{2},\frac{1}{2}\right),-\left(\frac{1}{2},0\right)=\left(\frac{1}{2},0\right),-\left(0,\frac{1}{2}\right)=\left(0,\frac{1}{2}\right).$$

Thus, if V_i, $i=1,\dots,4$, are disks around $(0,0),(1/2,1/2),(1/2,0),(0,1/2)$, respectively, of the same radius and contained in $\mathbb{T}^2\setminus\Lambda$, then $M:=(\mathbb{T}^2\setminus\bigcup_{i=1}^4 V_i)/(x\sim -x)$ is a smooth manifold, indeed a 2-sphere with four holes. Since $f_P(-x)=-f_P(x)$, it induces a map $f':M\to M$ which is smooth and injective. Filling $S^2\setminus M$ with four repellers (one fixed and one period-3 cycle) gives a diffeomorphism $\tilde{f}:S^2\to S^2$ with a hyperbolic attractor (obtained by projecting Λ onto M). This is the (discrete-time) *Plykin attractor* on S^2, shown in Figure 6.3.7.[4]

Furthermore, if we remove a well-chosen neighborhood of the fixed point we obtain a closed 2-disk D such that $f(D)\subset\operatorname{int}(D)$, and the nonwandering set of the

[4] Topologically, this is also interesting because the 3 pieces of the basin of the repelling periodic orbit are "Wada basins," an example of *lakes of Wada*: they are 3 disjoint connected open sets in the plane or sphere that share the same boundary—the Plykin attractor in this case. This notion was introduced by Kunizô Yoneyama who credits Takeo Wada with the idea (on page 60 of "Theory of Continuous Set of Points (not finished)," Tohoku Mathematical Journal **12**, 43–158, see `https://www.jstage.jst.go.jp/article/tmj1911/12/0/12_0_43/_pdf/-char/en`).


The Plykin attractor
Graphics by Yves Coudène
Folding instructions at http://www.qland.de/origami/tetra/index.html
QLand.de

Picture on left from [**241**], on right after Yves Coudène, from `https://www.lpsm.paris/pageperso/coudene/dyn1.html`

FIGURE 6.3.7. The Plykin map and a template for folding a realistic rendition onto a tetrahedron ≃ sphere (more simply, cut out the triangle and fold along the lines connecting the 3 periodic points to gather the white vertices at the top)

restriction to D consists of the attractor Λ and the repelling period 3-orbit. The suspension of this flow is a flow on a solid torus.

In both cases, the suspension is the attractor in Definition 6.3.8.

4. Markov partitions

We now construct Markov partitions for hyperbolic sets. The construction for flows and maps are similar, but there are nontrivial difficulties for flows due to the flow direction. The essential idea is to construct a fine mesh of many local sections and to consider the return map (Figure 0.1.1) to their union, which is a global section. The flow is a special flow over this return map, and we will construct an analog of a Markov partition for the return map. This establishes a connection with a symbolic system that is defined as a special flow (Definition 1.2.7) over a topological Markov chain with a roof function that corresponds to the "travel times" between the local sections for the smooth system. Due to the nature of this construction, these are also often called *Markov sections*.

If $D \subset M$ is a codimension-one disk of small diameter transverse to the flow and if $\Lambda \cap D \neq \varnothing$ then D is included in a ξ*-flow box* $U_\xi(D) \coloneqq \varphi^{[-\xi,\xi]}(D)$ for some small $\xi > 0$ (we use the shorthand $\varphi^A(B) \coloneqq \bigcup_{t\in A} \varphi^t(B)$). Within this flow box we define the projection to D by $\pi_D(\varphi^t(x)) \coloneqq x \in D$. This is smooth and induces a local product structure on D.

Definition 6.4.1. If D is such a transversal and $B \subset \Lambda \cap D$ is closed with $d(B, \partial D) > 0$ and with diameter sufficiently small (relative to $d(B, \partial D)$), then define

$$[\cdot,\cdot]_D \colon B \times B \to \Lambda \cap D, \quad [x, y]_D \coloneqq \pi_D([x, y]).$$

B is called a *rectangle* if $[B, B]_D \subset B$, in which case we can write $[\cdot,\cdot]_B \coloneqq [\cdot,\cdot]_D \restriction_{B\times B}$, and we define

$$\begin{aligned} W^s(x, B) &\coloneqq \{[x, y]_B \mid y \in B\} = B \cap \pi_D(U_\xi(D) \cap W^{ss}_\xi(x)), \\ W^u(x, B) &\coloneqq \{[z, y]_B \mid z \in B\} = B \cap \pi_D(U_\xi(D) \cap W^u_\xi(x)), \end{aligned}$$

provided $\operatorname{diam}(B)$ is small enough. B is said to be *proper* if $B = \overline{\operatorname{Int}_{\Lambda\cap D} B}$.

This gives local product charts defined by the homeomorphisms

$$P_x \colon W^u(x, B) \times W^s(x, B) \to B, \quad (u, v) \to [u, v]_B.$$

We write $\partial B = \partial^s B \cup \partial^u B$, where

$$\begin{aligned} \partial^s B &\coloneqq P_x(\partial W^u(x, B) \times W^s(x, B)), \\ \partial^u B &\coloneqq P_x(W^u(x, B) \times \partial W^s(x, B)), \end{aligned}$$

and $\partial W^i(x, B)$ is relative to $W^i(x, \Lambda \cap D)$ for $i = u, s$.

Definition 6.4.2. A *proper family* of size η is a finite collection $\{T_0, \dots, T_{n-1}\}$ of closed subsets of Λ such that $\Lambda = \varphi^{[-\eta,0]}(\mathscr{T})$, where $\mathscr{T} \coloneqq \bigcup_{i=0}^{n-1} T_i$ (this means $\mathscr{T}$ is a section and corresponds to being a cover in the discrete-time case) and there are differentiable transversals $D_0, \dots, D_{n-1}$ such that

1. $\dim D_i = \dim M - 1$,
2. $\operatorname{diam} D_i < \eta$,
3. $T_i = \overline{\operatorname{Int}_{\Lambda\cap D_i} T_i} \subset D_i$,
4. if $i \neq j$, then at least one of the sets $D_i \cap \varphi^{[0,\eta]}(D_j) = \varnothing$ or $D_j \cap \varphi^{[0,\eta]}(D_i) = \varnothing$.

Now for η sufficiently small (at least smaller than the expansive constant for the flow) and $\mathscr{T}$ a proper family of size η, each $x \in \mathscr{T}$ there admits a first positive $t(x) \le \eta$ such that $f^{t(x)}(x) \in \mathscr{T}$. Furthermore, since the D_i are pairwise disjoint, closed, transverse to the flow, and there are only finitely many of them, there exists some $\beta > 0$ such that $t(x) \ge \beta$ for all $x \in \mathscr{T}$.

Now let $F_{\mathcal{T}}:\mathcal{T}\to\mathcal{T}$ be defined by $F_{\mathcal{T}}(x)=f^{t(x)}(x)$. From the properties of the proper family we see that $F_{\mathcal{T}}$ is a bijection, but neither $t(x)$ or $F_{\mathcal{T}}(x)$ need be continuous on $\mathcal{T}$. However, the restriction of $F_{\mathcal{T}}$ to

$$\mathcal{T}^*:=\bigcap_{i\in\mathbb{Z}}F_{\mathcal{T}}^{-i}\Big(\bigcup_{i=0}^{n-1}\operatorname{Int}_{\Lambda\cap D_i}T_i\Big).$$

is continuous. By the Baire category theorem, $\mathcal{T}^*$ is dense in $\mathcal{T}$.

Definition 6.4.3. A proper family $\{R_0,\dots,R_{n-1}\}$ of sufficiently small size η is said to be a *Markov partition* if each R_i is a rectangle, $\mathcal{R}=\bigcup_{i=0}^{n-1}R_i$, and

(1) if $x\in U_{ij}:=\overline{\mathcal{R}^*\cap R_i\cap F_{\mathcal{R}}^{-1}(R_j)}$ then $W^s(x,R_i)\subset U_{ij}$,
(2) if $y\in V_{ki}:=\overline{\mathcal{R}^*\cap F_{\mathcal{R}}(R_k)\cap R_i}$ then $W^u(x,R_i)\subset V_{ki}$.

For $x\in\mathcal{R}$ let $i(x)$ be the unique $i\in\{1,\dots,n\}$ such that $x\in R_i$. By construction the map $i:\mathcal{R}\to\{1,\dots,n\}$ is continuous. Then the *itinerary map* map $I:\mathcal{R}^*\to\Sigma_n$ is continuous. Since α was chosen less than the expansive constant the map I is injective, but of course may not be bijective.

We now let

$$\Lambda_{\mathcal{R}}=\overline{\{I(x)\,:\,x\in\mathcal{R}^*\}}\subset\Sigma_n.$$

Since $\sigma(I(x))=I(F_{\mathcal{R}}(x))$ for $x\in\mathcal{R}^*$ we see that $\sigma(\Lambda_{\mathcal{R}})=\Lambda_{\mathcal{R}}$ and so $\Lambda_{\mathcal{R}}$ is a subshift of Σ_n.

It is useful to observe the following:

Lemma 6.4.4. *If R is a rectangle then $\partial_\Lambda R=\partial^s R\cup\partial^u R$.*

Proof. $x\in\operatorname{Int}_\Lambda R\Rightarrow x\in\operatorname{Int}_{\Lambda\cap W_\eta^{uu}(x)}(R\cap W_\eta^{uu}(x)\cap\Lambda)=\operatorname{Int}_{\Lambda\cap W_\eta^{uu}(x)}W_R^u(x)$ since R is a neighborhood of x in Λ. Thus $\partial^s R\subset\partial_\Lambda R$. Likewise $\partial^u R\subset\partial_\Lambda R$. If

$$x\in(\operatorname{Int}_{\Lambda\cap W_\eta^{ss}(x)}W_R^s(x))\cap(\operatorname{Int}_{\Lambda\cap W_\eta^{uu}(x)}W_R^u(x))$$

then by continuity of $[\cdot,\cdot]_D$ there is a neighborhood U of x in Λ such that for all $y\in U$ we have $[x,y]_D,[y,x]_D\in R$ hence

$$y':=[[y,x]_D,[x,y]_D]\in R\cap W_\eta^{ss}(x)\cap W_\eta^{uu}(y)\subset W_\eta^{ss}(x)\cap W_\eta^{uu}(y)\subset\{y\},$$

so $x\in\operatorname{Int}_\Lambda R$. □

Now we are ready to prove the anticipated result.

Theorem 6.4.5 (Markov partitions)**.** *Let Λ be a hyperbolic set and U be a neighborhood of Λ. Then there exists a hyperbolic set $\tilde\Lambda$ with $\Lambda\subset\tilde\Lambda\subset U$ that has a Markov partition. That is to say there is a there is a semiconjugacy from a symbolic flow to $\Phi\restriction_{\tilde\Lambda}$ that is finite-to-one and one-to-one on a residual set of points, where the roof function for the subshift of finite type corresponds to the travel times between the local sections for the smooth system.*

Proof. We may assume that U is sufficiently small so that Λ_U is a hyperbolic set. If not simply take a smaller neighborhood inside U. We also assume that we have an adapted metric on Λ_U.

There is an $r>0$ such that for all $x\in\Lambda_U$ we have $\exp(E^s_r(x)\oplus E^u_r(x))=D_x$ such that for all $y\in D_x$ the angle between T_yD and $\dot{\varphi}^t(y)$ is between $(\pi/2-\eta,\pi/2+\eta)$ for η sufficiently small and such that $\varphi^{(-r,r)}(D_x)$ is a flow box. The existence of such a $r>0$ follows from compactness of Λ_U and continuity of the exponential map and flow. For $r>0$ possibly smaller, $\bigcup_{x\in\Lambda}B_{2r}(x)\subset U$.

Fix $\delta\in(0,r)$ small so that δ is less than an expansive constant for Λ_U. Let $\epsilon\in(0,\delta/2)$ be given by the Shadowing Lemma for Λ_U. Fix $\gamma<\epsilon/2$ such that if $z_1,z_2\in\Lambda$ are distinct where $d(z_1,z_2)<\gamma$, and $x,y\in\Lambda_U\cap D_{z_1}$ with $d(x,y)<\gamma$, then for $x',y'\in D_{z_2}$ the corresponding points in the forward orbit of x and y that are the first intersection with D_{z_2}, then $d(x',y')<\epsilon/2$.

Choose a set $P:=\{p_0,\dots,p_{N-1}\}$ that is γ-dense in Λ and let $\mathscr{R}=\bigcup_{i=0}^{N-1}D_{p_i}$. Define $F_{\mathscr{R}}$ on the set $\mathscr{R}$ as described in the previous section. We let

$$\Sigma_P:=\{\omega\in\Sigma_N\ \mid\ d(F_{\mathscr{T}}(p_{\omega_i}),p_{\omega_{i+1}})<\epsilon\}.$$

Then Σ_P is a topological Markov chain.

Let $X:=\{y\in\mathscr{R}:F^i_{\mathscr{R}}(y)$ exists for all $i\in\mathbb{Z}\}$. The conditions listed above imply

$$\Lambda\cap\mathscr{R}\subset X\subset\Lambda_U\cap\mathscr{R}.$$

Let Int_X be the interior of X with the subspace topology given by the disks. By the Baire Category Theorem the set

$$X^*:=\{x\in X:F^i_{\mathscr{R}}(x)\in\mathrm{Int}_X\ \forall\, i\in\mathbb{Z}\}$$

is a dense G_δ in X.

For each $x\in X^*$ there exists a unique itinerary $\omega\in\Sigma_P$ such that $F^i_{\mathscr{R}}(x)\in D_{p_{\omega_i}}$ for all $i\in\mathbb{Z}$ and the set of such ω is a dense G_δ set in Σ_P denoted Σ^*_P. We define the itinerary map by $\beta^*:\Sigma^*_P\to X^*$. By continuity we can extend this map to a map β defined onto a set $\hat{\Lambda}\subset X$ such that $\beta:\Sigma_P\to\hat{\Lambda}$ is a semiconjugacy.

Also, for each $\omega\in\Sigma^*_P$ there exists a unique minimal $f(\omega)$ such that

$$F_{\mathscr{R}}(\beta(\omega))=\beta(\sigma(\omega))=\varphi^{f(\omega)}(\beta(\omega)).$$

Again by continuity we can extend the function $f(\omega)$ to all of Σ_P. Since the D_i are differentiable we see that t is Lipschitz (since the D_i are Lipschitz and the flow is smooth). So we have a function $f:\Sigma_P\to\mathbb{R}$ that is Hölder continuous with respect to the metric on Σ_A and is used to form the suspension flow, a hyperbolic symbolic flow.

The next claim follows directly from the construction and the choice of constants.

Claim 6.4.6. *β is given by the unique shadowing point to the pseudo orbit given by $g : \mathbb{R} \to M$ such that $g(t) = \varphi^s p_{\omega_i}$ where $s \in [0, f(\sigma^i \omega))$ and $\sum_{j=0}^{i} f(\sigma^j \omega) \le t < \sum_{j=0}^{i+1} f(\sigma^j \omega)$.*

For $\omega, \omega' \in \Sigma_P$ we follow the standard notation and let

$$[\omega, \omega']_i = \begin{cases} \omega_i & \text{for } i \ge 0, \\ \omega'_i & \text{for } i \le 0. \end{cases}$$

for any $\omega, \omega' \in \Sigma_P$ with $\omega_0 = \omega'_0$. Then $[\cdot, \cdot]$ commutes with β', that is,

$$\beta'([\omega, \omega']) = [\beta'(\omega), \beta'(\omega')]_{D_{p_{\omega_0}}}.$$

Define $R'_i := \{\beta'(\omega) \mid \omega_0 = i\}$. Next note that $\mathscr{R}' := \{R'_i \mid 0 \le i < N\}$ satisfies a condition similar to (1) in Definition 6.4.3. Namely, suppose that $x = \beta'(\omega)$ with $(\omega_0, \omega_1) = (i, j)$ and that $y = \beta'(\omega') \in W^s_{R'_i}(x)$ with $\omega'_0 = i$. Then $F_{\mathscr{R}}(y) \in W^s_\eta(F_{\mathscr{R}}(x))$ but also $y = [x, y] = [\beta(\omega), \beta(\omega')] = \beta([\omega, \omega'])$ and hence $F_{\mathscr{R}}(y) \in \beta(\sigma([\omega, \omega'])) \in R'_j$, so $F_{\mathscr{R}}(y) \in W^s_{R'_j}(F_{\mathscr{T}}(x))$. This proves half of the analog of (2) and the other half follows similarly, that is,

$$F_{\mathscr{R}}(W^s_{R'_i}(x)) \subset W^s_{R'_j}(F_{\mathscr{R}}(x)) \text{ and } W^u_{R'_j}(F_{\mathscr{R}}(x)) \subset F_{\mathscr{R}}(W^u_{R'_i}(x)). \tag{6.4.1}$$

Note also that by continuity of β' the R'_i are compact, hence closed. To obtain a Markov partition we need, however, proper rectangles with pairwise disjoint interiors. To that end we modify these rectangles.

If $R'_i \cap R'_j \ne \varnothing$ then we cut R'_i into four rectangles. Let

$$A := \{x \in \Lambda' \mid W^{ss}_\eta(x) \cap \partial^s R'_i = \varnothing,\ W^{uu}_\eta(x) \cap \partial^u R'_i = \varnothing \text{ for all } i\}$$

is open and dense. If $R'_i \cap R'_j \ne \varnothing$ then we cut R'_i into four rectangles as follows:

$$\begin{aligned} R(i,j,su) &:= R'_i \cap R'_j, \\ R(i,j,0u) &:= \{x \in R'_i \mid W^s_{R'_i}(x) \cap R'_j = \varnothing,\ W^u_{R'_i}(x) \cap R'_j \ne \varnothing\}, \\ R(i,j,s0) &:= \{x \in R'_i \mid W^s_{R'_i}(x) \cap R'_j \ne \varnothing,\ W^u_{R'_i}(x) \cap R'_j = \varnothing\}, \\ R(i,j,00) &:= \{x \in R'_i \mid W^s_{R'_i}(x) \cap R'_j = \varnothing,\ W^u_{R'_i}(x) \cap R'_j = \varnothing\}, \end{aligned} \tag{6.4.2}$$

and for $x \in A$ let

$$R(x) := \bigcap \{\operatorname{Int}_{\Lambda'} R(i,j,q) \mid x \in R'_i,\ R'_i \cap R'_j \ne \varnothing,\ x \in R(i,j,q),\ q \in \{su, 0u, s0, 00\}\}.$$

Then $\overline{R(x)}$ are rectangles covering $R'_i \cap A$ and the $R(x)$ are finitely many pairwise disjoint open rectangles, so

$$\mathscr{R} := \{\overline{R(x)} \mid x \in A\} =: \{R_0, \dots, R_{m-1}\}$$

is a finite cover of $\beta'(\Sigma_P)$ by proper rectangles with pairwise disjoint interiors. We show that this is the desired Markov partition by showing that the Markov condition (2) of Definition 6.4.3 holds. It suffices to show that $F_{\mathscr{R}}(W^s_{R_i}(x)) \subset W^s_{R_j}(F_{\mathscr{R}}(x))$ for $x \in R_i \cap F^{-1}_{\mathscr{R}}(R_j)$ since the second half then follows by considering $F^{-1}_{\mathscr{R}}$.

We begin by showing that

$$R(F_{\mathscr{R}}(x)) = R(F_{\mathscr{R}}(y)) \text{ for } x, y \in A' := A \cap F^{-1}_{\mathscr{R}}(A),\ y \in W^s_{R(x)}(x). \tag{6.4.3}$$

First notice that if $x = \beta'(\omega)$ with $(\omega_0, \omega_1) = (i, j)$ and $y \in W^s_{R(x)}(x)$ then

$$F_{\mathscr{R}}(y) \in F_{\mathscr{R}}(W^s_{R(x)}(x)) \subset F_{\mathscr{R}}(W^s_{R'_i}(x)) \subset W^s_{R'_j}(f(x)) \subset R'_j$$

by (6.4.1). Thus $F_{\mathscr{R}}(x), F_{\mathscr{R}}(y) \in R'_j$. Next suppose $R'_j \cap R'_k \neq \varnothing$. To show that

$$F_{\mathscr{R}}(x), F_{\mathscr{R}}(y) \in R(j, k, q)$$

for some $q \in \{su, 0u, s0, 00\}$ note that the stable condition in the case distinction (6.4.2) is settled because $W^s_{R'_j}(F_{\mathscr{R}}(x)) = W^s_{R'_j}(F_{\mathscr{R}}(y))$ from above. We check the unstable condition by showing that if $F_{\mathscr{R}}(z) \in W^u_{R'_j}(F_{\mathscr{R}}(x)) \cap R'_k$ then $\varnothing \neq W^u_{R'_j}(F_{\mathscr{R}}(y)) \cap R'_k$. We show

Claim 6.4.7. $[F_{\mathscr{R}}(z), F_{\mathscr{R}}(y)] \in W^s_{R'_k}(F_{\mathscr{R}}(z)) \cap W^u_{R'_j}(F_{\mathscr{R}}(y))$.

Proof. Write $x = \beta'(\omega)$, $(\omega_0, \omega_1) = (i, j)$, $z = \beta(\omega')$, $(\omega'_0, \omega'_1) = (l, k)$. Then $F_{\mathscr{R}}(z) \in W^u_{R'_j}(F_{\mathscr{R}}(x)) \subset F_{\mathscr{R}}(W^u_{R'_i}(x))$ and hence $z \in W^u_{R'_i}(x) \cap R'_l$. $R(x) = R(y)$ (assumed in (6.4.3)) implies $x, y \in R(i, j, q)$ for some q, so there exists $z' \in W^u_{R'_i}(y) \cap R'_l$, and $[z, y] \in W^s_{R'_l}(z) \cap W^u_{R'_i}(y)$. Now $z = \beta(\omega')$, so $F_{\mathscr{R}}(W^s_{R'_l}(z)) \subset W^s_{R'_k}(F_{\mathscr{R}}(z))$. Since $F_{\mathscr{R}}(y)$, $F_{\mathscr{R}}(z) \in R'_j$, a rectangle, $[F_{\mathscr{R}}(z), F_{\mathscr{R}}(y)] \in W^s_{R'_k}(F_{\mathscr{R}}(z)) \cap W^u_{R'_j}(F_{\mathscr{R}}(y))$. □

This proves (6.4.3). To obtain the Markov condition (2) let

$$C^s := \bigcup \{W^s_\zeta(x) \mid x \in \bigcup_i \partial^s R'_i\}, \qquad C^u := \bigcup \{W^{uu}_\zeta(x) \mid x \in \bigcup_i \partial^u R'_i\},$$

and

$$B := \Lambda \smallsetminus ((C^s \cup C^u) \cup F^{-1}_{\mathscr{R}}(C^s \cup C^u)).$$

If $x \in B$ then $W^s_{R(x)}(x) \cap X'$ is open and dense in $W^s_{\overline{R(x)}}(x)$, so $R(F_{\mathscr{R}}(y)) = R(F_{\mathscr{R}}(x))$ by (6.4.3) and therefore $F_{\mathscr{R}}(W^s_{\overline{R(x)}}(x)) \subset \overline{R(F_{\mathscr{R}}(x))}$, which implies that $F_{\mathscr{R}}(W^s_{\overline{R(x)}}(x)) \subset$

$W^s_{\overline{R(F_{\mathscr{T}}(x))}}(F_{\mathscr{R}}(x))$. It only remains to verify this condition for arbitrary x. But if $x \in \operatorname{Int} R_i \cap F_{\mathscr{R}}^{-1}(\operatorname{Int} R_j)$ then there exists $x' \in B \cap \operatorname{Int} R_i \cap F_{\mathscr{R}}^{-1}(\operatorname{Int} R_j)$ and

$$\begin{aligned} F_{\mathscr{R}}(W^s_{R_i}(x)) = F_{\mathscr{R}}(\{[x,y] \mid \ y \in W^s_{R_i}(x')\}) &= \{[F_{\mathscr{R}}(x), F_{\mathscr{R}}(y)] \mid \ y \in W^s_{R_i}(x')\} \\ &\subset \{[F_{\mathscr{R}}(x), z] \mid \ z \in W^s_{R_j}(F_{\mathscr{R}}(x'))\} \subset W^s_{R_j}(F_{\mathscr{R}}(x)). \quad \square \end{aligned}$$

This yields the existence of a semiconjugacy to a special flow over a topological Markov chain. It is also useful to note:

Lemma 6.4.8. *With the above notation $F_{\mathscr{R}}(\partial^s \mathscr{R}) \subset \partial^s \mathscr{R}$ and $\partial^u \mathscr{R} \subset F_{\mathscr{R}}(\partial^u \mathscr{R})$.*

Proof. For $x \in R_i$ there exist $j, x_n \in \operatorname{Int} R_i \cap f^{-1}(R_j))$ such that $x_n \to x$. and $x \in R_i \cap F_{\mathscr{R}}^{-1}(R_j)$, hence $W^u_{R_j}(F_{\mathscr{R}}(x)) \subset F_{\mathscr{R}}(W^u_{R_i}(x))$. Thus, if $x \notin \partial^s \mathscr{R}$, hence $W^u_{R_j}(F_{\mathscr{R}}(x))$ is a neighborhood of $F_{\mathscr{R}}(x)$ in $W^s_\eta(F_{\mathscr{R}}(x)) \cap \Lambda$, then $W^u_{R_i}(X)$ is a neighborhood of x in $W^s_\eta(x) \cap \Lambda$, so $x \notin \partial^s \mathscr{R}$. The other inclusion follows by considering $F_{\mathscr{R}}^{-1}$. □

For the rectangles $\mathscr{R}$ formed above let A be the associated transition matrix so

$$A_{ij} = \begin{cases} 1 & F_{\mathscr{R}}(\operatorname{Int}(R_i) \cap \operatorname{Int}(R_j) \neq \varnothing \\ 0 & \text{else} \end{cases}.$$

Let Σ_A be the associated subshift of finite type and $\beta' : \Sigma_A \to \Lambda'$ be the associated itinerary map as defined above.

Another intuitive and useful consequence is that topological transitivity are equivalent for the hyperbolic set and its Markov model:

Proposition 6.4.9. *If Λ is a hyperbolic set for the flow then the topological Markov chain Σ_A obtained from the coding above is topologically transitive if $F_{\mathscr{T}}\restriction_{\Lambda'}$ is topologically transitive.*

Proof. If $\varnothing \neq U, V \in \Sigma_P$ are open there exist $\omega, \omega' \in \Sigma_A$ and $m \in \mathbb{N}$ such that $U' := C^{-m,\dots,m}_{\omega_{-m},\dots,\omega_m} \subset U$ and $V' := C^{-m,\dots,m}_{\omega'_{-m},\dots,\omega'_m} \subset V$ (using cylinder sets, hence

$$U_{\Lambda'} := \operatorname{Int} \bigcap_{-m \le i \le m} F_{\mathscr{T}}^{-i}(\operatorname{Int} R_{\omega_i}) \neq \varnothing \neq V_{\Lambda'} := \operatorname{Int} \bigcap_{-m \le i \le m} F_{\mathscr{T}}^{-i}(\operatorname{Int} R_{\omega'_i})$$

and, of course, $(\beta')^{-1}(U_{\Lambda'}) \subset U' \subset U$ and $(\beta')^{-1}(V_{\Lambda'}) \subset V' \subset V$. Thus topological transitivityof $F_{\mathscr{R}}$ imply the corresponding property for σ restricted to Σ_A. □

That under the factor β' points of X' have only one preimage is one way of saying that the factor map is very close to injective. We push this a little further now by noting that points have at most finitely many preimages.

Proposition 6.4.10. *Suppose Λ is a hyperbolic set and $\mathscr{R} = \{R_0, \dots, R_{m-1}\}$ is a Markov partition of diameter less than half an expansivity constant. Then under the factor map β no point has more than m^2 preimages.*

PROOF. We first use the coding to define an equivalence relation on $\{0,\dots,m-1\}$ by $i \sim j :\Leftrightarrow R_i \cap R_j \neq \varnothing$. This induces an equivalence relation on the shift space Σ_A:

$$\omega \sim_A \omega' :\Leftrightarrow \omega_i \sim \omega'_i \text{ for all } i.$$

Then $\beta'(\omega) = \beta'(\omega') \Leftrightarrow \omega \sim_A \omega'$.

Claim 6.4.11. *If $N \in \mathbb{N}_0$ and $\omega, \omega \in \Sigma_A$ satisfy*

(1) $\omega_i \sim \omega'_i$ when $0 \le i \le N$,
(2) $\omega_0 = \omega'_0$,
(3) $\omega_N = \omega'_N$,

then $\omega_i = \omega'_i$ for $0 \le i \le N$.

To see that this implies the proposition, suppose $x \in \Lambda'$ and $\omega^1,\dots,\omega^K \in (\beta')^{-1}(\{x\})$ are pairwise distinct. Then there is an $l_{ij} \in Z$ for which $\omega^i_{l_{ij}} \neq \omega^j_{l_{ij}}$. For any $n \ge |l_{ij}|$ we then have $(\omega^i_{-n},\dots,\omega^i_n) \neq (\omega^j_{-n},\dots,\omega^j_n)$. Since $\omega^i \sim \omega^j$ the claim then implies that $(\omega^i_{-n},\omega^i_n) \neq (\omega^j_{-n},\omega^j_n)$.

Taking $n_0 := \max_{ij} |l_{ij}|$ we find that the K pairs $(\omega^i_{-n_0},\omega^i_{n_0}) \in \{0,\dots,m-1\}^2$ are all distinct, which implies that $K \le m^2$. □

PROOF OF CLAIM 6.4.11. $x = \beta'(\omega) \in \bigcap_{i=0}^N f^{-i}(\overline{\mathrm{Int}(R_{\omega_i})}) = \overline{\bigcap_{i=0}^N f^{-i}(\mathrm{Int}(R_{\omega_i}))}$, so after changing x slightly, if necessary, we may assume $x \in \bigcap_{i=0}^N f^{-i}(\mathrm{Int}(R_{\omega_i}))$. Note that by construction this leaves $(\omega_0,\dots,\omega_N)$ (the actual subject of the claim) unchanged. Likewise, we take $y = \beta'(\omega')$ such that $y \in \bigcap_{i=0}^N f^{-i}(\mathrm{Int}(R_{\omega'_i}))$.

To prove the claim we now take $z := [x,y]$ and show that $F^i_{\mathscr{R}}(z) \in \mathrm{Int}\, R_{\omega_i} \cap \mathrm{Int}\, R_{\omega'_i}$ (hence $\omega_i = \omega'_i$) for $0 \le i \le N$.

Since $x \in \mathrm{Int}\, R_{\omega_0}$ and $y \in \mathrm{Int}\, R_{\omega'_0} = \mathrm{Int}\, R_{\omega_0}$, we have

$$z = [x,y] \in R_{\omega_0} \cap W^{ss}_\epsilon(x) = W^s(x, R_{\omega_0}),$$

and actually, $z \in \mathrm{Int}\, W^s(x, R_{\omega_0})$.

For $0 \le i \le N$ we have $\omega_i \sim \omega'_i$, hence $R_{\omega_i} \cap R_{\omega'_i} \neq \varnothing$, so $d(F^i_{\mathscr{T}}(x), F^i_{\mathscr{T}}(y))$ is small enough to conclude that

$$F^i_{\mathscr{R}}(z) = F^i_{\mathscr{R}}([x,y]) = [F^i_{\mathscr{R}}(x), F^i_{\mathscr{R}}(y)].$$

Applying the Markov property recursively, we therefore conclude that indeed

$$F^i_{\mathscr{R}}(z) \in \mathrm{Int}\, W^s(F^i_{\mathscr{R}}(x), R_{\omega_i}) \subset \mathrm{Int}\, R_{\omega_i} \text{ when } 0 \le i \le N. \tag{6.4.4}$$

Working backwards in like manner from the starting point

$$F^N_{\mathscr{R}}(z) = [F^N_{\mathscr{R}}(x), F^N_{\mathscr{R}}(y)] \in \mathrm{Int}\, W^u(F^N_{\mathscr{R}}(y), R_{\omega'_N})$$

we obtain that

$$F_{\mathscr{R}}^{i}(z) \in \operatorname{Int} W^{u}(F_{\mathscr{R}}^{i}(y), R_{\omega'_i}) \subset \operatorname{Int} R_{\omega'_i} \text{ when } 0 \le i \le N. \tag{6.4.5}$$

Combining (6.4.4) and (6.4.5) shows $F_{\mathscr{R}}^{i}(z) \in \operatorname{Int} R_{\omega_i} \cap \operatorname{Int} R_{\omega'_i}$ and hence $\omega_i = \omega'_i$ for $0 \le i \le N$. □

Let us note a related consequence.

Proposition 6.4.12. *If $x \in \Lambda'$ is* bitransitive*, that is, x has dense positive and negative semiorbits then x has only one preimage under the coding.*

Proof. Given $\omega, \omega' \in (\beta')^{-1}(x)$ and $n < m \in \mathbb{Z}$ there are $n' < n$ and $m' > m$ such that $F_{\mathscr{T}}^{n'}(x) \in \operatorname{Int}(R_{\omega_{n'}})$, $F_{\mathscr{R}}^{n'}(x) \in \operatorname{Int}(R_{\omega'_{n'}})$, $F_{\mathscr{R}}^{m'}(x) \in \operatorname{Int}(R_{\omega_{m'}})$ and $F_{\mathscr{R}}^{m'}(x) \in \operatorname{Int}(R_{\omega'_{m'}})$. Thus $\omega_{n'} = \omega'_{n'}$ and $\omega_{m'} = \omega_{m'}$, so $\omega \sim_A \omega'$ implies $(\omega_n, \dots, \omega_m) = (\omega'_n, \dots, \omega'_m)$. Since n, m were arbitrary, this proves $\omega = \omega'$. □

Another easy consequence of Proposition 6.4.10 is

Corollary 6.4.13. *If x is periodic then so is every code, that is, every point of $(\beta')^{-1}(x)$.*

Shifting by common periods of preimages, this in turn implies

Corollary 6.4.14. *If x is periodic and $\omega, \omega' \in (\beta')^{-1}(x)$ satisfy $\omega_i = \omega_i$ for some i, then $\omega = \omega'$.*

Remark 6.4.15. By Theorem 4.2.13 the symbolic flow associated with any Markov partition for a hyperbolic set Λ of a flow Φ has the same topological entropy as $\Phi\restriction_{\Lambda}$.

With Proposition 6.4.10 and Proposition 6.4.12 in mind, one may ask when the semiconjugacy obtained is a conjugacy. Clearly it is necessary that Λ' be totally disconnected, because Σ_A is. By now we have enough machinery to show that this condition is indeed sufficient.

Proposition 6.4.16. *Let Λ be a totally disconnected hyperbolic set with a Markov partition. Then $F_{\mathscr{R}}\restriction_{\Lambda}$ is topologically conjugate to a topological Markov chain.*

Proof. We outline the proof, only skipping details that are easy to fill in.

Construct a cover by rectangles that are both open and closed. This is accomplished by taking closed open sets on stable and unstable manifolds of a point and using $[\cdot,\cdot]$ to produce such a rectangle. For each pair of such rectangles apply the cutting construction (6.4.2). If the diameters of the original rectangles are small enough then the rectangles thus obtained are also open and closed. Thus we have a partition of Λ into disjoint closed rectangles and hence there is a $\gamma > 0$ such that if two points are closer than γ then they are in the same rectangle and their bracket is in the same rectangle. Now code according to this partition. The coding map is an injective continuous map. The image is a closed invariant subset of the full shift which has a local product structure by the previous remark. Thus the image is a locally maximal subset of the full shift. (Even though this theorem was proved for hyperbolic sets it applies to shifts because they can be viewed as a horseshoe-type invariant set of a smooth system.) Finally, the image of Λ under the coding is an N-step topological Markov chain, and any n-step topological Markov chain is topologically conjugate to a topological Markov chain. □

5. Not locally maximal hyperbolic sets*

A long-standing question in smooth dynamical systems was whether every hyperbolic set Λ for a flow is included in a locally maximal hyperbolic set $\tilde{\Lambda}$. A stronger version of the question was whether, given any hyperbolic set Λ and any neighborhood U of Λ, there exists a locally maximal hyperbolic set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset U$ [**181**, p. 272]. Anosov called such sets *premaximal*.

By Theorem 6.4.5, given a hyperbolic set Λ and a neighborhood U there exists a hyperbolic set $\tilde{\Lambda}$ with $\Lambda \subset \tilde{\Lambda} \subset U$ that has a Markov partition. Although a locally maximal hyperbolic set has a Markov partition, this does not imply that any hyperbolic set with a Markov partition is locally maximal.

Anosov proved that one dimensional hyperbolic sets are premaximal: if Λ is a topologically one-dimensional hyperbolic set and U is a neighborhood of Λ, then there exists a locally maximal hyperbolic set $\tilde{\Lambda}$ with $\Lambda \subset \tilde{\Lambda} \subset U$.

In general, not only are hyperbolic sets not premaximal, but they may not even be included in *any* locally maximal hyperbolic set. The first example of this was due to Crovisier [**95**] and gave an open set of diffeomorphisms of the 3-torus each of which contains a hyperbolic set not contained in a locally maximal hyperbolic set. For flows we have the following.

Theorem 6.5.1. *Let M be a closed C^r manifold ($1 \le r \le \infty$) with $\dim M \ge 3$. Then there is a C^k-open set of C^k flows ($1 \le k \le r$) on M, each of which contains a hyperbolic set that is not contained in a locally maximal one.*

Proof. The basic strategy is to suspend an example constructed for diffeomorphisms (in [**116**]) as a modification of the Plykin attractor from Definition 6.3.8. We first review the construction of the hyperbolic set not included in any locally maximal set by modifying the Plykin attractor construction. Next we show how the suspension of this example leads to a counterexample on a 3-manifold. Finally, we show how this example can be extended to higher-dimensional settings.

Recall that the Plykin attractor from Definition 6.3.8 on the solid 2-torus is the suspension of a map of a closed disk D_0 (containing the discrete-time Plykin attractor Λ) strictly into itself, so the suspension is a smooth flow of a solid 2-torus T_0 with a hyperbolic attractor Λ_a (the suspension of the Plykin attractor) such that $\varphi^t(T_0) \subset \mathrm{int}(T_0)$ for all $t > 0$. Inside T_0 the nonwandering set consists of Λ_a and a repelling periodic orbit, of period 3, which is the suspension of a repelling period 3-orbit.

We now extend the flow to a larger torus T_1 so that $\phi^t(x) \to \Lambda_a$ as $t \to \infty$ for all $x \in \mathrm{int}(T_1)$, and so that every $x \in \partial T_1$ is 1-periodic. To do this it is sufficient to extend the map $f : D_0 \to D_0$ to a map f of a closed 2-disk D_1 containing D_0 in its interior. The extension is carried out by using a bump function so that the map remains the same in D_0 and such that the map is the identity on the boundary of D_1 while $f^n(x) \to \Lambda$ as $n \to \infty$ for each $x \in \mathrm{int}(D_1)$. The flow Φ on T_1 is simply the suspension of the map $f : D_1 \to D_1$ (with constant roof function 1).

We now thicken T_1 to a larger solid 2-torus T and extend the flow to T so that the flow is the identity in a neighborhood of the boundary of T. To do this we again first extend the flow so that in $T - T_1$ each point has period 1 and then use a bump function so that we obtain the desired flow on T. Then let D be a three dimensional disk such that T is contained in the interior of D and extend the flow to be the identity on $D \smallsetminus T$.

Fix some $p \in \partial T_1$ and an open neighborhood U of $\mathscr{O}(p)$ small enough to be disjoint from ∂T and Λ, and alter Φ in U so that p is a hyperbolic saddle periodic point with $W^{ss}(p) \subset U \cap \partial T_1$. Also, $W^{cu}(p) \setminus \mathscr{O}(p)$ contains two components, one of which is contained in $T - T_1$ and the other, denoted $W^*(p)$ is contained strictly in the interior of T_1.

Let $q \in \Lambda$ be a periodic point. Since $\overline{W^{cs}(q)} = \overline{W^s(\Lambda_a)} = \mathrm{int}(T_1)$ we know that given any $z \in W^*(p)$ there is a point in $W^{cs}(q)$ arbitrarily close to z. Perturb the flow in a neighborhood of z so that $z \in W^{cs}(q) \pitchfork W^*(p)$. This can be done since z is a wandering point for the flow. Indeed, for an arbitrarily small neighborhood U of z that contains a point of $W^{cs}(q)$ there is a $t > 0$ such that $\varphi^{-t}(U) \cap U = \varnothing$ since z is wandering, so if we perturb φ in $\varphi^{-t}(U)$ we can adjust $W^*(p)$ without changing $W^{cs}(q)$ in U.

Now we need two definitions. A hyperbolic set Λ for a C^1 flow has a *heteroclinic tangency* if there exist $x, y \in \Lambda$ such that $W^{ss}(x) \cap W^{uu}(y)$ contains a point of tangency. A point of *quadratic tangency* for a C^2 flow is defined as a point of heteroclinic tangency where the curvature of the stable and unstable manifolds differs at the point of tangency.

Take $w \in W^*(p)$ not in the orbit of z and $q' \in W^{uu}(q)$ not in the orbit of q. Another perturbation as above can ensure that $w \in W^*(p) \cap W^{cs}(q')$ and that, furthermore, $W^{uu}(p)$ and $W^{ss}(q')$ have a quadratic tangency at w. Let I be the

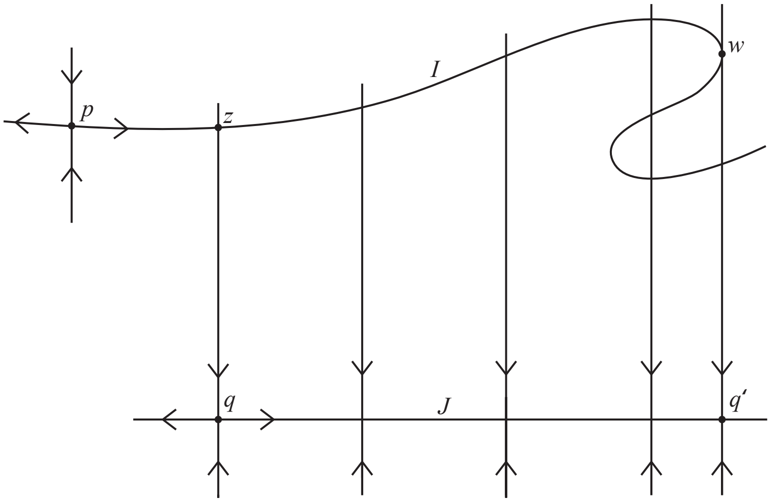


FIGURE 6.5.1. Intervals I and J.

segment of $W^{uu}(p)$ from z to w, and let J be the segment of $W^{uu}(q)$ from q to q'. Figure 6.5.1 shows a cross section of the constructed flow.

We will show that the resulting flow φ on the disk D contains a hyperbolic set that cannot be contained in a locally maximal set. Let $\Lambda = \Lambda_a \cup \mathcal{O}(p) \cup \mathcal{O}(z)$. The sets $\mathcal{O}(p)$, $\mathcal{O}(z)$, and Λ_a are invariant under ϕ, by definition. By construction, $\phi_t(z)$ converges to $\mathcal{O}(q)$ as $t \to \infty$ and converges to $\mathcal{O}(p)$ as $t \to -\infty$. Since these are both in Λ and Λ_a is closed, so is Λ.

For z we let $E^s_z = T_z W^{ss}(q)$ and $E^u_z = T_z W^{uu}(p)$. For $x = \varphi^t(z)$ for some $t \in \mathbb{R}$ we let $E^s_x = T_x W^{ss}(\varphi^t(q))$ and $E^u_x = T_x W^{uu}(\varphi^t(p))$. This gives an invariant splitting on the closed invariant set Λ and by the Inclination Lemma, this splitting varies continuously.

Let $\lambda_p, \lambda_a \in (0,1)$ be constants that guarantee hyperbolic behavior in $\mathcal{O}(p)$ and Λ_a, respectively. Let y be an arbitrary element of $\mathcal{O}(p)$. We assume an adapted metric, so for $t > 0$ and $v \in E^s(y)$ we have $\|D\phi_t(y)v\| \le \lambda_p^t \|v\|$, and for $t > 0$ and $v \in E^u(y)$ we have $\|D\phi_{-t}(y)r\| \le \lambda_p^t \|r\|$. Similarly, for Λ_a if $y \in \Lambda_a$ we have we have $\|D\phi_t(y)v\| \le \lambda_a^t \|v\|$, and for $t > 0$ and $v \in E^u(y)$ we have $\|D\phi_{-t}(y)r\| \le \lambda_p^t \|r\|$. Let $\lambda_{max} = \max\{\lambda_p, \lambda_a\}$ and $\lambda \in (\lambda_{max}, 1)$. Furthermore, there exists an $\epsilon > 0$ such that for all $x \in \Lambda$ where $d(x, \mathcal{O}(p)) < \epsilon$ we have contraction in E^s_x by at least λ and expansion in E^u_x by at least λ^{-1}. The same holds for points in Λ sufficiently close to $\mathcal{O}(q)$.

Since $\lim\limits_{t\to\infty}\phi_t(z) = \mathcal{O}(q)$ and $\lim\limits_{t\to\infty}\phi_{-t}(z) = \mathcal{O}(p)$, we know that given $\epsilon > 0$ there exists a $T > 0$ such that $\phi_t(z)$ is ϵ-close to $\mathcal{O}(q)$ and $\phi^{-t}(z)$ is ϵ-close to $\mathcal{O}(p)$ for all $t \geq T$. By the previous paragraph, we can guarantee hyperbolicity in ϵ-neighborhoods of $\mathcal{O}(p)$ and $\mathcal{O}(q)$ for ϵ sufficiently small. Now we only need guarantee hyperbolicity outside of those neighborhoods. For $t \in [-T, T]$ and $v \in \{a \in \mathbb{R}^3 : \|a\| = 1\}$, the function $\|D\phi_t(z)v\|$ is bounded because it is a continuous function on a compact domain. So, there exists a $C \geq 1$ such that, for any $x \in \Lambda$ and $t > 0$, we have $\|D\phi_t(x)v\| \leq C\lambda^t\|v\|$ for all $v \in E^s(x)$, and $\|D\phi_{-t}(x)v\| \leq C\lambda^t\|v\|$ for all $v \in E^u(x)$. Hence, Λ is hyperbolic.

Now suppose $\Lambda \subset \Lambda'$, where Λ' is a locally maximal hyperbolic set. We will see that this implies that $w \in \Lambda'$, but since $W^{ss}(p)$ and $W^{uu}(q')$ are not transverse at w this implies that the hyperbolic splitting for Λ' is not transverse, a contradiction. So to conclude the proof we show that w is necessarily in Λ'.

Fix $\epsilon > 0$ and $\delta > 0$ constants from the local product structure on Λ'. By construction, every point in I is in the stable manifold of exactly one point in J. For each $x \in I$ we then fix x' the unique point in J associated with x. Fix $T > 0$ such that $t \geq T$ implies $d(\varphi^t(x), \varphi^t(x')) < \delta/2$ for all $x \in I$. Such a T exists since I and J are compact and the stable manifolds connecting I to J depend continuously on points in I. Let $I' = I \cap \Lambda'$. Then $z \in I'$ and for $t \geq T$ we see that a small neighborhood of z in I is contained in I', to see this just take the bracket of $\varphi^t(z)$ and points in $\varphi^t(J)$ near $\varphi^t(q)$. Continuing to use the product structure we see that the entire interval I must be contained in Λ', a contradiction.

We now show the construction is robust under perturbation. Since transversality is trivially open, and hyperbolicity is open , it is sufficient to show that there remains a point $\tilde{w} \in W^{uu}(\tilde{p}) \cap W^{ss}(\tilde{u})$ for some $\tilde{u} \in W^{cu}_{(}\tilde{q})$ where $\tilde{p}$ is the continuation of p and $\tilde{q}$ is the continuation of q for the perturbed flow. By construction, the stable manifolds for all the $x \in W^{cu}_{(}\tilde{q})$ locally foliate the region, so there must exist a point $\tilde{u} \in W^{cu}_{\text{loc}}(\tilde{q})$ and a point $\tilde{w} \in W^{cs}(\tilde{u}) \cap W^{uu}(\tilde{p})$ such that the one-dimensional path $W^{uu}(\tilde{p})$ remains tangent to the two-dimensional plane $W^{cs}(\tilde{u})$ at $\tilde{w}$. Specifically, we have $T_{\tilde{w}}W^{uu}(\tilde{p}) \subsetneq T_{\tilde{w}}W^{cs}(\tilde{u})$.

For any 3-dimensional manifold we let simply embed the above example into a chart for the manifold and let the flow on the rest of the manifold be the identity. We then obtain an open set of flows on the manifold that contain a hyperbolic set that is never contained in a locally maximal hyperbolic set.

For higher dimensional examples we let $n = \dim(M)$ where $n > 3$. Then let D_n be a closed n-dimensional disk. For 3 dimensions we let the flow be defined as above. In the other directions we let the flow contract in the interior of T_1

stronger than any construction in the closed 3-disk and such that the closed 3-disk is invariant for the flow. Using a bump function we allow the flow to be the identity in a neighborhood of the boundary of D_n. One can check that for the resulting flow the set Λ is hyperbolic and has a stable splitting of dimension $n-2$ and a 1-dimensional unstable splitting. The previous construction shows that Λ can still not be included in a locally maximal hyperbolic set. Then as before we can show that Λ is never included in a locally maximal hyperbolic set. To include this in a manifold we again simply embed the flow in a local coordinate chart for the manifold and extend the flow to be the identity on the rest of the manifold. □

Exercises

6.1. Prove that for suspensions $t(x,y) \equiv 0$ in Proposition 6.2.2.

6.2. Describe the Bowen bracket (Proposition 6.2.2) explicitly for symbolic flows (see Exercise 1.17).

6.3. Show that the Smale horseshoe in Example 1.5.21 has local product structure.

6.4. Show that a hyperbolic attractor has local product structure.

6.5. Show that a compact factor of $\mathbb{H}$ (Section 2.4) has closed geodesics of incommensurate lengths.

6.6. For a compact factor Σ of $\mathbb{H}$ (Section 2.4) suppose v is a unit vector generating a closed geodesic γ_v. Show that there are unit vectors $v_i \to v$ such that each v_i generates a closed geodesic whose length is incommensurate with that of γ_v.

6.7. Show that the horocycle flow (Example 2.2.3) on a compact factor of $\mathbb{H}$ is minimal (Definition 1.6.21). (This replaces the use of Example 2.1.16 elsewhere.)

CHAPTER 7

Hölder structures and regularity*

This chapter studies a range of questions relating to the regularity of structures in hyperbolic dynamical systems. Some of these are technically useful statements about, for instance, the invariant foliations. Others are of interest in their own right, such as those leading to rigidity questions, which will encounter later.

1. Hölder regularity

The leading regularity notion in hyperbolic dynamics is Hölder continuity, and the essential reason is that both it and hyperbolicity are described by exponential behavior.

Definition 7.1.1. A map f between metric spaces is said to be Hölder continuous with exponent $\alpha \in (0,1]$ if $d(f(x), f(y)) \le (d(x,y))^\alpha$ for nearby x and y.

$f\colon \mathbb{R} \to \mathbb{R}$ is said to be β-*Hölder* continuous, written $f \in C^\beta$, if $f \in C^{[\beta]}$ and $f^{([\beta])}$ is Hölder continuous with exponent $\beta - [\beta]$. (In the literature this is usually denoted by $C^{[\beta],\beta-[\beta]}$ or $C^{[\beta]+(\beta-[\beta])}$; if $\beta < 1$ this coincides with the usual definition given above.)

$C^{\beta-} := \bigcap_{\epsilon>0} C^{\beta-\epsilon}$ is the space of functions which are Hölder of all orders less than β and $C^{\beta+} := \bigcup_{\epsilon>0} C^{\beta+\epsilon}$ denotes the functions which are $C^{\beta+\epsilon}$ for some ϵ. $C^{1,\mathrm{Lip}} \subset C^{2-\epsilon}$ is the space of functions with *Lipschitz* (that is, 1-Hölder) derivative.[1] A subbundle is said to be C^β (resp. $C^{\beta-}$, $C^{\beta+}$) if it is spanned by vector fields with C^β (resp. $C^{\beta-}$, $C^{\beta+}$) components in a chart.

The connection with exponential behavior is that exponentially small differences in the inputs result in exponentially small differences in the outputs of a Hölder continuous map. On one hand, this causes natural structures associated with hyperbolic dynamics to be Hölder continuous, and on the other hand, Hölder continuous functions play well with hyperbolic dynamics (see, for example, Proposition 8.3.1).

[1] See also Definition 4.2.32, Definition 12.1.1.

Section 7.2 showcases this interplay nicely by combining exponential closing and Hölder regularity to great effect. Section 7.3 establishes Hölder regularity of orbit-equivalences. In Section 7.4 we show that the stable and unstable subbundles are also Hölder continuous. This implies that an important function, the expansion rate, whose pressure (see Definition 4.3.1) we study in order to learn about smooth invariant measures, is Hölder-continuous. Later explorations seek the optimal degree of regularity: Section 7.5 begins with more of the same, tending to the joint strong subbundles rather than the individual ones, and then turns to rigidity issues by exploring more closely what regularity is optimal. Finally, Section 7.6 takes on the question of optimal regularity for the weak subbundles; this as well leads to issues that motivate rigidity theory.

2. Livshitz Theory

We have approached the abundance of periodic points in hyperbolic flows in a variety of ways. They are dense, and the Specification Property (Definition 8.3.2) says in essence that all conceivable orbit behaviors are exhibited by periodic orbits. We will also see their abundance in the context of ergodic theory—the measures discussed in Chapter 8 are all obtained as weak limits of measures on periodic orbits. Here, we present a much more geometric take on the abundance of periodic orbits, namely that by a combination of exponential closing and Hölder regularity, periodic data determine a global quantity. The fundamental theorem (previewed by Theorem 5.3.23) is the following:

Theorem 7.2.1 (Livshitz Theorem)**.** *Let Λ be a compact locally maximal hyperbolic set for a flow Φ generated by a vector field X on a manifold M and $a\colon \Lambda \to \mathbb{R}$ Hölder continuous such that $\varphi^T(x) = x \Rightarrow \int_0^T a(\varphi^t(x))\,dt = 0$. If there is a dense orbit in Λ, then there is a continuous $A\colon \Lambda \to \mathbb{R}$ such that $a = XA$, the derivative in the flow direction. Moreover, A is unique up to an additive constant and Hölder continuous with the same Hölder exponent as a. Furthermore, if $a \in C^1$, then $A \in C^1$.*

Proof. If $\Lambda = \overline{\mathcal{O}(x_0)}$, set $A(\varphi^t(x_0)) := \int_0^t a(\varphi^s(x_0))\,ds$.

Claim 7.2.2. *A is Hölder continuous on $\mathcal{O}(x_0)$ with the same Hölder exponent as a.*

Proof. Suppose $|a(x) - a(y)| \le Hd(x,y)^\alpha$ for small $d(x,y)$. If $t_1 < t_2$ are such that $\epsilon := d(\varphi^{t_1}(x_0), \varphi^{t_2}(x_0))$ is as in the Anosov Closing Lemma (Theorem 5.3.10) and as in Proposition 6.2.4, then a T-periodic orbit with $|T - t_2 + t_1| < \epsilon\ L\epsilon$-shadows $\varphi^{[t_1,t_2]}(x_0)$ and contains a point y such that

$$d(\varphi^{t_1+s}(x_0), \varphi^s(y)) \le C\epsilon\eta^{\min(s,T-s)} \quad \text{for} \quad s \in [0,T].$$

Thus,

$$\begin{aligned}\big|A(\varphi^{t_1}(x_0)) - A(\varphi^{t_1+T}(x_0))\big| &= \Big|\int_0^T a(\varphi^{T+s}(x_0))\,ds\Big|\\ &= \Big|\int_0^T a(\varphi^{T+s}(x_0)) - a(\varphi^s(y))\,ds + \underbrace{\int_0^T a(\varphi^s(y))\,ds}_{=0}\Big|\\ &\le \int_0^T \underbrace{\big|a(\varphi^{T+s}(x_0)) - a(\varphi^s(y))\big|}_{\le HC^\alpha\epsilon^\alpha\eta^{\alpha\min(s,T-s)}}\,ds\\ &\le 2HC^\alpha\epsilon^\alpha\int_0^T \eta^{\alpha s}\,ds\end{aligned}$$

On the other hand, $\big|A(\varphi^{t_2}(x_0)) - A(\varphi^{t_1+T}(x_0))\big| = \big|\int_{t_2}^{t_1+T} a(\varphi^s(x_0))\,ds\big| \le \epsilon\|a\|_\infty$. □

This implies that A is uniformly continuous and hence has a unique continuous extension to $\Lambda = \overline{\mathcal{O}(x_0)}$, which then clearly has the same Hölder exponent as well. Uniqueness: If $XA = XA'$, then $X(A - A') \equiv 0$, so $A - A'$ is a constant of motion (Definition 1.1.23), hence constant on the dense orbit, hence constant. Finally, a and XA are continuous and agree on a dense set and therefore coincide.

Now suppose $a \in C^1$. By the preceding, A is Lipschitz-continuous (hence differentiable a.e., but we need more). We show that the derivatives of A along stable (hence by symmetry unstable) leaves exist and are continuous. Since A is also continuously differentiable along orbits ($XA = a$), it is then C^1 by Lemma 7.2.3 below. If $y \in W^{ss}(x)$, then

$$\begin{aligned}A(y) - A(x) &= \lim_{T\to+\infty}\Big(-\int_0^T a(\varphi^s(y)) - a(\varphi^s(x))\,ds\Big) + A(\varphi^T(x)) - A(\varphi^T(y))\\ &= -\int_0^\infty a(\varphi^s(y)) - a(\varphi^s(x))\,ds.\end{aligned}$$

Differentiating with respect to $y = x + tv$ at $t = 0$ (in local coordinates) gives $D_vA(x) = -\int_0^\infty D_{v_s}a(\varphi^s(x))D_v(\varphi^s)(x)$ by the chain rule, where $v_s := D\varphi^s v$. Both factors of the integrand are exponentially small in s (since the v_s are exponentially small and a is C^1, and since v is a stable vector), so the improper integral converges uniformly, hence to a well-defined continuous function, which thus agrees with the derivative of the left-hand side. □

Lemma 7.2.3. *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is C^1 along the leaves of k continuous transverse foliations, then f is C^1.*

Proof. For y near x we need to find a linear map L_x such that $f(y) - f(x) = L_x(x - y)$ up to higher-order terms in $|y - x|$. We do this for 2 foliations A and B; for larger k the proof differs mainly by notation. For $z \in A(x) \cap B(y)$ we have

$$f(y) - f(x) = f(y) - f(z) + f(z) - f(x) = L_z^B(y-z) + L_x^A(z-x)$$

for linear maps L^A and L^B that depend continuously on the base point, so $L_z^B \to L_x^B$ as $y \to x$, hence $z \to x$ and thus $L_z^B(y-z) = L_x^B(y-z)$ up to higher order, and we can take L_x to be the linear map that restricts to L^A on $TA(x)$ and to L^B on $TB(x)$. □

Remark 7.2.4. We saw that any 2 solutions of the cohomological equation differ by a constant of motion (Proposition 1.6.5). If we drop the assumption that there is a dense orbit, then we can apply the Livshitz Theorem on each transitive component of the spectral decomposition, which gives a solution of the cohomological equation that is unique up to the addition of a function that is constant on each component.

Part of the interest in Theorem 7.2.1 and its proof is the regularity of the solution A of the cohomological equation. Since the existence of a bounded solution implies the vanishing of periodic data, we get a result that improves the regularity (and without using transitivity):

Corollary 7.2.5. *Let Λ be a compact hyperbolic set for a flow Φ generated by a vector field X on a manifold M and $A_0\colon M \to \mathbb{R}$ bounded. If $a := XA_0\restriction_\Lambda$ is Hölder continuous, then there is an $A\colon \Lambda \to \mathbb{R}$ (unique up to the addition of a constant of motion) such that $XA_0 = XA$, and A is Hölder continuous with the same Hölder exponent as a, and C^1 if a is.*

The nature of the uniqueness assertion is such that for *continuous* A_0, the conclusion is about A_0 itself:

Corollary 7.2.6. *Let Λ be a compact hyperbolic set for a flow Φ generated by a vector field X on a manifold M and $A\colon M \to \mathbb{R}$ continuous. If $XA\restriction_\Lambda$ is Hölder continuous or C^1, then so is A.*

To see an application of this theory, recall that we noted in Section 1.3 that conjugacy (flow-equivalence) of flows preserves the periods of closed orbits. For time-changes of hyperbolic flows, there is an easy converse: If the periods of closed orbits are unaffected by a time change, then the time-changed flow is conjugate to the original one.

Proposition 7.2.7. *Let Φ be a flow on a manifold M with a compact topologically transitive hyperbolic set Λ and Ψ a time change of Φ with a Hölder continuous α (as in Proposition 1.2.2). If the periods of all periodic orbits of Φ and Ψ agree then Φ and Ψ are conjugate on Λ via a homeomorphism which is Hölder continuous and C^1 if the time change is C^1.*

Proof. Recall from page 40 that a time change $\psi^t(x) = \varphi^{\alpha(t,x)}(x)$ arises from a conjugacy, that is, is trivial, if there is a function $\beta\colon \Lambda \to \mathbb{R}$, differentiable along orbits, such that $\alpha(t,x) - t = \beta(x) - \beta(\varphi^t(x))$. But by assumption the values of the cocycle on the left-hand side over periodic orbits are zero, so by the Livschitz Theorem 7.2.1 there is a Hölder solution β which is C^1 if α is. □

Theorem 7.2.8. *Suppose $\varphi^t\colon M \to M$ and $\psi^t\colon M' \to M'$ are orbit equivalent on hyperbolic sets Λ, Λ', respectively, and that the periods of corresponding periodic orbits in Λ and Λ' agree. Then φ^t and ψ^t are conjugate (Definition 1.3.1).*

Proof. By Theorem 7.3.3 below, the orbit equivalence h can be taken to be Hölder continuous. Thus $h \circ \varphi^t \circ h^{-1}$ is a Hölder-continuous time change of ψ^t with the same periods as ψ^t; hence by Proposition 7.2.7 it is Hölder conjugate to ψ^t and hence so is φ. □

Let us state without proof a remarkable strengthening of the regularity statement in Theorem 7.2.1 or Corollary 7.2.5:

Theorem 7.2.9 (Livshitz Regularity)**.** *Let Λ be a compact hyperbolic set for a C^∞ flow Φ generated by a vector field X on a manifold M and $A_0\colon M \to \mathbb{R}$ bounded. If $a := XA_0\restriction_\Lambda$ is C^∞, then there is a C^∞ function $A\colon \Lambda \to \mathbb{R}$ such that $XA_0 = XA$.*

Thus, Corollary 7.2.6 has a C^∞ counterpart, so we get high regularity "for free." (There are suitable C^k counterparts to this.) Here is an easy application.

Theorem 7.2.10. *If a C^∞ Anosov flow Φ preserves a continuous volume $\mu = \rho\, d\mathrm{vol}$, then $\rho \in C^\infty$.*

Proof. $X \log\rho = \eta$, where $\mathscr{L}_{\dot\varphi}(d\mathrm{vol}) = \eta\, d\mathrm{vol}$. □

Of course, the question of the existence of an invariant volume in the first place can be answered using Theorem 7.2.1:

Theorem 7.2.11 (Livshitz–Sinai)**.** *A transitive Anosov flow Φ has an invariant volume if and only if $\varphi^\tau(x) = x \Rightarrow \det d\varphi^\tau(x) = 1$.*

We also note that the Livshitz Theorem plays a central role in the classification of equilibrium states (Theorem 8.3.21).

Finally, there is a strengthening of the Livshitz Theorem complementary to that which obtains higher regularity of the solution: the existence of a merely measurable solution is enough.

Theorem 7.2.12 (Measurable Livshitz Theorem)**.** *If X generates a volume-preserving C^2 Anosov flow and $A_0\colon \Lambda \to \mathbb{R}$ is measurable and such that $a := XA_0$ is α-Hölder continous, then there is an α-Hölder $A\colon \Lambda \to \mathbb{R}$ such that $A = A_0$ a.e.*

3. Hölder continuity of orbit equivalence

In this section we will see that the class of Hölder-continuous functions on a hyperbolic set also arises rather naturally. One of the main points of studying such functions is that they enable us to study the ergodic theory of hyperbolic flows in greater detail. They enter through the definition of *pressure*, a generalization of entropy, and the study of pressure in turn gives us insights into the behavior of smooth invariant measures—showing their ergodicity, for example. On the other hand the class of Hölder-continuous functions is a natural class of functions to study, since the principal structures associated with hyperbolicity are Hölder continuous with respect to the smooth structure—although they usually do not possess any higher regularity (such as Lipschitz continuity or C^1).

We now show that an orbit-equivalence between hyperbolic sets of smooth flows is effected by a Hölder-continuous homeomorphism. This implies that the class of Hölder-continuous functions is an invariant of orbit-equivalence.

Here and elsewhere we obtain Hölder continuity by establishing Hölder continuity along the flow direction and stable and unstable leaves separately. This is sufficient because the stable and unstable manifolds are uniformly transverse continuously varying Lipschitz submanifolds, and we give this result as an abstract lemma about metric spaces, where we think of stable and unstable leaves as ("vertical" and "horizontal") equivalence classes. (Recursively, this also works for 3 or more foliations.)

Proposition 7.3.1. *Let Λ be a metric space with two equivalence relations $\sim_h$ and $\sim_v$ for which there exist $\epsilon > 0$, $K_1 \in \mathbb{R}$ such that if $x \sim_h y \sim_v z$ and $d(x,z) < \epsilon$, then*

$$d(x,y)^2 + d(y,z)^2 \le K_1 d(x,z)^2,$$

and such that for sufficiently nearby $x, y \in \Lambda$ there exists a w such that $x \sim_h w \sim_v y$. Let $\varphi \colon \Lambda \to X$ be a map to a metric space X with some $K_2, \alpha > 0$ for which $d(x,y) < \epsilon$ and $x \sim_h y$ or $x \sim_v y$ imply

$$d(\varphi(x), \varphi(y)) \le K_2 d(x,y)^\alpha.$$

Then $d(\varphi(x), \varphi(y)) \le 2K_1K_2 d(x,y)^\alpha$ for all sufficiently close $x, y \in \Lambda$.[2]

Proof. First note that for $(x,y) \in \mathbb{R}^2$ one has

$$(|x|^\alpha + |y|^\alpha)^{1/\alpha} \le 2^{1/\alpha}(x^2 + y^2)^{1/2}.$$

[2] Theorem 10.2.9 is a remarkable related result.

To see this note (for example, by drawing $\{(x,y)\in\mathbb{R}^2 \mid |x|^\alpha+|y|^\alpha=1\}$) that the discrepancy is greatest for $x=y$, in which case

$$\frac{(|x|^\alpha+|y|^\alpha)^{1/\alpha}}{(x^2+y^2)^{1/2}}=\frac{2^{1/\alpha}|x|}{2^{1/2}|x|}<2^{1/\alpha}.$$

For $x,y\in\Lambda$ take w such that $x\sim_h w\sim_v y$ and note that

$$\begin{aligned}d(\varphi(x),\varphi(y))&\le d(\varphi(x),\varphi(w))+d(\varphi(w),\varphi(y))\le K_2(d(x,w)^\alpha+d(w,y)^\alpha)\\&\le 2K_2(d(x,w)^2+d(w,y)^2)^{\alpha/2}\le 2K_1K_2d(x,y)^\alpha.\end{aligned}$$ □

Theorem 7.3.2. *Let Λ and Λ' be compact hyperbolic sets for diffeomorphisms f and f', respectively, and $h=f'hf^{-1}\colon \Lambda\to\Lambda'$ a topological conjugacy. Then both h and h^{-1} are Hölder continuous.*

Proof. Since f and f' appear symmetrically in the statement it suffices to check that h itself is Hölder continuous. Furthermore we just showed that it is indeed enough to show Hölder continuity of h along stable and unstable manifolds. Since h also conjugates f^{-1} and f'^{-1} (for which stable and unstable manifolds reverse roles) it is, in fact, enough to prove that $h\restriction_{W^{uu}(x)\cap\Lambda}$ is Hölder continuous for every $x\in\Lambda$ (with uniform constant and exponent).

To this end take $c<1<C$ such that C is a Lipschitz constant for f and c is a Lipschitz constant for $f'^{-1}\restriction_{W^{uu}}$ and let $\alpha>0$ be such that $cC^\alpha<1$. Fix $\epsilon_0>0$. Since Λ is compact and h is continuous, hence uniformly continuous, there exists $\delta_0>0$ such that $d(x,y)<\delta_0$ implies $d(h(x),h(y))<\epsilon_0$.

Now if $x,y\in\Lambda$, $y\in W^{uu}(x)$, and $\delta:=d(x,y)$ is sufficiently small, then there exists $n\in\mathbb{N}$ such that

$$d(f^n(x),f^n(y))\le C^n\delta<\delta_0\le C^{n+1}\delta.$$

Hence $d(h(f^n(x)),h(f^n(y)))<\epsilon_0$, by choice of δ_0, so using $cC^\alpha<1$ we have

$$\begin{aligned}d(h(x),h(y))&=d(f'^{-n}hf^n(x),f'^{-n}hf^n(y))<c^n\epsilon_0\\&=c^n\delta_0^\alpha\cdot\epsilon_0/\delta_0^\alpha\le(cC^\alpha)^nC^\alpha(\epsilon_0/\delta_0^\alpha)\delta^\alpha<C^\alpha(\epsilon_0/\delta_0^\alpha)(d(x,y))^\alpha.\end{aligned}$$ □

An analog of the preceding result applies to flows as well by similar reasoning. There is, however, a new aspect to be taken into account here, namely, the lack of uniqueness in the flow direction. We thus obtain the following result:

Theorem 7.3.3. *Let $\Lambda\subset M$, $\Lambda'\subset M'$ be compact hyperbolic sets for flows φ and ψ, respectively, and suppose that φ and ψ are orbit equivalent via $h\colon\Lambda\to\Lambda'$. Then arbitrarily C^0-close to h there is an orbit equivalence that is Hölder-continuous together with its inverse.*

Proof. We begin with a local construction of a Hölder orbit equivalence. Take small smooth transversals $\mathscr{T}$ at $p \in \Lambda$ and $\mathscr{T}'$ at $q = h(p) \in \Lambda'$. Then locally $h(\mathscr{T})$ projects canonically to $\mathscr{T}'$ along the orbits of ψ and the composition of h with this projection is Hölder continuous by the same arguments as in the proof of Theorem 7.3.2 using the intersections of $\mathscr{T}$ with weak unstable and weak stable foliations as the equivalence classes in Proposition 7.3.1.

Now fix some $\delta > 0$ and cover Λ by flow boxes whose floors are small smooth transversals and fix corresponding smooth transversals in Λ'. From the Hölder-continuous map between these transversals construct local conjugacies on the flow boxes by taking them to be time preserving. This gives local homeomorphisms from these flow boxes to Λ'. To assemble these into one global map take a smooth partition of unity on Λ subordinate to the covering by these flow boxes. Now all images of a point $x \in \Lambda$ lie on an orbit segment and thus one can take the average of the corresponding time parameters weighted by the values of the members of the partition of unity at x. This gives a well-defined Hölder-continuous map $\tilde{h}$ which is C^0-close to h and takes orbits of φ to orbits of ψ. $\tilde{h}$ is also differentiable along the orbits of φ. The remaining problem is that $\tilde{h}$ may not be monotone along orbits.

To find a homeomorphism with the desired properties we use the fact that $\tilde{h}$ is as C^0-close to the homeomorphism h as we please, so long as δ is sufficiently small. This implies that there is an $\eta > 0$ such that for any $x \in \Lambda$ and $t > \eta$ we have $\tilde{h}(\varphi^t(x)) = \psi^s(\tilde{h}(x))$ with $s > 0$. This implies that defining $h'(x) := (1/\eta)\int_0^\eta \tilde{h}(\varphi^t(x))\,dt$ (the integral interpreted as one involving the real parameter along the orbit of x) gives a homeomorphism h' with all desired properties. □

4. Regularity of the invariant subbundles

Proposition 5.1.4 noted that the defining subbundles are automatically continuous, and we now establish that they are indeed Hölder continuous. This result is formulated in discrete time in such a way as to be applicable to the time-1 map of a flow and to give information about both the weak- and strong-(un)stable subbundles. In this form it is due to Brin and Stuck.

Theorem 7.4.1. *Suppose $f\colon M \to M$ is a $C^{1+\alpha}$ diffeomorphism of a compact manifold and admits a (λ,μ)-splitting:*

$$\|Df^n(x)v^s\| \le C\lambda^n\|v^s\| \quad \textit{and} \quad \|Df^n(x)v^u\| \ge C^{-1}\mu^n\|v^u\|$$

for $v^s \in E^s(x)$, $v^u \in E^u(x)$ and $n \in \mathbb{N}$. If $b := b_1^2 + H$, where $b_1 := \max_{x\in M}\|Df(x)\| \ge 1$ and H is the Hölder coefficient of Df, then E^s is $\alpha\frac{\log\mu-\log\lambda}{\log b-\log\lambda}$-Hölder continuous.

Remark 7.4.2. We note that it is not necessary to assume $\lambda < 1$ or $\mu > 1$.

First we show that the "contracting" subspaces of close maps must be close.

Lemma 7.4.3. *If $L_n^i\colon \mathbb{R}^N \to \mathbb{R}^N$ are linear maps for $i = 1,2$, $n \in \mathbb{N}$ and there are $b > 0$, $\delta \in (0,1)$, $C > 1$, $\mu > \lambda < b$ and subspaces $E^1, E^2 \subset \mathbb{R}^N$ such that $\|L_n^1 - L_n^2\| \le \delta b^n$,*

$$\|L_n^i(v)\| \begin{cases} \le C\lambda^n \|v\| & \text{for } v \in E^i \text{, and} \\ \ge C^{-1}\mu^n \|v\| & \text{for } v \perp E^i, \end{cases}$$

then $d(E^1, E^2) \le 3C^2 \dfrac{\mu}{\lambda} \delta^{\frac{\log\mu - \log\lambda}{\log b - \log\lambda}}$.

Proof. Since $\gamma := \lambda/b < 1$ there is a unique $n \in \mathbb{N}$ for which $\gamma^{n+1} < \delta \le \gamma^n$. For $v \in E^2$ we then have

$$\|L_n^1(v)\| \le \|L_n^2(v)\| + \|L_n^1 - L_n^2\|\|v\| \le C\lambda^n\|v\| + \delta b^n\|v\| \le (C\lambda^n + (b\gamma)^n)\|v\| \le 2C\lambda^n\|v\|.$$

Writing $v = v^1 + v^\perp \in E^1 \oplus E^{1^\perp}$ gives

$$\|L_n^1(v)\| = \|L_n^1(v^1 + v^\perp)\| \ge \|L_n^1(v^\perp)\| - \|L_n^1(v^1)\| \ge C^{-1}\mu^n\|v^\perp\| - C\lambda^n\|v^1\|,$$

hence

$$\|v - v^1\| = \|v^\perp\| \le C\mu^{-n}(\|L_n^1(v)\| + C\lambda^n\|v^1\|) \le 3C^2\Big(\frac{\lambda}{\mu}\Big)^n\|v\|,$$

so $d(v, E^1) \le 3C^2\big(\lambda/\mu\big)^n\|v\|$, which implies $d(E^1, E^2) \le 3C^2\frac{\mu}{\lambda}\big(\frac{\lambda}{\mu}\big)^{n+1}$ by symmetry. Now $x := \frac{\log\mu - \log\lambda}{\log b - \log\lambda} \Rightarrow \frac{\lambda}{\mu} = \gamma^x \Rightarrow \big(\frac{\lambda}{\mu}\big)^{n+1} = (\gamma^{n+1})^x < \delta^x$, hence the claim. □

In preparation for our application of this lemma we note how the bound "δb^n" arises from a $C^{1+\alpha}$ diffeomorphism.

Lemma 7.4.4. *If f is a $C^{1+\alpha}$ diffeomorphism of a manifold $M \subset \mathbb{R}^N$, $\|x - y\| < 1$ and $n \in \mathbb{N}$, then*

$$\|Df^n(x) - Df^n(y)\| \le b^n\|x - y\|^\alpha$$

with $b := b_1^2 + H$, where $b_1 := \|Df(\cdot)\|_\infty \ge 1$ and H is the Hölder coefficient of Df.

Proof. First a remark about Hölder continuity: If $\|g(x) - g(y)\| \le H\|x - y\|^\alpha$ whenever $\|x - y\| \le 1$, then for $\|x - y\| > 1$ we subdivide a path from x to y into $\lfloor\|x - y\|\rfloor + 1$ equal pieces to get

$$\|g(x) - g(y)\| \le \sum_{i=0}^{\lfloor\|x-y\|\rfloor} \|g(x_i) - g(x_{i+1})\| \le \sum_{i=0}^{\lfloor\|x-y\|\rfloor} H \le H\|x - y\|,$$

so Hölder continuity is characterized by $\|g(x) - g(y)\| \le H\max(\|x - y\|^\alpha, \|x - y\|)$ without restrictions on $\|x - y\|$.

We now prove the claim by induction, noting that the case $n = 1$ is clear and that $\|f^n(x) - f^n(y)\| \le b_1^n \|x - y\|$. Then

$$\begin{aligned}
&\|Df^{n+1}(x) - Df^{n+1}(y)\| \\
&\le \|Df(f^n(x))\|\|Df^n(x) - Df^n(y)\| + \|Df(f^n(x)) - Df(f^n(y))\|\|Df^n(y))\| \\
&\le b_1 b^n \|x-y\|^\alpha + H\max((b_1^n\|x-y\|)^\alpha, b_1^n\|x-y\|)b_1^n \le \underbrace{[b_1 b^n + Hb_1^{2n}]}_{=b^n\left[b_1+H(b_1^2/b)^n\right]<b^n[b_1+H]\le b^n[b_1^2+H]=b^{n+1}}\|x-y\|^\alpha. \quad \square
\end{aligned}$$

Having intimated how the first inequality in Lemma 7.4.3 arises, hyperbolicity now uses the other 2 for the main result.

PROOF OF THEOREM 7.4.1. As implied by Lemma 7.4.4, we assume $M \subset \mathbb{R}^N$ (without loss of generality by the Whitney Embedding Theorem) and prove smoothness of $E^s \oplus TM^\perp$ by applying our lemmas to $L(x) := Df(x) \oplus 0\colon T_xM \oplus T_xM^\perp \to \mathbb{R}^N$ as follows. Set $L_n(x) := L(f^{n-1}(x)) \circ \cdots \circ L(f(x)) \circ L(x)$ and note that on one hand $L_n(x)\restriction_{T_xM} = Df^n(x)$ and on the other hand by compactness of M we can adjust C such that $v \perp E^s(x) \Rightarrow \|Df^n(x)(v)\| \ge C^{-1}\mu^n\|v\|$. Thus, by Lemma 7.4.4 we can apply Lemma 7.4.3 for $\delta = \|x_1 - x_2\|^\alpha < 1$ to $L_n^i := L_n(x_i)$ and $E^i := E^s(x_i)$. □

Theorem 7.4.1 is effective, but we next find the optimal Hölder exponent. That is, we now more closely study the regularity of the stable and unstable subbundles of hyperbolic flows in terms of local "spectral" control, namely bounds on ratios involving contraction and expansion rates of the flow (Definition 7.4.5). Theorem 7.4.14 provides such a regularity statement for the invariant subbundles (see also Theorem 7.5.3), and Theorem 7.6.2 shows that it is sharp. There are easy corollaries for codimension-one flows (Corollaries 7.4.15 and 7.4.16) as well as geodesic flows (Corollary 7.4.17). We also give a relatively easy instance of smooth rigidity in Theorem 10.3.1 and comment on related subjects in Remark 10.3.9.

Definition 7.4.5 (Bunching)**.** A compact hyperbolic set Λ for a flow Φ on a Riemannian manifold M with invariant splitting $T_\Lambda M = E^u \oplus E^c \oplus E^s$ admits $C, \epsilon > 0$ such that for all $p \in \Lambda$ there exist $\mu_f < \mu_s < 1 - \epsilon < 1 + \epsilon < \nu_s < \nu_f$ as follows: if $v \in E^s(p), u \in E^u(p)$ and $t > 0$, then

$$\frac{1}{C}\mu_f^t\|v\| \le \|D\varphi^t(v)\| \le C\mu_s^t\|v\| \text{ and } \frac{1}{C}\nu_f^{-t}\|u\| \le \|D\varphi^{-t}(u)\| \le C\nu_s^{-t}\|u\|.$$

We say that $p \in \Lambda$ is *α-bunched* if $\mu_s(p)\nu_s(p)^{-1} \le \big(\min(\mu_f(p), \nu_f(p)^{-1})\big)^\alpha$, and φ^t is *α-bunched* if $\sup_{p\in\Lambda}\mu_s\nu_s^{-1}\big(\min(\mu_f, \nu_f^{-1})\big)^{-\alpha} \le 1$. The *unstable bunching constant* is $B^u(\Phi) := \inf_{p\in\Lambda}(\log\mu_s - \log\nu_s)/\log\mu_f$.

Remark 7.4.6. Note that we chose the $\mu_f, \mu_s, \nu_s, \nu_f$ to depend on p, and the "min" above are taken over two numbers, they do *not* range over Λ. If μ_f and μ_s are chosen optimally they essentially give the maximal and minimal multipliers for the strong stable subbundle and likewise ν_f and ν_s give the maximal and minimal expansion rates for the unstable subbundle.

For *symplectic* Anosov flows, that is, flows on odd-dimensional manifolds with a flow-invariant 2-form that is nondegenerate on transversals to the flow, the *invariant symplectic form* for the flow) the notion of bunching is clearer since $\mu_i = \nu_i^{-1}$ $(i = f, s)$ and hence α-bunching implies $\mu_s^{2/\alpha} \le \mu_f \le \mu_s$, which bunches the "spectrum" of contracting multipliers increasingly as $\alpha \to 2$. The main examples of symplectic Anosov flows are geodesic flows of negatively curved manifolds. The Riemannian structure gives a canonical contact form which renders these flows symplectic. (Alternatively symplecticity is a consequence of the Hamiltonian structure of free particle motion on a manifold.) a-pinching of the sectional curvature of a Riemannian manifold implies $2\sqrt{a}$-bunching of the geodesic flow. This is related to the fact that the rate obtained in the proof of Lemma 5.2.5 is the square root of the curvature bound above (5.2.2).

Proposition 7.4.7. *$E^u \in C^{B^u(\Phi)}$ if $B^u(\Phi) \in (0,1)$, and if $B^u(\Phi) \ge 1$, then E^u has modulus of continuity $O(x|\log x|)$ (see Remark 3.2.18).*

Remark 7.4.8. Likewise for E^s and $B^s(\Phi) := \inf\limits_{p\in\Lambda} \dfrac{\log\nu_s - \log\mu_s}{\log\nu_f}$ (the *stable bunching constant*), and for both simultaneously with $B(\Phi) := \inf\limits_{p\in\Lambda} \dfrac{\log\nu_s - \log\mu_s}{\max(\log\nu_f, -\log\mu_f)}$ (the *bunching constant*). Moreover, we later obtain differentiability with Hölder continuous derivatives if $B^u(\Phi) > 1$ in Theorem 7.4.14. (See also Theorem 7.5.3.)

The proof is a careful application of the Hadamard graph-transform method.

Proof. First we introduce adapted coordinates.[3] For $p \in \Lambda$ take a hypersurface $\mathcal{T}_p$ transverse to $\dot\varphi$ of uniform size depending C^∞ on p. For each p let $W^u := W^{uu}(p) \cap \mathcal{T}_p$, $W^s := W^s(p) \cap \mathcal{T}_p$, $E^u := TW^u$ and $E^s := TW^s$. Take coordinates $\Xi\colon \Lambda \times [-\epsilon,\epsilon]^{k+l} \to M$ such that $\Xi_p\colon [-\epsilon,\epsilon]^{k+l} \xrightarrow{C^\infty} \mathcal{T}_p$ is continuous in p, $[-\epsilon,\epsilon]^k \times \{0\} \to W^u$, $\{0\} \times [-\epsilon,\epsilon]^l \to W^s$. Write the coordinates as (x,y) with $\Xi_p(x,0) \in W^u$ and $\Xi_p(0,y) \in W^s$. Denote the induced (return or Poincaré) map by $\phi^t\colon \mathcal{T}_p \to \mathcal{T}_{\varphi^t p}$ and represent $E^u(0,y)$ as the image of a linear map $\begin{pmatrix} I \\ D \end{pmatrix}\colon \mathbb{R}^k \to \mathbb{R}^{k+l}$. To obtain the desired regularity we give estimates along orbits in the stable manifold of a reference orbit. These estimates are uniform (in both reference orbit and the

[3] The construction here is quite simple; Lemma 7.5.4 further adapts to an invariant area, and further adapations are possible.

orbit at hand) and hence establish the desired regularity uniformly along stable manifolds. Smoothness in the unstable direction then shows that this is, in fact, the actual degree of regularity.

Lemma 7.4.9. *Given $p \in \Lambda$, $q \sim (0,y) \in W^s$, $(0,y_t) := \phi^t(0,y)$, there exist $C > 0$ and $C_t > 0$ such that $D\phi^t = \begin{pmatrix} A_t & 0 \\ B_t & C_t \end{pmatrix}$ with $\|A_t^{-1}\| < C\nu_s(q)^{-t}$, $\|C_t\| < C\mu_s(q)^t$, $\|C_t^{-1}\| < C\mu_f(q)^{-t}$, $\|B_t\| < C_t\|y\|$, $C\|y_t\| \geq \mu_f(q)^t\|y\|$.*

Proof. $\|A_t^{-1}\| < \nu_s(q,t)^{-1}$ in coordinates centered at q. Up to a distortion factor, uniformly bounded independently of t, the linear part of the coordinate change is of the form $\left(\begin{smallmatrix} I & O \\ D & I \end{smallmatrix}\right)$, so that up to a bounded factor the representations A_t^{-1} agree in both systems, as do the ones for C_t and C_t^{-1}. $\|B_t\| < C_t\|y\|$ since φ^t is a diffeomorphism with B_t differentiable and vanishing at the origin of the coordinate system. For the remaining claim it is slightly easier and by boundedness of coordinate changes clearly sufficient to show $\|y\| \leq C\mu_f(p,t)^{-1}\|\phi^t(y)\|$. To this end let $\gamma_t \colon [0,1] \to \mathscr{T}_{\varphi^t p}$ be a geodesic with $\gamma_t(0) = \phi^t(p)$, $\gamma_t(1) = \phi^t(q)$, where $q \sim (0,y)$. By the Inclination Lemma (Theorem 12.6.1) $\phi^{-t}\gamma_t$ converges to a smooth curve $c(\cdot) \subset \mathscr{T}_p$. If $\lim\limits_{n\to\infty} \|\phi^t(y)\|/\mu_f(p,t)\|y\| = 0$ then by the intermediate value theorem this holds for all $c(s)$, $s \in (0,1]$. Compactness of Λ controls higher derivatives and yields uniformity in s, so $\lim\limits_{n\to\infty} \|D\phi^t(v)\|\|v\|/\mu_f(p,t) = 0$ for $v = \dot{c}(0)$, contrary to the choice of μ_f. □

Let $z_t = (0, y_t)$. Then inductively

$$\begin{aligned} D(z_n) &= [B_1(z_{n-1}) + C_1(z_{n-1})D(z_{n-1})]A_1^{-1}(z_{n-1}) \\ &= \sum_{i=0}^{n-1} C_i(z_{n-i})B_1(z_{n-i-1})A_{i+1}^{-1}(z_{n-i-1}) + C_n(z_0)D(z_0)A_n^{-1}(z_0). \end{aligned}$$

So, if Φ is α-bunched, then $\mu_s^i\nu_s^{-i} \leq \mu_f^{\alpha i} = O(\|z_n\|/\|z_{n-i}\|)^\alpha$, and

$$\frac{\|D(z_n)\|}{\|z_n\|^\alpha} = O\Big(\sum_{i=0}^{n-1} \mu_s^i\nu_s^{-i}\frac{\|z_{n-i-1}\|}{\|z_n\|^\alpha}\Big) + O\Big(\frac{\mu_s^n\nu_s^{-n}}{\|z_n\|^\alpha}\Big) = O\Big(1 + \sum_{i=0}^{n-1} \|z_{n-i-1}\|^{1-\alpha}\Big).$$

If $\alpha < 1$ then the right hand side is uniformly bounded, so the unstable subbundle is α-Hölder along W^s. If $\alpha = 1$ then the right hand side is uniformly $O(n)$, so the unstable subbundle has modulus of continuity $O(x|\log x|)$ along W^s. Since the unstable subbundle is (uniformly) smooth along W^u it is as regular as along the stable foliation (Proposition 7.3.1 below). □

Remark 7.4.10. If Φ is α-bunched for some $\alpha > 1$, then the preceding estimates imply that the unstable subbundle is Lipschitz continuous.

As we just suggested, we can obtain stronger conclusions with stronger bunching. First among these is differentiability:

Proposition 7.4.11. *If* $\sup_{p\in M}\mu_s\nu_s^{-1}\mu_f^{-1} < 1$ *in Definition 1.1 then* $E^u\restriction_{W^s}$ *is differentiable.*

This follows from Lemma 7.4.13 below once we show that differentiability can be proved via similar estimates as before:

Claim 7.4.12. $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *is differentiable if for* $v_i \in \mathbb{R}^n$ *with* $v_1 + v_2 + v_3 = 0$

$$\frac{\big|h_2h_3f(x+v_1h_1)+h_1h_3f(x+v_2h_2)+h_1h_2f(x+v_3h_3)-(h_2h_3+h_1h_3+h_1h_2)f(x)\big|}{h_1h_2h_3} \to 0$$

uniformly as $(h_1, h_2, h_3) \to 0$ *(we write “*$o(h_1h_2h_3)$*”*[4]*). For a function on a manifold replace the arguments* $(x + v_ih_i)$ *by* $c_i(h_i)$*, where* c_i *is a curve with* $\dot c_i(0) = v_i$*.*

Proof. Setting $v_3 = 0$ shows existence of directional derivatives, then this condition shows that these depend linearly on the direction. Uniformity guarantees that this is sufficient. □

In adapted coordinates consider triples of vector fields on M as follows:

$\mathscr{B}{:=}\big\{(v_1, v_2, v_3)$ with $v_i(p) \in E^s(p)\cap T\mathscr{T}_p,\ v_1+v_2+v_3 = 0$, and $\|v_1\|+\|v_2\|+\|v_3\| = 1\big\}$.

If

$$\tilde v_i(p) = \frac{D\phi^t v_i(\varphi^{-t}p)}{\xi_t(v_1(\varphi^{-t}p), v_2(\varphi^{-t}p), v_3(\varphi^{-t}p))}$$

with $\xi_t \in \mathbb{R}$ such that $(\tilde v_1, \tilde v_2, \tilde v_3) \in \mathscr{B}$ then φ^t acts on $\mathscr{B}$ by $\mathscr{P}_t(v_1, v_2, v_3){:=}(\tilde v_1, \tilde v_2, \tilde v_3)$. The sufficient condition in Claim 7.4.12 and hence Proposition 7.4.11 follows from

Lemma 7.4.13. *For* $K > 0$ *there are* $T, \epsilon > 0$ *such that if* $\gamma_{v_i(p)}$ *denote geodesics in* $\mathscr{T}_p$ *with* $\dot\gamma_{v_i(p)}(0) = v_i(p)$ *and for* $p \in M, (v_1, v_2, v_3) \in \mathscr{B}, 0 < h_1, h_2, h_3 < \epsilon$

$$\|h_2h_3D(\gamma_{v_1(p)}(h_1)) + h_1h_3D(\gamma_{v_2(p)}(h_2)) + h_1h_2D(\gamma_{v_3(p)}(h_3))\| < Kh_1h_2h_3$$

then $\forall\, p \in M, t \in [T, 2T], (\tilde v_1, \tilde v_2, \tilde v_3) \in \mathscr{P}_t\mathscr{B}, 0 < h_1, h_2, h_3 < \epsilon$ *and with* $\tilde h_i \coloneqq \xi_t h_i$

$$\|\tilde h_2\tilde h_3D(\gamma_{\tilde v_1(p)}(\tilde h_1)) + \tilde h_1\tilde h_3D(\gamma_{\tilde v_2(p)}(\tilde h_2)) + \tilde h_1\tilde h_2D(\gamma_{\tilde v_3(p)}(\tilde h_3))\| < \frac{3}{4}K\tilde h_1\tilde h_2\tilde h_3.$$

Proof. For T such that $(\sup_{p\in M}\mu_s\nu_s\mu_f^{-1})^T < L^{-3}/3$ consider $\phi^t(\gamma_{v_i(\varphi^{-t}p)}(h_i))$ instead of $\gamma_{\tilde v_i(p)}(\xi_t h_i)$.[5] Let $\Gamma_i \coloneqq \gamma_{v_i(\varphi^{-t}p)}(h_i)$, $A_i \coloneqq A_t(\Gamma_i)^{-1}$, $B_i \coloneqq B_t(\Gamma_i)$, $C_i \coloneqq C_t(\Gamma_i)$,

[4] See Remark 3.2.18.

[5] These are tangent as $h_i \to 0$, so the error is $o(\tilde h_1\tilde h_2\tilde h_3)$ in the notation from Remark 3.2.18.

$h_iD_i := h_1h_2h_3D(\Gamma_i)$, $\tilde{h}_i\tilde{D}_i := \tilde{h}_1\tilde{h}_2\tilde{h}_3D(\phi^t(\Gamma_i)) = \tilde{h}_i\xi_t^2C_iD_iA_i + \tilde{h}_1\tilde{h}_2\tilde{h}_3B_iA_i$. Then

$$\begin{aligned}\|\tilde{D}_1+\tilde{D}_2+\tilde{D}_3\| &\le \xi_t^2\|C_1D_1A_1+C_2D_2A_2+C_3D_3A_3\| \\ &\qquad + \|\tilde{h}_2\tilde{h}_3B_1A_1+\tilde{h}_1\tilde{h}_3B_2A_2+\tilde{h}_1\tilde{h}_2B_3A_3\| \\ &\le \overbrace{\xi_t^2\underbrace{\|C_1\|}_{\le L\mu_s^t}\cdot\underbrace{\|D_1+D_2+D_3\|}_{\le Kh_1h_2h_3=K\xi_t^{-3}\tilde{h}_1\tilde{h}_2\tilde{h}_3}\cdot\underbrace{\|A_1\|}_{\le L\nu_s^{-t}}}^{L^2\mu_s^t\nu_s^{-t}\xi_t^{-1}K\tilde{h}_1\tilde{h}_2\tilde{h}_3} \\ &\qquad + \xi_t^2\underbrace{\|C_2-C_1\|}_{\in o(h_1h_2h_3)}\|D_2\|\|A_1\| + \xi_t^2\|C_2\|\|D_2\|\underbrace{\|A_2-A_1\|}_{\in o(h_1h_2h_3)} \\ &\qquad + \xi_t^2\underbrace{\|C_3-C_1\|}_{\in o(h_1h_2h_3)}\|D_3\|\|A_1\| + \xi_t^2\|C_3\|\|D_3\|\underbrace{\|A_3-A_1\|}_{\in o(h_1h_2h_3)} \\ &\qquad + \xi_t^2\underbrace{\|h_2h_3B_1A_1+h_1h_3B_2A_2+h_1h_2B_3A_3\|}_{\in o(h_1h_2h_3)}.\end{aligned}$$

$\xi_t^{-1} \le L\mu_f^{-t}$ after possibly adjusting L, so for $t > T$ the first term is at most $\frac{1}{3}K\tilde{h}_1\tilde{h}_2\tilde{h}_3$. Since $\|D_i\| \in O(h_1h_2h_3)$ by Remark 7.4.10, the other terms are $o(h_1h_2h_3)=o(\tilde{h}_1\tilde{h}_2\tilde{h}_3)$ uniformly for $t \in [T,2T]$. Take $\epsilon > 0$ such that their sum is at most $\frac{1}{3}K\tilde{h}_1\tilde{h}_2\tilde{h}_3$ for $t \in [T,2T]$. □

As mentioned previously, once the bunching constant exceeds 1 the derivatives are Hölder continuous with the Hölder exponent one would naturally expect.

Theorem 7.4.14. *$E^u \in C^{B^u(\Phi)}$ if $B^u(\Phi) \notin \mathbb{N}$, and $E^u \in C^{B^u(\Phi)-1+x|\log x|}$ if $B^u(\Phi) \in \mathbb{N}$.*

Proof. We limit ourselves to $B^u(\Phi) \le 2$ since this is as high as one can get for stable and unstable bunching simultaneously, and because beyond this point the reader could adapt the arguments given here if needed. Thus, Proposition 7.4.7 and Proposition 7.4.11 leave us to produce the right Hölder exponent for the derivative when $B^u(\Phi) > 1$.

Suppose, then, that Φ is α-bunched with $\alpha > 1$. Then D is C^1 by Proposition 7.4.11, and we write $D(y) = {}^LD(y) + {}^ND(y)$ with ${}^LD(y)$ linear in y, ${}^ND(y)$ differentiable in y, $D_y\,{}^ND(0) = 0$, and $A_t^{-1}(y) = A_t^{-1}(0) + {}^LA_t^{-1}(y) + {}^NA_t^{-1}(y)$, etc.,

$$N(z) := B_1(z)A_1^{-1}(z) - {}^LB_1(z)A_1^{-1}(0) + C_1(z)\,{}^LD(z)A_1^{-1}(z) - C_1(0)\,{}^LD(z)A_1^{-1}(0) \in O(\|z\|).$$

Note that $\|D_zN(z)\| \in O(\|z\|)$. As in the proof of Proposition 7.4.7 we inductively get

$$\begin{aligned}{}^ND(z_n) &= N(z_{n-1}) + C_1(z_{n-1})\,{}^ND(z_{n-1})A_1^{-1}(z_{n-1}) \\ &= \sum_{i=0}^{n-1} C_i(z_{n-i})N(z_{n-i-1})A_i^{-1}(z_{n-i-1}) + C_n(z_0)\,{}^ND(z_0)A_n^{-1}(z_0)\end{aligned}$$

and

$$\begin{aligned} D_{z_n}{}^N D(z_n) = \sum_{i=0}^{n-1} \Big[& (D_{z_n} C_i(z_{n-i})) N(z_{n-i-1}) A_i^{-1}(z_{n-i-1}) \\ & + C_i(z_{n-i})(D_{z_n} N(z_{n-i-1})) A_i^{-1}(z_{n-i-1}) \\ & + C_i(z_{n-i}) N(z_{n-i-1})(D_{z_n} A_i^{-1}(z_{n-i-1}))\Big] \\ + D_{z_n}\big[& C_n(z_0)\, {}^N D(z_0) A_n^{-1}(z_0)\big]. \end{aligned}$$

By the product and chain rule

$$\begin{aligned} D_{z_n} C_i(z_{n-i}) = D_{z_n} \sum_{j=1}^{i} C_1(z_{n-j}) &= \sum_{j=1}^{i} (\prod_{k=1}^{j-1} C_1(z_{n-k})) D_{z_n} C_1(z_{n-j}) (\prod_{k=j+1}^{i} C_1(z_{n-k})) \\ &= \sum_{j=1}^{i} C_{j-1}(z_{n-j+1})[D_{z_{n-j}} C_1(z_{n-j}) C_j^{-1}(z_{n-j})] C_{i-j}(z_{n-i}), \end{aligned}$$

so

$$\|D_{z_n} C_i(z_{n-i})\| = O\big(\sum_{j=1}^{i} \mu_s^{i-1} \mu_f^{-j}\big) = O(\mu_s^{i-1} \mu_f^{-i}).$$

Likewise $\|D_{z_n} A_i^{-1}(z_{n-i})\| \in O(\nu_s^{1-i} \mu_f^{-i})$ and similarly

$$\|D_{z_n} N(z_{n-i-1})\| \in O(\|z_{n-i-1}\|)\|C_{i+1}^{-1}(z_{n-i-1})\| = O(\|z_{n-i-1}\| \mu_f^{-i}).$$

Since we assume $\mu_s^i \nu_s^{-i} \le \mu_f^{\alpha i} = O(\|z_n\| / \|z_{n-i}\|)^\alpha$,

$$\begin{aligned} &\|D_{z_n}[C_n(z_0)\, {}^N D(z_0) A_n^{-1}(z_0)]\| = \\ &\|[D_{z_n} C_n(z_0)]\, {}^N D(z_0) A_n^{-1}(z_0) + C_n(z_0)[D_{z_n}{}^N D(z_0)] A_n^{-1}(z_0) + C_n(z_0)\, {}^N D(z_0) D_{z_n} A_n^{-1}(z_0)\| \\ &\qquad = \underbrace{O(\mu_s^{n-1} \mu_f^{-n} \nu_s^{-n}) + O(\mu_s^n \mu_f^{-n} \nu_s^{-n}) + O(\mu_s^n \nu_s^{1-n} \mu_f^{-n})}_{\in O(\mu_f^{(\alpha-1)n}) = O(\|z_n\|^{\alpha-1})}. \end{aligned}$$

Therefore, again using $\mu_s^i \nu_s^{-i} \le \mu_f^{\alpha i} = O(\|z_n\| / \|z_{n-i}\|)^\alpha$,

$$\begin{aligned} &\|D_{z_n}{}^N D(z_n)\| \\ &\quad = O\big(\sum_{i=0}^{n-1} [\mu_s^{i-1} \mu_f^{-i} \|z_{n-i}\| \nu_s^{-i} + \mu_s^i \|z_{n-i}\| \mu_f^{-i} \nu_s^{-i} + \mu_s^i \|z_{n-i}\| \nu_s^{1-i} \mu_f^{-i}]\big) + O(\|z_n\|^{\alpha-1}) \\ &\qquad = O\big(\sum_{i=0}^{n-1} \|z_{n-i}\| \Big(\frac{\|z_n\|}{\|z_{n-i}\|}\Big)^{\alpha-1}\big) + O(\|z_n\|^{\alpha-1}) = O\big(\big([1 + \sum_{i=0}^{n-1} \|z_{n-i}\|^{2-\alpha}] \|z_n\|^{\alpha-1}\big). \end{aligned}$$

If $\alpha < 2$ then the right hand side is $O(\|z_n\|^{\alpha-1})$, hence the unstable subbundle has $(\alpha-1)$-Hölder derivative along W^s. If $\alpha = 2$ then the right hand side is $O(n\|z_n\|)$, so the derivative of the unstable subbundle has modulus of continuity $O(x\,|\log x|)$ along W^s. It is easy to see that derivatives in the unstable direction have the same (and even higher) modulus of continuity along W^s. Being uniformly smooth along W^u the unstable subbundle has the claimed regularity (Proposition 7.3.1). □

Corollary 7.4.15. *If* $\operatorname{codim} E^u = 1$, *then* $E^u \in C^{1+\inf_{p\in M}\frac{\log\nu_s}{-\log\mu_f}-\epsilon} \subset C^1$.

Proof. By assumption $\mu_f = \mu_s$. □

Corollary 7.4.16. *If* φ *preserves volume and* $\operatorname{codim} E^u = 1$ *then the Anosov splitting is* $C^{1+\inf_{p\in M}\frac{\log\nu_s}{-\log\mu_f}-\epsilon}$. *In particular it is* $C^{1+\epsilon}$.

Proof. $\mu_s\nu_f \le 1$ so $\dfrac{\log\nu_s - \log\mu_s}{\log\nu_f} \ge \dfrac{\log\nu_s}{\log\nu_f} + 1 \ge 1 + \dfrac{\log\nu_s}{-\log\mu_s}$. Use Theorem 7.4.14 and Corollary 7.4.15. □

A compact Riemannian manifold N is said to be *a-pinched* if there is a $C < 0$ such that $C \le$ sectional curvature $\le aC$. a-pinching of the sectional curvature of a Riemannian manifold implies $2\sqrt{a}$-bunching of the geodesic flow, this is closely related to the way in which the curvature bound k^2 used in Lemma 5.2.5 yields an expansion rate $2k$ in that argument. *Horospheric foliations* of a geodesic flow are defined to be the foliations of the unit tangent bundle of the manifold by the stable and unstable leaves of the geodesic flow. Thus we have

Corollary 7.4.17. *For* $a \in [0,1)$, *a-pinched negatively curved manifolds have* $C^{2\sqrt{a}}$ *horospheric foliations.*

5. Longitudinal regularity

We next develop 2 new lines of inquiry: One is the regularity of the *strong* subbundles, which one expects to be similar but different (see, for example, Proposition 7.5.5), the other is how far one can push up the regularity by bunching conditions. It is illuminating and relatively easy to do both at the same time for volume-preserving Anosov flows of 3-manifolds. Here, the combination of volume-preservation and one-dimensional strong subbundles produces the maximum bunching constraint (there is only one contraction and expansion rate each and they are reciprocals), and in a way that produces higher regularity than the counterpart to Theorem 7.4.14. However, one rate drops out in graph-transform arguments for strong subbundles because the complementary direction includes the flow direction. Accordingly, we expect the dimension assumption to yield a slight improvement over modulus of continuity $x|\log x|$.

Definition 7.5.1. A continuous function $f\colon U\to\mathbb{R}$ on an open set $U\subset\mathbb{R}$ is said to be *Zygmund-regular* if there is $Z>0$ such that $|f(x+h)+f(x-h)-2f(x)|\le Z|h|$ for all $x\in U$ and sufficiently small h. To specify a value of Z we may refer to a function as being Z-Zygmund. The function is said to be "little Zygmund" if $|f(x+h)+f(x-h)-2f(x)|\in o(|h|)$.[6]

Remark 7.5.2. Zygmund regularity implies modulus of continuity $O(|x\log|x||)$ and hence H-Hölder continuity for all $H<1$. It follows from Lipschitz continuity and hence from differentiability. Being "little Zygmund" implies having modulus of continuity $o(|x\log|x||)$.

From Proposition 7.5.6 below we obtain:

Theorem 7.5.3 (Longitudinal Zygmund regularity)**.** *Let $k\ge 2$, Φ a C^k volume-preserving Anosov flow on a 3-manifold. Then $E^u\oplus E^s$ is Zygmund-regular.*

Our first ingredient are adapted coordinates that also incorporate the invariant volume and thereby require a finer argument than at the start of the proof of Proposition 7.4.7. (Lemma 7.6.4 goes further yet.)

Lemma 7.5.4. *There exist local coordinates adapted to the invariant foliations, that is, coordinate systems $\mathfrak{C}\colon M\times(-\epsilon,\epsilon)^3\to M$ such that $\mathfrak{C}_p:=\mathfrak{C}(p,\cdot)$ satisfies*

(1) $\mathfrak{C}_p$ is C^k for every $p\in M$.
(2) $\mathfrak{C}_p$ depends (Hölder-) continuously on p.
(3) $\mathfrak{C}_p$ preserves volume for each $p\in M$.
(4) $\mathfrak{C}_p(0)=p$.
(5) $\mathfrak{C}_p((-\epsilon,\epsilon)\times\{0\}\times\{0\})=W^{uu}_{loc}(p)\cap\mathfrak{C}_p((-\epsilon,\epsilon)^3)$.
(6) $\mathfrak{C}_p^{-1}(\varphi^\delta(\mathfrak{C}_p(u,t,s)))=(u,t+\delta,s)$ for $|\delta|<\epsilon$.
(7) $\mathfrak{C}_p(\{0\}\times\{0\}\times(-\epsilon,\epsilon))=W^{ss}_{loc}(p)\cap\mathfrak{C}_p((-\epsilon,\epsilon)^3)$.

Proof. Since W^u and W^s are continuous, there is a continuous family $p\mapsto\mathcal{T}_p$ of local C^k transversals, each containing $W^{uu}_{\mathrm{loc}}(p)\cup W^{ss}_{\mathrm{loc}}(p)$ (The family an be taken Hölder continuous once it is known that $p\mapsto W^i_{\mathrm{loc}}(p)$ are Hölder continuous). We produce local coordinates in each $U:=\mathcal{T}_p$ that are adapted to area and to $W^{uu}_{\mathrm{loc}}(p)\cup W^{ss}_{\mathrm{loc}}(p)$; building flow-boxes on these then gives the desired local coordinates.

Take $p\colon U\to\mathbb{R}$ such that $p\equiv 0$ on W^s and P defined by $dp=\alpha(P,\cdot)$ is transverse to W^u; here α is the area form on U. If P^t is the Hamiltonian flow generated by P, then $\forall\, z\in U\ \exists!\, y\in W^u, q(z)\in\mathbb{R}\colon\ z=P_1^{q(z)}(y)$, so $\{q_1,p_1\}=1$ (Poisson bracket), $q\restriction_{W^u}=0$, and p,q are the desired coordinates. □

[6] See Remark 3.2.18.

Since the subbundles are invariant under the flow, the coordinate representation of the flow preserves the axes of the local coordinate system as well as volume. Note that this flow produces maps between different local coordinate patches.

The differential of φ^T at points of the stable leaf (the third coordinate axis, or s-axis) therefore takes the following form:

$$D\varphi^T(0,0,s) = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ b_1 & 1 & 0 \\ b_2 & 0 & \alpha \end{pmatrix},$$

where $b_i(s) \in O(s)$ and $|\alpha(s)| < 1$. As a warmup and because we need it later, we put this to use for proving that the strong subbundles are Hölder continuous.

Proposition 7.5.5. *In the context of Theorem 7.5.3, for each $H \in (0,1)$ there is a $Z > 0$ such that the graph transform preserves the space of subbundles that are H-Hölder along E^s with constant Z in local coordinates.*

Thus the graph transform preserves Hölder continuity. More specifically, applying the graph transform to a subbundle that is H-Hölder with sufficiently large constant Z in local coordinates gives a subbundle with the same property (for the same Z and H). This holds for any $H < 1$. This implies immediately that E^u is H-Hölder for any $H < 1$, because it shows that the unique fixed point of the graph transform lies in the space of H-Hölder subbundles.

PROOF OF PROPOSITION 7.5.5. In adapted coordinates a subbundle transverse to $E^c \oplus E^s$ is represented by graphs of linear maps from E^u to $E^c \oplus E^s$. Using the canonical representation of the tangent bundle of $\mathbb{R}^3$ we can write any subspace transverse to the ts-plane as the *image* of a linear map given by a column matrix $\left(\begin{smallmatrix} 1 \\ e \\ \overline{e} \end{smallmatrix}\right)$. Accordingly, the restriction of such a subbundle to stable leaves is given locally by matrices $\left(\begin{smallmatrix} 1 \\ e(s) \\ \overline{e}(s) \end{smallmatrix}\right)$. The advantage of this representation is that applying the derivative amounts to simple composition. The image (in the coordinates at $\varphi^T(p)$ of the subspace is the represented as the image of the linear map with matrix

$$\begin{pmatrix} \alpha^{-1} & 0 & 0 \\ b_1 & 1 & 0 \\ b_2 & 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ e \\ \overline{e} \end{pmatrix} = \begin{pmatrix} \alpha^{-1} \\ b_1 + e \\ b_2 + \overline{e}\alpha \end{pmatrix},$$

which is also the image of the linear map with matrix

$$(7.5.1) \qquad \begin{pmatrix} 1 \\ \alpha(s)b_1(s) + \alpha(s)e(s) \\ \alpha(s)b_2(s) + \overline{e}(s)\alpha^2(s) \end{pmatrix} =: \begin{pmatrix} 1 \\ e(s_\varphi) \\ \overline{e}(s_\varphi) \end{pmatrix},$$

where $\varphi^T(0,0,s) =: (0,0,s_\varphi)$ in local coordinates and $|\alpha(s)s| = |s_\varphi + O(s^2)| = |s_\varphi|(1 + O(s))$ by the C^2 assumption.

To prove Proposition 7.5.5 assume $|e(s)| \le Z|s|^H$ with uniform Z and H. Then

$$\begin{aligned} |e(s_\varphi)| &= |\alpha(s)||b_1(s) + e(s)| \\ &\le |\alpha(s)|[O(s) + Z|s|^H] \\ &= |\alpha(s)s|^H[Z|\alpha|^{1-H} + |\alpha(s)|^{1-H}O(s^{1-H})] \\ &= |s_\varphi|^H(1 + O(s))[Z|\alpha|^{1-H} + |\alpha(s)|^{1-H}O(s^{1-H})] \le Z|s_\varphi|^H \end{aligned} \tag{7.5.2}$$

for sufficiently large Z. Likewise, $|\bar{e}(s)| \le Z|s|^H$ with uniform Z and $H \in (0,1]$ gives

$$\begin{aligned} |\bar{e}(s_\varphi)| &= |\alpha(s)b_2(s) + \bar{e}(s)\alpha^2(s)| \\ &= |\alpha(s)s|^H[|\alpha(s)|^{1-H}O(|s|^{1-H}) + Z|\alpha(s)|^{2-H}] \\ &= |s_\varphi|^H(1 + O(s))[|\alpha(s)|^{1-H}O(|s|^{1-H}) + Z|\alpha(s)|^{2-H}] \le Z|s_\varphi|^H \end{aligned}$$

for sufficiently large Z and sufficiently small s. □

We return to this argument later.

To prove Zygmund regularity of the strong unstable subbundle we vary the strategy slightly. Instead of showing that the space of Z-Zygmund subbundles is preserved for sufficiently large Z, we show that for each Z there is a $Z' > 0$ such that repeated application of the graph transform to a Z-Zygmund subbundle always gives Z'-Zygmund subbundles. Since the space of Z'-Zygmund subbundles is closed, this proves that the unique fixed point E^u is Z'-Zygmund.

Proposition 7.5.6. *For each $Z > 0$ there is a $Z' > 0$ such that all forward images under the graph transform of the space of Z-Zygmund subbundles transverse to $E^s \oplus E^c$ lie in the space of Z'-Zygmund subbundles.*

Proof. Consider a subbundle that is represented in local coordinates as $\begin{pmatrix} 1 \\ e \\ \overline{e} \end{pmatrix}$ with $|e(s) + e(-s)| \le Z|s|$ and $|\overline{e}(s) + \overline{e}(-s)| \le Z|s|$ for all s in any local coordinate system.

Then $|e(s_\varphi) + e(-(s_\varphi))| \le |e(s_\varphi) + e((-s)_\varphi)| + |e((-s)_\varphi) + e(-(s_\varphi))|$, where the last term is $O(s^{2H})$ for any $H < 1$ by Proposition 7.5.5 because $|s_\varphi + (-s)_\varphi| = \|\varphi(0,0,s) + \varphi(0,0,-s)\| \in O(s^2)$ since φ is C^2.

The other term is estimated as follows:

$$\begin{aligned}
|e(s_\varphi)+e((-s)_\varphi)| &= |\alpha(s)b_1(s)+\alpha(-s)b_1(-s)+\alpha(s)e(s)+\alpha(-s)e(-s)| \\
&\le |\alpha(s)||b_1(s)+b_1(-s)|+|b_1(-s)||\alpha(s)-\alpha(-s)| \\
&\qquad +|\alpha(s)||e(s)+e(-s)|+|e(-s)||\alpha(s)-\alpha(-s)| \\
&\le |\alpha(s)|O(s^2)+O(s)O(s)+|\alpha(s)|Z|s|+O(s^H)O(s) \\
&\le Z(1+O(s^H))|\alpha(s)s| \\
&= Z(1+O(s^H))(1+o(s))|s_\varphi| \\
&= Z(1+\kappa(s))|s_\varphi|,
\end{aligned}$$

with $\kappa(s)\in O(s^H)$ decreasing in Z.

Note that $Z' \coloneqq Z\prod_{i=0}^{\infty}(1+\kappa(s_{\varphi^i}))<\infty$ and all images of $\begin{pmatrix}1\\ e\\ \bar{e}\end{pmatrix}$ under the graph transform are Z'-Zygmund in local coordinates. □

The regularity in Theorem 7.5.3 is sharp: It is easy to see that differentiability of the strong stable or unstable foliation cannot be expected.

Proposition 7.5.7. *There is an obstruction to differentiability of the strong unstable subbundle.*

Proof. If p is a T-periodic point, then differentiating (7.5.1) at 0 gives $e'(0)\alpha(0) = \alpha(0)b_1'(0)+\alpha(0)e'(0)$, and hence $K(p,T)\coloneqq b_1'(0)=0$. □

This simple observation is the germ of a nice instance of *smooth rigidity*, which we demonstrate in Theorem 10.3.1: if this obstruction vanishes, then we get much higher smoothness.

6. Sharpness for transversely symplectic flows, threading

The preceding explorations drew structural conclusions from exceeding the maximum provable regularity of the invariant foliations. To complement this picture, we now return to the context of Proposition 7.4.7, where one can show that the Hölder exponent of the invariant subbundles resulting from bunching data is optimal, and in a way that produces (in symplectic contexts) a positive-codimension constraint for exceeding this regularity. Even though that constraint has not been exploited to yield structural information, it provides a strong way of establishing sharpness of Proposition 7.4.7:

Theorem 7.6.1 (Sharpness of Hölder exponent)**.** *For an open dense set of symplectic Anosov flows the regularity of the invariant subbundles is exactly that in Theorem 7.4.14.*

Proof. We carry out the proof for the context of Proposition 7.4.7. The necessary closed condition (7.6.1) for excessive regularity (Theorem 7.6.2) means that the invariant foliations have to "thread a needle" when the regularity is higher than asserted in Proposition 7.4.7 and fails for an open set of symplectic Anosov flows. Density of this set follows from Proposition 7.6.6 which finds a point where the perturbation in Proposition 7.6.10 to break the threading condition (7.6.1) can be implemented (the "dethreading"). □

The first step to establishing sharpness is to exhibit a rare circumstance that is necessary for excessive regularity.

Theorem 7.6.2 (Threading)**.** *If $\alpha \in (0,1)$ and a transversely symplectic flow Φ has C^α unstable subbundle E^u but is not α-bunched, then E^u satisfies the positive-codimension threading condition* (7.6.1)*.*

Remark 7.6.3. This means that if the functions μ_f, μ_s, ν_f, ν_s are chosen optimally, then the predicted regularity is as good as it can be. At periodic points the optimal choice is to take $\mu_f(p)$, $\mu_s(p)$, $\nu_f(p)$, $\nu_s(p)$ to be moduli of eigenvalues of the differential $D\Phi_p$ of the Poincaré map of a local section.

The threading condition in (7.6.1) consists of an identity which implies that part of the unstable subbundle (which is by construction determined by the "past") is determined by the "future" of the orbit. We will later use that these can be modified independently to break this relation.

We will use that periodic α-bunching implies α-bunching [**146**][7] by assuming that Φ has a periodic point p with $\mu_s(p)\nu_s(p)^{-1} > (\mu_f(p))^\alpha$ (this is the reverse of the bunching inequality in Definition 7.4.5), where $\mu_f(p)$, $\mu_s(p)$, $\nu_s(p)$ are as in Definition 7.4.5 and also moduli of eigenvalues of the differential $D\Phi_p$ of the Poincaré map of a local section.

Transverse symplecticity implies $\mu_s(p) = \nu_s(p)^{-1}$, so $\mu_s(p)\nu_s(p)^{-1} > (\mu_f(p))^\alpha$ becomes $\nu_s(p)^{-2} > (\mu_f(p))^\alpha$.

Before embarking on the proof, we produce a higher-dimensional symplectic counterpart to the adapted coordinates in Lemma 7.5.4.

Lemma 7.6.4. *Let $F\colon U \to (\mathbb{R}^{2n},\omega)$ symplectic, $F(0) = 0 \in U \subset \mathbb{R}^{2n}$ hyperbolic,* $0 \in$ *$W_1^j \subset W_2^j \subset \cdots \subset W_n^j = W^j(0)$ submanifolds,* $\dim W_i^j = i$*, $j = u, s$. Then there exist coordinates $(p_1,\dots,p_n,q_1,\dots,q_n)$ on U such that*

(1) $\omega = \sum dp_i \wedge dq_i$ ($\sim \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I is the (n,n)-identity),

[7] [**146**] invokes the proof of a lemma from elsewhere that does not work as intended, but the implication can also be obtained from [**169**, Theorem 1.4] or [**158**, Proposition 3.5].

(2) $W_i^s = \{(0,\dots,0,0,\dots,0,q_{n-i+1},\dots,q_n) \in U\}$
(3) $W_i^u = \{(0,\dots,0,p_{n-i+1},\dots,p_n,0,\dots,0) \in U\}$

Proof. Let $M_i \supset W_i^s \cup W_i^u$ be a $2i$-dimensional submanifold of U, $p_1 \colon U \to \mathbb{R}$ such that $p_1\restriction_{W^s \cup M_{n-1}} = 0$ and P_1 defined by $dp_1 = \omega(P_1,\cdot)$ is transverse to a hypersurface $N_n \supset W_n^u \cup M_{n-1}$. If P_1^t is the Hamiltonian flow generated by P_1 then $\forall\, z \in U$ $\exists!\, y \in N_n, q_1(z) \in \mathbb{R}$: $z = P_1^{q_1(z)}(y)$. Thus $\{q_1,p_1\} = 1$, $q_1\restriction_{N_n} = 0$ and $M_{n-1} = \{x \in U \mid p_1(x) = q_1(x) = 0\}$ since dp_1 and dq_1 are independent. If $n = 1$ this ends the proof, otherwise the $2n-2$-manifold $\big(M_{n-1},\omega\restriction_{M_{n-1}}\big)$ is symplectic because its tangent spaces are the skew complement of the span of P_1 and Q_1 (defined by $dq_1 = \omega(Q_1,\cdot)$)[8], hence has adapted coordinates $\{p_i,q_i\}_{i=2}^n$. $\{p_i,q_i\}_{i=1}^n$ are adapted: $\forall\, z \in U\ \exists!\, y \in M_{n-1}, s,t \in \mathbb{R}$: $z = P_1^s(Q_1^t(y))$, where Q_1^t is the Hamiltonian flow for q_1; (2), (3) hold by construction, (1) follows from standard arguments [**14**, §43E]. □

Proof of Theorem 7.6.2. In adapted coordinates on $\mathscr{T}_p$, E^u is the graph of $D \in gl(n,\mathbb{R})$ or the image of $\left(\begin{smallmatrix} I \\ D \end{smallmatrix}\right) \colon \mathbb{R}^n \simeq \mathbb{R}^n \times \{(0,0)\} \to \mathbb{R}^{2n}$. At $q = (0,\dots,0,z) =: z_0$ on the fast stable leaf through 0 the differential of Φ is $D\Phi_{|z_0} = \left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$ with $C = A^{t^{-1}}$ lower block triangular. To bring $D\Phi\left(\begin{smallmatrix} I \\ D \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)\left(\begin{smallmatrix} I \\ D \end{smallmatrix}\right) = \left(\begin{smallmatrix} A \\ B+CD \end{smallmatrix}\right)$ into the form $\left(\begin{smallmatrix} I \\ \tilde{D} \end{smallmatrix}\right)$, right multiply with A^{-1} and use symplecticity ($A^{-1} = C^t$):

$$\begin{pmatrix} I \\ \tilde{D} \end{pmatrix} = \Phi^* \begin{pmatrix} I \\ D \end{pmatrix} = \begin{pmatrix} A \\ B+CD \end{pmatrix} A^{-1} = \begin{pmatrix} I \\ (B+CD)C^t \end{pmatrix}.$$

Since E^u is invariant, $D(\Phi(z_0)) = (B + CD(z_0))C^t$. Denote the upper left k-blocks of A, B, C and D by a, b, c and d, respectively, where k is such that $\|(c(0))^{-n}\| \le C\nu_s^n$. Since C is lower block triangular, $\tilde{d} = c \cdot d \cdot c^t + b \cdot c^t$ or

$$d(\Phi(z_0)) = c(z_0) \cdot d(z_0) \cdot c^t(z_0) + b(z_0) \cdot c^t(z_0).$$

With d we have isolated the component of D on which the flow acts with the slowest expansion and contraction. Symplecticity allows us to “decouple” this block from the others and obtain this recursion relation.

If $z_n := \Phi^n(z_0)$, $\xi_{z_0}^n := \prod_{i=0}^{n-1} c(z_{n-i-1})$, $\Delta_{z_0}^n := -\sum_{i=0}^{n-1}(\xi_{z_0}^{i+1})^{-1} b(z_i)(\xi_{z_0}^i)^{t^{-1}}$ then

$$d(z_n) = \xi_{z_0}^n \cdot (d(z_0) - \Delta_{z_0}^n) \cdot \xi_{z_0}^{n\ t},$$

so $d(\cdot) \in C^\alpha \Rightarrow \|d(z_0) - \Delta_{z_0}^n\| \le \xi_{z_0}^{n\ -1} \cdot \|d(z_n)\| \cdot \xi_{z_0}^{n\ t^{-1}} \le C_2 \nu_s^{2n} \nu_f^{-\alpha n} \to 0$ since $\nu_s^2 < \nu_f^\alpha$ and $\|(c(0))^{-n}\| \le C\nu_s^n$. Thus

$$d(z) = \Delta_z^\infty. \tag{7.6.1}$$

□

[8] For $v \in T_x M_{n-1}$ we have $0 = dp_1(v) = \omega(P_1, v)$ and $0 = dq_1(v) = \omega(Q_1, v)$

Remark 7.6.5. Both this obstruction and that in Proposition 7.5.7 have positive codimension, but they are of a different nature. The latter is expressible in terms of derivatives of the flow, whereas (7.6.1) arises from the geometry of the invariant subbundles. Nonetheless, being of positive codimension, both imply that "typically" Anosov flows do not have more regular subbundles than the regularity theorems assert.

What we have obtained with the threading condition (7.6.1) is the following. If $\binom{I}{D}$ represents E^u then $d(z)$, like E^u, is determined by the past of the orbit of p. (7.6.1) shows that if excessively regular, then d is simultaneously determined by the future of the orbit of p, namely by Δ_z^∞. We shall see that the past and future can be modified independently, so this is a special situation.

Accordingly, we seek a point where the past and future can be disentangled—which will allow a perturbation to cause the threading condition (7.6.1) to fail.

Proposition 7.6.6. *Let M be a Riemannian manifold, Φ an Anosov flow, $p \in M$ periodic. Then the fast stable leaf of p contains a negatively nonrecurrent point.*

Remark 7.6.7. Density of nonrecurrent points on the stable leaf is clear since any point heteroclinic to a different periodic point has this property and periodic points are dense. But the existence of heteroclinic points in the *fast* stable leaf is not as clear. The proof we give actually yields *density* of nonrecurrent points on the fast stable leaf.

Proof. Let $C, \chi > 0$ such that $\|D\varphi^t{\restriction_{E^s}}\| \le C \cdot e^{-\chi t}$ for $t > 0$, $\mathscr{T}$ a hypersurface through p transverse to the flow direction, $W^{fs}(p)$ the fast stable leaf of p for the return map Φ, $q \in W^{fs}(p) \smallsetminus \{p\}$ and $B_1 = B_{\sqrt{\epsilon}}(q)$ where $\epsilon > 0$ is such that

$$T := \inf\{t > t_0 \mid \varphi^{-t} B_{\sqrt{\epsilon}}(q) \cap B_{\sqrt{\epsilon}}(q) \neq \varnothing\} > \chi^{-1} \log 4C,$$

and $t_0 := \inf\{t > 0 \mid \varphi^{-t} B_{\sqrt{\epsilon}}(q) \cap B_{\sqrt{\epsilon}}(q) = \varnothing\}$. $B_\epsilon(q)$ is the ϵ-ball around q in M.

For $x \in \mathscr{T}$, $A \subset \mathscr{T}$ let $\operatorname{rad}_x(A) := \sup\{d(x,y) \colon y \in A\}$, where d is the distance on $\mathscr{T}$. Let $W^{fs}(p) := \mathbb{C}\big(B_1 \cap (\text{fast stable leaf of } p), p\big)$ (as in Definition 1.6.13 and $W^{ss}(x) := \mathbb{C}\,(B_1 \cap (\text{stable leaf of } x), x)$ for $x \in B_1$. For ϵ small $B_2 := B_1 \cap \mathscr{T}$ has local product structure, that is, local stable and unstable leaves intersect in a point, so we introduce coordinates on $W^{uu}(p)$ and $W^{ss}(p)$ and represent $x \in B_2$ by (*coordinate of* $W^{ss}(x) \cap W^{uu}(p)$, *coordinate of* $W^{uu}(x) \cap W^{ss}(p)$).

Let D be an ϵ-ball in $\mathscr{T} \cap$ (weak unstable leaf of q), $U_1 := \bigcup_{x \in D} B^s_\epsilon(x)$, $Q_1 := \bigcup_{x \in D} B^s_{2\epsilon}(x)$, $V_1 := \bigcup_{x \in D} \overline{B^s_{4\epsilon}(x)}$ and $W_1 := \bigcup_{x \in D} B^s_{5\epsilon}(x)$, where "$\overline{}$" denotes closure and $B^s_\epsilon(x)$ is the ϵ-ball in $W^{ss}(x)$, $x \in B_1$. For $x \in B_1$ let $U_x := U_1 \cap W^{ss}(x)$, $V_x := V_1 \cap W^{ss}(x)$, $Q_x := Q_1 \cap W^{ss}(x)$, $W_x := W_1 \cap W^{ss}(x)$ and $S_x := V_x \smallsetminus Q_x$. Thus we may view D as a little essentially horizontal segment and U_1, Q_1, V_1, W_1 then are essentially "rectangles" $D \times B_\epsilon$, $D \times B_{2\epsilon}$, $D \times B_{4\epsilon}$, $D \times B_{5\epsilon}$. The subscript x denotes a vertical slice through these. The S_x are spherical shells in $W^{ss}(x)$.

It suffices to find a point in $U \coloneqq U_1 \cap W^{fs}(p)$ that does not return to U_1 in negative time. Define $t\colon B_2 \to \mathbb{R}^+ \cup \{\infty\}$, $x \mapsto t(x) \coloneqq \inf\{t > 0 \mid \varphi^{-t}x \in \overline{U}_1\}$ and take $x_0 \in U_2 \coloneqq \overline{U}$ such that $t(x_0) = \min\{t(x)\colon x_0 \in U_2\} > T$. There is a smooth function $\tau\colon W_2 \coloneqq W_{\varphi^{-t(x_0)}x_0} \to \mathbb{R}^+$ such that $\psi_1(x) \coloneqq \varphi^{\tau(x)}x \in \mathscr{T}$ for $x \in W_2$ and $\tau(\varphi^{-t(x_0)}x_0) = t(x_0)$. Thus $\psi_1\colon W_2 \to \mathscr{T}$ is a diffeomorphism onto its image and $\psi_1(W_2) \subset W^{ss}(p) = W^{ss}(x_0)$. The intersection $U \cap S_1$ of the spherical shell $S_1 = \psi_1(S_{\varphi^{-t(x_0)}x_0})\imath W^{ss}(p)$ with U consists of points not returning to U_1 for time t with $-T_1 \coloneqq -\max\{t(x)\colon x \in V_2\} \le t < 0$, where $V_2 \coloneqq V_{\varphi^{-t(x_0)}x_0}$.

Claim 7.6.8. *$U \cap S_1$ has a connected component U_3' such that $(\psi_1(W_2) \smallsetminus S_1) \cup U_3'$ is connected.*

Proof. U is connected. $\mathrm{rad}_{x_0}(U) \ge \epsilon$. $\mathrm{rad}_{x_0}(S_1) < \epsilon$ by choice of T. So U contains points outside S_1. Since $x_0 \in \overline{U}$ is inside the shell S_1 so are some points of U. □

Take $U_3 \subset U_3'$ closed and connected such that $(\psi_1(W_2) \smallsetminus S_1) \cup U_3$ is connected. $\psi_1^{-1}(U_3)$ connects the complement of $S_{\varphi^{-t(x_0)}x_0}$ in W_2 and is connected. Therefore, $\mathrm{rad}_x(\psi_1^{-1}(U_3)) \ge \epsilon$ for all $x \in \psi_1^{-1}(U_3)$. Let $t\colon W_2 \to \mathbb{R}^+ \cup \{\infty\}$, $x \mapsto t(x) \coloneqq \inf\{t > 0 \mid \varphi^{-t}x \in \overline{U}_1\}$ and take $x_1 \in \psi_1^{-1}(U_3)$ such that $t(x_1) = \min\{t(x)\colon x \in \psi_1^{-1}(U_3)\} > T$. There is a smooth function $\tau\colon W_3 \coloneqq W_{\varphi^{-t(x_1)}x_1} \to \mathbb{R}^+$ such that $\psi_2(x) \coloneqq \varphi^{\tau(x)}x \in \mathscr{T}$ for $x \in W_3$ and $\tau(\varphi^{-t(x_1)}x_1) = t(x_1)$. Thus $\psi_2\colon W_3 \to \mathscr{T}$ is a diffeomorphism onto its image and $\psi(W_3) \subset W^{ss}(x_1)$. The spherical shell $S_2 \coloneqq \psi_2(S_{\varphi^{-t(x_1)}x_1}) \subset W^{ss}(x_1)$ consists of points not returning to U_1 for time t with $-T_2 \coloneqq -\max\{t(x)\colon x \in V_3\} \le t < 0$, where $V_3 \coloneqq V_{\varphi^{-t(x_1)}x_1}$.

Claim 7.6.9. *$\psi_1^{-1}(U_3) \cap S_2$ has a connected component U_4' with $(\psi_2(W_3) \smallsetminus S_2) \cup U_4'$ connected.*

Proof. As before: $\psi_1^{-1}(U_3)$ is connected and $\mathrm{rad}_{x_1}(\psi_1^{-1}(U_3)) \ge \epsilon$. □

No points of $U_4 \coloneqq \psi_1(U_4') \subset U_3$ return to U_1 for time $t \in [-T_1 - T_2, 0) \supset [-2T, 0)$. Iterating this argument gives compact $U_1 \supset \cdots \supset U_{n+2} \supset \dots$ with return times $\ge nT$. The nonempty intersection consists of negatively nonrecurrent points. □

Proposition 7.6.10 (Dethreading)**.** *Failure of the threading condition* (7.6.1) *is dense among symplectic Anosov flows.*

Proof. Pick a negatively nonrecurrent (resp. nonreturning for geodesic flows) point q in the fast stable leaf of the periodic point p and inside the adapted coordinate patch. Translate the coordinates so that q is the origin. Construct a perturbation as follows: Let $\overline{\gamma}(s,t,u,v,w,z)=\frac{1}{2}\langle s,s\rangle=\frac{1}{2}\|s\|^2$ and $\gamma=\rho\overline{\gamma}$ where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^k$ and $\rho\in C^\infty(B_\epsilon,\mathbb{R})$ is C^k-small and such that $\rho=\epsilon^{k+1}$ on B_{ϵ^2} and $\rho=0$ on $B_\epsilon\smallsetminus B_{\epsilon(1-\epsilon)}$. The vector field X with $\omega(X,\cdot)=d\gamma$ generates a complete symplectic flow G_τ.

The vector field $\overline{X}(s,t,u,v,w,z)=(0,0,0,s,0,0)^t$ satisfies $\omega(\overline{X},\cdot)=d\overline{\gamma}$ and generates the flow $\overline{G}_\tau(s,t,u,v,w,z)=(s,t,u,v+\tau s,w,z)$ with

$$D\overline{G}_\tau=\begin{pmatrix} I & 0 \\ \begin{smallmatrix}\tau I & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{smallmatrix} & I\end{pmatrix}=:\begin{pmatrix} I & 0\\ \tau\overline{E} & I\end{pmatrix}.$$

Let $\eta\in C^\infty(\mathbb{R},\mathbb{R}^+)$ be C^k-small such that $\eta(x)=0$ $(x<0)$, $\eta(x)=\epsilon^{k+1}$ $(x>\epsilon)$. Redefine $\varphi^\tau\simeq(s,x)\mapsto(s+\tau,x)$ on $B=[0,\epsilon]\times B_\epsilon$ so that $\overline{\varphi}^\tau(0,x)=(\tau,G_{\eta(\tau)}(x))$. $\overline{\varphi}^\tau$ is a symplectic C^k-small perturbation of φ^τ. It causes (7.6.1) to fail:

$$D\overline{G}_\tau\begin{pmatrix} I\\ D\end{pmatrix}=\begin{pmatrix} I & 0\\ \tau\overline{E} & I\end{pmatrix}\begin{pmatrix} I\\ D\end{pmatrix}=\begin{pmatrix} I\\ \tau\overline{E}+D\end{pmatrix}.$$

Similarly for $DG_\tau=\left(\begin{smallmatrix} I & 0\\ E & I\end{smallmatrix}\right)$. This ends the proof by Claim 7.6.11. □

Claim 7.6.11. *$E+D$ is the unstable direction at q for the flow $\overline{\varphi}^\tau$.*

Proof. Take a subbundle $\mathscr{E}$ on $\{\varphi^t(q)\}_{t<0}$ close to E^u. Since q is nonrecurrent (resp. nonreturning for geodesic flows) $E^u(\varphi^{-\epsilon}q)=\lim_{t\to\infty}D\varphi^t(\mathscr{E}(\varphi^{-t-\epsilon}(q)))=\overline{E}^u(\varphi^{-\epsilon}q)$. $\overline{\varphi}^t$-invariance of $\overline{E}^u$ gives the claim. □

The failure of regularity tends to be pervasive through the phase space. One way to make this explicit is the following.

Theorem 7.6.12 ([**150**, **151**])**.** *For $\alpha\in(0,1)$ there is a nonempty C^1-open set of transversely symplectic C^∞ Anosov flows for which*

- *E^{cu} and E^{cs} are C^α only on a set whose complement is residual and has full measure for any fully supported ergodic invariant Borel probability measure, and*
- *the Bowen bracket is almost nowhere C^α.*

This sharpness result and the earlier ones suggest that systems whith excessive regularity of the invariant subbundles are "special" in some sense, although here that merely means that they are rare in the sense of Baire category. There are in some situations much stronger statements to the effect that the only exceptional systems are of a rather specific nature, usually algebraic in some sense. This phenomenon is called *rigidity*, and we explore it in Chapter 10 below.

7. Smooth linearization and normal forms

a. Differentiability in the Hartman–Grobman Theorem. We return to a context in which we previously noted that higher reguarity would be concretely useful. Remark 5.6.2 noted that the Hartman–Grobman Theorem provides a purely topological conclusion, but it can be refined to yield a local conjugacy that is differentiable at the fixed point. Although we are interested in flows, the proof of Theorem 5.6.1 shows that it suffices to prove this in discrete time because the conjugacy we obtain that way is by uniqueness the conjugacy between the flow and its linearization.

We can refine the Hartman–Grobman Theorem:

Theorem 7.7.1 (Differentiable Hartman–Grobman Theorem [**292**])**.** *If the flow in the Hartman–Grobman Theorem 5.6.1 is C^2 then the linearizing homeomorphism is differentiable at the fixed point, and its derivative at the fixed point is the identity.*[9]

Remark 7.7.2. Even the C^2 assumption is a little more than needed. It suffices for the flow to be C^1 near the equilibrium and for the differential to have sufficiently high Hölder exponent at the equilibrium. However, assuming C^1 only is definitely not enough. The proof we give here assumes that the flow is C^∞ in order to use a reduction to a dynamical system given by a (particular kind of) second-order polynomial.

Example 7.7.4 shows an application of this result. We mention a different improvement on the Hartman–Grobman Theorem with like applications:

Theorem 7.7.3 ([**293**])**.** *If the flow in the Hartman–Grobman Theorem 5.6.1 is C^∞ then for each $\alpha \in (0,1)$ there is a neighborhood of the fixed point on which the linearizing homeomorphism and its inverse are α-Hölder continuous.*

We will see in Corollary 10.1.11 that an asymptotically stable fixed point for a flow can have a smooth linearization in a neighborhood of the fixed point so long as the eigenvalues of the derivative at the fixed point satisfy certain conditions. However, this will not ensure a smooth linearization for a saddle point.

Proof outline. We outline the proof (from [**141**]) here; the rest of the section is taken up by the full proof. The reader may decide that this outline is sufficient and take a look at Example 7.7.4 to see how this result is used.

First of all, as we showed in proving Theorem 5.6.1 from its discrete-time counterpart, we can consider diffeomorphisms rather than flows. The conclusion of Theorem 7.7.1 is that the conjugacy is within $o(x)$ of the identity. We use a result of Bronstein and Kopanskiĭ that reduces the time-1 map to its second-order

[9] It was at the *ETH* (in conversations with Jürgen Pöschel) where the second author's interest in proving this result was sparked.

expansion via a C^1 conjugacy (Theorem 7.7.19), that is, it tells us that without loss of generality the diffeomorphism is a quadratic polynomial—of a particular kind.

The iterates of this map are polynomials, and we estimate their coefficients first recursively, and then definitively in Lemma 7.7.24. Unsurprisingly, we define the conjugacy via a natural correspondence between orbits under the linear and nonlinear dynamics; it has the product form $h \coloneqq (h_+, h_-)$ with the components defined in (7.7.9). To show that this produces a map within $o(x)$ of the identity (Lemma 7.7.30) requires us to control the difference between orbits for the linear and nonlinear maps by combining these coefficient estimates with an estimate as to how close to the origin a point must be in order to stay in a given neighborhood for a specified time (Lemmas 7.7.26, 7.7.27 and 7.7.28). □

Example 7.7.4. To illustrate the application of differentiability in the Hartman–Grobman Theorem in a simple context, we draw a phase portrait of the system of differential equations $\begin{cases} \frac{dx}{dt} = 2x - 2x^2 - xy \\ \frac{dy}{dt} = 4y - 2xy - y^2 \end{cases}$ as follows. Setting the right-hand side to 0 to find the equilibria gives

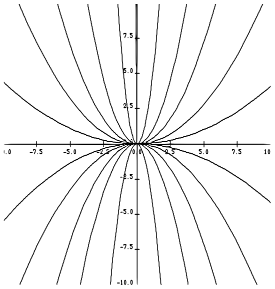
$$\begin{aligned} 0 &= 2x - 2x^2 - xy = x(2 - 2x - y), \\ 0 &= 4y - 2xy - y^2 = y(4 - 2x - y), \end{aligned}$$

so the equilibria are $(0,0)$, $(0,4)$, $(1,0)$. To linearize at each of these, note that the matrix of partial derivatives of the right-hand side is $\begin{pmatrix} 2-4x-y & -x \\ -2y & 4-2x-2y \end{pmatrix}$, so the linearizations are

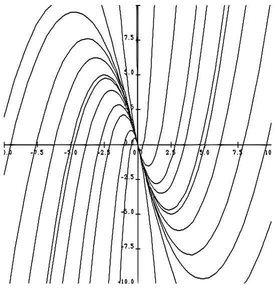
- $(0,0)$: $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ with eigenvalue-eigenvector pairs $\left(2, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $\left(4, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$, so for the linearization the origin is a repeller.
- $(0,4)$: $\begin{pmatrix} -2 & 0 \\ -8 & -4 \end{pmatrix}$ with eigenvalue-eigenvector pairs $\left(-2, \begin{pmatrix} 1 \\ -4 \end{pmatrix}\right)$, $\left(-4, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$; so the origin is an attractor.
- $(1,0)$: $\begin{pmatrix} -2 & -1 \\ 0 & 2 \end{pmatrix}$ with eigenvalue-eigenvector pairs $\left(-2, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $\left(2, \begin{pmatrix} 1 \\ -4 \end{pmatrix}\right)$, so this is a hyperbolic fixed point of saddle type.

Moreover, the phase portraits of the linearizations are as follows:

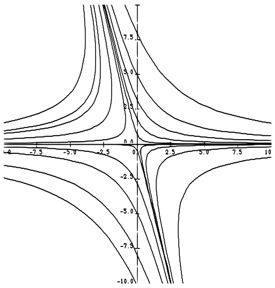
$(0,0):$



$(0,4):$



$(1,0):$



Because the linearizations show that the Hartman–Grobman Theorem applies to each equilibrium, we can now place each of these pictures on the respective fixed point as a thumbnail of the actual phase portrait and plausibly connect these. The resulting phase portrait is shown in Figure 7.7.1. Note that the local appearance near each equilibrium is *exactly* rather than just topologically that of the corresponding linearization—in contrast to Figure 5.6.1.

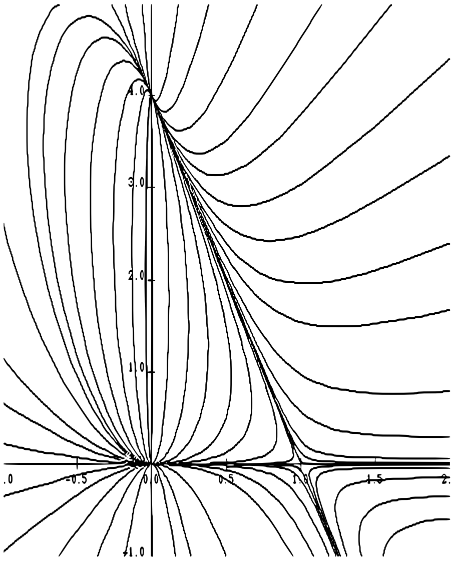


FIGURE 7.7.1. Phase portrait in the plane pieced together from Hartman–Grobman patches

b. Jets, formal power series, and smooth equivalence. In hyperbolic flows the linearized system locally serves as a model for the behavior of a nonlinear flow, thus implying that nonlinear terms constitute a perturbation which should be kept under control. A natural next step for local analysis is to consider the terms of order higher than linear in a more systematic way and determine more precisely the extent to which their influence has to be taken into account or whether it can be disregarded altogether. A hyperbolic periodic orbit is the best setting for such an analysis. The key phenomena here are "resonances" between the eigenvalues of the linearized map. Their presence or absence determines what higher-order terms must be taken into account.

In Example 1.3.8 we saw that eigenvalues at a fixed point are preserved under smooth conjugacy (since the differential of the conjugacy conjugates the linear parts of the maps). This motivates thinking about whether the conjugacy in the Hartman–Grobman Theorem 12.4.14 can be taken smooth. We introduced the refined Hartman–Grobman Theorem above, and we expand on these ideas now in order to both give a larger picture and present some of the ideas we use in the proof of differentiability of the Hartman–Grobman linearization.

First we should find out whether there are other *infinitesimal* invariants of local smooth conjugacy besides those coming from the linear part of the map at the fixed point. For that purpose fix local coordinates near a fixed point p of a map f and consider the coefficients of the kth Taylor polynomial of f for $k = 2, 3, \dots$. This set of coefficients is called the kth *jet* or k-*jet* $J^k_p(f)$ of f at p. Thus two C^k maps f and g have the same k-jet at a (not necessarily fixed) point p if $\|f(x) - g(x)\| = o(\|x - p\|^k)$. Obviously the first jet of a map is determined by the value of the map and its linear part. A real-analytic map is the limit of its kth Taylor polynomials as $k \to \infty$. For C^∞ maps one can write down a Taylor series, but it may not converge at more than one point. Thus we are led to consider *formal* power series, that is, a formal expression consisting of an infinite sum of monomials. The algebraic operations and substitutions are performed by applying the rules familiar from convergent power series.

Obviously the (formal) Taylor series of a C^∞ map at a point determines all jets at that point. k-jets of a map at a point can be identified with polynomial maps in a local coordinate system and their composition is defined by taking the usual composition and discarding higher-order terms. Furthermore near a fixed point a local C^k conjugacy between maps produces a conjugacy between the k-jets at the fixed point. Thus the conjugacy classes of these jets are the infinitesimal invariants we set out to find. For C^∞ maps C^∞ local conjugacy implies that the formal Taylor series at the reference point are conjugate, where the composition of the formal power series is obtained by substitution.

Now we can outline our strategy for solving the smooth conjugacy problem as follows. First we look for invariants of conjugacy of k-jets for any $k \in \mathbb{N}$. It turns out that all those invariants are completely determined by the linear part of the map, that is, its first jet. Let us say that two C^∞ maps f and g are C^∞ *tangent at p* if $J_p^k(f) = J_p^k(g)$ for all $k \in \mathbb{N}$, or $\|f(x) - g(x)\| = o(\|x - p\|^k)$ for all $k \in \mathbb{N}$, or equivalently if the formal Taylor series for f and g at p coincide. The second step is to show that if all jets of f and g are conjugate then there exists a C^∞ map h such that f and $f' := h^{-1} \circ g \circ h$ are C^∞ tangent. Finally, if the linear part of f at p is hyperbolic and f' is C^∞ tangent to f at p then f and f' are locally C^∞ conjugate via a local diffeomorphism C^∞ tangent to the identity. This last step can be carried out in various ways. If the linear part of the map f is contracting one can consider the conjugacy equation as the fixed-point equation $h = (f')^{-1} \circ h \circ f$ and use the Contraction Mapping Principle to show that the solution h is C^∞. In the properly hyperbolic case one can use a refined version of the fundamental-domain method called the Sternberg wedge method. We use a version of the homotopy trick, which was used to prove Theorem 2.6.11, to accomplish this final step (Theorem 7.7.13). This reduces the problem to analyzing the solution of a twisted cohomological equation.

c. Formal analysis of smooth conjugacy. Since we want to conveniently manipulate power series in n variables, we use multi-indices systematically. Thus for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ we define the *size* of k to be $|k| := \sum_{i=1}^n k_i$ and if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we let $x^k := \prod_{i=1}^n x_i^{k_i}$.

1. *Formal linearization in the nonresonance case.*

Definition 7.7.5. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. A relation of the form $\lambda_i = \lambda^k$ is called a *resonance* and correspondingly the condition "$\lambda_i \neq \lambda^k$ for all i and all $k \in \mathbb{N}_0^n$" is called a *nonresonance assumption.*

Proposition 7.7.6. *Consider a formal power series f given by*

$$f_i(x) = \sum_{k \in \mathbb{N}^n} f_{i,k} x^k$$

with vanishing constant term and whose linear part g is $\operatorname{diag}\lambda$ *(that is, the diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$) where λ satisfies the nonresonance assumption. Then there exists a formal power series h solving the conjugacy equation $h \circ f = g \circ h$.*

Proof. Note first that we can write

$$f_i(x) = \sum_{k \in \mathbb{N}^n} f_{i,k} x^k = \lambda_i x_i + \sum_{|k|>1} f_{i,k} x^k$$

as the linear part plus terms of higher order.

The nonresonance assumption for $|k| = 1$ implies in particular that the λ_i are pairwise distinct. Since the linear part of h commutes with that of f (because the latter coincides with that of g), the linear part of h must be diagonal, with eigenvalues α_i, say. Thus we write a candidate h as

$$h_i(x) = \sum_{k\in\mathbb{N}^n} h_{i,k}x^k = \alpha_i x_i + \sum_{|k|>1} h_{i,k}x^k.$$

The ith coordinate of the conjugacy equation $h \circ f(x) = g \circ h(x)$ now becomes

$$\sum_{k\in\mathbb{N}^n} h_{i,k}(f(x))^k = \lambda_i h_i(x),$$

or, by splitting off the linear parts,

$$\alpha_i f_i(x) + \sum_{|k|>1} h_{i,k}(\lambda_1 x_1 + \sum_{|j_1|>1} f_{1,j_1}x^{j_1})^{k_1}\cdots(\lambda_n x_n + \sum_{|j_n|>1} f_{n,j_n}x^{j_n})^{k_n} = \lambda_i h_i(x).$$

We want to solve this equation inductively in $m := |k|$ for the coefficients $h_{i,k}$ of h in terms of the coefficients $f_{i,k}$ and the λ_i. In other words, we want to construct a conjugacy for m-jets given a conjugacy for $(m-1)$-jets. For $m = 1$ the choices are arbitrary, for example, one can take $Dh = \mathrm{Id}$. Suppose $m \in \mathbb{N}$ and we have determined all $h_{i,k}$ as desired for all $|k| < m$. Then for any k such that $|k| = m$ consider the coefficients of the terms involving x^k. Comparing them yields

$$\alpha_i f_{i,k} + \lambda^k h_{i,k} = \lambda_i h_{i,k} + C_{i,k}, \tag{7.7.1}$$

where $C_{i,k}$ involves only coefficients with indices j of size $|j| < m$, which are thus entirely determined by the previous steps. Thus we solve for $h_{i,k}$ by taking

$$h_{i,k} = \frac{\alpha_i f_{i,k} - C_{i,k}}{\lambda_i - \lambda^k}, \tag{7.7.2}$$

which is possible by the nonresonance assumption. □

2. *Normal forms in the resonance case.* Now suppose we are in the situation of the previous proposition but there are $k \in \mathbb{N}^n$ such that $\lambda_i = \lambda^k$ for some i. Then terms with $h_{i,k}$ in (7.7.1) disappear, that is, one cannot remove the term involving x^k from the ith coordinate function. This observation leads to a study of *normal forms*, which are the natural generalizations of the linear part of a map.

Definition 7.7.7. If $\lambda_i = \lambda^k$ a nonzero term $c \cdot x^k$ in the ith coordinate function of a formal power series is called a *resonance term*.

A *formal normal form* of a power series f is a formal power series g with the same linear part whose power series contains only linear and resonance terms such that the conjugacy equation

$$h \circ f = g \circ h \tag{7.7.3}$$

holds for a formal power series h.

The previous process can be generalized to the situation where g has nonlinear terms, that is, $g_i(x) = \sum_{k\in\mathbb{N}^n} g_{i,k}x^k = \lambda_i x_i + \sum_{|k|>1} g_{i,k}x^k$. Then (7.7.1) becomes

$$\alpha_i f_{i,k} + \lambda^k h_{i,k} = \lambda_i h_{i,k} + \alpha^k g_{i,k} + C_{i,k}, \tag{7.7.4}$$

where the $C_{i,k}$ are determined by the previous steps and may involve lower-order terms of g. Suppose now that g has *only* resonance terms and the linear part of h is fixed, say, to be the identity. Then (7.7.2) still holds for the nonresonance terms and for resonance terms (7.7.4) implies $g_{i,k} = f_{i,k} - C_{i,k}$, that is, the resonance terms of g are uniquely defined if there is a formal conjugacy. Thus within a formal conjugacy class (with diagonal linear part) the normal form is indeed uniquely defined up to choice of linear part. To summarize:

Proposition 7.7.8. *For any formal power series with diagonal linear part there exists uniquely defined formal normal form.*

Even if f is an analytic map its formal normal form may not converge. There are however cases when it does even for formal power series. The first one is the non-resonance case. Another is contracting maps. In this case there are only finitely many resonances (namely, no more than $-\log r(Df^{-1})/\log r(Df)$, where r denotes spectral radius) and hence the normal form is a polynomial. Thus it makes sense to look for normal forms in the smooth or analytic categories. The analytic case is more straightforward

Definition 7.7.9. A *normal form* for an analytic map f is its formal normal form g whose power series converges and the solution h of the formal conjugacy equation (7.7.3) is given by a converging power series.

In the smooth case we need to take into account that a smooth function is not determined by its Taylor series.

Definition 7.7.10. A *normal form* for a smooth map f is its formal normal form g whose power series converges and the conjugacy equation $h \circ f = g \circ h$ has a smooth solution h.

3. *Examples of resonance cases.* Interest in these nonlinear normal forms is motivated by the fact that in several natural settings one has “built-in” resonances, that is, in certain natural classes of maps there is a natural collection of resonances exhibited by every one of these maps. The main examples are area-preserving and symplectic maps.

If p is a fixed point of a map preserving a positive absolutely continuous measure then the determinant of the differential Df_p is ±1 (cf. **??** which treats the orientation-preserving case). Thus, if $\lambda_1,\dots,\lambda_n$ are the eigenvalues of Df_p counted with multiplicities then

$$\lambda_1\cdots\lambda_n=\pm 1$$

and there are resonance relations

$$\lambda_i=\lambda_1\cdots\lambda_{i-1}\lambda_i^2\lambda_{i+1}\cdots\lambda_n$$

or

$$\lambda_i=\lambda_1^2\cdots\lambda_{i-1}^2\lambda_i^3\lambda_{i+1}^2\cdots\lambda_n^2.$$

Similarly, if p is a fixed point of a symplectic diffeomorphism then according to Proposition 2.6.8 the eigenvalues can be split into pairs of mutually reciprocal numbers, that is, the vector of eigenvalues can be arranged to look like

$$(\lambda_1,\lambda_1^{-1},\lambda_2,\lambda_2^{-1},\dots,\lambda_n,\lambda_n^{-1})$$

and hence there are n resonances

$$\lambda_i=\lambda_i^2\lambda_i^{-1},\qquad i=1,\dots,n.$$

In the two-dimensional situation where the notion of area-preservation and symplecticity coincide the normal form (**??**) is described by **??**. Note that it is more special than the normal form (7.7.5) for $p=q=1$.

Now we describe the possible resonances in the two-dimensional hyperbolic case. There are two eigenvalues which we denote by λ_- and λ_+ such that $|\lambda_-|<1<|\lambda_+|$. A resonance has the form $\lambda_-=\lambda_-^k\lambda_+^l$ or $\lambda_+=\lambda_+^k\lambda_-^l$, that is, $\lambda_-^{k-1}=\lambda_+^{-l}$ or $\lambda_+^{k-1}=\lambda_-^{-l}$, where $k,l\in\mathbb{N}_0$ are nonnegative integers. Thus if $\varphi_\pm=|\log|\lambda_\pm||$ then the resonance implies $\varphi_-/\varphi_+\in\mathbb{Q}$. Conversely it is easy to see that this implies that there is a resonance. If $\varphi_-/\varphi_+=p/q$ with $p,q\in\mathbb{N}$ relatively prime then we say that there is a (p,q)-resonance. To describe the normal forms in this case let us assume for simplicity that both eigenvalues are positive. Then $\lambda_-^q\lambda_+^p=1$ and if $\lambda_-=\lambda_-^k\lambda_+^l$ then $k=mq+1$, $l=mp$ for some $m\in\mathbb{N}$. Similarly $\lambda_+=\lambda_+^k\lambda_-^l$ implies $k=mp+1$, $l=mq$. Hence if f is a normal form then

$$f(x,y)=\Big(\lambda_-x\Big(1+\sum_{m=1}^{\infty}a_m(x^qy^p)^m\Big),\lambda_+y\Big(1+\sum_{m=1}^{\infty}b_m(x^qy^p)^m\Big)\Big).\tag{7.7.5}$$

Notice that in the area-preserving case there is a $(1,1)$-resonance. **??** provides a specialization of (7.7.5) for that case.

d. The hyperbolic smooth case. One way to justify the formal manipulations with power series is by showing that all power series involved have positive radius of convergence. Although the absence of resonances may not be sufficient for analytic linearization even in the hyperbolic case it is so in the C^∞ category. The basic idea is that for any formal power series there is a C^∞ function whose Taylor series coincides with the given power series.

Proposition 7.7.11. *For any sequence* $\{a_k\}_{k\in\mathbb{N}_0^n} \subset \mathbb{R}$ *there exists a* C^∞ *function* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *such that the* a_k *are the Taylor coefficients of* f.

Proof. First we introduce a notion that is useful in many places. By a *bump function* we mean a smooth nonzero function with compact support in an open set. An example on the real line is given by the function

$$b_1(x) := \begin{cases} e^{2-(x+1)^{-2}-(x-1)^{-2}} & \text{when } |x| \le 1, \\ 0 & \text{when } |x| > 1, \end{cases}$$

since b_1 vanishes to all orders at ± 1. Note that $b_2(x) := \int_{|x|}^{\infty} b_1(t-2)\,dt / \int_{-1}^{1} b_1(t)\,dt$ defines a bump function which is 1 on $[-1,1]$. Typically one uses bump functions of the second type.

Now we prove the proposition: Set

$$f(x) = \sum_{k\in\mathbb{N}_0^n} a_k x^k b_2(|k|! C_{|k|} \|x\|^2),$$

with $C_N := \sum_{l=0}^{N} \sum_{|i|=l} |a_i|$. Notice that this series converges since for each $x \ne 0$ there are only finitely many nonzero terms. Note that

(7.7.6)
$$|a_k x^k b_2(|k|!\, C_{|k|} \|x\|^2)| \le |a_k| \|x\|^{|k|} b_2(|k|!\, C_{|k|} \|x\|^2)| \le |a_k| \Big(\frac{2C_{|k|}}{|k|!}\Big)^{\frac{|k|}{2}} \le \Big(\frac{2}{|k|!}\Big)^{\frac{|k|}{2}},$$

since $|k|! C_{|k|} \|x\|^2 \le 2$ for all nonzero terms. Thus the sum converges uniformly and rapidly. To evaluate the derivatives of order N consider points x such that $|k|! C_{|k|} \|x\|^2 < 1$. For these points we have

$$f(x) = \sum_{|k|\le N} a_k x^k + \sum_{|k|>N} a_k x^k b_2(|k|! C_{|k|} \|x\|^2).$$

By (7.7.6) the second sum is a bounded multiple of the sum of its lowest-order terms (by factoring them out), so the remainder is of order higher than N and the derivatives up to order N yield the required coefficients. □

Corollary 7.7.12. *Suppose* f *is a* C^∞ *map with a fixed point* p *such that the linear part is* $\operatorname{diag}\lambda$ *(that is, the diagonal matrix with eigenvalues* $\lambda_1, \dots, \lambda_n$*) satisfying the* *nonresonance condition $\lambda_i \neq \lambda^k$ for all i and all $k \in \mathbb{N}_0^n$. Then there exists a local C^∞ map h such that $h \circ f \circ h^{-1}$ is C^∞ tangent to the linear part of f.*

Proof. Take the formal power series from Proposition 7.7.6 and construct from it a C^∞ map h using Proposition 7.7.11. □

Therefore local smooth linearization for hyperbolic fixed points with no resonances follows from

Theorem 7.7.13. *Let f be a C^∞ map with a hyperbolic fixed point p and g any C^∞ map C^∞ tangent to f. Then there is a neighborhood U of p and a C^∞ diffeomorphism h which is C^∞ tangent to the identity such that $h \circ f = g \circ h$.*

Proof. First, using Theorem 12.5.3 introduce adapted local coordinates with p at the origin such that the stable and unstable manifolds of p are the coordinate spaces $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively. Since the stable and unstable manifolds for g are C^∞ tangent to those for f one can conjugate g by a diffeomorphism C^∞ tangent to the identity such that the resulting stable and unstable manifolds coincide with those for f. Next, by the Extension Theorem 12.4.12 we can construct C^∞ diffeomorphisms of $\mathbb{R}^n$ fixing the origin that coincide with the coordinate representations of f and g, respectively, in a smaller neighborhood of the origin and with the linear part of f and g outside a larger neighborhood, preserve $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, and are C^1-close to the linear part. We still denote these maps by f and g. Then $\alpha := f - g$ has zero jets of all orders at the origin and vanishes outside some neighborhood of 0. Next we show that α can be decomposed as

$$\alpha = \alpha^+ + \alpha^-,$$

where α^+ and all its jets vanish on $\mathbb{R}^k$ and α^- and its jets vanish on $\mathbb{R}^{n-k}$. We construct conjugacies C^∞ tangent to the identity between f and $w := f + \alpha^-$ and between w and g.

To obtain α^- take a C^∞ function ρ on the unit sphere S such that $\rho \equiv 1$ on the intersection of S with the horizontal cone $H_{1/2}$ and $\rho = 0$ on the intersection $S \cap V_{1/2}$ and set

$$\alpha^-(x) = \alpha(x)\rho\Big(\frac{x}{\|x\|}\Big) \text{ for } x \neq 0$$

and $\alpha^-(0) = 0$. Then set $\alpha^+ = \alpha - \alpha^-$. Clearly these are as desired, except that we need to verify that both are C^∞ at the origin. Notice that for $k \in \mathbb{N}_0^n$ and $m \in \mathbb{N}$ $\|D^k\alpha(x)\| = o(\|c\|^m)$ and that the derivatives of ρ are bounded. Using the chain rule one sees that the expression for $D^k\alpha^-$ outside the origin is a polynomial in the derivatives of α, ρ, and $\|x\|^{-1}$, and that each monomial contains α or some of its derivatives. This implies that $\|D^k\alpha^-(x)\| = o(\|x\|^m)$ and hence α^- is a C^∞ function.

Now let $f_t := f + t\alpha^-$ for $t \in [0,1]$. We look for a family of C^∞ diffeomorphisms h_t such that

$$f_0 = h_t^{-1} \circ f_t \circ h_t.$$

This family is generated by the family of vector fields

$$v_t = \frac{d(h_s h_t^{-1})}{ds}|_{s=t}.$$

Differentiating the relation $h_s \circ h_t^{-1} \circ f_t = f_s \circ h_s \circ h_t^{-1}$ with respect to s we obtain

$$v_t \circ f_t - Df_t(v_t) = \alpha^- \text{ or } v_t - (f_t)_* v_t = \alpha^- \circ f_t^{-1},$$

where $f_* v = Df(v \circ F^{-1})$. Inverting the operator $\mathrm{Id} - (f_t)_*$ formally using the geometric series we obtain

$$v_t = \sum_{m=0}^{\infty} (f_t)_*^m \alpha^- \circ f_t^{-1} = \sum_{m=0}^{\infty} Df_t^m \alpha^- \circ f_t^{-m-1}.$$

To show that v_t is a C^∞ vector field in a neighborhood of the origin we need to show that this sum converges in the C^∞ topology, that is, that the sum of kth derivatives converges for every $k \in \mathbb{N}_0^n$. Such observations were first made in step 5 in the proof of the Hadamard–Perron Theorem 12.5.2. Note first that by the chain rule and product rule the kth derivative of an m-fold composition grows at a rate of at most $C^m m^{|k|}$, where C is an upper bound for the derivatives up to order $|k|$ of the individual terms. Thus the kth derivative of f^{-m-1} grows at most exponentially with m. Next consider the kth derivative of $\alpha^- \circ f_t^{-m-1}$. By the chain rule this is a polynomial in derivatives of α^- and f_t^{-n-1} and each term contains a derivative of α^- or α^- itself, evaluated at f^{-m-1}, that is, exponentially close to $\mathbb{R}^k$. Thus these factors are superexponentially small by construction of α^- and hence the kth derivative of $\alpha^- \circ f_t^{-m-1}$ converges to zero superexponentially as $m \to \infty$. Again, the kth derivative of the entire summand is a polynomial whose terms each contain a derivative of $\alpha^- \circ f_t^{-m-1}$. Thus each term is superexponentially small (because it consists of a superexponentially small factor and m bounded factors), so in fact the kth derivatives of the summands go to zero superexponentially.

Thus we obtain the desired family h_t and hence the conjugacy between f and $f + \alpha^-$. The second conjugacy between $f + \alpha^-$ and g is constructed similarly using positive iterates of f and α^+ instead of α^-. □

Thus we have completed the proof of the Sternberg Linearization Theorem:

Theorem 7.7.14. *Suppose f is a C^∞ diffeomorphism with a hyperbolic fixed point p such that the linear part of f at p has no resonances. Then near p, f is C^∞ conjugate to its linear part.*

In fact, the previous arguments give results about C^∞ conjugacy even in the presence of resonances:

Theorem 7.7.15. *Suppose that f is a C^∞ diffeomorphism with a hyperbolic fixed point p such that the linear part of f at p is diagonal and the normal form of f near p is a convergent power series. Then f is locally C^∞ conjugate to its normal form.*

Proof. First, Proposition 7.7.8 gives a formal conjugacy between f and its normal form, so by Proposition 7.7.11 we obtain a conjugacy to a map C^∞ tangent to the normal form, which then by Theorem 7.7.13 gives the result. □

Even for analytic maps the normal form may not be convergent. As we pointed out before for contracting maps the formal normal form is always a polynomial and hence converges. In particular, we can rule out all resonances altogether by a *bunching assumption*

Corollary 7.7.16. *Suppose that f is a C^∞ diffeomorphism with a fixed point p such that $Df_{\restriction p}$ is diagonal. Then f is C^∞ conjugate to a polynomial normal form.*

If in addition $-\log r(Df^{-1})/\log r(Df) < 2$ then f is C^∞ linearizable at p.

This condition is sharp (the map $(x,y) \mapsto (\lambda x, \lambda^2 y + ax^2)$ cannot be C^2 linearized for any $\lambda \in (0,1)$ and $a \neq 0$). In particular equality cannot be allowed in the bunching condition.

The standing assumption of diagonalizability of the linear part was only used in the formal part of our arguments and these can, in fact, be modified to work in the case of nontrivial Jordan normal forms in the linear part. In particular Corollary 7.7.16 holds without the diagonalizability assumption.

The arguments for Theorem 7.7.15 can be used to provide useful information even if the normal form does not converge. To this end we use Proposition 7.7.11 to get a local normal form.

Definition 7.7.17. A *local normal form* for a smooth map f is a smooth map g whose Taylor series at the fixed point coincides with the formal normal form of f.

Theorem 7.7.18. *Suppose that f is a C^∞ diffeomorphism with a hyperbolic fixed point p such that the linear part of f at p is diagonal. Then f is locally C^∞ conjugate to its local normal form.*

Proof. First, Proposition 7.7.8 gives a formal conjugacy between f and its normal form, so by Proposition 7.7.11 applied to the formal conjugacy and the normal form we obtain a conjugacy to a map C^∞ tangent to the local normal form, which then by Theorem 7.7.13 gives the result. □

Let us remark in closing that the arguments of this section can be carried out for maps of finite differentiability as well, but with a loss of several degrees of differentiability. This in itself is not a surprising observation, but finding optimal results where this loss is minimal requires much more careful estimates.

e. Differentiability in the Hartman–Grobman Theorem. We now present the full proof of Theorem 7.7.1. The initial step uses normal-form theory well beyond the level of the preceding introduction, but in a way that can be used as a "black box." This is a reduction to quadratic maps.

Theorem 7.7.19 (Bronstein–Kopanskiĭ)**.** *If $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is a C^∞ diffeomorphism such that $f(0) = 0$ and $L := Df(0)$ is hyperbolic, then on a neighborhood of 0 there is a C^1 conjugacy $H^+ = \mathrm{Id} + \Delta h$ with $\|\Delta h(x)\| \in o(\|x\|)$*[10] *between f and a quadratic polynomial $\mathfrak{f}(x) := Lx + Q(x)$ with only weakly resonant terms (Definition 7.7.20).*

Proof. This follows from [**66**, Theorem 11.9] once one verifies that monomials of order greater than 2 satisfy their condition $\mathscr{A}(1)$. Per [**66**, Remark 7.6], $\mathscr{A}(1)$ in turn follows from their condition $S(1)$ (see (7.7.7)), and this is relatively straightforward to check [**66**, bottom p. 191]. Indeed, denote by $-\lambda_l < \cdots < -\lambda_1 < 0 < \mu_1 < \cdots < \mu_m$ the distinct values of $\log|\nu|$, where ν is an eigenvalue of L. For a multiindex $\tau = (\alpha^l, \dots, \alpha^1, \beta^1, \dots, \beta^m)$ the condition $S(1)$ [**66**, p. 111], is that

$$\text{either } \sum_{i=1}^{r} \alpha^i \lambda_i > \lambda_r \text{ for some } r \le l \text{ or } \sum_{j=1}^{s} \beta^j \mu_j > \mu_s \text{ for some } s \le m. \tag{7.7.7}$$

For terms of order at least 3 the multiindex-exponent satisfies $|\tau| := \sum \alpha^i + \sum \beta^j \ge 3$ and hence $S(1)$ because either $\sum \alpha^i \ge 2$, in which case $r = \max\{i \mid \alpha^i > 0\}$ works, or else $s = \max\{j \mid \beta^j > 0\}$ does.[11] □

Definition 7.7.20. Q contains only *weakly resonant* terms if (in coordinates adapted to the decomposition into generalized eigenspaces or *root spaces*[12]) the ith component of Q contains a term $a \cdot xy$ with $a \neq 0$ then the eigenvalues λ and μ associated with the root spaces corresponding to x and y, respectively, are related to the eigenvalue η associated with the root space containing the ith unit vector by $|\eta| = |\lambda||\mu|$.

[10] See Remark 3.2.18.

[11] As we mentioned, invoking the results from [**66**] is the only reason we make the C^∞ assumption, and, even in this context it can be replaced by a C^k assumption, where k depends on the spectrum of the linear part. As this dependence is complicated, we assume C^∞, but a keen reader may decide to study the precise assumptions of [**66**, Theorem 11.9]—or the proof in [**292**], which assumes far less.

[12] The root space associated to an eigenvalue λ of a linear map L is the maximal subspace on which $(L - \lambda\,\mathrm{Id})^n = 0$ for some $n \in \mathbb{N}$.

PROOF OF THEOREM 7.7.1. That Q contains only weakly resonant terms implies that if we consider the root space decomposition of $L := Df(0)$ and define a linear map D to be a scaling by a on each root space for an eigenvalue of absolute value a, then we have the following result.

Lemma 7.7.21. $Q\circ D = DQ$.

PROOF. This restates that Q contains only weakly resonant terms according to Theorem 7.7.19. By definition, this means that if (in coordinates adapted to the root space decomposition) the ith component of Q contains a term $a\cdot xy$ with $a\neq 0$ then the eigenvalues λ and μ associated with the root spaces corresponding to x and y, respectively, are related to the eigenvalue η associated with the root space containing the ith unit vector by $|\eta| = |\lambda||\mu|$. Note that this is exactly the claim. □

Define J by $L = DJ$ and note (for example via the Jordan normal form of L) that $DJ = JD$, all eigenvalues of J have absolute value 1, and that the entries of J^n are polynomials in n. We assume our coordinate system is adaped to the root space decomposition, so J is block diagonal. We occasionally write $Q(x)$ for $Q(x,x)$.

Remark 7.7.22. The iterates $\mathfrak{f}^n$ of $\mathfrak{f}$ are polynomials, as are the $h_n := L^{-n}\mathfrak{f}^n =: \mathrm{Id} + \mathrm{NL}_n$.[13] The conjugacy will be constructed from the h_n, and accordingly, we wish to estimate the coefficients of the nonlinear terms. To that end denote by $b_{n,m}$ the sum of the absolute values of all mth-order coefficients in the coordinate representation of h_n (this is the *height* of the mth-order term of h_n). The first step is a recursive estimate.

Lemma 7.7.23. $\exists\, N,k\in\mathbb{N}\ \forall\, n,m\in\mathbb{N}\colon\ b_{n+1,m}\le b_{n,m} + Nn^{k-1}\sum_{p=1}^{m-1} b_{n,p}b_{n,m-p}$.

PROOF. $\mathrm{Id}+\mathrm{NL}_{n+1} = h_{n+1} = L^{-n-1}\mathfrak{f}\circ\mathfrak{f}^n = L^{-n-1}(L+Q)L^n(\mathrm{Id}+\mathrm{NL}_n)$, so

$$\begin{aligned}
\mathrm{NL}_{n+1} &= \mathrm{NL}_n + \phantom{J^{-n-1}}L^{-n-1}Q(L^n(\mathrm{Id}+\mathrm{NL}_n))\\
&= \mathrm{NL}_n + J^{-n-1}D^{-n-1}Q(D^nJ^n(\mathrm{Id}+\mathrm{NL}_n)) \qquad (7.7.8)\\
\text{Lemma 7.7.21 and } DJ=JD \Rightarrow\quad &= \mathrm{NL}_n + D^{-1}J^{-n-1}Q(J^n(\mathrm{Id}+\mathrm{NL}_n)).
\end{aligned}$$

To bound $b_{n+1,m}$ note first that terms of a given order m in a coordinate representation of $J^n(\mathrm{Id}+\mathrm{NL}_n)$ come in linear combinations with polynomial coefficients of mth order terms in $\mathrm{Id}+\mathrm{NL}_n$. These polynomial coefficients arise from entries of J^n, and the form of the linear combinations is otherwise independent of n. Thus, the sum $\beta_{n,p}$ of the absolute values of all pth-order coefficients in the coordinate

[13] Here, "NL" stands for "nonlinear terms."

representation of $J^n(\mathrm{Id}+\mathrm{NL}_n)$ is at most $P_1(n)b_{n,p}$ for some polynomial P_1 that is independent of p because it encodes only the action of J^n.

Likewise, in a coordinate representation of $D^{-1}J^{-n-1}Q(J^n(\mathrm{Id}+\mathrm{NL}_n))$ terms of a given order m come in linear combinations with polynomial coefficients of mth order terms in $Q(J^n(\mathrm{Id}+\mathrm{NL}_n))$. These coefficients arise from entries of J^{-n-1}, and the form of the linear combinations is otherwise independent of n. Thus, the sum of mth-order coefficients in $D^{-1}J^{-n-1}Q(J^n(\mathrm{Id}+\mathrm{NL}_n))$ is bounded in terms of that in $Q(J^n(\mathrm{Id}+\mathrm{NL}_n))$ by including a polynomial multiplier $P_2(n)$.

Sorting by the order m of terms, the ith component of $Q(J^n(\mathrm{Id}+\mathrm{NL}_n))$ is

$$\sum_{j,l} a_{ijl}\big[J^n(\mathrm{Id}+\mathrm{NL}_n)\big]_j\big[J^n(\mathrm{Id}+\mathrm{NL}_n)\big]_l = \sum_m\sum_{p=1}^{m-1}\sum_{j,l} a_{ijl}\sum_{|\rho|=p}\alpha_{j,n,\rho}x^\rho\sum_{|\tau|=m-p}\alpha_{l,n,\tau}x^\tau.$$

Thus, the absolute values of all mth order coefficients in $Q(J^n(\mathrm{Id}+\mathrm{NL}_n))$ sum to

$$\sum_i\sum_{p=1}^{m-1}\sum_{j,l}|a_{ijl}|\big[\sum_{|\rho|=p}|\alpha_{j,n,\rho}|\big]\big[\sum_{|\tau|=m-p}|\alpha_{l,n,\tau}|\big] \le c_0\sum_p \beta_{n,p}\beta_{n,m-p},$$

using $\sum_{|\rho|=p,|\tau|=m-p}|\alpha_{j,n,\rho}\alpha_{l,n,\tau}| = \sum_{|\rho|=p}|\alpha_{j,n,\rho}|\sum_{|\tau|=m-p}|\alpha_{l,n,\tau}|$. Take $N,k\in\mathbb{N}$ such that $P_2(n)c_0P_1^2(n)\le Nn^{k-1}$ for all $n\in\mathbb{N}$ to get the claim by (7.7.8). □

This recursion relation allows us to bound the coefficients inductively.

Lemma 7.7.24. *For any $\alpha>0$ there is a $C>0$ such that $b_{n,m}\le e^{\alpha(n+C)m}$.*

Proof. If $\alpha C\ge\sup_n 2\log N+2k\log n-\alpha n$ then $(N^2n^{2k})^m\le e^{\alpha(n+C)m}$, so it suffices to show inductively that if N,k are as in Lemma 7.7.23 then $b_{n,m}\le(Nn^k)^{2m-1}$. This is clear for $n=1$, and if true for $b_{i,m}$ with $i\le n$ and $m\in\mathbb{N}$ then by Lemma 7.7.23

$$\begin{aligned}b_{n+1,m}&\le(Nn^k)^{2m-1}+Nn^{k-1}\sum_{p=1}^{m-1}(Nn^k)^{2p-1}(Nn^k)^{2(m-p)-1}\\&=N^{2m-1}n^{(2m-1)k}+Nn^{k-1}(m-1)N^{2m-2}n^{2(m-2)k}\\&\le N^{2m-1}(n^{(2m-1)k}+(2m-1)kn^{(2m-1)k-1})\le N^{2m-1}(n+1)^{(2m-1)k}\end{aligned}$$

by recognizing the leading terms in the binomial expansion of $(n+1)^{(2m-1)k}$. □

Remark 7.7.25. These arguments establish these same results also for the case where the coefficients are bounded functions; one merely replaces the coefficients by their bounds.

Denote now by $-\lambda_l<\cdots<-\lambda_1<0<\mu_1<\cdots<\mu_m$ the distinct values of $\log|\nu|$, where ν is an eigenvalue of L, and define

$$R_+ := \{(\alpha^1,\dots,\alpha^l,\beta^1,\dots,\beta^m)\ \big|\ \sum_{i=1}^m\beta^i\mu_i-\sum_{i=1}^l\alpha^i\lambda_i=\mu_s \text{ for some } s\le m\}.$$

Lemma 7.7.26. *There exists $p \in (0,1/2)$ such that $\sum_{i=1}^m \beta_i \ge p|\tau|$ for any $\tau \in R_+$.*

Proof. $\sum\limits_{i=1}^m \beta^i \mu_i - \sum\limits_{i=1}^l \alpha^i \lambda_i = \mu_s > 0$ gives $\lambda_1 \sum\limits_{i=1}^l \alpha^i \le \sum\limits_{i=1}^l \alpha^i \lambda_i < \sum\limits_{i=1}^m \beta^i \mu_i \le \mu_m \sum\limits_{i=1}^m \beta^i$ and $|\tau| = \sum\limits_{i=1}^l \alpha^i + \sum\limits_{i=1}^m \beta^i < (1+\dfrac{\mu_m}{\lambda_1})\sum\limits_{i=1}^m \beta^i$. Thus, any $p \le \dfrac{1}{1+\frac{\mu_m}{\lambda_1}}$ is as required. □

These estimates now control how close orbits are to the origin.

Lemma 7.7.27. *There are $H, \gamma_1 > 0$ such that if δ is sufficiently small and $\|\mathfrak{f}^j(x)\| < \delta$ for $j = 0, \dots, n$ then $|x_i| < e^{H^2\log\delta - \gamma_1 n}$ for each expanding coordinate x_i.*

Proof. Denote by $(\cdot)_\pm$ the coordinate projections to the expanding and contracting directions, respectively. We will use that the conjugacy h to $\mathfrak{f}$ is H-Hölder continuous together with its inverse (Remark 5.6.2) and preserves the contracting subspace. Let $(x_+, x_-) = h((x'_+, x'_-))$. If $\|\mathfrak{f}^j(x)\| < \delta$ for $j = 0, \dots, n$ then $\|L^j(x')\| < \delta^H$ for $j = 0, \dots, n$. For the linear part of $\mathfrak{f}$ we have $\|x'_+\| < \delta^H e^{-\mu_1 n}$, so there is a $\gamma_1 > 0$ such that

$$\|x_+\| \le \|h((x'_+, x'_-)) - h((0, x'_-))\| < \delta^{H^2} e^{-\gamma_1 n}. \qquad \square$$

Lemma 7.7.28. *There exists $\gamma > 0$ such that if $\|\mathfrak{f}^j(x)\| < \delta$ for $j = 0, \dots, n$ then*

$$c_{n,m}(x) := \max\bigl\{|x^\tau| \bigm| \tau \in R_+, |\tau| = m\bigr\} \le e^{-\gamma(n+C_\delta)m}\|x\| \textit{ for } m \ge 2,$$

where $C_\delta \to \infty$ as $\delta \to 0$.

Proof. x^τ has over pm expanding components (counted with multiplicity) by Lemma 7.7.26. Applying Lemma 7.7.27 to these and $|x_i| \le \|x\|$ to the remaining $m - (\lfloor pm \rfloor + 1) > m - (\frac{m}{2} + 1) \ge \frac{m}{2} - 1 \ge 0$ gives $|x^\tau| < e^{-pm(\gamma_1 n - H^2\log\delta)}\|x\|$. □

Remark 7.7.29. By Lemma 7.7.24 there exists $C > 0$ such that $b_{n,m} \le e^{\gamma(n+C)m/2}$, and we henceforth take $\delta > 0$ such that $C_\delta > C$.

We now construct the desired conjugacy, starting with the Bronstein–Kopanskiĭ conjugacy H^+ from Theorem 7.7.19 of the given map to $\mathfrak{f}$.

Consider a nonincreasing C^∞ "bump" function $\varphi\colon [0,1] \to [0,1]$ with $\varphi(1/4) = 1$ and $\varphi(3/4) = 0$, and multiply the quadratic terms of $\mathfrak{f}$ by $\varphi(\|x\|/\delta)$. Near 0 this new map $\tilde{\mathfrak{f}}$ is conjugate to $\mathfrak{f}$ by the identity. If $\|x\| \ge \delta$ then $\tilde{\mathfrak{f}}(x) = Lx$.

Take $\delta' < \delta$ sufficiently small, and for $\|x\| < \delta'$ define

$$n_+(x) := 1 + \max\{n \in \mathbb{N} \bigm| \|\tilde{\mathfrak{f}}^i(x)\| < \delta \text{ for } 0 \le i \le n\}.$$

Then $\lim_{\|x\|\to 0} n_+(x) = \infty$ and $n_+(\tilde{\mathfrak{f}}(x)) = n_+(x) - 1$. Since the linear part of f is hyperbolic, n_+ is finite off the contracting direction. For $\|x\| < \delta'$ and with $h_n :=$

$L^{-n}\tilde{\mathfrak{f}}^n$ as in Remark 7.7.22 we set $h_+ = 0$ on the contracting direction and

$$h_+(x) \coloneqq (h_{n_+(H^+(x))}(H^+(x)))_+, \tag{7.7.9}$$

Continuity on the contracting direction follows from the last sentence in the proof of Lemma 7.7.30. Discontinuities of n_+ do not produce discontinuities of h_+ because $h_{n_+(H^+(x))}(H^+(x)) = h_{n_+(H^+(x))+1}(H^+(x))$.

Theorem 7.7.19 applied to f^{-1} yields a conjugacy to a quadratic polynomial. To this our intermediate results also apply, and hence we can define n_- and h_- analogously to n_+ and h_+ and set $h \coloneqq (h_+, h_-)$. The next two results for h_+ combined with the analogous ones for h_- (which we omit) show that h is differentiable at 0 and is a conjugacy between f and its linear part L.

Lemma 7.7.30. *$Dh_+(0) = \mathrm{Id}$, that is, $\|(h_+(x) - x)_+\| \in o(\|x\|)$.*[14]

Proof. Since $\|H^+(x) - x\| = \|\Delta h(x)\| \in o(\|x\|)$ by Theorem 7.7.19, and $o(\|H^+(x)\|) \subset o(\|x\|)$ we show $\|(h_{n_+(x)}(x) - x)_+\| \in o(\|x\|)$. With $c_{n,m}$ from Lemma 7.7.28

$$\begin{aligned}\|(h_{n_+(x)}(x) - x)_+\| &\le \sum_{\substack{\text{expanding}\\ \text{coordinates } i}} \sum_{\substack{\tau\in R_+\\ |\tau|\ge 2}} |\alpha_{i,n_+,\tau}| \cdot |x^\tau| \le \sum_{m\ge 2} \sum_{\substack{|\tau|=m\\ \tau\in R_+}} |\alpha_{i,n_+,\tau}| \cdot c_{n_+,m}(x)\\ &\le \sum_{m\ge 2} c_{n_+,m}(x) b_{n_+,m} \le \sum_{m\ge 2} e^{-\gamma(n_+ + C_\delta)m} e^{\gamma(n_+ + C)m/2} \|x\|\end{aligned}$$

by Remark 7.7.25 and Lemma 7.7.24 with $\alpha = \gamma/2$. Remark 7.7.29 and $q \coloneqq e^{-\gamma(n_+ + C)/2}$ give $\|(h_{n_+(x)}(x) - x)_+\| \le \sum_{m\ge 2} q^m \|x\| = \|x\| q^2/(1-q)$. If $\|x_+\| \to 0$ then $n_+(x) \to \infty$ and $q \to 0$, so $\|(h_{n_+(x)}(x) - x)_+\| \in o(\|x\|)$, hence $\|(h(x) - x)_+\| \in o(\|x\|)$. □

Analogously to our other notations we write $L = L_+ \oplus L_-$. Then

$$\begin{aligned}(h(f(x)))_+ &= (L^{-n_+(H^+(f(x)))}\, \tilde{\mathfrak{f}}^{n_+(H^+(f(x)))} (H^+(f(x))))_+\\ &= (L^{-n_+(\tilde{\mathfrak{f}}(H^+(x)))}\, \tilde{\mathfrak{f}}^{n_+(\tilde{\mathfrak{f}}(H^+(x)))}\, (\tilde{\mathfrak{f}}(H^+(x))))_+\\ &= (L^{-n_+(H^+(x))+1}\, \tilde{\mathfrak{f}}^{n_+(H^+(x))} \quad (H^+(x)))_+\\ &= L_+^{-n_+(H^+(x))+1}\, (\tilde{\mathfrak{f}}^{n_+(H^+(x))} \quad (H^+(x)))_+\\ &= L_+(L^{-n_+(H^+(x))}\tilde{\mathfrak{f}}^{n_+(H^+(x))} \quad (H^+(x)))_+ = L_+ h_+(x) = (Lh(x))_+.\end{aligned}$$

This observation and its counterpart for h_- give $h \circ f = L \circ h$ for small x, that is, h conjugates f and L in a neighborhood of 0. Since Lemma 7.7.30 and its counterpart for h_- imply invertibility near 0, this completes the proof of Theorem 7.7.1. □

[14] See Remark 3.2.18.

CHAPTER 8

Ergodic theory of hyperbolic sets

In this chapter we investigate invariant measures for hyperbolic sets. First, we show by use of the Hopf argument that any volume preserving Anosov flow on a compact manifold is ergodic with respect to the volume form.

We then study how the ergodic notions (entropy, pressure, and equilibrium states) from Chapter 4 play out in this context. We first prove that if Λ is a transitive locally maximal hyperbolic set, then there is a unique measure of maximal entropy, which is therefore ergodic. We prove this using a construction due to Bowen that centers on periodic points as the carriers of information on entropy. Next, we apply the Bowen construction more generally to the study of equilibrium states for Hölder continuous potentials, notably by establishing uniqueness. This naturally leads to the investigation of the Sinai–Ruelle–Bowen measure, which is the equilibrium state for a natural dynamical potential, the geometric potential Definition 8.4.1, and which also provides a "physical measure" in that it determines the asymptotic distribution of *Lebesgue*-a.e. orbit.

We then give a construction of the measure of maximal entropy by Margulis, which is based on a homogeneous scaling property of the conditionals of this measure and which, together with the properties of the Bowen construction, gives a much more precise asymptotic of the growth rate for the number of periodic orbits.

The final section of this chapter is optional, and examines connections between entropy and fractal dimension.

1. The Hopf argument, absolute continuity, mixing

In Chapters 3 and 4 we developed ergodic theory, and we are now prepared to bring it to bear on hyperbolic dynamical systems beyond the homogeneous examples we have been able to treat so far. As indicated in the introduction, ergodicity (of volume) was a major motivation for studying hyperbolic flows, and we present the 2 main ways of establishing it as well as stronger mixing properties. The first of these is the Hopf argument, the original method for establishing ergodicity of volume in hyperbolic dynamical systems that are not of an algebraic nature. Indeed, Hopf grasped the fundamental way in which hyperbolicity provides the very mechanism that produces ergodicity (Theorem 8.1.27), and we will furthermore see that it produces mixing properties as well (Corollary 8.1.11, Theorem 8.1.13, Theorem 8.1.29). The other method is the construction of equilibrium states, whose intimate connection with periodic orbits produces ergodicity (Theorem 8.3.6) and indeed strong mixing (Remark 8.3.19). Both methods retain their importance. The Hopf method has proved useful as research developed beyond uniform hyperbolicity more quickly than the theory of equilibrium states, while the latter retains its central role in producing a multitude of interesting, tractable and mixing measures for hyperbolic dynamical systems.

The essential idea of the Hopf argument is to use Theorem 3.3.10 by showing that Birkhoff averages of continuous functions are constant on stable leaves and on unstable leaves, and then to show that this implies that any invariant function is constant almost everywhere.

For a metric space X with a Borel probability measure μ and a μ-preserving flow Φ, the stable and unstable partitions of Φ are defined by

$$\begin{aligned} W^{ss}(x) &:= \{y \in X \mid d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t\to+\infty]{} 0\}, \\ W^{uu}(x) &:= \{y \in X \mid d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t\to-\infty]{} 0\}. \end{aligned} \tag{8.1.1}$$

Definition 8.1.1. A function $f\colon X \to \mathbb{R}$ is *subordinate* to W^{ss} or W^{ss}*-saturated* if there is a set $G \subset X$ with $\mu(G) = 1$ such that $x, y \in G$ and $y \in W^{ss}(x)$ imply $f(x) = f(y)$. Likewise for W^{uu}.

Remark 8.1.2. In this case $f^s(x) := \begin{cases} 0 & \text{if } W^{ss}(x) \cap G = \varnothing \\ f(y) & \text{if } y \in G \cap W^{ss}(x) \end{cases} \overset{\text{ae}}{=} f$ is (everywhere!) constant on stable sets.

Theorem 8.1.3 (Hopf Argument I)**.** *If* (X, μ) *is a metric Borel probability space,* Φ μ*-preserving, then any* Φ*-invariant* $f \in L^1(\mu)$ *is* W^{ss}*-saturated and* W^{uu}*-saturated.*

Proof. Suppose f is uniformly continuous. Then f_Φ is W^{ss}-saturated: If $f_\Phi(x)$ exists and $y \in W^{ss}(x)$, then

$$\frac{1}{T}\int_0^T f(\varphi^t(x))\,dt - \frac{1}{T}\int_0^T f(\varphi^t(y))\,dt = \frac{1}{T}\int_0^T f(\varphi^t(x)) - f(\varphi^t(y))\,dt \xrightarrow[T\to\infty]{} 0$$

since $d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t\to+\infty]{} 0$ and f is uniformly continuous. Thus $f_\Phi(y) = f_\Phi(x)$, as claimed. (In particular, f_Φ is *defined* and constant on $W^{ss}(x)$.)

If $f \in L^1(\mu)$ there are uniformly continuous g^n such that $\|f - g^n\|_p < 1/n$ and hence $\|f_\Phi - g^n_\Phi\|_p < \epsilon$;[1] that is, $g^n_\Phi \xrightarrow[t\to+\infty]{L^1} f_\Phi$, so a subsequence of the g^n_Φ converges

[1] Both f_Φ and g^n_Φ exist a.e. by the Birkhoff Ergodic Theorem 3.2.16.

a.e. to f_Φ. Since the g^n_Φ are W^{ss}-saturated, so is f_Φ. W^{us}-saturation follows by reversing time. □

Corollary 8.1.4. *If (X,μ) is a metric Borel probability space, Φ μ-preserving such that any Φ-invariant W^{ss}- and W^{uu}-saturated $f \in L^1(\mu)$ is constant, then Φ is ergodic.*

We first put this to use with a "traditional" application—to the suspension of a hyperbolic toral automorphism.

Proposition 8.1.5. *If $A \in GL(m,\mathbb{Z})$ is hyperbolic, then the suspension of the induced automorphism F_A of $\mathbb{T}^m$ is ergodic with respect to Lebesgue measure.*

Proof. For $q \in \mathbb{T}^m$ the *stable subspace* $W^{ss}(q \times \{t\})$ in (8.1.1) is $W^{ss}(q \times \{t\}) = \pi(E^- + q) \times \{t\}$, where E^- is the contracting subspace of A and $\pi\colon \mathbb{R}^m \to \mathbb{T}^m$ is the projection. Likewise, $W^u(q \times \{t\}) = \pi(E^+ + q) \times \{t\}$.

To apply Corollary 8.1.4 consider a Φ-invariant $f \in L^1$ for which there is a set $G \subset \mathbb{T}^n$ of measure 1 with $x, y \in G \times \{t\}$, $y \in W^{ss}(x) \Rightarrow f(x) = f(y)$ and $x, y \in G \times \{t\}$, $y \in W^u(x) \Rightarrow f(x) = f(y)$. If we can conclude that $f \overset{\text{ae}}{=} \text{const.}$, then Corollary 8.1.4 implies ergodicity.

Let $D^\pm \subset E^\pm$ be small disks and $q \in \mathbb{T}^m$. Then q has a neighborhood that is up to rotation and translation of the form $D^- \times D^+$, and

$$C \coloneqq G \cap (D^- \times D^+)$$

has full Lebesgue measure in $D^- \times D^+$, that is, if $\mu^\pm$ denotes the normalized Lebesgue measure on $D^\pm$ and $\mu = \mu^- \times \mu^+$, then $\int_{D^- \times D^+} \chi_C \, d\mu = 1$. By the Fubini Theorem

$$1 = \int_{D^- \times D^+} \chi_C \, d\mu = \int_{D^-} \int_{D^+} \chi_C \, d\mu^+ \, d\mu^-, \text{ so } \int_{D^+} \chi_C(u,\cdot) \, d\mu^+ \overset{\mu^-\text{-a.e.}}{=\!=\!=} 1.$$

Fix such a $u_0 \in D^-$, and note that by construction $C^- \coloneqq D^- \times \big(C \cap (\{u_0\} \times D^+)\big)$ has full Lebesgue measure. If $(u,v), (u',v') \in C^- \cap C$, a set of full measure, then

$$f(u,v,t) = f(u_0,v,t) = f(u_0,v',t) = f(u',v',t).$$

This applies to any such neighborhood of an arbitrary $q \in \mathbb{T}^n$, so $f \overset{\text{ae}}{=} \text{const.}$ □

Remark 8.1.6. This application of the Hopf argument yields a weaker conclusion than Proposition 3.3.7, and with more effort. However, the simplicity of the proof of Proposition 3.3.7 relied entirely on the linearity of this system, whereas the Hopf argument does not and supports other applications. The argument above uses the Fubini Theorem, however, and to that end relies on smooth local charts in which Lebesgue measure is a product measure. In Section 8.1 we develop ways to sidestep this use of linearity, leading to Theorem 8.1.27.

We now amplify the Hopf argument in the direction of mixing. The underlying technical ingredient is a Hilbert-space lemma that "upgrades" weak convergence to pointwise convergence:

Theorem 8.1.7 (Banach–Saks)**.** *If $x_n \xrightarrow{\text{weakly}} x$ in a Hilbert space, then there is a subsequence $y_k := x_{n_k}$ such that $z_n := \frac{1}{n}\sum_{k=1}^{n} y_k \xrightarrow{\|\cdot\|} x$. In L^2, z_{n^2} then converges a.e. (Borel–Cantelli).*

Proof. Passing to $x_n - x$ assume $x_n \xrightarrow{\text{weakly}} 0$ and hence recursively choose $y_1 = x_1$ and y_k such that $|\langle y_k, y_i\rangle| < 1/k$ for $1 \le i < k$. Then $\|y_k\| \le C$ for some $C \in \mathbb{R}$ (weakly convergent⇒norm-bounded), and

$$\Big\|\frac{1}{n}\sum_{k=1}^{n} y_k\Big\|^2 = \frac{1}{n^2}\sum_{k=1}^{n}\sum_{i=1}^{n}\langle y_k, y_i\rangle = \frac{1}{n^2}\Big[\underbrace{\sum_{k=1}^{n}\|y_k\|^2}_{\le nC^2} + 2\underbrace{\sum_{k=1}^{n}\overbrace{\sum_{1\le i<k}\underbrace{\langle y_k, y_i\rangle}_{<1/k}}^{<1}}_{<n}\Big] \in O(\frac{1}{n}). \qquad \square$$

The Banach–Saks Theorem gives the following amplified Hopf argument.

Theorem 8.1.8. *If X is a metric space, Φ a continuous flow on X, μ a Φ-invariant Borel probability measure, $f_i \in L^2(\mu)$, then any weak accumulation point $F_n = \prod_{i=1}^{N} f_i \circ \varphi^{t_{i,n}} \xrightarrow[n\to\infty]{\text{weakly}} F$ of $\prod_{i=1}^{N} f_i \circ \varphi^{t_i}$ is W^{ss}-subordinate.*

Proposition 3.4.29 gives a strong immediate consequence of Theorem 8.1.8 to the effect that a far stronger assumption than joint ergodicity (Remark 8.1.12) gives multiple mixing:

Corollary 8.1.9. *Φ is multiply mixing if every W^{ss}-subordinate $f \in L^2$ is constant a.e.*

Proof of Theorem 8.1.8 (Coudène)**.** By the Banach–Saks Theorem 8.1.7 an accumulating subsequence $F_n \xrightarrow[n\to\infty]{L^2\text{-weakly}} F$ admits sequences m_l, n_{i_k} with

$$\mathscr{F}_l := \frac{1}{m_l}\sum_{k=0}^{m_l-1} F_{n_{i_k}} \xrightarrow{\text{a.e.}} F.$$

We passed to pointwise convergence because this is W^{ss}-subordinate for bounded uniformly continuous f_i: If $p^l_{ij} := f_i(\varphi^{(t_i)_l}(x_j))$ for $j = 1,2$ with $x_2 \in W^s(x_1)$, then

$$\prod_{i=1}^{N} p^l_{i2} - \prod_{i=1}^{N} p^l_{i1} = \sum_{j=1}^{N} \underbrace{\prod_{i<j} p^l_{i2}}_{\text{bounded}} \underbrace{(p^l_{j2} - p^l_{j1})}_{\xrightarrow[l\to\infty]{} 0} \underbrace{\prod_{i>j} p^l_{i1}}_{\text{bounded}} \xrightarrow[l\to\infty]{} 0.$$

Now L^2-approximate bounded L^2 functions f_i^0 by bounded uniformly continuous functions f_i^k within $1/k$ and this time let $p_{ij}^l \coloneqq f_i^j \circ \varphi^{(t_i)_l}$ to find that weak limits (of subsequences if necessary) satisfy

$$\|F - F^k\| \le \varliminf_{l\to\infty} \Big\| \prod_{i=1}^N p_{ik}^l - \prod_{i=1}^N p_{i0}^l \Big\| \le \sum_{j=1}^N \underbrace{\prod_{i<j} \|p_{ik}^l\|_\infty}_{\text{bounded}} \underbrace{\|p_{jk}^l - p_{j0}^l\|_2}_{\xrightarrow[k\to\infty]{} 0} \underbrace{\prod_{i>j} \|p_{i0}^l\|_\infty}_{\text{bounded}} \xrightarrow[k\to\infty]{} 0$$

so, passing to a subsequence, $F^k \xrightarrow{\text{a.e.}} F$, which is hence W^{ss}-subordinate. □

Theorem 8.1.10 (Babillot–Hopf Argument)**.** *If (X,μ) is a metric Borel probability space, Φ μ-preserving, $f \in L^2(\mu)$, then weak-accumulation points of $\{f \circ \varphi^t \mid t \ge 0\}$ are W^{ss}-saturated and W^{uu}-saturated.*

Proof (Babillot–Coudène)**.** If $f \perp I \subset L^2(\mu)$, the (closed) subspace of functions subordinate to W^u, and $f \circ \varphi^{t_i} \xrightarrow[i\to\infty]{\text{weakly}} g$, then Theorem 8.1.8 applied to φ^{-t} gives a subsequence $t_{i_k} \to \infty$ with $g \circ \varphi^{-t_{i_k}} \xrightarrow[k\to\infty]{\text{weakly}} g' \in I$, so $\langle g, g\rangle = \lim_{k\to\infty}\langle f \circ \varphi^{t_{i_k}}, g\rangle = \lim_{k\to\infty}\langle f, g \circ \varphi^{-t_{i_k}}\rangle = \langle f, g'\rangle = 0$, that is, $g = 0$, so $f \circ \varphi^t \xrightarrow[t\to\infty]{\text{weakly}} 0$.

For an arbitrary $f = f_I + f^\perp \in I \oplus I^\perp = L^2$ we then have $f^\perp \circ \varphi^t \xrightarrow[t\to\infty]{\text{weakly}} 0$, so the accumulation points of $f \circ \varphi^t$ are accumulation points of $f_I \circ \varphi^t \in I$, hence W^{ss}-saturated and W^{uu}-saturated. □

Since mixing is characterized in terms of weak convergence (Proposition 3.4.28), we obtain:

Corollary 8.1.11 (Hopf Argument II)**.** *If (X,μ) is a metric Borel probability space, Φ μ-preserving such that any W^{ss}- and W^{uu}-saturated $f \in L^2(\mu)$ is constant, then Φ is mixing.*

Remark 8.1.12. The condition that any W^{ss}- and W^{uu}-saturated $f \in L^2(\mu)$ is constant is referred to as *joint ergodicity* of W^{ss} and W^{uu}, and Corollary 8.1.11 says that this implies mixing. We say that W^{ss} is ergodic if any W^{ss}-saturated $f \in L^2(\mu)$ is constant, and Corollary 8.1.9 shows that this implies multiple mixing. We now apply this to geodesic flows, with Corollary 3.3.20 as the main ingredient—by Theorem 3.4.44, W^{ss} is ergodic, and Corollary 8.1.9 implies:

Theorem 8.1.13. *The geodesic flow on a compact surface of constant negative curvature is multiply mixing.*

Remark 8.1.14. This strengthens Corollary 3.3.17 and Theorems 3.4.43, and 3.4.32. Stronger mixing follows from Theorem 8.4.17 in greater generality than here. This is analogous to Theorems 6.2.12 and 9.1.1 in the context of topological dynamics. Since the Liouville measure is positive on open sets, this in particular strengthens

Theorem 2.4.4: this geodesic flow is topologically mixing (see also Exercise 2.7 and Corollary 9.1.4).

The use of the Fubini Theorem in the proof of Proposition 8.1.5 was pivotal, and this requires looking for ways to check the assumed implication in Corollary 8.1.4, that is, to more generally see that "constant on stable and unstable ⇒ constant" from hyperbolicity without having the Fubini Theorem available. The reason is that the Fubini Theorem often does fail in that the foliations are not sufficiently smooth to apply the theorem. We will see this in Example 8.1.24. In particular, as we briefly mentioned in the introduction, Hopf ran into problems proving ergodicity for certain geodesic flows since he could not establish the foliations were C^1. It took over 30 years for Anosov to find a mechanism to establish ergodicity. In doing so he observed that even if the foliations are not C^1 there is a property they satisfy that is sufficient to establish ergodicity. The property that replaces the use of the Fubini Theorem is absolute continuity of the invariant foliations with respect to Riemannian volume: that a set of full measure intersects almost every stable leaf in a set of full leaf-measure.

Definition 8.1.15. A C^r d-dimensional *foliation* of an n-dimensional manifold M consists of a covering by open sets U_i and maps $h_i : U_i \to \mathbb{R}^n$ such that the transition functions $h_{ij} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $h_{ij} = h_j h_i^{-1}$ are of the form

$$h_{ij}(x,y) = (h^1_{ij}(x), h^2_{ij}(x,y))$$

where x is the first $n-d$ coordinates and y denotes the last d coordinates. So

$$h^1_{ij} : \mathbb{R}^{n-d} \to \mathbb{R}^{n-d}$$
$$h^2_{ij} : \mathbb{R}^n \to \mathbb{R}^d$$

where the h^1_{ij} are C^r and the h^2_{ij} are continuous. In the chart U_i the strips $x =$ const. align with the corresponding strips in U_j. The strips connect together from chart to chart to form a leaf of the foliation which is a maximal connected immersed submanifold.

Definition 8.1.16. A *foliation box* of a foliation $\mathscr{F}$ with d-dimensional leaves is the domain of a foliation chart, that is, it is the image O of a homeomorphism $\mathbb{R}^{n-d} \times \mathbb{R}^d \to M$ that sends each $\{x\} \times \mathbb{R}^d$ into a leaf of $\mathscr{F}$; the image that contains $z \in O$ is denoted by $\mathscr{F}_{\text{loc}}(z)$. For $\mathscr{F} = W^s$ or $\mathscr{F} = W^u$ we will always choose foliation boxes small enough that these images are *local leaves* of $\mathscr{F}$.

Remark 8.1.17. This definition of a foliation differs from the one in geometry in that the h^2_{ij} are only taken to be continuous and not smooth. Additionally, the holonomy maps are defined differently from what is usually done in geometry, where parallel translation along loops is involved [**139**]. Therefore, one must take care when trying to apply results about foliations and holonomy maps in geometry directly to flows.

The first notion of absolute continuity is of a transverse nature.

Definition 8.1.18. The holonomies coming from a foliation $\mathscr{F}$ are locally defined by maps h_{τ_1,τ_2} between two nearby smooth transversals τ_1 and τ_2 that send $x \in \tau_1$ to the unique intersection point of τ_2 with the local leaf of $\mathscr{F}$ through x.

A foliation $\mathscr{F}$ with smooth leaves is *transversely absolutely continuous with bounded Jacobians* if for any $\alpha \in (0, \pi/2]$ there are $C, R > 0$ such that for any foliation box O of diameter at most R and any smooth transversals τ_1 and τ_2 to $\mathscr{F}$ that lie in O and make an angle at least α with $\mathscr{F}$ we have

$$m_{\tau_1}(A)/C \le m_{\tau_2}(h_{\tau_1,\tau_2}(A)) \le C m_{\tau_1}(A)$$

for every m_{τ_1}-measurable set $A \subset \tau_1$. Here m_N denotes the volume on a smooth submanifold $N \subset M$ induced by the Riemannian metric. Equivalently, there is a bounded positive measurable function J_h such that for every m_{τ_1}-measurable $A \subset \tau_1$ we have

$$m_{\tau_2}(h_{\tau_1,\tau_2}(A)) = \int_A J_h(x)\, dm_{\tau_1}(x).$$

We now define absolute continuity (without "transversely").

Definition 8.1.19. We say that a foliation $\mathscr{F}$ with smooth leaves is *absolutely continuous with bounded Jacobians* if for any $\alpha \in (0, \pi/2]$ there are $C, R > 0$ such that for any foliation box O of diameter at most R and any smooth transversal τ to $\mathscr{F}$ that lies in O and makes an angle at least α with $\mathscr{F}$ we have

$$\frac{m(A)}{C} \le \int_\tau m_{\mathscr{F}_{\text{loc}}(z)}(A \cap \mathscr{F}_{\text{loc}}(z))\, dm_\tau(z) \le C m(A)$$

for every measurable $A \subset O$. More precisely, there is a measurable family of functions (called *conditional densities*) $\delta_z \colon \mathscr{F}_{\text{loc}}(z) \to [1/C, C]$ such that

$$m(A) = \int_\tau \int_{\mathscr{F}_{\text{loc}}(z)} \chi_A(z, y) \delta_z(y)\, dm_{\mathscr{F}_{\text{loc}}(z)}(y)\, dm_\tau(z)$$

for every measurable $A \subset O$.

Remark 8.1.20. Theorem 8.1.26 below establishes this for hyperbolic flows, and by Corollary 8.1.4 this implies that Anosov flows are ergodic (Theorem 8.1.27). (This is further strengthened in Theorem 8.1.13, Theorem 8.1.29, Remark 8.1.30, and Theorem 8.4.17.)

There are important measures that are "product-like" in a way which directly implies this property (see also Remark 8.1.22). For instance, the Margulis measure (Definition 8.6.19) has this property by (8.6.6).

Proposition 8.1.21. *If $\mathscr{F}$ is transversely absolutely continuous with bounded Jacobians then $\mathscr{F}$ is absolutely continuous with bounded Jacobians.*

Proof. Take τ and O as in Definition 8.1.19 and include τ in a C^1-foliation $\mathscr{G}$ of O, that is, fixing $z \in \tau$ we have $\mathscr{G}_{\mathrm{loc}}(z) = \tau$ and $O = \bigcup_{y \in \mathscr{F}_{\mathrm{loc}}(z)} \mathscr{G}_{\mathrm{loc}}(y)$. Since $\mathscr{G}$ is a C^1 foliation it is absolutely continuous and transversely absolutely continuous (both with bounded Jacobians), and indeed there are continuous conditional densities $\delta_y(\cdot)$ for which

$$m(A) = \int_{\mathscr{F}_{\mathrm{loc}}(z)} \int_{\mathscr{G}_{\mathrm{loc}}(y)} \chi_A(y,x)\delta_y(x)\, dm_{\mathscr{G}_{\mathrm{loc}}(y)}(x)\, dm_{\mathscr{F}_{\mathrm{loc}}(z)}(y). \tag{8.1.2}$$

Since we are using y as the parameter for leaves of $\mathscr{G}$, we now denote by h_y the holonomy from τ to $\mathscr{G}_{\mathrm{loc}}(y)$ and by J_y its Jacobian, that is to say, the inner integral above can be written as

$$\int_{\mathscr{G}_{\mathrm{loc}}(y)} \chi_A(y,x)\delta_y(x)\, dm_{\mathscr{G}_{\mathrm{loc}}(y)}(x) = \int_\tau \chi_A(h_y(s))J_y(s)\delta_y(h_y(s))\, dm_\tau(s).$$

This is where we use the transverse absolute continuity hypothesis.

Substituting the right-hand side into (8.1.2) gives

$$m(A) = \int_{\mathscr{F}_{\mathrm{loc}}(z)} \int_\tau \chi_A(h_y(s))J_y(s)\delta_y(h_y(s))\, dm_\tau(s)\, dm_{\mathscr{F}_{\mathrm{loc}}(z)}(y).$$

This is an integral with respect to a product measure, so we can exchange the order of integration to get

$$m(A) = \int_\tau \int_{\mathscr{F}_{\mathrm{loc}}(z)} \chi_A(h_y(s))J_y(s)\delta_y(h_y(s))\, dm_{\mathscr{F}_{\mathrm{loc}}(z)}(y)\, dm_\tau(s). \tag{8.1.3}$$

Using transverse absolute continuity of $\mathscr{G}$ we now rewrite the new inner integral. To that end denote by $\bar{h}_s$ the holonomy along leaves of $\mathscr{G}$ from $\mathscr{F}_{\mathrm{loc}}(z)$ to $\mathscr{F}_{\mathrm{loc}}(s)$ for $s \in \tau$, and let $\bar{J}_s$ be its Jacobian. Note that $h_y(s) = \bar{h}_s(y)$, so if we write $r := h_y(s)$ then $y = \bar{h}_s^{-1}(r)$ and we can make the corresponding change of variables:

$$\int_{\mathscr{F}_{\mathrm{loc}}(z)} \chi_A(h_y(s))J_y(s)\delta_y(h_y(s))\, dm_{\mathscr{F}_{\mathrm{loc}}(z)}(y) = \int_{\mathscr{F}_{\mathrm{loc}}(s)} \chi_A(r)J_y(s)\delta_y(r)\bar{J}_s^{-1}(r)\, dm_{\mathscr{F}_{\mathrm{loc}}(s)}(r).$$

Inserting this into (8.1.3) gives

$$m(A) = \int_\tau \int_{\mathscr{F}_{\mathrm{loc}}(s)} \chi_A(r)J_y(s)\delta_y(r)\bar{J}_s^{-1}(r)\, dm_{\mathscr{F}_{\mathrm{loc}}(s)}(r)\, dm_\tau(s).$$

This proves absolute continuity with density functions $J_y(s)\delta_y(\cdot)\bar{J}_s^{-1}(\cdot)$. □

Remark 8.1.22. By fixing $r = r_0$ in this last integral, we see that locally, m (and likewise, the SRB-measure in Definition 8.4.1 below) is equivalent, in the sense of mutual absolute continuity (with bounded Jacobians), to a product measure.

Example 8.1.23. A simple example may serve to illuminate some distinctions here. Consider the square $[0,1]^2$ with the foliation $\mathscr{F}$ defined as follows: Each leaf is a line segment with endpoints $(x,0)$ and $(h(x),1)$, where $h\colon [0,1] \to [0,1]$ is the homeomorphism that sends the ternary Cantor set C to a "fat" Cantor set with positive Lebesgue measure and endpoints 0 and 1, and which is linear on the gaps. This foliation is not transversely absolutely continuous: Take $\tau_1 = [0,1] \times \{0\}$, $\tau_2 = [0,1] \times \{1\}$ and $A = C$ in Definition 8.1.18. Note, though, that it only "barely" fails to be so because if τ_1 and τ_2 are both disjoint from $[0,1] \times \{0\}$, then the conclusion of Definition 8.1.18 holds.

Taking a closer look at Definition 8.1.19 we note that when A is taken to be the $\mathscr{F}$-saturation of C, it has positive Lebesgue measure while the integral in the first version of this definition is 0 if we take the "pathological" transversal $\tau = [0,1] \times \{0\}$. However, the second (and "official") version applies here; indeed $dm_{\mathscr{F}_{\text{loc}}}$ could be represented by the natural measure on a generic smooth transversal. Accordingly, this is an example of an absolutely continuous foliation that is not transversely absolutely continuous—albeit only just.

Absolute continuity is not a consequence of the definition of a foliation:

Example 8.1.24 (Katok's foliation foiling Fubini [**214**])**.** If $p \in (0,1)$, then

$$f_p\colon [0,1] \to [0,1], \quad x \mapsto \begin{cases} x/p & \text{for } x \in [0,p), \\ (x-p)/(1-p) & \text{for } x \in [p,1), \end{cases}$$

preserves Lebesgue measure on $[0,1]$ and is ergodic, so $(\chi_{[0,p)})_{f_p}(x) = p$ for a.e. x (time average) by Corollary 3.3.11, that is, almost every x lands in $[0,p)$ about p of the time. Therefore the set

$$A \coloneqq \big\{(p,x) \;\big|\; (\chi_{[0,p)})_{f_p}(x) = p\big\} \subset S \coloneqq \big\{(p,x) \;\big|\; p \in (0,1),\ x \in [0,1]\big\}$$

has Lebesgue measure 1. On the other hand, there is a homeomorphism $h\colon S \to S$ such that $h_p \coloneqq h(p,\cdot)$ is increasing, and $f_p \circ h_p = h_p \circ E_2$, where $E_2 \coloneqq f_{1/2}\colon x \mapsto 2x$ (mod 1) is the doubling map. (In particular, $h_p(1/2) = p$.) Then

$$f_p^n(h_p(x)) < p \Leftrightarrow h_p(E_2^n(x)) < p \Leftrightarrow E_2^n(x) < 1/2,$$

so $(\chi_{[0,p)})_{f_p}(h_p(x)) = (\chi_{[0,1/2)})_{E_2}(x)$ is independent of p. Thus, $A_z \coloneqq \text{graph}(h(\cdot,z))$ intersects A in at most one point, while the A_z are leaves of a continuous foliation of S by continuous curves.

Remark 8.1.25. Katok's original example was created in 1992, possibly on the spot, and had a 2-dimensional Anosov base. It was distilled down to a 1-dimensional expanding base by Burns and Flaminio. From Burns the idea traveled via Wilkinson, Shub and Pugh to Milnor, who published it with this explicit choice of expanding base and noted that Yorke independently had an example as well. In Section 12.7

we produce the original Katok example as written down and distilled by Keith Burns.

Theorem 12.7.5 and Theorem 12.7.6 apply to the time-1 map of a flow restricted to a hyperbolic set to give:

Theorem 8.1.26 (Absolute continuity)**.** *The strong and weak stable and unstable foliations of a $C^{1+\alpha}$ hyperbolic flow are (transversely) absolutely continuous with bounded Jacobians. (And the orbit foliation is so because it is differentiable.)*

Corollary 8.1.4 then implies:

Theorem 8.1.27. *Volume-preserving Anosov flows are ergodic.*

A pertinent dichotomy will later prove useful:

Theorem 8.1.28. *For an continuous n-form θ invariant under a C^2 Anosov flow Φ on a compact n-manifold M there are two possibilities:*

- $\theta \equiv 0$ *or*
- *The measure determined by θ on M is equivalent to the Riemannian volume and ergodic for Φ.*

Proof [**240**, Corollary 4.6]. Unless $\theta \equiv 0$, there is an $x \in M$ at which $\theta_x \neq 0$, and then $x \in NW(\Phi)$ by the Poincaré Recurrence Theorem 3.2.1, so $NW(\Phi)$ has nonempty interior, and hence $NW(\Phi) = M$ (Theorem 5.3.51). Thus Φ is topologically transitive (Theorem 5.3.50), and $\theta_x \neq 0$ on an open dense set, so it defines a volume equivalent to the Riemannian volume; being invariant and finite, it is ergodic (Theorem 8.1.27). □

To get stronger mixing properties from, say, Corollary 8.1.11 one needs to exclude suspensions in a suitable way. Theorem 8.1.13 is an instance of this, and without proof, we give here a general result based on this insight.

Theorem 8.1.29. *Weakly mixing volume-preserving Anosov flows are mixing.*

Remark 8.1.30. Multiple mixing actually holds in this case: the *Anosov alternative* [**10**], a measurable counterpart to the Plante alternative (Theorem 9.1.1) for volume-preserving Anosov flows, is that the strong (un)stable foliation is either of suspension type, that is, the Anosov flow is not weakly mixing, or ergodic (Remark 8.1.12), in which case the flow is multiply mixing (Corollary 8.1.9). Note that contact Anosov flows are always of this latter kind.

It is reasonable to hope that for hyperbolic billiards (Theorem 5.2.18) one obtains ergodic properties in the same fashion as for geodesic flows, and this is not far off. However, that the regular set is neither compact nor a manifold makes for extra technical work. The preceding ergodicity arguments and those to come rely on the expansion of unstable manifolds under the flow, and while in the case of billiards, that may be differentiably true, the manifolds themselves are truncated by encounters with singularities. Balancing this against the local expansion is delicate indeed, and it takes a book to build a suitable machinery [**184**]. The core insight is easy to state: that the measure of a neighborhood of the singularity set must be bounded by a power of its thickness. Theorem 8.4.17 below does work in that context to give strong mixing properties of finite-horizon dispersing billiards [**19**, **221**, **275**]. The questions for the billiard flow versus the associated billiard map are, of course, closely related. Some results for billiard maps presented in the spirit of this section can be found in [**147**].

Remark 8.1.31. It is far harder to see that a gas of hard particles (Subsection 0.2c) is ergodic. While this is a billiard system (Theorem 5.2.31), considering nonspherical particles is a non-starter because the resulting billiards are not hyperbolic [**93**]. For spherical particles this ergodicity question is known as the Boltzmann–Sinai hypothesis, and its solution was announced by Sinai in 1963 but only eventually proved by generations of others 50 years later [**273**]. The daunting challenges involved are far beyond the scope of this book. The issue that is most easy to spot is that while 2 particles collide, all others can move freely to some extent, so the scatterers in the configuration space are cylindrical and hence only semi-dispersing.

2. Stable ergodicity*

Theorem 8.1.27 invites a digression to a rather modern subject in dynamical systems that combines topological and ergodic concerns. Questions of stability and persistence are usually raised in topological and smooth dynamics, such as the persistence of the Anosov property (Corollary 5.1.11). Ergodic theory tends to study a system at hand, or relations between given systems. Stable ergodicity concerns the robustness of an ergodic property under perturbations of a smooth system. It is nicely illustrated by

Proposition 8.2.1. *Volume-preserving Anosov flows are stably ergodic: volume-preserving C^1 perturbations of an Anosov flow are ergodic (with respect to volume).*

Proof. Volume-preserving Anosov flows are ergodic (Theorem 8.1.27), and so are volume-preserving C^1-perturbations because they are themselves volume-preserving Anosov flows. □

This simple argument hides a few subtle aspects of this issue. First of all, structural stability is more powerful than persistence of hyperbolicity, but it does not produce stable ergodicity by itself because on one hand it gives an orbit-equivalence rather than a conjugacy and on the other hand the orbit-equivalence is a homeomorphism and may not be absolutely continuous (Remark 10.2.8). Both of these mean that the conjugacy cannot be expected to send volume to volume, so it produces ergodicity of an unspecified other invariant measure for the perturbation. The underlying issue is the abundance of invariant measures and our focus on a specific one.

The question about stable ergodicity we speak to here is whether hyperbolicity can produce stable ergodicity when it is less complete than in the Anosov situation. Weakening hyperbolicity from Anosov to our definition of a hyperbolic flow does not broaden the perspective: volume-preservation means that the chain-recurrent set is the whole manifold, so our definition of hyperbolicity of a flow (Definition 5.3.48) implies that the flow is an Anosov flow.

Instead, dynamicists turned to weakening hyperbolicity to partial hyperbolicity (Definition 5.5.2), where the center direction in the definition of a hyperbolic set (Definition 5.1.1) is allowed to have dimension higher than 1, that is, it is a subbundle of vectors that experience less expansion and contraction than stable and unstable vectors. Partial hyperbolicity and stable ergodicity are active research areas, but beyond our intended scope. This section presents highlights of this research area, and we recomnend, for instance [**77**, **236**, **290**] for further reading.

We formalized partial hyperbolicity for discrete time in Definition 5.5.2.

Remark 8.2.2. Each of the subbundles E^u, E^s, E^c, E^{cu}, E^{cs} in Definition 5.5.2 is Hölder continuous (Theorem 7.4.1). The center subbundle turns out to be quite more fragile than the others, however. The Hirsch–Pugh–Shub theory of normal hyperbolicity [**159**] helps control both the (moderate) regularity of its leaves and provide some robustness under perturbation—once a center foliation is known to exist. Existence is a rather delicate matter and is known only under several rather stringent assumptions, while nonexistence is an open property. The Katok example in Section 12.7 shows that if there is a center foliation at all it may fail to be absolutely continuous. It turns out that this is not at all exceptional, and it is conjectured that the center foliation of a "typical" partially hyperbolic diffeomorphism with negative central exponents is not absolutely continuous[2] and that moreover, arbitrarily close to a partially hyperbolic dynamical system whose central Lyapunov exponents are zero, there are partially hyperbolic dynamical systems with negative central exponents. For instance, in a volume-preserving

[2] Mañé proved that if the center foliation is one-dimensional and has compact leaves then this foliation is not absolutely continuous provided the Lyapunov exponent in the center direction is nonzero on a set of positive measure. (The only record of this appears to be a fax with statement and proof he sent to Michael Shub on September 13, 1993, of which we received a scan from Amie Wilkinson; it is notable that 1993 is in the very early era of this subject.)

perturbation of the time-one map of the geodesic flow of a compact surface with negative curvature the Liouville measure either has atomic disintegration along the center foliation, or the perturbation is itself the time-one map of a smooth volume-preserving flow (and hence the disintegration is Lebesgue) [**18**, **160**, **263**].

Example 8.2.3. The time-1 map of an Anosov flow is partially hyperbolic. Note that this is obviously not ergodic when the flow is a suspension.

This weakening thus poses challenges for the Hopf argument because it makes it much easier for the now lower-dimensional stable and unstable leaves to fail to intersect. This problem does not go away by restricting to flows, because the product of a volume-preserving Anosov flow with the identity is a nonergodic volume-preserving partially hyperbolic flow. The presence of such intersections (or some irreducibility condition that implies it) thus becomes a necessary additional assumption. But while partial hyperbolicity persists under C^1 perturbation (by a cone criterion or by showing that the Mather spectrum—the spectrum of $\mathscr{F}$ in Theorem 5.5.5—various continuously), this *accessibility* will not obviously do so.

Let us explore the complementary situation first.

Definition 8.2.4. The foliations W^u and W^s (or the subbundles E^u and E^s) are said to be *jointly integrable* if every point of M is contained in a local product neighborhood for which

$$\mathscr{H}_{y,z}(W^{ss}(u) \cap W^{cs}_{\delta'}(y)) \subset W^{ss}(\mathscr{H}_{y,z}(u)) \cap W^{cs}_{\delta}(z),$$

where $\mathscr{H}_{y,z}\colon W^{cs}_{\delta}(y) \to W^{cs}(z)$ is the *holonomy* along strong unstable leaves defined by

$$\mathscr{H}_{y,z}(a) \in W^{uu}_{\mathrm{loc}}(a) \cap W^{cs}(z)$$

for $z \in W^{uu}_{\mathrm{loc}}(y)$ and $a \in W^{cs}_{\delta}(y)$.

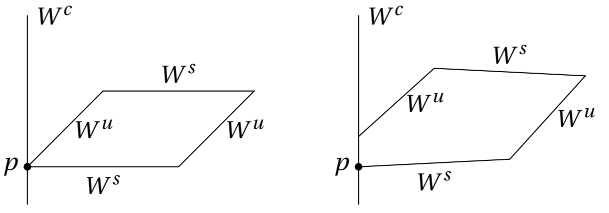


FIGURE 8.2.1. Joint integrability versus not joint integrability

A characterization of joint integrability versus not joint integrability can be seen by su-quadrilaterals. Fix a point $p \in M$. First move to a point $q_0 \in W^{ss}(p)$, next a point $q_1 \in W^{uu}(q_0)$, then a point $q_2 \in W^{ss}(q_1)$ such that $\mathscr{O}(p) \cap W^{uu}(q_2) \neq \varnothing$. If we finish the quadrilateral and move back to the original point p then the stable and unstable foliations are jointly integrable. If we finish the quadrilateral and move back to another point on the orbit of p (in the center manifold $W^c(p)$), then the stable and unstable foliations are not jointly integrable.

Example 8.2.5 (Frame flows)**.** Example 2.2.7 pointed out that parallel-translating a vector along a geodesic can be construed to define a flow, and combined with just tangent and normal vectors this defines a partially hyperbolic flow as follows. Consider the bundle of oriented orthonormal 2-frames on a surface Σ. This produces a fiber bundle $\pi\colon N \to M := S\Sigma$, where the natural projection π takes a frame into its first vector. Over n-dimensional manifolds we get a fiber bundle $\pi\colon N \to M$ where the associated structure group $SO(n-1)$ acts on fibers by rotating the frames, keeping the first vector fixed. Therefore, we can identify each fiber N_x with $SO(n-1)$. The *frame flow* Φ^t acts on frames by moving their first vectors by the geodesic flow and moving the other vectors by parallel translation along the geodesic defined by the first vector. Thus, $\pi \circ \Phi^t = g^t \circ \pi$ for each t. The frame flow preserves the measure that is locally the product of the Liouville measure with normalized Haar measure on $SO(n-1)$. The frame flow is a partially hyperbolic flow. The center bundle has dimension $1 + \dim SO(n-1)$ and is spanned by the flow direction and the fiber direction.

With this, the question of stable ergodicity of volume-preserving partially hyperbolic flows becomes one of accessibility and stable accessibility. One needs to home in on a notion of accessibility (or *essential accessibility*) that is strong enough to make the Hopf argument work and robust enough to (often) be stable under C^1-perturbations—while not being altogether so restrictive as to be unsatisfying. Making the Hopf argument work required significant breakthroughs in geometric measure theory, while stability of accessibility involves subtle hyperbolic dynamics. Before pursuing this more carefully, let us note another need when considering partially hyperbolic sets rather than the "partial" counterpart of an Anosov flow: for this line of reasoning we need partial hyperbolicity to persist under perturbation. There are 2 other general situations in which this is the case. One is that of partially hyperbolic attractors because attractors are stable under perturbation. The other is that of a partially hyperbolic set that is a normally hyperbolic manifold [**236**, Remark 2.20].[3]

[3] We rely here on a prior allusion to cone arguments, but the actual argument is semicontinuity of the Mather spectrum [**236**, Theorem 3.4].

Let us briefly examine the issues that have to be addressed in this approach. First, to use the Hopf argument, one needs absolute continuity of the stable and unstable foliations in order to use the Fubini argument analogously to the way have already done. Fortunately, our Theorem 12.7.5 is for the partially hyperbolic case.[4] Next, accessibility needs to be defined in order to support the Hopf argument, which in this case needs to look like our arguments for weak mixing, because invariance of a function does not make it constant on the now larger center foliation (additional reasons to bypass the center are that there may be no center foliation, without expansion or contraction Birkhoff averages need not be constant on it, and it is often not absolutely continuous). Accordingly, while in hyperbolic flows any 2 points are connected by a heteroclinic one (the strong unstable manifold of one of them intersects the weak-stable of the other or vice versa, it is sufficient if any 2 points are connected by an "*su*-path," that is, a path concatenated from segments, each of which lies in a stable or an unstable leaf. Put differently, call these 2 points equivalent when this happens, and assume that there is only one equivalence class. As in the hyperbolic case, the formal definition does not refer to actual paths.

Definition 8.2.6. Let Φ be a partially hyperbolic flow on a compact Riemannian manifold M.

Two points $p, q \in M$ are said to be *accessible*, if there are points $z_i \in M$ with $z_0 = p$, $z_\ell = q$, such that $z_i \in W^\sigma_{\text{loc}}(z_{i-1})$ for $i = 1, \dots, \ell$ and $\sigma = s$ or u. The collection of points $z_0, z_1, \dots, z_\ell$ is called the *us-path* connecting p and q and is denoted variously by $[p,q]_\Phi = [p,q] = [z_0, z_1, \dots, z_\ell]$.[5]

Accessibility is an equivalence relation and the collection of points accessible from a given point p is called the *accessibility class* of p.

A flow Φ is said to have the *accessibility property* if any two points are accessible (so the sole accessibility class is the whole manifold).

We say that Φ has the *essential accessibility property* if the partition of M by the accessibility classes is trivial in the measure-theoretical sense, i.e., any measurable set that consists of accessibility classes has measure zero or one.

If Φ has the accessibility property then the subbundle $E^s \oplus E^u$ is not integrable (so the stable and unstable foliations are not jointly integrable). Otherwise, the accessibility class of any $p \in M$ would be the leaf of the corresponding foliation passing through p.

We digress to mention that essential accessibility is also interesting with respect to topological dynamics:

[4] It is stated for a diffeomorphism and, applied to the time-1 map, gives the same conclusion.

[5] There is an actual path from p to q that consists of pieces of smooth curves on local stable or unstable manifolds with the z_i as endpoints.

Theorem 8.2.7 ([**71**])**.** *An essentially accessible volume-preserving partially hyperbolic diffeomorphism is topologically transitive.*

Is (essential) accessibility stable? Does it hold often?

Definition 8.2.8. A partially hyperbolic flow Φ is said to be *stably accessible* if there is a C^1-neighborhood $\mathscr{U}$ of Φ (possibly in the space off flows preserving a Φ-invariant Borel probability measure ν) such that any flow $\Psi \in \mathscr{U}$ has the accessibility property.

Corollary 6.2.23 (with Theorem 6.2.22) is encouraging; it even shows genericity of having dense strong leaves in our context. Our line of reasoning does not apply since it uses incommensurability of periodic orbits, which is explicitly limited to helping in the orbit direction, but some of what we did elsewhere is suggestive. The *quadrilateral formula* (2.2.4) we used in an algebraic context suggests geometric arguments outside of the algebraic context: it says that a quadrilateral with $h_\pm$-sides about ϵ causes a $2\epsilon^2$ displacement along a geodesic, and for an Anosov flow one would wish to show that small *su*-quadrilaterals produce some displacement in the flow direction.

In partially hyperbolic dynamics, this *Brin quadrilateral argument* establishes robustness of accessibility as follows: if a small such quadrilateral from x ends on $W^c(x) \smallsetminus \{x\}$, then homotoping it (by shrinking each edge) to length 0 gives a continuous curve in $W^c(x)$ of endpoints that ends at x, and this circumstance should be robust.

Let us repeat this more carefully, assuming for simplicity that the central subbundle E^c is integrable. At $p \in M$ consider a 4-legged us-path $[p = z_0, z_1, z_2, z_3, z_4]$ and connect z_{i-1} with z_i by a geodesic γ_i in the corresponding stable or unstable manifold to obtain the curve $\Gamma_p = \bigcup_{1\le i\le 4} \gamma_i$. We parameterize it by $t \in [0,1]$ with $\Gamma_p(0) = p$.

If the subbundle $E^s \oplus E^u$ were integrable (and hence, f not accessible), the endpoint $z_4 = \Gamma_p(1)$ would lie on the leaf through p of the corresponding foliation. Therefore, one can hope to achieve accessibility if one can arrange a 4-legged us-path in such a way that $\Gamma_p(1) \in W^c(p)$ and $\Gamma_p(1) \neq p$. In this case the path Γ_p can be homotoped through 4-legged us-paths originating at p to the trivial path so that the endpoints stay in $W^c(p)$ during the homotopy and form a continuous curve. Such a situation is usually persistent under small perturbations of Φ and hence leads to stable accessibility.

What Corollary 6.2.23 (with Theorem 6.2.22) suggests, is actually true: genericity of accessibility.

Theorem 8.2.9 ([**106**][6])**.** *Let $q \ge 1$, Φ a partially hyperbolic C^q flow (possibly preserving a smooth measure ν on M). Then in every C^1-neighborhood $\mathscr{U}$ (possibly of μ-preserving flows) of Φ there is a* stably *accessible C^q flow Ψ. Indeed, if the center bundle is 1-dimensional, then accessibility is open dense in the C^2-topology [**101**].*

Before getting to the main ergodicity theorem, let us remark on time-t maps. For an Anosov flow Φ on a compact smooth Riemannian manifold M stable accessibility of the time-1 map depends on whether the subbundle $E^s \oplus E^u$ is integrable, that is, whether the stable and unstable foliations, of the time-1 map are jointly integrable.

To make this notion precise, fix $\varepsilon > 0$ and for a point $x \in M$ consider a local smooth submanifold

$$\Pi(x) = \bigcup_{y \in B^u(x,\varepsilon)} \bigcup_{-\varepsilon \le \tau \le \varepsilon} \varphi^\tau(y)$$

through x. For $x, x' \in M$ let $\pi_{x,x'} \colon \Pi(x) \to \Pi(x')$ be the holonomy map generated by the family of local stable manifolds. The foliations W^s and W^u are *jointly integrable* if for every $y \in \Pi(x)$ the image of the local unstable leaf $W^u_{\text{loc}}(y)$ under $\pi_{x,x'}$ is the local unstable leaf $W^u_{\text{loc}}(\pi_{x,x'}(y))$.

Theorem 8.2.10 ([**74**])**.** *If the stable and unstable foliations of an Anosov flow are not jointly integrable, then the time-*1 *map is stably accessible.*

This suggests that being a "suspension" is the sole obstruction to stable ergodicity in this context.

Corollary 8.2.11. *The time-*1 *map is stably accessible for*

*(1) geodesic flows of negatively curved manifolds (indeed, contact Anosov flows, Theorem 9.1.2) [**173**];*

*(2) C^2 volume-preserving Anosov flows on compact 3-manifolds that are not suspensions [**74**].*

By now we see that the expected ingredients for stable ergodicity are in place for a substantial class of dynamical systems. It is time to state the ergodicity theorem (as yet without "stable").

Theorem 8.2.12 (Grayson–Pugh–Shub–Burns–Wilkinson Ergodicity Theorem)**.** *A C^2 volume-preserving partially hyperbolic essentially accessible center-bunched diffeomorphism is ergodic.*

Remark 8.2.13. As we noted, one can replace "center-bunched" by "$\dim E^c = 1$."

[6] An outline of the argument for one-dimensional center direction can be found in [**236**, Theorem 8.5].

Grayson, Pugh and Shub [**136**] proved ergodicity for small perturbations of the time-1 map of the geodesic flow on a surface of constant negative curvature. Wilkinson's thesis extended this to small perturbations of the time-1 map of the geodesic flow on a surface of possibly variable negative curvature. Pugh and Shub [**249**–**251**] then proved ergodicity of dynamically coherent diffeomorphisms under a stronger center-bunching condition. Burns and Wilkinson weakened the needed center-bunching condition (to Definition 8.2.14) [**76**] and then removed the need for dynamical coherence [**77**].

Definition 8.2.14 ([**76**])**.** We say that a partially hyperbolic diffeomorphism f is *center-bunched* if we can take $\max\{\mu_1, \lambda_3^{-1}\} < \lambda_2/\mu_2$ in (see Definition 5.5.1)

$$\begin{aligned}\lambda_1 \le \lfloor\!\lfloor d_x f \restriction E^s(x) \rfloor\!\rfloor \le \| d_x f \restriction E^s(x)\| \le \mu_1,\\ \lambda_2 \le \lfloor\!\lfloor d_x f \restriction E^c(x) \rfloor\!\rfloor \le \| d_x f \restriction E^c(x)\| \le \mu_2,\\ \lambda_3 \le \lfloor\!\lfloor d_x f \restriction E^u(x) \rfloor\!\rfloor \le \| d_x f \restriction E^u(x)\| \le \mu_3.\end{aligned}$$

More precisely, we merely require "pointwise center-bunching":

$$\max\{\mu_1(p), \lambda_3^{-1}(p)\} < \lambda_2(p)/\mu_2(p) \text{ for every } p,$$

where $\lambda_i(p)$ and $\mu_i(p)$ are *pointwise* bounds on rates of expansion and contraction. (This is automatic if $\dim E^c = 1$.)

This Burns–Wilkinson center-bunching imposes a much weaker constraint than earlier versions.

Although it is no longer needed for this result, we provide the definition of dynamical coherence in its most economical form; it guarantees not only integrability of the central subbundle, but its unique integrability:

Definition 8.2.15 (Burns–Wilkinson coherence)**.** A partially hyperbolic embedding is said to be *dynamically coherent* if E^{cs} and E^{cu} are integrable to foliations W^{cs} and W^{cu}, respectively.

Consider the hypotheses of Theorem 8.2.12 in light of our desire for stable ergodicity. Most of them are robust: partial hyperbolicity is a C^1-open condition, and so is center-bunching by semicontinuity (though this is not trivial). We are restricting to volume-preserving perturbations, so stability of accessibility is the only additional assumption we need.

Theorem 8.2.16 (GraysonPughShubBurnsWilkinson Stable Ergodicity Theorem)**.** *A C^2 volume-preserving partially hyperbolic stably (essentially) accessible center-bunched diffeomorphism is stably ergodic.*

The need to assume *stable* accessibility motivates two conjectures about accessibility.

Conjecture 8.2.17. *A partially hyperbolic dynamical system with the accessibility property is stably accessible.*

An example by Brin shows that this conjecture fails if one replaces accessibility by essential accessibility [**73**].

Conjecture 8.2.18. *The space of stably accessible partially hyperbolic dynamical systems is open and dense in the C^r topology for any $r \geq 1$. (This is known for $r = 1$ by [**106**].)*

Fortunately, we established stable accessibility in a few salient situations. For instance, Theorem 8.2.10 now becomes

Theorem 8.2.19 ([**74**])**.** *If the stable and unstable foliations of a volume-preserving Anosov flow are not jointly integrable, then the time-*1 *map is stably ergodic.*

In the special case of flows on 3-manifolds one can strengthen this result:

Theorem 8.2.20. *The time-one map of a volume-preserving Anosov 3-flow is stably ergodic if and only if the flow is not the suspension of an Anosov diffeomorphism.*

In particular,

Proposition 8.2.21. *The time-1 map of a volume-preserving topologically mixing C^2 Anosov 3-flow is stably ergodic.*

This theory applies to the aforementioned frame flows (Example 8.2.5) as well.

Theorem 8.2.22. *Let Φ^t be the frame flow on an n-dimensional compact smooth Riemannian manifold with sectional curvatures between $-\Lambda^2$ and $-\lambda^2$. Then in each of the following cases the flow is ergodic (indeed, Bernoulli Theorem 8.4.17), and the time-one map of the frame flow is stably ergodic (and stably K-mixing Remark 8.3.19):*

*(1) if the curvature is constant [**63**];*
*(2) for a set of metrics of negative curvature which is open and dense in the C^3 topology [**63**];*
*(3) if n is odd and $n \neq 7$ [**61**];*
*(4) if n is even, $n \neq 8$, and $\lambda/\Lambda > 0.93$ [**62**];*
*(5) if $n = 7$ and $\lambda/\Lambda > 0.99023\ldots$ [**72**];*
*(6) if $n = 8$ and $\lambda/\Lambda > 0.99023\ldots$ [**72**].*

Ergodicity of the frame flow was proved by the respective cited authors; [**72**] pointed out K-mixing and the Bernoulli property and used [**75**, Corollary 1.2] (which relies on [**64**]) to deduce those of the time-1-maps across all cases.

We now give an indication what is involved in proving Theorem 8.2.12.

We begin with recasting the Hopf argument for the present situation. Let f be a partially hyperbolic C^2 diffeomorphism of a smooth compact Riemannian manifold M that preserves a smooth measure ν. To recast the Hopf argument say that $x, y \in M$ are stably equivalent if

$$\rho(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty,$$

and unstably equivalent if

$$\rho(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty.$$

Stable and unstable equivalence classes induce two partitions of M, and we denote by $\mathscr{S}$ and $\mathscr{U}$, the Borel σ-algebras they generate. Recall that for an algebra $\mathscr{A} \subset \mathscr{B}$:=the Borel σ-algebra, its saturated algebra is the set

$$\operatorname{Sat}(\mathscr{A}) = \{B \in \mathscr{B} : \text{ there exists } A \in \mathscr{A} \text{ with } \nu(A \triangle B) = 0\}.$$

The Hopf argument says that f is ergodic if

$$\operatorname{Sat}(\mathscr{S}) \cap \operatorname{Sat}(\mathscr{U}) = \mathscr{T} := \text{the trivial algebra.} \tag{8.2.1}$$

If f is an Anosov diffeomorphism, then the stable equivalence class containing a point x is its stable leaf $W^s(x)$. Similarly, the unstable equivalence class containing x is its unstable leaf $W^u(x)$. The σ-algebra $\mathscr{S}$ consists of those Borel sets S for which $W^s(x) \subset S$ whenever $x \in S$, and the σ-algebra $\mathscr{U}$ consists of those Borel sets U for which $W^u(x) \subset U$ whenever $x \in U$. The relation (8.2.1) holds by absolute continuity of stable and unstable foliations, which proves ergodicity of Anosov diffeomorphisms.

If f is partially hyperbolic, then the stable and unstable foliations W^s and W^u of M also generate Borel σ-algebras $\mathscr{M}^s$ and $\mathscr{M}^u$, respectively, so $\mathscr{S} \subset \mathscr{M}^s$ and $\mathscr{U} \subset \mathscr{M}^u$ (note that stable and unstable sets containing a point x may be larger than $W^s(x)$ and $W^u(x)$ due to possible contractions and expansions along the central directions). It follows that

$$\operatorname{Sat}(\mathscr{S}) \cap \operatorname{Sat}(\mathscr{U}) \subset \operatorname{Sat}(\mathscr{M}^s) \cap \operatorname{Sat}(\mathscr{M}^u).$$

If f is accessible then $\operatorname{Sat}(\mathscr{M}^s \cap \mathscr{M}^u) = \mathscr{T}$ (in fact, it suffices to assume essential accessibility (Definition 8.2.6)), and ergodicity would follow from

$$\operatorname{Sat}(\mathscr{M}^s) \cap \operatorname{Sat}(\mathscr{M}^u) \subset \operatorname{Sat}(\mathscr{M}^s \cap \mathscr{M}^u) \tag{8.2.2}$$

(the opposite inclusion is obvious). We now discuss how the assumptions of Theorem 8.2.12 guarantee this.

The way to establish (8.2.2) without absolute continuity of the center-stable and center-unstable foliation is through the use of a collection of special sets at every point $x \in M$ called *juliennes*, $J_n(x)$.[7] We describe a construction of these sets which assures that

(J1) $J_n(x)$ form a basis of the topology.

(J2) $J_n(x)$ form a basis of the Borel σ-algebra. More precisely, let Z be a Borel set; a point $x \in Z$ is said to be *julienne dense* if

$$\lim_{n\to+\infty} \frac{\nu(J_n(x)\cap Z)}{\nu(J_n(x))} = 1.$$

Let $D(Z)$ be the set of all julienne dense points of Z. Then

$$D(Z) = Z \pmod 0.$$

(J3) If $Z \in \mathrm{Sat}(\mathscr{M}^s)\cap \mathrm{Sat}(\mathscr{M}^u)$, then $\mathscr{D}(Z) \in \mathrm{Sat}(\mathscr{M}^s \cap \mathscr{M}^u)$.

Properties (J1)–(J3) imply (8.2.2).

Note that the collection of balls $B(x, 1/n)$ satisfies requirements (J1) and (J2) but not (J3). Juliennes can be viewed as balls "distorted" by the dynamics in the following sense. Fix an integer $n \geq 0$, a point $x \in M$ and numbers τ, σ such that $0 < \tau < \sigma < 1$. Denote by

$$B_n^s(x,\tau) = \{y \in W^s(x) \mid \rho(f^{-k}(x), f^{-k}(y)) \leq \tau^k\},$$
$$B_n^u(x,\tau) = \{y \in W^u(x) \mid \rho(f^{k}(x), f^{k}(y)) \leq \tau^k\}.$$

and define the julienne

$$J_n(x) := [J_n^{cs}(x) \times B_n^u(x,\tau)] \cap [B_n^s(x,\tau) \times J_n^{cu}(x)],$$

where the local foliation products

$$J_n^{cs}(x) = B_n^s(x,\tau) \times B^c(x,\sigma^n), \quad J_n^{cu}(x) = B_n^u(x,\tau) \times B^c(x,\sigma^n)$$

are the *center stable* and *center unstable juliennes*, and $B^c(x,\sigma^n)$ is the ball in $W^c(x)$ centered at x of radius σ^n. One may think of $J_n(x)$ as a substitute for $B_n^s(x,\tau) \times B^c(x,\sigma^n) \times B_n^u(x,\tau)$, which is only well-defined if the stable and unstable foliations are jointly integrable.

The proof of (J1)–(J3) is based on the the following properties of juliennes:

(1) *scaling*: If $k \geq 0$ then $\nu(J_n(x))/\nu(J_{n+k}(x))$ is bounded, uniformly in $n \in \mathbb{N}$;

(2) *engulfing*: there is $\ell \geq 0$ such that, for any $x, y \in M$, if $J_{n+\ell}(x) \cap J_{n+\ell}(y) \neq \varnothing$ then $J_{n+\ell}(x) \cup J_{n+\ell}(y) \subset J_n(x)$;

[7] They resemble slivered vegetables as used in consommé Julienne, said to be attributed to the chef Jean Julien in 1722 by François Massialot (Le nouveau cuisinier royal et bourgeois ou cuisine moderne, reprint 2003 by Eibron Classics).

(3) *quasi-conformality*: there is $k \geq 0$ such that if $x, y \in M$ are connected by an arc on an unstable manifold that has length ≤ 1 then the holonomy map $\pi : W^{cs}_{\text{loc}}(x) \to W^{cs}_{\text{loc}}(y)$ generated by the family of local unstable manifolds satisfies $J^{cs}_{n+k}(y) \subset \pi(J^{cs}_n(x)) \subset J^{cs}_{n-k}(y)$.

The properties (1) and (2) hold for the family of balls in Euclidean space, and they underlie the proof of the Lebesgue Density Theorem. One can use these properties to show that juliennes are density bases. The center-unstable juliennes are a density basis on $W^{cu}(x)$ with respect to the smooth conditional measure $\nu_{W^{cu}}$ on $W^{cu}(x)$, the center-stable juliennes are a density basis on $W^{cs}(x)$ with respect to the smooth conditional measure $\nu_{W^{cs}}$ on $W^{cs}(x)$, and the juliennes are a density basis on M with respect to the smooth measure ν.

Juliennes, $J_n(x)$, are small but highly eccentric sets in the sense that the ratio of their diameter to their inner diameter increases with n (the inner diameter of a set is the diameter of the largest ball it contains). In general, sets of such shape may not form density bases, but juliennes do because their elongation and eccentricity are controlled by the dynamics; in particular, they nest in a way similar to balls.

Quasi-conformality is what is needed to prove Property (J3). Roughly speaking it means that the holonomy map (almost) preserves the shape of juliennes.

3. Specification, uniqueness of equilibrium states

We now return to the study of equilibrium states, which began in Section 4.3. First we show that for hyperbolic flows the class of Bowen-bounded functions, and indeed that of Walters-continuous functions (Definition 4.3.17) is substantial: it contains all Hölder-continuous functions.

Proposition 8.3.1. *Let M be a Riemannian manifold, Φ a smooth flow, and $\Lambda \subset M$ a locally maximal hyperbolic set for Φ. Then every Hölder-continuous function on Λ is Walters-continuous, hence Bowen-bounded (Definition 4.3.17).*

Proof. Suppose f is α-Hölder: $|f(x) - f(y)| \leq cd(x,y)^\alpha$ and $\epsilon > 0$. With C', η as in Proposition 6.2.4 choose $\delta > 0$ such that $2\delta\|f\|_\infty + 2c(C'\delta)^\alpha \int_0^\infty \eta^{\alpha s}\, ds < \epsilon$. If $d^\Phi_t(x,y) < \delta$, then Proposition 6.2.2 gives

$$|S_t f(x) - S_t f(y)| \leq \underbrace{|S_t(f - f \circ \varphi^\tau)(x)|}_{=|S_\tau f(x) - S_\tau f(\varphi^t(x))| \leq 2\delta\|f\|_\infty} + \underbrace{|S_t f \circ \varphi^\tau(x) - S_t f(y)|}_{< c\int_0^t (C'(\eta^s + \eta^{t-s})\delta)^\alpha ds \leq 2c(C'\delta)^\alpha \int_0^\infty \eta^{\alpha s} ds} < \epsilon. \qquad \square$$

As we have seen in Chapter 7, the class of Hölder-continuous functions is invariant under orbit-equivalence of hyperbolic sets and hence a natural class of functions to consider.

Section 4.3 established existence of equilibrium states, and we now establish uniqueness for hyperbolic flows, with specification as a central tool, albeit in a stronger form than Definition 5.3.56 which we here obtain assuming topological mixing rather than mere transitivity as in Theorem 5.3.59. The essential reason for this improvement is Proposition 6.2.18, though we do not actually invoke density of strong leaves in the proof.

Definition 8.3.2 (Strong Specification)**.** A flow on Φ on a compact metric space X satisfies *specification* if for any $\epsilon > 0$ there is a T_ϵ such that for $x_0,\dots,x_n \in X$ and $t_0,\dots,t_{n+1} \in [0,\infty)$[8] with $t_{i+1} \ge t_i + T_\epsilon$ there is a y with

$$d(\varphi^t(y),\varphi^{t-t_i}(x_i)) < \epsilon \text{ for } t \in [t_i, t_{i+1} - T_\epsilon], \tag{8.3.1}$$

and we can take $y \in \mathbb{P}_\epsilon(t_{n+1} - t_0)$ (Definition 4.2.22).

Remark 8.3.3.

- A compact locally maximal hyperbolic set with this property is necessarily topologically mixing because it produces incommensurate periods (Theorem 6.2.12 and Proposition 6.2.19).
- Of course, a bare-handed argument verifies in full generality that the strong specification property implies topological mixing: For any two ϵ-balls pick x_0 in one, x_1 in the other, $t_0 = 0$, and $t_1 \ge T_\epsilon$ arbitrary to obtain the statement of Definition 1.6.31.
- By choosing $t_i \in \tau\mathbb{Z}$, this property directly implies the corresponding property for the time-τ map of Φ except that one might not be able to choose y periodic.

Theorem 8.3.4 (Specification Theorem)**.** *If Λ is a topologically mixing compact locally maximal hyperbolic set for a flow Φ, then $\Phi\restriction_\Lambda$ has the (strong) specification property (Definition 8.3.2).*

Remark 8.3.5. This result is interesting even if the shadowing orbit is not required to be closed. The reader will note that in this case the proof even works when the first or last specified segment is infinite. Also, the arguments here implicitly reprove the conclusion of Proposition 6.2.18; while redundant, this line of reasoning provides more intuitive and direct control of the construction of the shadowing orbit.

Proof. For $\epsilon > 0$ and $L \ge 1$ as in Theorem 5.3.10 take $\delta \coloneqq \epsilon/8L$ and $T_\epsilon > 0$ such that if $x, y \in \Lambda$ then for an adapted metric (Proposition 5.1.5) $\varphi^t(B(x,\delta)) \cap B(y,\delta) \neq \varnothing$ for all $t \ge T_\epsilon$ (Remark 1.6.32) and furthermore, if $y \in W^u_\epsilon(x)$ and $t \ge T_\epsilon$, then $d(\varphi^{-t}(x),\varphi^{-t}(y)) < \frac{1}{2}d(x,y)$. Let $y_0 \coloneqq \varphi^{-t_0}(x_0)$.
The choice of T_ϵ, gives a $z \in \varphi^{T_\epsilon}\big(B(\varphi^{t_1-T_\epsilon}(y_0),\delta)\big) \cap B(x_1,\delta)$, hence y_1 with

$$\varphi^{t_1-T_\epsilon}(y_1) \in W^u_\delta(\varphi^{t_1-T_\epsilon}(y_0)) \cap W^{cs}_\delta(\varphi^{-T_\epsilon}(z)).$$

[8] Note that the t_i play a different role here than in Definition 5.3.56; they are the prescribed start times for each segment.

The choice of T_ϵ, gives a $z \in \varphi^{T_\epsilon}\big(B(\varphi^{t_2-T_\epsilon}(y_1),\delta)\big) \cap B(x_2,\delta)$, hence y_2 with

$$\varphi^{t_2-T_\epsilon}(y_2) \in W^u_\delta(\varphi^{t_2-T_\epsilon}(y_1)) \cap W^{cs}_\delta(\varphi^{-T_\epsilon}(z))\dots$$

Eventually this gives a $z \in \varphi^{T_\epsilon}\big(B(\varphi^{t_{n+1}-T_\epsilon}(y_n),\delta)\big) \cap B(x_0,\delta)$, hence y_{n+1} with

$$\varphi^{t_{n+1}-T_\epsilon}(y_{n+1}) \in W^u_\delta(\varphi^{t_{n+1}-T_\epsilon}(y_n)) \cap W^{cs}_\delta(\varphi^{-T_\epsilon}(z)).$$

Then $\varphi^{[t_1,t_{n+1}]}(y_{n+1})$ closes within $\epsilon/2L$ and shadows the specification within $\epsilon/4L$. The closed orbit from Theorem 5.3.10 is within $\epsilon/2$ of $\varphi^{[t_1,t_{n+1}]}(y_{n+1})$ and hence shadows the specification within ϵ, as required. □

We will show that for each Bowen-bounded potential function f there is a measure μ_f with the Gibbs property (Definition 4.3.22), where $P = P(\Phi, f)$ (Definition 4.3.1, Proposition 8.3.14). By Theorem 4.3.23 this is an equilibrium state, and we will establish further that μ_f is the unique equilibrium state for Φ with respect to f. Specification is central to this because the measures whose weak limit is the equilibrium state, in (8.3.3), are defined on periodic orbits.

Theorem 8.3.6 (Bowen)**.** *If Φ is an expansive flow with specification and with finite topological entropy on a compact metric space, $f \in V(\Phi)$, then the equilibrium state for Φ associated with f is unique and weakly mixing.*[9]

Corollary 8.3.7. *If Φ is a smooth flow on a compact manifold and Λ a topologically mixing locally maximal hyperbolic set and $f \in V(\Phi_{\restriction\Lambda})$,*[10] *then there exists a unique equilibrium state for $\Phi_{\restriction\Lambda}$ associated with f, and it is weakly mixing—indeed, K-mixing (Remark 8.3.19).*

Before proceeding to the proof of the theorem (ultimately, on page 401), we develop a number of auxiliary results. Throughout, we assume that Φ is an expansive flow with expansivity constant δ_0 and with the specification property and that $f \in V(\Phi)$ with ϵ, K as in the definition of $V(\Phi)$ (Definition 4.3.17).

The first step is careful control of the growth of statistical sums (Proposition 8.3.9). Next, the same control for statistical sums over periodic points, defined in (8.3.2), is obtained in Proposition 8.3.12. Because these 2 quantities grow in lockstep, the candidate measure from (8.3.3) is a Gibbs measure (Proposition 8.3.14), hence an equilibrium state (Theorem 4.3.23). Moreover, this allows us to show ergodicity, and that in turn reduces the proof of uniqueness to showing that any invariant Borel probability measure that is singular with respect to the measure from (8.3.3) has lower pressure.

[9] In particular, Φ is not a suspension (Proposition 3.4.9).

[10] for example, f is Hölder continuous; see Proposition 8.3.1.

Lemma 8.3.8. *For* $\epsilon \in (0, \delta_0/3)$ *there are* $k_\epsilon K_\epsilon > 0$ *such that if* $t_1, t_2 > 0$*, then*

$$k_\epsilon N_d(\Phi, f, \epsilon, t_1) N_d(\Phi, f, \epsilon, t_2) \le N_d\big(\Phi, f, \epsilon, t_1 + t_2\big) \le K_\epsilon N_d(\Phi, f, \epsilon, t_1) N_d(\Phi, f, \epsilon, t_2).$$

Proof. (i) If E is (t_1+t_2, ϵ)-separated and F_j a maximal $(t_j, \epsilon/2)$-separated set, then for $x \in E$ there is a unique $z(x) := (z_1(x), z_2(x)) \in F_1 \times F_2$ such that $d^\Phi_{t_1}(x, z_1(x)) \le \epsilon/2$ and $d^\Phi_{t_2}(\varphi^{t_1}(x), z_2(x)) \le \epsilon/2$, and $z(\cdot)$ is injective. Since furthermore

$$|S_{t_1+t_2} f(x) - S_{t_1} f(z_1(x)) - S_{t_2} f(z_2(x))| \le |S_{t_1} f(x) - S_{t_1} f(z_1(x))| \\ + |S_{t_2} f(\varphi^{t_1}(x)) - S_{t_2} f(z_2(x))| \le 2K,$$

we have

$$\sum_{x \in E} \exp(S_{t_1+t_2} f(x)) \le e^{2K} N_d(\Phi, f, \frac{\epsilon}{2}, t_1) N_d(\Phi, f, \frac{\epsilon}{2}, t_2).$$

Now invoke Proposition 4.2.18.

(ii) If E_j is $(t_j, 3\epsilon)$-separated, $a_1 = 0$, $a_2 = t_1 + T_\epsilon$, with T_ϵ as in the specification property, and $I_j = [a_j, a_j + t_j]$, then specification implies that for $x := (x_1, x_2) \in E_1 \times E_2$ there is a $z = z(x)$ such that $d^\Phi_{t_j}(\varphi^{a_j}(z), x_j) < \epsilon$. Note that by construction $E := \{z(x) \mid x \in E_1 \times E_2\}$ is $(a_2 + t_2, \epsilon)$-separated. Since furthermore

$$S_{a_2+t_2} f(z(x)) \ge -T_\epsilon \|f\|_{C^0} - 2K + S_{t_1} f(x_1) + S_{t_2} f(x_2),$$

we have

$$N_d(\Phi, f, \epsilon, a_2 + t_2) \ge e^{-T_\epsilon \|f\|_{C^0} - 2K} N_d(\Phi, f, 3\epsilon, t_1) N_d(\Phi, f, 3\epsilon, t_2).$$

Meanwhile, $a_2 + t_2 = t_1 + t_2 + T_\epsilon$ and part (i) yield

$$N_d(\Phi, f, \epsilon, a_2 + t_2) \le K_\epsilon N_d(\Phi, f, \epsilon, t_1 + t_2) N_d(\Phi, f, \epsilon, T_\epsilon),$$

so by Lemma 4.3.18

$$N_d(\Phi, f, \epsilon, t_1+t_2) \ge N_d(\Phi, f, 3\epsilon, t_1) N_d(\Phi, f, 3\epsilon, t_2) \big/ K_\epsilon N_d(\Phi, f, \epsilon, T_\epsilon) e^{T_\epsilon \|f\|_{C^0} + 2K}. \quad \square$$

Proposition 8.3.9. *Let X be a compact metric space, Φ a flow with expansivity constant δ_0 and with the specification property. If $0 < \epsilon < \delta_0/3$, $t \in \mathbb{N}$, and k_ϵ and K_ϵ are as in Lemma 8.3.8, then*

$$\frac{1}{K_\epsilon} e^{tP(f)} \le N_d(\Phi, f, \epsilon, t) \le \frac{1}{k_\epsilon} e^{tP(f)}$$

Proof. Proposition 4.3.20 and (4.3.1) give $P(f) = \lim_{t \to \infty} \frac{1}{t} \log N_d(\Phi, f, \epsilon, t)$ for $\epsilon < \frac{\delta_0}{2}$. By Lemma 8.3.8 we can apply the Bowen–Fekete Lemma 4.2.7 to $\log K_\epsilon N_d(\Phi, f, \epsilon, t)$ and $-\log k_\epsilon N_d(\Phi, f, \epsilon, t)$. □

Unlike in Proposition 4.3.12, we define the desired equilibrium state using statistical sums over *periodic* points.

Definition 8.3.10. With the notations of Definition 4.2.22 define

$$\mu_{\epsilon,t} := \frac{1}{P(\Phi, f, t, \epsilon)} \sum_{\gamma \in \mathbb{O}_\epsilon(t)} e^{S_{\pi(\gamma)} f(x)} \delta_\gamma \in \mathfrak{M}(\Phi),$$

where δ_γ is the Lebesgue measure on γ (that is, induced from Lebesgue measure on $[0, \pi'(\gamma)]$ by $t \mapsto \varphi^t(x)$ for $x \in \gamma$, and

$$P(\Phi, f, t, \epsilon) := \sum_{\gamma \in \mathbb{O}_\epsilon(t)} \pi'(\gamma) e^{S_{\pi(\gamma)} f(x)} \tag{8.3.2}$$

is the normalizing factor. Weak*-compactness of $\mathfrak{M}(\Phi)$ gives an accumulation point

$$\mu_f = \lim_{t_i \to \infty} \mu_{\epsilon, t_i} \in \mathfrak{M}(\Phi). \tag{8.3.3}$$

We will show that this is an equilibrium state and that it is the only one (so, a posteriori, this is a proper limit).

We first use the characterization Theorem 1.7.5(3) of expansivity to connect periodicity and separation.

Proposition 8.3.11. *If ϵ and α are as in Theorem 1.7.5(3), $q \ge \epsilon$, $\rho \le \alpha/2$, $\theta \le \alpha$, then $x, y \in \mathbb{P}_\rho(t)$ and $x \notin \varphi^{[-q,q]}(y) \Rightarrow x$ and y are (t,θ)-separated.*

Proof. Otherwise, $x, y \in \mathbb{P}_{\alpha/2}(t)$ are not (t,α)-separated, so there are $a, b \in [t - \frac{\alpha}{2}, t + \frac{\alpha}{2}]$ with $\varphi^a(x) = x$ and $\varphi^b(y) = y$, then

$$d(\varphi^{t_{pm+q}}(x), \varphi^{u_{pm+q}}(y)) = d(\varphi^{q\alpha}(x), \varphi^{q\alpha}(y)) \le \alpha$$

for $0 \le q < m := 1 + \lfloor \frac{1}{\alpha}(t - \frac{\alpha}{2}) \rfloor$, where $t_{pm+q} := pa + q\alpha$ and $u_{pm+q} := pb + q\alpha$, so $y = \varphi^t(x)$ with $|t| < \epsilon$ by Theorem 1.7.5(3). □

Proposition 8.3.12. *Let X be a compact metric space, Φ an expansive flow with the specification property, $f \in V(\Phi)$ and $\epsilon > 0$ as in Definition 4.3.17 sufficiently small. Then there exist $c_1, c_2 > 0$ such that for sufficiently large t*

$$c_1 e^{tP(f)} \le P(\Phi, f, t, \epsilon) \le c_2 e^{tP(f)}.$$

Remark 8.3.13. With $f \equiv 0$ (and the notations of Definition 8.3.10) this becomes

$$c_1 e^{t h_{\text{top}}(\Phi)} \le \sum_{\gamma \in \mathbb{O}_\epsilon(t)} \pi'(\gamma) \le c_2 e^{t h_{\text{top}}(\Phi)},$$

so $p(\Phi) = h_{\text{top}}(\Phi)$ (Definition 4.2.1).

Proof. If $q > 0$ is such that $x, y \in \mathbb{P}_\epsilon(t)$ are (t,ϵ)-separated unless $x \in \varphi^{[-q,q]}(y)$ (see Proposition 8.3.11), then we can find a (t,ϵ)-separated subset E_γ of $\gamma \in \mathbb{O}_\epsilon(t) =:$

$\{\gamma_1,\dots,\gamma_r\}$ with $\pi'(\gamma) \le 2q \operatorname{card} E_\gamma$, so

$$P(\Phi, f, t, \epsilon) \le \sum_{i=1}^{r} 2q \sum_{z\in E_{\gamma_i}} e^{S_{\pi(\gamma_i)} f(z)} \le 2q e^{\epsilon\|f\|} \sum_{z\in E} e^{S_t f(z)} \underset{\text{Proposition 8.3.9}}{\underbrace{\le}} \frac{2q e^{\epsilon\|f\|}}{k_\epsilon} e^{tP(f)}$$

since $E := \bigcup_{i=1}^{r} E_{\gamma_i}$ is (t,ϵ)-separated. This gives the upper bound.

If E is a $(t - T_\epsilon, 3\epsilon)$-separated set with T_ϵ as in specification, and $x \in E$ there exists $z = z(x) \in \mathbb{P}_\epsilon(t)$ with $d^{\Phi}_{t-T_\epsilon}(x,z) \le \epsilon$. $z(\cdot)$ is "close enough" to injective: If $\beta > 0$ is such that $|s| < 3\beta \Rightarrow d(x, \varphi^s(x)) < \epsilon$ for all $x \in X$, then

$$x \ne x' \Rightarrow \varphi^{[-\beta,\beta]}(z(x)) \cap \varphi^{[-\beta,\beta]}(z(x')) = \varnothing,$$

because $x \ne x' \Rightarrow d(\varphi^s(z(x)), \varphi^s(z(x'))) > \epsilon$ for some $s \in [0, t - T_\epsilon]$ by the triangle inequality, so $z(x') \notin \varphi^{[-3\beta,3\beta]}(z(x))$. This implies that

$$P(\Phi, f, t, \epsilon) \ge \sum_{x\in E} 2\beta \exp\big(\underbrace{S_{\pi(\gamma)} f(z(x))}_{\ge S_{t-T_\epsilon} f(x) - K - (T_\epsilon+\epsilon)\|f\|}\big) \ge 2\beta e^{-K-(T_\epsilon+\epsilon)\|f\|} \underbrace{N_d(\Phi, f, t - T_\epsilon, 3\epsilon)}_{\ge N_d(\Phi, f, t-T_\epsilon, \epsilon)/C_{\epsilon,3\epsilon}}$$

by Lemma 4.3.18 with K as in (4.3.6). Proposition 8.3.9 gives the lower bound. □

We can now show that μ_f is a Gibbs measure (Definition 4.3.22) for $P(f)$:

Proposition 8.3.14. *Let (X,d) be a compact metric space, Φ an expansive flow with specification, $f \in V(\Phi)$, μ_f as in (8.3.3), and $\epsilon > 0$ as in Definition 4.3.17. Then there are $A_\epsilon, B_\epsilon > 0$ such that for $x \in X$ and $t > 0$ (with the notation from (4.2.1)) we have*

$$A_\epsilon e^{S_t f(x) - tP(f)} \le \mu_f(B_\Phi(x, \epsilon, t)) \le B_\epsilon e^{S_t f(x) - tP(f)}.$$

We recall that by Theorem 4.3.23 μ_f is then an equilibrium state for Φ with respect to f.

Proof. If s is (very) large and Q_s is (s,ϵ)-separated with

$$\sum_{y\in Q_s} e^{S_s f(y)} = N_d(\Phi, f, s, \epsilon) \ge e^{sP(f)}/K_\epsilon,$$

(by Proposition 8.3.9) and $\alpha < \epsilon/6$ is such that

$$d(\varphi^u(z), z) < \epsilon/3 \text{ whenever } |u| \le 2\alpha \text{ and } z \in X, \tag{8.3.4}$$

then specification gives

$$w(y) \in \mathbb{O}_{2\alpha}(t + s + 2M_\alpha) \text{ with } \begin{cases} d(\varphi^r(w(y)), \varphi^r(x)) < \alpha \text{ for } r \in [0,t] \\ d(\varphi^{p_v+\sigma}(w(y)), \varphi^v(y)) < \alpha \text{ for } v \in [0,s],\ p_v := v + t + M_\alpha, \end{cases}$$

with $|\sigma| < \alpha$. Then (8.3.4) implies that $y \ne y' \in Q_s \Rightarrow \varphi^{[0,\alpha]}(w(y)) \cap \varphi^{[0,\alpha]}(w(y')) = \varnothing$: there is a $v \in [0,s]$ such that $d(\varphi^v(y), \varphi^v(y')) > \epsilon$ and hence

$$d(\varphi^{p_c+\sigma}(w(y)), \varphi^{p_v+\sigma}(w(y'))) > \epsilon - 2\alpha > 2\epsilon/3$$

by the triangle inequality, so (8.3.4) implies $d(\varphi^{p_v+u}(w(y)),\varphi^{p_v}(w(y'))) > 0$ for $|u| \le \alpha$ and hence $\varphi^{p_v+[0,\alpha]}(w(y)) \cap \varphi^{p_v+[0,\alpha]}(w(y')) = \varnothing$, which implies the claim.

Furthermore, $d(\varphi^r(\varphi^u(w(y)),\varphi^r(x)) < \epsilon/3+\alpha < \epsilon/2$ for $u \in [0,\alpha]$, $r \in [0,t]$, so $\bigcup_{y\in Q_s}\varphi^{[0,\alpha]}(w(y)) \subset B_\Phi(x,\epsilon,t)$, and

$$\mu_{\alpha,t+s+2M_\alpha} \ge \underbrace{\frac{1}{P(\Phi,f,t+s+2M_\alpha,\epsilon)}}_{\ge e^{-P(f)(t+s+2M_\alpha)}/c_2}\sum_{y\in Q_s}\alpha\exp(\underbrace{S_{\pi(w(y))}f(w(y))}_{\ge S_tf(x)-(2M_\alpha+\alpha)\|f\|-2K+S_sf(y)})) \ge A_\epsilon e^{S_tf(x)-tP(f)}$$

by Proposition 8.3.12 with K as in (4.3.6). Now we let $s\to\infty$:

$$\mu_f(B_\Phi(x,\epsilon,t)) \ge \lim_{i\to\infty}\mu_{\alpha,t_i}(B_\Phi(x,\epsilon,t)) \ge \underline{\lim}_{s\to\infty}\mu_{\alpha,t+s+2M_\alpha}(B_\Phi(x,\epsilon,t)) \ge A_\epsilon e^{S_tf(x)-tP(f)}.$$

The following with $V = B_\Phi(x,\epsilon,t)$, $\theta = 2\epsilon$, $\rho=\epsilon$ gives the reverse inequality:

Lemma 8.3.15. $\forall\theta,\rho,\eta>0\ \exists S>0\ \forall t'>0, V\subset X\ \exists A\subset V\cap\mathbb{P}_\rho(t')$ *and* $R\colon A\to[-\eta,\eta]$ *such that* $\varphi^{R_x}(x)\subset V$, $\varphi^s(A)$ *is* (t',θ)*-separated for any* s *(so, if* $V=B_\Phi(x,\epsilon,t)\supset A$, *then* $\varphi^t(A)$ *is moreover* $(t-t',2\epsilon)$*-separated), and*

$$\frac{1}{2(S+1)}\mu_{\rho,t'}(V) \le \mu_{\rho,t'}(\bigcup_{x\in A}\varphi^{R_x}(x)) = \overbrace{\frac{1}{P(\Phi,f,t',\rho)}}^{\le e^{-t'P(f)}/c_1}\underbrace{\sum_{x\in A}\lambda(R_x)\exp(S_{\pi(x)}f(x))}_{\le 2\eta\sup\left\{\sum_{z\in B}e^{S_{\pi(z)}f(z)}\ \middle|\ B\ (t'-t,2\epsilon)\text{-separated}\right\}\le 2\eta e^{S_tf(z)+\epsilon\|f\|}N_d(\Phi,f,t'-t,2\epsilon)}.$$

Thus, Proposition 8.3.9 gives $\mu_{\epsilon,t'}(B_\Phi(x,\epsilon,t)) \le B_\epsilon e^{S_tf(z)-tP(f)}$, and

$$\mu_f(B_\Phi(x,\epsilon,t)) = \lim_{i\to\infty}\mu_{\epsilon,t_i}(B_\Phi(x,\epsilon,t)) \ge \underline{\lim}_{t'\to\infty}\mu_{\epsilon,t'}(B_\Phi(x,\epsilon,t)) \le B_\epsilon e^{S_tf(x)-tP(f)}.\quad\square$$

Proof of Lemma 8.3.15. By Theorem 1.7.5(3) there is a $q>0$ such that $x,y\in\mathbb{P}_\rho(t')$, $x\notin\varphi^{[-q,q]}(y)\Rightarrow x,y$ are (t',θ)-separated. Let $\eta' := \min(\eta,q)$, $S := \lceil 2q/\eta'\rceil$, and $A := \bigcup_{\gamma\in\mathbb{P}_\epsilon(t')}A_\gamma$, where the choice of A_γ will now be described.

Partition $\gamma\in\mathbb{P}_\rho(t')$ into closed segments $I_1,\dots,I_m$ of length $l\in(\eta'/2,\eta)$. Then $I_i\cap\varphi^{[-q,q]}(I_j)=\varnothing$ whenever $|i-j|>S$ in the sense of cyclic ordering. To get separation and at the same time control the percentage of time spent in V group these i into $E_1,\dots,E_{2(S+1)}$ such that $i,j\in E_k$, $i\ne j\Rightarrow|i-j|>S$ (by taking $E_1=\{1,\dots,S\}$, $E_2=\{S+1,\dots,2S\}$ and so on but putting the last at most $S+1$ indices into singletons E_k). Then

$$\mu_{\rho,t'}(V\cap\gamma) = \sum_{k=1}^{2(S+1)}\mu_{\rho,t'}(V\cap\gamma\cap\bigcup_{i\in E_k}I_i),\ \text{so}\ \mu_{\rho,t'}(V\cap\gamma\cap\bigcup_{i\in E_{k^*}}I_i) \ge \frac{1}{2(S+1)}\mu_{\rho,t'}(V\cap\gamma)$$

for some k^*, and we define A_γ to consist of one $x_i\in V\cap I_i$ for each $i\in E_{k^*}$, and

$$R_{x_i} := \left\{t\ \middle|\ \varphi^t(x_i)\in V\cap I_i\right\}\subset[-\eta,\eta].$$

If $x \in A :=\bigcup_{\gamma\in\mathbb{P}_\epsilon(t')} A_\gamma$, then $\varphi^{R_x}(x) \subset V$, and $\varphi^s(A)$ is (t',θ)-separated for all s. And

$$\begin{aligned}\mu_{\rho,t'}\big(\bigcup_{x\in A}\varphi^{R_x}(x)\big) &= \frac{1}{P(\Phi,f,t',\rho)}\sum_{x\in A}\lambda(R_x)e^{S_{\pi(x)}f(x)}\\ &= \sum_{\gamma\in\mathbb{P}_\rho(t')}\underbrace{\frac{1}{P(\Phi,f,t',\rho)}\sum_{x\in A_\gamma}\lambda(R_x)e^{S_{\pi(x)}f(x)}}_{=\mu_{\rho,t'}(V\cap\gamma\cap\bigcup_{i\in E_{k^*}} I_i)\geq\frac{1}{2(S+1)}\mu_{\rho,t'}(V\cap\gamma)}\\ &\geq \frac{1}{2(S+1)}\mu_{\rho,t'}(V).\end{aligned}$$

□

Proposition 8.3.16. *The measure* μ_f *in* (8.3.3) *is ergodic (indeed, weakly mixing).*

Proof. As in the proof of Proposition 3.4.16, we derive this from something like the lower bound in that result. Specifically, we show that there is a $c > 0$ with

$$c\mu_f(P)\mu_f(Q) \leq \varliminf_{r\to\infty}\mu_f(P\cap\varphi^{[-r-\beta,-r+\beta]}(Q)) \tag{8.3.5}$$

for all measurable sets $P, Q \subset X$. This implies ergodicity because if $Q = X \smallsetminus P$ is a Φ-invariant set of intermediate measure, then the intersection on the right-hand side is empty, so (8.3.5) fails. We note that as in the proof of Proposition 3.4.16, this argument gives ergodicity of $\Phi\times\Phi$ and hence weak mixing by Proposition 3.4.19.

It suffices to show $\varliminf_{r\to\infty}\mu(\bar{U}_1\cap\varphi^{[-r-\beta,-r+\beta]}(\bar{U}_2)) \geq c\overline{\lim}_{t\to\infty}\mu_{\epsilon,t}(V_1)\mu_{\epsilon,t}(V_2)$ for $\delta > 0$, $V_1, V_2 \subset X$ compact, and δ-neighborhoods U_1, U_2 of V_1, V_2, respectively (Borel regularity).

To do so, let $\alpha, \beta > 0$ and $\eta \in (0, \beta/4)$ such that

$$\sup\{d(x,\varphi^s(x)) \mid x\in X,\ |s|\leq 4\eta\} < \alpha,$$

$\alpha^* < \alpha$ such that $2\alpha^*$ is an expansivity constant for η (Definition 1.7.1), $\epsilon\in(0,\alpha^*/2)$ such that

$$\sup\{d(x,\varphi^s(x)) \mid x\in X,\ |s|\leq 6\epsilon\} < \alpha^*,$$

$c^* > 0$ such that $\sup\{d(\varphi^s(x),\varphi^s(y)) \mid d(x,y)\leq c^*,\ |s|\leq\eta\} < \delta$, and (Proposition 1.7.4) $T > 0$ such that

$$d(\varphi^s(x),\varphi^s(y)) \leq 2\alpha^* \text{ for } |s| < T \Rightarrow d(\varphi^s(x),y) \leq c^* \text{ for some } s\in[-\eta,\eta].$$

From now suppose $t \geq 2T$ and $\tau_1, \tau_2 > 0$ (at the end of the proof these will tend to ∞), and let

$$t_0 := -T,\quad t_1 := t_0 + t + T_\epsilon,\quad t_2 := t_1 + \tau_1 + T_\epsilon,\quad t_3 := t_2 + t + T_\epsilon,\quad t_4 := t_3 + \tau_2 + T_\epsilon,$$

where T_ϵ comes from specification (Definition 8.3.2, Theorem 8.3.4). Let E_i be $(\tau_i, 3\alpha)$-separated sets for $i = 1, 2$, and A_i the sets (and R_x the function) obtained from Lemma 8.3.15 applied to V_i with $\theta = 3\alpha$, $\rho = 3\epsilon$.

For $\mathbf{z} = (z_0, \dots, z_3) \in E := \varphi^{-T}(A_1) \times E_1 \times \varphi^{-T}(A_2) \times E_2$ there is a $w(\mathbf{z}) \in \mathbb{P}_{3\epsilon}(t_4 - t_0)$ with $d(\varphi^{t_i+u+s(t_i+u)}(w), \varphi^u(z_i)) < \epsilon$ for $u \in [0, t_{i+1} - t_i - T_\epsilon]$ (notation as in (8.3.1)).

Claim 8.3.17. *$\varphi^{-T}(\{w(\mathbf{z}) \colon \mathbf{z} \in E\})$ is $(t_4 - T_\epsilon + 3\epsilon, \alpha)$-separated.*

Proof. If $\mathbf{z} \neq \mathbf{z}'$, then $z_k \neq z'_k$ for some k and hence a $u \in [0, t_{k+1} - t_k - T_\epsilon]$ such that (with $t = t_k + u \in [-T, t_4 - T_\epsilon]$)

$$\begin{aligned} 3\alpha &< d(\varphi^u(z_k), \varphi^u(z'_k)) \\ &\le \underbrace{d(\varphi^u(z_k), \varphi^{t+s(t)}(w(\mathbf{z})))}_{\le\epsilon} + d(\varphi^{t+s(t)}(w(\mathbf{z})), \varphi^{t+s(t)}(w(\mathbf{z}'))) \\ &\quad + \underbrace{d(\varphi^{t+s(t)}(w(\mathbf{z}')), \varphi^{t+s'(t)}(w(\mathbf{z}')))}_{\le\alpha^*} + \underbrace{d(\varphi^{t+s'(t)}(w(\mathbf{z}')), \varphi^u(z'_k))}_{\le\epsilon}, \end{aligned}$$

that is, $d(\varphi^{t+s(t)}(w(\mathbf{z})), \varphi^{t+s(t)}(w(\mathbf{z}'))) \ge 3\alpha - \alpha^* - 2\epsilon > \alpha$. □

If $w = w(\mathbf{z})$, $x = \varphi^T(z_0) \in A_1$, then $d(\varphi^p(w), \varphi^p(x)) < \epsilon < 2\alpha^*$ whenever $|p| \le T$ by choice of T, so $d(\varphi^{u(w)}(w), x) \le c^*$ with $|u(w)| \le \eta$.

Claim 8.3.18. *The $\varphi^{u(w)+R_x}(w)$ are pairwise disjoint (for different $w = w(\mathbf{z})$ with $\mathbf{z} \in E$) subsets of $U_1 \cap \varphi^{[-r-\beta, -r+\beta]}(U_2)$, where $r := t + \tau_1 + 2T_\epsilon$.*

Proof. $\varphi^{u(w)+R_x}(w)$ is contained in a δ-neighborhood of $\varphi^{R_x}(x)$ and hence in U_1 since $R_x \subset [-\eta, \eta]$ and by choice of c^*.

If $x_2 := \varphi^T(z_2) \in A_2$, and $|p| \le T$, then $d(\varphi^{p+r+s(p+r)}(w), \varphi^p(x_2)) < \epsilon$, so

$$d(\varphi^{p+r}(w), \varphi^p(x_2)) < \epsilon + \alpha^* < 2\alpha^*$$

by choice of ϵ since $|s(p+r)| \le 3\epsilon$. Thus, $f_r(w) \in f_{[-\eta,\eta]} \underbrace{B_{c^*}(V_2)}_{c^*\text{-neighborhood of } V_2 \subset U_2} \subset \varphi^{[-\eta,\eta]}(U_2)$. Since $u(w) \in [-\eta, \eta]$ and $R_x \subset [-\eta, \eta]$, this implies $\varphi^r(\varphi^{u(w)+R_x}(w)) \subset \varphi^{[-3\eta, 3\eta]}(U_2)$, hence, since $\beta \ge 3\eta$,

$$\varphi^{u(w)+R_x}(w) \subset \varphi^{[-r-\beta, -r+\beta]}(U_2).$$

To see disjointness, suppose $\varphi^{u(w)+R_x}(w) \cap \varphi^{u(w')+R'_x}(w') \neq \varnothing$. Then $w' = \varphi^v(w)$ with $|v| \le 4\eta$, hence $d(\varphi^s(w), \varphi^s(w')) = d(\varphi^s(w), \varphi^s(\varphi^v(w))) < \alpha$ for all s by definition of η, so $w = w'$ since the w are (T, α)-separated. □

This gives the desired lower bound: if $t' \coloneqq t_4 - t_0 = \tau_1 + \tau_2 + 2T + 4T_\epsilon$, then

$$\begin{aligned}
&\mu_{3\epsilon,t'}(U_1 \cap \varphi^{[-r-\beta,-r+\beta]}(U_2)\\
&\quad\geq \frac{1}{P(\Phi,f,t',3\epsilon)} \sum_w \lambda(R_x)\exp(S_{\pi(w)}f(w))\\
&\quad\geq \frac{e^{-4K-4T_\epsilon\|f\|-3\epsilon}}{P(\Phi,f,t',3\epsilon)} N_d(\Phi,f,\tau_1,3\alpha) \underbrace{\sum_{z_2} e^{S_{\pi()}f(z_2)}}_{\geq\frac{1}{2\eta}P(\Phi,f,t,3\alpha)M\mu_{3\alpha,t}(V_2)} N_d(\Phi,f,\tau_2,3\alpha) \underbrace{\sum_{z_0} e^{S_{\pi()}f(z_0)}\lambda(R_x)}_{\geq P(\Phi,f,t,3\alpha)M\mu_{3\alpha,t}(V_1)}\\
&\quad\geq c\mu_{3\alpha,t}(V_1)\mu_{3\alpha,t}(V_2),
\end{aligned}$$

where $M \coloneqq \frac{1}{2(S+1)}$ is as in Lemma 8.3.15. Fixing $t \geq 2T$ and τ_1, let $\tau_2 \to \infty$ (with $t' = t_i$) to get

$$\begin{aligned}
\mu_f(\bar{U}_1 \cap \varphi^{[-r-\beta,-r+\beta]}(\bar{U}_2) &\geq \overline{\lim_{i\to\infty}} \mu_{3\epsilon,t_i}(U_1 \cap \varphi^{[-r-\beta,-r+\beta]}(U_2)\\
&\geq c\mu_{3\alpha,t}(V_1)\mu_{3\alpha,t}(V_2).
\end{aligned}$$

Letting now $\tau_1 \to \infty$ (with $r = s_k$) gives

$$\underline{\lim_{k\to\infty}} \mu_f \bar{U}_1 \cap \varphi^{[-s_k-\beta,-s_k+\beta]}(\bar{U}_2) \geq c\mu_{3\alpha,t}(V_1)\mu_{3\alpha,t}(V_2),$$

which, as $t \to \infty$, becomes

$$\underline{\lim_{r\to\infty}} \mu_f \bar{U}_1 \cap \varphi^{[-r-\beta,-r+\beta]}(\bar{U}_2) \geq c\,\overline{\lim_{t\to\infty}} \mu_{3\alpha,t}(V_1)\mu_{3\alpha,t}(V_2), \qquad \square$$

PROOF OF THEOREM 8.3.6. We show $P_\nu(\Phi,f) = P(\Phi,f) \Rightarrow \nu = \mu$; this implies $P_\mu(\Phi,f) = P(\Phi,f)$ and that there is only one accumulation point. By Proposition 8.3.16, μ in (8.3.3) is ergodic, so it suffices to show $\nu \perp \mu \Rightarrow P_\nu(\Phi,f) < P(\Phi,f)$.

For $t \in \mathbb{R}^+$ and a maximal $(t,2\epsilon)$-separated set E_t take Borel sets β_x such that $B_\Phi(x,\epsilon,t) \subset \beta_x \subset B_\Phi(x,2\epsilon,t)$, $\mathfrak{B}_t \coloneqq \{\beta_x \mid x \in E_t\}$ is a partition, and $(\mu+\nu)(\partial\mathfrak{B}_t) = 0$. Since Φ is expansive, $\operatorname{diam}\varphi^{-t/2}(\mathfrak{B}_t) \xrightarrow[t\to\infty]{} 0$ so if $\varphi^t(B) = B \subset X$ such that $\mu(B) = 0$ and $\nu(B) = 1$ then there exist finite unions C_t of elements of $\mathfrak{B}_t$ such that

$$(\mu+\nu)(C_t \bigtriangleup B) = (\mu+\nu)(\Phi^{-t/2}(C_t) \bigtriangleup B) \xrightarrow[t\to\infty]{} 0.$$

Furthermore, if $\epsilon < \delta_0/2$ then $\mathfrak{B}_t$ is generating for Φ^t, that is, $th_\nu(\Phi) = h_\nu(\varphi^t) = h_\nu(\varphi^t,\mathfrak{B}_t) \leq H_\nu(\mathfrak{B}_t)$. By possibly shrinking ϵ we can ensure that $S_t\varphi \leq K + S_t\varphi(x)$

on $B_f(x,2\epsilon,t)$ and hence on β_x (see Definition 4.3.17). This yields

$$\begin{aligned} tP_\nu(\Phi,f) &\le -\sum_{x\in E_t}\big(\nu(\beta_x)\log(\nu(\beta_x)) + \int_{\beta_x} S_t\varphi\, d\nu\big) \\ &\le K + \sum_{x\in E_t;\beta_x\subset C_t} \nu(\beta_x)(S_t\varphi(x) - \log\nu(\beta_x)) \\ &\qquad + \sum_{x\in E_t;\beta_x\cap C_t=\varnothing} \nu(\beta_x)(S_t\varphi(x) - \log\nu(\beta_x)) \\ &\le K + \nu(C_t)\log \sum_{x\in E_t;\beta_x\subset C_t} e^{S_t\varphi(x)} \\ &\qquad + \nu(X\smallsetminus C_t)\log \sum_{x\in E_t;\beta_x\cap C_t=\varnothing} e^{S_t\varphi(x)} + \frac{2}{e}, \end{aligned}$$

where the last estimate used (11.2.12). We now apply Proposition 8.3.14 to get

$$\begin{aligned} t\big(P_\nu(\Phi,f) - P(\Phi,f)\big) - \frac{2}{e} - K &\le \nu(C_t)\log\Big(\sum_{x\in E_t;\beta_x\subset C_t} e^{S_t\varphi(x)-tP(\Phi,f)}\Big) \\ &\qquad + \nu(X\smallsetminus C_t)\log\Big(\sum_{x\in E_t;\beta_x\cap C_t=\varnothing} e^{S_t\varphi(x)-tP(\Phi,f)}\Big) \\ &\le \nu(C_t)\log(A_\epsilon^{-1}\mu(C_t)) + \nu(X\smallsetminus C_t)\log(A_\epsilon^{-1}\mu(X\smallsetminus C_t)) \\ &\xrightarrow[t\to\infty]{} -\infty, \end{aligned}$$

since $\nu(C_t)\to 1$ and $\mu(C_t)\to 0$. Thus $P_\nu(\Phi,f) < P(\Phi,f)$, hence uniqueness. □

Remark 8.3.19. Theorem 8.3.6 asserts that the equilibrium state is weakly mixing, and stronger mixing properties are suggested by Remark 3.4.4. Indeed, Remark 3.4.4 allows us to establish K-mixing as follows ([**198**, Proposition 1.4]; see also [**221**] and Theorem 8.4.17). If $\Phi' := \Phi$ on $X' := X$, then $\Phi\times\Phi'$ is expansive with specification and hence has a unique equilibrium state for $F(x,x') := f(x) + f(x') \in C^{\Phi\times\Phi'}(X\times X')$. Define a probability measure on $X\times X'$ by extending

$$\bar\mu(A\times B) := \int_A \underbrace{E(\chi_B \mid \pi(\Phi))}_{\text{conditional expectation from Corollary 3.2.11 with respect to the Pinsker partition of }\Phi} d\mu,$$

and check that

$$h_{\bar\mu}(\Phi\times\Phi') = 2h_\mu(\Phi) = h_{\mu\times\mu}(\Phi\times\Phi') \text{ and } \int F\,d\bar\mu = 2\int f\,d\mu = \int F\,d\mu\times\mu',$$

so $\bar\mu = \mu\times\mu$ by uniqueness, hence $\pi(\Phi)$ is trivial, which characterizes K-mixing.

Proposition 4.3.15 about equilibrium states of special flows can be formulated more nicely in the context of the (discrete-time counterpart of) Theorem 8.3.6:

Proposition 8.3.20. *On a compact metric space X, let F be an expansive homeomorphism with specification, $h_{\text{top}}(F) < \infty$, $0 < r \in V(F)$, Φ_r the special flow on X_r. Then*

- *$h_{\text{top}}(\Phi_r)$ is the unique solution of $P(F, -cr) = 0$.*
- *If m is the unique equilibrium state of F for $-h_{\text{top}}(\Phi_r)r$, then m_r is the unique measure of maximal entropy for Φ_r on X_r.*

If $G \in V(\Phi_r)$, and $g(x) := \int_0^{r(x)} G(x,t)\,dt \in V(F)$, then

- *$P(\Phi_r, G)$ is the unique solution of $P(F, g - cr) = 0$.*
- *If m is the unique equilibrium state for $g - cr$, then m_r (from* (3.6.3)*) is the unique equilibrium state of G for Φ_r.*

Uniqueness in Theorem 8.3.6 gives a map $f \mapsto \mu_f$ that associates to a Hölder potential the corresponding equilibrium state. Is it injective? There are different potentials that give rise to the same equilibrium state: Adding a constant to a function f changes $P(f)$ and $P_\mu(f)$ by the same additive constant and thus produces the same equilibrium state, that is, $\mu_{f+c} = \mu_f$ for any constant c. Furthermore, one sees directly from Definition 8.3.10 that two potentials f, g give rise to the same equilibrium state if their statistical sums $S_{\pi(\gamma)}f(x)$ and $S_{\pi(\gamma)}g(x)$ coincide for all periodic orbits, which is the case when f and g are cohomologous (see Proposition 1.3.17, Theorem 4.3.11). However, by the Livshitz Theorem, these are the only ways in which two potentials give rise to the same equilibrium state.

Theorem 8.3.21 (Classification of equilibrium states)**.** *Let Λ be a topologically mixing compact locally maximal hyperbolic set for a flow Φ generated by X and $f, g\colon \Lambda \to \mathbb{R}$ Hölder continuous. Then $\mu_f = \mu_g$ if and only if $g(x) = f(x) + c + Xk$ for some Hölder continuous $k\colon \Lambda \to \mathbb{R}$, that is, if and only if f and g are cohomologous up to a constant.*

Proof. Adjust g by an additive constant such that $P(f) = P(g)$ (Remark 4.3.3). By the Livshitz Theorem 7.2.1 and by symmetry, it suffices to show that $S_t f(x) \le S_t g(x)$ whenever $\varphi^t(x) = x$. The Gibbs property (Proposition 8.3.14) implies that $A_\epsilon^f e^{S_t f(x)} \le B_\epsilon^g e^{S_t g(x)}$ hence $S_t f(x) + \log A_\epsilon^f \le S_t g(x) + \log B_\epsilon^g$ whenever $\varphi^t(x) = x$, so $S_t f(x) = \lim_{n\to\infty} S_{nt} f(x)/n \le \lim_{n\to\infty} S_{nt} g(x)/n = S_t g(x)$ since $\varphi^{nt}(x) = x$. □

Let us add another observation about equilibrium states for Anosov flows—equilibrium states vary continuously with the Anosov flow.

Theorem 8.3.22 (Weak continuity of equilibrium states)**.** *On a manifold M consider C^1 functions f_n and Anosov flows Φ_n such that $\Phi_n \xrightarrow[n\to\infty]{C^1} \Phi$ and $f_n \xrightarrow[n\to\infty]{C^1} f$. Then $\mu_n \xrightarrow[n\to\infty]{\text{weakly}} \mu$, where the μ_n are the equilibrium states for f_n of Φ_n and μ is the equilibrium state for f of Φ.*

Proof. If $\mu^* = \lim_{k\to\infty} \mu_{n_k}$ is any weak accumulation point of the μ_n, then

$$P(\Phi, f) \overset{(4.3.4)}{\geq} h_{\mu^*}(\Phi) + \int f\, d\mu^* \overset{\substack{\text{Remark}\\ 4.3.16}}{\geqq} \varlimsup_{k\to\infty} \underbrace{h_{\mu_{n_k}}(\Phi_{n_k}) + \int f_{n_k}\, d\mu_{n_k}}_{=P(\Phi_{n_k}, f_{n_k})} \overset{\substack{\text{Structural}\\ \text{Stability}}}{=\!=\!=} P(\Phi, f),$$

so μ^* is an equilibrium state and by uniqueness the weak limit of the μ_n. □

Although this uniqueness proof seems like a long slog through estimates, the core approach of using expansivity and specification is quite elegant—while also covering the smooth and symbolic situations at the same time. Expansivity, and even more so specification, are closely connected to the uniformity of our hyperbolicity assumption, and this approach fell somewhat to the wayside in research beyond uniform hyperbolicity. A 1980 paper by Katok [**178**] is a striking exception which combines the Bowen approach with Pesin theory to great effect. The results were indeed remarkable (such as, positive entropy in any 2-dimensional diffeomorphism is entirely attributable to horseshoes) and this paper is still cited frequently.[11] Yet, it remained singular, and it focused on shadowing rather than specification. Only much more recently a renaissance of Bowen's approach began with a collaboration of Climenhaga and Thompson, whose feat of imagination and technical mastery revealed that Bowen's approach, suitably adapted and combined with other advances since, turns out to provide a powerful machinery ready for application in nonuniformly hyperbolic dynamics [**87**]. The underlying idea is to decompose every orbit segment into "good" and ”bad” parts, where the "good" parts satisfy Bowen's conditions, and the "bad" parts carry smaller topological pressure than the whole flow.

To illustrate, consider the geodesic flow on a compact (rank-1) nonpositively curved Riemannian manifold. The rank-1 condition on the Riemannian metric is that there is a geodesic for which the tangent vector field is the only parallel Jacobi field, and the (open dense invariant) regular set is the set of tangent vectors to such geodesics; the singular set is its complement (and empty only in the well-understood case of negative curvature). This kind of geodesic flow is the original exemplar which motivated the development of the theory of nonuniform hyperbolicity because here one obtains (in a nonuniform way) complete hyperbolicity. 20 years prior, Gerhard Knieper had been able to establish in this context that there is a unique measure of maximal entropy [**185**]; this was a huge breakthough and only possible at the time because Knieper was able to fully utilize all the structure provided by the geometry of this flow. Now, a mere byproduct of the applications of the Climenhaga–Thompson technique is a direct and purely dynamical proof of

[11] Indeed, Katok was prouder of this paper than any other.

this fact. To single out a specific issue, it is a corollary of Knieper's uniqueness proof that the singular set carries less topological entropy than the regular set—but this does not produce a direct constructive proof of the entropy gap. The Climenhaga–Thompson machinery produces such an argument: approximate singular orbit segments by regular orbit segments having the specification property, and use these to build a collection of orbits with greater topological entropy than the singular set; this reproves the result by Knieper using dynamical methods [**70**]. More generally, they also obtain uniqueness of equilibrium states in this context.

Theorem 8.3.23 (Burns–Climenhaga–Fisher–Thompson [**70**])**.** *The geodesic flow of a compact rank-1 nonpositively curved Riemannian manifold has a unique equilibrium state for any Hölder continuous potential with a* pressure gap*, i.e., if the singular set does not carry full pressure, meaning that the restriction of the geodesic flow to it has smaller pressure than on the complement or, equivalently, in total. (This is automatic for potentials that are locally constant on a neighborhood of the singular set.) The unique equilibrium state is hyperbolic, fully supported, and is the weak* limit of weighted regular closed geodesics.*

The pressure gap is essential: without it, there is at least one equilibrium state supported on the singular set. On the other hand, only equilibrium states supported on the regular set are hyperbolic.

4. Sinai–Ruelle–Bowen measures

We have studied invariant measures in some generality, with particular attention to equilibrium states. These were motivated by the measure of maximal entropy on one hand, which we study further in the next section. On the other hand, in the case of smooth flows it is natural to be interested in smooth invariant measures. These 2 measures are special when invariant measures abound. However, not all smooth dynamical systems preserve a smooth measure, and we now introduce an equilibrium state that is always present and is smooth when there is a smooth invariant measure and of unique interest for attracting hyperbolic sets (Remark 8.4.8). It is thus the natural generalization of a smooth invariant measure. This realizes one aim in developing the theory of equilibrium states to the extent that we have done so.

Definition 8.4.1. If Λ is a compact hyperbolic attractor for a flow Φ, then the *Sinai–Ruelle–Bowen measure* or *SRB-measure* for Φ (on Λ) is the equilibrium state $\mu_{\mathrm{SRB}} := \mu_J$ for the *geometric potential*

$$J := J^u := -\frac{d \log j_t}{dt}\big|_{t=0} = -\frac{d j_t}{dt}\big|_{t=0},$$

where $j_t(x)$ is the Jacobian of $d\varphi^t\colon E^u(x) \to E^u(\varphi^t(x))$ with respect to the volume defined by the Riemannian metric.

Remark 8.4.2. J is Hölder-continuous since E^u is. Since $j_{s+t}(x) = j_t(\varphi^s(x))\,j_s(x)$, we have $-\log j_t(\varphi^s(x)) = -\log j_{s+t}(x) + j_s(x)$, hence $J(\varphi^s(x)) = -\frac{d\log j_\tau(x)}{d\tau}\big|_{\tau=t}$, and $\int_0^t J(\varphi^t(x))\,dt = -\log j_t(x)$.

We assume that the Riemannian metric is adapted to Φ (Proposition 5.1.5).

This particular measure is of great interest with respect to the study of attractors[12], and the main theorem (Theorem 8.4.7) is that *Lebesgue*-a.e. point near an attractor will be a typical point for its SRB measure. This does, in effect, say that the toner density produced by printing a computed orbit starting near an attractor will reflect the density of the SRB measure. For an Anosov flow, the entire manifold is a global attractor, so any smooth invariant measure is, in fact, the SRB measure (Theorem 8.4.10) and hence unique and (provided the flow is topologically mixing) mixing by Theorem 8.3.6 (actually, K-mixing by Remark 8.3.19, and we explain at the end that one indeed obtains the Bernoulli property (Theorem 8.4.17)). Its entropy then is the average infinitesimal unstable volume-expansion rate (the Pesin entropy formula, Remark 8.4.11). If no smooth measure is invariant, then the flow itself dissipates a volume to a forward and a backward SRB measure (Theorem 8.4.14). In the midst of this we will also demonstrate the utility of this measure with respect to questions that appear to have no connection to statistical methods at all (Theorem 8.4.13).

Proposition 8.4.3 (Volume Lemma)**.** *If Λ is a compact locally maximal hyperbolic set for a C^2 flow Φ and $\epsilon > 0$ is sufficiently small then there exist $C_\epsilon, D_\epsilon > 0$ such that*

$$D_\epsilon \le j_t(x)\lambda(B_\Phi(x,\epsilon,t)) \le C_\epsilon \tag{8.4.1}$$

for all $t \ge 0$, where λ denotes the Riemannian volume.[13]

Proof. Let $m = \dim M$. We first replace the balls $B_\Phi(x,\epsilon,t)$ by sets that are easier to handle. On a neighborhood V_x of each $x \in M$ introduce adapted coordinates $\psi_x\colon \underbrace{B_\epsilon(0)}_{\subset T_xM} \to M$ in which $\psi(\underbrace{E^u(x)}_{\sim\mathbb{R}^k\times\{0\}}) \subset W^u(x)$, $\psi(\underbrace{E^{cs}(x)}_{\sim\{0\}\times\mathbb{R}^{m-k}}) \subset W^{cs}(x)$, and $\varphi_x \coloneqq \psi^{-1}_{\varphi^1(x)} \circ \varphi^1 \circ \psi_x$ is tangent to $D_x\varphi^1$ at 0. If $y \in V_x$ is parameterized by $(y_1, y_2) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ then there are constants $c_1, c_2 > 0$ independent of x and y with

$$c_1 \max(\|y_1\|, \|y_2\|) \le d(x,y) \le c_2 \max(\|y_1\|, \|y_2\|).$$

[12] to which Theorems 5.3.27 and 5.3.25 are pertinent

[13] Compare (8.4.1) to the Gibbs property (Definition 4.3.22).

Let

$$B_\epsilon(x) := \{ y \in V_x \mid V_{\varphi^\tau(x)} \ni \varphi^\tau(y) = (y_1^{(\tau)}, y_2^{(\tau)}),\ \max(\| y_1^{(\tau)} \|, \| y_2^{(\tau)} \|) \le \epsilon \text{ for } 0 \le \tau \le t \}.$$

Then $B_{\epsilon/c_2}(x) \subset B_\Phi(x, \epsilon, n) \subset B_{\epsilon/c_1}(x)$ and in order to prove (8.4.1) it suffices to prove a like estimate for $\lambda(B_\epsilon(x))$. Furthermore the measure λ in V_x is given by a density with bounded logarithm. Thus instead of estimating $\lambda(B_\epsilon(x))$ we estimate the volume $\mathrm{vol}^m(B_\epsilon(x))$ of $B_\epsilon(x)$ in adapted coordinates. Similarly the Jacobian, which is taken with respect to a fixed Riemannian metric, can be replaced by the Jacobian of φ^{-t} from adapted local coordinates near $\varphi^t(x)$ to those near x.

$B_\epsilon(x)$ contains a piece S of the stable manifold $W_\epsilon^s(x)$ of size ϵ in adapted coordinates, whose $(m-k)$-dimensional volume then is of order ϵ^{m-k}. In local coordinates fix $y = (0, y_2) \in S$, and let $U_{y_2} := \{ z \in B_\epsilon(x) \mid z = (y_1, y_2) \}$, that is, the horizontal "slice" of $B_\epsilon(x)$ through y. Then

$$\mathrm{vol}^m(B_\epsilon(x)) = \int_S \mathrm{vol}^k(U_{y_2})\, dy_2. \tag{8.4.2}$$

To estimate the k-dimensional volume of the sets U_{y_2} note that for small enough ϵ the tangent space $T_z U_{y_2}$ is close to the unstable subspace E_z^u and in particular inside an invariant cone family around E^u. This implies that if in local coordinates around $\varphi^\tau(x)$ we write $\varphi^\tau(z) = (z_1^{(\tau)}, z_2^{(\tau)})$, then $\| z_1^{(\tau)} \| \ge \mu^r \| z_1^{(\tau - r)} \|$ for some $\mu > 1$ whenever $0 \le r \le \tau$. Hence $\| z_1^{(\tau)} \|$ is maximal for $\tau = t$, and the image $\varphi^t(U_{y_2})$ is the graph of a Lipschitz function in the adapted coordinates around $\varphi^t(x)$ defined over the whole ϵ-ball around the origin. Thus $\mathrm{vol}^k(\varphi^t(U_{y_2}))$ is of order ϵ^k. Now if ω is the k-dimensional volume element on $\varphi^t(U_{y_2})$ then

$$\mathrm{vol}^k(U_{y_2}) = \int_{\varphi^t(U_{y_2})} \tilde{j}_{-t}\, d\omega,$$

where $\tilde{j}_t$ is the Jacobian of $d\varphi^t \colon TU_{y_2} \to TU_{z_2^{(\tau)}}$.

Together with (8.4.2) this means that Proposition 8.4.3 follows from Proposition 8.3.1: at $w := \varphi^t(z) \in \varphi^t(U_{y_2})$ the ratio of the Jacobian of φ^{-t} on $T\varphi^t(U_{y_2})$ and the Jacobian of φ^{-t} on $E^u_{\varphi^t(x)}$ (which is $j_{-t}(\varphi^t(x))$) is bounded from above and below by positive constants. □

Proposition 8.4.4. *If Λ is a compact locally maximal hyperbolic set for a C^2 flow Φ, $\epsilon > 0$ sufficiently small and $B_\Phi(\epsilon, t) := \bigcup_{x \in \Lambda} B_\Phi(x, \epsilon, t)$, then*

$$P(J) = \lim_{t \to \infty} \frac{1}{t} \log \lambda(B_\Phi(\epsilon, t)).$$

Proof. Fix $\delta \le \epsilon$ and a maximal (t, δ)-separated set $E \subset \Lambda$ (see Section 4.2). If $x \in \Lambda$ then $x \in B_\Phi(y, \delta, n)$ for some $y \in E$ by maximality, and hence

$$B_\Phi(y, \delta/2, t) \subset B_\Phi(x, \epsilon, t) \subset B_\Phi(y, \delta + \epsilon, t).$$

Since the $B_\Phi(y,\delta/2,t)$ are pairwise disjoint, Proposition 8.4.3 yields

$$D_{\delta/2}\sum_{y\in E} j_{-t}(\varphi^t(x)) \le \lambda(B(\epsilon,n)) \le C_{\delta+\epsilon}\sum_{y\in E} j_{-t}(\varphi^t(x)).$$

Thus, $P(\varphi) = \lim_{n\to\infty}\frac{1}{n}\log\lambda(B(\epsilon,n))$ by (4.3.2). □

Corollary 8.4.5. *Under the hypotheses of Proposition 8.4.4 we have* $P(J) \le 0$*, and if* $\lambda(W^s_\epsilon(\Lambda)) > 0$ *then* $P(J) = 0$*, so* $h_{\mu_{SRB}}(\Phi) = -\int J\,d\mu_{SRB}$*.*

Proof. $\lambda(B(\epsilon,n)) \le \lambda(M) < \infty \Rightarrow P(J) = \lim_{t\to\infty}\frac{1}{t}\log\lambda(B(\epsilon,t)) \le 0$, while

$$0 < \lambda(W^s_\epsilon(\Lambda)) \le \lambda(B(\epsilon,t)) \Rightarrow 0 = \lim_{t\to\infty}\frac{1}{t}\log\lambda(B(\epsilon,t)) = P(J) = h_{\mu_{\mathrm{SRB}}}(\Phi) + \int J\,d\mu_{\mathrm{SRB}}. \ \square$$

Remark 8.4.6 (Pesin entropy formula)**.** The second case of Corollary 8.4.5 is worth emphasizing because it says something of interest about volume entropy: the measure-theoretic entropy of a volume-preserving hyperbolic flow is the average volume-expansion rate in the unstable subbundle. Indeed, since equilibrium states (hence, in this case, volume, for which we could also invoke the Hopf argument) are ergodic, the word "average" can be omitted or replaced by "almost-everywhere value of." This fact turns out to hold in rather greater generality and is known as the *Pesin entropy formula* (Remark 8.4.11), which in turn has been further extended by Ledrappier and Young [**193**–**195**].

We now come to the main theorem of this section.

Theorem 8.4.7 (Sinai–Ruelle–Bowen measure)**.** *Suppose* Λ *is a compact hyperbolic attractor of a* C^2 *flow* Φ *on* M *and* $f\colon M\to\mathbb{R}$ *is continuous. Then*

$$\lim_{T\to\infty}\int_0^T f(\varphi^t(x)) = \int f\,d\mu_{SRB}$$

for λ*-almost all* $x\in W^s(\Lambda)$*.*

Remark 8.4.8. The conclusion of this result looks much like that of the Birkhoff Ergodic Theorem 3.2.16. Indeed, for μ_{SRB}-almost every point this is a restatement of the Birkhoff Ergodic Theorem. The crucial difference is that, while μ_{SRB} is supported on the attractor, the conclusion here holds for almost every point with respect to the *Riemannian measure* on a neighborhood of the attractor. This means that a point picked "at random" (with respect to the *Riemannian volume*) in a neighborhood of the attractor will be generic for the *Sinai–Ruelle–Bowen measure*, that is, its distribution will represent the Sinai–Ruelle–Bowen measure exactly. In short, Lebesgue-a.e. point is SRB-generic, or, for Lebesgue-a.e. point x the empirical measure (Remark 3.2.22) converges weakly to the SRB-measure. This is why physicists call a measure with this property an *observable* or *physical measure*.

Indeed, with some effort along the lines of what we have already done, one can show that for a hyperbolic flow, λ-almost all points approach some attractor. Accordingly, Theorem 8.4.7 tells us that in this case almost every point on the ambient manifold will reflect an SRB-measure in its asymptotics.

We should note as well, that the mere existence of Birkhoff averages is a nontrivial part of this result, as noted already in Remark 3.2.26. To emphasize, we have shown that for a hyperbolic flow Lebesgue-a.e. point has a Birkhoff average (or, in the terminology of Remark 3.2.26, almost no point is historic).

Proof. We restate the result as follows. For some $\delta > 0$, we let

$$\begin{aligned}
f(T,x) &:= \frac{1}{T}\int_0^T f(\varphi^t(x))\,dt \quad \text{and} \quad \bar{f} := \int f\,d\mu_{\text{SRB}} \\
C_T(f,\delta) &:= \{x \in M \mid |f(T,x) - \bar{f}| > \delta\} \\
C(f,\delta) &:= \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} C_n(f,\delta) \\
&= \{x \in M \mid |f(n,x) - \bar{f}| > \delta \text{ for infinitely many } n \in \mathbb{N}\},
\end{aligned}$$

and aim to show $\lambda(C(f,\delta)\cap W^s(\Lambda)) \to 0$ for all continuous f.

Take $\epsilon > 0$ such that $|f(x) - f(y)| < \delta$ whenever $d(x,y) < \epsilon$, and fix $N \in \mathbb{N}$. We construct $(n,2\epsilon)$-separated sets recursively in a way that keeps "old" points when passing to larger n: For $n \geq N$ let $R_n \subset \Lambda \cap C_n(f,2\delta)$ be a maximal set for which

(1) $B(x,\epsilon,n)\cap B(y,\epsilon,n) = \varnothing$ if $x \in R_n$, $y \in R_k$ and $N \leq k < n$,
(2) $B(x,\epsilon,n)\cap B(y,\epsilon,n) = \varnothing$ if $x,y \in R_n$ and $x \neq y$.

Set $V_{N,\epsilon} := \bigcup_{k=N}^{\infty}\bigcup_{x\in R_k} B(x,\epsilon,k)$.

Claim 8.4.9. $\bigcup_{n=N}^{\infty} C_n(f,3\delta)\cap W^s_\epsilon(\Lambda) \subset V_{N,2\epsilon}$

Proof. If $x \in C_n(f,3\delta)\cap W^s_\epsilon(\Lambda)$ for some $n \geq N$ then there is a $z \in \Lambda$ such that $x \in W^s_\epsilon(z)$ and hence $z \in C_n(f,2\delta)$ by choice of ϵ. Since R_n is maximal, there is a $y \in R_k$ for some k between N and n such that $B(z,\epsilon,n)\cap B(y,\epsilon,k) \neq \varnothing$ and hence $x \in B(x,\epsilon,n) \subset B(z,2\epsilon,k)$. □

This claim and Proposition 8.4.3 imply that

$$\lambda\Big(\bigcup_{n=N}^{\infty} C_n(f,3\delta)\cap W^s_\epsilon(\Lambda)\Big) \leq C_{2\epsilon}\sum_{k=N}^{\infty}\sum_{x\in R_k} j_{-t}(\varphi^t(x)) \leq C_{2\epsilon}\mu_{\text{SRB}}(V_{N,\epsilon})/A_\epsilon \tag{8.4.3}$$

using Corollary 8.4.5, Proposition 8.3.14 and that $V_{N,\epsilon}$ is defined by a disjoint union (by maximality of R_n).

If $x \in R_k \subset C_k(f,2\delta)$ then $B(x,\epsilon,k) \subset C_k(f,\delta)$ and hence $V_{N,\epsilon} \subset \bigcup_{k=N}^{\infty} C_k(f,\delta)$. Ergodicity of μ_{SRB} (Theorem 8.3.6) and the Birkhoff Ergodic Theorem then imply

$$0 = \mu_{\text{SRB}}(C(f,\delta)) = \lim_{N\to\infty} \mu_{\text{SRB}}(\bigcup_{n=N}^{\infty} C_n(f,\delta)) \geq \lim_{N\to\infty} \mu_{\text{SRB}}(V_{N,\epsilon}).$$

(8.4.3) then gives $\lambda(\bigcup_{n=N}^{\infty} C_n(f,3\delta) \cap W_\epsilon^s(\Lambda)) \xrightarrow[N\to\infty]{} 0$, so $\lambda(C(f,3\delta) \cap W_\epsilon^s(\Lambda)) = 0$.

To replace $\lambda(W_\epsilon^s(\Lambda))$ by $\lambda(W^s(\Lambda))$ note that if $\delta' \leq 3\delta$ then

$$C(f,\delta') \cap \varphi^{-n}(W_\epsilon^s(\Lambda)) \subset \varphi^{-n}(C(f,3\delta) \cap W_\epsilon^s(\Lambda)),$$

which is a set of Riemannian measure 0 since φ^n is a diffeomorphism and hence maps λ-null set to λ-null sets. Definition 1.5.5 then implies that

$$\lambda(C(f,\delta') \cap W_\epsilon^s(\Lambda)) \leq \sum_{n=0}^{\infty} \lambda(C(f,\delta') \cap \varphi^{-n}(W_\epsilon^s(\Lambda))) = 0.$$

This is the desired result. Indeed, since δ' is arbitrary, this implies that

$$\lim_{n\to\infty} f(n,x) = \bar{f} \text{ for all } x \in W^s(\Lambda) \text{ outside a } \lambda\text{-null set } N(f).$$

If $f \in C(M) = \overline{\{f_n\}_{n\in\mathbb{N}}}$ then $\lim_{n\to\infty} f(n,\cdot) = \bar{f}$ on $W^s(\Lambda) \smallsetminus \bigcup_{n\in\mathbb{N}} N(f_n)$. □

We now describe the ergodic theory of smooth invariant measures for Anosov flows by using the results about the Sinai–Ruelle–Bowen measure.

Theorem 8.4.10. *Let M be a compact connected smooth Riemannian manifold and Φ a mixing Anosov flow. Then Φ has at most one smooth invariant measure: If ν is a smooth invariant measure then ν is the Sinai–Ruelle–Bowen measure for Φ (hence is mixing by Theorem 8.3.6), and $h_\nu(\Phi) = -\int \log J\, d\nu$ (the* Pesin entropy formula*).*

Remark 8.4.11 (Pesin Entropy Formula)**.** The minus sign in front of the integral that gives entropy combines with that in the definition of J, so the entropy of the SRB measure, in particular of an invariant volume, is given by the average infinitesimal unstable volume-expansion rate. Example 4.1.11 illustrates this.

Proof. If Φ has a smooth invariant measure ν then the Poincaré Recurrence Theorem 3.2.1 implies that ν-a.e. point is nonwandering. Since $NW(\Phi)$ is closed and ν is positive on open sets, this means that $NW(\Phi) = M$.

If $f\colon M \to \mathbb{R}$ is continuous, the Birkhoff Ergodic Theorem 3.2.16 gives

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f\circ\varphi^t \overset{\nu\text{-a.e.}}{=\!=\!=} f_{\mathscr{I}_\Phi},$$

while Theorem 8.4.7 implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f\circ\varphi^t \overset{\lambda\text{-a.e.}}{=\!=\!=} \int f\, d\mu_{\text{SRB}}.$$

Since $\nu \ll \lambda$, this latter equality holds ν-a.e., so $f_{\mathscr{I}_\Phi} \overset{\nu\text{-a.e.}}{=\!=\!=} \int f\, d\mu_{\text{SRB}}$, and hence

$$\int f\, d\nu = \int f_{\mathscr{I}_\Phi}\, d\nu = \int f\, d\mu_{\text{SRB}}.$$

That this holds for all continuous φ means that $\nu = \mu_{\text{SRB}}$.

The entropy is given by Corollary 8.4.5. □

Theorem 8.4.12 ([**205**, Corollary 4.4])**.** *For a C^k Anosov 3-flow the densities of the SRB-measure in each unstable (or stable) leaf are C^k.*

Proof. The densities are defined up to a multiplicative constant and if $y \in W^u(x)$ then one can show that

$$\frac{\rho(y)}{\rho(x)} = \lim_{n\to\infty} \frac{J^u f^{-n}(x)}{J^u f^{-n}(y)},$$

and the latter expression is in fact C^k along the leaves, and all derivatives are continuous. A similar argument holds for the stable leaves. □

We now study Anosov flows that preserve a contact structure (see Section 6.2). We consider the lowest-dimensional situation (flows on 3-manifolds), and the role of the linear model is played by geodesic flows on factors of the hyperbolic plane.

Theorem 8.4.13. *Orbit-equivalent contact Anosov flows on a three-dimensional manifold are C^1 conjugate if the periods of corresponding periodic orbits coincide.*

Proof. The flows are Hölder conjugate by Theorem 7.2.8. The conjugacy preserves the strong stable and unstable foliations and the orbits. The weak stable and unstable foliations are C^1 by Corollary 7.4.15. The strong stable and strong unstable subbundles are the intersections of the weak stable and weak unstable subbundles with the kernel of the contact form by Lemma 9.1.3. Since the contact form is C^1 this implies that the strong foliations are C^1.

By Theorem 8.4.10 the equilibrium state for the unstable Jacobian is the invariant volume induced by the contact form. This implies that the conjugacy preserves volume, and since it preserves the three one-dimensional foliations (orbits and the strong foliations), which are C^1, it preserves the conditional measures on those foliations and hence it is C^1. □

Even for flows without a smooth invariant measure, the SRB measure provides information about the way in which volume fails to be invariant: it dissipates towards the SRB measure:

Theorem 8.4.14. *Suppose Λ is a mixing attractor for a C^2 flow Φ and ν is a probability measure with support in $W^s(\Lambda)$. Then $\varphi^{t*}(\nu) \xrightarrow[t\to\infty]{\text{weak*}} \mu_{SRB}$, that is, $\int f \circ \varphi^t\, d\nu \xrightarrow[t\to\infty]{} \int f\, d\mu_{SRB}$ for continuous $f\colon M \to \mathbb{R}$.*

Proof. Approximation arguments show that we can assume without loss of generality that $\nu = \rho\lambda$ for a bounded density $\rho \geq 0$ and that the support of μ is in an open set whose closure is in $W^s(\Lambda)$. Then for a given $\epsilon > 0$ there is a τ such that if $t \geq \tau$, then $\operatorname{supp} \varphi^{t^*}(\nu)$ is in an ϵ-neighborhood of Λ, in particular, given a maximal (T,ϵ)-separated set $E \subset \Lambda$ for $\Phi_{\restriction_\Lambda}$, $\operatorname{supp} \varphi^{t^*}(\nu) \subset \bigcup_{x\in E} B(x,2\epsilon,T)$. Now the probability measure

$$\nu_{\epsilon,T} \coloneqq \sum_{x\in E} \Big(\frac{\int \psi_x \, d(\varphi^{t(\epsilon)^*}(\nu))}{\int \chi_{B(x,\epsilon,T)} \, d\mu}\Big)\chi_{B(x,\epsilon,T)}\mu_{\text{SRB}} \ll \mu_{\text{SRB}},$$

where $\{\psi_x\}_{x\in E}$ is a measurable partition of unity on $\operatorname{supp} \varphi^{t^*}(\nu)$ subordinate to $\{B(x,2\epsilon,T)\}_{x\in E}$, can be written as $\nu_{\epsilon,T} = \rho_{\epsilon,T}\mu_{\text{SRB}}$ with $\|\rho_{\epsilon,T}\|_\infty$ bounded independently of T because

$$\frac{\int \psi_x \, d(\varphi^{t(\epsilon)^*}(\nu))}{\int \chi_{B(x,\epsilon,T)} \, d\mu} \leq \|r_{t(\epsilon)}\|_\infty \underbrace{\frac{\lambda(B(x,2\epsilon,T))}{\mu_{\text{SRB}}(B(x,\epsilon,T))}}_{\text{bounded by Proposition 8.3.14, Proposition 8.4.4, Proposition 8.4.3}},$$

if $\varphi^{t(\epsilon)^*}(\nu) = r_{t(\epsilon)}\lambda$. This approximates the effect of the flow on ν in the following sense. $\nu_{\epsilon,T}$ redistributes the weight of $\varphi^{t(\epsilon)^*}(\nu)$ in such a way that everything that goes into $B(x,\epsilon,T)$ comes from $B(x,2\epsilon,T)$. Likewise, $\varphi^{t^*}(\nu_{\epsilon,T})$ redistributes the weight of $\varphi^{t+t(\epsilon)^*}(\nu)$ such that what goes into $\varphi^t(B(x,\epsilon,T))$ comes from $\varphi^t(B(x,2\epsilon,T))$, and for $t \in [0,T]$ the diameter of $\varphi^t(B(x,2\epsilon,T))$ is at most 4ϵ. Thus, for a given closed weak neighborhood N of 0 we can choose ϵ such that

$$\varphi^{t+t(\epsilon)^*}(\nu) - \varphi^{t^*}(\nu_{\epsilon,T}) \in N \quad \text{whenever} \quad t \in [0,T].$$

Since $\|\rho_{\epsilon,T}\|_\infty$ is bounded independently of T, we can find $T_n \to \infty$ such that $\rho_{\epsilon,T} \to \rho_\epsilon$ weakly in L^∞ (as the dual of L^1), so

$$\varphi^{t+t(\epsilon)^*}(\nu) - \varphi^{t^*}(\rho_\epsilon\mu_{\text{SRB}}) \in N \quad \text{whenever} \quad t \geq 0.$$

On the other hand, μ_{SRB} is mixing, so $\rho_\epsilon \circ \varphi^t \xrightarrow[t\to\infty]{\text{weakly}} \int \rho_e \, d\mu_{\text{SRB}} = 1$ by Proposition 3.4.28, that is, there is a $t_N \in \mathbb{R}$ such that $t \geq t_N \Rightarrow \varphi^{t^*}(\rho_\epsilon\mu_{\text{SRB}}) - \mu_{\text{SRB}} \in N$, hence

$$\varphi^{t^*}(\nu) - \mu_{\text{SRB}} \in 2N \quad \text{for} \quad t \geq t(\epsilon) + t_N. \qquad \square$$

We add another result about Sinai–Ruelle–Bowen measure without proof.

Theorem 8.4.15 ([**58**, Proposition 5.4, Theorem 5.6])**.** *The SRB measure of a basic set Λ depends continuously (in the weak topology) on the C^2 flow (in the C^1 topology),*[14] *as do its entropy and the pressure of the unstable Jacobian, and the following are equivalent:*

- *Λ is an attractor,*

[14] The dependence is C^1 for C^3 Axiom-A flows when viewed as a functional on C^2 functions [**262**].

- $\mu_{SRB}(W^s_\Lambda) > 0$,
- $P(\Phi\restriction_\Lambda, J^u) = 0$.

Corollary 8.4.16. *If Φ is a C^2 hyperbolic flow on a compact manifold M, then the closures of the basins of the attractors cover M, and if Λ is a basic set with $\mu_{SRB}(\Lambda) > 0$, then Λ is a connected component of M, and $\Phi\restriction_\Lambda$ is an Anosov flow.*

PROOF. Hyperbolicity implies that $M = \bigcup\{W^s(\Lambda) \mid \Lambda \text{ is a basic set}\}$, so the complement of the basins of the attractors is $\bigcup\{W^s(\Lambda) \mid \Lambda \text{ is not an attractor}\}$, which is a null set and hence has empty interior.

$\mu_{\mathrm{SRB}}(\Lambda) > 0$ implies $\mu_{\mathrm{SRB}}(W^s(\Lambda)) > 0$, so Λ is an attractor, hence $W^u(\Lambda) = \Lambda$ (Theorem 5.3.27). Likewise, $\mu_{\mathrm{SRB}}(W^u(\Lambda)) > 0$, so Λ is a repeller, hence $W^u(\Lambda) = \Lambda$ is open; being also closed and connected (since $W^{cs}(p)$ is dense for periodic $p \in \Lambda$), it is a connected component. □

Finally, we explain how one obtains the Bernoulli property from K mixing and hyperbolicity [**85**, **221**].

Theorem 8.4.17. *The Sinai–Ruelle–Bowen measure is Bernoulli. In particular, volume-preserving Anosov flows*[15] *are Bernoulli.*

PROOF OUTLINE [**221**]. By Theorem 3.4.47 we need to check the very weak Bernoulli property from Definition 3.4.45. Let ξ be a finite generating (or very fine) partition. We will show that it has the desired property by an analysis of suitable small local-product neighborhoods. Consider a local-product neighborhood of the type unstable×center stable on which the SRB measure μ_{SRB} is equivalent to a product measure $\bar{\mu}$ (Remark 8.1.22) with bounded Radon–Nikodym derivative r. Being measurable, r is close to a step function over measurable rectangles, so for $\epsilon > 0$ there is a smaller local-product neighborhood $P \sim P^u \times P^{cs}$ and an $r_0 > 0$ such that

$$B := \{x \in P \mid |r_0 - r(x)| \geq \epsilon^2\}$$

satisfies $(\mu_{\mathrm{SRB}})_P(B) \leq \epsilon^2$ and $\bar{\mu}_P(B) \leq \epsilon^2$; in a possibly smaller B we can further assume that most center-stable fibers have almost the same conditional measure. This means that there are ϵ-measure-preserving maps $\theta_z \colon P^u \times \{z\} \times [0,1] \to P$, $(p, z, x) \mapsto (p, \vartheta(x))$ along weak-stable leaves, where $\vartheta \colon [0,1] \to P^{cs}$ (with normalized measure) is a measurable isomorphism.

We will use these θ_z to construct the map θ required by Definition 3.4.45. In anticipation, we note that if θ_z were to slide along *strong* stable leaves, we would be assured that future itineraries match as required by Definition 3.4.45, but the construction of θ_z ensures only that the forward orbits of a point and its image stay

[15] And, more generally, equilibrium states for Anosov flows [**255**]

close rather than being positively asymptotic. This is sufficient for our purposes: take P to be thin enough in the flow direction (depending on the chosen partition ξ) such that the set of times where future itineraries mismatch has density less than ϵ (by ergodicity).

We now map P by iterates of $f := \varphi^t$ to get local product neighborhoods $P_n = f^n(P)$ with induced ϵ-measure-preserving maps θ_n from almost every unstable leaf in P_n onto P_n. By ergodicity, there is an $N \in \mathbb{N}$ such that each point is contained in about equally many P_n with $1 \le n \le N$, and by K-mixing there is an $M \in \mathbb{N}$ such that if $m \ge M$, then ϵ-a.e. atom E of $\bigvee_{j=-m}^{-M} f^{-j}(\xi)$ intersects each P_n in a set of $(\mu_{\text{SRB}})_E$-measure close to $\mu_{\text{SRB}}(P)$ and further, $\sum \chi_{P_n}$ is ϵ-close to $N\mu(P)$ ϵ-a.e. on E. Then $X \times [0,1]$ is ϵ-approximated by $\bigcup_{n=0}^{N} P_n \times J_n$, where the $J_n \subset [0,1]$ are disjoint intervals of length $1/N\mu(P)$, and the restriction of the various θ_n to $E \times [0,1]$ gives, modulo these approximations, the required map in Definition 3.4.45. □

Remark 8.4.18. This indicates that the Bernoulli property is central to the smooth ergodic of hyperbolic flows, and this conclusion is quite broadly true, that is, even when the hyperbolicity is nonuniform or the system has singularities [**221**, §1]. More remarkably yet, hyperbolic techniques can be brought to bear beyond situations in which hyperbolicity per se is assumed:

Theorem 8.4.19 ([**197**])**.** *Every equilibrium measure of a contact 3-flow for a Hölder continuous potential has at most countably many ergodic components with positive entropy, and each of them is Bernoulli.*

This implies strong stochastic properties of geodesic flows on surfaces in a generality quite beyond Theorem 5.2.8:

Corollary 8.4.20 ([**197**])**.** *The geodesic flow of a nonflat compact smooth orientable closed surface of nonpositive curvature is Bernoulli with respect to its (unique) measure of maximal entropy.*

In Theorem 8.3.22 we noted that equilibrium states depend continuously on the dynamical system, and this section focuses on a preferred equilibrium state. Even though in some ways the Sinai–Ruelle–Bowen measure is harder to obtain than some equilibrium states for other more regular potentials, it depends more nicely on the dynamical system than stated in Theorem 8.3.22, in part because the potential itself is tied to the dynamics:

Theorem 8.4.21 ([**80**, **262**])**.** *The Sinai–Ruelle–Bowen varies C^1 with an Axiom-A flow.*

At the end of our discussion at the end of Section 8.3 about extensions of the Bowen techniques beyond uniform hyperbolicity, we stated a result about Hölder potentials on rank-1 manifolds (Theorem 8.3.23). The same work also produced insights related to Sinai–Ruelle–Bowen measures:

Theorem 8.4.22 (Burns–Climenhaga–Fisher–Thompson)**.** *For $q < 1$ and the geometric potential qJ^u (Definition 8.4.1), the geodesic flow of a closed rank-1 surface has a unique equilibrium state μ_q, which is hyperbolic, Bernoulli (Theorem 8.4.19), fully supported, and the weak* limit of weighted regular closed geodesics. The function $q \mapsto P(qJ^u)$ is C^1.*

Remark 8.4.23. It is conspicuous that SRB-measure is excluded in Theorem 8.4.22 because q cannot be 1. This is because in that case the pressure is $h(\mu) - \int \varphi^u = 0$ because (for a surface) the integrand is zero a.e., and so is the entropy on the singular set. This means that the restriction to the singular set is h-expansive and hence has an equilibrium state (Remark 4.2.20)—which, being supported on the singular set, differs from the equilibrium state on the regular set.

5. Rates of mixing*

We pick up from Remark 3.4.27 to explore quantitative measures of mixing. Proposition 3.4.25 expresses the various mixing properties in terms of functions (or observables or random variables) by way of covariance (Definition 3.4.23). Specifically, mixing means that $\operatorname{cov}(U^t_\Phi(f), g) \xrightarrow[t\to\infty]{} 0$ for any $f, g \in L^2$, and we now consider the question of how rapid this convergence might be. Since covariance is closely related to correlation, this is known as the rate of *decay of correlations*.

We first explore this issue in the context of a toy problem in discrete time to show what conclusions one might expect and want to reach. When we consider the matter for flows, we will concentrate on pointing out the issues that make this situation different from this simple example. These are twofold. On one hand, the toy problem is purely expanding, which makes it much simpler but noninvertible. The correct discrete-time counterpart of a flow is an invertible map. This necessitates dealing with contracting directions in addition to expanding ones, which alone adds significant difficulty. Next, we have already noted that the notion of mixing for flows is quite sensitive to timing issues; for instance, being a suspension precludes mixing altogether. Accordingly, subtleties in the orbit direction caused this matter to make substantial progress (in the case of flows) only recently.

We now attend to our introductory discrete-time toy problem.

Proposition 8.5.1. *Consider the expanding endomorphism $E_m\colon x \mapsto mx \bmod 1$ on S^1 for $|m| \ge 2$ with Lebesgue measure and suppose $\varphi, \psi\colon [0,1] \to \mathbb{R}$ are α-Hölder-continuous functions with coefficient L, that is, $|\psi(x) - \psi(y)| \le L|x-y|^\alpha$. Then* *correlations decay with the exponential rate $m^{-\alpha}$:*

$$|\operatorname{cov}(U^n_{E_m}(\varphi),\psi)| \le Lm^{-\alpha n}\|\varphi\|.$$

Remark 8.5.2. We note that the parameters that affect the decay rate of correlations are the expansion rate of the transformation as well as the Hölder exponent of the functions under consideration. In particular, for Lipschitz-continuous functions the decay rate is the reciprocal of the expansion rate of the transformation.

Proof. Assume without loss of generality that $\varphi \perp 1$. Then

$$\begin{aligned}|\operatorname{cov}(U^n_{E_m}(\varphi),\psi)| &= \Big|\sum_{k=0}^{m^n-1}\int_{\frac{k}{m^n}}^{\frac{k+1}{m^n}}\varphi\circ E^n_m\cdot\bar{\psi}\Big|\\ &\le \Big|\sum_{k=0}^{m^n-1}\int_{\frac{k}{m^n}}^{\frac{k+1}{m^n}}\varphi\circ E^n_m\cdot\overline{\big(\psi-\int_{\frac{k}{m^n}}^{\frac{k+1}{m^n}}\psi\big)}\Big| + \Big|\sum_{k=0}^{m^n-1}\int_{\frac{k}{m^n}}^{\frac{k+1}{m^n}}\varphi\circ E^n_m\cdot\overline{\int_{\frac{k}{m^n}}^{\frac{k+1}{m^n}}\psi}\Big|\\ &\le \frac{L}{m^{\alpha n}}\sum_{k=0}^{m^n-1}\int_{\frac{k}{m^n}}^{\frac{k+1}{m^n}}|\varphi\circ E^n_m| + \Big|\overline{\int_{\frac{k}{m^n}}^{\frac{k+1}{m^n}}\psi}\sum_{k=0}^{m^n-1}\underbrace{\int_0^1\frac{\varphi}{m^n}}_{=0}\Big| = Lm^{-\alpha n}\|\varphi\|. \qquad\Box\end{aligned}$$

For smoother functions we get even more rapid decay of correlations:

Proposition 8.5.3. *Consider the expanding endomorphism E_m for $|m|\ge 2$ with Lebesgue measure and suppose $\varphi,\psi\colon [0,1]\to\mathbb{R}$ are C^r functions. Then correlations decay with the exponential rate m^{-r}: $\operatorname{cov}(U^n_{E_m}(\varphi),\psi) = O(m^{-rn})$. (See Remark 3.2.18.) In particular, analytic functions have superexponential decay of correlations.*

Proof. If $\varphi(x) = \sum_{k\in\mathbb{Z}}\varphi_k\exp(2\pi ikx)\perp 1$ and $\psi(x) = \sum_{l\in\mathbb{Z}}\psi_l\exp(2\pi ilx)\perp 1$, then

$$\Big|\int U^n_{E_m}\varphi\cdot\psi\Big| = \Big|\sum_{k,l\in\mathbb{Z}}\varphi_k\psi_l\int\exp(2\pi i(m^nk+l)x)\Big| = \sum_k\varphi_k\psi_{-m^nk},$$

and $|\psi_l|\le C|l|^{-r}$, hence $|\psi_{-m^nk}|\le C(m^n|k|)^{-r} = Cm^{-rn}|k|^{-r}$, so

$$\Big|\sum_k\varphi_k\psi_{-m^nk}\Big| \le \sqrt{\sum|\varphi_k|^2}\sqrt{\sum|\psi_{-m^nk}|^2} \le \|\psi\|_2\cdot C'm^{-rn}. \qquad\Box$$

We note from these that rates of mixing depend on regularity of the functions involved; the converse to speeding up the rate of mixing by looking at smoother functions is that in, say, L^2 there is no hope to get a specific rate. This dependence also shows that rates of mixing are associated with a smooth flow rather than a class modulo measurable isomorphism. (This is also a reason to express these rates in terms of functions rather than sets.) On the other hand, we note that the rate of correlation decay in both results is exponential, regardless of the class of functions under consideration, so long as the class is contained in that of Hölder continuous functions. In particular, it is fair to say that this discrete-time dynamical system has exponential decay of correlations for Hölder functions. And this means that having exponential decay of correlations (or failing to do so) is an interesting notion for hyperbolic flows since the class of Hölder continuous functions is invariant under orbit-equivalence (Theorem 7.3.3).

In both of these results one can see expansion as an essential mechanism manifested in either the stretching out of subintervals to $[0,1]$ or in the "stretching" of Fourier coefficients. Trying to apply the same steps to a hyperbolic map immediately fails, and not only do the expressions fail to improve that need to be controlled, but they indeed get much worse from the contraction. The reader may try to see how this works out in Example 1.5.23.

As this subject came to the fore, the principal approach was to use a Markov partition in order to represent a hyperbolic diffeomorphism as a shift and then to use a result that one can replace a function on the shift (modulo cohomology) by one that depends only on the past, that is, a function of the expanding direction only. For such functions, arguments like the preceding ones again get traction.

There is a cost to the symbolic approach in that one potentially loses geometric information when passing to the shift; this invites the question of whether a direct approach might yield sharper results. These issues also arise for flows.

There is a much more serious difficulty for flows, illustrated by a problem with the earlier statement that "having exponential decay of correlations is an interesting notion for hyperbolic flows since the class of Hölder continuous functions is invariant under orbit-equivalence (Theorem 7.3.3)": A mixing flow can be orbit-equivalent to a suspension, which is never mixing. While this statement can be fixed by invoking a conjugacy rather than an orbit-equivalence, the underlying sensitivity to timing issues is an essential and profound problem: while hyperbolicity helps mix up sets in the phase space, this is only fully effective if the structure of the flow provides a mechanism by which this can force mixing in the flow direction as well. Simple explicit examples illustrate the need for this:

Proposition 8.5.4 ([**260**])**.** *The flow from Example 8.7.16 with $r_0/r_1 \notin \mathbb{Q}$ is mixing, but, for functions independent of the base variable, not exponentially so.*

The flow-counterpart to our toy example, or rather to its invertible cousin in Example 1.5.23, is the geodesic flow of a surface of constant negative curvature (Chapter 2), and exponential mixing was indeed established for these in the 1980s [**88**, **215**, **256**]. The pertinent counterpart to Fourier analysis is representation theory. The 3-dimensional case was then treated in the 1990s by Pollicott [**246**] who established the existence of a spectral gap of the associated transfer operators as a sufficient condition for exponential mixing. These developments notwithstanding, one can say that for 3 decades there was no fundamental progress on the question of decay of correlations. Meanwhile, however, Chernov introduced uniform nonintegrability of the strong foliations (which is illustrated by Proposition 10.2.3 below) as the mechanism that forces mixing in the flow direction even in the absence of a complete algebraic description, which led to (subexponential) rates of mixing for geodesic flows of surfaces of *variable* negative curvature [**83**, **84**], and he conjectured that the correlation decay is exponential for these. This restarted this research area, and it was then transformed by Dolgopyat [**103**].[16] Among other things, this allowed him to prove:

Theorem 8.5.5 (Dolgopyat)**.** *Mixing Anosov flows with C^1 foliations have exponential decay of correlations [**103**], C^∞ Axiom A flows are rapidly mixing if there are two periodic orbits whose periods periods have a Diophantine ratio [**104**], and suspensions over shifts are generically exponentially mixing [**105**].*

To outline the ideas involved, we follow [**204**] and first describe the state of the art from the 1980s [**244**]. Consider the *transfer operator* on probability measures (viewed as functionals) associated to a flow Φ by

$$\mu \mapsto \mathcal{L}^t(\mu) \text{ with } \mathcal{L}^t(\mu)(f) = \mu(f \circ \varphi^t).$$

Clearly, invariant measures are fixed points. Moreover, if $f \in C^r(M)$ with $m(f) = 1$, where m is Lebesgue measure, then $\mathcal{L}^t(m_f) \xrightarrow[t\to\infty]{} \mu_{\text{SRB}}$, and we aim to show that this happens at an exponential rate.

$t \mapsto \mathcal{L}^t$ is a strongly continuous semigroup and hence has a generator Z (a bounded linear operator) with resolvent (Definition 12.3.1)

$$R(z) := (z\,\mathrm{Id} - Z)^{-1} = \int_0^\infty e^{-zt}\mathcal{L}^t\,dt \tag{8.5.1}$$

in terms of which one can write (with $z = a + ib$ and large enough a)

$$\mathcal{L}^t = \lim_{L\to\infty}\int_{-L}^{L} e^{zt}R(z)\,db \tag{8.5.2}$$

Studying R instead of the transfer operator itself directly addresses the problem of the absence of hyperbolicity in the flow direction; the time-integral in (8.5.1) smoothes functions out along the flow direction, and R can be handled in ways similar to the transfer operators in discrete time where the challenge in the flow direction is absent. (8.5.2) then produces the required exponential convergence rate for the transfer operator itself.

[16] Anong the important other work that followed is the development of methods to study correlation decay directly for the flow in hand rather than a symbolic model [**203**].

For $z \in (0,\infty) + \mathbb{R}i$ there is a $\sigma_z > 0$ such that if $f, g \in C^r(M)$ with $m(f) = 1$, then

$$R(z)m_f(g) = \mu_{\text{SRB}}(g)/z + \hat{R}(z)m_f(g), \text{ with } |\hat{R}^n(z)^n m_f(g)| \leq C_{z,f,g}(\Re(z) + \sigma_z)^{-n},$$

so $z \mapsto \hat{R}(z)m_f(g)$ is analytic near 0 and on $(0,\infty) + i\mathbb{R}$, that is, for $L > M \geq 0$ there is an $\omega_M > 0$ such that the path

$$\gamma_{L,M}(s) := \begin{cases} a + i(s + a + \omega_m) & -L - a - \omega_M \leq s \leq -M - a - -\omega_M \\ -M - \omega_M - s - iM & -M - a - \omega_M \leq s \leq -M \\ -\omega_M + is & -M \leq s \leq M \\ -M - \omega_m + s + iM & M \leq s \leq M + a + \omega_M \\ a + i(s - a - \omega_M) & M + a + \omega_M \leq s \leq L + a + \omega_M \end{cases}$$

is in the domain of analyticity of $\hat{R}(\cdot)$ and hence (writing again $z = a + ib$, and using $R(z) = \sum_{k=0}^{r} z^{-k-1}Z^k + R(z)Z^r$)

$$\begin{aligned} \mathscr{L}^t(m_f(g)) &= \mu_{\text{SRB}}(g) + \lim_{L\to\infty} \int_{-L}^{L} e^{zt}\hat{R}(z)m_f(g)\,db \\ &= \mu_{\text{SRB}}(g) + \lim_{L\to\infty} \int_{\gamma_{L,M}} e^{zt}\hat{R}(z)m_f(g)\,dz \\ &= \mu_{\text{SRB}}(g) + \lim_{L\to\infty} \int_{\gamma_{L,M}} \frac{e^{zt}}{z^r}\hat{R}(z)Z^r m_f(g)\,dz \\ &= \mu_{\text{SRB}}(g) + e^{-\omega_M t} \lim_{L\to\infty} \int_{-M}^{M} \frac{e^{ibt}}{-\omega_M + ib}\hat{R}(-\omega_M + ib)Z^r m_f(g)\,db \\ &\quad + \lim_{L\to\infty} \int_{M\leq|b|\leq L} \frac{e^{at+ibt}}{(a+ib)^r}\hat{R}(a+ib)Z^r m_f(g)\,db \\ &\quad - \int_{-\omega_M}^{a} \frac{e^{iMt}}{(x+iM)^r}\hat{R}(x+iM)Z^r m_f(g)\,dx \\ &\quad + \int_{-\omega_M}^{a} \frac{e^{-iMt}}{(x-iM)^r}\hat{R}(x-iM)Z^r m_f(g)\,dx. \end{aligned}$$

Therefore, there is a $C_{M,f,g,r}$ such that

$$|\mathscr{L}^t m_f(g) - \mu_{\text{SRB}}(g)| \leq C_{M,f,g,r}(M^{-r+1} + e^{-\omega_M t}).$$

Although the right-hand side suggests exponential decay, the fact that we do not know how $C_{M,f,g,r}$ and ω_M depend on M means that a substantial new idea is needed. A method that yields a positive lower bound for ω_M combined with a lower bound M^α for $C_{M,f,g,r}$ would suffice. Overcoming this challenge opened up this subject at the end of the 20th century:

Theorem 8.5.6 (Dolgopyat inequality [**103**])**.** *If the stable and unstable foliations are C^1 and uniformly not jointly integrable, then there are $a, \alpha, \beta > 0$ such that*

$$R(a+ib)^{\beta \log|b|} m_f(g) = O(|b|^{-\alpha}|f|_u|g|_s)$$

for large $|b|$, where $|f|_u = |f|_\infty + |\partial_u f|_\infty$, $|g|_s = |g|_\infty + |\partial_s g|_\infty$ with ∂_u, ∂_s the derivatives in the strong unstable and stable directions.

With moderate effort this implies

Corollary 8.5.7. $|R(-\omega+ib)m_f(g)| \le C_{f,g}|b|^{\beta \log(\alpha+\omega)}$.

Using this in the previous estimates implies exponential decay of correlations.

We continue with a decription of the new ideas that produce Theorem 8.5.6. The core insight is that the claim as akin to a quantitative version of the Riemannian–Lebesgue Lemma, because the assertion is that the contribution for large b, which are the "high-frequency contributions," are small [**283**].

The starting point is

$$R(z)^n \nu_\psi(\varphi) = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-zt}\nu_\psi(\varphi\circ\phi_t)\,dt.$$

The Stirling formula (11.3.5) shows that the contribution of the integral from 0 to cn is $\lesssim \frac{(cn)^n}{n!} \lesssim \left(\frac{c}{e}\right)^n$, hence negligible for small c. For large n we can then assume that φ is essentially constant along stable leaves and ν_ψ is essentially constant along strong unstable leaves, and one can furthermore disintegrate ν_ψ along unstable leaves to reduce the problem to estimates of

$$\frac{1}{(n-1)!}\int_{cn}^\infty t^{n-1}e^{-zt}\int_W \varphi\circ\phi_t,$$

where W is a small local strong unstable leaf. Finally, partitioning the integral over time into time-intervals of fixed length gives integrals

$$\int_{W_c} e^{-zt}\varphi\circ\phi_l$$

with W_c a local center-unstable leaf (of a fixed size) and $l \ge cn$. By changing variable, the above integral in turn becomes an integral over $\phi_l W_c$, which is a large manifold in the strong unstable direction. Partitioning it into manifolds W_i of fixed size gives

$$\int_{W_c} e^{-zt}\varphi\circ\phi_l = \sum_i \int_{W_i} e^{-zt}\varphi J_i,$$

where the J_i reflect the Jacobian of the change of variables and a partition of unity is used to smoothly subdivide the integral. The essential improvement over prior developments is that the integral on the right-hand side can be shown to be $O(|z|^{-1})$ when the stable and unstable foliations are uniformly not jointly integrable.

For a finite cover, we can group the W_i according to which element of the covering each meets, and we aim to bound the sum over each such subcollection. We consider a covering by balls, and for large n each ball U will be "packed" by many tightly stacked W_i, each of which is related to one leaf W_U of this collection near the center of U. The integral over each of these W_i can then be reexpressed as an integral over W_U as follows. The unstable and flow coordinates v, s on W_U are sent to the corresponding coordinates $(u, t) = \Psi_i(v, s)$ on W_i by the strong-stable holonomy Ψ_i in such a way that (because the holonomy is assumed C^1) $t \sim s + a_i v$ with a_i proportional to the distance between W_U and W_i by a factor that is nonzero by the assumption of uniform nonintegrability (a quadrilateral from $p \in W_U$ to $\Psi_i(p) \in W_i$, then along the strong-unstable direction, then back to W_U by Ψ_i^{-1} and back to the orbit of p along the strong-unstable direction produces a uniform displacement in the flow direction). Thus, with a suitable Jacobian $\tilde{J}_i$,

$$\sum_i \int_{W_i} e^{-zt} \varphi J_i = \sum_i \int_{W_U} e^{-z(s+a_i w)} \varphi \tilde{J}_i + \mathscr{O}(|\partial_s \varphi|_\infty)$$

[Cauchy–Schwartz inequality] $$\leq |\varphi|_\infty \sqrt{\sum_{i,j \in \mathscr{W}_U} \int_{W_U} e^{-z(a_i - a_j)w} \tilde{J}_j \tilde{J}_i} + \mathscr{O}(|\partial_s \varphi|_\infty) = O(|z|^{-1/2})$$

since

$$\int_{W_U} e^{-z(a_i - a_j)w} \tilde{J}_j \tilde{J}_i \sim |\tilde{J}_j \tilde{J}_i|_{C^1} |z|^{-1} |a_i - a_j|^{-1}$$

and subject to controlling the $a_i - a_j$ by bounding how closely the W_i can be packed. This gives the desired control in Theorem 8.5.6 for large imaginary parts of z.

6. Margulis measure*

This section presents an alternative construction due to Margulis of the unique measure of maximal entropy in the case of topologically mixing Anosov flows. Unlike the Bowen construction from Theorem 8.3.6 which produces this measure as a limit distribution of periodic orbits, the Margulis construction considers limits of normalized Lebesgue measure on long pieces of unstable leaves. Naturally this construction also works in the discrete-time case, but there it does not lead to any particularly interesting new results. In the flow case, however, it allows us in the next section to obtain the most precise asymptotic known of the growth rate for the number of periodic orbits. Thus throughout this section we assume that Φ is a topologically mixing Anosov flow on M (that is, a regionally recurrent Anosov flow that is not a suspension, see Theorem 9.1.1). Definition 8.6.19 defines the Margulis measure, and Lemma 8.6.12, Lemma 8.6.10, Proposition 8.6.18, and Theorem 8.6.20 are the principal properties. We first establish some notation.

Definition 8.6.1. We write $A \subset W^{cu}$ when $A \subset W^{cu}(p)$ for some $p \in M$ and use the notions of openness, compactness, continuity, and measurability for sets and functions defined on an unstable leaf; sometimes we write W^{cu}*-open*, and so on. Thus a W^{cu}*-neighborhood* of a point $p \in M$ is a W^{cu}-open set containing p. We let $C(W^{cu}) := \{f\colon M \to \mathbb{R} \mid \operatorname{supp}(f) \subset W^{cu} \text{ compact}, f\restriction_{\operatorname{supp}(f)} \text{ continuous}\}$. The distance function on a leaf $W^{cu}(p)$ induced by the Riemannian structure of this leaf is denoted by d^{cu}, and λ^{cu} is the (Lebesgue) measure on each unstable leaf $W^{cu}(p)$ induced by its Riemannian volume. $B^{cu}(p,r) := \{q \in W^{cu}(p) \mid d^{cu}(p,q) < r\}$ is the r-ball around p in $W^{cu}(p)$. This definition carries over to the other foliations (W^u, W^{cs}, W^s) as well. As in Definition 8.1.18, if $x, y \in \Lambda$ are sufficiently close, then there is a well-defined *holonomy map* $\mathscr{H}\colon B^{cu}(x,\epsilon) \to W^{cu}(y), z \mapsto W^s(z) \cap B^{cu}(y,\delta)$, where ϵ and δ depend on x and y. We say that $A, B \subset W^{cu}$ are ϵ*-equivalent* if there is a well-defined holonomy $\mathscr{H}$ from A to B and $d^s(x, \mathscr{H}(x)) < \epsilon$ for all $x \in A$. We say that $f, g \in C(W^{cu})$ are ϵ*-equivalent* if supp(f) and supp(g) are ϵ-equivalent via $\mathscr{H}$ and $f = g \circ \mathscr{H}$.

Theorem 8.6.2 (Bowen–Margulis measure)**.** *For a C^2 Anosov flow with dense strong stable and strong unstable leaves the Bowen measure coincides with the* Margulis measure*, an invariant Borel probability measure μ with full support such that the conditionals μ^u_z on unstable leaves $W^u(z)$ are positive on open and finite on compact sets, and $\mu^u_{\varphi^t(z)} \circ \varphi^t = e^{h_{\text{top}}(\Phi)t} \cdot \mu_z$.*

If $p_1, p_2 \in M$ and $r > 0$, then $W^s(p_1) \cap B^{cu}(p_2, r) \neq \varnothing$ (Theorem 9.1.1), so a compactness argument shows that for open $A \subset W^{cu}$ there are $\epsilon(A), r(A) > 0$ such that for all $p \in M$ the ball $B^{cu}(p,r)$ is $\epsilon(A)$-equivalent to a subset of A. This yields

Lemma 8.6.3. *If $A \subset W^{cu}$ is open then there exists $C(A)$ such that for $p \in M, t \geq 0$*

$$\lambda^{cu}(\varphi^t B^{cu}(p, r(A))) < C(A)\lambda^{cu}(\varphi^t(A)).$$

Proof. If $T > 0$ the claim holds for all $p \in M$ and $t \in [0,T]$ since the holonomy maps establishing ϵ-equivalence of $\varphi^t(B^{cu}(p, r(A)))$ to a subset C of $\varphi^t(A)$ are a uniformly equicontinuous family of local homeomorphisms with uniformly equicontinuous inverses for $t \in [0,T]$, $s \in [0, \epsilon(A)]$.

If T is large enough (depending on A) and $t \geq T$ then $\varphi^t(B^{cu}(p, r(A)))$ is ϵ-equivalent to a $C \subset \varphi^t(A)$ with ϵ small enough that $\lambda^{cu}(\varphi^t(B^{cu}(p, r(A)))) <$ const.$\cdot\lambda^{cu}(C)$ with a bounded constant depending on A. (Possible because the curvature of the boundary of $\varphi^t(B^{cu}(p, r(A)))$ is bounded independently of t.) □

Lemma 8.6.4. *If $0 \le f \in C(W^{cu})$ and $K \in W^{cu}$ compact then there exists $C(K,f) > 0$ such that for any bounded W^{cu}-measurable g with support in K and for $t \ge 0$*

$$\int g \circ \varphi^{-t}\, d\lambda^{cu} < C(K,f)\|g\|_\infty \int f \circ \varphi^{-t}\, d\lambda^{cu},$$

where $\|\cdot\|_\infty$ is the essential-supremum norm.

Proof. Let $A = f^{-1}((\epsilon,\infty))$ and cover K by balls $B^{cu}(x_i, r(A))$, $i = 1,\dots,N$. Lemma 8.6.3 gives

$$\begin{aligned}\int g \circ \varphi^{-t}\, d\lambda^{cu} &\le \lambda^{cu}(\varphi^t(K))\|g\|_\infty < \sum_{i=1}^N \lambda^{cu}(B^{cu}(x_i, r(A)))\|g\|_\infty \\ &< NC(A)\lambda^{cu}(\varphi^t(A))\|g\|_\infty < \underbrace{NC(A)/\epsilon}_{=:C(K,f)}\|g\|_\infty \int f \circ \varphi^{-t}\, d\lambda^{cu}. \quad \square\end{aligned}$$

For $p \in M$ we define a function $f_p\colon M \to \mathbb{R}$ by

$$f_p(x) := \begin{cases} \big(1 + \lambda^{cu}\big(B^{cu}\big(p, d^{cu}\big(p,x\big)\big)\big)\big)^{-2} & \text{if } x \in W^{cu}(p), \\ 0 & \text{otherwise.}\end{cases}$$

Let $r_p(s)$ be such that $\lambda^{cu}(B^{cu}(p, r_p(s))) = s$ and set

$$U_p^i := \{x \in W^{cu}(p) \mid i \le \lambda^{cu}(B^{cu}(p, d^{cu}(p,x))) < i+1\} = B^{cu}(p, r_p(i+1)) \smallsetminus B^{cu}(p, r_p(i))$$

for $i \in \mathbb{N}_0$. Then $\lambda^{cu}(U_p^i) \le 1$ and hence

$$\int f_p(x)\, d\lambda^{cu} < \sum_i \int_{U_p^i} f_p(x)\, d\lambda^{cu} \le \sum_i 1/(i+1)^2 < 2. \tag{8.6.1}$$

For $A \subset W^{cu}$ open, $p \in M$, and $\chi_{p,A} := \chi_{\overline{B^{cu}(p,r(A))}}$ (the characteristic function of the closed $r(A)$-ball),

$$g_{p,A}(x) := \int \chi_{x,A}(y) f_p(y)\, d\lambda^{cu}(y)$$

is W^{cu}-continuous and positive on $W^{cu}(p)$.

Lemma 8.6.5. *If $A \subset W^{cu}$ open, $q \in M$, $t \ge 0$, and $C(A)$ as in Lemma 8.6.3, then*

$$\int g_{p,A}(\varphi^{-t}(x))\, d\lambda^{cu}(x) < 2C(A)\lambda^{cu}(\varphi^t(A)).$$

Proof. The Fubini Theorem, Lemma 8.6.3, and (8.6.1) yield

$$\begin{aligned} \int g_{p,A}(\varphi^{-t}(x))\,d\lambda^{cu}(x) &= \iint \chi_{\varphi^{-t}(x),A}(y) f_p(y)\,d\lambda^{cu}(y)\,d\lambda^{cu}(x) \\ \chi_{p,A}(x)=\chi_{x,A}(p)\Rightarrow \quad &= \iint \chi_{y,A}(\varphi^{-t}(x))\,d\lambda^{cu}(x) f_p(y)\,d\lambda^{cu}(y) \\ &= \int \lambda^{cu}(\varphi^t(B^{cu}(\varphi^{-t}(y), r(A)))) f_p(y)\,d\lambda^{cu}(y) \\ &< 2C(A)\lambda^{cu}(\varphi^t(A)). \end{aligned}$$

□

Lemma 8.6.6. *If $f_1 \in C(W^{cu})$ and $\epsilon > 0$ then there exists $\delta > 0$ (depending continuously on f_1 in the C^0 topology) such that if f_2 is δ-equivalent to f_1 then*

$$\left|\int f_1\,d\lambda^{cu} - \int f_2\,d\lambda^{cu}\right| < \epsilon \int |f_1|\,d\lambda^{cu}.$$

Proof. There are step functions $\underline{\xi}$ and $\overline{\xi}$ representing upper and lower Riemann sums for $\int f_1\,d\lambda^{cu}$ that are accurate to within $\frac{\epsilon}{2}\int |f_1|\,d\lambda^{cu}$. Then it suffices to show the result for $\underline{\xi}$ and $\overline{\xi}$ because if f_2 is δ-equivalent to f_1 then the corresponding δ-equivalent step functions give upper and lower bounds for f_2. In each case it suffices to show that for a given open $O \subset W^{cu}$ and $\alpha > 0$ there exists η such that any η-equivalent set O' has the same volume up to α, that is, $|\operatorname{vol} O - \operatorname{vol} O'| < \alpha$. This follows from the fact that the holonomies converge to isometries as $\eta \to 0$. □

Henceforth fix a W^{cu}-open set $\mathcal{K}$ with W^{cu}-compact closure, and a function

$$f_{\mathcal{K}} > \chi_{\mathcal{K}}$$

in $C(W^{cu})$, where $\chi_{\mathcal{K}}$ is the characteristic function of $\mathcal{K}$.

Definition 8.6.7. $f_1, f_2 \in C(W^{cu})$ are said to be *ϵ-close* if there exist $\tilde{f}_1, \tilde{f}_2 \in C(W^{cu})$ and $x_1, x_2 \in M$ such that

(1) $\tilde{f}_1, \tilde{f}_2$ are ϵ-equivalent,
(2) $|f_i(x) - \tilde{f}_i(x)| < \epsilon g_{x_i,\mathcal{K}}(x)$ for all $x \in M$.

Lemma 8.6.8. *If $0 \le f_1 \in C(W^{cu})$ and $\epsilon > 0$ then there exists $\delta(\epsilon, f_1)$ such that for $f_2 \in C(W^{cu})$ δ-close to f_1 we have*

$$\left|\int f_1 \circ \varphi^{-t}\,d\lambda^{cu} - \int f_2 \circ \varphi^{-t}\,d\lambda^{cu}\right| < \epsilon \int f_{\mathcal{K}} \circ \varphi^{-t}\,d\lambda^{cu}.$$

PROOF. By definition

$$\begin{aligned}\left|\int f_1\circ\varphi^{-t}\,d\lambda^{cu}-\int f_2\circ\varphi^{-t}\,d\lambda^{cu}\right| \le & \underbrace{\left|\int f_1\circ\varphi^{-t}\,d\lambda^{cu}-\int \tilde f_1\circ\varphi^{-t}\,d\lambda^{cu}\right|}_{\le\delta\int g_{x_1,\mathcal{K}}\circ\varphi^{-t}(x)\,d\lambda^{cu}\le\delta 2C(A)\lambda^{cu}(\varphi^t\mathcal{K})\text{ by Lemma 8.6.5}}\\ &+\underbrace{\left|\int (\tilde f_1\circ\varphi^{-t}-\tilde f_2\circ\varphi^{-t})\,d\lambda^{cu}\right|}_{\le\frac{\epsilon}{2\|\tilde f_1\|_\infty C(\mathcal{K},f_1)}\int|\tilde f_1|\circ\varphi^{-t}\,d\lambda^{cu}}\\ &+\underbrace{\left|\int \tilde f_2\circ\varphi^{-t}\,d\lambda^{cu}-\int f_2\circ\varphi^{-t}\,d\lambda^{cu}\right|}_{\le\delta\int g_{x_2,\mathcal{K}}\circ\varphi^{-t}(x)\,d\lambda^{cu}\le\delta 2C(A)\lambda^{cu}(\varphi^t\mathcal{K})\text{ by Lemma 8.6.5}},\end{aligned}$$

where the middle term was estimated using Lemma 8.6.6 for sufficiently small δ and that $\tilde f_1\circ\varphi^{-t}$ and $\tilde f_2\circ\varphi^{-t}$ are $\delta\lambda^t$-equivalent ($\lambda<1$ as in Definition 5.1.1). With $\delta<\epsilon/(8C(A))$, Lemma 8.6.4 shows that $\epsilon\int f_{\mathcal{K}}\circ\varphi^{-t}\,d\lambda^{cu}$ is indeed an upper bound. □

Although $C(W^{cu})$ is not closed under addition), we say that $F\colon C(W^{cu})\to\mathbb{R}$ is *linear* or a *linear functional* if $F(\alpha f)=\alpha F(f)$ and $F(f+g)=F(f)+F(g)$ whenever $f,g,f+g\in C(W^{cu})$, $\alpha\in\mathbb{R}$. The space C^* of functionals on $C(W^{cu})$ has a topology induced by the natural embedding into $\prod_{f\in C(W^{cu})}\mathbb{R}_f$, where $\mathbb{R}_f$ is a copy of $\mathbb{R}$. This product topology is the topology of pointwise convergence (which over a linear space is the weak* topology).

Now define $\{F_t\}_{t\in\mathbb{R}}\subset C^*$ by $F_t(f):=\int f\circ\varphi^{-t}\,d\lambda^{cu}$, and

$$C_0^*:=\Big\{F\in C^*\;\Big|\;F=\sum_{i=1}^m c_iF_{t_i}\text{ for some }c_i,t_i\ge 0\text{ and }F(f_{\mathcal{K}})=1\Big\}.$$

Lemma 8.6.9. *(1) For $f\in C(W^{cu})$ there is a $C_1(f)$ such that $|F(f)|\le C_1(f)$ for all $F\in\overline{C_0^*}$.*

(2) For $0\le f\in C(W^{cu})\smallsetminus\{0\}$ there is a $C_2(f)$ such that $|F(f)|\ge C_2(f)$ for all $F\in\overline{C_0^*}$.

(3) For $f\in C(W^{cu})$ and $\epsilon>0$ there is a $\delta>0$ such that if $g\in C(W^{cu})$ is δ-close to f then $|F(f)-F(g)|<\epsilon$ for all $F\in\overline{C_0^*}$.

PROOF. Lemmas 8.6.4 and 8.6.8 imply (1)–(3) with $A:=\big\{\hat F_t:=F_t/F_t(f_{\mathcal{K}})\;\big|\;t\in\mathbb{R}\big\}$ in place of $\overline{C_0^*}$, hence for the convex hull C_0^* of A and its closure. □

Let $\varphi^{t*}(F)(f):=F(f\circ\varphi^{-t})$. Φ acts on $\overline{C_0^*}$ by $\widehat{\varphi^t}^*(F)(f):=\dfrac{F(f\circ\varphi^{-t})}{F(f_{\mathcal{K}}\circ\varphi^{-t})}$.

Lemma 8.6.10. *There exist* $\mathfrak{m} \in \overline{C_0^*}$ *and* $h^u > 0$ *such that*

$$\varphi^{t*}\mathfrak{m} = e^{h^u t}\mathfrak{m}. \tag{8.6.2}$$

Remark 8.6.11. This gives the uniform-expansion property that distinguishes the Margulis measure.[17]

Proof. Lemma 8.6.9(1) implies that $\overline{C_0^*}$ is compact in the topology of pointwise convergence.[18] By the Tychonoff Fixed-Point Theorem[19] there is an $\mathfrak{m} \in \overline{C_0^*}$ with $\widehat{\varphi^t}^*\mathfrak{m} = \mathfrak{m}$, hence (8.6.2), for $t \geq 0$ and thus for $t \in \mathbb{R}$.

To see that $h^u > 0$ let $0 \leq f \in C(W^{cu}) \setminus \{0\}$ and $t_1, t_2 \geq 0$. Then

$h^u > 0$: If $0 \leq f \in C(W^{cu}) \setminus \{0\}$, $t_1, t_2 \geq 0$, λ as in Definition 5.1.1, then

$$\varphi^{t_1*}F_{t_2}(f) = F_{t_2}(f \circ \varphi^{-t_1}) = \int (f \circ \varphi^{-t_1}) \circ \varphi^{-t_2}\, d\lambda^{cu} \geq \lambda^{-t_1} \int f \circ \varphi^{-t_2}\, d\lambda^{cu} = \lambda^{-t_1} F_{t_2}(f).$$

Thus $\varphi^{t*}F = \lambda^{-t}F$ for $F \in A$, hence for $F \in \overline{\text{co}}(A) = \overline{C_0^*}$. Therefore $h^u > 0$. □

Lemma 8.6.12. *If* f, g *are* ϵ*-equivalent then* $\mathfrak{m}(f) = \mathfrak{m}(g)$.

Remark 8.6.13. This gives holonomy invariance, another property that characterizes Margulis measure [**57**, Theorem 3.7].[20]

Proof. By considering positive parts and using linearity we may assume that f and g are nonnegative. Since $f \circ \varphi^{-t}$ and $g \circ \varphi^{-t}$ are $\lambda^t\epsilon$-equivalent, Lemma 8.6.6 shows that $\lim_{t\to\infty} F_t f / F_t g = 1$ so for $\eta > 0$ there exists $T_\eta > 0$ such that

$$|F_t(f) - F_t(g)| \leq \eta F_t(g)$$

for $t \geq T_\eta$. Thus if $F = \sum c_i F_{t_i}$ with $c_i, t_i \geq 0$ then

$$|\varphi^{t*}F(f) - \varphi^{t*}F(g)| \leq \sum c_i |\underbrace{\varphi^{t*}F_{t_i}(f) - \varphi^{t*}F_{t_i}(g)}_{=F_{t_i+t}(f) - F_{t_i+t}(g)}| \leq \eta \sum c_i F_{t_i+t}(g) = \eta\varphi^{t*}F(g).$$

The same estimate then holds for $F \in A$, hence for all $F \in \overline{\text{co}}(A) = C_0^*$, hence for $\mathfrak{m}$. Thus by Lemma 8.6.10 we indeed have

$$\frac{\mathfrak{m}(f)}{\mathfrak{m}(g)} = \lim_{t\to\infty} \frac{C_{cu}^t \mathfrak{m}(f)}{C_{cu}^t \mathfrak{m}(g)} = \lim_{t\to\infty} \frac{\varphi^{t*}\mathfrak{m}(f)}{\varphi^{t*}\mathfrak{m}(g)} = 1.$$ □

[17] This uniform-expansion property also follows from holonomy-invariance (Lemma 8.6.12) [**57**, p. 58].

[18] By the Tychonoff Theorem: The product of compact spaces is compact.

[19] If K is compact and convex in a locally convex topological vector space, then every continuous map $f\colon K \to K$ has a fixed point.

[20] In fact, Bowen and Marcus prove that on a topologically mixing basic set of an Axiom A flow the stable and unstable foliations are uniquely ergodic, that is, admit only one transverse holonomy-invariant measure; see also [**132**, Theorems 2.9 and 3.2].

To show that $\mathfrak{m}$ corresponds to a family of measures on leaves of W^{cu}, let $OC(W^{cu}(p))$ be the collection of open sets in $W^{cu}(p)$ with compact closure. If $U \in OC(W^{cu}) := \bigcup_{p\in M} OC(W^{cu}(p))$ let $C_U(W^{cu}) := \{f \in C(W^{cu}) \mid \operatorname{supp}(f) \subset \bar{U}\}$ with the supremum norm $\|\cdot\|_\infty$. By Lemma 8.6.9(3) $\mathfrak{m}$ is a continuous linear functional on $C_U(W^{cu})$ and extends to the space $C(\bar{U})$ of continuous functions on $\bar{U}$ by the Hahn–Banach Theorem. The Riesz Representation Theorem 3.1.10 gives a measure μ_U on $\bar{U}$ such that

$$\mathfrak{m}(f) = \int f\,d\mu_U$$

for $f \in C_U(W^{cu})$. If $U_1 \subset U_2$ in $OC(W^{cu})$ then there exist $\{f_j\}_{j\in\mathbb{N}} \subset C_U(W^{cu})$ such that $f_i \nearrow \chi_{U_1}$ and hence $\mu_{U_2}(U_1) = \mathfrak{m}(\chi_{U_1}) = \lim_{j\to\infty} \mathfrak{m}(f_j)$, so we have

Theorem 8.6.14. *There is a map $\mu^{cu}\colon OC(W^{cu}) \to \mathbb{R}$ such that*

(1) $\mu^{cu}\restriction_{OC(W^{cu}(p))}$ extends to a measure on $W^{cu}(p)$;
(2) $\mu^{cu}(\varphi^t(U)) = e^{h^u t}\mu^{cu}(U)$ for $U \in OC(W^{cu})$, $t \in \mathbb{R}$;
(3) if $\varnothing \neq U \in OC(W^{cu})$ then $0 < \mu^{cu}(U) < \infty$;
(4) if $U_1, U_2 \in OC(W^{cu})$ are ϵ-equivalent then $\mu^{cu}(U_1) = \mu^{cu}(U_2)$.

We will see in Lemma 8.6.22 that $h^u = h_{\text{top}}(\Phi)$.

Replacing φ^t by φ^{-t} we obtain a measure μ^{cs} for which the same results hold, except that in (2) we obtain a constant $h^s < 0$. (In fact, $h^s = -h^u$; see (8.6.7).)

Adapting the notation to W^u, W^{cs}, and W^s gives $\bigcup_{t_1<t<t_2} \varphi^t(U) \in OC(W^{cu})$ when $t_1 < t_2$ and $U \in OC(W^u)$. Furthermore there are $r_0, t_0 > 0$ with

$$\varphi^{t_1}(U) \cap \varphi^{t_2}(U) = \varnothing \text{ if } 0 \le t_1 < t_2 \le t_0,\ U \subset B^i(p, r_0) \text{ open, } i = u, s. \tag{8.6.3}$$

Thus for $i = u, s$, $U \subset B^i(p, r_0)$, $\mu^i(U) := \mu^{0i}(\bigcup_{0<t<t_0} \varphi^t(U))$, induces a measure on $B^i(p, r_0)$, and indeed on $OC(W^i)$, with

$$\mu^i(\varphi^t(U)) = e^{h^i t}\mu^i(U) \text{ and } 0 < \mu^i(U) < \infty \text{ for } \varnothing \neq U \in OC(W^i) \tag{8.6.4}$$

For t_0, r_0 as in (8.6.3), Theorem 8.6.14(2) then yields

$$\mu^{0i}\big(\bigcup_{t_1<t<t_2} \varphi^t(U)\big) = \frac{\int_{t_1}^{t_2} e^{h^i t}dt}{\int_0^{t_0} e^{h^i t}dt}\mu^i(U) \text{ if } 0 \le t_1 < t_2 \le t_0,\ U \subset B^i(p, r_0) \text{ open.} \tag{8.6.5}$$

With the notion of ϵ-equivalence adapted to the strong foliations W^i, we get

Lemma 8.6.15. *For all $\epsilon > 0$ there is a $\zeta > 0$ such that for all $A_1, A_2 \subset W^i$ measurable ($i = u, s$) and A_1, A_2 ζ-equivalent we have*

$$\left|\frac{\mu^i(A_1)}{\mu^i(A_2)} - 1\right| < \epsilon.$$

Proof. If A_1, A_2 are ζ-equivalent then there are partitions $A_l = \bigcup_k A_l^k$ with $A_l^k \subset B^i(p_l^k, r_0)$ $(l = 1,2)$ and A_1^k, A_2^k ζ-equivalent. Thus, $A_l \subset B^i(p_l, r_0)$ without loss of generality. But for any $\alpha > 0$ there exists a $\zeta > 0$ such that if $A_1, A_2 \subset W^i$ are ζ-equivalent and $A_l \subset B^i(p_l, r_0)$, then $\bigcup_{\alpha<t<t_0-\alpha} \varphi^t(A_2)$ is ζ-equivalent to a subset of $\bigcup_{0<t<t_0} \varphi^t(A_1)$ and vice versa $\bigcup_{\alpha<t<t_0-\alpha} \varphi^t(A_1)$ is ζ-equivalent to a subset of $\bigcup_{0<t<t_0} \varphi^t(A_2)$. For these sets, (8.6.5) and Theorem 8.6.14(4) yield

$$\mu^i(A_1) = \frac{\int_0^{t_0} e^{h^u t}dt}{\int_\alpha^{t_0-\alpha} e^{h^u t}dt} \underbrace{\mu^{0i}\Big(\bigcup_{\alpha<t<t_0-\alpha} \varphi^t(A_1)\Big)}_{\le \mu^{0i}\left(\bigcup_{0<t<t_0} \varphi^t(A_2)\right) = \mu^i(A_2)} \le \frac{\int_0^{t_0} e^{h^u t}dt}{\int_\alpha^{t_0-\alpha} e^{h^u t}dt} \mu^i(A_2),$$

and vice versa, which in turn implies the claim. □

Corollary 8.6.16. *For $r > 0$ there is a $C > 1$ such that if A_1, A_2 are r-equivalent then*

$$\frac{1}{C} < \frac{\mu^i(A_1)}{\mu^i(A_2)} < C.$$

From these measures on leaves we now construct a finite Φ-invariant measure on M. We do this by locally defining a *weighted product measure* as follows. Every $p \in M$ has a neighborhood $U(p)$ which is a *local product cube*, that is, using the local product structure we can write $U(p)$ as $U^{cu}(p) \times U^s(p)$, where $U^{cu}(p) \subset W^{cu}(p)$ and $U^s(p) \subset W^s(p)$. If $O \subset U(p)$ let

$$f_O(q) := \mu^s((\{q\} \times U^s(p)) \cap O) \quad (q \in U^{cu}(p)).$$

Lemma 8.6.17. *f_O is upper semicontinuous (hence locally integrable).*

Proof. For $x \in U^{cu}$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\mu^s\left((\pi^s(O \cap (\{x\} \times U^s)) \times U^{cu}) \cap (\{y\} \times U^s)\right)}{\mu^s((\{y\} \times U^s) \cap O)} > 1 - \epsilon$$

when $d^{cu}(x, y) < \delta$, where π^s is the projection to W^s. Now apply Lemma 8.6.15. □

For $q \in U^s(p)$, $A \subset U^{cu}(p)$ let $\mu_q(A) := \mu^{cu}(A \times \{q\})$ wherever defined. By Theorem 8.6.14(4) this is independent of $q \in U^s(p)$. Together with Lemma 8.6.17 this shows that

$$\mu(O) := \int f_O(x)\, d\mu_q(x) \tag{8.6.6}$$

is well defined.

Proposition 8.6.18. *The measure on M obtained from* (8.6.6) *by extending to Borel sets is finite and Φ-invariant.*

Proof. Finiteness is clear since M is compact and local product cubes have finite measure. Theorem 8.6.14(2), (8.6.4) and (8.6.6) show that

$$\mu(\varphi^t(A)) = e^{(h^u+h^i)t}\mu(A)$$

for all measurable $A \in M$. Setting $A = M$ thus yields

$$h^u = -h^s =: h. \tag{8.6.7}$$ □

Proposition 8.6.18 allows us to define the probability measure of interest.

Definition 8.6.19. The Φ-invariant Borel probability measure μ obtained from (8.6.6) by normalization (by proper choice of $\mathscr{K}$, for example) is called the *Margulis measure* for Φ.

Theorem 8.6.20. *For a C^2 Anosov flow with dense strong leaves, the Bowen measure and the Margulis measure coincide.*

Proof. We show equality via volume estimates for d_t^Φ ϵ-balls.

Lemma 8.6.21. $E_\epsilon e^{-th^u} \le \mu(B_\Phi(x,\epsilon,t)) \le F_\epsilon e^{-th^u}$ *for some constants* E_ϵ, F_ϵ.

Proof. It suffices to show this for "boxes" $B_\Phi^1(x,\epsilon,t) := B_\Phi^{cs}(x,\epsilon,t)\times B_\Phi^u(x,\epsilon,t)$ since for $\epsilon > 0$ there exist $\epsilon_1,\epsilon_2 > 0$ such that $B_\Phi^1(x,\epsilon_1,t) \subset B_\Phi(x,\epsilon,t) \subset B_\Phi^1(x,\epsilon_2,t)$. But $B_\Phi^1(x,\epsilon,t) = B^{cs}(x,\epsilon)\times\varphi^{-t}(B^u(\varphi^t(x),\epsilon))$, which immediately yields the claim by the uniform-expansion property of μ^u. □

Lemma 8.6.22. $h^u = h_{\text{top}}(\varphi)$.

Proof. If E is a maximal d_t^Φ-ϵ-separated set then $M = \bigcup_{x\in E} B_\Phi(x,\epsilon,t)$ and hence $1 \le \sum_{x\in E}\mu(B_\Phi(x,\epsilon,t)) \le \text{const.}\, e^{t(h_{\text{top}}(\varphi)-h^u)}$ by the previous lemma and Proposition 8.3.9. Thus $h^u \le h_{\text{top}}(\varphi)$. Conversely $B_\Phi(x,\epsilon/2,t)$ are pairwise disjoint, so $1 \ge \sum_{x\in E}\mu(B_\Phi(x,\epsilon/2,t)) \ge \text{const.}\, e^{t(h_{\text{top}}(\varphi)-h^u)}$ and $h^u \ge h_{\text{top}}(\varphi)$. □

The preceding two lemmas imply that Margulis measure is absolutely continuous with respect to Bowen measure by Proposition 8.3.14. Since Bowen measure is ergodic this implies the claim. □

Remark 8.6.23. We write h for $h_{\text{top}}(\varphi)$ in the sequel.

Entropy is closely connected with fractal dimensions (Definition 4.2.34), and we first saw a connection in Theorem 4.2.36. The connections go much deeper than there, and we will only minimally explore these. An illuminating instance arises in connection with the Pesin Entropy Formula (Remark 8.4.6) according to which the entropy of volume (if invariant) is given by the sum of the positive Lyapunov exponents (Remark 8.4.11). For other invariant measures, this can be extended, provided each Lyapunov exponent is weighted with the fractal dimension of the measure conditioned on the corresponding unstable subleaf [**193**–**195**]. For Lebesgue measure, this is just the multiplicity, but for other measures this puts the fractal dimension at the center of connecting expansion and complexity.

We will in a simple way illustrate this kind of connection with another description of the Margulis measure. Its stable and unstable conditionals turn out to be the h-dimensional *spherical measure* for a suitable dynamically defined distance on each leaf, and the associated dimension coincides with the entropy.[21]

Throughout, $\lambda = e^a$ is as in Definition 5.1.1, except that we assume that the Riemannian metric is adapted to the dynamics as in Proposition 5.1.5, that is, that for all $t > 0$ and all $x \in M$ we have

$$\|D\varphi^t\restriction_{E^s_x}\|^* \le e^{-at} \quad \text{and} \quad \|D\varphi^{-t}\restriction_{E^u_x}\|^* \le e^{-at}.$$

Definition 8.6.24 (Spherical measure)**.** Fix $R \in \mathbb{R}$, and for $x, y \in W^u(z)$ define

$$\eta(x,y) \coloneqq \eta_{z,R}(x,y) \coloneqq e^{-\sup\left\{t\in\mathbb{R} \mid\ d_{\varphi^t(z)}(\varphi^t(x),\varphi^t(y))\le R\right\}}.$$

For $x \in W^u(z)$ denote by $B_\eta(x,\epsilon)$ the ϵ-ball for η around x, and for $S \subset W^u(z)$ let

$$\sigma_\epsilon(S) \coloneqq \inf\Big\{\sum_{j\in\mathbb{N}}\epsilon_j^h \;\Big|\; S \subset \bigcup_{j\in\mathbb{N}} B_\eta(x_j,\epsilon_j) \text{ with } x_j \in W^u(z),\ \epsilon_j \le \epsilon\Big\},\ \sigma(S) \coloneqq \sigma_z(S) \coloneqq \sup_{\epsilon>0}\sigma_\epsilon(S).$$

Theorem 8.6.25. *If Φ is as in Theorem 8.6.20, then the spherical measure σ from Definition 8.6.24 coincides on every unstable leaf with the conditional Margulis measure μ^u from* (8.6.4)*.*

Remark 8.6.26. σ is the h-dimensional spherical measure on W^u associated with η. Technically, η might not be a distance, and one can either view σ as the h/a-dimensional spherical measure associated with the distance η^a or perform a constant rescaling of time after which we can take $a = 1$, making η a distance. We note that for geodesic flows of Riemannian metrics whose curvature is bounded above by -1, η is a proper distance, that is, we can indeed take $a = 1$.

Lemma 8.6.27. *$\eta \circ \varphi^t = e^t\eta$, $\eta_z = \eta_{z'}$ for $z' \in W^u(z)$, and η^a is a distance on $W^u(z)$.*

Proof. The triangle inequality $\eta(x_1,x_2)^a \le \eta(x_1,y)^a + \eta(y,x_2)^a$ is the only nontrivial item. This is clear if any $r_i \coloneqq d_{\varphi^t(z)}(\varphi^t(x_i),\varphi^t(y)) > R$ for $i = 1,2$, where $t \coloneqq -\log\eta(x_1,x_2)$, because then $\eta^a(x_i,y) > \eta^a(x_1,x_2)$.

If both $r_i < R$, then $\eta^a(x_1,x_2) = e^{-at} \underset{r_1+r_2\ge d_{\varphi^t(z)}(\varphi^t(x_1),\varphi^t(x_2))=R}{\le} \frac{r_1+r_2}{R}e^{-at} \underset{\text{Claim 8.6.28 with } \Delta=R,\ \delta=r_i,\ x=x_i}{\le} \eta^a(x_1,y) + \eta^a(x_2,y)$. □

[21] We should add that spherical measures are not much used otherwise—they are well-adapted to this particular situation in which the expansion is isotropic. The box dimension (Definition 4.2.34) or Hausdorff dimension are in general much better suited to interact well with dynamical properties.

Claim 8.6.28. $\eta_\Delta(x,y) \ge (\delta/\Delta)^{1/a}e^{-t}$, *where* $\delta := d_{\varphi^t(z)}(\varphi^t(x),\varphi^t(y))$.

Proof. $d_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}(x),\varphi^{t+\tau}(y)) \ge \delta e^{a\tau} \ge \Delta$ if $e^{a\tau} \ge \frac{\Delta}{\delta}$, so $\eta_\Delta(x,y) \ge e^{-(t+\tau)} = \big(\frac{\delta}{\Delta}\big)^{1/a}e^{-t}$. □

Remark 8.6.29. $\eta_R \le \eta_r \le (R/r)^{1/a}\eta_R$ by Claim 8.6.28 with $\delta = r \le \Delta = R$, $t = -\log\eta_r(x,y)$.

The first step towards identifying σ and μ^u is

Lemma 8.6.30. $\log\mu^u(B_\eta(x,\epsilon)) - h\log\epsilon$ *is bounded.*

Proof. It suffices to show that $0 < \alpha_1 \le \mu^u(B_\eta(x,1) \le \alpha_2 < \infty$ because

$$\mu^u(B_\eta(x,\epsilon)) = \mu^u(\varphi^{\log\epsilon}(B_\eta(\varphi^{-\log\epsilon}(x),1))) = \epsilon^h\mu^u(B_\eta(\varphi^{-\log\epsilon}(x),1)).$$

Suppose to the contrary that $\mu^u(B_\eta(x_i,1) \xrightarrow[i\to\infty]{} 0$ for suitable $x_i \xrightarrow[i\to\infty]{} x \in M$ by compactness of M. Then $S := \bar{B}_\eta(x,\frac{1}{2})$ is ϵ-equivalent to some $S' \subset B_\eta(x_i,1)$ for large enough i, hence $\mu^u(B_\eta(x_i,1)) \ge \mu^u(S') \ge \frac{1}{2}\mu^u(S) > 0$, since μ^u is positive on open sets, a contradiction. Likewise, finiteness of μ^u on compact sets gives an upper bound. □

Lemma 8.6.31. $1/\alpha_2\mu^u \le \sigma \le (2^{h/a}/\alpha_1)\mu^u$.

Proof. Let $S \subset W^u(z)$, $\epsilon,\delta > 0$. By definition of σ_ϵ there is a covering

$$S \subset \bigcup_{j\in\mathbb{N}} B_\eta(x_j,\epsilon_j),\ \epsilon_j \le \epsilon,\ x_j \in W^u(z)$$

such that $\sigma_\epsilon(S) + \delta \ge \sum_{j\in\mathbb{N}}\epsilon_j^h \ge \alpha_2^{-1}\sum_{j\in\mathbb{N}}\mu^u(B_\eta(x_j,\epsilon_j)) \ge \alpha_2^{-1}\mu^u(S)$. Conversely, let $S \subset W^u(z)$ compact, $\epsilon > 0$, $S_\epsilon := \big\{x \in W^u(z) \mid \eta(x,y) < \epsilon/2^{1/a}$ for some $y \in S\big\}$ and $\{x_j\}_{j=1}^m \subset S$ a maximal subset such that the $B_\eta(x_j,\epsilon/2^{1/a})$ are pairwise disjoint. Then $S \subset \bigcup_{j=1}^m B_\eta(x_j,\epsilon)$, so

$$\sigma_\epsilon(S) \le \sum_{j=1}^m \epsilon_j^h \le 2^{h/a}\alpha_1^{-1}\sum_{j=1}^m \mu^u(B_\eta(x_j,\epsilon/2^{1/a})) \le 2^{h/a}\alpha_1^{-1}\mu^u(S_\epsilon).$$ □

Analogously to Definition 8.6.1 we say that $S \subset W^u(z)$ and $S' \subset W^u(z')$ are ϵ-equivalent if there is a continuous $H\colon S\times[0,1] \to M$ with $H(\cdot,0) = \mathrm{Id}$, $\mathscr{H} := H(\cdot,1)\colon S \to S'$ a homeomorphism, and $\mathscr{H}(x,[0,1]) \subset W^{cs}(x)$ a curve of length $< \epsilon$ for all $x \in S$.

Lemma 8.6.32. *For* $\delta > 0$ *there is an* $\epsilon > 0$ *such that if* $S \subset W^u(z)$ *and* $S' \subset W^u(z')$ *are* ϵ*-equivalent, then* $(1-\delta)\sigma(S) \le \sigma(S') \le (1+\delta)\sigma(S)$.

Proof. By symmetry it suffices to prove $\sigma(S') \le \big(\frac{R+\theta(C\epsilon)}{R}\big)^{h/a}\sigma(S)$ with $\theta(\epsilon) \xrightarrow[\epsilon\to 0]{} 0$. To that end suppose $S \subset \bigcup_{j\in\mathbb{N}} B_{\eta_z}(x_j,\delta_j)$ with $\sum_j \delta_j^h \le \sigma(S)+\delta$ and $\delta_j < \delta < 1$. Now, if $\{x\},\{x'\}$ are ϵ-equivalent, then $x'' := W^{cu}(x') \cap W^s(x) = \varphi^\tau(x')$ for some $\tau \in \mathbb{R}$, so there is a (uniform) $C \in \mathbb{R}$ for which $\varphi^t(x), \varphi^t(x')$ are $C\epsilon$-equivalent for $t > 0$. Thus, if $y \in S \cap B_{\eta_z}(x,\delta)$, then $d_{\varphi^{-\log\delta}(z)}(\varphi^{-\log\delta}(x), \varphi^{-\log\delta})(y)) < R$ and

$$d_{\mathscr{H}(\varphi^{-\log\delta}(z))}(\mathscr{H}(\varphi^{-\log\delta}(x)), \mathscr{H}(\varphi^{-\log\delta}(y))) < R+\theta(C\epsilon)$$

with $\theta(\epsilon) \xrightarrow[\epsilon\to 0]{} 0$ by uniform continuity of E^u, so Remark 8.6.29 implies

$$\eta_{\mathscr{H}(z)}(\mathscr{H}(x),\mathscr{H}(y)) < \Big(\frac{R+\theta(C\epsilon)}{R}\Big)^{1/a}\delta.$$

Then $\mathscr{H}(S \cap B_{\eta_z}(x_j,\delta_j)) \subset S' \cap B_{\eta_{\mathscr{H}(z)}}(\mathscr{H}(x_j), \big(\frac{R+\theta(C\epsilon)}{R}\big)^{1/a}\delta_j)$, hence the claim. □

Proof of Theorem 8.6.25. σ has (uniquely defined) bounded measurable densities $\rho_z\colon W^u(z) \to \mathbb{R}$ with respect to μ^u (Lemma 8.6.31) such that $\rho\colon M \to \mathbb{R}, z \mapsto \rho_z(z)$ is Φ-invariant (since $\mu^u \circ \varphi^t = e^{ht}\mu^u$ and $\sigma \circ \varphi^t = e^{ht}\sigma$) and measurable by Lemma 8.6.32 and its counterpart for Margulis measure, so $f \overset{\text{ae}}{=}$ const. by ergodicity of μ, that is, $f_z \equiv 1$ μ^u-a.e. on each $W^u(z)$ after normalization. □

It is interesting to note a few simple consequences of Theorem 8.6.25.

Proposition 8.6.33. *The topological and Liouville entropies of an m-manifold of curvature -1 (Chapter 2) are both $m-1$.*

Proof. The Liouville measure and the Margulis measure coincide, which implies equality of the entropies: The Liouville measure is the Sinai–Ruelle–Bowen measure and hence the equilibrium state of the unstable Jacobian of the geodesic flow, which is constant in this case. Thus, its equilibrium state is the measure of maximal entropy, and moreover, each μ^u is the Riemannian measure on its unstable leaf, which is Lebesgue measure, that is, the $m-1$-dimensional Hausdorff measure for the Riemannian distance because $m-1$ is the (topological) dimension of strong unstable leaves. Because the flow is conformal, this is the same as the $m-1$-dimensional spherical measure for η, so $h_{\text{top}} = h = m-1$ by Theorem 8.6.25. □

Remark 8.6.34. In the 2-dimensional case it turns out that the topological entropy of any other metric on the same surface (normalized so the average curvature is -1) is strictly bigger than 1, while the Liouville entropy is strictly smaller—provided the Riemannian metric has no focal points (Theorem 10.4.2).

Remark 8.6.35. The main ingredient in this proof is that the unstable Jacobian is constant, and this is more generally the case for geodesic flows of locally symmetric Riemannian manifolds (because the isometry group is transitive). However, Proposition 8.6.33 plays out differently for locally symmetric spaces of nonconstant curvature. The spherical dimension exceeds the topological dimension of unstable leaves because there is additional expansion missed in the definition of η. Specifically, when curvature is normalized to have maximum -1, the entropy of (the geodesic flow of) complex hyperbolic m-space is $2m$ because there is a 1-dimensional direction with expansion rate 2, and this adds 1 to the entropy[22]. For quaternionic hyperbolic m-spaces it is $4m+2$ because 3 of the $4m-1$ expanding directions have a rate of 2, and for the Cayley plane it is 22, accounting for 6 of 16 directions having expansion rate 2.

Dimension theory interacts in substantial ways with hyperbolic dynamics, and the preceding is just a small sample. In particular, the spherical dimension invoked here is different from the Hausdorff and other more commonly used notions of fractal dimensions. Numerous issues related to these are of interest in their own right and can provide additional information about a dynamical system, and others are of direct utility beyond themselves; an instance is the proper generalization of the Pesin Entropy Formula (Remark 8.4.11). There are very good books on the subject [**235**].

7. Asymptotic orbit growth*

Following Charles H. Toll, we establish a *multiplicative asymptotic* of the growth of periodic orbits for flows, that is, finer bounds than a determination of the exponential rate (Theorem 8.7.9).[23] The main ingredient is a description of "local product flow boxes" and their *full components* of intersection. These provide the context in which the equality of Bowen measure and Margulis measure as well as the earlier estimates give the multiplicative asymptotic of the growth of periodic points.

Throughout this section we again assume that Φ is a topologically mixing Anosov flow on a compact Riemannian manifold M.

Local product flow boxes are simple to describe and involve two size parameters ϵ and δ. We first obtain several technically useful properties which depend on δ being sufficiently small with respect to ϵ.

Take $0<\epsilon<1/2\min\{1,\delta_0\}$, where δ_0 is an expansivity constant of Φ (see Definition 1.7.2), that is, if $d(\varphi^t(x),\varphi^t(y))<\delta_0$ for all $t\in\mathbb{R}$ then $y\in\mathcal{O}(x)$. Assume furthermore that ϵ is less than the least period of orbits of Φ. Given $p\in M$ let $C\coloneqq B^{cu}(p)\coloneqq\bigcup_{0\le t\le\epsilon}\varphi^t(\overline{B^u(p,\delta)})$. If ϵ is sufficiently small then $B\coloneqq\bigcup_{z\in C}B^s(z)$ is

[22] ...of the Liouville measure by the Pesin Entropy Formula (Remark 8.4.11)

[23] A finer asymptotic in greater generality was contemporaneous with Toll's work [**228**].

a local product cube, where $B^s(z) \subset \overline{B^s(z,\delta)}$. For $x \in B$ denote by $B^u(x)$ the connected component of $W^u(x) \cap B$ containing x and similarly define $B^{cu}(x)$ and $B^s(x)$. Define π_C by $C \cap B^s(x) = \{\pi_C(x)\}$. If $z \in C$ then clearly B contains an orbit segment of (parameter) length ϵ. This is true for all $x \in B$:

Lemma 8.7.1. *If $x \in B$ then there exists $t_0 \in [0,\epsilon]$ such that $\{\varphi^t(x) \mid t \in [t_0 - \epsilon, t_0]\} \subset B$.*

Proof. Let $z = \pi_C(x)$. If $\varphi^t(z) \in C$ then $\varphi^t(B^s(x)) = B^s(\varphi^t(x))$, hence $\{\varphi^t(x)\} = B^s(\varphi^t(z)) \cap B^{cu}(x) \subset B$. □

From Section 6.2 we recall

Theorem 8.7.2. *There are η, $\gamma > 0$ such that if $d(x,y) < \eta$ then there is a unique $\theta = \theta(x,y) \in (-\gamma,\gamma)$ such that $\varnothing \neq B^s(x,\gamma) \cap B^u(\varphi^\theta(y)) =: \{[x,y]\}$. θ and $[\cdot,\cdot]$ are continuous on $\{(x,y) \in M \times M \mid d(x,y) < \eta\}$.*

Define $\tau \colon B \to [0,\epsilon]$ by $z \in \varphi^{\tau(z)}(B^u(p))$ when $z \in C$ and $\tau(x) := \tau(\pi_C(x))$ for $x \in B$. For sufficiently small δ uniform continuity of the unstable foliation yields

$$\tau(B^u(x)) \subset [\tau(x) - \epsilon^2, \tau(x) + \epsilon^2] \tag{8.7.1}$$

for $x \in B$. We furthermore assume that δ is also small enough that if $y \in B^u(x)$ then $B^s(x)$ and $B^s(y)$ are ζ-equivalent (this being defined analogously to Definition 8.6.1), where ζ is as in Lemma 8.6.15 applied with our choice of ϵ. Thus

$$\left| \frac{\mu^s(B^s(x))}{\mu^s(B^s(y))} - 1 \right| < \epsilon.$$

Lemma 8.7.3. *There exists $K > 0$ (independent of ϵ and δ) such that if $x, y \in B$ then*

$$\left| \frac{\mu^s(B^s(x))}{\mu^s(B^s(y))} - 1 \right| < K\epsilon.$$

Proof. Suppose $\tau(x) = t$, $\tau(y) = t'$, $\tau(w) = 0$. Then

$$\frac{\mu^s(B^s(x))}{\mu^s(B^s(y))} = \frac{\mu^s(B^s(x))}{\mu^s(B^s(\varphi^t(w)))} \underbrace{\frac{\mu^s(B^s(\varphi^t(w)))}{\mu^s(B^s(\varphi^{t'}(w)))}}_{=e^{h(t-t')}} \frac{\mu^s(B^s(\varphi^{t'}(w)))}{\mu^s(B^s(y))}$$

The remaining fractions differ from 1 by at most ϵ, so we obtain the claim. □

We write $x \sim y$ if $x, y \in B$ lie on a common orbit segment contained in B and let $[x] := \{y \in B \mid y \sim x\}$. Relation (8.7.1) shows that if $y \in B^{cu}(x)$ and $\tau(x), \tau(y) \in (\epsilon^2, \epsilon - \epsilon^2)$ then $\bigcup_{z \in B^u(x)} [z] = \bigcup_{z \in B^u(y)} [z]$. For $x \in B$ we set $\Delta(x) := \sup\{r > 0 \mid B^u(x,r) \subset B\}$. For $T > 0$ there exists $r(T)$ (exponentially decreasing as $T \to \infty$) such that if $x, \varphi^T(x) \in B$, $\Delta(x) > r(T)$ then $B^u(\varphi^T(x)) \subset \varphi^T(B)$.

Definition 8.7.4. Let $B^\circ(T) := \{x \in B \mid \epsilon^2 < \tau(x) < \epsilon - \epsilon^2,\ \Delta(x) > r(T)\}$. If Δ_0 is a connected component of $B^\circ(T) \cap \varphi^T(B^\circ(T))$ then $\Delta := \bigcup_{x\in\Delta_0}[x] \cap \varphi^T(B)$ is called a *full component of intersection.*

An important observation is that full components of intersection essentially correspond bijectively to periodic orbits. This is the content of the next two lemmas.

Lemma 8.7.5. *If Δ is a full component of intersection of $B \cap \varphi^T(B)$ then Δ intersects a unique orbit of period in $[T-\epsilon, T+\epsilon]$.*

Proof. Consider the action of the iterates of φ^T on the projection of Δ to $B/\sim$ to obtain a unique fixed point which corresponds to the desired orbit. □

Conversely it is easy to check

Lemma 8.7.6. *Each orbit segment in $B^\circ(T)$ of length $\epsilon - 2\epsilon^2$ that belongs to a periodic orbit of period in $[T-(\epsilon-2\epsilon^2), T+(\epsilon-2\epsilon^2)]$ intersects a unique full component of intersection of $B \cap \varphi^T(B)$.*

Next we estimate the number of full components of intersection:

Proposition 8.7.7. *Let $\Delta(T)$ be the number of full components of intersection in $B \cap \varphi^T(B)$. Then $\Delta(T) = 2e^{hT}\mu(B)(1+O(\epsilon))(1+o(T^0))$, with $O(\epsilon)$ independent of B.*

Proof. We will calculate $\mu(\varphi^T(B))$ in (8.7.3) and $\mu(B \cap \varphi^T(B))$ in (8.7.6). Since μ is mixing (Theorem 8.3.6), this yields the claim. If $x, y \in C$ then $B^s(x)$ and $B^s(y)$ are ζ-equivalent, hence $\mu^s(B^s(x)) = (1+O(\epsilon))\mu^s(B^s(y))$, and by (8.6.4) $\mu^s(\varphi^T(B^s(x))) = (1+O(\epsilon))\mu^s(\varphi^T(B^s(y)))$ as well and hence

$$\mu^s(\varphi^T(B^s(x))) = C(T)(1+O(\epsilon)), \tag{8.7.2}$$

with $C(T)$ independent of $x \in B$. Since $\mu^{cu}(\varphi^T(C)) = e^{hT}\mu^{cu}(C)$ we get

$$\mu(\varphi^T(B)) = e^{hT}\mu^{cu}(C)C(T)(1+O(\epsilon)), \tag{8.7.3}$$

with $O(\epsilon)$ independent of B.

To calculate $\mu(B \cap \varphi^T(B))$ we first show that it suffices to consider full components of intersection. Any point of $B \cap \varphi^T(B)$ not contained in a full component of intersection is in

$$A_T := \Big(\varphi^T\big(\bigcup_{0\le t\le\epsilon^2} B_t\big) \cap \bigcup_{\epsilon-\epsilon^2\le t\le\epsilon} B_t\Big) \cup \Big(\varphi^T\big(\bigcup_{\epsilon-\epsilon^2\le t\le\epsilon} B_t\big) \cap \bigcup_{0\le t\le\epsilon^2} B_t\Big) \cup (\{x \in B \mid \Delta(x) \le r(x)\} \cap \varphi^T(B_t)),$$

because it is too close to the boundary of B either in the time direction or in an unstable leaf. By mixing, each of the first two sets has measure $\epsilon^2\mu(B)^2(1+o(T^0))$. The measure of the third set decreases exponentially with T, so by absorbing it into the error we have

$$\mu(A_T) \le 2\epsilon^2\mu(B)^2(1+o(T^0)). \tag{8.7.4}$$

To prove the proposition it is clearly useful to calculate the measure of full components of intersection via $\Delta(T)$ and their average measure. To calculate the average measure note that a full component of intersection Δ is of the form $\Delta = B \cap \varphi^T(\Delta')$, where $\Delta' = \bigcup\{[x] \mid x \in B,\ \varphi^T(x) \in \Delta\}$ and either $\varphi^T(\Delta' \cap B_\epsilon) \subset \Delta$ (a "front intersection component") or $\varphi^T(\Delta' \cap B_0) \subset \Delta$ (a "back intersection component"). In the first case we define the thickness of Δ by

$$\theta(\Delta) := \inf\{\tau(x) \mid x \in \varphi^T(\Delta' \cap B_\epsilon)\},$$

and in the second by

$$\theta(\Delta) := \epsilon - \sup\{\tau(x) \mid x \in \varphi^T(\Delta' \cap B_0)\}.$$

By (8.7.1) every orbit segment in Δ has (parameter) length $\epsilon(1 + O(\epsilon))$, so (8.7.2) gives

$$\mu(\Delta) = \frac{1}{\epsilon}\theta(\Delta)\mu^{cu}(C)C(T)(1 + O(\epsilon)). \tag{8.7.5}$$

Naturally we have

Lemma 8.7.8. *The average thickness of full components of intersection of $B \cap \varphi^T(B)$ is $(\epsilon/2)(1 + O(\epsilon))(1 + o(T^0))$.*

Proof. Partition B into $n := [1/\epsilon]$ sets $S_i := \bigcup\{B_t \mid \epsilon j \le tn \le \epsilon(j+1)\}$ $(0 \le j < n)$. For sufficiently large T the number of components of $\varphi^T(S_{n-1}) \cap S_i$ is independent of j by mixing, so the average thickness is $(\epsilon/2)(1 + O(1/n))(1 + o(T^0))$ as claimed. □

Since the error in (8.7.4) can be absorbed into $O(\epsilon)(1 + o(T^0))$ we obtain from (8.7.5) and Lemma 8.7.8

$$\mu(B \cap \varphi^T(B)) = \frac{1}{2}\Delta(T)C(T)\mu^{cu}(C)(1 + O(\epsilon))(1 + o(T^0)). \tag{8.7.6}$$

Since μ is mixing, (8.7.3) yields

$$\mu(B \cap \varphi^T(B)) = \mu(B)\mu^{cu}(C)e^{hT}C(T)(1 + O(\epsilon))(1 + o(T^0)),$$

which, with (8.7.6), yields the claim. □

We now give the promised multiplicative asymptotic for the growth of the number $P_t(\Phi)$ of periodic orbits of Φ with period at most t.

Theorem 8.7.9 (Multiplicative orbit-growth asymptotic)**.** *If Φ is a topologically mixing Anosov flow on a compact Riemannian manifold M, then*

$$\lim_{t\to\infty} t h_{\text{top}}(\Phi) P_t(\Phi) e^{-t h_{\text{top}}(\Phi)} = 1, \quad \textit{that is,} \quad P_t(\Phi) \sim \frac{e^{t h_{\text{top}}(\Phi)}}{t h_{\text{top}}(\Phi)}.$$

Remark 8.7.10. That $e^{th_{\mathrm{top}}(\Phi)}/t$ is the growth rate of $P_t(\Phi)$ was previously known [**53**] but Margulis was the first to show that $\lim_{t\to\infty} th_{\mathrm{top}}(\Phi)P_t(\Phi)e^{-th_{\mathrm{top}}(\Phi)}$ exists [**212**]; Toll was the first to determine this limit.

Proof. We write $P_{T,\epsilon} \coloneqq \operatorname{card} \mathbb{O}_t(T)$ (Definition 4.2.22) and let μ_B denote the Bowen measure. Then $\mu_B(B) = \mu_B(B^\circ(T))(1 + O(\epsilon))$ for sufficiently large T, and Lemmas 8.7.5 and 8.7.6 imply

$$\begin{aligned}\frac{\epsilon\Delta(T)}{P_{T,\epsilon}} &= \frac{1}{P_{T,\epsilon}} \sum_{\mathscr{O}\in\mathbb{O}_\epsilon(T)} \delta_{\mathscr{O}}(B)\\ &\le \mu_B(B)(1+o(T^0)) = \mu_B(B^\circ(T))(1+O(\epsilon))(1+o(T^0))\\ &= \frac{1}{P_{T,\epsilon-2\epsilon^2}} \sum_{\mathscr{O}\in\mathbb{O}_{\epsilon-2\epsilon^2}(T)} \delta_{\mathscr{O}}(B^\circ(T))(1+O(\epsilon)) \le \frac{\epsilon\Delta(T)}{P_{T,\epsilon-2\epsilon^2}}(1+O(\epsilon)).\end{aligned}$$

By Proposition 8.7.7 this yields

$$P_{t,\epsilon-2\epsilon^2} \le \frac{2\epsilon e^{hT}\mu(B)}{\mu_B(B)}(1+O(\epsilon))(1+o(T^0)) \le P_{t,\epsilon}.$$

Replacing ϵ by ϵ' with $\epsilon' - 2\epsilon'^2 = \epsilon$ introduces another factor of $1 + O(\epsilon)$ and by Theorem 8.6.20 we get

$$P_{T,\epsilon} = 2\epsilon e^{hT}(1+O(\epsilon))(1+o(T^0)). \tag{8.7.7}$$

Since $P_{T,\epsilon}$ is the number of periodic orbits with *a* period in $(T-\epsilon, T+\epsilon]$, we have

$$P_{T,\epsilon} = T_1(P_{T+\epsilon}(\Phi) - P_{T-\epsilon}(\Phi)) + T_2(P_{(T+\epsilon)/2}(\Phi) - P_{(T-\epsilon)/2}(\Phi)) + \cdots,$$

where $iT_i \in [T-\epsilon, T+\epsilon]$. By (8.7.7) this simplifies to $P_{T,\epsilon} = T_1(P_{T+\epsilon}(\Phi) - P_{T-\epsilon}(\Phi))(1+o(T^0))$. Using $T_1 = T(1+o(T^0))$ and (8.7.7) we find

$$P_{T+\epsilon}(\Phi) - P_{T-\epsilon}(\Phi) = \frac{2\epsilon}{T}e^{hT}(1+O(\epsilon))(1+o(T^0)). \tag{8.7.8}$$

Now fix $T_0 > 1/h$ such that $|o(T^0)| < \epsilon$ for all $T \ge T_0$ on the right-hand side. Writing

$$\begin{aligned}P_{T+\epsilon}(\Phi) = (P_{T+\epsilon}(\Phi) - P_{T-\epsilon}(\Phi)) + (P_{T-\epsilon}(\Phi) - P_{T-3\epsilon}(\Phi))\\ + \cdots + (P_{T-2j\epsilon+\epsilon}(\Phi) - P_{T-2j\epsilon-\epsilon}(\Phi)) + P_{T-2j\epsilon-\epsilon}(\Phi)\end{aligned}$$

for $T-(2j+1)\epsilon \le T_0 < t - 2j\epsilon$ and estimating the differences by (8.7.8) gives

$$P_{T+\epsilon}(\Phi) = \frac{e^{hT}}{T}S(T)(1+O(\epsilon))(1+o(T^0)),$$

where $S(T){:=}2\epsilon\Big(1{+}\dfrac{T}{T-2\epsilon}e^{-2\epsilon h}{+}\cdots{+}\dfrac{T}{T-2j\epsilon}e^{-2j\epsilon h}\Big)$ and we absorbed $P_{T-2j\epsilon-\epsilon}(\Phi)$ into $(1+o(T^0))$. Observe now that $S(T)$ is a Riemann sum for

$$\underbrace{\int_0^{T-T_0}\frac{T}{T-x}e^{-hx}\,dx}_{=Te^{-hT}\int_{T_0}^{T}\frac{e^{hu}}{u}du} = Te^{-hT}\Big(\frac{e^{hT}}{hT}-\frac{e^{hT_0}}{hT_0}-T\int_{T_0}^{T}\frac{e^{hu}}{hu^2}du\Big)=\frac{1}{h}(1+o(T^0)).$$

The integrand decreases on $[0,T-T_0]$, so $S(T)=\dfrac{1}{h}(1+O(\epsilon))(1+o(T^0))$ and hence

$$P_T(\varphi)=P_{T+\epsilon}(\Phi)(1{+}O(\epsilon))=\frac{e^{hT}}{T}S(T)(1{+}O(\epsilon))(1{+}o(T^0))=\frac{e^{hT}}{hT}(1{+}O(\epsilon))(1{+}o(T^0)),$$

that is, $\lim_{T\to\infty}\dfrac{P_T(\varphi)}{e^{ht}/hT}=(1+O(\epsilon))$, proving the claim. □

Theorem 8.7.11 (Margulis)**.** *Let M be a compact Riemannian manifold of negative sectional curvature, $G(t)$ the number of different closed geodesics of length at most t, and h the topological entropy of the geodesic flow. Then $G(t)2the^{-th}\xrightarrow[t\to\infty]{}1$.*

Proof. Each closed geodesic of length t defines exactly two period-t orbits of the geodesic flow. The geodesic flow is a contact Anosov flow, hence topologically mixing by Theorem 9.1.2. Thus Theorem 8.7.9 applies. □

While this is a remarkably fine asymptotic, the understanding of the growth of periodic orbits has become even more refined. The published version of the Margulis asymptotic includes a broad survey of related investigations [**8**, **130**, **212**]. The dynamical zeta-function encodes all periods of a flow in one function and is an important ingredient of these refinements of orbit-counting as well as an important tool in several areas of hyperbolic dynamics [**8**, **130**, **212**, **212**, **229**].

Definition 8.7.12 (ζ-function)**.** The *ζ-function* is defined by

$$\zeta_\Phi(s)=\prod_\gamma\Big(1-e^{-s\,l(\gamma)}\Big)^{-1}, \tag{8.7.9}$$

where the product is taken over all (nonfixed) closed orbits γ of the flow and $l(\gamma)$ is the smallest positive period ("length") of γ.

If $P_T(\Phi)<\infty$ (Definition 4.2.1) for all T, then a standard convergence criterion for infinite products (or studying its logarithm) shows that this product converges for $\operatorname{Re}s>p(\Phi)$ and has singularities on the critical line $\operatorname{Re}s=p(\Phi)$.

Remark 8.7.13 (Discrete-time ζ-function)**.** We digress to connect this form of the ζ-function to the one more commonly used for discrete-time systems. To that end, let $P_n(f) := \operatorname{card}\operatorname{Fix}(f^n)$ be the number of periodic points of a map f, and define the ζ-function for f by

$$\zeta_f(z) = \exp \sum_{n\in\mathbb{N}} \frac{P_n(f)}{n} z^n,$$

where $z \in \mathbb{C}$. By the ratio test this converges for $|z| < \exp(-p(f))$ and has singularities on the circle $|z| = \exp(-p(f))$. The coefficient $\frac{P_n(f)}{n}$ is the number of n-periodic *orbits*, so

$$\zeta_f(z) = \exp \sum_{n\in\mathbb{N}} \sum_{\substack{n\text{-periodic}\\ \text{orbits}}} z^n.$$

Here, for instance, a 3-periodic orbit appears again as a 6-periodic orbit, a 9-periodic orbit, and so on, and this invites reexpressing the sums in terms of sums over *prime* (or shortest) orbits:

$$\sum_{n\in\mathbb{N}} \sum_{\substack{n\text{-periodic}\\ \text{orbits}}} z^n = \sum_{\gamma \text{ prime}} \sum_{k\in\mathbb{N}} \frac{z^{kl(\gamma)}}{k},$$

where the denominator accounts for the overcounting in this rearranged sum, so

$$\zeta_f(z) = \exp \sum_{\gamma \text{ prime}} \underbrace{\sum_{k\in\mathbb{N}} z^{kl(\gamma)}/k}_{=-\log(1-z^{l(\gamma)})} = \prod_{\gamma \text{ prime}} \exp(-\log(1 - z^{l(\gamma)})) = \prod_{\gamma \text{ prime}} (1 - z^{l(\gamma)})^{-1}.$$

Taking here $z = e^{-s}$ gives (8.7.9). And this form does not require the periods to be integers. Moreover, for flows it is natural to work with prime orbits, because these are the canonical representatives of the orbits as geometric objects.

Remark 8.7.14. (8.7.9) shows the analogy with the Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \overset{\text{Euler}}{\underset{\substack{\text{product}\\ \text{formula}}}{=\!=}} \prod_{p \text{ prime}} (1 - p^{-s})^{-1},$$

where the Euler product formula is seen by an argument due to Euler:

$$\frac{1}{2^s}\zeta(s) = \sum_{n \text{ even}} \frac{1}{n^s}, \text{ so } (1 - \frac{1}{2^s})\zeta(s) = \sum_{2\nmid n} \frac{1}{n^s}; \text{ likewise } (1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\zeta(s) = \sum_{2\nmid n,\ 3\nmid n} \frac{1}{n^s},$$

and so on through all primes, giving $\prod_p (1 - p^{-s})\zeta(s) = 1$.

The numbers $P_T(\Phi)$ and the ζ-function are obviously invariants of conjugacy. In general, they are *not* invariant under orbit equivalence. Although an orbit equivalence takes periodic orbits into periodic orbits, it may change their period. However, a cruder property survives a time change.

Proposition 8.7.15. *Let X, Y be compact metric spaces and $\varphi^t\colon X \to X$ and $\psi^t\colon Y \to Y$ continuous flows without fixed points. Suppose that the flows are orbit equivalent and $p(\Psi) = 0$. Then $p(\Phi) = 0$.*

Proof. Let $h\colon X \to Y$ be a homeomorphism that maps orbits of the flow φ^t onto orbits of ψ^t. Then $h(\varphi^1(x)) = \psi^{\alpha(x)} h(x)$ with α continuous and positive; hence it is bounded from above, say $\alpha(x) \le M$. Thus the image of any orbit segment for Φ of length 1 is an orbit segment of Ψ of length at most M. Hence the image of a segment of length at most T has length less than $(T+1)M$ which for $T \ge 1$ is less than $2MT$. That means that the image of any periodic orbit of period $\le T$ has period $\le 2MT$, that is,

$$P_{2MT}(\Psi) \ge P_T(\Phi),$$

and $p(\Psi) \ge p(\Phi)/2M$. □

Example 8.7.16 ([**260**])**.** For the flow Φ under the function r over the full 2-shift defined by $r = r_i$ on the cylinder $\omega_0 = i$ for $i = 1, 2$

$$\begin{aligned}\zeta(s) &= \exp - \sum_{n\in\mathbb{N}} \frac{1}{n} \sum_{\omega=\sigma^n(\omega)} \exp\Big(\sum_{k=0}^{n-1} r(\sigma^k(\omega))\Big)\\ &= \exp - \sum_{n\in\mathbb{N}} \frac{1}{n} \sum_{p=0}^{n} \binom{n}{p} \exp(-s(pr_0 + (n-p)r_1)) = 1 - e^{-r_0 s} - e^{-r_1 s}.\end{aligned}$$

We remark briefly how the ζ-function has been used to count periodic orbits. Initially, this was done for flows over subshifts of finite type with a roof function such that the periods are incommensurate. The pertinent "weight" is $N(\gamma) = e^{h\ell(\gamma)}$, where ℓ is the (least) period of the periodic orbit γ and $h = h_{\text{top}}(\Phi)$. Then

$$\pi(T) := \operatorname{card}\big\{\gamma \bigm| \gamma \text{ periodic with } N(\gamma) \le T\big\},$$

that is, $\pi(T) \log T / T \xrightarrow[T\to\infty]{} 1$, if r is locally constant [**230**] or Hölder continuous [**227**], which via coding then applies to hyperbolic flows (with incommensurate periods), including mixing Anosov flows, thus generalizing Theorem 8.7.9.

The proof of the prime number theorem (that the number of primes of size up to N is asymptotically $N/\log N$) uses analytic properties of the Riemann ζ-function, and in the present context this is mirrored by establishing analogous properties of the dynamical ζ-function. For locally constant roof functions $r(x) = r(x_0, x_1)$ over a subshift Σ_A this is easy to describe: for $s \in \mathbb{C}$ let $P^s_{ij} := A_{ij} e^{-sr(i,j)}$ to get $\zeta(s) = \det(\mathrm{Id} - P^s)^{-1}$. For Hölder continuous roof functions, one instead uses Ruelle transfer operators—which also play a central role in studying correlation decay (Section 8.5), a subject which is thereby closely connected to orbit-counting.

CHAPTER 9

Anosov flows

One of the purposes of this book is to examine those features of hyperbolic systems with continuous time that are in particularly sharp relief with the discrete-time counterparts. Anosov systems provide rich material in this respect. While there are many parallels in their fundamental aspects—provided one watches out for longitudinal effects in flows (Section 9.1)—there are fundamental issues that play out differently to an astonishing extent.

For instance, is an Anosov system topologically transitive? In discrete time this question was answered in the affirmative long ago—for the Anosov diffeomorphisms known to date. However, a useful answer is not expected any time soon. This is related to whether one can hope to grasp in some way the scope of Anosov systems, and in discrete time there has been little progress for decades. It is so far fair to say, then, that Anosov diffeomorphisms seem to be rather rigid objects from the point of view of topological conjugacy in that all known examples are topologically equivalent to an algebraic diffeomorphism, specifically a hyperbolic toral automorphism or a counterpart on infranilmanifolds while at the same time we have little idea whether this is all there is..

For Anosov flows the state of affairs could hardly be more different. While the question about their transitivity took some efforts to resolve (Section 9.3), the answer is no, and this is but one manifestation of the proliferation of examples of Anosov *flows* that have arisen, in no small part by surgery constructions. This, indeed, may be one of the fundamental distinctions at play here: surgery constructions are naturally suited for the creation of new flows from old and have no equally natural discrete-time counterpart. Accordingly, some sections of this chapter are devoted to surgeries that produce new kinds of Anosov flows. First, even among geodesic flows of negatively curved Riemannian manifolds (Theorem 5.2.4) there are instances where the fundamental group is different from that of any locally symmetric space, so the phase space of the geodesic flow is also topologically distinct from the unit tangent bundle of any locally symmetric space and hence the geodesic flows are not orbit equivalent to an algebraic system. Furthermore, already in dimension 3 there are compact manifolds that admit Anosov flows without being homeomorphic to a mapping torus (suspension manifold) or the unit tangent bundle of any surface; Section 9.2 gives such an example. Among those examples there are even Anosov flows with nowhere dense nonwandering sets which are therefore dynamically quite different from any volume-preserving flow (Section 9.3). In this context it is the wealth of examples that poses a challenge for any classification, but we present a range of insights pertinent to structural analysis of Anosov flows, and in contrast to the discrete-time context, one arguably sees progress towards a comprehensive understanding of Anosov flows, even though that goal still seems far out.

As noted, Anosov flows can be topologically transitive or not, which is established by our prior examples combined with those from Section 9.3. There is another dichotomy among Anosov flows: they may be $\mathbb{R}$-*covered* (Definition 9.4.4) or not, and in this dichotomy as well, Section 9.3 provides the new kind of Anosov flow. These dichotomies overlap partially—being $\mathbb{R}$-covered implies topological transitivity (Proposition 9.4.6)—but not fully. Sections 9.4 and 9.5 introduce not only this latter dichotomy and the pertinent terminology but also substantial tools for structural analysis.

Section 9.6 finally explores an altogether separate agenda by extending to Anosov flows in more generality and depth the observations we used about the geodesic and horocycle flows of compact factors of the hyperbolic plane in order to develop their ergodic properties. Their "commutation" relation was the central tool, and we show how much further one can push it. This also shows up interesting interactions with other subjects we have discussed, and it is also particular to continuous time. It may be of special interest here that hyperbolic dynamics provides the tools for studying a parabolic flow, but that this is by a deep entanglement of the flows in the arguments rather than by an application of results from hyperbolic dynamics, and that this in turn more deeply illuminates some of the reasoning we encountered early on in this book. Particularly in this respect it is coherent with the other sections in this chapter, because several of them as well are centered on geodesic flows as a starting point for a broader and deeper understanding of other flows.

The style of this chapter is a little different from previous chapters in that some results are presented more in the style of a survey. We define the necessary notions, and give full proofs of many of the results, but we are a little more liberal in stating results without proof.

1. Suspensions versus mixing

As a start to embarking on the program described in the introduction, we begin with a more basic issue relative to longitudinal effects, which is the relationship between being topologically transitive versus topologically mixing. In discrete time these coincide in the Anosov case: transitive Anosov diffeomorphisms are mixing, and this situation corresponds to density of unstable (or stable) manifolds of periodic points. There is a corresponding fact for transitive Anosov flows—weak unstable manifolds of periodic points are dense (Corollary 6.2.10)—but for the suspension of a transitive Anosov diffeomorphism only *weak* unstable manifolds are dense (by Theorem 6.2.12 and Example 1.6.34). Indeed, density of weak unstable manifolds versus density of strong unstable manifolds is the very difference that corresponds to the difference between the flow being transitive versus mixing. This is the subject of the present section: transitive Anosov flows are mixing unless they are suspensions.

Theorem 9.1.1 (Plante)**.** *For a transitive Anosov flow either*

(1) *each strong stable and each strong unstable manifold is dense (so the flow is topologically mixing by Theorem 6.2.12), or*
(2) *the flow is a suspension of an Anosov diffeomorphism (and hence not topologically mixing by Example 1.6.34).*

This raises the question of how one rules out a suspension to obtain mixing for a flow. We here give an important example that includes geodesic flows.

Theorem 9.1.2. *Contact Anosov flows are topologically mixing.*

Proof. The contact form θ provides a smooth φ^t-invariant measure via the volume element $\theta \wedge (d\theta)^{m-1}$, so $NW(\varphi^t) = M$ by the Poincaré Recurrence Theorem 3.2.1. The suspension scenario in Theorem 9.1.1 is ruled out by

Lemma 9.1.3. $\ker\theta = E^+ \oplus E^-$.

Proof. The contact form θ vanishes on (strong) stable vectors: the unit tangent bundle of M is compact, so by continuity θ is bounded on it, that is, if v is any vector then $|\theta(v)| \le \text{const.}\,\|v\|$. If $v \in E^-(x)$, then φ^t-invariance of θ yields $|\theta(v)| = |\varphi^t_*\theta(v)| = |\theta(\varphi^t(v))| \le \text{const.}\,\|\varphi^t(v)\| \xrightarrow[t\to\infty]{} 0$, so $\theta(v) = 0$. Likewise, $\theta = 0$ on unstable vectors. □

Now, if Φ is a suspension, then $\ker\theta = E^+ \oplus E^-$ is tangent to the foliation into level sets of t of the suspension manifold, hence $\theta = f\,dt$ with Φ-invariant, hence constant smooth f (Proposition 1.6.4), and $d\theta = f \cdot ddt = 0$. But then $\theta \wedge d\theta \equiv 0$, contrary to the contact property. □

Corollary 9.1.4. *Geodesic flows of compact negatively curved Riemannian manifolds are topologically mixing.*

Proof. They are contact Anosov flows (Proposition 2.6.28). □

We note a complementary theorem by Plante.

Theorem 9.1.5. *If Φ is a C^2 Anosov 3-flow whose invariant splitting is C^1, then Φ is topologically transitive and either a suspension (of an Anosov diffeomorphism of $\mathbb{T}^2$ because $E^s \oplus E^u$ is integrable) or a contact flow in the sense that the canonical invariant 1-form A defined by $A(\dot\varphi) = 1$, $A_{\restriction_{E^u \oplus E^s}} = 0$ defines an invariant ergodic measure equivalent to the Riemannian volume.*

Proof [**240**, Theorem 4.7]. By assumption, the canonical form is C^1, so $\theta := A \wedge dA$ is a well-defined Φ-invariant 3-form. Unless $\theta \equiv 0$, Theorem 8.1.28 implies the second conclusion (and transitivity). If $\theta \equiv 0$, then $\ker A = E^s \oplus E^u$ is integrable [**145**], that is, tangent to a C^1 foliation, and the argument after Lemma 9.1.9 shows that Φ is the suspension of an Anosov diffeomorphism (of $\mathbb{T}^2$), hence topologically transitive. □

In contrast to its ergodic counterpart (Definition 8.1.1) the present context does not allow exceptional null sets with respect to saturation:

Definition 9.1.6. A set is said to be *saturated* by a foliation if it is a union of leaves.

Proof of Theorem 9.1.1. Proposition 6.2.14 shows that if there is a periodic point whose strong unstable manifold is not dense, then the flow is a suspension. By Proposition 6.2.18 the only other possiblity is that all strong unstable manifolds are dense (not just those of periodic points). It remains to show that the suspension obtained in the first case is not just topological, that is, to show that the set over which we suspend is indeed a C^1 submanifold.

To that end recall that for a suspension flow each strong unstable leaf densely fills a codimension-one submanifold that is (up to a shift by the flow) the manifold over which we suspend. Since the same goes for strong stable leaves, joint integrability (Definition 8.2.4) is clearly necessary for being a suspension.

Lemma 9.1.7. *W^u and W^s are jointly integrable (Definition 8.2.4) unless all strong unstable and all strong stable leaves are dense.*

Proof. If any one strong unstable or strong stable leaf fails to be dense, then by Proposition 6.2.18 this is the case for a leaf of a periodic point, and then the flow is a suspension of a map $f = \varphi^s{}_{\restriction_K}$ of the W^u-saturated set K in Proposition 6.2.14.

Claim 9.1.8. *K is W^s-saturated.*

Proof. If $x \in K$, $y \in W^{ss}(x)$ take t such that $y \in \varphi^t(K)$. To see that $t = 0$ note that K and $\varphi^t(K)$ are φ^s-invariant and compact while $d((\varphi^s)^n(x), (\varphi^s)^n(y)) \xrightarrow[n\to\infty]{} 0$. □

The claim implies that $W^{ss}(y) \cap W^{cs}_{\delta'}(y)$ lies entirely in some $\varphi^t(K)$, and moreover, that $\mathcal{H}_{y,z}(W^{ss}(y) \cap W^{cs}_{\delta'}(y))$ also lies in $\varphi^t(K)$.

If the desired inclusion fails, that is,

$$\mathcal{H}_{y,z}(W^{ss}(u) \cap W^{cs}_{\delta'}(y)) \not\subset W^{ss}(\mathcal{H}_{y,z}(u)) \cap W^{cs}_{\delta}(z),$$

the mismatch can only be in the time direction, which means that there is an open interval of $s \in \mathbb{R}$ for which $\varphi^s(K) \cap K \neq \varnothing$, contrary to Proposition 6.2.14. □

It is now easy to see that this defines a smooth structure:

Lemma 9.1.9. *If W^u and W^s are jointly integrable, then $E^u \oplus E^s$ is tangent to a C^1 foliation.*[1]

Remark 9.1.10. It is important that the foliation in question does not just have smooth leaves but C^1 foliation charts.

Proof. One can see that through each point there is a C^1 submanifold to which $E^u \oplus E^s$ is tangent. To see that the resulting foliation has C^1 foliation charts we use that for any ball B in a leaf we have a local parametrization $(-\delta, \delta) \times B \to M$, $(s, x) \mapsto \varphi^s(x)$. Once one identifies B with a euclidean ball, this collection of parametrizations define the required C^1-atlas of M. □

To conclude the proof of Theorem 9.1.1 we need to show that the leaves of the C^1-foliation in the preceding lemma are the $\varphi^t(K)$. To that end take a leaf L and note first that since L contains a strong unstable manifold it is dense in some $\varphi^t(K)$. This reduces the problem to showing that L is a compact submanifold of M.

If this is not so, then there are points in L that are far apart in the intrinsic distance, yet close is the metric on M. Since L is a codimension-one submanifold for which the projection π from Proposition 6.2.14 is locally injective, this means that these points are separated by a small time shift. However, this gives small values of s for which $\varphi^{t+s}(K) \cap \varphi^t(P) \neq \varnothing$, contrary to Proposition 6.2.14. □

2. Foulon–Handel–Thurston surgery

In this section, we modify the geodesic flow of a compact surface of constant negative curvature (Chapter 2) by a surgery construction to obtain flows of an entirely new topological type (Theorem 9.2.3). These are transitive (indeed, mixing) and hence do not add directly to the investigation of the transitivity question, but

[1] This does not imply that E^u or E^s are C^1, only that $E^u \oplus E^s$ is.

they introduce us to a much broader class of $\mathbb{R}$-covered flows, as we will call them, than geodesic flows and suspensions alone.

It may help the intuition not only for this section but also for the structural analysis later in the chapter to revisit some features of those geodesic flows with particular dynamical aspects in mind. This discussion is not needed in a technical sense, so a hurried reader can skip ahead to Definition 9.2.2 and Theorem 9.2.3.

Early on we introduced sections for a flow (Figure 0.1.1), and when studying geodesic flows on the sphere, Birkhoff introduced a local section that can also be helpful in our context. He considered those tangent vectors at points on the equator of the sphere that point into the northern hemisphere, say. In the unit tangent bundle this is an annulus (S^1 in the base times an open interval of angles, and at times it is convenient to consider the closure, which additionally includes 2 periodic orbits as the boundary components (the equator traveled east and west, respectively).

Definition 9.2.1. A *Birkhoff annulus* is an embedded annulus whose interior is transverse to the flow and whose boundary consists of 2 periodic orbits.

Birkhoff annuli abound for geodesic flows of a negatively curved surface because this construction works for any geodesic, the annulus being those unit vectors along it whose inner product with a chosen normal vector field is nonnegative. Although for the geodesic flow this may seem overcomplicated, let us give a purely dynamical description of this construction. To that end, consider the upper half-plane model $\mathbb{H}$ and suppose the geodesic in question is the imaginary axis I (with upward vectors). Taking the rightward horizontal unit vector field to determine the Birkhoff annulus means that we select those points whose orbits limit in positive time on ∞ or a nonnegative real point. The following alternative description eliminates the need for the normal vector field as a starting point: For every such orbit, its unstable leaf (outward unit vectors on a circle tangent to $\mathbb{R}$ at the nonnegative positive-time limit) contains a point of the stable manifold of I *to the right of* I. Likewise, each stable leaf of such a point contains a point in the right half of the unstable leaf of I. In an Anosov 3-flow the weak-stable or weak-unstable leaf of an orbit is bisected by the orbit; choosing one half for each gives a description of the Birkhoff annulus construction without appeal to an underlying geometric structure. (Not every pairing gives a nontrivial set; consider the right unstable and left stable halves of I, for example.)

Returning to the choice of the right half of the stable manifold of I and the right half of the unstable manifold of I, the set of orbits it encompassed can be viewed as a "rectangle" with these halves as its sides because every orbit is characterized by its pair of (weak) stable and unstable leaves. This rectangle includes I and its reverse as vertices, but the 2 other vertices are missing because there is no orbit that is positively asymptotic to I and negatively asymptotic to its reverse because there is no geodesic with both (asymptotic) end points at 0 (Figure 9.5.5).

This description was framed with $S\mathbb{H}$ in mind, but as noted in the discussion represented in Figure 2.4.2, the proper context is the universal cover of $S\mathbb{H}$ (or of $S\mathbb{D}$). Here one can pass to the orbit space and obtain literal rectangles from this construction. In either context, one can now translate the joining of a Birkhoff annulus defined by a closed geodesic to the one defined by its reverse, and in this rectangle picture, this corresponds to producing like rectangle that shares exactly the vertex that corresponds to the reverse closed geodesic. In $S\mathbb{H}$ the vertices corresponding to I are identified, but notably, in the universal cover of $S\mathbb{H}$ they are not, so this unfolds to a bi-infinite string of such rectangles joined by vertices (Figure 9.5.6).

Beyond the context of geodesic flows one might hope to obtain a Birkhoff annulus from a pair of isotopic periodic orbits, but one needs to find an isotopy through circles transverse to the flow. For a geodesic flow, moreover, Birkhoff annuli come in pairs corresopnding to the 2 choices of normal vector field, and taken together, such a pair gives a *Birkhoff torus*, which consists of all the unit tangent vectors to a closed geodesic and is transverse to the geodesic flow except on the 2 periodic orbits.

Whether there are embedded *transverse* tori is an entirely different matter. We will see that geodesic flows have none, while the Franks–Williams flow (Section 9.3) does—as the central piece of the construction.

With this in mind, one can describe the essential purpose of the present section as performing surgery on a Birkhoff annulus that produces new (contact) Anosov flows. In contrast, Section 9.3 modifies an Anosov flow in such a way that there is a transverse torus (which breaks the Anosov property, so surgery then restores it to give an Anosov flow with a transverse torus).

Definition 9.2.2. An Anosov flow on a 3-manifold is said to be of *algebraic type* if it is finitely covered by the geodesic flow of a compact surface (Section 2.3) or the suspension of a diffeomorphism of the 2-torus (Example 1.5.23).

The Reeb flow of a contact form (Definition 2.2.5) is *nondegenerate* if all its periodic orbits are transverse (1 is not an eigenvalue of the differential of the first-return map) and a *contact Anosov flow* if it is an Anosov flow.

Geodesic flows of negatively curved surfaces are the primary example of contact Anosov flows (on 3-manifolds), and here we describe a construction to obtain topologically new flows of this type.

Theorem 9.2.3 ([**119**])**.** *Consider a negatively curved oriented surface Σ and a closed geodesic $\mathfrak{c}\colon S^1 \to \Sigma$, $s \mapsto \mathfrak{c}(s)$. For $q \in \mathbb{Z}$ there is a $(1, q)$-Dehn surgery (described below)* *that produces a 3-manifold that (unless $q = 0$) is not homeomorphic to the unit tangent bundle of* any *surface or to a suspension manifold, and a volume-preserving flow on it that is therefore not orbit-equivalent to any algebraic flow. This flow is Anosov if and only if $q \geq 0$. If $\mathfrak{c}$ is* filling *and* indivisible,[2] *then for all but finitely many $q > 0$ the resulting 3-manifold is hyperbolic, and each free homotopy class of closed orbits of the new flow is (countably) infinite (Theorem 9.5.1).*[3]

Remark 9.2.4. Given our focus on hyperbolic flows, this surgery os most interesting when $q > 0$, but we note that the flows obtained for $q < 0$ are of interest as well, even though we will not study them here.

Proof. Let $s \mapsto \gamma(s)$ be the unit vector field perpendicular to the closed geodesic $\mathfrak{c}\colon S^1 \to \Sigma$, $s \mapsto \mathfrak{c}(s)$ given by $\theta = \pi/2$, that is, rotated in the positive direction. If $\mathfrak{c}$ is simple and separating as in Figure 9.2.1, we denote the unit tangent bundles of the two components of the surface by M_1 and M_2, and the common boundary of M_1 and M_2 is a torus $S^1 \times S^1$ parametrized by the parameter s of the geodesic $\mathfrak{c}$ and the angle θ with the tangent vector of the geodesic. Either way we change the geodesic flow g^t on the unit tangent bundle $S\Sigma$ in a neighborhood Λ of γ. To parametrize Λ,

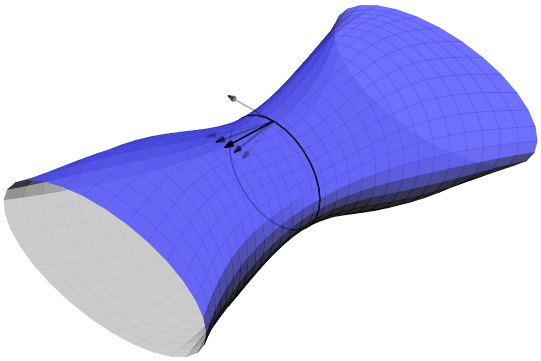


FIGURE 9.2.1. Near-normal vectors to a geodesic

[2] A closed curve c in a surface *fills* the surface if $\alpha \cap c \neq \varnothing$ whenever α is a closed curve that is not null-homotopic. It (and, more generally, a closed orbit) is said to be *indivisible* if it is not the same geodesic traversed more than once.

[3] For algebraic flows, free homotopy classes of closed orbits have finite cardinality, for geodesic flows no 2 (parametrized) orbits are homotopic, though by rotating the tangent vector though π, each is isotopic to its flip, which has the same *image* as another orbit (the same geodesic run backwards), and only in suspensions are all free homotopy classes of *images* of orbits singletons [**32**].

linearize the angle θ with the tangent vector field to $\mathfrak{c}$ by taking $w \coloneqq \dfrac{\ell}{2\pi}\cos\theta$ for θ near $-\pi/2$, where ℓ is the length of $\mathfrak{c}$. This gives parameters

$$(t, s, w) \in \Omega \coloneqq (-\eta, \eta) \times S^1 \times (-\epsilon, +\epsilon),$$

where $t \in (-\eta, \eta)$ parametrizes the flow direction and $s \in S^1$ is the parameter along $\mathfrak{c}$. γ is parametrized by $\{0\} \times S^1 \times \{0\}$.

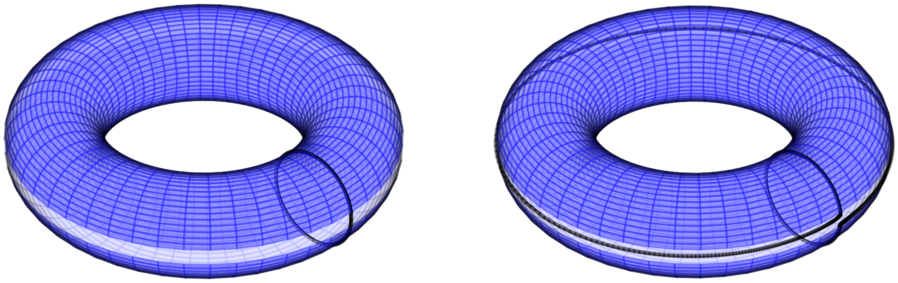


FIGURE 9.2.2. Surgery annulus before and after surgery ($q = 1$)

Lemma 9.2.5. *The standard contact form A in this chart satisfies*

$$A = dt + w\,ds, \quad dA = dw \wedge ds \quad \textit{and} \quad A \wedge dA = dt \wedge dw \wedge ds.$$

PROOF. If g_s denotes the Riemannian metric at $(0, s) \in \Sigma$ and we write $(0, s, \theta) = (x, u) \in S\Sigma$, then $d\pi(\partial/\partial\theta) = 0$ implies that for a vector $Z = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s} + c\frac{\partial}{\partial\theta}$ we have

$$A_{(0,s,\theta)}(Z) = g_s(u, d\pi(Z)) = g_s(u, a\,d\pi(\frac{\partial}{\partial t}) + b\,d\pi(\frac{\partial}{\partial s})) = a + bg_s(u, \frac{\partial}{\partial s}).$$

Taking S^1 to have length 2π we necessarily obtain $\|\partial/\partial s\| = \ell/2\pi$. Since a priori $A_{(t,s,\theta)} = dt + g\,ds + h\,d\theta$ with functions g and h, this implies $A_{(t,s,\theta)} = dt + \frac{\ell}{2\pi}\cos\theta\,ds$, that is, $A = dt + w\,ds$. The other claims immediately follow. □

These conventions are natural with respect to the canonical framing from page 106 by X, the vertical vector field $V = \partial/\partial\theta$ and the horizontal vector field $H = [V, X]$ in that $dw(Y) > 0$, $dw(V)ds(H) = dw \wedge ds(V, H) = dA(V, H) = 1$ and E^+ is spanned by a vector $V + H$ in the first quadrant (because $[X, V + H] = [X, V] + [X, H] = H + V$).

The $(1, q)$-Dehn surgery can be described as splitting apart an annulus in the manifold and gluing both copies of this annulus back together with a shear, that is, by identifying the 2 annuli via the shear map. Although this shear map is a homeomorphism of the annulus, this defines a discontinuous operation since the resulting space is no longer homeomorphic to the original one. It is important that the surgery yield a smooth manifold and that the resulting vector field is well-defined on the surgered manifold.

Lemma 9.2.5 gives an annulus that *contains* γ and is uniformly transverse to the flow. We split the flow-box chart from Lemma 9.2.5 into 2 one-sided flow-box neighborhoods of the surgery annulus, and while the initial transition map between these on $\{0\}\times S^1\times(-\epsilon,+\epsilon)$ is the identity, the surgered manifold is defined by imposing the desired shear as the transition map on this annulus:

$$F\colon S^1\times(-\epsilon,\epsilon)\to S^1\times(-\epsilon,\epsilon),\quad (s,w)\mapsto(s+f(w),w) \tag{9.2.1}$$

with $f\colon[-\epsilon,\epsilon]\to S^1$, $w\mapsto\exp(iqg(w/\epsilon))$, $q\in\mathbb{N}$, $g\colon\mathbb{R}\to[0,2\pi]$ nondecreasing smooth, $0\le g'\le 4$ even, and $g((-\infty,-1])=\{0\}$, $g([1,\infty))=\{2\pi\}$. The use of flow-box charts ensures that the original vector field defining the contact Anosov flow defines a smooth vector field on the surgered manifold, that is, that the orbits are reglued to smooth curves.

Proposition 9.2.6. *The new flow preserves the Liouville volume defined by $A\wedge dA$.*

Proof. Since $d(s+f(w))=ds+f'(w)\,dw$ we have

$$\begin{aligned}F_*A&=dt+w\,d(s+f(w))=A+w\,f'(w)\,dw\\ F_*dA&=dw\wedge d(s+f(w))=dw\wedge ds=dA,\end{aligned}$$

and

$$F_*A\wedge dA=A\wedge dA,$$

so $A\wedge dA$ is a well-defined volume on M_F. The vector field X on M induces a vector field X_F on M_F. Its flow clearly preserves $A\wedge dA$. □

It remains to show that for $q>0$ the new flow is an Anosov flow, and we use the formulation of hyperbolicity in terms of suitable Lyapunov–Lorentz metrics as described in Proposition 5.1.9; this is a reformulation of the usual cone criterion for hyperbolicity.

By Proposition 5.1.9 there is a pair of Lyapunov–Lorentz metrics for φ^t. We deform these to work as needed for φ_h^t. First, as in the proof of Theorem 5.2.4, we arrange for the Lyapunov–Lorentz metrics for φ^t to have the form

$$Q^\pm=\pm dw\,ds-c\,dt^2$$

in Λ, where c is chosen sufficiently small to ensure that the positive $Q^\pm$-cone contains $E^\pm$. Choose $\beta\colon\mathbb{R}\to\mathbb{R}^+$ smooth with $\beta((-\infty,0])=\{1\}$, $\beta([\eta,\infty))=\{0\}$ and $\beta'<0$ on $(-\eta,\eta)$ to obtain:

Claim 9.2.7. *Taking $Q_0^\pm$ and $Q_1^\pm$ to be the old Lyapunov–Lorentz metrics outside Λ and*

$$Q_i^\pm := \pm(dw\,ds - i\beta(t)f'(w)\,dw^2) - c\,dt^2$$

inside defines Lyapunov–Lorentz metrics for φ_h^t. Here, $i = 0$ on one side of the surgery and $i = 1$ on the other.

Proof. Our choice of f and β ensures that these are smooth metrics.

These choices fit together, that is, F sends the choice on one side to that on the other, because for $t = 0$:

$$\begin{aligned} F_* Q_0^\pm = F_*\big(\pm\,dw\,ds - c\,dt^2\big) &= \pm dw\,d(s - f(w)) - c\,dt^2 \\ &= \pm(dw\,ds - f'(w)\,dw\,dw) - c\,dt^2 \\ &= \pm(dw\,ds - \beta(0)f'(w)\,dw^2) - c\,dt^2 = Q_1^\pm. \qquad \square \end{aligned}$$

Of the required properties in Proposition 5.1.9, (2) and (3) are clear. We wish to check that (1) and (4) in Proposition 5.1.9 are inherited from the same properties for φ^t. Outside of Λ this is given since $\varphi = \varphi_h$.

On Λ we have been using a flow-box chart for φ_t, so the flow is represented by a shift in time. This makes it a $Q_0^\pm$-isometry. We need to see how the surgery and the choice of β affect invariance, that is, Proposition 5.1.9(4). To that end it is helpful to restrict attention to the trace of these cones in the sw-plane. Here, the $Q_0^\pm$-cones show as quadrants since

$$0 = Q_0^\pm(a\frac{\partial}{\partial s} + b\frac{\partial}{\partial w}) = ab$$

implies $a = 0$ or $b = 0$. On the immediate other side of the surgery ($t = 0$) they are given by

$$0 = Q_1^\pm(a\frac{\partial}{\partial s} + b\frac{\partial}{\partial w}) = ab - f'(w)b^2 = (a - f'(w))b,$$

which implies $a = f'(w)b$ or $b = 0$; since $f' \le 0$, this describes a subcone of the first and third quadrant that shares the horizontal axis. Since Λ is a flow-box, φ_h leaves these cones exactly invariant, which means that strict monotonicity of β produces a strictly invariant cone field that connects smoothly at $t = \eta$. This gives Proposition 5.1.9.(4).

To obtain Proposition 5.1.9(1) note that, Λ being a flow-box chart,

$$\pm Q_1^\pm(D\varphi_h^t\big(a\frac{\partial}{\partial s} + b\frac{\partial}{\partial w}\big)) = \pm Q_1^\pm(a\frac{\partial}{\partial s} + b\frac{\partial}{\partial w}) = ab - \beta(t)f'(w)b^2$$

is increasing in t. Combined with the exponential growth outside Λ, this yields Proposition 5.1.9(1).

With Proposition 9.2.9 below, this completes the proof of Theorem 9.2.3 $\square$

We now show that the flows thus obtained are not topologically orbit equivalent to an algebraic flow.

Definition 9.2.8. A 3-manifold is said to be Seifert-fibered if it admits a decomposition into a disjoint union of circles (the fibers) such that each fiber has a tubular neighborhood diffeomorphic (in a fiber-preserving way) to the torus $D^2 \times S^1$ obtained from $D^2 \times [0,1]$ by identifying $D^2 \times \{0\}$ and $D^2 \times \{1\}$ via a rational rotation.

Proposition 9.2.9. *When the surgery is carried out using a separating curve, the resulting flow is not topologically orbit equivalent to an algebraic Anosov flow. More strongly [**281**, page 419], no finite cover of the surgered manifold is a Seifert-fibered manifold (much less a sphere bundle) or a torus bundle over a circle.*

Proof. We study finite covers of the surgered manifold M by examining their fundamental group. The two pertinent facts are

(1) The fundamental group of a torus bundle over a circle is solvable; thus we wish to show that $\pi_1(M)$ is not virtually solvable, that is, has no solvable finite-index subgroup.

(2) The fundamental group of a Seifert-fibered manifold contains an infinite normal cyclic subgroup generated by a regular fiber [**266**, page 432]; thus we want to show that no finite-index subgroup of $\pi_1(M)$ contains an infinite cyclic normal subgroup.

One observation that we will use for both of these items is the following.

Remark 9.2.10. If a group contains a finite-index subgroup H and a free subgroup F, then H contains a subgroup of F that is isomorphic to F.

By the van Kampen Theorem [**152**, Theorem 1.20], we have

$$\pi_1(M) = \pi_1(M_1) \underset{\pi_1(\partial M_1)}{*} \pi_1(M_2) = \pi_1(M_1) \underset{\pi_1(\partial M_2)}{*} \pi_1(M_2),$$

using the isomorphism $F_*\colon \pi_1(\partial M_1) \to \pi_1(\partial M_2)$ induced by F from (9.2.1).

Puncturing a surface of genus g and retracting the remainder to its skeleton (a string of $2g$ circles) shows that the fundamental group is a free group with $2g$ generators. Thus, we see that $\pi_1(M_i) = F_i \oplus \mathbb{Z}$ for $i = 1,2$, where F_1 and F_2 are free groups.

If $H < \pi_1(M)$ has finite index, then, as remarked above, it contains a free group inherited from F_1 or F_2, and since this holds recursively, H is not solvable, hence item (1).

For item (2) suppose $\langle g\rangle < H$ is an infinite cyclic normal subgroup. This means that for every $h \in H$ there is a $p_h \in \mathbb{Z}$ such that $hgh^{-1} = g^{p_h}$. Clearly, $p_{h_1h_2} = p_{h_1}p_{h_2}$ for any $h_1, h_2 \in H$, so $p_{\mathrm{Id}} = 1$ implies that for each $h \in H$ we have $p_h \in \{\pm 1\}$ and $p_h = p_{h^{-1}}$. Thus, after possibly passing to the index-2 subgroup $\{h \in H \mid p_h = 1\}$, we may assume without loss of generality that H is in the centralizer of g, that is, $gh = hg$ for all $h \in H$.

As remarked above, there is a free group $F_H \subset F_1 \cap H$ isomorphic to F_1 (we only need that it is large enough). We can write $g = wr^k$ with w a word in generators of F_1 and F_2 only and r the generator of the S^1-factor of $\pi_1(M_1)$: writing one of the generators of $\pi_1(\partial M_2)$ as $\alpha_a r = a = \omega_a r'^k$ with $\alpha_a \in F_1$, $\omega_a \in F_2$ and r' the generator of the S^1-factor of $\pi_1(M_2)$, we find that any occurrence of $r\omega$ with $\omega \in F_2$ can be rewritten as $\alpha_a^{-1}\omega_a\omega\omega_a^{-1}\alpha_a r$; one applies this recursively to get the claim. We thus find that for any $h \in F_H \subset F_1$ we get

$$whr^k = wr^k h = gh = hg = hwr^k,$$

that is, $wh = hw$. But for $w \neq \mathrm{Id}$, this only holds when h is a power of w. Since F_H is not a subgroup of a cyclic group, we must have $g = r^k$ for some $k \in \mathbb{Z}$.

The same reasoning using F_2 shows that $g = r'^\ell$ for some $\ell \in \mathbb{Z}$, where r' is the generator of the S^1-factor of $\pi_1(M_2)$. This is incompatible with the earlier observation that $g = r^k \sim (sr')^k$, where s represents the word in F_2 corresponding to the slope of the surgery—unless $s = \mathrm{Id}$, so the surgery is trivial. Hence (2). □

What we have presented so far is the Handel–Thurston surgery in modern form. A subsequent refinement by Foulon modifies it in such a way that the resulting flow is a contact flow.

Theorem 9.2.11 (Foulon surgery [**119**]). *A suitable time-change of the flow obtained in Theorem 9.2.3 is a contact flow.*

Proof. The problem we have to address appears in the proof of Proposition 9.2.6: $F_* A = dt + w\,d(s + f(w)) = A + wf'(w)\,dw$ implies that there is no well-defined contact form after surgery. The additive nature of the discrepancy here also suggests a deformation which produces a contact form after surgery: take $A_h^\mp = A \mp dh$ for $\pm t \geq 0$, where

$$h(t,w) \coloneqq \frac{1}{2}\underbrace{\lambda(t)}_{\lambda\colon \mathbb{R}\to[0,1]\text{ is a smooth bump function}}\int_{-\epsilon}^{w} xf'(x)\,dx \text{ on } (-\eta,\eta)\times(-\epsilon,\epsilon) \text{ and } h = 0 \text{ outside.}$$

satisfies $dh = \frac{1}{2}wf'(w)dw$ on the surgery annulus and $h \equiv 0$ for t close to $\pm\eta$. Hence $F^*(A_h^+) = A_h^-$ and $A_h^\pm$ induces a contact form A_A because

$$F_*(A_h) = F_*(A - dh) = F_* A - F_* dh = (A + 2dh) - dh = A + dh = A_h.$$

Its Reeb field is a time-change

$$X_h \coloneqq \frac{X}{1 \pm dh(X)}$$

of X because clearly $A_h(X_h) \equiv 1$. This is well-defined by Lemma 9.2.12. □

Lemma 9.2.12. *If* $0 < \epsilon < \frac{\eta}{2\pi q}$, *then* $|dh(X)| < 1$.

Proof. $|f'(w)| = q\left|\frac{d}{dw} g\left(\frac{w}{\epsilon}\right)\right| = \frac{q}{\epsilon}\left|g'\left(\frac{w}{\epsilon}\right)\right|$, $|\lambda'| \le \pi/\eta$, $0 \le g' \le 4$ give

$$\left|\frac{\partial h}{\partial t}\right| = \left|\frac{1}{2}\lambda'(t)\int_0^w x f'(x)\,dx\right| \le \left|\frac{q\pi}{2\eta}\right|\left|\int_0^\epsilon x\frac{|g'|}{\epsilon}\,dx\right| \le \left|\frac{q\pi}{2\eta}\epsilon\frac{4}{\epsilon}\int_0^\epsilon dx\right| = \frac{2q\pi}{\eta}\epsilon < 1. \quad \square$$

We note that by contraposition, Theorem 10.4.9 below gives:

Proposition 9.2.13. *The Bowen–Margulis measure of the contact flow from Theorem 9.2.11 is singular with respect to the Liouville measure, that is, the topological entropy is strictly larger than the Liouville entropy.*

3. Anomalous Anosov flows

Our examples of Anosov flows so far are topologically transitive, and we will see that this is tied deeply to their structure. Unlike in the context of diffeomorphisms, we have instances of Anosov flows of entirely different types, and this section explores constructions of such. These are distinguished by having closed transversals; this is both a device in the construction as well as an essential means to making sure they are not topologically transitive, while the ℝ-covered Anosov flows we study in Section 9.5 and turned out to have built in Section 9.2 are transitive and are either suspensions or akin to geodesic flows ("skewed"). The constructions in this chapter originated with Franks and Williams, with whose example we begin this section. A later construction by Bonatti and Langevin [**51**] turns out to be of the same type, and both are generalized by the Béguin–Bonatti–Yu construction presented at the end of this section. ℝ-covered Anosov flows arise from surgeries first introduced by Handel and Thurston [**144**] and soon generalized by Goodman [**133**]; in the context of contact Anosov flows this is the construction we presented in Section 9.2. We mention here a different kind of surgery due to Fried [**125**] (which shows that every smooth transitive Anosov 3-flow is obtained by Dehn surgery of the suspension of a pseudo-Anosov homeomorphism) and the fact that Bonatti, Barbot and Fenley have not only produced altogether the most complete studies of Anosov 3-flows but also a range of additional constructions.

The examples of Anosov flows that we have seen so far invite the conjecture that Anosov flows are topologically transitive. While the counterpart for Anosov diffeomorphisms remains a question, this section presents Anosov flows that are not topologically transitive, and we show how this construction produces an infinite variety of new Anosov flows.

Theorem 9.3.1 (Franks–Williams)**.** *There is an Anosov flow on a 3-manifold that is not topologically transitive, that is, whose chain-recurrent set is not the whole manifold.*

The construction in fact produces a flow with an attractor-repeller pair, and consists of gluing together a flow with an attractor and a time-reversed copy of the same flow. That this flow is an Anosov flow will be a consequence of either Theorem 6.2.25 or the Alekseev cone criterion.

An Anosov flow of a 3-manifold is topologically transitive if the splitting $TM = E^u \oplus E^s \oplus \mathbb{R}X$ is C^1 (Theorem 9.1.5), so this splitting is not C^1 for the Franks–Williams flow.

In preparation for surgery we remove from the DA flow (Definition 6.3.4) a cylindrical tube around the repelling orbit of the origin. The flow on the remainder is then well-defined for positive time because it goes "inward" from the void left by the tube. Specifically, represent a neighborhood of the repelling orbit as $D \times [0,1]$, where D is a disk with polar coordinates (r,θ) such that $r = 0$ represents points of the periodic orbit and chosen such that in these coordinates the identification map in the suspension construction has the form

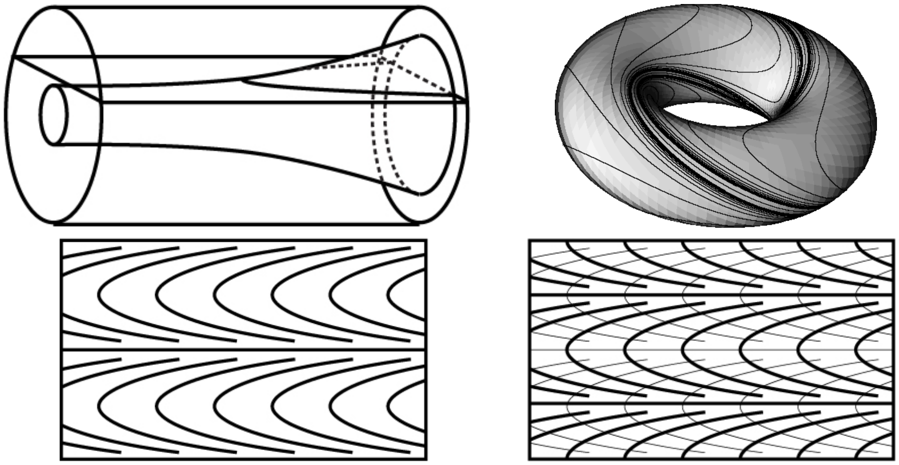
$$(r,\theta,t) \mapsto (er,\theta,t+1),$$

possibly after rescaling t (to make the radial scale factor exactly e). This means that the set B' parametrized by $(\epsilon e^t,\theta,t)$ is invariant under this identification, hence a well-defined torus in the suspension manifold. We remove the interior of this torus, including the orbit of 0.

A preliminary step ensures that gluing the remaining manifold together with a time-reversed copy produces a smooth flow on the resulting manifold. To deform the flow so the generating vector field is normal to the boundary and has unit speed there, glue a thickened torus $\mathbb{T}^2 \times [0,1]$ to the boundary B' by identifying $\mathbb{T}^2 \times \{1\}$ with B'. Isotopically extend φ^t through this neighborhood in such a way that it is transverse to each toral layer $\mathbb{T}^2 \times s$ and for $s \in [0,1/2]$ is furthermore normal to these layers and has unit speed. In particular, this is so on the boundary $B := \mathbb{T}^2 \times \{0\}$. Finally, use the flow itself to extend the local coordinates (r,θ,t) through this collar. This gives the "attracting half" of our system.

The "repelling half" is an identical copy of this attracting half, but with reversed flow direction.

Lemma 9.3.2. *Gluing the attracting and repelling halves together along the boundary tori B and $\bar{B}$ by the identification $t = \bar{t}$ and $\theta = \bar{\theta} - \pi/2$ ensures that the stable and unstable leaves of the resulting flow are transverse at points of the common boundary $B \sim \bar{B}$.*



Animation at

http://www.tsuboiweb.matrix.jp/showroom/public_html/animations/gif/stereograph3e/stereograph3e.html

FIGURE 9.3.1. The excised tube, the foliation on it (shown 2 ways, including one by Tsuboi), and the complementary foliations

This establishes the hypotheses of the Mañé criterion: The chain-recurrent set of the resulting flow is the union of the attractor and the repeller, hence hyperbolic; the dimension of the stable and unstable manifolds is constant, and they are transverse at a point of each orbit: For orbits in the attractor and in the repeller, this is hyperbolicity, and all other orbits go through B, where the lemma establishes transversality. Another method of proof is to show that the construction satisfies the cone criterion in Proposition 5.1.7.

PROOF OF THE LEMMA. The stable leaves of boundary points are those of positive semiorbits in the attracting half. These agree with those of the Anosov diffeomorphism from which the DA map was constructed, and they are therefore planes parallel to the one defined by $\theta \in \{0, \pi\}$ (up to possibly changing the choice of θ by an additive constant). The distance from the line $r = 0$ of such a plane is given by $d = r|\cos\theta|$, so its intersection with the boundary $B' = \{r = \epsilon e^t\}$ is given by $\epsilon e^t = d/|\cos\theta|$ or $\log\epsilon + t - \log d + \log|\cos\theta| = 0$. Thus, the unstable subbundle on B' is the kernel of the 1-form $dt + \tan\theta\, d\theta$, and since the coordinates and foliations extend from here to B via the flow, the same holds on B. In like manner, the unstable subbundle on $\bar{B}$, which is defined by the negative semiorbits in the repelling part, is the kernel of the 1-form $d\bar{t} + \tan\bar{\theta}\, d\bar{\theta} = dt + \cot\theta\, d\theta$. This is orthogonal to

$dt + \tan\theta\, d\theta$, which makes stable and unstable leaves orthogonal on $B \sim \bar{B}$ and hence transverse. □

By construction, the torus along which we glued the 2 pieces of the construction separates the manifold into pieces each of which contains a basic set. It turns out that this is indicative of the structure of any nontransitive Anosov diffeomorphism, in which case the spectral decomposition comes with a corresponding topological decomposition.

Theorem 9.3.3 ([**68**])**.** *If Φ is a nontransitive Anosov flow of a 3-manifold M, then there is a finite set of disjoint (incompressible) tori in the complement of $NW(\Phi)$ that decompose M into components each of which contains exactly one of the basic sets in the spectral decomposition of $NW(\Phi)$.*

The flow direction is important in the construction. Indeed, if we try a similar construction for a map of a 3-torus we would have one of the stable or unstable splittings 2-dimensional and another 1-dimensional. The gluing procedure can then not be done while keeping the foliations transverse (there will be a point of tangency). In low dimensions and codimension-one (so one of the splittings is dimension one) it is known that any Anosov diffeomorphism is transitive. As mentioned previously, for higher dimensions it is not known if there exist nontransitive Anosov diffeomorphisms.

Moreover, the Franks–Williams construction has come to be seen as but an instance of a toolkit for the construction of Anosov flows on 3-manifolds. The 2 pieces of the construction are instances of the following (recall Remark 5.3.39 here).

Definition 9.3.4. A *plug* is a pair (M, V) of a 3-manifold M with boundary and a vector field V on M that is transverse to the boundary ∂M. It is *attracting* if the *exit boundary* $\partial^{\text{out}} M \subset \partial M$ (the part of ∂M where V points outward) is empty, *repelling* if the *entrance boundary* $\partial^{\text{in}} M$ (where V points inward) is empty, and *hyperbolic* if $\Lambda \coloneqq \bigcap_{t\in\mathbb{R}} \varphi^t(M)$ is hyperbolic with 1-dimensional stable and unstable subbundles and no fixed points; here φ^t is the flow generated by V. In the latter case, $W^s(\Lambda)$ intersects $\partial^{\text{in}} M$ transversely (and is disjoint from $\partial^{\text{out}} M \subset \partial M$), so $\mathscr{L}_V^s \coloneqq W^s(\Lambda) \cap \partial^{\text{in}} M$ is a 1-dimensional lamination, called the *entrance lamination*. Likewise, one obtains the *exit lamination* $\mathscr{L}_V^u \coloneqq W^u(\Lambda) \cap \partial^{\text{out}} M$.

If (M, V) is a hyperbolic attracting plug, then $\mathscr{L}_V^s$ is a foliation of $\partial^{\text{in}} M$—by lines due to transversality—so every component of $\mathscr{L}_V^s$ is a 2-torus.[4]

Proposition 9.3.5 (Béguin–Bonatti–Yu)**.** *The entrance and exit laminations are* Morse–Smale laminations*, that is, they have finitely many compact leaves, every*

[4] Or possibly a Klein bottle if M is not orientable.

half noncompact leaf is asymptotic to a compact leaf, and each compact leaf can be oriented so as to be "attracting" (the contracting orientation).

With this terminology, the arguments in the Franks–Williams construction imply:

Theorem 9.3.6. *If (M,V) and (N,W) are hyperbolic plugs, B^{out} is a union of connected components of $\partial^{out}M$, B^{in} is a union of connected components of $\partial^{in}N$, and $f\colon B^{out}\to B^{in}$ is a diffeomorphism such that $f_*(\mathscr{L}^u_V)$ is transverse to $\mathscr{L}^s_W$, then (P,X) is a hyperbolic plug, where X is the vector field induced by V and W on $P \coloneqq M\cup N/f$ (in particular, there is a differentiable structure on P that is compatible with M and N and such that X is differentiable).*

As in the Franks–Williams construction hyperbolicity is established in the above situation by Proposition 5.1.7 or Theorem 6.2.25. The Franks–Williams construction uses an attracting plug and a repelling plug with one boundary component each, so the resulting plug has no boundary.

An additional property of the entrance and exit laminations is reminiscent of the DA construction and central to the creation of Anosov flows via these ideas.

Definition 9.3.7. A Morse–Smale lamination is said to be *filling* if every connected component of its complement is a "strip" as in Figure 6.3.5, that is, a topological disk bounded by two distinct noncompact leaves that are asymptotic to each other at both ends.[5]

This name reflects the fact that such laminations can in an obvious way be "filled in" to foliations; as a consequence, a surface with such a lamination is a torus. Less obviously, if one of the entrance or exit laminations of a hyperbolic plug is filling, then so is the other. In that case we speak of a hyperbolic plug with filling Morse–Smale laminations.

One more needed strengthening of the notions so far:

Definition 9.3.8. Two laminations $\mathscr{L}_1$, $\mathscr{L}_2$ of a surface S are said to be *strongly transverse* if they are transverse and if each connected component of $S\smallsetminus(\mathscr{L}_1\cup\mathscr{L}_2)$ is a "rectangle" whose boundary consists of 2 segments each from $\mathscr{L}_1$ and $\mathscr{L}_2$. A diffeomorphism $f\colon B^{\text{out}}\to B^{\text{in}}$ as in Theorem 9.3.6 is said to be strongly transverse if $f_*(\mathscr{L}^u_V)$ is strongly transverse to $\mathscr{L}^s_W$.

Proposition 9.3.9. *If $\mathscr{L}^u_V$ and $\mathscr{L}^s_W$ are filling and f is strongly transverse in Theorem 9.3.6, then P has filling Morse–Smale laminations.*

[5] "Bounded by" is a little more subtle than the topological notion of boundary, which in this case gives the whole complement; we refer to the *accessible boundary*, which is the collection of points in the topological boundary that are end-points of open segments in the set.

While Theorem 9.3.6 suggests an additive approach, where examples are built in a step-by-step assembly of plugs, Anosov flows can be obtained even if the 2 plugs in Theorem 9.3.6 are the same. Here, the step from "hyperbolic" to "Anosov" is nontrivial and necessitates an additional hypothesis to make sure that the gluing diffeomorphism can be chosen well.

Also, while based on the Franks–Williams idea, this is a rather different assembly because of the self-gluing and the absence of attractors and repellers. Nonetheless, this construction can also produce nontransitive Anosov flows, and there is a remarkable criterion for transitivity in the spirit of spectral decomposition and the no-cycles condition.

Theorem 9.3.10 (Béguin–Bonatti–Yu)**.** *If (M,V) is a hyperbolic plug with filling Morse–Smale laminations whose maximal invariant set contains no attractors or repellers, then a strongly transverse gluing diffeomorphism $f\colon \partial^{out}M \to \partial^{in}M$ can be isotoped in such a way that the vector field X induced on M/f by V is Anosov.*

X is transitive if and only if any 2 vertices of the following directed graph can be joined by a directed path: the vertices are the basic sets in the spectral decomposition of V, and an edge goes from Λ_i to Λ_j if either $W^u_V(\Lambda_i)\cap W^s_V(\Lambda_j)\neq\varnothing$ or $f_*(W^u_V(\Lambda_i)\cap \partial^{out}M)\cap(W^s_V(\Lambda_j)\cap\partial^{in}M)\neq\varnothing$.

Theorem 9.3.10 (or a several-plugs counterpart of it based on Theorem 9.3.6) produces a veritable machine or "Lego set" for recursively building new Anosov flows from known ones if one prepends the first part of the Franks–Williams construction—making a plug from an Anosov flow.

One implementation of this "machine" or "construction game" is the Béguin–Bonatti–Yu blow-up–excise–glue construction starting from a transitive Anosov flow:

- "Blow up" 2 closed orbits, one each by a repelling DA construction and an attracting DA construction to get a flow with these periodic orbits and a saddle hyperbolic set Λ as basic pieces.
- Excise tubular neighborhoods of the 2 closed orbits chosen such that a hyperbolic plug remains.
- Glue the 2 boundary tori as in Theorem 9.3.10 to get a new transitive Anosov flow; it turns out that its restriction to Λ has the initial flow as a factor.

The last item implies that this construction can be applied repeatedly to produce "ever more complicated" transitive Anosov flows. Here are some remarkable examples of what variants of this construction can produce.

Theorem 9.3.11 (Béguin–Bonatti–Yu)**.** *There is a closed 3-manifold that carries both a transitive Anosov flow and a nontransitive Anosov flow.*

For each $n \in \mathbb{N}$ there is a 3-manifold that carries n Anosov flows no two of which are orbit-equivalent.

There is a transitive Anosov flow on a 3-manifold with infinitely many pairwise nonisotopic transverse tori.[6]

The construction of a closed 3-manifold that carries both a transitive Anosov flow and a nontransitive Anosov flow goes as follows. Starting with an Anosov flow (such as the suspension of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$) select 2 closed orbits and perform blow-up–excise–glue surgery in two ways. First, perform DA modifications on both periodic orbits so as to make both attracting, and likewise on an identical copy, but making both expanding. Then delete tubular neighborhoods of each and glue the resulting manifold together along these tori analogously to the Franks–Williams construction; this also gives a nontransitive Anosov flow. Second, instead perform DA surgeries to make one each of these 2 orbits attracting and repelling, and the same in reverse on an identical copy, then excise and glue these 2 manifolds together to get a transitive Anosov flow.

Remark 9.3.12. Theorem 9.3.11 suggests the question of whether there is a manifold that carries infinitely many distinct Anosov flows. This has not been answered to date but is being investigated.

In light of the complementary universe of Anosov flows we study and produce in Sections 9.5 and 9.2 one might further ask about the extent to which Anosov 3-flows of these different kinds can coexist on a given manifold. This indicates a rich field of study at the boundary of 3-manifold topology and dynamical systems.

We close with an observation on the title of the present section. “Anomalous” was chosen to echo the title of the seminal work in this realm [**124**], but it also reflects the fact that the flows constructed here are of a more deeply different nature from suspensions and geodesic flows than the $\mathbb{R}$-covered flows in Section 9.2 discussed more generally in Section 9.5 below. And while “nontransitive” is the feature for which the Franks–Williams construction was first undertaken, we repeat that it does not characterize the Béguin–Bonatti–Yu flows.

4. Codimension-one Anosov flows

There is something jarring about the possibility that Anosov flows need not be topologically transitive (Theorem 9.3.1). This is a reason Franks and Williams

[6] This is interesting (with respect to decomposition ideas as in Theorem 9.3.3): given the construction of these examples from repeated addition of hyperbolic plugs, one might hope for a complete decomposition into basic standard pieces along finitely many embedded tori which are unique modulo isotopy—this is too much to hope for.

referred to their examples as anomalous Anosov flows. There are, of course, natural sufficient conditions for topological transitivity of Anosov flows. For instance, the flows constructed in Section 9.2 are also exotic (topologically different from algebraic ones), but since they are volume-preserving and hence ergodic (see Section 8.1), they are topologically transitive. While volume-preservation is a restrictive assumption, it actually suffices to assume that the flow in question is orbit-equivalent to a volume-preserving one because orbit-equivalence preserves transitivity—albeit also the fact that a some smooth measure is preserved (Proposition 3.5.1). At any rate, it is natural as well to seek topological conditions for transitivity, and this section presents one of them: If the (strong) stable or unstable foliation of an Anosov flow is 1-dimensional, then the flow is topologically transitive—provided the underlying manifold has dimension greater than 3, because the Franks–Williams flow is a counterexample.[7] While this is the major result of this section we take some time discussing codimension-one Anosov flows. Among the reasons is that these are the Anosov flows one can most hope to understand to any general extent from a topological point of view.

Definition 9.4.1. An Anosov flow is said to be a codimension-1 Anosov flow if either the strong stable or the strong unstable leaves are 1-dimensional.

To fix ideas we assume throughout this section that strong unstable leaves are 1-dimensional. We note that the codimension-1 foliation is continuously differentiable by Corollary 7.4.15. Let us take a first look at a "transverse" approach to studying these kinds of Anosov flows, that is, an approach that focuses on the space of orbits.

Definition 9.4.2. For a manifold M, denote by $\pi\colon \tilde{M} \to M$ the universal cover. The lifts $\widetilde{\mathscr{W}}^i$ to $\tilde{M}$ of the foliations $\mathscr{W}^i$ with $i = cs, cu, s, u$ for an Anosov flow Φ on M are called the *lifted foliations*, and the lift $\widetilde{\Phi}$ of Φ to $\tilde{M}$ is called the *lifted flow*.

The *leaf space* of a foliation is the identification space of $\tilde{M}$ whose elements are leaves. Denote by $\mathscr{O}^\Phi$, $\mathscr{L}^s$ and $\mathscr{L}^u$ the leaf spaces of the lifted orbit foliation and of $\widetilde{\mathscr{W}}^{cs}$ and $\widetilde{\mathscr{W}}^{cu}$, respectively, with canonical projections $\pi^\Phi\colon \tilde{M} \to \mathscr{O}^\Phi$, $\pi^s\colon \tilde{M} \to \mathscr{L}^s$, $\pi^u\colon \tilde{M} \to \mathscr{L}^u$.

Since the natural action on $\tilde{M}$ of the fundamental group Γ of M permutes the lifted leaves, it induces action on the 3 leaf spaces. This action interacts with the flow: a point $x \in \widetilde{\mathscr{W}}^{cs}$ is a leaf $\widetilde{W}^{cs}$, which projects to a leaf W^{cs} of $\mathscr{W}^s$ in M, and the Γ-stabilizer of x is the fundamental group of W^{cs} (which is trivial except for periodic x, when it is $\mathbb{Z}$). Along similar lines one proves:

[7] By contrast, whether all Anosov *diffeomorphisms* are topologically transitive is an open question that is deemed exceedingly hard.

Proposition 9.4.3. *Let Φ be a codimension-1 Anosov flow. Then*

- *The Γ-stabilizer of a point in $\mathscr{O}^{\Phi}$, $\mathscr{L}^u$ or $\mathscr{L}^s$ is either trivial or cyclic,*
- *If $\gamma \in \Gamma$ fixes $O \in \mathscr{O}^{\Phi}$, then O is a hyperbolic fixed point of γ,*
- *If $\gamma \in \Gamma$ fixes a point x of $\mathscr{L}^u$ or $\mathscr{L}^s$, then x is an attracting or repelling fixed point of γ,*
- *(Franks) The set of points in $\mathscr{L}^u$ (or $\mathscr{L}^s$) with nontrivial Γ-stabilizer is dense*,[8]
- *(Verjovsky) Each leaf of $\mathscr{W}^u$ intersects each leaf of $\mathscr{W}^{cs}$ in at most one point and likewise for $\mathscr{W}^s$ and $\mathscr{W}^{cu}$, hence*
- *$\mathscr{L}^s$ (or $\mathscr{L}^u$) is a connected and simply connected 1-manifold, that is, each point disconnects it; it is a separable topological space in which each point has a neighborhood homeomorphic to an interval,* but not necessarily Hausdorff.

The possible failure of the Hausdorff separation axiom in the last item suggests to call $x \in \mathscr{L}^u$ *nonseparated* if there is a $y \in \mathscr{L}^u$ such that every neighborhood of x overlaps with every neighborhood of y.

Definition 9.4.4. A codimension-1 Anosov flow is said to be $\mathbb{R}$-*covered* (or *covered by an* $\mathbb{R}$*-foliation*) if the leaf space $\mathscr{L}^s$ is Hausdorff (hence homeomorphic to $\mathbb{R}$).

Proposition 9.4.5. *The Franks–Williams flow (Theorem 9.3.1) is not $\mathbb{R}$-covered.*

Proof. Figure 9.3.1 shows Reeb components of the stable/unstable foliation (the middle picture there shows a pair of those), and these produce nonseparated leaves.[9] □

Proposition 9.4.6. *An $\mathbb{R}$-covered Anosov flow is topologically transitive.*

Proof. Otherwise, there is a nondense Γ-orbit in $\mathscr{L}^s$; denote by I a connected component of the complement of its closure and let a be an end-point. I has a dense set of points with nontrivial Γ-stabilizer, and each $\gamma \in \Gamma$ with a fixed point in I fixes I and hence without loss of generality a. The generator of the stabilizer of a then fixes all points of I with nontrivial stabilizer, contrary to the fixed-point set of any $\gamma \in \Gamma$ being hyperbolic, hence discrete. □

This is the second way to see that the Franks–Williams flow is not $\mathbb{R}$-covered.

[8] That is to say, the set of (un)stable leaves of periodic points is dense—even when periodic points themselves are not, as in Theorem 9.3.1.

[9] While Figure 9.3.1 shows leaves rather than the leaf space, this being a picture of a compact transversal that meets every orbit at most once implies that the same configuration appears in the leaf space. Note that a similar picture can occur for the orbits of a flow (Figure 1.5.3) and thus produce a non-Hausdorff orbit space—so having a "nice" orbit space in the present context is nontrivial (Proposition 9.4.20).

Corollary 9.4.7. *The Franks–Williams flow (Theorem 9.3.1) is not* $\mathbb{R}$*-covered.*

By contrast, Figure 2.4.2 leads to:

Proposition 9.4.8. *The geodesic flow of a compact negatively curved surface is* $\mathbb{R}$*-covered.*

PROOF. Using horocycles, stable leaves are (homeomorphically) represented by boundary points of the Poincaré disk $\mathbb{D}$ or equivalently, by unit tangent vectors at $0 \in \mathbb{D}$. Passing to the universal cover of the *phase space* "unrolls" these tangent circles to the fibers of a line bundle, and hence represents $\mathscr{L}^s$ homeomorphically as $\mathbb{R}$. □

More generally:

Proposition 9.4.9 ([**24**, Theorem A])**.** *Contact Anosov 3-flows are* $\mathbb{R}$*-covered.*

Remark 9.4.10. Though our present focus is not ergodic theory it is worth noting that the Margulis measure (Theorem 8.6.2) induces a Γ-invariant measure on $\mathscr{L}^s$ or $\mathscr{L}^u$, whichever is 1-dimensional, that scales homothetically under Φ.

Although this is not needed, we mention a foundational theorem by Verjovsky:

Theorem 9.4.11 ([**285**])**.** *For a codimension-1 Anosov flow on an n-manifold M,*

(1) *the closed orbits are homotopically nontrivial,*
(2) *the lifts of the codimension-1 leaves to the universal cover are diffeomorphic to* $\mathbb{R}^{n-1}$,
(3) *the universal cover of M is diffeomorphic to* $\mathbb{R}^n$ *[**226**, Corollary 5].*[10]

PROOF OUTLINE. (1) is a consequence of a theorem of Haefliger about codimension-1 foliations and implies (2). □

Oddly, it is not known whether periodic orbits of Anosov flows are always homotopically nontrivial (or whether there could be contractible ones).

We now come to the famous theorem of Verjovsky, which obtains topological transitivity without assuming that the flow is $\mathbb{R}$-covered:

Theorem 9.4.12 (Verjovsky)**.** *A codimension-1 Anosov flow on a manifold with dimension at least 4 is topologically transitive.*

We prove this by showing that every leaf of W^{cs} is dense (see Theorem 6.2.11), so we first introduce a few notions pertinent to foliations. We assume that the weak-stable foliation W^{cs} is transversely oriented, that is, TM/TW^{cs} is oriented. If

[10] The underlying fact is that $\mathbb{R}^n$ is the only simply connected n-manifold that has a foliation with leaves diffeomorphic to $\mathbb{R}^{n-1}$ [**226**, Corollary 3].

need be, this can be achieved by passing to a suitable double cover, and topological transitivity on the double cover implies topological transitivity on the manifold itself, so this is no loss of generality. The transverse orientation of W^{cs} yields a flow Θ along leaves of W^u, which is hence transverse to W^{cs}. Open saturated sets can then be studied with the following:

Definition 9.4.13 (Dippolito completion)**.** Let E be an open saturated set for a foliation of a Riemannian manifold M, and let d_E be the distance function on E induced by restricting the Riemannian metric on M to E. Then the completion $\hat{E}$ of E with respect to the metric d_E is called the *Dippolito completion* of E, and we denote by p the canonical projection $\hat{E} \to \overline{E}$ to the topological closure, by $\hat{\mathscr{F}}$ the foliation of $\hat{E}$ induced by $\mathscr{F}$, by $\hat{\Theta}$ the flow induced by the transverse flow Θ, and by $\delta E = \delta^+ E \cup \delta^- E$ the boundary and the boundary components where Θ flows outward, respectively inward (these are unions of leaves).

To clarify this, note that d_E is not the same as the distance induced by the distance on M. For instance, if E is the complement of a leaf, then Dippolito-completing E adds 2 copies of that leaf. Compactness of M implies:

Proposition 9.4.14 (Dippolito)**.** *If V^- is a leaf in $\delta^- E$, then there is a compact $A^- \subset V^-$ and a $T > 0$ such that for all $x \in V^- \smallsetminus A^-$ there is a $t \in (0, T)$ with $\hat{\theta}^t(x) \in \delta^+ E$.*

Proof. M is covered by the interiors of finitely many sets $B_i \times I_i$ such that $B_i \times \{t\}$ is a closed ball in a leaf of $\mathscr{F}$ and $\{x\} \times I_i$ is a closed orbit segment of Θ. We identify I_i and $\{c_i\} \times I_i$, where c_i is the center of B_i. With finitely many exceptions, the connected components of $E \cap I_i$ are intervals in the interior of I_i. Set $A_i \coloneqq \varnothing$ if either $I_i \subset E$ or the last point q_i of the θ-orbit segment I_i is not in E. If q_i but not I_i is contained in E, let $A_i \coloneqq \{a_i\}$, where a_i is the point of $I_i \cap \partial E$ nearest q_i in I_i. Then $A^- \coloneqq V^- \cap p^{-1}(\bigcup_i B_i \times A_i)$ is as desired. □

Proof of Theorem 9.4.12 (Matsumoto [**213**]). A *minimal* set for a foliation $\mathscr{F}$ of a manifold is a minimal element (with respect to inclusion) of the collection of nonempty, closed, $\mathscr{F}$-saturated sets (see Definition 9.1.6); it is *exceptional* if it is neither a single leaf nor the whole manifold. The existence of minimal sets is proved analogously to Proposition 1.6.27 or using Zorn's Lemma. In fact, Proposition 1.6.27 is a special case, where the foliation is the orbit foliation. In the case at hand, a minimal set of the weak-stable foliation cannot be a single leaf since minimal sets are compact. If it is the whole manifold, then all weak-stable leaves are dense, and the flow is transitive by Theorem 6.2.11. Therefore we assume that there is an exceptional minimal set and derive a contradiction.

Each leaf of W^{cs} is homeomorphic to either $\mathbb{R}^{n-1}$ or to $S^1 \times \mathbb{R}^{n-2}$, where $n = \dim M$. The latter occurs if and only if the leaf contains a periodic orbit, and we call such leaves periodic leaves.

Any connected component E of the *complement* of the exceptional minimal set is open and saturated by weak-stable leaves and hence has a Dippolito completion $\hat{E}$. The (positive) normal vector field to the weak-stable subbundle defines a transverse flow $\Theta = \{\theta^t\}_{t\in\mathbb{R}}$.

We will have to argue 2 cases separately, and in either one, we produce an inconsistency with the dynamics of Φ.

The first and significantly simpler case is the one in which there is a leaf V^- in $\delta^- E$ (or in $\delta^+ E$—reversing θ interchanges these) which is nonperiodic (and hence homeomorphic to $\mathbb{R}^{n-1}$). In that case take A^- as in Proposition 9.4.14 and assume (by enlarging it if necessary) that it is homeomorphic to a ball D^{n-1}. Then

$$V^- \smallsetminus A^- \subset U^- := \{x \in V^- \mid \hat{\theta}^t(x) \in \delta^+ E \text{ for some } t =: \tau(x) > 0\}$$

by Proposition 9.4.14. The continuous $\tau\colon U^- \to \mathbb{R}$ introduced hereby defines a map

$$h\colon U^- \to \delta^+ E, \quad x \mapsto \hat{\theta}^{\tau(x)}(x)$$

that nicely intertwines with the Anosov flow: V^- and $\delta^+ E$ are $\hat{\Phi}$-invariant, and $\hat{\Phi}$ sends orbits of $\hat{\Theta}$ to orbits of $\hat{\Theta}$, so U^- is $\hat{\Phi}$-invariant and

$$h(\hat{\varphi}^t(x)) = \hat{\varphi}^t(h(x)).$$

This implies $U^- = V^-$ because U^- is $\hat{\Phi}$-invariant and no orbit is contained in A^-. But then τ is bounded by Proposition 9.4.14, contrary to the fact that $\hat{\Phi}$ expands $\hat{\theta}$-orbits.

We now come to the rather harder case in which every leaf V^- in $\delta^- E$ (and in $\delta^+ E$) is periodic, that is, V^- contains a unique periodic orbit $\mathcal{O}(p)$, which we view as $S^1 \times \{0\}$ oriented by the slow direction and enlarge to a tubular neighborhood $S^1 \times D^{n-2}$ that contains the set A^- from Proposition 9.4.14 and such that

(1) $\hat{\Phi}$ is transverse to the boundary $S^1 \times S^{n-3}$ and flows inward,
(2) each fiber $\{t\} \times D^{n-2}$ lies in a strong stable leaf (and is hence transverse to $\hat{\Phi}$.

Because $\mathcal{O}(p)$ is the only $\hat{\Phi}$-periodic orbit in this neighborhood, the argument that $U^- = V^-$ in the first case shows that when we define U^-, τ and h as before, we now have $V^- \smallsetminus \mathcal{O}(p) \subset U^-$. If $U^- = V^-$, then we are done by the argument at the end of the first case so we henceforth assume $U^- = V^- \smallsetminus \mathcal{O}(p)$. Thus U^- is connected because $n - 2 \geq 2$,[11] and therefore $h(U^-)$ is in a single leaf V^+ of $\delta^+ E$ which is homeomorphic to $S^1 \times \mathbb{R}^{n-2}$ (because otherwise we are in the previous case). Applying the argument that $U^- = V^- \smallsetminus \mathcal{O}(p)$ to $h(U^-)$ with the flow $\hat{\theta}$ reversed implies that $h(U^-) = V^+ \smallsetminus \mathcal{O}(q)$ for a unique periodic orbit $\mathcal{O}(q)$ and that h is a homeomorphism.

[11] This is where we use that $n \geq 4$.

To cover these periodic leaves by nonperiodic ones, the following auxiliary notion becomes nontrivial and helpful:

Definition 9.4.15 (Leaf holonomy)**.** For a loop γ in a leaf of W^{cs}, the transverse flow lets us define a holonomy as follows. The map

$$\Gamma\colon [0,1]\times[-\eta,\eta]\to M, \quad (s,t)\mapsto\theta^t(\gamma(s))$$

induces a foliation $\Gamma^*\mathscr{F}$ on the rectangle $[0,1]\times[-\eta,\eta]$ whose leaves are components of the preimages of leaves of $\mathscr{F}$, and they are transverse to the "vertical" foliation $s=\text{const.}$ $\gamma\sim[0,1]\times\{0\}$ is one of them, so for some δ, each leaf through $(0,t)$ with $|t|<\delta$ contains a point $(1,g(t))\in\{1\}\times[-\eta,\eta]$, and this defines an orientation-preserving homeomorphism $\mathrm{Hol}_\gamma\colon(-\delta,\delta)\to[-\eta,\eta]$ with 0 as a fixed point. It (or its germ H_γ at 0.[12]) is called the *leaf holonomy along* γ

Fix $x^-(a,b)\in S^1\times S^{n-3}$, and let I be the $\hat\theta$-orbit segment from x^- to $x^+:=h(x^-)$. Then the leaf holonomy Hol_γ of the loop $\gamma=S^1\times\{b\}$ is defined on I and surjective with x^- as an expanding, hence isolated, fixed point. We assume that there are no fixed points of Hol_γ other than x^- and x^+ in I; otherwise replace x^+ by the fixed point nearest X^-. Then for $y\in\operatorname{int}I$ we have $\mathrm{Hol}_\gamma^n(y)\xrightarrow[n\to\pm\infty]{}x^\pm$, and for any

$$z\in U:=\{z\in V:=\hat W^{ss}(y)\ \big|\ \ z=\hat\theta(x)\text{ for some }t>0,\ x\in U^-\}$$

there is a unique $x=:\pi(z)\in U^-$ with $z=\hat\theta^t(x)$, and $\pi\colon U\to U^-$ is a covering map that intertwines with $\hat\Phi$.

The complement of $\mathbb{R}\times S^{n-3}=\pi^{-1}(S^1\times S^{n-3})$ in V has two connected components, the *exterior* $\pi^{-1}(V^-\smallsetminus S^1\times D^{n-2})$ and the *interior* $\mathscr{I}$. By π-invariance, $\hat\Phi$ flows into $\mathscr{I}$.

Now extend π (albeit possibly not as a covering map) to

$$W:=\{z\in V\ \big|\ \ z=\hat\theta(x)\text{ for some }t>0,\ x\in V^-\}\subset V$$

and denote by J the positive $\hat\theta$-semiorbit of the Φ-periodic point p from the start of this case.

There are points $p'\in V\cap J$ arbitrarily close to p because $V\cap I$ contains points $\mathrm{Hol}_\gamma^{-n}(y)$ arbitrarily close to $x^-\in V^-$. Then $p'\subset B^-\subset V$ such that $\pi\restriction_{B^-}$ is a homeomorphism onto $\{a\}\times D^{n-2}\ni p$. Then B^- separates $\mathscr{I}$ into 2 components (corresponding to $y>a$ and $t<a$), one of which accumulates on V^-, and the closure of the other of which we denote by C_-. $\hat\Phi$ flows into C_-: It is transverse to B_- because it is transverse to $\{a\}\times D^{n-2}$, and the forward orbit of p', being in the weak-unstable leaf of p, can't accumulate on V^-. Since $\hat\Phi$ flows into $\mathscr{I}$, the orbit of p' must instead enter C_-.

[12] that is, equivalence class of maps modulo agreeing on a neighborhood of 0

We now argue likewise with $q \in V^+$ to obtain a disk B_+ bounding a component C_+ that does not accumulate on V^+ and such that $\hat{\Phi}$ flows (transversely) into C_+. But this is impossible: $C \coloneqq C_- \cap C_+ \subset V$ is a ball[13] into which $\hat{\Phi}$ flows, contrary to the dynamics on a weak-stable leaf. This completes the proof of Theorem 9.4.12. □

We remark on types of Anosov flows to which transitivity of codimension-1 Anosov flows is pertinent.

If a transitive Anosov flow has a global section and is hence (topologically equivalent to) a special flow, that is, topologically equivalent to a suspension, then the base transformation is a transitive Anosov diffeomorphism, and transitive codimension-1 Anosov diffeomorphisms are topologically conjugate to an algebraic Anosov diffeomorphism [**123**]. Verjovsky wondered whether in this situation there always is a global section, and he proved it when the phase space is a manifold with solvable fundamental group.

Theorem 9.4.16 (Verjovsky)**.** *A codimension-1 Anosov flow on a manifold with solvable fundamental group has a global section and is hence a suspension over an algebraic Anosov diffeomorphism.*

Barbot instead gave a dynamical criterion for being a suspension:

Theorem 9.4.17 ([**21**, Théorème 2.7])**.** *A codimension-1 Anosov flow in which every (1-dimensional) strong unstable leaf meets every weak-stable leaf is topologically orbit equivalent to a suspension.*

Verjovsky conjectured that neither this hypothesis nor the topological hypothesis of Theorem 9.4.16 is needed:

Conjecture 9.4.18 (Verjovsky Conjecture)**.** *A codimension-1 Anosov flow (in dimension greater than 3) has a global section and is hence a suspension over an algebraic Anosov diffeomorphism.*

This has been proved by assuming for instance that the invariant foliations are C^1 (or Lipschitz-continuous [**274**]), but remains open as stated. (In light of Theorem 9.4.12 one can without loss of generality assume volume-preservation: a topologically transitive codimension-one Anosov flow on a closed manifold is topologically equivalent to a C^∞ Anosov flow with a C^∞ invariant volume [**15**].) Recent work explored possible ways of constructing Anosov flows that are not topologically conjugate to algebraic flows (that is, geodesic flows or suspensions of algebraic Anosov diffeomorphisms), and this led to a stronger conjecture:

[13] The boundary is an $(n-2)$-sphere (a cylinder plus 2 disks) and the complement is noncompact; this implies that it is a ball by the generalization of a theorem of Schönflies: if $E\colon S^{n-1} \to S^n$ is a topological embedding and A is the closure of a component of $S^n \smallsetminus E(S^{n-1})$, then $A \approx D^n$ if A is a manifold.

Conjecture 9.4.19 (Barthelmé–Bonatti–Gogolev–Hertz [**31**])**.** *Anosov flows whose stable and unstable subbundles do not have the same dimension are orbit-equivalent to the suspension of an Anosov diffeomorphism.*[14]

As we mentioned at the start of this section, codimension-1 Anosov flows are more amenable to topological analysis than Anosov flows are in full generality. We now outline a way in which one can determine whether 2 of them are orbit equivalent by checking conjugacy of the fundamental group actions on the orbit space. This is interesting in part because the homeomorphism involved in orbit equivalence is determined up to a shift along orbits, whereas one can hope to find a conjugacy between group actions by the contraction principle.

To that end we first show that the orbit space is topologically nice.

Proposition 9.4.20. *If Φ is a codimension-1 Anosov flow on an n-manifold M, then its orbit space $\mathcal{O}^\Phi$ is diffeomorphic to $\mathbb{R}^{n-1}$, and the canonical projection $\pi\colon \tilde{M} \to \mathcal{O}^\Phi$ is a locally trivial $\mathbb{R}$-principal fibration.*

Proof outline. First, $\mathcal{O}^\Phi$ is a (possibly not Hausdorff) topological $m-1$-manifold: Each orbit $\tilde{\mathcal{O}}$ of $\tilde{\Phi}$ admits a local transversal that meets any other orbit of $\tilde{\Phi}$ in at most one point, and this gives local transversals whose intersections with leaves of the lifted weak foliations are connected; this yields the claim.

Next, we show by contraposition that $\mathcal{O}^\Phi$ is Hausdorff—if $\mathcal{O}_1$ and $\mathcal{O}_2$ are orbits of $\tilde{\Phi}$ that are not separated by $\tilde{\Phi}$-invariant open sets, then they coincide. For $i = 1,2$, let U_i be the $\tilde{\mathscr{W}}^s$ saturation of $F_i \coloneqq \tilde{W}^{cu}(\tilde{\mathcal{O}}_i)$. If these are not disjoint, then there is a $\tilde{\mathscr{W}}^s$-leaf S that meets F_i in a unique point x_i for $i = 1,2$. If $x_1 \neq x_2$, then there are disjoint S-neighborhoods of the x_i, and their $\tilde{\mathscr{W}}^{cu}$-saturations separate $\mathcal{O}_1$ and $\mathcal{O}_2$. If $x_1 = x_2$, then $\mathcal{O}_1$ and $\mathcal{O}_2$ are on the same unstable leaf, so if the preceding arguments with s and u interchanged do not produce separating neighborhoods, then they lead to $\mathcal{O}_1 = \mathcal{O}_2$.

Thus, $\mathcal{O}^\Phi$ is a Hausdorff topological $n-1$-manifold, and $\pi(\tilde{\mathscr{W}}^{cu})$ is a foliation by codimension-1 planes, so $\mathcal{O}^\Phi$ is diffeomorphic to $\mathbb{R}^{n-1}$.[15] That π is locally trivial follows because local transversals produce local sections of π. □

Theorem 9.4.21. *If for $i = 1,2$, Φ_i is a codimension-1 Anosov flow on a manifold M_i with fundamental group Γ_i, then Φ_1 is orbit-equivalent to Φ_2 or its reverse if and only if there are an isomorphism $I\colon \Gamma_1 \to \Gamma_2$ and a homeomorphism $h\colon \mathcal{O}^{\Phi_1} \to \mathcal{O}^{\Phi_2}$ such that $h(\gamma\mathcal{O}) = I(\gamma)h(\mathcal{O})$ for all $\gamma \in \Gamma_1$ and $\mathcal{O} \in \mathcal{O}^{\Phi_1}$.*

[14] The claims in [**124**] to flows incompatible with this conjecture turn out to be unsupported—which prompted the work leading to this conjecture.

[15] The only simply connected manifolds that admit plane foliations are, up to diffeomorphism, the euclidean spaces $\mathbb{R}^n$ [**226**, Corollary 3].

In this case, moreover, h can be chosen so as to lift to a homeomorphism $H\colon \tilde{M}_1 \to \tilde{M}_2$ with $\pi_{\Phi_2} \circ H = h \circ \pi_{\Phi_1}$, where $\pi_{\Phi_i}\colon \tilde{M}_i \to \mathscr{O}^{\Phi_i}$ are the projections.

PROOF OUTLINE. That orbit-equivalence implies equivalence of the group actions is clear. For the converse we first produce the lift H in the last assertion. Choose a global (not necessarily connected) section T of Φ_1 in such a way that every connected component of its lift $\tilde{T}$ to $\tilde{M}_1$ meets every $\widetilde{\Phi}_1$-orbit in at most one point and each $\widetilde{\Phi}_1$-orbit meets a finite (nonzero) number of connected components of $\tilde{T}$.

Since π_{Φ_1} is a locally trivial fibration, the restriction of $h \circ \pi_{\Phi_1}$ to any connected component $\tilde{C}$ of $\tilde{T}$ lifts to a map $h_{\tilde{C}}\colon \tilde{C} \to \tilde{M}_2$ in an I-equivariant way. Now, for every $\tilde{X} \in \tilde{M}_1$ there is a connected component $\tilde{C}$, a $\tilde{y} \in \tilde{C}$ and a $t \in \mathbb{R}$ such that $\tilde{x} = \tilde{\varphi}_1^t(\tilde{y})$, and we "define" $H(\tilde{x}) \coloneqq \varphi_2^t(h_{\tilde{C}}(\tilde{y}))$.

This is not quite well-defined because possibly more than one connected component could be chosen here, so this method produces finitely many points—but they all lie on the same orbit; replacing this collection by a weighted average (along the orbit), where the weights come from a partition of unity on M_1, now produces a well-defined continuous equivariant map H such that $\pi_{\Phi_2} \circ H = h \circ \pi_{\Phi_1}$.

The orbit-equivalence is now produced by using equivariance of H to get an induced map $\mathfrak{h}\colon M_1 \to M_2$, which sends orbits to orbits. If injective, this is indeed the orbit-equivalence, but similarly to the application of Claim 1.7.6 we need to ensure monotonicity in the flow direction.

More precisely, we have $\mathfrak{h}(\varphi_1^t(x)) = \varphi_2^{\alpha(t,x)}(\mathfrak{h}(x))$ for a cocycle α, and injectivity is equivalent to strict monotonicity of $t \mapsto \alpha(t,x)$ for fixed x. A suitable "diffusion" due to Ghys and Gromov along orbits as in the proof of Proposition 1.3.27 (or a transversality argument by Matsumoto and Tsuboi) modifies $\mathfrak{h}$ in such a way as to make α strictly monotone in t. Whether α is increasing or decreasing determines whether $\mathfrak{h}$ connects Φ_1 to Φ_2 or to the reverse of Φ_2. □

Remark 9.4.22. Unsurprisingly, a criterion for whether one has an orbit equivalence between the flows versus between one and the reverse of the other is that $\mathfrak{h}$ sends stable leaves of Φ_1 to stable leaves of Φ_2 and likewise for unstable ones, or whether it sends stable leaves to unstable ones and vice versa. This can also be checked via the analogous dichotomy for the action of h on the projections of the foliations to the orbit space.

5. ℝ-covered Anosov 3-flows

We continue the study of codimension-1 Anosov flows by returning our attention to flows on 3-manifolds. The central ingredient are the notions from the beginning of Section 9.4. This is quite complementary to topological results using assumptions on the fundamental group (Theorem 9.4.16 is an example, though not the best).[16] We repeat that the Franks–Williams flows in Section 9.3, not being topologically transitive, are fundamentally different from the $\mathbb{R}$-covered Anosov 3-flows we henceforth attend to.

For what follows, the discussion of Birkhoff annuli at the beginning of Section 9.2 may provide useful intuition; we will here see manifestations of those ideas beyond geodesic flows.

The foundational result of the global topological theory of $\mathbb{R}$-covered Anosov 3-flows is the following, and its main case distinction recalls the difference between the situations of Remark 1.5.25 and Figure 2.4.2:

Theorem 9.5.1 (Barbot, Fenley)**.** *If an orientable Anosov 3-flow Φ is $\mathbb{R}$-covered (Definition 9.4.4), then the leaf space $\mathscr{L}^s$ is also Hausdorff. The embedding $\mathscr{O}^\Phi \to \mathscr{L}^u \times \mathscr{L}^s$ sending each orbit to its (center-)stable and (center-)unstable leaf is equivariant with respect to the diagonal action of Γ, and its image $\mathscr{O}$ is open and either*

- *$\mathscr{L}^u \times \mathscr{L}^s$, in which case we say that the flow is a product flow (and it is orbit-equivalent to the suspension (Theorem 9.4.17) of a linear toral automorphism (Example 1.5.24), which is transitive but not mixing), or*
- *the open set bounded by the graphs of homeomorphisms $\alpha, \beta\colon \mathscr{L}^u \to \mathscr{L}^s$ (with $\alpha^{-1} \circ \beta$ and $\beta \circ \alpha^{-1}$ increasing), in which case we say that Φ is a* skewed *$\mathbb{R}$-covered Anosov flow.*

The action of a deck transformation g on the (un)stable leaf space either has

- *no fixed point,*
- *exactly one fixed point (and Φ is of product type), or*
- *infinitely many fixed points (and Φ is skewed).*

In the latter cases, g is associated with a periodic orbit in the sense that the loop to which it corresponds is freely homotopic to a closed orbit.

If Φ is skewed and M is a hyperbolic manifold, then every closed orbit is freely homotopic to infinitely many others.

Theorem 9.5.1 gives a clear description of the orbit space of a skewed $\mathbb{R}$-covered Anosov 3-flow. We can picture it as a strip between the graphs of 2 increasing (say) functions that is foliated by horizontal and vertical line segments (Figure 9.5.1).

Remark 9.5.2. Of these 2 cases, product flows can be deemed completely understood since toral automorphisms are.

Proposition 9.5.3. *Geodesic flows of negatively curved surfaces and more generally, contact Anosov 3-flows are skewed $\mathbb{R}$-covered Anosov flows*

[16] This is also complementary to rigidity theory, for example, the exploitation of smoothness assumptions on the foliation such as in Theorem 10.3.14

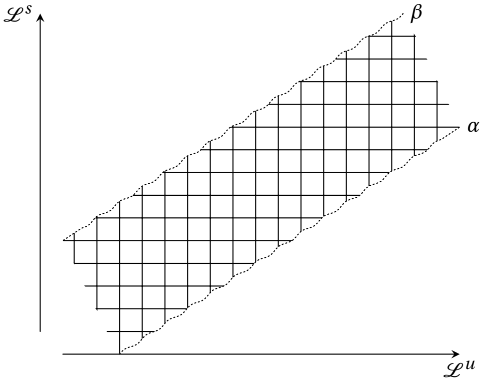


FIGURE 9.5.1. The orbit space of a skewed ℝ-covered Anosov 3-flow

PROOF. They are ℝ-covered by Proposition 9.4.9 (and Proposition 9.4.8), and skewed since they are not orbit-equivalent to suspensions (suspensions have a global section S, 3-flows preserving a contact form A can not: $0 \neq \int_s dA = \int_{\partial S} A = 0$ when $\partial S = \varnothing$). □

Conjecturally this is an equivalence:

Conjecture 9.5.4 (Barthelmé). *ℝ-covered Anosov 3-flows are topologically orbit-equivalent to a contact Anosov flow.*

Some observations about the dynamics of skewed ℝ-covered Anosov flows:

Proposition 9.5.5. *For a skewed ℝ-covered Anosov flow Φ on a 3-manifold M:*

(1) M is orientable,
(2) Φ is orbit-equivalent to its reverse,
(3) no compact (boundaryless) surface is transversely immersed in M,
(4) Φ is topologically mixing.

The impossibility of a transversal implies (again):

Corollary 9.5.6. *The Franks–Williams flow (Theorem 9.3.1) is not ℝ-covered.*

Corollary 9.5.7. *The Béguin–Bonatti–Yu flows (for example, Theorem 9.3.11) are not ℝ-covered.*

Remark 9.5.8. Corollary 9.5.7 shows that a flow can be topologically transitive without being ℝ-covered. This motivates "anomalous" (rather than "nontransitive") in the title of Section 9.3. A complementary fact is that there are manifolds that support both an $\mathbb{R}$-covered Anosov flow and an Anosov flow that is not $\mathbb{R}$-covered [**23**].

PROOF OF PROPOSITION 9.5.5. If p^u and p^s are the coordinate projections in $\mathscr{L}^u \times \mathscr{L}^s$, then

$$\tau := (\tau^u, \tau^s) := (\alpha^{-1} \circ p^s, \beta \circ p^u) \tag{9.5.1}$$

is a Γ-equivariant homeomorphism of $\mathscr{L}^u \times \mathscr{L}^s$ that preserves $\mathscr{O}$ and exchanges the stable and unstable foliations. (Here α and β are as in Theorem 9.5.1.) Theorem 9.4.21 and Remark 9.4.22 then yield the orbit-equivalence to the reverse flow. A transversely immersed surface defines a cohomology class that that has nonnegative values on orbits, and this is imcompatible with orbit-equivalence to the reverse flow. Finally, since the orbit-equivalence to the reverse interchanges the stable and unstable foliations, they are both or neither transversely orientable. Since orbits are oriented, so is M. Finally, mixing follows from Theorem 9.1.1 since Φ is not a suspension by (3). □

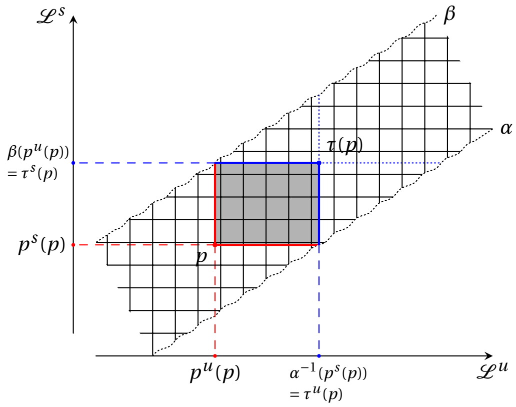


FIGURE 9.5.2. The proof of Proposition 9.5.5: illustration of (9.5.1)

A much harder recent result is that the second item is a characterization:

Theorem 9.5.9 (Barthelmé–Gogolev [**34**])**.** *If Φ is an Anosov 3-flow for which one of the weak foliations is transversely orientable, then Φ is orbit equivalent to its reverse by a homeomorphism isotopic to the identity if and only if Φ is skewed $\mathbb{R}$-covered.*

The inherent symmetry of the argument for Proposition 9.5.5 implies that "half" of the data (the Γ-action on $\mathscr{L}^u$ together with τ^u from (9.5.1)) determine the flow already: $\mathscr{O}$, hence $\mathscr{O}^\Phi$, is equivariantly identified with the subset of $\mathscr{L}^u \times \mathscr{L}^u$ bounded by the graphs of τ^u and the identity.

Corollary 9.5.10. *Two skewed* $\mathbb{R}$*-covered Anosov 3-flows* Φ, Ψ *are orbit-equivalent if and only if there is a homeomorphism* $f\colon \mathscr{L}^u(\Phi) \to \mathscr{L}^u(\Psi)$ *that is equivariant by the fundamental-group actions and such that* $f \circ \tau^u_\Phi = \tau^u_\Psi \circ f$.

PROOF OF THEOREM 9.5.1 [115]. If Φ is of product type then the conclusion of Theorem 9.5.1 follows. Therefore, we assume that Φ is orientable (after possibly passing to a double cover) and not of product type. We will show that if $\mathscr{L}^u$ is Hausdorff then so is $\mathscr{L}^s$ and that Φ is skewed in that case—by constructing a "lozenge" as in Figure 9.5.2, albeit in reverse, because are building rather than using the skewed structure. To fix ideas, identify $\mathscr{O}^\Phi$ with $H := (-1,1) \times \mathbb{R} \subset \mathbb{R}^2$ so the product and skewed situations look as in Figure 9.5.3 with $\widetilde{\mathscr{W}}^{cu} = \{W_t \mid t \in \mathbb{R}\}$ consisting of horizontal segments and t the vertical coordinate. (Note that the right picture is isotopic to Figure 9.5.1, which matches the description in the statement and will later be more convenient.)

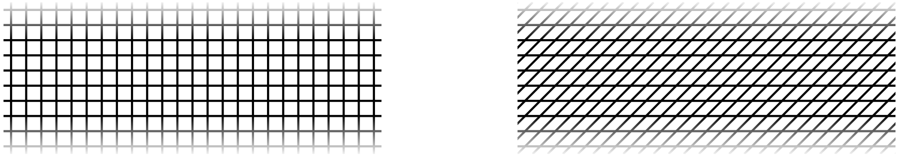


FIGURE 9.5.3. Product and skewed structures on $H := (-1,1) \times \mathbb{R}$

Since Φ is not of product type, there is a $V \in \widetilde{\mathscr{W}}^s$ that does not intersect all W_t; say it misses W_u and meets W_t for $t \in (u, \bar{u})$. Its corresponding boundary point must lie on the boundary of H and (since u is finite) either on the line $x = 1$ ("positive boundary") or on the line $x = -1$ ("negative boundary"). To fix ideas and in line with the right half of Figure 9.5.3 suppose the former (Figure 9.5.4).

Let $v \in (u, u_0]$ and choose a periodic orbit p near $V \cap W_v$ that lies right of V and below W_v. Then $p = A \cap W_a$ with $A \in \widetilde{\mathscr{W}}^u$ and $u < a < v$, so $A \cap W_u = \varnothing$, and the "negative half" of A determined by p, that is, its part below p, intersects W_t for $b < t < a$ for some $b \geq u$. Since p is periodic, there is an indivisible deck transformation α that fixes A and hence $\{W_t \mid b < t < a\}$; since α fixes W_a (because it contains p—and because this leaf space is known to be $\mathbb{R}$), α fixes W_b. Here, as in Theorem 9.2.3, a closed orbit is said to be *indivisible* if it is not a repeated closed orbit, and in that case the associated deck transformation is also said to be indivisible.

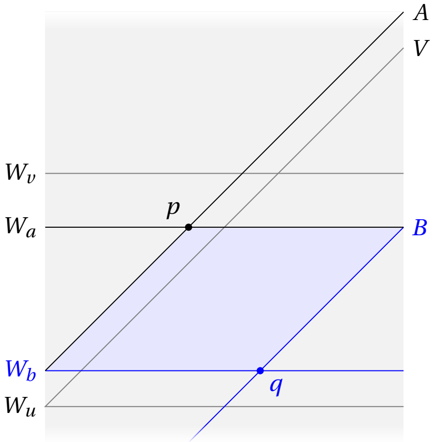


FIGURE 9.5.4. Building a lozenge

This implies that there is a closed orbit $q \in W_b$.

Claim 9.5.11. *q is freely homotopic to the reverse of p (Lemma 9.5.14).*

Since α expands or contracts the stable leaf B through q and leaves W_a invariant, it follows that $B \cap W_a = \varnothing$ and that α fixes no other $\widetilde{\mathscr{W}}^u$-leaf between W_a and W_b. (We note that this argument shows that in a product flow a deck transformation has at most one fixed point on the stable or unstable leaf space.) Thus, the positive ("upper") half of B meets exactly the leaves W_t with $t \in (b,a)$ and ends on the negative boundary. The quadrilateral thus determined (with sides being the respective halves of A, B, W_a and W_b and vertices being p, q and one point on each boundary component) and like ones for other periodic points serve as the skeleton to derive the picture on the right of Figure 9.5.3—which shows that $\mathscr{L}^u$ is also $\mathbb{R}$-covered.

Remark 9.5.12. Such a quadrilateral is called a *lozenge*[17] with corners p and q, which for this terminology need not be periodic; the arguments here show that if one corner is periodic, then so is the other: a deck transformation that fixes one corner must also fix the other. Lozenges are a useful notion for pushing these

[17] From Wikipedia: "A lozenge (◊), often referred to as a diamond, is a form of rhombus. The definition of lozenge is not strictly fixed, and it is sometimes used simply as a synonym (from the French losange) for rhombus."

arguments further. For instance, one can show this way that any Anosov 3-flow (not necessarily ℝ-covered!) that is not a suspension has periodic orbits p, q such that q is freely homotopic to p^{-1}. We emphasize as well that a lozenge is a "rectangle" but with only 2 vertices because the orbit space is open, so the vertices on its edge in Figure 9.5.2 are "at infinity." Thus, lozenges are more often drawn to look like in Figure 9.5.5.

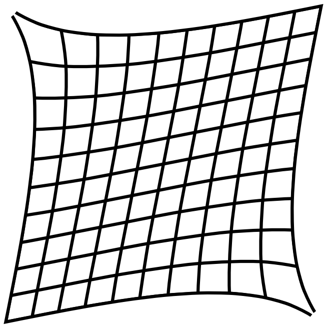


FIGURE 9.5.5. A lozenge by itself

But first we digress to note that one could instead consider the negative ("lower") part of B: it similarly ends on another fixed leaf W_c, which must contain a periodic point, and so on. (Likewise, when one moves upward from A.) This proves that in the skewed case a deck transformation has countably many fixed points on either leaf space. With our previous parenthetical, this proves the triple-alternative in the second half of Theorem 9.5.1. Claim 9.5.11 shows that in the skewed case, these countably many orbits γ_i form two infinite free homotopy classes (even and odd indices, respectively), and the last assertion of Theorem 9.5.1 follows from arguments in 3-manifold topology to the effect that in a hyperbolic manifold no closed orbit is nontrivially freely homotopic to itself (basically, because that would produce a nontrivial 2-torus, while hyperbolicity of the manifold implies atoroidality [**115**, Lemma 4.3]), so the orbits we obtained are distinct. This is illustrated in Figure 9.5.6.

To put lozenges to use, consider any $\tilde{\mathscr{W}}^s$-leaf C and $r \in C$. Although it is not needed for the theorem we now invoke topological transitivity as a shortcut: it implies that the projection of B is dense, so there are deck transformations α_i such that $\alpha_i(B)$ near γ and to either side of it, so $\{t \in \mathbb{R} \mid C \cap W_t \neq \varnothing\}$ is bounded above and below, and C necessarily goes to the positive boundary at the bottom end and to the negative boundary at the upper end, which produces the same arrangement for all $\tilde{\mathscr{W}}^s$-leaves and thus shows that $\mathscr{L}^s$ is also Hausdorff.

We next show that Φ is skewed, that is, we produce the full picture on the right-hand side of Figure 9.5.3. We show that no two distinct stable leaves limit on the same unstable leaf on either boundary; together with the same result for unstable leaves limiting on distinct stable ones, this shows that every W_t is the upper/lower end of a complementary leaf, which defines the requisite pair of homeomorphisms.

For purposes of contradiction suppose W_s is the upper limit of both A and B with A to the right of B. Then A, B intersect W_t for $w \le t < s$ (but not W_s) and we can choose periodic orbits $p \in W_a$, $q \in W_b$ between A and B such that $w < a < b < s$ and with stable leaves A', B', respectively, which then necessarily also limit on W_s. Hence, A intersects W_b, and we will show that this is impossible.

One can argue as before that the indivisible deck transformation α associated with p fixes W_s, which thus contains a closed orbit r freely homotopic to the reverse of p. The very same goes for q, so p and q are freely homotopic and hence associated to the same deck transformations, which implies that α fixes W_b. Since it expands or contracts A, we must have $A \cap W_b = \varnothing$, a contradiction. □

Remark 9.5.13. Because of its central role let us describe once more the construction of a periodic lozenge in Figure 9.5.4, which is at the same time a description of (9.5.1) and Figure 9.5.2: Draw complementary half-leaves from a periodic point p to the boundary of $\mathscr{O}^\Phi$, then draw the complementary half leaves back inwards from the end-points; they intersect in a periodic point q.

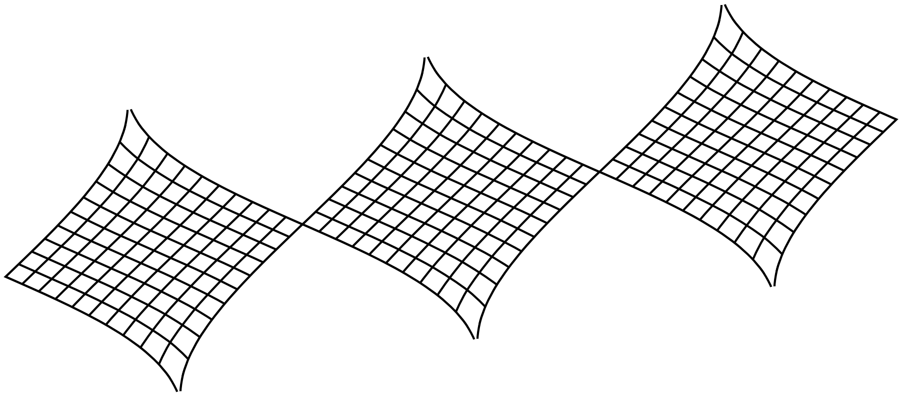


FIGURE 9.5.6. A string of lozenges

Claim 9.5.11 above follows from:

Lemma 9.5.14. *If p, q are indivisible closed orbits and q^n is freely homotopic to p^m (we write $q^n \simeq p^m$), then $p \simeq q$ or $q \simeq p^{-1}$. (And the deck transformation associated with an indivisible closed orbit is indivisible in the fundamental group.)*

Proof. The orbits lift to $\tilde{q} \in W_t$ and $\tilde{p} \in W_s$, and the deck transformation γ associated with q clearly preserves W_t, and γ^n preserves W_s. Since Φ is orientable, γ induces an orientation-preserving homeomorphism on $\mathscr{L}^u$, so γ itself preserves W_s, and therefore q is freely homotopic to a closed curve in the projection $\pi(W_s)$, whose fundamental group is generated by $[p]$. Thus $q \simeq p^i$ for some $i \in \mathbb{Z}$ and likewise, $p \simeq q^j$ for some $j \in \mathbb{Z}$, so $q \simeq q^{ij}$, and $ij = 1$. □

We remark that these arguments are but a sampling ideas that are being pushed much further at the time of this writing; the lozenges in the proof have supported a much deeper understanding of these Anosov flows [**33**].[18]

To pick up some of the earlier discussion and contrast it with the present situation, let us mention the possibility of lozenges being adjacent in the sense of sharing a side rather than solely a vertex. This happens in an Anosov flow that has a transverse torus. Unless the flow is a suspension, the weak stable and weak unstable foliations have some closed leaves on this transverse torus. Each of these corresponds to periodic orbits that are freely homotopic (up to orientation), and a coherent lift of these orbits (that is, including a lift of the free homotopy) has lozenges sharing sides. The interior of their union is the image of the transverse torus. The Franks–Williams example has 4 distinct orbits in the free homotopy class, and the Béguin–Bonatti–Yu construction gives more complicated examples.

6. Horocycle and unstable flows*

We have seen various close interactions between the geodesic flow and the horocycle flow of a compact negatively curved surface, and there have been hints that this is part of a larger picture. We are now in a position to show how the structural theory of hyperbolic systems we developed in previous chapters can make such connections in a broader context to support the analysis of flows that generalize the horocycle flow from Chapter 2. We saw in Proposition 3.3.19 (say) how the basic renormalization commutation relation (2.2.3) between the geodesic flow g^t and the horocycle flow h_+^t on a surface of constant negative curvature,

$$g^t h_+^s g^{-t} = h_+^{se^t},$$

[18] The only comprehensive exposition known to us is a set of lecture notes for the School on contemporary dynamical systems, held in July 2017 at Montréal at `https://sites.google.com/site/thomasbarthelme/Anosov_flows_in_3_manifolds.pdf`.

can be used for the study of horocycle flows, and in some more general situations. Our methods there were algebraic and analytic in nature suitably for the systems of algebraic nature we were dealing with.

We now leave the domain of algebraic dynamics and consider flows associated with the expanding foliations of hyperbolic systems. Even in the situations where those objects are defined unambiguously such as horocycle flows on surfaces of variable curvature they possess only moderate regularity, in the latter case they are C^1 but never C^2 (Theorem 10.3.10). In more general situations there is also a question of choosing a suitable parameterization that does not influence the description of invariant measures but may affect mixing and other properties. Two natural candidates are smooth parameterizations and the singular but dynamically well defined Margulis parameter introduced in Section 8.7. As it turns out, a variety of dynamical properties can be effectively studied for both types of parameterizations.

a. Definition of horocycle and unstable flows. To define horocycle flows we generalize the considerations from Subsection 2.1c to variable negative curvature, although we will concentrate on 2-dimensional manifolds.

Geodesic flows can be defined on any Riemannian manifold, and if the sectional curvature of the manifold is negative, then the geodesic flow is an Anosov flow (Theorem 5.2.4). The stable and unstable foliations project to horospheric foliations on the manifold itself (Section 6.2), which means that in the case of surfaces one can represent the situation on the universal cover pictorially in much the same way as in the constant-curvature situation of Figure 2.1.3. A horocycle flow is a flow whose orbits are the horocycles.

Definition 9.6.1. Consider an oriented negatively curved surface N. A flow on $M = SN$ whose orbits are the unstable manifolds (horospheres) of the geodesic flow is called a *horocycle flow*. More generally, given an Anosov flow on a manifold M with 1-dimensional orientable unstable foliation, a flow whose orbits are the unstable leaves is called an *unstable flow*.

Remark 9.6.2. Note that this definition does not specify a parameterization, so even for any one geodesic flow this describes a class of flows (that are related by time changes or reversals). The existence of such flows is clear: Orientability of the unstable foliation gives a unit vector field along it which defines a parametrization of a flow of the desired kind. In other words, we actually study a 1-dimensional oriented foliation, but our notions and tools are best adapted to viewing it as a flow.

In the case of geodesic flows Remark 6.2.1 provides this orientation. We remark that in this case the flow is a little more regular than required. Corollary 7.4.15 shows that the weak unstable subbundle is C^1, and the strong unstable subbundle is given by intersecting the weak unstable subbundle with the kernel of the contact form, which is smooth. So the weak unstable subbundle of the geodesic flow on a surface is C^1. (In fact, Theorem 7.4.14 shows that the unstable subbundle is $C^{1+\alpha}$ for all $\alpha < 1$, but no more by Theorem 10.3.10.)

The definition implies that such flows have no fixed points.

Some properties of these flows depend on a parameterization, others do not. In the former case we need to specify a parameterization.

Remark 9.6.3. Our further investigations rely on *renormalization* relations between the Anosov flow and the corresponding unstable flow. If φ^s is an unstable flow for an Anosov flow g^t then there is a function

$$\mathfrak{s}\colon \mathbb{R}\times\mathbb{R}\times M \to \mathbb{R} \quad \text{such that} \quad g^t(\varphi^s(x)) = \varphi^{\mathfrak{s}(t,s,x)}(g^t(x)). \tag{9.6.1}$$

This can be taken to measure the rate at which g^t expands unstable manifolds as measured by the time parameter of φ^s.

We define three distinguished choices of parametrization.

Definition 9.6.4. The *Margulis parametrization* or the *conformal parametrization* of a horocycle flow is the parametrization ψ^s for which $\mathfrak{s}(t,s,x) = h^t s$ in (9.6.1).

The horocycle flow for which

$$\frac{d}{ds}|_{s=0}\mathfrak{s}(t,s,x) = Y_x(t),$$

the unstable Jacobi field (Remark 5.2.6) along the geodesic defined by $x \in SM$, is called the *standard horocycle flow*.

The horocycle parametrized with unit speed is called the *unit-speed horocycle flow*.

For the Margulis parametrization the interplay between the horocycle flow and the geodesic (or Anosov) flow is particularly direct. We will show below that such a parametrization exists; the reason for the name is that this uses the Margulis measure (Definition 8.6.19).

Remark 9.6.5. Since g^t contracts unstable leaves when $t < 0$ it is easy to see that for $t < 0$ and $x \in M$ we have

$$\lim_{s\to\infty} \mathfrak{s}(t,s,x) - s = -\infty.$$

However, this also follows from the much stronger result that asymptotically we always have $s(t,s,x) = h^t s$ (Lemma 9.6.15 below).

We now explicitly describe the Margulis parametrization. The central ingredient is the Margulis measure from Definition 8.6.19, or, more to the point, its conditionals μ^u on unstable leaves. They first appeared in (8.6.4), which summarizes the pertinent property for us now—that they scale conformally under the geodesic flow. This determines a parametrization for a flow ψ^s by taking the direction determined by the orientation on W^u and the speed just such that the μ^u-measure of the arc between $\psi^t(x)$ and $\psi^s(x)$ is $|t-s|$. This gives an injective parametrization of leaves of W^u since μ^u is positive on open sets and unbounded on a leaf. Moreover, $s \mapsto \psi^s(s)$ is continuous since μ^u is a nonatomic Borel measure. We will obtain continuity of ψ^s as a byproduct of holonomy-compatibility.

We can now see that this gives the desired parametrization: By (8.6.4), we have

$$g^t \circ \psi^s = \psi^{h^t s} \circ g^t, \tag{9.6.2}$$

which is exactly (9.6.1) with the condition of Definition 9.6.4.

Next, we check the promised holonomy-compatibility. Note that for $x, y \in M$ close enough and $s \in \mathbb{R}$ small enough to be in a local product neighborhood we can write

$$\psi^{k_{x,y}(s)}(x) = [\psi^s(y), x],$$

(Bowen bracket, Proposition 6.2.2). The function $k_{x,y}$ thus defined measures how holonomy affects the parametrization. We claim:

(1) $k_{x,y}(\cdot)$ is strictly increasing,
(2) $k_{x,y}(\cdot)$ is Lipschitz continuous,
(3) $\lim_{y\to x} k_{x,y}(s) = s$ uniformly in s,
(4) $\lim_{y\to x} \frac{d}{ds} k_{x,y}(s) = 1$ uniformly in s.

For (4) note that the derivative is defined almost everywhere by (2); set it to 1 where not otherwise defined.

(1) follows from the fact that μ^u is positive on open sets.

(2) and (4) follow because Lemma 8.6.15 implies

$$\left| \frac{k_{x,y}(t) - k_{x,y}(s)}{t-s} - 1 \right| < \epsilon$$

for sufficiently small $s \neq t$. Indeed, this also implies (3) because

$$\lim_{y\to x} \psi^{k_{x,y}(0)}(x) = \lim_{y\to x} [y, x] = [x, x] = x$$

tells us that $\lim_{y\to x} k_{x,y}(0) = 0$.

To check that $(x, s) \mapsto \psi^s(x)$ is continuous we write $\psi^s(x) = [\psi_{k_{x,y}(s)}(x), y]$ and note that $[\cdot,\cdot]$ and $t \mapsto \psi^t(x)$ are continuous as is $(y, s) \mapsto k_{x,y}(s)$ by (3) because

$$|k_{x,y}(s) - k_{x,x}(t)| = |k_{x,y}(s) - t| \leq |k_{x,y}(s) - s| + |s - t|.$$

This concludes the description of this parametrization.

b. Minimality, entropy, unique ergodicity, and mixing. Minimality of horocycle flows is easy:

Proposition 9.6.6. *Horocycle flows are minimal.*

Proof. Geodesic flows are contact (Proposition 2.6.28) and Anosov (Theorem 5.2.4), hence topologically mixing (Theorem 9.1.2), so all unstable manifolds are dense (Theorem 9.1.1), which is the claim. □

The commutation relation (9.6.1) makes it easy to obtain the entropy of unstable flows.

Theorem 9.6.7. *Horocycle flows have zero entropy.*

Proof. The standard horocycle flow has finite entropy (Corollary 4.2.37), hence so does the Margulis parametrization Ψ (Theorem 4.3.14), and

$$h_{\text{top}}(\psi) = h_{\text{top}}(\psi^1) \overset{\text{Proposition 4.2.12(3)}}{=\!=\!=} \frac{1}{n} h_{\text{top}}(\psi^n) \overset{(9.6.4)}{=\!=} \frac{1}{n} h_{\text{top}}(\underbrace{\psi^1}_{<\infty}) \xrightarrow[n\to\infty]{} 0.$$

The claim follows from Theorem 4.3.14. □

Theorem 9.6.8. *A horocycle flow is uniquely ergodic.*

Remark 9.6.9. Indeed, we prove this for any unstable flow for a mixing C^2 Anosov flow with oriented 1-dimensional unstable subbundle. (Note that mixing follows from not being a suspension, see Theorem 9.1.1.)

Proof. Because unique ergodicity is independent of the parametrization (Corollary 3.5.5), we work with the Margulis parametrization.

To prove that this horocycle flow is uniquely ergodic we use (9.6.2) and Theorem 3.3.32: We will show that Birkhoff averages uniformly converge to a constant.

The first major step in this direction is the following:

Lemma 9.6.10. *If f is continuous on M then*

$$E_n f(x) := \frac{1}{2^n} \int_0^{2^n} f(\psi^s(g^{t_n}(x)))\, ds = \int_0^1 f(g^{t_n}(\psi^s(x)))\, ds$$

converges uniformly to a constant as $n \to \infty$. Here t_n is defined by $h^{t_n} = 2^n$ with h as in (9.6.2).

To see that this implies unique ergodicity, note that it implies

$$\forall f \in C(M)\ \exists c \in \mathbb{R}\ \forall \epsilon > 0\ \exists n \in \mathbb{N}\ \forall y \in M \quad |E_n f(y) - c| < \epsilon.$$

To apply this we want to move base points a little: For $j \in \mathbb{N}$ and $x \in M$ the quantity

$$\frac{1}{2^n j} \int_0^{2^n j} f(\psi^s(x))\, ds$$

is the average of $\{E_nf(g^{-t_n}(\psi^{2^n i}(x))\}_{i=0}^{j-1}$ and hence lies in $(c-\epsilon, c+\epsilon)$. But since $2^n j \xrightarrow[j\to\infty]{} \infty$ in an arithmetic progression, this implies that for large enough (but uniform) t we have

$$\Big|\frac{1}{t}\int_0^t f(\psi^s(x))\,ds - c\Big| < 2\epsilon,$$

that is, $\frac{1}{t}\int_0^t f(\psi^s(x))\,ds \to c$ uniformly, as claimed. □

Proof of Lemma 9.6.10. The definition of E_n implies that

$$E_{n+m}f = \frac{1}{2^m}\sum_{j=0}^{2^m-1} E_nf\circ\psi^j\circ g^{t_m}, \tag{9.6.3}$$

so $c_n := \min_{x\in M} E_nf(x)$ is nondecreasing, and we can take $c := \lim_{n\to\infty} c_n$.

Claim 9.6.11. *$\{E_nf\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded.*

By the Arzela–Ascoli Theorem there is a subsequence n_k such that $E_{n_k}f \to h \in C(M)$ uniformly. Then $\min h = c$, and (9.6.3) shows that given any $m\in\mathbb{N}$ we have

$$E_{n_k+m}f \to \bar h := \frac{1}{2^m}\sum_{j=0}^{2^m-1} h\circ\psi^j\circ g^{t_m},$$

and $\min\bar h = c$ as well.

The point is that if $\bar h(x_0) = c$, then $h(\psi^j(g^{t_m}(x_0))) = c$ for $0\le j<2^m$. Since m above was arbitrary, $h = c$ on a δ-dense set (Proposition 1.6.25), so $h\equiv c$ by continuity, and $E_{n_k}f\to c$ uniformly. But then (9.6.3) implies that $E_nf\to c$ uniformly, as claimed. □

Proof of Claim 9.6.11. We use that $\{g^t\restriction_{W_\eta^{0s}(x)}\}_{t\ge 0}$ is equicontinuous since it contracts. Specifically, given $x\in M$ and $\epsilon>0$, if $y\in M$ is close enough to x then for $|s|<1$ and $t\ge 0$ we have

$$|f(g^t(\psi^s(y))) - f(g^t([\psi^s(y),x]))| < \epsilon$$

as well as

$$|k_{x,y}(s) - s| < \epsilon \text{ and } |\frac{d}{ds}k_{x,y}(s) - 1| < \epsilon.$$

To prove the claim we show that for all $n\in\mathbb{N}$ we have

$$|E_nf(x) - E_nf(y)| < \epsilon(1+3\|f\|).$$

To see this we step from $E_nf(y)$ to $E_nf(x)$ via

$$\int_0^1 f(g^{t_n}([\psi^s(y),x]))\,ds = \int_0^1 f(g^{t_n}(\psi^{k_{x,y}(s)}(x)))\,ds$$

and

$$\int_0^1 \big(\frac{d}{ds}k_{x,y}(s)\big)\cdot(f(g^{t_n}(\psi^{k_{x,y}(s)}(x))))\,ds = \int_{k_{x,y}(0)}^{k_{x,y}(1)} f(g^{t_n}(\psi^s(x)))\,ds$$

(change of variables using that $k_{x,y}$ is Lipschitz, hence absolutely continuous).

The three resulting errors are

$$\Big|E_n f(y) - \int_0^1 f(g^{t_n}([\psi^s(y),x]))\,ds\Big| = \Big|\int_0^1 f(g^{t_n}(\psi^s(y))) - f(g^{t_n}([\psi^s(y),x]))\,ds\Big| < \epsilon,$$

$$\Big|\int_0^1 f(g^{t_n}(\psi^{k_{x,y}(s)}(x)))\,ds - \int_0^1 \big(\frac{d}{ds}k_{x,y}(s)\big)\cdot(f(g^{t_n}(\psi^{k_{x,y}(s)}(x))))\,ds\Big| < \epsilon\|f\|$$

and

$$\Big|\int_{k_{x,y}(0)}^{k_{x,y}(1)} f(g^{t_n}(\psi^s(x)))\,ds - E_n f(x)\Big| < \|f\|\cdot|k_{x,y}(1) - 1 + k_{x,y}(0)| < 2\epsilon\|f\|.$$

The sum of these is bounded by $\epsilon(1+3\|f\|)$, as claimed. □

While Theorem 9.6.8 holds independently of the parametrization chosen, we wish to point out that for the Margulis parametrization we used in the proof, the Margulis measure is not just ergodic—this extends Theorem 3.4.44.

Proposition 9.6.12. *The Margulis measure is weakly mixing (Definition 3.4.37) for the horocycle flow with the Margulis parametrization.*

PROOF. The Margulis measure is the sole invariant measure of the horocycle flow with the Margulis parametrization, so it is ergodic (Corollary 3.3.28). This implies that some time-t map of this horocycle flow is ergodic (Theorem 3.3.13). Since the Margulis measure is invariant under g^t, (9.6.2) rewritten as

$$g^{\log_h(t/s)} \circ \psi^s = \psi^t \circ g^{\log_h(t/s)} \tag{9.6.4}$$

establishes measure-theoretic isomorphisms between any two time-t maps for $t > 0$. This implies that all time-t maps are ergodic, which implies weak mixing (Proposition 3.4.40). □

This result may be surprising because our first observation about horocycle (and unstable) flows was that they are minimal. Minimality and unique ergodicity appeared first for rotations and toral translations, and none of those are mixing (Corollary 3.4.10). Our primary examples of mixing dynamical systems were of a rather different nature; mixing was due to hyperbolicity. Proposition 9.6.12 suggests that minimality and mixing are not mutually exclusive, and indeed, they are not:

Theorem 9.6.13. *Unstable flows (Definition 9.6.1) are topologically mixing.*

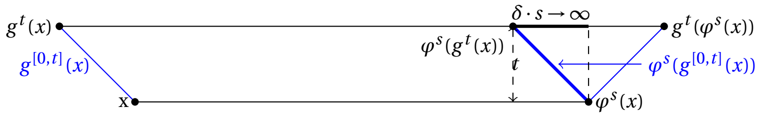


FIGURE 9.6.1. Stretching of a geodesic segment to a horocyclic one

The idea is that a geodesic segment $g^{[0,t]}(x)$ is (slowly) stretched by Φ to be ever more dense because it tracks the Φ-orbit of $g^t(x)$ within about t for length $\delta s \to \infty$ (where δ depends on t), hence within about ϵt for length $\epsilon\delta s \to \infty$, and this Φ-orbit is dense by minimality (Figure 9.6.1).

PROOF. If $U, V \subset M$ open choose a disk whose closure D is in V, and (by minimality) an $s_1 > 0$ such that

$$\varphi^{[0,s_1]}(x) \cap D \neq \varnothing \text{ for } x \in M. \tag{9.6.5}$$

(Here, $\varphi^A(B) := \bigcup_{s\in A} \varphi^s(B)$.) Also choose $x_0 \in U$ and $a > 0$ such that

$$g^{[0,a]}(x_0) \subset U \text{ and } \sup\{d(g^t(x), x) \mid x \in M,\ 0 \le t \le a\} < d(D, M \smallsetminus V). \tag{9.6.6}$$

Then $\varphi^s(U) \cap V \neq \varnothing$ for all sufficiently large s because

$$\varphi^s(g^{[0,a]}(x_0)) \cap V \neq \varnothing, \tag{9.6.7}$$

by Claim 9.6.14 below. □

Claim 9.6.14. *For large enough s there is a $t \in [0, a]$ for which $\varphi^s(g^t(x_0)) \in V$.*

PROOF. $\mathfrak{s}(-a, s, g^a(x_0)) < s - s_1$ for large s (Remark 9.6.5). Since $\mathfrak{s}(0, s, g^0(x_0)) < s$, this implies that $[s, s - s_1] \subset \mathscr{S} := \bigcup_{t\in[0,a]} \mathfrak{s}(-t, s, g^t(x_0))$. (9.6.5) gives a $t \in [0, a]$ with

$$\varphi^{\mathfrak{s}(-t,s,g^t(x_0))}(x_0) \in D.$$

By (9.6.6) this implies $\varphi^s(g^t(x_0)) = g^t(\varphi^{\mathfrak{s}(-t,s,g^t(x_0))}(x_0)) \in V$ as claimed. □

Our next object is to establish that horocycle flows are, in fact, measure-theoretically mixing (with respect to to the sole invariant measure). While the above proof of mixing demonstrates the essential ideas needed for the measure-theoretic result, there is a need for substantial refinements, and therefore we now digress to two technical results regarding the parameter change in (9.6.1). We already mentioned the first of these:

Lemma 9.6.15 (Expanding lemma)**.** *If φ^s is an unstable flow (Definition 9.6.1), $\mathfrak{s}$ is as in* (9.6.1)*, h is as in* (9.6.2) *and $a < b \in \mathbb{R}$, then*

$$\mathfrak{s}(t, s, x)/s \xrightarrow[s\to\infty]{} h^t$$

uniformly in $(x,t) \in M \times [a,b]$.

Proof. Fix $[a,b] \subset \mathbb{R}$. For the Margulis parametrization ψ^s we have equality by definition, and it is useful to compare the given parametrization φ^s with the Margulis parametrization. As in Definition 1.2.1, denote the time change by α: $\varphi^s(x) = \psi^{\alpha(s,x)}(x)$. This and (9.6.2) imply

$$\alpha(\mathfrak{s}(t,s,x), g^t(x)) = h^t \alpha(s,x), \tag{9.6.8}$$

and hence

$$\frac{\mathfrak{s}(t,s,x)}{s} = h^t \frac{\frac{\alpha(s,x)}{s}}{\frac{\alpha(\mathfrak{s}(t,s,x),g^t(x))}{\mathfrak{s}(t,s,x)}}. \tag{9.6.9}$$

Claim 9.6.16. $\alpha(s,x)/s \xrightarrow[s\to\infty]{} \alpha_0 \neq 0$ *uniformly in* $x \in M$.

The claim implies first that the numerator on the right-hand side of (9.6.9) tends to α_0. Together with (9.6.8) it also implies that $\mathfrak{s}(t,s,x) \xrightarrow[s\to\infty]{} \infty$ uniformly in $x \in M$, $t \in [a,b]$. But then the denominator on the right-hand side of (9.6.9) is of the form $\alpha(s^*,x)/s^*$ with $s^* \to \infty$, hence goes to α_0 uniformly in $x \in M$, $t \in [a,b]$ as well. □

Proof of Claim 9.6.16. Since a is continuous, we can use unique ergodicity of Φ. It suffices to prove this for an arithmetic progression of s, so we take t such that φ^t is uniquely ergodic (Theorem 3.3.34) and write

$$\frac{\alpha(nt,x)}{n} = \frac{\sum_{i=0}^{n-1} a(\varphi^{nt}(x))}{n},$$

where $a(x) := \alpha(t,x)$. This is a Birkhoff average of a continuous function and hence converges uniformly to a constant (Proposition 3.3.33). □

The next result is a rather finer version of this statement.

Lemma 9.6.17. *If* φ^s *is an unstable flow such that* $\dfrac{\partial^2 \mathfrak{s}}{\partial t \partial s}$ *is continuous in* (t,s,x), *then* $u_{s,x}(t) := \mathfrak{s}(-t,s,g^t(x))$ *satisfies*

$$\frac{1}{s}\frac{d}{dt}\Big|_{t_0} u_{s,x}(t) \xrightarrow[t_0\to 0]{s\to\infty} -\log h$$

uniformly in $x \in M$, *where* h *is as in* (9.6.2).

Proof. It is useful to note that $u_{s,x}(t)$ is the inverse of $\mathfrak{s}$ in that

$$s = \mathfrak{s}(t, u_{s,x}(t), x) \tag{9.6.10}$$

To see this, write $z = g^t(x)$ to get

$$\begin{aligned}\varphi^s(z) = \varphi^s(g^t(x)) &= g^t[g^{-t}(\varphi^s(g^t(x)))]\\ &= g^t[\varphi^{u_{s,x}(t)}(x)] = \varphi^{\mathfrak{s}(t,u_{s,x}(t),x)}(g^t(x)) \qquad\qquad \text{by (9.6.1)}\\ &= \varphi^{\mathfrak{s}(t,u_{s,x}(t),x)}(z).\end{aligned}$$

The point of (9.6.10) is that differentiating with respect to t gives

$$0 = \frac{d}{dt}|_{t_0}\mathfrak{s}(t,u_{s,x}(t_0),x) + \Big[\frac{d}{ds}|_{u_{s,x}(t_0)}\mathfrak{s}(t_0,s,x)\Big]\Big[\frac{d}{dt}|_{t_0}u_{s,x}(t)\Big],$$

so

$$\frac{d}{dt}|_{t_0}u_{s,x}(t) = -\frac{\frac{d}{dt}|_{t_0}\mathfrak{s}(t,u_{s,x}(t_0),x)}{\frac{d}{ds}|_{u_{s,x}(t_0)}\mathfrak{s}(t_0,s,x)}.$$

For the denominator check that

$$\begin{aligned}\frac{d}{ds}|_{u_{s,x}(t_0)}\mathfrak{s}(t_0,s,x) &= \frac{d}{ds}|_0\mathfrak{s}(t_0,s,\varphi^{u_{s,x}(t_0)}x)\\ &\xrightarrow[t_0\to 0]{} \frac{d}{ds}|_0\mathfrak{s}(0,s,\varphi^{u_{s,x}(0)}x) = \frac{d}{ds}|_0\mathfrak{s}(0,s,x) = 1\end{aligned}$$

uniformly in (s,x). Thus, Lemma 9.6.17 follows from the next claim. □

Claim 9.6.18. $\frac{1}{s}\frac{d}{dt}|_{t_0}\mathfrak{s}(t,u_{s,x}(t_0),x) \xrightarrow[s\to\infty]{} \log h$ *uniformly in* $(x,t)\in M\times[-\epsilon,\epsilon]$.

Proof. $h_{s,x}(t) := \dfrac{\mathfrak{s}(t,u_{s,x}(t_0),x)}{u_{s,x}(t_0)} \xrightarrow[s\to\infty]{} h^t$ uniformly in (x,t) (Lemma 9.6.15), and

$$\begin{aligned}h'_{s,x}(t) = \frac{\frac{d}{dt}|_{t_0}\mathfrak{s}(t,u_{s,x}(t_0),x)}{u_{s,x}(t_0)} &= \frac{\int_0^{u_{s,x}(t_0)}\frac{\partial^2}{\partial s\partial t}|_{t=t_0,s=0}\mathfrak{s}(t,s,\varphi^s x)\,ds}{u_{s,x}(t_0)}\\ &\xrightarrow[s\to\infty]{} \int\frac{\partial^2}{\partial s\partial t}|_{t=t_0,s=0}\mathfrak{s}(t,s,x)\,d\mu(x) =: h(t)\end{aligned}$$

uniformly in (x,t) by continuity and unique ergodicity (here we use the continuity hypothesis on $\frac{\partial^2}{\partial s\partial t}|_{t=t_0,s=0}\mathfrak{s}(t,s,x)$, and μ is the invariant measure).

Since $h_{s,x}$ and $h'_{s,x}$ converge uniformly we can exchange limit and differentiation to conclude that $h(t) = \frac{d}{dt}h^t = h^t\log h$. This finally gives

$$\frac{1}{s}\frac{d}{dt}|_{t_0}\mathfrak{s}(t,u_{s,x}(t_0),x) = h'_{s,x}(t)\cdot\frac{s}{u_{s,x}(t_0)} \xrightarrow[s\to\infty]{} h^t\log h\cdot\frac{1}{h^t}$$

uniformly in $(x,t)\in M\times[-\epsilon,\epsilon]$ by Lemma 9.6.15. □

We now exhibit the essential ingredient of the proof of mixing with respect to the invariant measure. In the proof that these flows are topologically mixing we needed to show that for a given $a > 0$ the image $\varphi^s(g^{[0,a]}(x_0))$ under φ^s of an unstable arc is ϵ-dense for large enough s (see, for example, (9.6.7)). The idea is the same for establishing that these flows are measure-theoretically mixing, but the requirement now is that long arcs of this type be almost uniformly distributed in order to adequately reflect suitable measures of intersections of sets.

Lemma 9.6.19. *Under the hypotheses of Lemma 9.6.17, suppose $f\colon M \to \mathbb{R}$ is continuous and $b > 0$. Then*

$$\frac{1}{b}\int_0^b f(\varphi^s(g^t(x)))\,dt \xrightarrow[s\to\infty]{} \int f\,d\mu$$

uniformly in $x \in M$.

Proof. Fix $\epsilon > 0$; we need to show that

$$\max_x \left|\frac{1}{b}\int_0^b f(\varphi^s(g^t(x)))\,dt - \int f\,d\mu\right| < \epsilon$$

for all sufficiently large s. First note that it suffices to do so for $b < b_0(\epsilon)$: Writing

$$\frac{1}{b}\int_0^b f(\varphi^s(g^t(x)))\,dt = \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{b/n}\int_0^{b/n} f(\varphi^s(g^t(g^{ib/n}(x))))\,dt$$

shows that it suffices to bring each summand in the average to within ϵ of $\int f\,d\mu$.

Choose $b_0(\epsilon)$ such that $\sup_{t\in[0,b_0(\epsilon)]}\|f - f\circ g^t\| < \epsilon/2$, and restrict attention to $b \in [0, b_0(\epsilon)]$, for which

$$\frac{1}{b}\int_0^b f(\varphi^s(g^t(x)))\,dt = \frac{1}{b}\int_0^b f(g^t(\varphi^{u_{s,x}(t)}(x)))\,dt$$

is within $\epsilon/2$ of

$$\begin{aligned}\frac{1}{b}\int_0^b f(\varphi^{u_{s,x}(t)}(x))\,dt &\underset{\text{[change of variables } u=u_{s,x}(t)]}{=} \frac{1}{b}\int_s^{u_{s,x}(b)} f(\varphi^u(x))\Big[\frac{1}{\frac{d}{dt}\big|_{u_{s,x}^{-1}(u)} u_{s,x}(t)}\Big]\,du\\ &\underset{\text{[Mean-Value Theorem; } t_0\in[0,b]]}{=} \frac{1}{b}\,\frac{1}{\frac{d}{dt}\big|_{t_0} u_{s,x}(t)}\int_s^{u_{s,x}(b)} f(\varphi^u(x))\,du\\ &= \Bigg[\frac{\frac{u_{s,x}(b)-s}{b}}{\frac{d}{dt}\big|_{t_0} u_{s,x}(t)}\Bigg]\cdot\Bigg[\frac{1}{u_{s,x}(b)-s}\int_s^{u_{s,x}(b)} f(\varphi^u(x))\,du\Bigg].\end{aligned}$$

It remains to bring this within $\epsilon/2$ of $\int f\,d\mu$, and the point of these manipulations was that the very last term is an ergodic average hence by Remark 9.6.5 and unique ergodicity converges uniformly to $\int f\,d\mu$.

Thus, we only have to show that the first factor can be brought close to 1. To that end note that by the Mean-Value Theorem

$$\frac{u_{s,x}(b)-s}{b}=\frac{d}{dt}|_{t_1}u_{s,x}(t)$$

with $t_1\in[0,b]$. Thus, we need to control

$$\frac{\frac{d}{dt}|_{t_1}u_{s,x}(t)}{\frac{d}{dt}|_{t_0}u_{s,x}(t)}\quad\text{for } t_0,t_1\in[0,b]\tag{9.6.11}$$

uniformly in x by taking b small and then s large. This is a delicate problem and the reason for Lemma 9.6.17, which now tells us that

$$\frac{\frac{d}{dt}|_{t_1}u_{s,x}(t)}{\frac{d}{dt}|_{t_0}u_{s,x}(t)}=\frac{\frac{1}{s}\frac{d}{dt}|_{t_1}u_{s,x}(t)}{\frac{1}{s}\frac{d}{dt}|_{t_0}u_{s,x}(t)}\xrightarrow[t_0,t_1\to 0]{s\to\infty}\frac{-\log h}{-\log h}=1,$$

so small b and large s make (9.6.11) as close to 1 as needed—uniformly in x. □

The following is straightforward to verify.

Theorem 9.6.20. *Consider a flow Φ on a measure space (X,μ) for which there is a $\lambda>0$ such that $\mu(\varphi^t(A))=\lambda^t\mu(A)$ for each measurable set A and every $t\in\mathbb{R}$. Suppose $S\subset X$ is measurable and such that $\Phi\restriction_{[0,a]\times S}$, $(t,x)\mapsto\varphi^t(x)$ is injective for some $a>0$. Then $S_a:=\Phi([0,a]\times S)$ is called a* flow box *over S, and*

$$\int f\,d\mu=\int_0^a\int_S\lambda^t f(\varphi^t(x))\,d\mu_S(x)dt,$$

for measurable $f\colon S_a\to\mathbb{R}$, where $\mu_S(A):=\dfrac{\mu(\varphi^{[t_1,t_2]}(A))}{\int_{t_1}^{t_2}\lambda^t dt}$ with $0\le t_1<t_2\le a$.

Note that the case $\lambda=1$ corresponds to invariant measures.

Theorem 9.6.21. *If φ^s is an unstable flow such that $\dfrac{\partial^2\mathfrak{s}}{\partial t\partial s}$ is continuous in (t,s,x), then the invariant measure for φ^s is mixing.*

Proof. We will show that $\displaystyle\int_N f\circ\varphi^s\,d\mu\xrightarrow[s\to\infty]{}\mu(N)\int f\,d\mu$ whenever $f\colon M\to\mathbb{R}$ is continuous and N is a local product neighborhood. Since continuous functions and linear combinations of characteristic functions of product neighborhoods are dense in $L^1(\mu)$, this implies that μ is mixing.

For a product neighborhood $N:=\varphi^{[0,a]}(g^{[0,b]}(W^s_\epsilon(z)))$ Theorem 9.6.20 applied twice gives a measure μ^s on $W^s_\epsilon(z)$ such that the natural parametrization $(s,t,x)\mapsto$

$\varphi^s(g^t(x)))$ gives

$$\int_N f\,d\mu = \int_{W^s_\epsilon(z)} \int_0^b \int_0^a h^{-t} f(\varphi^\sigma(g^t(x)))\,d\sigma\,dt\,d\mu^s(x)$$

for any continuous $f\colon N \to \mathbb{R}$. In fact, the measure μ^s, being transverse to the unstable flow, is independent of the parametrization chosen and hence coincides with the measure μ^s from Section 8.6 (except that the roles of the stable and unstable manifolds are here interchanged).

This, plus the Fubini Theorem, allows us to rewrite

$$\int_N f\circ\varphi^s\,d\mu = \int_{W^s_\epsilon(z)} \int_0^a \int_0^b h^{-t} f(\varphi^{s+\sigma}(g^t(x)))\,dt\,d\sigma\,d\mu^s(x).$$

Claim 9.6.22. $\int_0^b h^{-t} f(\varphi^{s+\sigma}(g^t(x)))\,dt \xrightarrow[s\to\infty]{} \Big[\int_0^b h^{-t}\,dt\Big]\Big[\int f\,d\mu\Big]$.

Proof. This is essentially Lemma 9.6.19, except for the factor h^{-t}. Lemma 9.6.19 gives $\int_0^b f(\varphi^{s+\sigma}(g^t(x)))\,dt \xrightarrow[s\to\infty]{} \big[\int_0^b 1\,dt\big]\big[\int f\,d\mu\big]$, and the trick is that it does so for arbitrarily small b. Therefore we subdivide $[0,b]$ into small subintervals on which h^{-t} is sufficiently close to constant. Piecing together the corresponding applications of Lemma 9.6.19 yields the claim. □

The claim finally yields

$$\begin{aligned}\int_N f\circ\varphi^s\,d\mu &= \int_{W^s_\epsilon(z)} \int_0^a \int_0^b h^{-t} f(\varphi^{s+\sigma}(g^t(x)))\,dt\,d\sigma\,d\mu^s(x)\\ &\xrightarrow[s\to\infty]{} \int_{W^s_\epsilon(z)} \int_0^a \Big[\int_0^b h^{-t}\,dt\Big]\Big[\int f\,d\mu\Big]\,d\sigma\,d\mu^s(x) = \mu(N)\int f\,d\mu. \quad \square\end{aligned}$$

We now check the smoothness assumption in Theorem 9.6.21 for the three parametrizations in Definition 9.6.4.

Corollary 9.6.23. *An unstable flow with the Margulis parametrization is mixing.*

Proof. $\dfrac{\partial^2 \mathfrak{s}}{\partial t \partial s} = \dfrac{\partial^2}{\partial t \partial s} h^t s$ is continuous in (t,s,x). □

Corollary 9.6.24. *Standard horocycle flows are mixing.*

Proof. Using the definition and the Jacobi equation (5.2.2) we have

$$\frac{\partial^s \mathfrak{s}}{\partial t \partial s} = \dot{Y}_{\varphi^s(x)}(t) = -\int_{-\infty}^t K(g^\tau(\varphi^s(x)))\,Y_{\varphi^s(x)}(\tau)\,d\tau$$

is continuous in (t,s,x) because K is continuous and $Y_\cdot(t) \xrightarrow[t\to-\infty]{} 0$ exponentially and uniformly. □

Corollary 9.6.25. *Unstable flows with unit-speed parametrization are mixing.*

Proof. Differentiate $g^t(\varphi^s(x)) = \varphi^{\mathfrak{s}(t,s,x)}(g^t(x))$ with respect to s to get

$$\|Dg^t(\frac{\partial}{\partial s}\varphi^s(x))\| = \frac{\partial}{\partial s}\mathfrak{s}(t,s,x)\|\frac{\partial}{\partial s}\varphi^s(g^t(x)))\| = \frac{\partial}{\partial s}\mathfrak{s}(t,s,x),$$

since the parametrization is with unit speed. To see that the t-derivative is continuous note that Dg^t is C^1 in t, and $x \mapsto \frac{\partial}{\partial s}\varphi^s(x)$ is continuous. □

c. Multiple mixing. Unstable flows with the Margulis parametrization are, in fact, mixing of all orders (Definition 3.4.37). This is an ergodic result that requires little structural information except for the renormalization relation (9.6.2).

Theorem 9.6.26. *Let M be a manifold and μ a Borel probability measure on M. If G and Ψ are continuous fixed point free μ-preserving flows on M such that*

(1) Ψ is ergodic with respect to μ,
(2) (9.6.2) *holds for some $h > 1$,*

then Ψ is mixing of all orders.

We actually prove the following: for a μ-preserving flow ϕ^s on X define

$$P(N) :\Leftrightarrow \forall\{f_1,\dots,f_n\} \subset C_c(X) \colon \frac{1}{n-m}\int_m^n \prod_{i=1}^N f_i(\phi^{K_i u}(x))\,du \xrightarrow[\substack{n-m\to\infty \\ (K_i-K_{i-1})n\to\infty \\ 1=K_1<K_2<\cdots<K_N}]{L^2} \prod_{i=1}^N \int f_i\,d\mu.$$

Prior to proving this (for all N), we check that $P(N)$ implies mixing of order N.

To see this, we will verify mixing with respect to small flow boxes. Thus we first localize $P(N)$ as follows.

Lemma 9.6.27. *If $\mathscr{S}$ is a collection of local sections S (Theorem 9.6.20) for the flow g^t in Theorem 9.6.26 and*

- *$\mu_S(S) \le 1$ for all $S \in \mathscr{S}$*
- *$\alpha \in [0,1)$,*
- *ψ^s satisfies $P(N)$, and*
- *$\{f_1,\dots,f_n\} \subset C_c(X)$,*

then

$$\frac{1}{n-\alpha n}\int_{\alpha n}^n \int_S \prod_{i=1}^N f_i(\phi^{K_i u}(y))\,d\mu_S(y)\ du \xrightarrow[\text{uniformly in } S\in\mathscr{S}]{\substack{n\to\infty \\ (K_i-K_{i-1})n\to\infty \\ 1=K_1<K_2<\cdots<K_N}} \mu_S(S)\prod_{i=1}^N \int f_i\,d\mu$$

uniformly in $S \in \mathscr{S}$.

Proof. We first note that by recognizing an L^2-product

$$\frac{1}{b}\int_0^b \underbrace{\frac{1}{n-\alpha n}\int_{\alpha n}^{n}\int_S \prod_{i=1}^{N} f_i(\psi^{K_i u}(g^t(y)))\,d\mu_S(y)\,du}_{=:h_{n,K_1,\dots,K_N,S}(t)}\,dt$$

$$= \left\langle \frac{1}{b}\chi_{S_b}, \frac{1}{n-\alpha n}\int_{\alpha n}^{n}\prod_{i=1}^{N} f_i\circ\psi^{K_i u}\,du\right\rangle \xrightarrow[\substack{(K_i-K_{i-1})n\to\infty\\ 1=K_1<K_2<\cdots<K_N}]{n\to\infty} \mu_S(S)\cdot\prod_{i=1}^{N}\int f_i\,d\mu$$

by hypothesis, and convergence is uniform in S (Cauchy–Schwarz inequality using $\|\chi_{S_b}/b\|_2 \le b^{-1/2}$).

To obtain the lemma from this note that (9.6.2) and a change of variable allow us to rewrite the integrand as

$$h_{n,K_1,\dots,K_N,S}(t) = \frac{h^t}{n-\alpha n}\int_{\alpha n/h^t}^{n/h^t}\int_S \prod_{i=1}^{N} f_i(g^t(\psi^{K_i w}(y)))\,d\mu(y)\,dw,$$

so $h_{n,K_1,\dots,K_N,S}(t)$ is equicontinuous at 0. As b (and t) go to 0 we obtain from the above that

$$h_{n,K_1,\dots,K_N,S}(0) \xrightarrow[\substack{(K_i-K_{i-1})n\to\infty\\ 1=K_1<K_2<\cdots<K_N}]{n\to\infty} \mu_S(S)\prod_{i=1}^{N}\int f_i\,d\mu$$

uniformly in $S\in\mathscr{S}$, which is the conclusion of the lemma. □

We now verify mixing of order N by checking the conditions of Definition 3.4.37 for $f_0 = \chi_{S_a}$ and $f_1,\dots,f_N \in C_c(X)$ such that $\max_{1\le i\le N}\|f_i\|_\infty \le 1$, $\mu_S(S)\le 1$ and $s_0 = 0$. (Linear combination and approximation then yield the general statement.)

It is enough to check “thin” flow boxes, that is, to consider S_a for small a:

$$\frac{1}{a}\int\prod_{i=0}^{N} f_i(\psi^{s_i}(x))\,d\mu = \frac{1}{a}\int f_0(x)\prod_{i=1}^{N} f_i(\psi^{s_i}(x))\,d\mu$$
$$= \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{a/n}\int \chi_{g^{ia/n}(S)_{a/n}}\prod_{i=1}^{N} f_i(\psi^{s_i}(x))\,d\mu.$$

Therefore, we check mixing of order N by proving the

Claim 9.6.28. $\displaystyle\lim_{b\to 0}\ \overline{\lim}_{s_i-s_{i-1}\to\infty}\left|\frac{1}{b}\int\chi_{T_b}(x)\prod_{i=1}^{N} f_i(\psi^{s_i}(x))\,d\mu - \mu_T(T)\prod_{i=1}^{N}\int f_i\,d\mu\right| = 0$ *uniformly in* $T\in\{g^t(S)\mid S\in\mathscr{S},\ t\in[0,a]\}$.

PROOF. For $\epsilon > 0$ we can choose b small enough such that the leftmost item,

$$\begin{aligned}\frac{1}{b}\int \chi_{T_b}(x)\prod_{i=1}^{N} f_i(\psi^{s_i}(x))\,d\mu &= \frac{1}{b}\int_0^b \int_T \prod_{i=1}^{N} f_i(\psi^{s_i}(g^t(y)))\,d\mu(y)\,dt\\ &= \frac{1}{b}\int_0^b \int_T \prod_{i=1}^{N} f_i(g^t(\psi^{s_i/h^t}(y)))\,d\mu(y)\,dt\end{aligned}$$

(we used Theorem 9.6.20 and (9.6.2)) is within ϵ of

$$\mathscr{X} := \frac{1}{b}\int_0^b \int_T \prod_{i=1}^{N} f_i(\psi^{s_i/h^t}(y))\,d\mu(y)\,dt.$$

We change to the variable $u = s_1/h^t$ and use the Mean-Value Theorem to find a $u_0 \in [s_1/h^b, s_1]$ for which

$$\begin{aligned}\mathscr{X} &= \frac{1}{b}\frac{1}{u_0 \log h}\int_{s_1/h^b}^{s_1}\int_T \prod_{i=1}^{N} f_i(\psi^{s_i u/s_1}(y))\,d\mu(y)\,du\\ &= \Big[\underbrace{\frac{h^b-1}{b\log h}}_{\xrightarrow[b\to 0]{}\frac{dh^b/db}{\log h}=1}\Big]\Big[\underbrace{\frac{s_1/h^b}{u_0}}_{\in[h^{-b},1]}\Big]\Big[\frac{1}{s_1-h^{-b}s_1}\int_{s_1/h^b}^{s_1}\int_T \prod_{i=1}^{N} f_i(\psi^{s_i u/s_1}(y))\,d\mu(y)\,du\Big].\end{aligned}$$

This is close to $\mu_T(T)\prod_{i=1}^{N}\int f_i\,d\mu$ the claim. Lemma 9.6.27 with $\alpha = h^{-b}$. □

PROOF OF THEOREM 9.6.26. We now check inductively that ψ^s satisfies $P(N)$ for all $N \in \mathbb{N}$. For $N = 1$ this is the case by ergodicity and the Mean Ergodic Theorem 3.2.4. Thus suppose N is such that ψ^s satisfies $P(n)$ for all $n < N$.

To reduce the clutter in limits, we choose sequences of parameters as follows: For $l \in \mathbb{N}$ take $n_l, m_l, K_{1,l}, \dots, K_{N,l}$ such that

- $n_l - m_l \xrightarrow[l\to\infty]{} \infty$,
- $(K_{i,l} - K_{i-1,l})n_l \xrightarrow[l\to\infty]{} \infty$,
- $1 = K_{1,l} < K_{2,l} < \cdots < K_{N,l}$,
- $\lim_{l\to\infty} K_{i,l} \in [0,\infty]$ exists for each i.

The last item can be arranged by passing to a subsequence.

Claim 9.6.29. $\displaystyle \frac{1}{n_l - m_l}\int_{m_l}^{n_l}\prod_{i=1}^{N} f_i(\psi^{K_{i,l}u}(x))\,du \xrightarrow[l\to\infty]{L^2(\mu)} \prod_{i=1}^{N}\int f_i\,d\mu.$

In proving this claim we consider two cases separately: in the first case one of the $K_{i,l} \xrightarrow[l\to\infty]{} \infty$ and we let j be the smallest such index. We will see that we can split the indices into those below j and those above, and we can then use the inductive hypothesis on each piece (after rescaling the "top" part).

Write $\mathscr{F}_u^n(x) \coloneqq \prod_{i=1}^n f_i(\psi^{K_{i,l}u}(x))$. Then for sufficiently large $p \in \mathbb{N}$ we have

$$|\mathscr{F}_{u_1}^{j-1}(x) - \mathscr{F}_{u_2}^{j-1}(x)| < \epsilon \quad \text{when } |u_1 - u_2| < 1/p$$

and

$$\Big|\frac{1}{p}\sum_{=1} p\mathscr{F}_{q/p}^{j-1}(x_i) - \int_0^1 \mathscr{F}_u^{j-1}(x_i)\,du\Big| < \epsilon,$$

hence

$$\Big|\frac{1/p}{n_l - m_l}\sum_{q=m_l p+1}^{n_l p} \mathscr{F}_{p/q}^{j-1}(x) - \frac{1}{n_l - m_l}\int_{m_l}^{n_l} \mathscr{F}_u^{j-1}(x)\,du\Big| < \epsilon. \tag{9.6.12}$$

The split of indices works out as follows. In the expression $\dfrac{1}{n_l - m_l}\displaystyle\int_{m_l}^{n_l} \mathscr{F}_u^N(x)\,du$ of the claim we replace the terms below j by using (9.6.12). The product of the top terms, $\displaystyle\prod_{i=j}^N f_i(g^{K_{i,l}q/p}(x))$, is approximated by $b_{q,l}(x) \coloneqq p\displaystyle\int_{(q-1)/p}^{q/p}\prod_{i=j}^N f_i(g^{K_{i,l}u}(x))\,du$, so

$$\Big|\frac{1}{n_l - m_l}\int_{m_l}^{n_l} \mathscr{F}_u^N(x)\,du - \frac{1/p}{n_l - m_l}\sum_{q=m_l p+1}^{n_l p} \mathscr{F}_{p/q}^{j-1}(x)\cdot b_{q,l}(x)\Big| < \epsilon. \tag{9.6.13}$$

We first deal with the "top end." A change of variable shows that

$$b_{q,l}(x) = \frac{1}{K_{j,l}/p}\int_{K_{j,l}\frac{q-1}{p}}^{K_{j,l}\frac{q}{p}} \prod_{i=j}^N f_i(\psi^{K_{i,l}w/K_{j,l}}(x)\,dw,$$

and because $K_{j,l}/p \xrightarrow[l\to\infty]{} \infty$ and $\Big(\frac{K_{i,l}}{K_{j,l}} - \frac{K_{i-1,l}}{K_{j,l}}\Big)K_{j,l}n_l \xrightarrow[l\to\infty]{} \infty$, we can apply the induction hypothesis to this expression and therefore replace $b_{q,l}(x)$ in (9.6.13) by $\prod_{i=j}^N \int f_i\,d\mu$ up to a small error. To the resulting expression

$$\frac{1/p}{n_l - m_l}\sum_{q=m_l p+1}^{n_l p} \mathscr{F}_{p/q}^{j-1}(x)\cdot\prod_{i=j}^N\int f_i\,d\mu$$

we can again apply (9.6.12) (now backwards) to instead get

$$\frac{1}{n_l - m_l}\int_{m_l}^{n_l} \mathscr{F}_u^{j-1}(x)\,du\cdot\prod_{i=j}^N\int f_i\,d\mu = \frac{1}{n_l - m_l}\int_{m_l}^{n_l}\prod_{i=1}^{j-1} f_i(\psi^{K_{i,l}u}(x))\,du\cdot\prod_{i=j}^N\int f_i\,d\mu$$

and a slightly larger error.

The induction hypothesis for the terms with $i < j$ makes the last expression (and hence $\frac{1}{n_l - m_l}\int_{m_l}^{n_l}\prod_{i=1}^N f_i(\psi^{K_{i,l}u}(x))\,du$) L^2-close to $\prod_{i=1}^N \int f_i\,d\mu$.

This gives the claim (and hence $P(N)$) when some $K_{j,l} \to \infty$.

In the complementary case in which all the $K_{i,l}$ are bounded, a compactness argument (in the parameter space) allows us to produce limits that are uniform in the possible choices of the $K_{i,l}$. We then conclude the inductive proof of $P(N)$ using the following general lemma.

For a manifold M denote by $\mathscr{F}$ the set of continuous flows with the topology of uniform convergence on compact sets. For a Borel probability measure μ we say that $\mathscr{E}\subset\mathscr{F}$ is μ-ergodic if μ is an ergodic invariant measure for every $\varphi\in\mathscr{E}$.

Lemma 9.6.30. *Suppose $\mathscr{E}\subset\mathscr{F}$ and there are*

- $c\colon\mathscr{E}\to\mathbb{R}^+$
- *Borel probability measures μ and ν*
- *a closed $\mathscr{E}$-invariant algebra $\mathscr{A}\subset C_c(M)$*

such that

- *$\mathscr{E}$ has compact closure $\bar{\mathscr{E}}$*
- *$\bar{\mathscr{E}}$ is μ-ergodic*
- *$\mathscr{E}$ is $(\nu,c,\mathscr{A})$-*generic*, that is, if $f\in\mathscr{A}$, then*

$$\frac{1}{n-m}\int_m^n\int f(\varphi^u(x))\,d\nu(x)\,du\xrightarrow[c(\varphi)\cdot n\to\infty]{n-m\to\infty}\int f\,d\mu.$$

Then $\dfrac{1}{n-m}\displaystyle\int_m^n f(\varphi^u(x))\,du\xrightarrow[c(\varphi)\cdot n\to\infty]{n-m\to\infty}\int f\,d\mu$ *for all $f\in\mathscr{A}$.*

To apply this in the case of bounded $K_{i,l}$ take an interval $E^*\subset[1,\infty)$ such that $K_{i,l}\in E^*$ for all i,l. Let $M:=X^N$, ν the diagonal measure, and $\mathscr{E}$ the set of flows on M of the form

$$\varphi^u(x_1,\dots,x_n)=(\psi^{K_1u}(x_1),\dots,\psi^{K_Nu}(x_N))$$

with $1=K_1<\cdots<K_N\in E^*$ (this clearly has compact closure) and $c(\varphi):=\min_{1<i\le N}K_i-K_{i-1}$. For $\mathscr{A}$ take the closure of the algebra generated by products $\prod_{i=1}^N f_i$ with $f_i\in C_c(X)$. $\bar{\mathscr{E}}$ is also ergodic for the product measure μ^N by Proposition 3.4.19.

$(\nu,c,\mathscr{A})$-genericity of $\mathscr{E}$ follows from our induction hypothesis because we showed that it implies mixing of order $N-1$, that is, we get the even stronger statement

$$\int\prod_{i=1}^N f_i(\varphi^{K_in}(x))\,d\mu\xrightarrow[(K_i-K_{i-1})n\to\infty]{}\prod_{i=1}^N\int f_i\,d\mu.$$

Now note that the conclusion of the lemma with the present terminology is exactly $P(N)$. This proves Theorem 9.6.26. □

Proof of Lemma 9.6.30. Assume $f\perp 1$; the lemma follows by adding a constant.

Ergodicity plus the Mean Ergodic Theorem 3.2.4 show that for every $\varphi\in\bar{\mathscr{E}}$ there is an m_φ such that $\int(\frac{1}{m_\varphi}\int_0^{m_\varphi}f(\varphi^u(x))\,du)^2\,d\mu(x)<\epsilon$. By compactness there is a uniform choice: there are finitely many numbers m_j such that for all $\varphi \in \bar{\mathscr{E}}$ there is a j for which $h_\varphi(x) := \frac{1}{m_j}\int_0^{m_j} f(\varphi^u(x))\,du$ satisfies $\int (h_\varphi(x))^2 d\mu(x) < \epsilon$.

Since the $g_\varphi := h_\varphi^2 \in \mathscr{A}$ have common compact support and compact closure, the genericity assumption implies that there is an $M_0 \in \mathbb{N}$ such that if $n - m \geq M_0$ and $\varphi \in \mathscr{E}$ is such that $c(\varphi)n \geq M_0$ then

$$\Big|\frac{1}{n-m}\int_m^n \int g_\varphi(\varphi^u(x))\,d\nu\,du\Big| < \epsilon.$$

At the same time, for sufficiently large $n - m$ we have

$$\Big|\frac{1}{n-m}\int_m^n f(\varphi^u(x)) - h_\varphi(\varphi^u(x))\,du\Big| < \epsilon$$

for all $\varphi \in \mathscr{E}$ and $x \in M$. Combining these last two equations gives

$$\Big\|\frac{1}{n-m}\int_m^n f(\varphi^u(x))\,du\Big\|_{L^2}$$
$$\leq \underbrace{\Big\|\frac{1}{n-m}\int_m^n f(\varphi^u(x)) - h_\varphi(\varphi^u(x))\,du\Big\|_{L^2}}_{<\epsilon} + \underbrace{\sqrt{\frac{1}{n-m}\int_m^n \int g_\varphi(\varphi^u(x))\,d\nu\,du}}_{<\sqrt{\epsilon}}. \qquad \square$$

Since this has been a long proof at the end of a long development through this Section 9.6, let us recapitulate: We just established Theorem 9.6.26 (page 490), which says that if continuous fixed-point free μ-preserving flows G and Ψ satisfy the renormalization relation (9.6.2) for some $h > 1$ and Ψ is ergodic with respect to μ, then Ψ is mixing of all orders, the desired corollary being that the Margulis parametrization of an unstable flow is mixing of all orders. Recall that we worked our way up to this through intermediate results: The Margulis measure is weakly mixing (Proposition 9.6.12) and mixing (Corollary 9.6.23) for the horocycle flow.

These strong stochastic properties are particularly remarkable because at the same time unstable flows have zero entropy (Theorem 9.6.7) and are uniquely ergodic (Theorem 9.6.8).

Topologically as well, there is a contrast with other examples because properties coexist here that otherwise seem to belong to rather distinct categories: these flows are topologically mixing (Theorem 9.6.13) while at the same time being minimal (Proposition 9.6.6).

Figure 9.6.1 serves to show that, of course, while mixing happens in these flows, it happens much more slowly than we have come to expect in hyperbolic flows, instead of being exponential, it is linear. The renormalization relation (9.6.2) is the key for transferring the strong stochastic properties of the Anosov flow to the horocycle/unstable flow, but the tradeoff is that exponential rates are translated to linear ones.

The purposes of this section included applying hyperbolic dynamics to study a closely intertwined family of parabolic flows and to illustrate by the resulting examples that topological and ergodic properties (minimality and unique ergodicity) we otherwise only saw in uncomplicated algebraic examples can coexist with strong topological and ergodic properties (topological mixing and multiple mixing) we otherwise only know from hyperbolic flows, albeit in subexponential incarnations.

CHAPTER 10

Rigidity

Rigidity is a phenomenon such as one observes in connection with functions of a complex variable: Once differentiable, such functions are infinitely differentiable, an instance of smooth rigidity. Moreover, in that case they are no longer deformable, which is geometric rigidity: The values on any set with an accumulation point determine the function completely. Or, the value at the center of a circle is determined by the average over the circle.

In dynamical systems, structural stability of Anosov flows (Theorem 5.3.7, Corollary 5.4.7) is also called local *topological rigidity* because it means that the topological type of an Anosov flow cannot be changed by C^1-perturbation. A classical instance of *global* topological rigidity is that every Anosov diffeomorphism of a torus is topologically conjugate to a linear one [**122**,**210**], [**181**, Theorem 18.6.1]. A continuous-time result of similar flavor says that algebraic Anosov flows are globally topologically rigid:

Theorem 10.0.1 (Ghys–Plante)**.** *If M is a closed manifold that admits a Riemannian metric of constant negative curvature and Φ, Ψ are Anosov geodesic flows for Riemannian metrics on M, then they are topologically orbit-equivalent [**126**].*

*Indeed, every Anosov flow on a circle bundle over a surface is topologically equivalent, up to finite covers, to the geodesic flow of a hyperbolic Riemannian surface, and this holds more generally on 3-manifolds whose fundamental group has nontrivial center [**22**]; see also [**23**, Theorem B].*

*Every Anosov flow on a torus bundle over the circle is topologically equivalent to the suspension of a toral automorphism; indeed that manifold supports only one other Anosov flow, the suspension of the inverse [**239**].*

Remark 10.0.2. The manifolds in this theorem support essentially only one Anosov flow; this is quite different for the manifolds in Theorem 9.3.11.

The Livshitz Theorem 7.2.1 produced a precursor of smooth rigidity in dynamical systems: under the right circumstances a measurable solution can be replaced by a continuous one, and this then turns out to be C^∞.

We explained that the homeomorphism that establishes the orbit-equivalence in structural stability is usually not differentiable, so likewise, nonobvious sufficient conditions for its smoothness are naturally interesting, and such a situation is referred to as *smooth rigidity*. Topological *conjugacy* (rather than orbit-equivalence) provides an instance: a topological *conjugacy* between geodesic flows of Riemannian surfaces is a C^1 diffeomorphism (Theorem 10.2.1) and a C^1 conjugacy between C^k Anosov 3-flows is $C^{k-\epsilon}$ (Theorem 10.2.7), so as *smooth* dynamical systems, and not just topologically, these two flows are indistinguishable. For geodesic flows, this prompts the question whether the underlying manifolds must therefore be isometric, which means that the flows are *geometrically* indistinguishable; this would be an instance of *geometric rigidity*. Indeed, if one of the metrics has constant curvature, then the conclusion implies that for the other geodesic flow the Liouville and topological entropies also coincide, so something close to this is indeed the case: Theorem 10.4.1 below gives constant curvature for both metrics. Moreover, Theorem 10.5.2 actually produces an isometry.

We begin this chapter with a sample of higher-rank rigidity: there are essentially no faithful $\mathbb{R}^2$-actions that include hyperbolic flows, that is, usually no flow commutes with a hyperbolic flow (Corollary 10.1.4, Theorem 10.1.22). If one suitably relaxes hyperbolicity (to the right kind of partial hyperbolicity), then there are faithful $\mathbb{R}^2$-actions, but the algebraic ones are locally smoothly rigid (Theorem 10.1.24).

Next, we encounter *smooth rigidity* of Anosov 3-flows related to conjugacies (Theorem 10.2.1 and Theorem 10.2.7), and to smoothness of the invariant foliations (Theorem 10.3.1). The proof of the latter illustrates the methods used to prove a transverse counterpart (Theorem 10.3.10), which does, moreover, also include *geometric rigidity*: a system is algebraic once an associated parameter (here, the regularity of the invariant subbundles) is extreme, and it is a seminal instance of this because ultimately it led to the definitive higher-dimensional results (Theorem 10.3.14 and Theorem 10.3.15), which we present to complete this circle of ideas.

However, *entropy rigidity* (Theorem 10.4.1) is the first geometric-rigidity result we prove, and we present much of the state of the art of this subject, including Theorem 10.4.9 for contact Anosov 3-flows as well as the astonishing Lyapunov-spectrum rigidity theorem of Butler (Theorem 10.4.11).

We conclude with two simple geometric-rigidity results whose proofs introduce possible tools for establishing such results (Theorem 10.5.2 and Theorem 10.5.1). These are *Godbillon–Vey invariants* (a novel tool we develop fully except that we restrict attention to 3-manifolds) and the Bott–Kanai connection (a central tool for the original proofs of the important results in this area, but one we only describe here but do not develop).

As in the previous chapter we present some material in a manner similar to a survey of the topic. Specifically, we present some results without proof to convey an impression of the pertinent landscape. While we include some higher-dimensional results, it is a matter of conjecture how other rigidity questions play out in higher dimension.

1. Multidimensional time: commuting flows

While the focus of this book is flows, we now touch upon a major subject in modern rigidity theory: the action of groups other than $\mathbb{R}$ (and $\mathbb{Z}$) with some hyperbolicity. In the main we study the question of whether there can be (faithful) $\mathbb{R}^k$-actions (with $k > 1$) by hyperbolic flows, that is, whether a hyperbolic flow can commute with another flow; this can be viewed as having symmetries. We will see that typically this does not happen: Anosov flows never commute with another flow (Corollary 10.1.4; this result appears to be new to the literature), and hyperbolic flows rarely do (Theorem 10.1.22). Indeed, the nucleus of the problem is immediately manifested at periodic points (Lemma 10.1.7).

While these results are definitive, they may be taken as indicating that "complete" hyperbolicity is too strict a notion and should be relaxed to hyperbolicity normal to the orbit foliation (by immersed copies of $\mathbb{R}^k$). We give an indication of the state of the art on such actions—they exist but are rigid in ways analogous to what we see in other contexts later: where $\mathbb{R}$-actions are (locally) topologically rigid, these actions are *smoothly* so (Theorem 10.1.24).

We start with the notion of a centralizer.

Definition 10.1.1. Consider a C^r flow Φ on a closed manifold M with $0 \le r \le \infty$. A C^r diffeomorphism $f\colon M \to M$ *commutes with* Φ if $f\varphi^t = \varphi^t f$ for all $t \in \mathbb{R}$. A C^r flow Ψ commutes with Φ if all ψ^s do, that is, $\psi^s\varphi^t = \varphi^t\psi^s$ for all $s, t \in \mathbb{R}$. The set of such flows Ψ is called the *C^r-centralizer $Z^r(\Phi)$* of Φ. We say that the C^r-centralizer of Φ is trivial if $Z^r(\Phi)$ consists of all constant-time reparameterizations of Φ, that is, $\Psi \in Z^r(\Phi) \Rightarrow \psi^t = \varphi^{ct}$ for some $c \in \mathbb{R}$ and all $t \in \mathbb{R}$.

Note that flows can be viewed as 1-parameter subgroups of the diffeomorphism (or homeomorphism) group of a manifold, and this notion of centralizer is consistent with the group-theoretic notion—except that we restrict here to commuting flows rather than commuting elements of the diffeomorphism group, which is the discrete-time counterpart.

A homeomorphism or diffeomorphism commuting with a flow is a C^r (time-preserving!) conjugacy of the flow with itself (in particular, it maps orbits to orbits), so it also reflects the extent to which a conjugacy with another flow may fail to be unique. This plays a role in classifying flows up to topological or differentiable conjugacy. Note that $Z^{r_1}(\Phi) \subset Z^{r_2}(\Phi)$ when $r_1 \ge r_2$.

The existence of a nontrivial C^r-centralizer for a C^r-flow (or of commuting diffeomorphisms) implies the existence of symmetries in the dynamical system, and for hyperbolic flows this is rare. Indeed, it has been conjectured in a number of ways that typical flows have trivial centralizer, where typical can mean a generic set, open and dense set, or Lebesgue almost every point of a family of flows given by a finite number of parameters—typically one does not expect nontrivial symmetries to be present in the dynamical system. Let us note, however, some obvious instances: The geodesic flow on the usual genus-2 surface has a finite symmetry group that includes reflection and rotation isometries of the double torus (but there are 1-parameter familes of symmetries as in Theorem 2.6.21). The suspension of $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$ has the symmetry coming from $x \mapsto -x$.

Expansivity ensures that centralizers are "transversely discrete":

Proposition 10.1.2. *Suppose Φ is a continuous flow on X and f a homeomorphism that commutes with Φ.*

(1) If Φ is expansive, ϵ an expansivity constant, $d_{C^0}(f, \mathrm{Id}) < \epsilon$, then $f(x) \in \mathcal{O}(x)$ for all $x \in X$.

(2) $f(x) \in \mathcal{O}(x) \Rightarrow \exists \tau = \tau(\mathcal{O}(x)) \in \mathbb{R}\colon f\restriction_{\overline{\mathcal{O}(x)}} = \varphi^\tau \restriction_{\overline{\mathcal{O}(x)}}$.

(3) In the context of (1), $x \mapsto \tau(\mathcal{O}(x))$ *from* (2) *can be chosen continuously on X.*

(4) If Ψ is a continuous flow on X, then $\forall \epsilon > 0\ \exists \delta_0 > 0\colon |\delta| < \delta_0 \Rightarrow d_{C^0}(\psi^\delta, \mathrm{Id}) < \epsilon$.

*(5) If Φ, Ψ are continuous flows with $\{\psi^t \mid t \in \mathbb{R}\} = \{\varphi^t \mid t \in \mathbb{R}\} \neq \{\mathrm{Id}\}$, then $\exists c \in \mathbb{R}\ \forall t \in \mathbb{R} \quad \psi^t = \varphi^{ct}$ [**1**, §2.1].*

Thus, if Φ is a topologically transitive expansive continuous flow on X, ϵ an expansivity constant, f a homeomorphism that commutes with Φ, $d_{C^0}(f, \mathrm{Id}) < \epsilon$, then $f = \varphi^\tau$ for some τ.

Moreover (Proposition 1.6.5), if a homeomorphism f commutes with an expansive flow Φ on a connected space with countably many chain-components, each topologically transitive, and ϵ is an expansivity constant, then $d_{C^0}(f, \mathrm{Id}) < \epsilon \Rightarrow f = \varphi^\tau$ for some τ.

Proof. (1): Contraposition: If $f(x) \notin \mathcal{O}(x)$ for some $x \in X$, then expansivity gives a t with $\epsilon \le d(\varphi^t(f(x)), \varphi^t(x)) = d(f(\varphi^t(x)), \varphi^t(x))$, so $d_{C^0}(f, \mathrm{Id}) \ge \epsilon$.

(2): Writing $f(x) = \varphi^\tau(x)$ gives $f(\varphi^t(x)) = \varphi^t(f(x)) = \varphi^{t+\tau}(x) = \varphi^\tau(\varphi^t(x))$, so $f\restriction_{\mathcal{O}(x)} = \varphi^\tau\restriction_{\mathcal{O}(x)}$, and the claim follows by continuity of f and φ^τ.

(3): At fixed points x this is vacuous because they are isolated points of X by expansivity, same for isolated periodic orbits. Elsewhere, consider a flow-box neighborhood of $\varphi^{[0,2\tau]}(x)$ (and the canonical choice for periodic orbits). □

With Definition 10.1.1 considering time-δ-maps of Ψ implies:

Theorem 10.1.3. *A transitive expansive flow has trivial C^0-centralizer*[1] *as does, more generally, an expansive flow on a connected space with at most countably many chain-components, all of which are topologically transitive.*

Corollary 10.1.4. *Anosov flows have trivial centralizer.*

Theorem 5.4.21 implies

Corollary 10.1.5. *A quasitransverse (Definition 5.4.22) hyperbolic flow without fixed points on a connected space has trivial centralizer.*

From Proposition 10.1.2 we deduce in particular:

Proposition 10.1.6. *Let Φ be a C^r Axiom A flow on a closed manifold M and let $\epsilon > 0$ be an expansive constant for $\Phi\restriction_{NW(\Phi)}$. If $f \in \mathrm{Diff}(M)$ commutes with Φ and $d_0(f, \mathrm{Id}) < \epsilon$, where Id is the identity map on M, then $f(x) \in \mathcal{O}(x)$ for all $x \in NW(\Phi)$.*

Theorem 10.1.21 extends Proposition 10.1.6 beyond the nonwandering set.

Difficulties arise when the nonwandering set is not the entire manifold. As Proposition 10.1.6 suggests, for Axiom-A flows we obtain triviality of the centralizer restricted to the nonwandering set without too much difficulty; however, on the wandering set the dynamics are much harder to control, similarly to Section 5.4. In both cases, this is because if we take a sufficiently small neighborhood of a wandering point one can change the dynamics of the unstable manifold without changing the dynamics in the stable manifold and vice versa. Hence, one can change the dynamics in the past or future without affecting the other. Here and in Section 5.4 this causes difficulties, while in Proposition 7.6.6 it is being put to use.

As suggested by Corollary 10.1.4, the majority of results establishing triviality of the C^r-centralizer are for hyperbolic flows. Our main result is Theorem 10.1.22: Axiom A flows with the *strong transversality condition* (Definition 10.1.17) generically have trivial centralizer.

We begin with a number of preliminary results as we work our way from implications for periodic points, to their invariant manifolds, to attractors and repellers, to their basins, and ultimately to the whole manifold. The first result states that any element of the centralizer maps periodic orbits to periodic points whose derivative is conjugate.

Lemma 10.1.7. *Let Φ be a C^r-flow and suppose $f \in \mathrm{Diff}^r(M)$ commutes with Φ. If $p \in M$ is a fixed point or periodic point for Φ with period T, then so is $f(p)$, and the derivatives of φ^T at p and $f(p)$ are (linearly) conjugate.*

[1] For Anosov flows this was asserted previously to be well-known and elementary [**127**, p. 262].

Proof. If $\varphi^t(p) = p$, then $f(p) = f(\varphi^t(p)) = \varphi^t(f(p))$. If $p \in M$ is a fixed point, then this holds for all $t \in \mathbb{R}$ and so $f(p)$ is a fixed point for Φ. If $p \in M$ is T-periodic, then this holds for $t = T$, so $f(p)$ is T-periodic. Differentiation gives

$$D\varphi^T(f(p))Df(p) = Df(\varphi^T(p))D\varphi^T(p) = Df(p)D\varphi^T(p).$$

Hence, $D\varphi^T(f(p)) = Df(p)D\varphi^T(p)[Df(p)]^{-1}$. □

Specifically, this says that the spectrum of $D\varphi^T(p)$ and $D\varphi^T(f(p))$ are the same; later this will be an important step in establishing triviality of the centralizer.

Proposition 1.3.26 implies that any element in the centralizer of an Axiom A flow maps stable (unstable) manifolds to stable (unstable) manifolds.

Lemma 10.1.8. *Let Φ be a C^r Axiom A flow on a closed manifold M. If $f \in \mathrm{Diff}(M)$ commutes with Φ and $x \in M$, then*

$$f(W^{ss}(x,\Phi)) = W^{ss}(f(x),\Phi) \text{ and } f(W^{uu}(x,\Phi)) = W^{uu}(f(x),\Phi).$$

These lemmas are fundamental tools for working with the centralizer of a flow.

One of the important tools in the study of centralizers is a strengthening of the Hartman–Grobman Theorem 5.6.1 so that the conjugacy is differentiable (beyond Theorem 7.7.1!). In general, it is not possible to have a differentiable conjugacy to a hyperbolic matrix, but we develop sufficient “nonresonance” conditions below for such a linearization to exist.

As we will only be interested in applying the theorem to stable hyperbolic matrices, we will formulate the result in this setting. For more general formulation see [**268**].

Definition 10.1.9. Denote the spectrum of an $n \times n$ matrix A by $\Sigma(A) = \{\lambda_1, \dots, \lambda_n\}$, the eigenvalues of A repeated with multiplicity. A is said to be *stable hyperbolic* if $\mathrm{Re}\lambda < 0$ for all $\lambda \in \Sigma(A)$, in which case we let

$$\rho := \rho(A) := \frac{\max\{|\mathrm{Re}\lambda| : \lambda \in \Sigma(A)\}}{\min\{|\mathrm{Re}\lambda| : \lambda \in \Sigma(A)\}}$$

and say that the function

$$(\lambda, m) \mapsto \gamma(\lambda, m) := \lambda - (m_1\lambda_1 + \cdots + \lambda_n m_n)$$

satisfies the *Sternberg condition of order* $N \geq 2$ if $\mathrm{Re}\gamma(\lambda, m) \neq 0$ for all $\lambda \in \Sigma(A)$ and $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ with $|m| := m_1 + \cdots m_n = N$. The *$Q$-smoothness of A* is $K := \lfloor Q/\rho \rfloor$.

We need the following linearization theorem for controlling the centralizer of a flow. We do not include the proof of this result (see [**268**] for a proof).

Theorem 10.1.10 (Sternberg's Theorem)**.** *Let $Q \geq 2$ and R be C^{2Q} on an open set $U \subset \mathbb{R}^n$ containing the origin. If $D^k R(0) = 0$ for $k = 0, 1$ and A is a stable hyperbolic matrix such that A satisfies the Sternberg condition of order Q, then the flow Φ on $\mathbb{R}^n$ generated by $x' = Ax + R(x)$ admits a C^K-linearization near 0, where K is the Q-smoothness of A*

We remark that a similar result holds for an unstable hyperbolic matrix simply by taking the inverse of the flow.

We say that a stable hyperbolic matrix is *nonresonant* if $\mathrm{Re}\gamma(\lambda, m) \neq 0$ for any m where $|m| \geq 2$ and any $\lambda \in \Sigma(A)$. The next corollary is an immediate consequence of the above theorem and will be the main application of Sternberg's Theorem.

Corollary 10.1.11. *If $f \in C^\infty$ and $x' = f(x) = Ax + R(x)$ where A is a nonresonant stable hyperbolic matrix, then there exists a C^∞-smooth linearization.*

Besides examining sinks and sources for Axiom A flows we will need to examine hyperbolic attractors (repellers). In each case we will want to linearize the stable (or unstable) manifold of a periodic point in the attractor (repeller). We then need a version of Sternberg's Theorem for maps since we will look at a periodic point p with period T and then φ^T maps $W^{ss}(p)$ to $W^{ss}(p)$ and we want a smooth linearization for this map.

There is a corresponding notion of nonresonance in this case, and it is related to the previous nonresonance notion by exponentiating the eigenvalues of the linear part of a generator to obtain the eigenvalues of the linearization of the map; thus the factors attached to eigenvalues in the nonresonance condition for generators now are exponents on eigenvalues of the linearization of a map. Which of these versions is pertinent is usually clear from context.

For a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ we say that the matrix is nonresonant if $\mathrm{Re}\lambda_i \neq \mathrm{Re}\lambda_1^{m_1} \cdots \mathrm{Re}\lambda_n^{m_n}$ where each m_j is a nonnegative integer and $\sum m_j \geq 2$.

Theorem 10.1.12. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ diffeomorphism and the origin is a hyperbolic sink for f with $Df(0)$ nonresonant, then there exists a C^∞-smooth linearization of f.*

We now further demonstrate the usefulness of nonresonance assumptions by establishing that the centralizer of a nonresonant linear system consists of linear maps (Theorems 10.1.13, 10.1.14), which implies that the smooth linearization from Theorem 10.1.12 for the stable manifold of a sink simultaneously smoothly linearizes any smooth map in the centralizer.

Theorem 10.1.13. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a nonresonant stable hyperbolic matrix and Φ_A be the linear flow generated by A. If g is a C^∞ homeomorphism such that $g\varphi_A^t = \varphi_A^t g$ for all $t \in \mathbb{R}$, then g is linear.*

Proof. Let $\overline{\lambda} = \max\{\mathrm{Re}\lambda : \lambda \in \Sigma(A)\}$ and $\underline{\lambda} = \min\{\mathrm{Re}\lambda : \lambda \in \Sigma(A)\}$. So $\underline{\lambda} < \overline{\lambda} < 0$. Fix $N \in \mathbb{N}$ to be the smallest natural number such that $N\overline{\lambda} < \underline{\lambda}$. Let $g = P + R$ where P is a polynomial of degree less than N and R is the remainder term such that $\|R(x)\|/\|x\|^N$ is bounded in a neighborhood of the origin. For any $t \in$ we know that

$$g = \varphi_A^{-t} g \varphi_A^t = \varphi_A^{-t} P \varphi_A^t + \varphi_A^{-t} R \varphi_A^t$$

by the linearity of the flow. Also, since Φ_A is linear, $P = \varphi_A^{-t} P \varphi_A^t$ and $R = \varphi_A^{-t} R \varphi_A^t$.

Since $\|R(x)\| = \|x\|^N h(x)$, where h is bounded as x approaches infinity,

$$\begin{aligned}\|\varphi_A^{-t} R(\varphi_A^t x)\| &\le \|\varphi_A^{-t}\| \cdot \|\varphi_A^t x\|^N h(\varphi_A^t x)\\ &\le \|\varphi_A^{-t}\| \cdot \|\varphi_A^t\|^N \|x\|^N h(\varphi_A^t x).\end{aligned}$$

Since $N\overline{\lambda} - \underline{\lambda} < 0$ we have

$$\lim_{t\to\infty} \|\Phi_A^{-t}\| \|\Phi_A^t\|^N = \lim_{t\to\infty} e^{(N\overline{\lambda}-\underline{\lambda})t} = 0.$$

Since Φ_A is an stable linear flow and h approaches zero as $x \to 0$ we know that $h(\Phi_A^t x) \to 0$ as $t \to \infty$. Hence, $R(x) = 0$ and g is a polynomial of degree less than N.

Now we use the nonresonant conditions on the eigenvalues of A to show that the P is in fact linear. First of all, the nonresonant condition implies that there are n distinct eigenvalues for A and so the real Jordan form of A is block diagonal. Also, the nonresonant condition implies that if $\alpha_i = \mathrm{Re}\lambda_i$ for each i, then

$$e^{\alpha_i} \neq e^{\alpha_1 m_1} \cdots e^{\alpha_n m_n}$$

where each m_j is a nonnegative integer and $\sum_{m_j} \geq 2$. Then A can be written in the form

$$A(x_1, x_1', \dots, x_k, x_k', x_{k+1}, \dots, x_\ell) = (\Lambda_1(x_1, x_1'), \dots, \Lambda_k(x_k, x_k'), \lambda_{k+1} x_{k+1}, \lambda_\ell, x_\ell)$$

where $\Lambda_j = \begin{bmatrix} \alpha_j & -\mu_j \\ \mu_j & \alpha_j \end{bmatrix}$ for each eigenvalue $\lambda_j = \alpha_j + \mu_j$ of A where $\mu_j > 0$. Then the linear flow Φ_A is represented by

$$e^{At}(x_1, x_1', ..., x_k, x_k', x_{k+1}, ..., x_\ell) = (e^{\Lambda_1 t}(x_1, x_1'), ..., e^{\Lambda_k t}(x_k, x_k'), e^{\lambda_{k+1} t} x_{k+1}, ..., e^{\lambda_\ell t} x_\ell)$$

where

$$e^{\Lambda_j t} = \begin{bmatrix} e^{\alpha_j t} \cos \mu_j t & -e^{\alpha_j t} \sin \mu_j t \\ e^{\alpha_j t} \sin \mu_j t & e^{\alpha_j t} \cos \mu_j t \end{bmatrix}.$$

Now a straightforward, but somewhat long computation, shows that if $e^{-At} P e^{At} = P$ for all $t \in \mathbb{R}$ and the eigenvalues are nonresonant, then P has to be linear. □

A similar proof as above shows the following result for maps.

Theorem 10.1.14. *Let $A:\mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that the origin is a stable hyperbolic fixed point for the map and assume that the eigenvalues of A are nonresonant. If g is a C^∞ homeomorphism such that $gA = Ag$, then g is linear.*

The above results yield the following:

Proposition 10.1.15. *Let $A:\mathbb{R}^n \to \mathbb{R}^n$ be a nonresonant stable hyperbolic matrix and Φ_A be the linear flow generated by A.*

(1) If

$$A(x_1, x_1', \dots, x_k, x_k', x_{k+1}, \dots, x_\ell) = (\Lambda_1(x_1, x_1'), \dots, \Lambda_k(x_k, x_k'), \lambda_{k+1}x_{k+1}, \lambda_\ell x_\ell)$$

and $\lambda_\ell > \alpha_i$ for all $1 \le i < \ell - 1$ and $\alpha_i = \mathrm{Re}\lambda_i$, and g is a C^∞ homeomorphism of $\mathbb{R}^n$ such that $ge^{At} = e^{At}g$ for all $t \in \mathbb{R}$ and g preserves the orbits of the flow e^{At}, then $c \neq 0$ and Σ_c the hyperplane $x_\ell = c$ there is an induced map $\bar{g}:\Sigma_c \to \Sigma_c$ defined by $\bar{g}(x) = g(\mathcal{O}(x) \cap \Sigma_c$ and $\bar{g}$ is of the form

$$\bar{g}(x_1, x_1', \dots, x_k, x_k', x_{k+1}, \dots, x_{\ell-1}) = (B_1(x_1, x_1'), \dots, B_k(x_k, x_k'), a_{k+1}x_{k+1}, \dots, a_{\ell-1}, x_{\ell-1}),$$

where

$$B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix} \text{ for all } 1 \le j \le k.$$

(2) If

$$A(x_1, \dots, x_{k-1}, x_k, x_k', \dots, x_\ell, x_\ell') = (\lambda_1 x_1, \dots, \lambda_{k-1}x_{k-1}, \Lambda_k(x_k, x_k'), \dots, \Lambda_\ell(x_\ell, x_\ell'))$$

and $\alpha_\ell > \alpha_i$ for all $1 \le i < \ell - 1$ and $\alpha_i = \mathrm{Re}\lambda_i$, and g is a C^∞ homeomorphism of $\mathbb{R}^n$ such that $ge^{At} = e^{At}g$ for all $t \in \mathbb{R}$ and g preserves the orbits of the flow e^{At}, then for $r > 0$ and $C(r)$ the cylinder $x_\ell = r\cos\theta, x_\ell' = r\sin\theta$ there is an induced map $\bar{g}: C(r) \to C(r)$ defined by $\bar{g}(x,\theta) = g(\mathcal{O}(x,\theta) \cap C(r)$ and $\bar{g}$ is of the form

$$\bar{g}(x_1, x_{k-1}, x_k, x_k', \dots, x_{\ell-1}, x_{\ell-1}', \theta) = (a_1 k_1, \dots, a_{k-1}x_{k-1}, B_k(x_k, x_k'), \dots, B_{\ell-1}(x_{\ell-1}, x_{\ell-1}'), \theta + \theta_0),$$

where

$$B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix} \text{ for all } k \le j \le \ell - 1.$$

Proof. Since g commutes with A, Theorem 10.1.13 implies that g is linear. Furthermore, since the matrix A is hyperbolic we can modify the proof of Proposition 10.1.2 to show that g takes orbits of the linear flow e^{At} to orbits.

Let $\mathbb{R}^n = \bigoplus_{j=1}^{\ell} E_j$ where E_j is 1 or 2-dimensional depending on if λ_j has zero imaginary part or nonzero imaginary part, respectively. Then $g_j(x) = a_j x$ where $a_j \in \mathbb{R}$ if $\lambda_j \in \mathbb{R}$, or $g_j(x) = B_j x$ where

$$B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix} \text{ if } \lambda_j \notin \mathbb{R}.$$

Now assume that $\lambda_\ell \in \mathbb{R}$. Fix $c \in \mathbb{R}$ where $c \neq 0$ and let $\bar{g}$ be the induced map on Σ_c. Then the trajectory of e^{At} through the point

$$(\bar{x}, c) = (x_1, x_1', \ldots, x_k, x_k', x_{k+1}, \ldots, x_{\ell-1}, c)$$

takes the form $x_\ell(t, \bar{x}) = ce^{\lambda_\ell t}$ and if $1 \le j \le \ell - 1$, then $x_j(t, \bar{x}) = e^{\lambda_j t} x_j$ if $\lambda_j \in \mathbb{R}$ and

$$x_j(t, \bar{x}) = e^{\alpha_j t} B(\beta_j t)(x_j, x_j')$$

when $\lambda_j = \alpha_j + i\beta_j$ where $\beta_j \neq 0$, and

$$B(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Now if $x_\ell = ce^{\lambda_\ell t}$, then $t = \dfrac{\ln x_\ell - \ln c}{\lambda_\ell}$.

If $\lambda_j \in \mathbb{R}$ let $\bar{\lambda}_j = \lambda_j / \lambda_\ell$. If $\lambda_j = \alpha_j + i\beta_j$ is not real, then let $\bar{\alpha}_j = \alpha_j / \lambda_\ell$ and $\bar{\beta}_j = \beta_j / \lambda_\ell$. Then we can define $\bar{x}_j(x_\ell, \bar{x})$ by

$$\bar{x}_j(x_\ell, \bar{x}) = \begin{cases} \left(\frac{x_\ell}{c}\right)^{\bar{\lambda}_j} x_j & \text{when } \lambda_j \in \mathbb{R}, \\ \left(\frac{x_\ell}{c}\right)^{\bar{\alpha}_j} B\left(\bar{\beta}_j \ln(x_\ell / c)\right)(x_j, x_j') & \text{when } \lambda_j \text{ is not real.} \end{cases}$$

Let

$$\tau(x_\ell, \bar{x}) = (x_1(x_\ell, \bar{x}), \ldots, x_{\ell-1}(x_\ell, \bar{x}), x_\ell).$$

Then we have $g(\tau(x_\ell, \bar{x})) = \tau(a_\ell x_\ell, \bar{g}(\bar{x}))$. This implies that

$$g_j(\tau(x_\ell, \bar{x})) = a_j(x_j(x_\ell, \bar{x})) = a_\ell \left(\frac{x_\ell}{c}\right)^{\bar{\lambda}_j} x_j = x_j(a_\ell x_\ell, \bar{g}(\bar{x})) = \left(\frac{a_\ell x_\ell}{c}\right)^{\bar{\lambda}_j} \bar{g}_j(\bar{x})$$

for $\lambda_j \in \mathbb{R}$. Then $\bar{g}_j(\bar{x}) = a_\ell^{1-\bar{\lambda}_j} x_j$. Similarly, we have

$$\begin{aligned} g_j(\tau(x_\ell, \bar{x})) &= B_j(x_j(x_\ell, \bar{x})) \\ &= B_j\left(\left(\frac{x_\ell}{c}\right)^{\bar{\alpha}_j} B\left(\bar{\beta}_j \ln(x_\ell / c)\right)(x_j, x_j')\right) \\ &= x_j(a_\ell x_\ell, \bar{g}(\bar{x})) \\ &= \left(\frac{a_\ell x_\ell}{c}\right)^{\bar{\alpha}_j} B(\bar{\beta}_j \ln(a_\ell x_\ell / c)) \bar{g}_j(\bar{x}) \end{aligned}$$

for $\lambda_j \notin \mathbb{R}$. Solving for $\bar{g}_j(\bar{x})$ we see that $\bar{g}_j$ is linear and of the desired form.

Now we assume that $\lambda_\ell \notin \mathbb{R}$. Then the orbit of e^{At} for the point

$$(\bar{x}, \theta) = (x_1, x_{k-1}, x_k, x_k', \ldots, x_{\ell-1}, x_{\ell-1}', \theta)$$

takes the form $x_\ell(t,(\bar{x},\theta)) = e^{\alpha_\ell t}B(\beta_j t)(r\cos\theta, r\sin\theta)$ and if $1 \le j \le \ell - 1$ we have $x_j(t,(\bar{x},\theta)) = e^{\lambda_j t}x_j$ if $\lambda_j \in \mathbb{R}$ and

$$x_j(t,(\bar{x},\theta)) = e^{\alpha_j t}B(\beta_j t)(x_j, x_j')$$

when $\lambda_j = \alpha_j + i\beta_j$ where $\beta_j \neq 0$, and

$$B(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Now as above we see that $t = \frac{\ln\|(x_\ell, x_\ell')\|}{\alpha_\ell}$. Let $s = \|(x_\ell, x_\ell')\|$. Then we have $x_\ell(s,(\bar{x},\theta)) = sB(\bar{\beta}_j \ln s)(r\cos\theta, r\sin\theta)$ where $\bar{\beta}_j = \beta_\ell/\alpha_\ell$. Furthermore, we have $x_j(s,(\bar{x},\theta)) = s^{\bar{\alpha}_j}x_j$ if $\lambda_j \in \mathbb{R}$ and $\bar{\alpha}_j = \lambda_j/\alpha_\ell$, and $x_j(s,(\bar{x},\theta)) = s^{\bar{\alpha}_j}B(\bar{\beta}_j \ln s)(x_j, x_j')$ if $\lambda_j \notin \mathbb{R}$ where $\bar{\alpha}_j = \alpha_j/\alpha_\ell$ and $\bar{\beta}_j = \beta_j/\alpha_\ell$. Arguing similarly to the previous case we can show that $\bar{g}$ is linear and of the desired form. □

We now prove two useful facts that extend local information about the centralizer to more global information. The first is that commuting diffeomorphisms that agree on an open subset of the basin of a hyperbolic attractor agree on the basin; the next shows that commuting maps that agree on an open set coincide on the whole space.

Theorem 10.1.16. *Let Φ be a C^∞ flow on a manifold M and $\Lambda \subset M$ be a transitive hyperbolic attractor containing a fixed or periodic point p that is nonresonant. If $f_1, f_2 \in \mathrm{Diff}^\infty(M)$ such that f_1 and f_2 commute with Φ, and there exists an open set $V \subset W^s(\Lambda)$ such that $f_1\restriction_V = f_2\restriction_V$, then $f_1\restriction_{W^s(\Lambda)} = f_2\restriction_{W^s(\Lambda)}$.*

Proof. We first suppose that p is a nonresonant stable hyperbolic fixed point for Φ. Then there exists a neighborhood U of p and $\Phi\restriction_U$ is linear with the appropriate smooth coordinate system by Corollary 10.1.11. Let $f = f_1 \circ f_2^{-1}$. Then f is a C^∞ homeomorphism of M and $f\varphi^t = \varphi^t f$ for all $t \in \mathbb{R}$. Hence, $f\restriction_U$ is linear using the same smooth coordinate system by Theorem 10.1.13.

There exists some $t > 0$ such that $\varphi^t(V) \cap U$ is an open set. By hypothesis, $f\restriction_{\varphi^t(V)}$ is the identity. So using the local coordinate system we have f a linear diffeomorphism on U that is the identity on a nonempty open subset of U. Hence, f is the identity on U. Now for any $y \in W^{ss}(p)$ there exists some t such that $\varphi^t(y) \in U$. Then $f(y) = (\varphi^{-t}f\varphi^t)(y) = y$ and $f\restriction_{W^{ss}(p)}$ is the identity. Hence, $f_1 = f_2$ on $W^{ss}(p)$.

More generally, we let p be a nonresonant hyperbolic periodic point contained in Λ. Since $W^{ss}(\mathscr{O}(p))$ is dense in $W^s(\Lambda)$ by the In-Phase Theorem (Theorem 5.3.25) and the second proof of the Spectral Decomposition Theorem, there exists some

$T_0 > 0$ such that $W^{ss}(\varphi^{T_0}(p)) \cap V$ contains an open set in $W^{ss}(\varphi^{T_0}(p)$. Let T be the period of p. As above we define $f = f_1 \circ f_2^{-1}$. Then f is a C^∞ homeomorphism of M and $f\varphi^t = \varphi^t f$ for all $t \in \mathbb{R}$.

By Theorem 10.1.12 there exists a neighborhood U of $\varphi^{T_0}(p)$ in $W^{ss}(\varphi^{T_0}(p))$ and a smooth coordinate system such that φ^T is linear on U. Then there is some $n \in \mathbb{N}$ such that $\varphi^{nT}(V)$ contains an open set in U. Then, as above, f is the identity in U. Hence, f is the identity on $W^{ss}(\varphi^{T_0})(p)$.

Now for $y \in W^{ss}(\mathscr{O}(p))$ there exists some t such that $\varphi^t(y) \in W^{ss}(\varphi^{T_0}(p))$. Then $f(y) = (\varphi^{-t} f \varphi^t)(y) = y$. Since $W^{ss}(\mathscr{O}(p))$ is dense in $W^s(\Lambda)$ and f is the identity on $W^{ss}(\mathscr{O}(p))$ we see that f is the identity on $W^s(\Lambda)$ so $f_1 = f_2$ on $W^s(\Lambda)$. □

Definition 10.1.17 (Strong transversality)**.** For a closed manifold M let $\mathscr{A}_S^r(M)$ be the set of C^r Axiom A flows on M with the *strong transversality* condition: $W^{ss}(x)$ and $W^{uu}(x)$ are transverse for all $x \in M$.

Remark 10.1.18. It is not hard to see that strong transversality implies the no-cycles condition for an Axiom A flow.

We need the next lemma concerning Axiom A flows with the strong transversality condition to link basins of attractors and repellers in a manner that allows us to obtain global results from results in the basins of the attractors and repellers.

Lemma 10.1.19. *Let Λ, Λ' be two transitive hyperbolic attractors for $\Phi \in \mathscr{A}_S^1(M)$ such that $\overline{W^s(\Lambda)} \cap \overline{W^s(\Lambda')} \neq \varnothing$. Then there exists some repeller Λ'' such that $W^s(\Lambda) \cap W^u(\Lambda'') \neq \varnothing \neq W^s(\Lambda') \cap W^u(\Lambda'')$.*

Proof. Let $x \in \overline{W^s(\Lambda)} \cap \overline{W^s(\Lambda')}$. Then $x \in W^s(\Lambda_1)$ for some basic set Λ_1 for Φ. Then there exists some sequence $x_n \to x$ as $n \to \infty$ where each $x_n \in W^s(\Lambda)$. Then $\varphi^t(x_n) \to \varphi^t(x)$ for each t as $n \to \infty$. Then for sufficiently large $t > 0$ we see that $\varphi^t(x)$ is closer to Λ_1 then the constants from the local products structure. So $W_\epsilon^{ss}(\varphi^t(x_n))$ intersects the center-unstable manifold of a point in Λ_1 for t and n sufficiently large where ϵ is the constant from the local product structure. Since $W^s(\Lambda)$ is invariant this says that $\overline{W^s(\Lambda)} \cap W^u(\Lambda_1) \neq \varnothing$.

Now since satisfies Axiom A there exists some basic set Λ_2 such that $W^s(\Lambda_2) \cap W^u(\Lambda_1) \neq \varnothing$ and modifying the above argument we have $W^s(\Lambda_2) \cap \overline{W^s(\Lambda)} \neq \varnothing$. Continuing this process there exists some Λ_j where the process stops since Φ has the no-cycles property and if Λ_j were not an attractor we could continue the process.

Then Λ_j is an attractor where $W^s(\Lambda_j) \cap \overline{W^s(\Lambda)} \neq \varnothing$, but this implies that $W^s(\Lambda_j) \cap W^s(\Lambda) \neq \varnothing$ since the basin of an attractor is open. This implies that $\Lambda = \Lambda_j$. Since $W^s(\Lambda_i) \cap W^u(\Lambda_{i-1}) \neq \varnothing$ for $1 \leq i \leq j$ the Inclination Lemma implies that $W^s(\Lambda) \cap W^u(\Lambda_1) \neq \varnothing$ and $W^s(\Lambda_1) \subset \overline{W^s(\Lambda)}$. Similarly, we have $W^s(\Lambda_1) \subset \overline{W^s(\Lambda')}$.

Then by the strong transversality condition, $W^s(\Lambda_1)$ intersects $W^u(\Lambda'')$ for some repeller Λ''. □

Theorem 10.1.20. *There exists an open and dense set* $\tilde{\mathscr{A}}^\infty_S(M) \subset \mathscr{A}^\infty(M)$ *such that if* $\Phi \in \tilde{\mathscr{A}}^\infty_S(M)$, $f_1, f_2 \in \mathrm{Diff}^\infty(M)$ *commute with* Φ *and* $f_1\restriction_U = f_2\restriction_U$ *on some open set* $U \subset M$, *then* $f_1 = f_2$.

Proof. For any $T > 0$ and any Axiom A C^∞ flow there are only a finite number of periodic orbits with period less than T. Then there is an open dense set of $\mathscr{A}^\infty(M)$ denoted $\mathscr{A}^\infty_T(M)$ such that for each $\Phi \in \mathscr{A}^\infty_T(M)$ and each periodic point p for Φ with period $t < T$ we have $D\varphi^t(p)\restriction_{E^i(p)}$ is nonresonant for $i = u, s$ and if q is another periodic point of period T for Φ, then $D\varphi^t(p)$ and $D\varphi^t(q)$ are conjugate if and only if $q \in \mathscr{O}(p)$.

From above we know that if

(1) $\Phi \in \mathscr{A}^\infty_T(M)$,
(2) $g \in \mathrm{Diff}^\infty(M)$ commutes with Φ, and
(3) $p \in M$ is periodic for Φ with period less then T,

then $g(p) \in \mathscr{O}(p)$.

For $\Phi \in \mathscr{A}^\infty(M)$ there exists some $T_0 > 0$ such that each attractor or repeller contains a periodic point of period less than T_0. By structural stability of Axiom A flows there exists an open set $\mathscr{U}$ in $\mathscr{A}^\infty(M)$ such that for each $\Psi \in \mathscr{U}$ and each attractor or repeller for Φ contains a periodic point of period less than T_0. Now let $\mathscr{V}$ be a connected component of $\mathscr{U} \cap \mathscr{A}^\infty_{T_0}(M)$. We will establish the conclusion of the theorem on $\mathscr{V}$.

Let $\Phi \in \mathscr{V}$ and $f_1, f_2 \in \mathrm{Diff}^\infty(M)$ commute with Φ such that there is an open set $U \subset M$ where $f_1\restriction_U = f_2\restriction_U$. Then since there is an open and dense set of points in M that are contained in an the basin of an attractor for Φ, there exists an attractor Λ_1 such that $U \cap W^s(\Lambda)$ is an open set. By Theorem 10.1.16, $f_1\restriction_{W^s(\Lambda_1)} = f_2\restriction_{W^s(\Lambda_1)}$.

Now let Λ_2 be a repeller such that $W^u(\Lambda_2) \cap W^s(\Lambda_1)$ contains an open set. Then $f_1\restriction_{W^u(\Lambda_2)} = f_2\restriction_{W^u(\Lambda_2)}$. Since there are only a finite number of attractors and repellers we can continue this process and from Lemma 10.1.19 we see that f_1 and f_2 agree on an open and dense set in M. Then from continuity we see that $f_1 = f_2$. □

We can now prove the first major result of this section, an Axiom-A counterpart to Proposition 10.1.2 and the needed extension of Proposition 10.1.6 beyond the nonwandering set.

Theorem 10.1.21. *There is an open and dense subset* $\tilde{\mathscr{A}}^\infty_S(M)$ *of* $\mathscr{A}^\infty_S(M)$ *such that for each* $\Phi \in \tilde{\mathscr{A}}^\infty_S(M)$ *there exists a* C^0*-neighborhood* $\mathscr{V}$ *of* Id *in* $\mathrm{Diff}^\infty(M)$ *such that* *if $g \in \mathcal{V}$ commutes with Φ, then $g(x) \in \mathcal{O}(x)$ for each $x \in M$, where $\mathcal{O}(x)$ is the Φ-orbit of x.*

Proof. By Proposition 10.1.6, given $\Phi \in \mathscr{A}_S(M)$ there exists some $\epsilon > 0$ such that if $g \in \mathrm{Diff}^\infty(M)$ that commutes with Φ, then there exists some $\epsilon > 0$ such that if $d_0(g, \mathrm{Id}) < \epsilon$, then $g(x) \in \mathcal{O}(x)$ for all $x \in \mathrm{NW}(\Phi)$. We need to use Sternberg's Linearization Theorem together with properties of attractors and repellers for Axiom A flows to finish the proof by extending this result to all points in the manifold.

For any $T > 0$ and any Axiom A C^∞ flow there are only a finite number of periodic orbits with period less than T. Then there is an open dense set of $\mathscr{A}_S^\infty(M)$ denoted $(\mathscr{A}_T)_S^\infty(M)$ such that for each $\Phi \in (\mathscr{A}_T)_S^\infty(M)$ and each periodic point p for Φ with period $t < T$ we have $D\varphi^t(p)\restriction_{E^i(p)}$ is nonresonant for $i = u, s$ and if q is another periodic point of period T for Φ, then $D\varphi^t(p)$ and $D\varphi^t(q)$ are conjugate if and only if $q \in \mathcal{O}(p)$.

From above we know that if

(1) $\Phi \in (\mathscr{A}_T)_S^\infty(M)$,
(2) $g \in \mathrm{Diff}^\infty(M)$ commutes with Φ, and
(3) $p \in M$ is periodic for Φ with period less then T,

then $g(p) \in \mathcal{O}(p)$.

For $\Phi \in \mathscr{A}_S^\infty(M)$ there exists some $T_0 > 0$ such that each attractor or repeller contains a periodic point of period less than T_0. By structural stability of Axiom A flows there exists an open set $\mathscr{U}$ in $\mathscr{A}_S^\infty(M)$ such that for each $\Psi \in \mathscr{U}$ each attractor or repeller contains a periodic point of period less than T_0. Now let $\mathcal{V}$ be a connected component of $\mathscr{U} \cap (\mathscr{A}_{T_0})_S^\infty(M)$. We will establish the conclusion of the theorem on $\mathcal{V}$.

We now divide the proof into two cases. First, if a transitive attracting or repelling set contains a periodic orbit. Second, if the attracting or repelling set is a fixed point.

Let $\Phi \in \mathcal{V}$ and Λ be a transitive attractor for Λ such that Λ contains a periodic point. We can choose the neighborhood of the identity map such that if p is a periodic point of Λ with period T and g is C^0-close to the identity with g commuting with Φ that $g(p) = \varphi^t(p)$ for $0 \le t < T/2$. Then $\varphi^{-t}g$ maps $W^{ss}(p)$ to $W^{ss}(p)$ and commutes with Φ. To conclude the proof in this case it is sufficient to show that $\varphi^{-t}g$ is the identity in a neighborhood of p in $W^{ss}(p)$. To see this notice that this implies that g is the identity on the orbits of Φ in a neighborhood of p in $W^{ss}(p)$. Since g commutes with Φ we see that this implies that $\varphi^{-t}g$ is the identity on the space of orbits for all points in $W^{ss}(\mathcal{O}(p))$. Since $W^{ss}(\mathcal{O}(p))$ is dense in $W^s(\Lambda)$ this implies that $\varphi^{-t}g$ is the identity on the orbits of Φ for $W^s(\Lambda)$.

We now show that there is a neighborhood of p in $W^{ss}(p)$ such that $\varphi^{-t}g$ is the identity. Since p is nonresonant, $\varphi^T\restriction_{W^{ss}(p)}$ satisfies the Sternberg conditions and so there is a neighborhood U of p in $W^{ss}(p)$ and a C^∞ linearizing coordinate system ξ. Then $\varphi^{-t}g$ commutes with φ^T and so is simultaneously linearized by ξ. Furthermore, since p is nonresonant we may assume that the linear system given by the matrix A has a block diagonal Jordan form with distinct eigenvalues and since $\varphi^{-t}g$ and φ^T commute, the matrix B representing the linear action for $\varphi^{-t}g$ is simultaneously block-diagonalizable with the same form.

If A has the form

$$A(x_1,\dots,x_{k-1},x_k,x'_k,\dots,x_\ell,x'_\ell)=(\lambda_1x_1,\dots,\lambda_{k-1}x_{k-1},\Lambda_k(x_k,x'_k),\dots,\Lambda_\ell(x_\ell,x'_\ell)),$$

then B has the form

$$B(x_1,\dots,x_{k-1},x_k,x'_k,\dots,x_\ell,x'_\ell)=(\mu_1x_1,\dots,\mu_{k-1}x_{k-1},M_k(x_k,x'_k),\dots,M_\ell(x_\ell,x'_\ell))$$

where each Λ_j and M_j has the form

$$\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix} \text{ for all } 1\le j\le k.$$

Since g is chosen close to the identity we may assume that each of the entries for B are much smaller in modulus than the entries for A. Now we take each coordinate at a time and make an arbitrarily small perturbation to the vector field generating Φ so that in each coordinate B must be the identity. From Proposition 10.1.15 we know that there is an $(\ell-1)$-dimensional Lie group of matrices that commute with A. Fix a set of generators $\{B_1,\dots,B_{\ell-1}\}$ of the Lie group. We choose $x_1,\dots,x_{\ell-1}$ points in U and sufficiently small neighborhoods $U_1,\dots,U_{\ell-1}$ such that $x_i\in U_i$ for each $i\in\{1,\dots,\ell-1\}$. Now we modify the vector field in each U_i so that none of the matrices in the 1-dimensional subspace spanned by B_i can be the linearization of a flow that commutes with Φ.

Let h be the smooth linearization in U. It is important that h only be given by the flow Φ. Furthermore, if we make a small modification to the vector field in U_i this does not modify the matrix A, but does make a small change to the conjugacy h. Now we work in the linearized flow. Let $s\neq 0$ such that $e^{B_is}h(x_i)\in\partial h(U_i)$. Fix $t\neq 0$ such that both $e^{At}h(x_i)\notin h(U_i)$ and $e^{At}e^{B_is}h(x_i)\notin h(U_i)$, see Figure 10.1.1 . Now modify the generating vector field V in a small region about x_1 such that

$$h^{-1}(e^{-B_is}e^{-At}e^{B_is}e^{At}h(x_1))\neq x_1.$$

As stated above the important component of this is that h is not defined by B_i, but only by Φ and A.

Furthermore, the perturbations above give an open condition. Namely, that if Ψ is a flow that commutes with Φ then restricted to a neighborhood of p is the

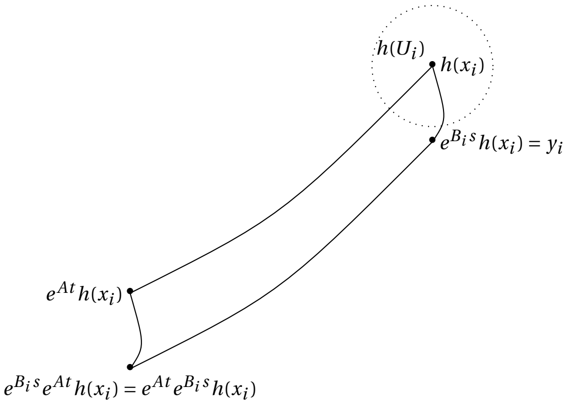


FIGURE 10.1.1. Commuting linear flows

strong-stable manifold we know that the linearization of Ψ is given by a multiple of A. Hence, orbits or Φ are preserved by Ψ.

We now let p be a nonresonant stable hyperbolic fixed point for Φ and use Proposition 10.1.15 and the induced linear action on either the hyperplanes or cylinders. Then we can perturb the vector field in a small neighborhood similar to the method described above such that the induced action must be the identity.

Hence, there is an open and dense set of flows in $\mathcal{V}$ such that any C^∞ diffeomorphism in this set that is C^0-close to the identity and commutes with the flow must map points into their own orbits. □

Theorem 10.1.22. *Axiom A flows with the* strong transversality condition *generically have trivial centralizer: there is an open and dense subset $\tilde{\mathscr{A}}^\infty_S(M)$ of the set $\mathscr{A}^\infty_S(M)$ of C^∞ Axiom A flows on M with the* strong transversality condition *(Definition 10.1.17) such that each $\Phi \in \tilde{\mathscr{A}}^\infty_S(M)$ has trivial centralizer.*

PROOF. Let Φ be in the open and dense set of $\mathscr{A}^\infty_S(M)$ from Theorem 10.1.21. Let $\Psi \in Z^\infty(\Phi)$. Then for t sufficiently small, ψ^t maps orbits of Φ to orbits of Φ by Theorem 10.1.21. Then $\psi^t(\varphi^s(x)) = \varphi^{\alpha(s,t,x)}(x)$ for all $s \in \mathbb{R}$. Let Y be the vector field generating Ψ and X be the vector field generating Φ. Then there exists a function $\rho : M \smallsetminus \mathrm{Fix}(\Phi) \to \mathbb{R}$ such that $Y = \rho X$.

Now let $x \in M \smallsetminus \mathrm{Fix}(\Phi)$ and $y, y' \in W^{ss}(x)$. Since $\psi^t(W^{ss}(x)) = W^{ss}(\psi^t(x))$ we see that ρ is constant on $W^{ss}(x)$. This also holds for $W^{uu}(x)$ for any $x \in M \smallsetminus \mathrm{Fix}(\Phi)$. Now we see that for any point $x \in M \smallsetminus \mathrm{Fix}(\Phi)$ that since $XY - YX = 0$ that $X\rho = 0$.

Therefore, ρ is constant on $W^{ss}(\mathscr{O}(x))$. For an transitive attractor Λ that is not a fixed point this implies that ρ is constant on $W^s(\Lambda)$ since $W^{ss}(\mathscr{O}(x))$ is dense in $W^s(\Lambda)$ for any periodic point x.

For any stable fixed point or unstable fixed point the fixed point satisfies Sternberg's condition so from the linearization of the fixed point we see that since ψ^t is simultaneously linearized for all sufficiently small t and then we see that ρ is constant on the stable or unstable set for the fixed point.

Now, ρ is constant on the basin of any attractor or repeller. Since this forms an open and dense set and Lemma 10.1.19 shows that the basins intersect, ρ is continuous on $M \smallsetminus \mathrm{Fix}(\Phi)$, so $Y = cX$ for some constant $c \in \mathbb{R}$. □

We now explain what rigidity results when the hyperbolicity assumption is adapted to allow for nontrivial $\mathbb{R}^k$-actions.

The basic examples of Anosov flows are algebraic $\mathbb{R}$-actions: geodesic flows of locally symmetric spaces (such as surfaces of constant negative curvature) and suspensions of algebraic Anosov diffeomorphisms. Structural stability means that these actions are locally topologically rigid and in a strong sense: the orbit-equivalence can be chosen near the identity and is transversely unique among such orbit-equivalences. For $\mathbb{R}^k$-actions, we instead get local *smooth* rigidity which furthermore is time-preserving up to an algebraic time-change. This is dramatically different than for flows. We begin with the definition of such actions.

Definition 10.1.23. Suppose $\mathbb{R}^k \subset H$ is a subgroup of a connected Lie group H and Λ a lattice in H. If $\mathbb{R}^k$ acts on a compact quotient H/Λ by left translations and C is a compact subgroup of H which commutes with $\mathbb{R}^k$, then the $\mathbb{R}^k$-action on H/Λ descends to an action on $C\backslash H/\Lambda$. An *algebraic* $\mathbb{R}^k$*-action* ρ is a finite factor of such an action. Let $\mathfrak{c}$ be the Lie algebra of C. The *linear part* of ρ is the representation of $\mathbb{R}^k$ on $\mathfrak{c}\backslash\mathfrak{h}$ induced by the adjoint representation of $\mathbb{R}^k$ on the Lie algebra $\mathfrak{h}$ of H.

For our purposes, the *standard* $\mathbb{R}^k$*-actions* are actions by infranilmanifold-automorphisms, Weyl chamber flows (Definition 2.5.7), twisted Weyl chamber flows and some further extensions (as explicitly described in [**182**, Section 2.2]).

An action of a Lie group G on a compact manifold is *Anosov* if some element $g \in G$ acts normally hyperbolically with respect to the orbit foliation, i.e., it is partially hyperbolic with the G-orbit direction as the center bundle (Definition 5.5.2).

Theorem 10.1.24 ([**183**, Corollary 5])**.** *The standard algebraic Anosov actions of* $\mathbb{R}^k$ *for* $k > 1$ *with semisimple linear part are locally* C^∞*-rigid. Moreover,*

- *the* C^∞*-conjugacy* φ *between the action composed with an automorphism* ρ *and a perturbation can be chosen* C^1*-close to the identity;*
- *the automorphism* ρ *is unique and also close to the identity;*

- *φ is unique among conjugacies close to the identity up to translations in the acting group.*

Remark 10.1.25 ([**25**])**.** We note that there has been work on Anosov actions in different directions. If the Anosov element is in the nilradical of (the not necessarily abelian or nilpotent acting Lie group) G, then its stable/unstable foliations are G-invariant, and if G is nilpotent, then there is a Spectral-Decomposition Theorem analogous to Theorem 5.3.35. If the Anosov element has 1-dimensional unstable subbundle and G acts on a manifold of dimension at least $\dim(G)+3$, then the action is topologically transitive, analogously to Theorem 9.4.12.

In another digression beyond the abelian context, we note a classification.

Definition 10.1.26 (Nilpotent algebraic action)**.** Suppose G is a Lie group, $\Gamma < G$ a torsion-free uniform lattice, $K < G$ compact, $\mathfrak{h}$ a nilpotent subalgebra of the Lie algebra of G which normalizes and trivially intersects the Lie algebra of K. Then the action of the simply-connected Lie group H with Lie algebra $\mathfrak{h}$ on the compact manifold $\Gamma\backslash G/K$ by right translations is called a nilpotent algebraic action.

Theorem 10.1.27 (Barbot–Maquera–Tomter [**26**,**282**])**.** *Up to finite covers and central extensions, nilpotent algebraic Anosov actions are nil-suspensions of either*

- *a suspension of a $\mathbb{Z}^k$ Anosov action on a nilmanifold ($G = N \rtimes \mathbb{R}^k$, $\Gamma = \Lambda \rtimes \mathbb{Z}^k$, $K = \{e\}$, $H = \mathbb{R}^k$), or*
- *a modified Weyl chamber action: G is noncompact semisimple with finite center, K a compact subgroup which centralizes a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of the Lie algebra of G and contains the semisimple part of $Z_G(\mathfrak{a})$, $\mathfrak{h} \supset \mathfrak{a}$ is an abelian subalgebra with $\mathrm{Lie}(Z_G(\mathfrak{a})) = \mathfrak{h} \oplus \mathrm{Lie}(K)$.*

Remark 10.1.28. As a complement we mention symmetric Anosov flows of mixed type [**69**,**281**,**282**]. “Symmetric” here includes suspensions of toral automorphisms (“solvable type”) and geodesic flows of locally symmetric spaces (“semisimple type”). “Mixed” ones combine both. For instance, Tomter constructs an Anosov flow on a compact 7-manifold with both a “semisimple” and a “solvable” part, which can be viewed as combination of the geodesic flow on a compact Riemannian surface of constant negative curvature and the suspension of a 4-dimensional toral Anosov diffeomorphism. It is obtained by letting $\mathrm{SL}(2,\mathbb{R})$ act suitably on $\mathbb{R}^4$ and taking a semidirect product. Specifically, realize $\mathrm{SL}(2,\mathbb{R})$ as the group of unit quaternions in the even-dimensional Clifford algebra corresponding to an anisotropic quadratic form on $\mathbb{R}^3$, i.e. a quadratic diophantine equation without integer solutions. Finite-volume quotients are easy to find. It takes more effort to find a uniform, discrete subgroup (find a uniform discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ that preserves an integer lattice of $\mathbb{R}^4$ using the theory of arithmetic subgroups) [**281**].

2. Time-preserving conjugacies

We have already seen that timing constraints may result in smooth rigidity; Theorem 8.4.13 is a prime example. Now we turn to smooth-rigidity results arising from the presence of a conjugacy rather than merely an orbit-equivalence, that is, from the preservation of time, Theorem 10.2.1 and Theorem 10.2.7. By contrast, we note an astonishing theorem of Ghys: An Anosov 3-flow on circle bundle is topologically orbit-equivalent to the geodesic flow of a Riemannian metric on the base with constant negative curvature [**126**, Théorème A].

Theorem 10.2.1 ([**114**, Theorem 5.2])**.** *A topological* conjugacy *between geodesic flows of Riemannian surfaces with negative curvature is a* C^1 *diffeomorphism.*

Remark 10.2.2. The step to differentiability is significant. This implies, for instance, that all geometric data associated with periodic points, such as eigenvalues of the return map, are preserved by the conjugacy, because the linear parts of the return maps are linearly conjugate. By contraposition, this means that changing a geodesic flow in such a way as to affect such data at even one periodic point precludes the existence of a *topological* conjugacy to the original geodesic flow.

Proof. Our proof follows the original arguments [**114**]. We begin by building counterparts to the various flows we developed in the case of constant negative curvature in Section 2.2. Let X be the geodesic vector field, $H_+ \neq 0$ an expanding vector field, and $H_- \neq 0$ a contracting vector field. By Corollary 7.4.15, we can take $H_\pm$ to be C^1. Analogously to the case of constant curvature, $[X, H_\pm]$ is a scalar multiple of $H_\pm$, so the orbits of g and of the horospheric flow h_- generated by H_- span the weak stable foliation, and those of g and h_+ the unstable foliation. The presence of an invariant contact form means that $a \neq 0$ in

$$[H_-, H_+] = aX + bH_+ + cH_-.$$

We assume that X, H_+, H_- is *properly oriented*, that is, $a > 0$, and introduce functions σ, τ, ρ of $x \in M$ and small $s, t \in \mathbb{R}$ by

$$g^\rho(h_-^\tau(h_+^s(x))) = h_+^\sigma(h_-^t(x)); \tag{10.2.1}$$

this relation reflects the fact that $W^{ss}(h_+^s(x)) \cap W^{cu}(h_-^t(x))$ and $W^{uu}(h_-^t(x)) \cap W^{cs}(h_+^s(x))$ lie on the same g-orbit, and it implies that $\operatorname{sign}\sigma = \operatorname{sign} s$ and $\operatorname{sign}\tau = \operatorname{sign} t$. The next two lemmas give the following fact, which is needed for (10.2.3).

Proposition 10.2.3. $\dfrac{\rho}{st} \xrightarrow[s,t\to 0]{\text{uniformly on } M} a.$

Lemma 10.2.4. $\sqrt{(\sigma - s)^2 + \rho^2 + (\tau - t)^2} \in O(|st|).$

Proof. We will use that if vector fields X, Y generate flows φ^t, ψ^t, and $f\colon M \to \mathbb{R}$ is C^2, then uniformly in x the second directional derivaties are limits as follows:

$$X(Yf)(x) - Y(Xf)(x) = \lim_{s,t\to 0} \frac{1}{st}\big(f(\psi^t(\varphi^s(x))) - f(\varphi^s(\psi^t(x)))\big). \tag{10.2.2}$$

For $\Delta > 0$ there is an $A > 0$ such that $|s|,|t| < \Delta \Rightarrow d(h_+^s(h_-^t(x)), h_-^t(h_+^s(x))) \le A|st|$ for all $x \in M$, while transversality of H_+, H_-, X and compactness yield a $\Delta' \le \Delta$ and $A' \ge A$ such that

$$|s|,|t|,|r| < \Delta' \Rightarrow\ \forall x \in M\ d(h_-^t(g^r(h_+^s(y))), y) > \sqrt{T^2 + r^2 + s^2}/A'.$$

If s, t are small enough that $\max(|s|,|t|,|\sigma - s|,|\tau - t|,|\rho|) < \Delta'$, then

$$\sqrt{(\sigma-s)^2+\rho^2+(\tau-t)^2} < A'd(\overbrace{h_-^{t-\tau}(g^{-\rho}(\underbrace{h_+^{\sigma-s}\big(h_+^s(h_-^t}_{=h_+^\sigma h_-^t = g^\rho h_-^\tau h_+^s}(x))\big)))}^{=h_-^t(h_+^s(x))}, h_+^s(h_-^t(x))) \le AA'|st|. \qquad \square$$

Lemma 10.2.5. $\dfrac{\rho}{st}X + \dfrac{s-\sigma}{st}H_+ + \dfrac{\tau-t}{st}H_- \xrightarrow[s,t\to 0]{\text{uniformly in } x} [H_-, H_+] = aX + bH_+ + cH_-.$

Proof. For $f \in C^2(M)$ the Mean-Value Theorem and (10.2.2) give[2]

$$\boxed{\rho Xf} + \underbrace{\rho}_{\in O(st)}\big[\underbrace{Xf(g^{\rho_1}(h_-^\tau(h_+^s(\cdot)))) - Xf}_{\in o(st)}\big] = \rho Xf(g^{\rho_1}(h_-^\tau(h_+^s(\cdot))))$$

$$= f(\underbrace{h_+^\sigma(h_-^t}_{=g^\rho h_-^\tau h_+^s}(\cdot))) - f(h_-^\tau(h_+^s(\cdot)))$$

$$= \underbrace{\Big(\overbrace{f(h_+^s(h_-^t(\cdot)))}^{\text{cancellation 1}} - \overbrace{f(h_-^t(h_+^s(\cdot)))}^{\text{cancellation 2}}\Big)}_{=\boxed{[H_-,H_+]fst}+o(st)}$$

$$+ \underbrace{\Big(f(h_+^\sigma(h_-^t(\cdot))) - \overbrace{f(h_+^s(h_-^t(\cdot)))}^{\text{cancellation 1}}\Big)}_{= \int_s^\sigma H_+f(h_+^{s'}(h_-^t(\cdot)))\,ds' = (\sigma-s)H_+f(h_+^{s_1}(h_-^t(\cdot))) = \underbrace{(\sigma-s)}_{\in O(st)}\big(\underbrace{H_+f(h_+^{s_1}(h_-^t(\cdot))) - H_+f}_{\in o(st)}\big) + \boxed{(\sigma-s)H_+f}}$$

$$- \underbrace{\Big(f(h_-^\tau(h_+^s(\cdot))) - \overbrace{f(h_-^t(h_+^s(\cdot)))}^{\text{cancellation 2}}\Big)}_{= \int_t^\tau H_-f(h_-^{t'}(h_+^s(\cdot)))\,dt' = (\tau-t)H_-f(h_-^{t_1}(h_+^s(\cdot))) = \underbrace{(\tau-t)}_{\in O(st)}\big(\underbrace{H_-f(h_-^{t_1}(h_+^s(\cdot))) - H_-f}_{\in o(st)}\big) + \boxed{(\tau-t)H_-f}}.$$

Rearrange to get $[H_-, H_+] = \dfrac{\rho}{st}X + \dfrac{s-\sigma}{st}H_+ + \dfrac{\tau-t}{st}H_- + o(1)$ uniformly on M. $\square$

[2] See Remark 3.2.18.

We now turn to the conjugacy problem. Denote by h the conjugacy between the geodesic flows g and g'. (We assume that g and g' are both normalized (by scaling the metric) to have topological entropy 1.) The counterparts for g' of the various vector fields, functions and flows just described are denoted in the same way but with primes added.

Since h preserves the strong invariant foliations, we can use local dynamical coordinates to write

$$h(h_-^t(h_+^s(g^r(x)))) = {h'}_-^{t'}({h'}_+^{s'}({g'}^{r'}(x')))$$

with $x' = h(x)$, $r = r'$, $s' = S(g^r(x), s)$, $t' = T(h_+^s(g^r(x)), t)$, where S, T are defined by ${h'}_+^{S(x,s)}(h(x)) = h(h_+^s(x))$ and ${h'}_-^{T(x,t)}(h(x)) = h(h_-^t(x))$. Since (r, s, t) and (r', s', t') are local C^1 coordinates, it will suffice to check continuous differentiability of

$$(r, s, t) \mapsto (r', s', t') = (r, S(g^r(x), s), T(h_+^s(g^r(x)), t)).$$

Lemma 10.2.6. *$s \mapsto S(\cdot, s)$ and $t \mapsto T(\cdot, t)$ are C^1.*

Proof. Uniformly on M we have $S(\cdot, s) \xrightarrow[s\to 0]{} 0$, $T(\cdot, t) \xrightarrow[t\to 0]{} 0$, $\frac{\rho(\cdot,s,t)}{sta} \xrightarrow[s,t\to 0]{} 1$ (Proposition 10.2.3) and $\frac{\rho'(\cdot,s',t')}{s't'a'} \xrightarrow[s',t'\to 0]{} 1$, and we apply this to both sides of

$$\rho(x, s, t) = \rho'(h(x), S(x, s), T(x, t)):$$

For $\epsilon > 0$ choose $\delta > 0$ such that $|s|, |t| < \delta \Rightarrow \left|\dfrac{S(\cdot, s)T(\cdot, t)a' \circ h}{sta} - 1\right| < \epsilon$ or

$$\left|\log \frac{S(\cdot, s)}{s} + \log \frac{T(\cdot, t)}{t} - \log \frac{a}{a' \circ h}\right| < \epsilon. \tag{10.2.3}$$

With fixed $|t| \in (0, \delta)$ this implies that $s \mapsto \log S(\cdot, s)/s$ is within ϵ of a continuous function. Since ϵ is arbitrary, $\log S(\cdot, s)/s$ uniformly converges to a continuous function as $s \to 0$, hence $s \mapsto S(\cdot, s)/s$ uniformly converges to a nonzero continuous function, and $s \mapsto S(\cdot, s)$ is differentiable at $s = 0$.

To show continuity of the derivative note that the cocycle condition $S(x, s_0 + s) = S(x, s_0) + S(h_+^{s_0}(x), s)$ implies that $s \mapsto S(\cdot, s)$ is differentiable for all s and with

$$\frac{dS(\cdot, s)}{ds}\Big|_{s=s_0} = \frac{dS(h_+^{s_0}(x), s)}{ds}\Big|_{s=0}.$$

Arguing as above from (10.2.3) at $h_+^{s_0}(\cdot)$ rather than $(\cdot)$ shows that $S(h_+^{s_0}(\cdot), s)/s$ uniformly converges as $s \to 0$ to a function that is continuous in s_0.

Continuous differentiability of $t \mapsto T(\cdot, t)$ follows in like manner from (10.2.3). □

From this, we finally show that the conjugacy h is C^1; by symmetry, this implies that h is a C^1 diffeomorphism. To check that

$$(r, s, t) \mapsto (r', s', t') = (r, S(g^r(x), s), T(h_+^s(g^r(x)), t))$$

is C^1, we begin with $(r,s) \mapsto s' = S(g^r(x), s)$, the problem being the r-dependence in the first entry. We circumvent this by using the commutation relations to rewrite this as a composition of C^1 maps.

The reparametrization $\alpha(\cdot,\cdot,r)$ of h_+ defined by $g^r(h_+^s(x)) = h_+^{\alpha(x,s,r)}(g^r(x))$ for fixed r is C^1, as is the reverse reparametrization $\tilde{\alpha}$ with $\tilde{\alpha}(x,\alpha(x,s,r),r) = s$. Likewise for $\alpha', \tilde{\alpha}'$ on the side of g'. The rendering of an (s,r)-coordinate patch in Figure 10.2.1 suggests how to build the desired composition: each map is C^1 in

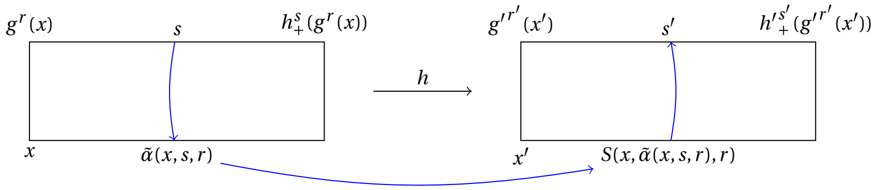


FIGURE 10.2.1. Smoothness of the conjugacy along unstable leaves

$$(s,r) \mapsto \tilde{\alpha}(x,s,r) \mapsto S(x,\tilde{\alpha}(x,s,r),r) \mapsto \alpha'(x', S(x,\tilde{\alpha}(x,s,r),r),r) = s'.$$

The same argument with T shows that $(r,s) \mapsto T(g^r(x),t)$ is C^1. This is useful, but it remains to show that $(r,s,t) \mapsto T(h_+^s(g^r(x)),t)$ is C^1, which uses a similar tactic. The function τ from (10.2.1) is C^1, as is its counterpart τ' as well as the inverse reparametrization $\tilde{\tau}$ of $\tau(x,s,\cdot)$ given by $\tilde{\tau}(x,s,\tau(x,s,t)) = t$ and its counterpart $\tilde{\tau}'$. Taking now our cue from the (r,s,t)-patch in Figure 10.2.2, we find that

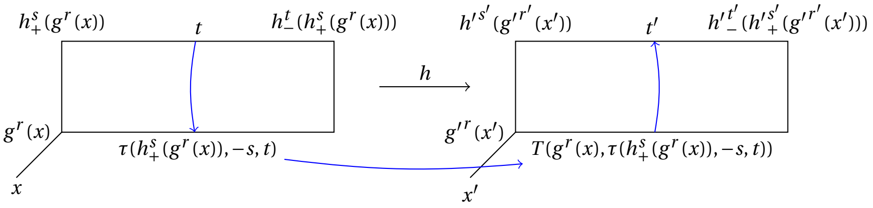

$h_+^s(g^r(x))$
t
$h_-^t(h_+^s(g^r(x)))$
$h'^{s'}(g'^{r'}(x'))$
t'
$h'^{t'}_-(h'^{s'}_+(g'^{r'}(x')))$
h
$g^r(x)$
$\tau(h_+^s(g^r(x)),-s,t)$
$g'^r(x')$
$T(g^r(x),\tau(h_+^s(g^r(x)),-s,t))$
x
x'

FIGURE 10.2.2. Smoothness of the conjugacy along stable leaves

$$(r,s,t) \mapsto t' = \tilde{\tau}'(h'^{s'}_+(g'^{r}(x')), -s, T(g^r(x), \tau(h_+^s(g^r(x)), -s, t)), t)$$

is C^1 by the chain rule because we noted already that $(r,s) \mapsto T(g^r(x),t)$ is C^1. □

Theorem 10.2.7. *A C^1 conjugacy between C^k Anosov 3-flows is $C^{k-\epsilon}$.*

Remark 10.2.8. A stronger result actually holds, with the same proof: A conjugacy between C^k Anosov 3-flows is $C^{k-\epsilon}$ if it is continuous and absolutely continuous.

Proof [**205**, Theorem 1.1]. A C^1 conjugacy preserves the unstable Jacobian and hence sends SRB-measure to SRB-measure, so it respects the densities of SRB-measure on unstable leaves and on stable leaves. Locally, along a leaf we thus have

$$\int_0^s \rho_0(s')\,ds' = \int_{h(0)}^{h(s)} \rho_{h(0)}(s')\,ds',$$

which upon differentiation with respect to s becomes a differential equation whose coefficient functions are C^k by Theorem 8.4.12, and so h is (uniformly) C^k when restricted to any leaf. To conclude smoothness in the (say) unstable direction without restriction, note that when pieces of unstable manifolds converge to a limiting piece, the restrictions of h to these are a C^k-bounded sequence, hence have a convergent subsequence in the $C^{k-\epsilon}$-topology; since h is continuous, this is a proper limit rather than just that of a subsequence. Thus, h is $C^{k-\epsilon}$ in the stable, unstable and flow directions. This implies the claim by Theorem 10.2.9. □

Theorem 10.2.9 ([**168**]). *If f is uniformly $C^{k+\alpha}$ along the leaves of 2 continuous transverse foliations with smooth leaves, then f is $C^{k+\alpha}$.*

Remark 10.2.10. Remark 10.2.2 noted that Theorem 10.2.1 fixes geometric data rigidly. Theorem 10.2.7 (together with Theorem 10.2.1) goes further in that a topological conjugacy is necessarily essentially as smooth as the flows, so they are completely indistinguishable as smooth dynamical systems.

As we will do in subsequent sections, let us close with a remark on *geometric rigidity* as opposed to smooth rigidity. We take an illustrative example from the theory of magnetic flows (Definition 5.1.13).

Theorem 10.2.11 ([**138**, Théorème 7.3]). *Suppose the Gauss curvature K of a closed Riemannian surface S satisfies $-k_2^2 \le K \le -k_1^2 < 0$ and a magnetic flow with $\|\mathfrak{m}\|_\infty^2 + \|\nabla\mathfrak{m}\|_\infty < k_1$ is topologically conjugate to the geodesic flow of a negatively curved metric on S with the same total area. If the conjugacy is absolutely continuous, then $\mathfrak{m} \equiv 0$ and the metrics are isometric.*

3. Smooth invariant foliations

We begin with smooth rigidity by building on Proposition 7.5.7.

Theorem 10.3.1 (Smooth longitudinal rigidity). *Let M be a 3-manifold, $k \ge 2$, Φ a C^k volume-preserving Anosov flow. Then $E^u \oplus E^s$ is Zygmund-regular, and there* *is an obstruction to higher regularity that can be described geometrically as the curvature of the image of a transversal under a return map. This obstruction defines the cohomology class of a cocycle (the longitudinal KAM-cocycle), and the following are equivalent:*

(1) $E^u \oplus E^s$ is "little Zygmund" (see Definition 7.5.1).
(2) The longitudinal KAM-cocycle is a coboundary.
(3) $E^u \oplus E^s$ is Lipschitz.
(4) $E^u \oplus E^s \in C^{k-1}$.

Remark 10.3.2. If $E^u \oplus E^s \in C^{k-1}$, then Φ is a suspension or contact flow (Theorem 9.1.5), so there is structural information in the conclusion of this theorem, and it can thus be viewed as a weak geometric rigidity result. We note that the last item clearly implies all the previous ones (the second one thanks to Proposition 7.5.7). Accordingly, the following results show that each of these items implies the next.

Theorem 10.2.7 is another instance of smooth rigidity.

The first step is to show that the obstruction from Proposition 7.5.7 arises from the cocycle promised in Theorem 10.3.1. To that end note that $K(p,T)$ is naturally defined as a second order partial derivative of φ^T in adapted coordinates at any point $p \in M$. There is a natural geometric interpretation. Consider the transversals $\Delta := \psi_p((-\epsilon,\epsilon)\times\{0\}\times(-\epsilon,\epsilon))$ and $\Delta' := \psi_{\varphi^T(p)}((-\epsilon,\epsilon)\times\{0\}\times(-\epsilon,\epsilon))$. Then $\Delta' \cap \varphi^T(\Delta)$ contains local strong stable and unstable manifolds of $\varphi^T(p)$, but the two transversals are not usually identical. As one sees from the coordinate represenatation of the flow, the obstruction gives the off-diagonal term in the Hessian of the map $(-\delta,\delta)^2 \to \mathbb{R}$ that gives the lengths of the orbit segments between Δ' and $\varphi^T(\Delta')$. This can be viewed as the "relative curvature" of image transversal versus original transversal. Lemma 10.3.6 shows that if the obstruction vanishes we can choose transversals such that the image transversals agree to third order.

Lemma 10.3.3. *K is an additive cocycle (Definition 1.2.3).*

We call K the *longitudinal KAM-cocycle*.[3]

Proof. To show that $K(p,T+S) = K(\varphi^T(p),S) + K(p,T)$ for all $p \in M$, $T,S \in \mathbb{R}$ write $D\varphi^T(0,0,s) = \begin{pmatrix}\alpha^{-1} & 0 & 0\\ b_1 & 1 & 0\\ b_2 & 0 & \alpha\end{pmatrix}$ at p and $D\varphi^S(0,0,s) = \begin{pmatrix}\bar\alpha^{-1} & 0 & 0\\ \bar b_1 & 1 & 0\\ \bar b_2 & 0 & \bar\alpha\end{pmatrix}$ at $\varphi^T(p)$. Then

[3] Here, "K" is for Katok because this cocycle is modeled on the one that plays a central role in Theorem 10.3.10, "A" is for Anosov, who first noted a counterpart of Proposition 7.5.7 for the weak subbundles, and "M" is for Moser, whose normal form contains that item as the "resonance" term.

$D\varphi^{T+S}(0,0,s) = \begin{pmatrix} * & 0 & 0 \\ \bar{b}_1\alpha^{-1}+b_1 & 1 & 0 \\ * & 0 & * \end{pmatrix}$. Using $\bar{b}_1(0) = 0$ this gives

$$\begin{aligned} K(p,T+S) = \frac{d}{ds}(\bar{b}_1\alpha^{-1} + b_1)_{|s=0} &= \big(\frac{d}{ds}\bar{b}_1(0)\big)\alpha^{-1}(0) + \frac{d}{ds}b_1(0) \\ &= \bar{b}_1'(0) + b_1'(0) = K(\varphi^T(p),S) + K(T,p). \quad \square \end{aligned}$$

Lemma 10.3.4. *The cohomology class of the longitudinal KAM-cocycle is unaffected by coordinate changes.*

Proof. Consider coordinate changes to coordinates with our desired properties. To see how the longitudinal KAM-cocycle changes we examine the change in the differential of φ^T entailed by the coordinate change. We need only study points on the stable leaf. To do the coordinate calculations we agree that the coordinate change transforms variables $(\tilde{u},\tilde{t},\tilde{s})$ to (u,t,s). Variables in coordinates at $\varphi^T(p)$ are marked by a subscript T. At a point $(0,0,\tilde{s})$ an allowed coordinate change has differential

$$\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ * & 0 & a^{-1} \end{pmatrix}$$

and the inverse in coordinates at $\varphi^T(p)$ is

$$\begin{pmatrix} a_T^{-1} & 0 & 0 \\ -a_T^{-1}b_T & 1 & 0 \\ * & 0 & a_T, \end{pmatrix}$$

with entries evaluated at $\tilde{s}_T$. Note that $a = \dfrac{d\tilde{s}}{ds}$. In these new coordinates the differential of φ^T at $(0,0,\tilde{s})$ becomes

$$\begin{pmatrix} a_T^{-1} & 0 & 0 \\ -a_T^{-1}b_T & 1 & 0 \\ * & 0 & a_T \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 & 0 \\ b_1 & 1 & 0 \\ b_2 & 0 & \alpha \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ * & 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ -a\alpha^{-1}a_T^{-1}b_T + ab_1 + b & 1 & 0 \\ * & * & a^{-1}\alpha a_T \end{pmatrix}.$$

Note that therefore $\dfrac{d\tilde{s}_T}{d\tilde{s}} = a^{-1}\alpha a_T$. This gives

(10.3.1)
$$\begin{aligned} \tilde{K}(p,T) \coloneqq \frac{d}{d\tilde{s}}\tilde{b}_{1|\tilde{s}=0} &= -a(0)\alpha^{-1}(0)a_T^{-1}(0)\frac{d}{d\tilde{s}}b_T(0) + a(0)\frac{d}{d\tilde{s}}b_1(0) + \frac{d}{d\tilde{s}}b(0) \\ &= \frac{d}{ds}b_1(0) + \frac{d}{d\tilde{s}}b(0) - \frac{d\tilde{s}}{d\tilde{s}_T}\frac{d}{d\tilde{s}}b_T(0) = K(p,T) + b'(0) - b_T'(0), \end{aligned}$$

which is cohomologous to K. $\square$

Actually, K is an obstruction to $E^u \oplus E^s$ being more regular than Zygmund.

Proposition 10.3.5. *If $E^u \oplus E^s$ is "little Zygmund" (Definition 7.5.1) then the longitudinal KAM-cocycle is null cohomologous.*

Proof. Let p be any T-periodic point. Then in our usual coordinates (7.5.1) gives

$$e(s_\varphi) = \alpha(s)(b_1(s) + e(s)) = \alpha(0)(1 + o(s))(b_1'(0)s + o(s) + e(s)).$$

If $\alpha := \alpha(0)$ then $\alpha s = s_\varphi + O(s^2)$. Since e is "little Zygmund", it is H-Hölder for all $H \in (1/2, 1)$, so

$$e(\alpha s) = e(s_\varphi) + \underbrace{o(s^{2H})}_{=o(s)} = \alpha(1 + o(s))(b_1'(0)s + o(s) + e(s)) = \alpha(K(p,T)s + e(s) + o(s)).$$

Recursively, this gives

$$e(\alpha^n s) = \alpha^n (nK(p,T)s + e(s) + \sum_{i=0}^{n-1} \alpha^{-i} o(\alpha^i s)).$$

Since the terms of the sum converge to 0 we get $\sum_{i=0}^{n-1} \alpha^{-i} o(\alpha^i s)/n \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned} 0 = \lim_{n\to\infty} & \frac{e(\alpha^n s)}{\alpha^n s \log(\alpha^n s)} \\ &= \lim_{n\to\infty} \Big[\frac{nK(p,T)}{n\log\alpha + \log s} + \frac{e(s)}{s\log(\alpha^n s)} + \frac{\sum_{i=0}^{n-1} \alpha^{-i} o(\alpha^i s)}{s\log(\alpha^n s)} \Big] = \frac{K(p,T)}{\log\alpha}. \quad \square \end{aligned}$$

We now study what happens when the longitudinal KAM-cocycle is trivial. As a first step we show that this allows more perfectly adapted coordinate systems.

Lemma 10.3.6. *If the longitudinal KAM-cocycle is a coboundary then there are nonstationary local coordinates in which it vanishes identically.*

Proof. If the longitudinal KAM-cocycle is null-cohomologous then there is a smooth $k\colon M \to \mathbb{R}$ such that $K(p,T) = k(\varphi^T(p)) - k(p)$ for all $p \in M$, $T \in \mathbb{R}$. With the notations from the proof of Lemma 10.3.4 define a coordinate change at p by

$$\begin{pmatrix} u \\ t \\ s \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{t} + k(p)\tilde{u}\tilde{s} \\ \tilde{s} \end{pmatrix}.$$

In these new coordinates $\tilde{K}(p,T) = 0$ for all $p \in M$, $t \in \mathbb{R}$ by (10.3.1). □

The first direct consequence of the existence of such coordinates is that in this case $E^u \oplus E^s$ is Lipschitz continuous.

Proposition 10.3.7. *If the longitudinal KAM-cocycle is a coboundary then $E^u \oplus E^s$ is Lipschitz continuous.*

Proof. We combine the arguments of Proposition 7.5.5 and Proposition 7.5.6, using that in our new coordinates the cocycle is trivial, i.e., $b_1'(0) = 0$ in (7.5.1), so $b_1(s) \in O(s^2)$. Assume that $|e(s)| \le Z|s|$ with uniform Z. Then the first calculation, (7.5.2), in Proposition 7.5.5 becomes

$$|e(s_\varphi)| = |\alpha(s)||b_1(s)+e(s)| \le |\alpha(s)|(O(s^2)+Z|s|) = (Z+O(s))|\alpha(s)s| = Z(1+\kappa(s))|s_\varphi|,$$

with $\kappa(s) \in O(s)$ decreasing in Z. With the argument from Proposition 7.5.5 for $\bar{e}$, which works for $H = 1$, this implies Lipschitz continuity as in Theorem 7.5.3. □

We finally show that Lipschitz continuity implies smoothness.

Proposition 10.3.8. *If the volume-preserving Anosov flow is C^k and the subbundle $E^u \oplus E^s$ is Lipschitz continuous, then $E^u \oplus E^s$ is C^{k-1}.*

Proof. By the Livshitz Theorem there is a C^k invariant volume form Ω (Theorem 7.2.10). The canonical invariant 1-form A associated to the flow by $A(\dot{\varphi}) = 1$, $A\upharpoonright_{E^u \oplus E^s} = 0$ is Lipschitz continuous, and we aim to prove that it is C^{k-1}.

In local charts as in Lemma 7.5.4 we have

$$A = dt + \lambda(u,s)ds + \beta(u,s)du,$$

where λ and β are Lipschitz-continuous functions independent of the variable t. Then the Anosov vector field $X = \dot{\varphi}\,(= \partial_t)$ is in the kernel of the integrable 2-form dA and the L^1 3-form $A \wedge dA$ is also flow invariant. Ergodicity gives an $\eta \in \mathbb{R}$ such that

$$A \wedge dA \overset{\text{ae}}{=} \eta\Omega,$$

with respect to the invariant measure associated to Ω,that is, $dA \overset{\text{ae}}{=} w = \eta\Omega(X,\cdot) = \eta i_X \Omega$, a C^k invariant 2-form. Invariance implies $0 = \mathscr{L}_X(w)/\eta = i_X\,dw + di_X\,w$, so $dw = 0$ (it is a 3-form that vanishes on X). Moreover, w is exact by the Stokes Theorem for Lipschitz continuous forms: Let c_θ, $\theta \in [-\epsilon, \epsilon]$ be a family of regular disjoint 2-cycles such that $[c_\theta] = c \in H^2(M)$, and set $\Delta = \bigcup_{t\in[-\epsilon,\epsilon]} c_\theta$. By the Fubini Theorem w is exact:

$$2\epsilon \cdot \omega(c) = \int_{-\epsilon}^{\epsilon}\Big(\int_{c_\theta} w\Big)d\theta = \int_\Delta d\theta \wedge w = \int_\Delta d\theta \wedge dA = \int_{-\epsilon}^{\epsilon}\Big(\int_{c_\theta} dA\Big)d\theta = 0.$$

Thus, there is a C^k one-form $\tilde{A}$ such that $w = d\tilde{A}$. Then the Lipschitz-continuous 1-form $A - \tilde{A}$ is almost everywhere closed. Choose a reference Riemannian metric g. Using the same type of Fubini–Stokes argument establishes the de Rham Hodge Theorem (there is a harmonic form in each cohomology class) in this context: there exist a g-harmonic 1-form H in the same cohomology class and a 1-form μ such that

$$A - \tilde{A} = H + \mu, \quad d\mu \overset{\text{ae}}{=} 0, \quad \int_\gamma \mu = 0$$

for any 1-cycle γ. We aim to show that μ is exact, but instead of using the same trick as before we will be more explicit to get higher regularity.

In adapted local coordinates at $p \in M$ denote by $[0, x]$, $x = (u, t, s)$, the image of the segment by the corresponding local chart. Introduce for any $c \in \mathbb{R}$ a local Lipschitz contiuous function

$$f_p^c(x) := c + \int_{[0,x]} \mu.$$

For every x, v and small ϵ the Stokes Theorem gives

$$f_p^c(x + \epsilon v) - f_p^c(x) = \int_{[x, x+\epsilon v]} \mu - \int_{T_\epsilon} d\mu,$$

where T_ϵ is the oriented triangle $(0, x, x + \epsilon v)$. Then $df_p^c(x)(v) = \mu(x)(v)$ for almost every x and v. Since the integral of μ along any closed curve vanishes there is a global function f that locally coincides with one of the functions f_p^c, that is,

$$A \overset{\text{ae}}{=} \tilde{A} + H + df. \tag{10.3.2}$$

We conclude the proof by showing that df is C^{k-1}.

Using the Anosov vector field and the definition of the canonical 1-form A we write

$$1 - \tilde{A}(X) - H(X) \overset{\text{ae}}{=} df(X).$$

The terms on the left are C^k, so by Theorem 7.2.12, there exists a C^k boundary $b\colon M \to \mathbb{R}$ such that

$$1 - \tilde{A}(X) - H(X) = db(X).$$

This in particular means that the Lipschitz-continuous function $\rho := b - f$ satisfies

$$d\rho(X) \overset{\text{ae}}{=} 0. \tag{10.3.3}$$

We show that this implies flow-invariance of ρ. Choose $p \in M$ and an adapted chart in which the Anosov vector field X is expressed as ∂_t. Assume there are $x_0 = (u_0, t_0, s_0)$ and $t \in \mathbb{R}$ such that $\rho(\varphi_t(x_0)) - \rho(x_0) \neq 0$. By continuity there is an open set U in the transversal $\{(u, t_0, s)\}$ such that

$$0 < \int_U (\rho(\varphi_t(u, t_0, s)) - \rho(u, t_0, s))\, du\, ds = \int_U \Big(\int_{t_0}^{t_0 + t} d\rho(X(w)) dw \Big)\, du\, ds,$$

contrary to (10.3.3).

Invariance and ergodicity imply that ρ is constant (everywhere by continuity), so $db \equiv df$ and hence $A = \tilde{A} + H + db$ because both sides of (10.3.2) are continuous. Thus A is C^{k-1}, and hence so is its kernel $E^u \oplus E^s$. □

Remark 10.3.9. Theorem 10.3.1 provides a lovely instance of smooth rigidity,[4] and Remark 10.3.2 explains how its conclusion implies structural information. However, the conclusion of Theorem 10.3.1 holds for all suspensions of Anosov diffeomorphisms and for all contact Anosov flows, so this does not provide a basis for finding an instance of *geometric rigidity*. The situation is quite different when we return to considerations of the *weak* stable or unstable foliation. A volume-preserving Anosov 3-flow satisfies the strongest bunching condition possible, so Theorem 7.4.14 implies that the weak foliations are $C^{1+x|\log x|}$. Analogously to Theorem 7.5.3 it turns out that in this case the *derivatives* are Zygmund-regular and that being little Zygmund implies smooth rigidity, that is, that the weak foliations are smooth. So far, this is quite analogous to Theorem 10.3.1. However, in this case one also obtains geometric rigidity: The flow must then be smoothly conjugate to either the suspension of a toral automorphism or to the geodesic flow of a surface of constant curvature, that is, the system is algebraic up to smooth conjugacy:

Theorem 10.3.10 (Ghys–Hurder–Katok [**165**])**.** *The weak stable and unstable subbundles of a volume-preserving C^{k+3} Anosov 3-flow are $C^{1+Zygmund}$, and if either of them is $C^{1+zygmund}$, then both are C^k. If, furthermore, the flow is a geodesic flow, then the metric has constant negative curvature.*

Remark 10.3.11. The rigidity conclusion also holds under the assumption that the derivative of one or the other of the weak subbundles has modulus of continuity $o(x|\log x|)$ [**165**] or is of bounded variation [**140**].

If the flow is a geodesic flow, a theorem by Ghys gives a smooth conjugacy to the geodesic flow of a surface of constant curvature, and Theorem 10.4.1 then implies that the metric has constant curvature (Theorem 10.5.1). (And Remark 10.3.16 is pertinent here as well.) We remark that while this structural information was obtained in the original preprint by Hurder and Katok, they assumed closeness to the algebraic model; the published version invokes his result. It is interesting here to spell out the theorem by Ghys because it does not assume the topology of a geodesic flow.

Theorem 10.3.12 ([**127**, **128**])**.** *A volume-preserving C^∞ Anosov 3-flow with C^2 weak foliations is either a suspension or C^∞-conjugate (modulo Remark 10.3.16) to the geodesic flow of a constantly curved surface.*

[4] Theorem 10.2.7 is another

Remark 10.3.13 ([**129**, Théorème 4.6])**.** The C^2-hypothesis can be replaced by assuming Lipschitz-continuous derivative, and for C^r-flows ($r \geq 2$) one gets a C^r-conjugacy to the geodesic flow. One can even drop volume-preservation in the assumptions, but then the “algebraic” models include exotic “hybrids” (Example 10.3.21).

Theorem 10.3.10 predates smooth longitudinal rigidity (Theorem 10.3.1), and we presented the proof of the latter to illustrate the approach in the proof of Theorem 10.3.10. Indeed, the counterpart here to Proposition 7.5.7 is a much earlier observation by Anosov that there is an obstruction to the weak subbundles being C^2. In this case it involves third derivatives of the flow (while the obstruction in Proposition 7.5.7 involves second derivatives). Of course, Theorem 10.3.10 also carries structural information while Theorem 10.3.1 does not.

Following Guysinsky [**140**] one can explain the Anosov cocycle using local normal forms. For a smooth area-preserving Anosov diffeomorphism on $\mathbb{T}^2$ DeLatte [**97**, **98**] showed that one can find local smooth coordinate systems around each point that depend continuously (actually C^1) on the point and bring the diffeomorphism f into the *Moser normal form* [**217**]

$$f(x,y) = \begin{pmatrix} \lambda_p^{-1} x/\varphi_p(xy) \\ \lambda_p y \varphi_p(xy) \end{pmatrix},$$

where (x,y) are in local coordinates around a point p and the expression on the right is in coordinates around $f(p)$. The terms involving φ_p that depend on the product xy correspond to the natural resonance $\lambda_p \lambda_p^{-1} = 1$ that arises from area-preservation (actually from the family of resonances $\lambda_p = \lambda_p^{n+1}\lambda_p^{-n}$). The function φ_p is as smooth as f, and $\varphi_p(0) = 1$. Now we suppress the (continuous) dependence of λ and φ on p and apply this to the return map f of a local cross-section at a periodic point of a flow. For a point $(0,y)$ we then have

$$Df = \begin{pmatrix} \lambda^{-1} xy(1/\varphi)'(xy) + \lambda^{-1}/\varphi(xy) & \lambda^{-1} x^2 (1/\varphi)'(xy) \\ \lambda y^2 \varphi'(xy) & \lambda xy \varphi'(xy) + \lambda \varphi(xy) \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ \lambda y^2 \varphi'(0) & \lambda \end{pmatrix}.$$

In these local coordinates the (center-) unstable direction at a point $(0,y)$ on the (center-) stable leaf of p in the section is spanned by a vector $(1, a(y))$. Since this subbundle is invariant under Df and since $f(0,y) = (0, \lambda y)$, the coordinate representation of Df from above gives $a(\lambda y) = \lambda^2 y^2 \varphi'(0) + \lambda^2 a(y)$. If the unstable subbundle is C^2 then differentating this relation twice with respect to x at 0 gives $\lambda^2 a''(0) = 2\lambda^2 \varphi'(0) + \lambda^2 a''(0)$, that is, $\varphi'(0) = 0$. This means that the Anosov obstruction is $\varphi'(0)$, where φ arises from the nonstationary Moser-deLatte normal form.

Hurder and Katok established that this obstruction arises from a cocycle which (cohomologically) vanishes if and only if (one and hence both of) the stable and unstable subbundles are $C^{1+\mathrm{zygmund}}$. In that latter case, the cocycle lends itself to showing that the regularity is indeed C^3, from where a bootstrap kicks in to give C^∞ subbundles. Thus far, this is a smooth-rigidity result, and work of Ghys then produces a geometric-rigidity result: that the flow is smoothly conjugate to one or the other of the algebraic ones.

The geometric rigidity of Theorem 10.3.10 (indeed, of Remark 10.3.11) has been extended to higher dimension, albeit with a "regularity gap." Specifically, Theorem 7.6.1 shows that in higher dimension the invariant subbundles are more than a little less regular than C^2. Conversely, at present, regularity somewhat above C^2 is needed for rigidity phenomena.

Kanai introduced the essential tool for these rigidity results, the Bott–Kanai connection (Proposition 10.5.25) and proved a rigidity result for geodesic flows with a curvature-pinching assumption. Feres and Katok built on this by increasingly removing pinching assumptions, and then the following result altogether removed the need to start with a geodesic flow in the first place.

Theorem 10.3.14 (Benoist–Foulon–Labourie [**43**])**.** *A contact Anosov flow with sufficiently smooth*[5] *foliations is smoothly conjugate to the geodesic flow of a locally symmetric Riemannian manifold. (Modulo Remark 10.3.16 below.)*

Indeed, Theorem 10.4.6 below then implies

Theorem 10.3.15. *A Riemannian metric whose geodesic flow has smooth invariant foliations is* isometric *to a locally symmetric metric.*

Remark 10.3.16. The statement of Theorem 10.3.14 is not quite correct. There are two modifications one can make to a contact Anosov flow, including geodesic flows, that affect neither the contact Anosov property nor smoothness of the invariant foliations, so the statement is to be understood modulo these additional modifications. One of these is to pass to finite covers (or quotients) of the phase space. The other is a *canonical time-change* which, unlike most time changes, preserves the contact property. We introduce this time-change next, and Proposition 1.3.19 makes it natural.

Definition 10.3.17. A *canonical time-change* is defined using a closed 1-form α by replacing the generator X of the flow by the vector field $X/(1+\alpha(X))$, provided α is such that the denominator is positive.

Proposition 10.3.18 (Cohomology class)**.** *If α and β are cohomologous closed 1-forms with* $1+\alpha(X)>0$ *and* $1+\beta(X)>0$ *then the associated canonical time-changes of X are smoothly conjugate.*

[5] This depends on either the dimension or bunching information

Proof. Writing $\beta=\alpha+df$ with smooth f lets us use Proposition 1.3.19:

$$\frac{X}{1+\beta(X)}=\frac{X}{1+\alpha(X)+df(X)}=\frac{\frac{X}{1+\alpha(X)}}{1+df(\frac{X}{1+\alpha(X)})}=\Big(\frac{X}{1+\alpha(X)}\Big)_f. \qquad \square$$

In the context of Definition 5.1.1, the *canonical 1-form* is the invariant 1-form A associated to Φ by

$$A(X)=1,\quad A(E^u\oplus E^s)=0.$$

Being a contact Anosov flow is equivalent to the canonical 1-form being smooth and nondegenerate ($A\wedge dA$ is a volume). Picking up on Definition 10.3.17 and Proposition 10.3.18 we have:

Proposition 10.3.19 (Regularity)**.** *Suppose X_0 generates an Anosov flow, and α is a closed 1-form such that* $1+\alpha(X_0)>0$. *If A_0 denotes the canonical form for X_0 then $A\coloneqq A_0+\alpha$ is the canonical form for $X\coloneqq\frac{X_0}{1+\alpha(X_0)}$.*

Remark 10.3.20. In particular, this shows that canonical time-changes with smooth closed forms do not affect the regularity of the canonical form.

Proof. Two invariant 1-forms for an Anosov flow are proportional because both are constant on X, and a continuous invariant 1-form that vanishes on X is trivial (because an invariant 1-form A is trivial on E^s—and likewise on E^u—by invariance: $A(v)=A(\underbrace{\varphi^t(v)}_{\xrightarrow[t\to\infty]{}0})\to 0$ for $v\in E^s$).

Since α is closed we have $dA=dA_0$. Also $A(X)=\dfrac{A_0(X_0)+\alpha(X_0)}{1+\alpha(X_0)}=1$, which implies that $\mathscr{L}_XA=0$, that is, A is X-invariant and hence proportional to the canonical 1-form of X—and indeed equal to it since $A(X)\equiv 1$. $\square$

Looking back at Theorem 10.3.10, we note a rather striking feature: The regularity assumption is imposed on the *weak* foliations, which are unaffected by time-changes, while the conclusion gives a conjugacy, which rigidly fixes the longitudinal behavior. Furthermore, nothing else in the assumption gives any indication that there is lngitudinal control at all. This underscores the subtlety of these kinds of results.[6]

[6] In this regard it is particularly impressive, that a rigidity theorem along the lines of that by Benoist–Foulon–Labourie is possible for transversely symplectic (rather than contact) flows, in which there is no control of timing. Without defining the notions involved, we state this recent result: a topologically mixing transversely symplectic Anosov flow whose weak stable and weak unstable subbundles are C^∞ and whose Hamenstädt metrics are sub-Riemannian is up to finite covers and a constant change of time scale C^∞ conjugate (not just orbit-equivalent!) to the geodesic flow of a locally symmetric space [**113**].

It turns out that volume-preservation is essential not only for the arguments, but for the conclusion. This is demonstrated by a construction of Ghys, which introduces a few interesting ideas.

Example 10.3.21 (The Ghys quasi-Fuchsian flows [**128**])**.** That time-changes preserve the Anosov property (Theorem 5.1.16) and the weak foliation motivates thinking about an Anosov flow as a 1-dimensional foliation equipped with suitable complementary foliations. On a 3-manifold, this would be a pair of foliations with 2-dimensional leaves, and in the present context, smooth.

To "hybridize" geodesic flows of a given surface Σ, replace the unit tangent bundle (which depends on a choice of Riemannian metric) by the *homogeneous bundle* $H\Sigma$ of half-lines in $T_x\Sigma$ for each $x \in \Sigma$. For any one Riemannian metric this is naturally identified with the unit tangent bundle, but it allows to consider different geodesic flows on the same manifold $H\Sigma$. Specifically, for any Riemannian metric g on Σ denote by W_g^s and W_g^u the weak-stable and weak-unstable foliations of the geodesic flow on $H\Sigma$. If now 2 smooth such Riemannian metrics g_1, g_2 of curvature -1 are sufficiently C^3 close, then the hybrid pair $W_{g_1}^s$ and $W_{g_2}^u$ is transverse, and the intersection is close to the orbits of either geodesic flow, with any smooth choice of parametrization defining an Anosov flow on this 3-manifold with smooth weak foliations (which is not conjugate to any such geodesic flow).[7] Volume-preservation is the missing ingredient here.

In closing, we also note something of interest with respect to Conjecture 10.4.3 below:

Theorem 10.3.22 ([**50**])**.** *Applying a canonical time-change to the geodesic flow of a negatively curved locally symmetric space does not change the Liouville entropy, but for small enough* ϵ*, the time change* $X \mapsto X/(1+\epsilon\alpha(X))$ *(combined with a volume renormalization) increases the topological entropy.*

4. Entropy and Lyapunov exponents

Next, we turn to entropy-rigidity. In the variational principle for entropy it is natural to expect that the inequality is "generically" strict. This is trivially the case in hyperbolic flows if one considers varying the invariant measure, because the measure of maximal entropy is unique, so the inequality is always strict for other measures. A more interesting question arises for hyperbolic flows preserving a smooth measure. In that case one may ask whether that smooth invariant measure has maximal entropy, or equivalently, one can ask whether the Bowen–Margulis

[7] Ghys calls these flows quasi-Fuchsian because they are parametrized by a *pair* of metrics rather than a single one, in which case "Fuchsian" would recall the underlying Fuchsian (fundamental) group.

measure is a smooth measure, that is, absolutely continuous with respect to an ambient volume. Even without having exhibited the Bowen–Margulis measure in many cases, and without having computed the entropy of many dynamical systems, one would expect this coincidence to be rare because it seems to impose rather special symmetries. If the Bowen measure is absolutely continuous, then periodic orbits are equidistributed with respect to volume, that is, geometrically. At the same time, absolute continuity of the Margulis measure implies strong homothetic behavior along the invariant foliations. We present a result of this very type that led to an important conjecture along these lines.

Theorem 10.4.1 (Katok Entropy-Rigidity [**179**])**.** *If the Bowen–Margulis measure for the geodesic flow of a negatively curved metric on a compact surface is absolutely continuous, then the curvature is constant.*

We outline a proof of Theorem 10.4.2, which implies this. (See also Theorem 10.4.9.)

For a negatively curved C^∞ Riemannian metric g on a surface M, there is a unique C^∞ function $\rho\colon M \to (0,\infty)$ such that $g_0 \coloneqq \rho g$ has constant curvature and the same volume as g, that is, $\int \rho \, d\mu_g = 1$, where μ_g is the Riemannian measure on (M,g), which is the projection of the Liouville measure λ_g on the g-unit tangent bundle S_gM of M. The Cauchy–Schwarz inequality implies that the *conformal coefficient*

$$\rho_g \coloneqq \int \sqrt{\rho}\, d\mu_g \le 1$$

with equality if and only if ρ, and hence the curvature of g, is constant.

Theorem 10.4.2 ([**179**])**.** *If g is a negatively curved C^∞ Riemannian metric on a surface, then*

$$h^g_{\text{vol}}/\rho_g \le h^{g_0}_{\text{top}} \le \rho_g h^g_{\text{top}},$$

where h^g_{vol} is the Liouville entropy of the geodesic flow of g, and h_{top} is the topological entropy of the respective geodesic flow. (Either inequality is strict unless g has constant curvature.)

This implies Theorem 10.4.1: If $h^g_{\text{vol}} = h^g_{\text{top}}$, then $\rho_g^2 \ge 1$, hence, as noted above, $\rho \equiv$ const., and g is constantly curved.

We note that in Theorem 10.4.2 the existence of ρ and the left inequality do not use the negative-curvature assumption and hold for any smooth metric g. The right inequality holds when g has no focal points.

Proof outline for Theorem 10.4.2 [180]. We first use that the Liouville measure for g_0 is the Bowen measure, that is, closed geodesics are equidistributed. Specifically, given a continuous function $f\colon M \to \mathbb{R}$ and $\epsilon > 0$ there is a $T > 0$ such that for $1-\epsilon\%$ of the geodesics of length at most T, the average along each of f is within ϵ of the space average $\int_{S_{g_0}M} f\,d\lambda_{g_0}$. We restate and apply this with $f=\rho^{-1/2}$: For $\epsilon>0$ there is a $T>0$ and a set E of cardinality at least $(1-\epsilon)P_T(g_0)$ of closed g_0-geodesics of length at most T such that the average of $\rho^{-1/2}$ along each is within ϵ of the λ_{g_0}-average

$$\int_{S_{g_0}M} \rho^{-1/2}\,d\lambda_{g_0} = \int_M \rho^{-1/2}\,d\mu_{g_0} = \int_M \sqrt{\rho}/\rho\,d\mu_{g_0} = \int_M \sqrt{\rho}\,d\mu_g = \rho_g$$

of ρ. Here, $P_T(g_0)$ is the number of closed g_0-geodesics of length at most T (compare with Definition 4.2.1).

This helps because the g-length of any $\gamma_0\in E$ is its g_0-length times the average of $\rho^{-1/2}$ along γ_0 and hence less than $T(\rho_g+\epsilon)$. Also, each such γ_0 represents a unique free homotopy class, which in turn contains a unique closed g-geodesic which, being the g-shortest curve in that class has g-length less than the g-length of γ_0 and hence less than $T(\rho_g+\epsilon)$. This shows that

$$P_{T(\rho_g+\epsilon)}(g) \ge (1-\epsilon)P_T(g_0),$$

with $P_\bullet(g)$ defined analogously to $P_\bullet(g_0)$ above.

Since these are pairwise separated (due to expansivity, but here simply because they are in pairwise distinct homotopy classes), this determines entropy:

$$h^g_{\text{top}} \ge \overline{\lim_{T\to\infty}} \frac{\log P_{T(\rho_g+\epsilon)}(g)}{T(\rho_g+\epsilon)} \ge \overline{\lim_{T\to\infty}} \frac{\log(1-\epsilon)P_T(g_0)}{T(\rho_g+\epsilon)} = \frac{h^{g_0}_{\text{top}}}{\rho_g+\epsilon}.$$

Since ϵ was arbitrary, we obtain $h^{g_0}_{\text{top}} \le \rho_g h^g_{\text{top}}$.

To prove the other inequality we imitate this argument with g and g_0 interchanged, and in order to make the Liouville entropy appear, we use its orbit-counting definition (Definition 4.1.2). □

The 1982 paper containing Theorem 10.4.2 [**179**] includes computations of the Liouville and topological entropies for the geodesics of all compact negatively curved locally symmetric spaces, which shows that in all these cases, both entropies coincide. Katok then remarked “It looks like a reasonable conjecture that those are the only cases of manifolds of negative curvature for which the Liouville measure has maximal entropy.”

Conjecture 10.4.3 (Katok Entropy Conjecture)**.** *If the topological and Liouville entropies of the geodesic flow of a negatively curved manifold coincide, then the manifold is a locally symmetric space.*

While we do not seem at this time to be much closer to settling it than decades ago, this conjecture engendered a great amount of mathematical progress (including Theorem 5.4.26), and some of the major results established in the interim are well worth stating. Theorem 10.4.2 suggests a slightly different line of inquiry. As we noted parenthetically, both inequalities in Theorem 10.4.2 are strict unless the curvature is constant, and this gives rise to questions in arbitrary dimension: With normalized total volume, is the Liouville entropy uniquely maximized by locally symmetric metrics? It is not—there are same-volume perturbations of compact locally symmetric spaces of negative curvature with larger Liouville entropy [**117**]. However, the inequality and rigidity for topological entropy does indeed generalize, and this is a special case of a major result of Besson, Courtois and Gallot.

Theorem 10.4.4 (Besson–Courtois–Gallot [**44, 45**])**.** *Minimizers of topological entropy with prescribed volume are locally symmetric spaces.*

Remark 10.4.5. The minimization is among homotopy-equivalent negatively curved Riemannian manifolds, and Besson–Courtois–Gallot conjectured (correctly) that one can broaden this to Finsler manifolds [**49**]; it has since also been pushed to Hilbert geometries [**2**].

Another consequence of their main result is worth stating. (Yet another is Mostow Rigidity, which is outside the scope of this book.)

Theorem 10.4.6. *A Riemannian manifold whose geodesic flow is C^1 conjugate to that of a compact locally symmetric manifold M is isometric to M.*

This is not just remarkable in itself but a booster of other rigidity results whose conclusion is the smooth conjugacy, and which are hereby amplified to yield isometry. Here is an instance (with the amplification built in):

Theorem 10.4.7 (Foulon–Labourie [**120**])**.** *A compact negatively curved Riemannian manifold whose horospheres have constant mean curvature is locally symmetric.*

In fact, together with a result of Ledrappier, this leads to a rigidity statement that has a rather similar flavor to the Katok Entropy Rigidity Conjecture in that the coincidence of the Bowen–Margulis measure with another preferred measure implies rigidity:

Theorem 10.4.8 (Ledrappier–Foulon–Labourie–Besson–Courtois–Gallot [**196**])**.** *A compact orientable Riemannian manifold whose Bowen–Margulis measure and harmonic measure coincide is locally symmetric.*

We do not define harmonic measure here but merely point out that it is connected with the asymptotics of Brownian motion on the manifold. Ledrappier showed that the coincidence of these measures implies asymptotic harmonicity, that is, that horospheres have constant mean curvature, from which the Foulon–Labourie result deduces smooth conjugacy to the geodesic flow of a locally symmetric space, and Theorem 10.4.6 then produces an isometry.

There is an altogether different direction into which one could widen the purview of Theorem 10.4.1: instead of looking for analogs in higher dimension one can ask whether Anosov flows for which the Bowen–Margulis measure is absolutely continuous have to be smoothly conjugate to the geodesic flow of a surface of constant curvature. Since the conclusion implies that the flow is a contact flow, the following is a natural answer to the question in dimension 3:

Theorem 10.4.9 (Foulon [**118**])**.** *A C^∞ contact Anosov flow on a closed 3-manifold whose measure-theoretic entropy is equal to its topological entropy, is, up to finite covers, C^∞-conjugate to the geodesic flow of a closed surface of constant negative curvature.*

It should be noted here that even topological conjugacy is a highly nontrivial assertion: the entropy assumption here determines even the topological nature of the flow, whereas in the preceding results, this was explicitly given in advance.

The entropy of a hyperbolic flow is intimately related to the contraction and expansion rates in the flow (Remark 8.4.11). Accordingly, we now indicate ways in which information about these gives rise to rigidity phenomena. A first instance is that Theorem 10.2.7 (C^1 conjugacies are smooth) is a corollary of a stronger result: instead of assuming that the conjugacy is C^1, it suffices to assume the coincidence of contraction and expansion rates: If two transitive Anosov flows on a compact 3-manifold are topologically conjugate and the contraction and expansion rates of corresponding periodic orbits agree, then the conjugating homeomorphism is analytic [**206**, Theorem 1.5.],[**205**, Theorem 1.1]. One can, however, go much further and obtain rigidity from such priodic data without first assuming a topological (time-preserving) conjugacy. Geodesic flows of locally symmetric spaces (Section 2.5) provide the most astonishing instance. To make this explicit, define the *Lyapunov spectrum* of a periodic orbit as follows.

Definition 10.4.10 (Lyapunov spectrum)**.** If Φ is a flow on an n-dimensional manifold and $\varphi^T(p) = p$, then let $\vec{\lambda}(p)$ be the n-vector whose entries are $\frac{1}{T}\log|\lambda_i|$ in increasing order, where the λ_i are the eigenvalues of $D\varphi^T{}_{|_p}$ repeated according to their respective multiplicities.[8]

For the geodesic flow on a locally symmetric space (X, g_0), this is independent of p and hence denoted by $\vec{\lambda}(X, g_0)$.

Theorem 10.4.11 (Butler local Lyapunov spectrum rigidity [**78**, **79**])**.** *A closed negatively curved locally symmetric space* (X, g_0) *with* $\dim X \geq 3$ *has a neighborhood* U

[8] These are the Lyapunov exponents of p.

such that each $Y \in U$ is isometric to (X, cg_0) for some $c > 0$ if[9] *for each periodic point p of the geodesic flow for Y there is a $\xi(p) > 0$ such that $\vec{\lambda}(p) = \xi(p)\vec{\lambda}(X)$.*

A posteriori $\xi(\cdot)$ is constant, but this is not assumed to establish the isometry. When (X, g_0) has constant curvature, then U can be taken as the space of metrics with strictly $1/4$-pinched negative curvature. Otherwise it is a suitable C^2-neighborhood of g_0.

5. Godbillon–Vey invariants

The purpose of this section is to introduce a tool from foliation theory that extends an invariant which was itself of interest in the context of the result from Remark 10.3.11:

Theorem 10.5.1 ([**165**])**.** *Negatively curved surfaces with C^2 horospheric foliations are constantly curved.*

Our proof of this will introduce the Bott–Kanai connection and also use the Godbillon–Vey invariants we present in this section.

To indicate further how these can be used to produce rigidity results in straightforward ways, we also prove another geometric-rigidity result, which follows from Theorem 10.4.1:

Theorem 10.5.2. *Suppose the geodesic flows φ^t and ψ^t of Riemannian surfaces M and S, respectively, are topologically conjugate. If S has constant curvature -1, then so does M.*

While the assumptions imply that the conjugacy is smooth (Theorem 10.2.7), the point is that smooth conjugacy controls the geometry, and that this is easy with the Godbillon–Vey invariants introduced here).

We now introduce Godbillon–Vey invariants for suitable foliations in contact 3-manifolds, study their interaction with the canonical flow associated to the contact form, explore consequences of this flow being Anosov, compute the top invariant among these for geodesic flows, and provide the main properties that underlie the rigidity results.

Specifically, if (M, A) is a contact 3-manifold and $\mathscr{F}$ a C^2 maximal isotropic foliation, we define *Godbillon–Vey invariants GV_i* for $i = 0, 1, 2$ in Definition 10.5.11. We lead up to that definition with work that introduces notions needed for the definition itself as well as for the proof that these are well-defined (Proposition 10.5.12).

[9] And obviously only if

GV_0 is (by definition) the volume of the manifold, and for a contact Anosov flow and the associated weak-stable foliation, GV_1 is the Liouville entropy (Proposition 10.5.15).

For geodesic flows of surfaces, we compute GV_2 in Proposition 10.5.18 (the Mitsumatsu formula), and the first rigidity result is the computation of the Godbillon–Vey invariants for contact Anosov flows with absolutely continuous Margulis measure (Theorem 10.5.19), and its application to geodesic flows (Theorem 10.5.21) then follows from (merely!) the Cauchy–Schwarz inequality. This then implies both rigidity results above.

Remark 10.5.3. What we construct here as Godbillon–Vey invariants extend the classical Godbillon–Vey class and invariant, but we do not build on the classical construction. Here is a condensed outline how that goes. If ω is a completely integrable nonsingular 1-form, then there is a 1-form η such that $d\omega = \omega \wedge \eta$ (Frobenius Theorem), and $0 = dd\omega = d(\omega \wedge \eta) = \omega \wedge d\eta$, so there is a 1-form ξ with $d\eta = \omega \wedge \xi$, hence $\eta \wedge d\eta$ is closed, and its de Rham cohomology class is independent of such choice of η—another choice must be of the form $\eta' = \eta + u\omega$ for a function u, and then $\eta' \wedge d\eta' = \eta \wedge d\eta + d(ud\omega)$. Indeed, this depends only on the codimension-one foliation F defined by complete integrablility of ω, for, any ω' defining the same foliation is a scalar multiple of ω. The cohomologycohomologous class of $\eta \wedge d\eta$ is called the Godbillon–Vey class of F, and if $\dim M = 3$, then $\int \eta \wedge d\eta$ is called the Godbillon–Vey invariant of F; it is a characteristic class, depends only on the foliated cobordism class of (M, F), is nontrivial, and varies continuously and nontrivially with F. By contrast, we show that the combination of a contact structure and an orientable maximal isotropic foliation gives rise to a *sequence* of what we call Godbillon–Vey invariants. The point is that as a *sequence* they are of interest for rigidity results.

Definition 10.5.4 (Isotropic, normal bundle)**.** Let M be a contact 3-manifold. A subspace $V \subset T_xM$ is said to be *isotropic* if $dA_x\restriction_V = 0$ and *maximal isotropic* if it furthermore has dimension 2. A subbundle is said to be (maximal) isotropic if it is so at each point, and a foliation is (maximal) isotropic if its tangent bundle is so. If M is a smooth manifold and F is a subbundle of TM, then we define the *normal bundle*

$$\mathcal{N}(F) := \{\omega \in T^*M \mid \iota_\xi \omega = 0 \text{ whenever } \xi \in F\}.$$

Lemma 10.5.5. *If F is integrable and $\omega \in \mathcal{N}(F)$, then $d\omega\restriction_F = 0$.*

Proof. If Z_1, Z_2 are tangent to F, then $0 \equiv \omega(Z_1) \equiv \omega(Z_2) \equiv \omega([Z_1, Z_2])$, so

$$d\omega(Z_1, Z_2) = \mathcal{L}_{Z_1}\omega(Z_2) - \mathcal{L}_{Z_2}\omega(Z_1) + \omega([Z_1, Z_2]) = 0 + 0 + 0. \qquad \square$$

We define the Godbillon–Vey invariants in terms of a 1-form transverse to the maximal isotropic foliation $\mathscr{F}$ (with tangent bundle F) in question. Specifically, since we assume F to be orientable, we henceforth fix an everywhere nonzero $\alpha \in \mathscr{N}(\mathscr{F})$. We will assume $\alpha \in C^2$ (specifically in the proof of Lemma 10.5.9), and hence that $\mathscr{F} \in C^2$. From here through Lemma 10.5.10 we study a 1-form β that is the key ingredient to defining our Godbillon–Vey invariants.

Proposition 10.5.6. *If α is C^1, then $d\alpha = \beta \wedge \alpha$ for a 1-form β.*

Proof. $\mathscr{N}(\mathscr{F})$ is 1-dimensional and contains both α and $\iota_Z d\alpha$ for any $Z \in F$ (Lemma 10.5.5), so the fact that α vanishes nowhere yields a $\beta(Z)$ for which $\iota_Z d\alpha = \beta(Z)\alpha$, and β is a 1-form on F. Now consider an extension of β to any 1-form. Then $\beta \wedge \alpha$ and $d\alpha$ can be evaluated on any pair of vectors by decomposing each vector with respect to a basis that contains a basis of F. For both $d\alpha$ and $\beta \wedge \alpha$ the only nonzero expressions that thus arise are those that include precisely one vector in F, and we just showed that $d\alpha = \beta \wedge \alpha$ for such pairs. □

Remark 10.5.7. That β is uniquely defined on $\mathscr{F}$ means that it is well-defined modulo $\mathscr{N}(\mathscr{F})$, that is, we uniquely defined $[\beta] := \{\beta + \omega \mid \omega \in \mathscr{N}(\mathscr{F})\}$.

Proposition 10.5.8. *The cohomology class $[\beta]$ is well-defined independently of the choice of α: $\alpha' = e^f \alpha$ with $f \colon M \to \mathbb{R}$ produces $\beta' = \beta + df$.*

Proof. $\beta' \wedge \alpha' = d\alpha' = d(e^f \alpha) = de^f \wedge \alpha + e^f d\alpha = e^f df \wedge \alpha + e^f \beta \wedge \alpha = (df + \beta) \wedge e^f \alpha = (df + \beta) \wedge \alpha'$. □

Accordingly, we write $[[\beta]] := \big\{\beta + df + \omega \mid f \colon M \to \mathbb{R}, \quad \omega \in \mathscr{N}(\mathscr{F})\big\}$.

Lemma 10.5.9. *$d\beta \wedge \alpha = 0$ and $d\beta\restriction_{\mathscr{F}} = 0$.*

Proof. $0 = dd\alpha = d\beta \wedge \alpha + \beta \wedge d\alpha = d\beta \wedge \alpha + \beta \wedge \beta \wedge \alpha = d\beta \wedge \alpha$. If $Z_1, Z_2 \in \mathscr{F}$, then $0 = \iota_{Z_1,Z_2} 0 = \iota_{Z_1,Z_2} d\beta \wedge \alpha = d\beta(Z_1, Z_2)\alpha$ because $\alpha \in \mathscr{N}(\mathscr{F})$. Then $d\beta(Z_1, Z_2) = 0$ because α is nowhere zero. □

Lemmas 10.5.5 and 10.5.9 serve to give Proposition 10.5.12 via:

Lemma 10.5.10. *$\omega \wedge (d\beta)^i \wedge d\omega^{p-i} \wedge dA^{1-p} = 0$ for $0 \le i \le p \le 1$ and $\omega \in \mathscr{N}(\mathscr{F})$.*

Proof. Evaluating this form on 3 linearly independent vectors and decomposing these with respect to a basis that contains a basis for $\mathscr{F}$ gives a linear combination of expressions each of which contains at least 2 elements of $\mathscr{F}$. Inserting a vector from $\mathscr{F}$ into $\omega \in \mathscr{N}(\mathscr{F})$ gives 0, and if 2 elements of $\mathscr{F}$ are inserted into $(d\beta)^i \wedge d\omega^{p-i} \wedge dA^{1-p}$, we get 0 because one gets 0 whenever more than one such vector is inserted into $d\beta$ (Lemma 10.5.9), $d\omega$ (Lemma 10.5.5), or dA. □

Since $[[\beta]]$ is intrinsically defined, we can now define the Godbillon–Vey invariants.

Definition 10.5.11 (Godbillon–Vey invariants)**.** If (M, A) is a contact 3-manifold and $\mathscr{F}$ a C^2 maximal isotropic foliation, define the *Godbillon–Vey invariants* by

$$\begin{aligned} GV_0 &= \int_M A \wedge dA =: \operatorname{vol}_A(M) \quad \text{(the contact volume)} \\ GV_1 &= \int_M \beta \wedge dA \\ GV_2 &= \int_M \beta \wedge d\beta \end{aligned}$$

Proposition 10.5.12. *The Godbillon–Vey invariants are well-defined.*

Proof. We need to show that $GV_{p+1} = \int_M \beta \wedge d\beta^p \wedge dA^{1-p}$ is constant on $[[\beta]]$, that is, that replacing β by $\beta + df + \omega$ and therefore $d\beta$ by $d\beta + d\omega$ has no effect. Replacing β by $\beta + df$ makes no difference:

$$\begin{aligned} &\int_M (\beta + df) \wedge d(\beta + df)^p \wedge dA^{1-p} - \int_M \beta \wedge d\beta^p \wedge dA^{1-p} \\ &\qquad = \int_M df \wedge d\beta^p \wedge dA^{1-p} = \int_M d\big(f \cdot d\beta^p \wedge dA^{1-p}\big) = 0. \end{aligned}$$

To see the effect of adding ω, expand $d(\beta + \omega)^p$:

$$\begin{aligned} (\beta + \omega) \wedge d(\beta + \omega)^p \wedge dA^{1-p} &= \beta \wedge d(\beta + \omega)^p \wedge dA^{1-p} + \underbrace{\omega \wedge d(\beta + \omega)^p \wedge dA^{1-p}}_{=0 \text{ by Lemma 10.5.10}} \\ &= \beta \wedge d\beta^p \wedge dA^{1-p} + \sum_{i=1}^{p} c_i \beta \wedge d\beta^{p-i} \wedge d\omega^i \wedge dA^{1-p}, \end{aligned}$$

so

$$\begin{aligned} &\int_M (\beta + \omega) \wedge d(\beta + \omega)^p \wedge dA^{1-p} - \int_M \beta \wedge d\beta^p \wedge dA^{1-p} \\ &\qquad = \sum_{i=1}^{p} c_i \underbrace{\int_M \beta \wedge d\beta^{p-i} \wedge d\omega^i \wedge dA^{1-p}}_{= \underbrace{\int_M d(\omega \wedge \beta \wedge d\beta^{p-i} \wedge d\omega^{i-1} \wedge dA^{1-p})}_{=0} - \int_M \underbrace{\omega \wedge d\beta^{p-i+1} \wedge d\omega^{i-1} \wedge dA^{1-p}}_{=0 \text{ by Lemma 10.5.10}}} = 0 \quad \square \end{aligned}$$

While GV_0 is volume, we now identify GV_1.

Proposition 10.5.13. *$GV_1 = \int \beta(X) A \wedge dA$, where X is the* Reeb field *of A defined uniquely by $\iota_X A = 1$ and $\iota_X dA = 0$.*

Proof. $\iota_X dA = 0$ implies that

- X is tangent to $\mathscr{F}$, so $\mathscr{F}$ is invariant under the flow generated by X,
- by duality there are a vector field η and a function λ with $\beta = \lambda A + \iota_\eta dA$,
- inserting X gives $\lambda = \beta(X)$, and
- the 3-form $\iota_\eta dA^2$ vanishes whenever X is in any slot.

Thus $\beta \wedge dA = \beta(X) A \wedge dA + \iota_\eta dA \wedge dA = \beta(X) A \wedge dA^m$. □

Now we return to dynamics, specifically the Reeb flow of A, which is the flow Φ generated by the Reeb vector field X of A, and from now we assume that this is an Anosov flow. Then $\mathscr{F} \coloneqq \mathbb{R}X \oplus E^-$ is integrable to a continuous foliation $\mathscr{F}$ with smooth leaves, the weak-stable foliation, which is maximal isotropic:

Lemma 10.5.14. $dA\restriction_{\mathscr{F}} = 0$, *that is,* $dA(Z_1, Z_2) = 0$ *if* $Z_1, Z_2 \in \mathbb{R}X \oplus E^-$.

Proof. $\iota_X dA = 0$ reduces this to the case $Z_1, Z_2 \in E^-$, where

$$dA(Z_1, Z_2) = dA(d\varphi^t(Z_1), d\varphi^t(Z_2)) \xrightarrow[t\to+\infty]{} 0$$

since A, hence dA, is φ^t-invariant and $\|d\varphi^t(Z_i)\| \xrightarrow[t\to+\infty]{} 0$. □

Proposition 10.5.15. *If* $\mathscr{F}$ *is the weak-stable foliation of a contact Anosov flow, then* $GV_1 = h_{\mathrm{vol}} \operatorname{vol}_A(M)$*, where* h_{vol} *is Liouville entropy.*

Proof. Choose $\beta = 0$ on E^+. Then $\mathscr{L}_X \alpha = \iota_X d\alpha = \beta(X)\alpha$, that is, $\beta(X)$ is the infinitesimal relative change of the unstable volume under the flow. Rescale A so $\operatorname{vol}_A(M) = 1$. Then the time average of $\beta(X)$, hence by ergodicity (Theorem 8.1.27) its space average GV_1, is then the average unstable infinitesimal volume-expansion, and by the Pesin Entropy Formula (Remark 8.4.11), this is h_{vol}. □

We next compute GV_2 for geodesic flows of surfaces. Denote the standard vertical vector field by V. Then $H \coloneqq [V, X]$ and X are horizontal, and

$$1 = A \wedge dA(X, V, H) = A(X) dA(V, H) = dA(V, H). \tag{10.5.1}$$

If K is the curvature, then the structural equations are

$$[X, V] = -H, \qquad [H, V] = X, \qquad [X, H] = KV.$$

If the invariant line bundle $F \cap \ker A$ is spanned by the vector field

$$\xi = uV + H,$$

then comparing coefficients in $(\dot u + K)V - uH = [X, \xi] = f\xi = fuV + fH$ implies $f = -u$ and $-u^2 = fu = \dot u + K$, which gives the *Riccati equation* $\dot u + u^2 + K = 0$.

Lemma 10.5.16. *If we choose* $\alpha = \iota_\xi dA$*, then* $\alpha(H) = u$*,* $\alpha(V) = -1$*,* $\alpha(X) = 0 = \alpha(\xi)$*.*

Proof. $\underbrace{\alpha(V)}_{=dA(\xi,V)=dA(uV+H,V)} = dA(H, V) \underbrace{= -1}_{(10.5.1)}$, $\alpha(X) = 0 = \alpha(\xi) = \alpha(uV + H) = -u + \alpha(H)$. □

Lemma 10.5.17. *If we choose $\beta(V)=0$, then $\beta(X)=-u$, $\beta(H)=\mathscr{L}_V u$.*

Proof. $\beta(X)\alpha(H)=d\alpha(X,H)=\underbrace{\mathscr{L}_X\alpha(H)}_{=\mathscr{L}_X u=\dot{u}}-\mathscr{L}_H\underbrace{\alpha(X)}_{\equiv 0}+\underbrace{\alpha([H,X])}_{=-K\alpha(V)=K}=-u^2=-u\alpha(H)$ (Riccati equation) and $\underbrace{\beta(\xi)\alpha(H)}_{=d\alpha(\xi,H)}=\underbrace{\mathscr{L}_\xi\alpha(H)}_{=\mathscr{L}_\xi u=u\mathscr{L}_V u+\mathscr{L}_H u}-\mathscr{L}_H\underbrace{\alpha(\xi)}_{\equiv 0}+\alpha(\underbrace{[H,\xi]}_{=u[H,V]+(\mathscr{L}_H u)V})=\underbrace{\alpha(H)}_{=u}(\mathscr{L}_V u)$. □

Proposition 10.5.18 (Mitsumatsu Formula)**.** *$GV_2=\int_M u^2+3(\mathscr{L}_V u)^2\ A\wedge dA$ for maximal isotropic foliations invariant by geodesic flows of surfaces. The* Mitsumatsu defect $\int_M 3(\mathscr{L}_V u)^2\ A\wedge dA$ *is the deviation of GV_2 from its value for constant curvature.*

Proof. We show $\int u^2+3(\mathscr{L}_V u)^2=\int\lambda$ with $\lambda\colon M\to\mathbb{R}$ such that $\beta\wedge d\beta=\lambda A\wedge dA$, so

$$\lambda\underbrace{A\wedge dA(X,V,H)}_{=1}=\beta\wedge d\beta(X,V,H)=\beta(X)d\beta(V,H)+\beta(H)d\beta(X,V)+\underbrace{\beta(V)}_{=0}d\beta(H,X).$$

Here

$$\begin{aligned} d\beta(V,H)&=\mathscr{L}_V\underbrace{\beta(H)}_{=\mathscr{L}_V u}-\mathscr{L}_H\underbrace{\beta(V)}_{\equiv 0}-\underbrace{\beta([V,H])}_{=\beta(-X)=u}=\mathscr{L}_V^2 u-u,\\ d\beta(X,V)&=\mathscr{L}_X\underbrace{\beta(V)}_{\equiv 0}-\mathscr{L}_V\underbrace{\beta(X)}_{=-u}-\underbrace{\beta([X,V])}_{=\beta(-H)=-\mathscr{L}_V u}=2\mathscr{L}_V u, \end{aligned}$$

so $\lambda=\underbrace{\beta(X)}_{=-u}\underbrace{d\beta(V,H)}_{=\mathscr{L}_V^2 u-u}+\underbrace{\beta(H)}_{=\mathscr{L}_V u}\underbrace{d\beta(X,V)}_{=2\mathscr{L}_V u}=u^2+2(\mathscr{L}_V u)^2-u\mathscr{L}_V^2 u$ by (10.5.1). It remains to show that $\int(\mathscr{L}_V u)^2+u\mathscr{L}_V^2 u=0$ (integration by parts):
$\underbrace{(A\wedge d\iota_V dA)}_{=-d(A\wedge\iota_V dA)=d\iota_V(A\wedge dA)=\mathscr{L}_V A\wedge dA}(X,V,H)=d(\iota_V dA)(V,H)=\underbrace{-\iota_V dA([V,H])}_{=dA(-V,[V,H])}=dA(V,X)=0$ implies

$$0=\int_M\mathscr{L}_V\big(u\mathscr{L}_V u\ A\wedge dA\big)=\int_M\mathscr{L}_V u\,\mathscr{L}_V u\ A\wedge dA+\int_M u\mathscr{L}_V\mathscr{L}_V u\ A\wedge dA.\qquad\square$$

Lastly, we compute specific values for the Godbillon–Vey invariants in the special case the geometric-rigidity results focus on.

Theorem 10.5.19. *$GV_i=h^i\operatorname{vol}_A(M)$ for contact Anosov flows with absolutely continuous Margulis measure, where h is topological entropy.*

Remark 10.5.20. This applies to geodesic flows of negatively curved locally symmetric spaces, in particular, of surfaces with constant negative curvature.

Proof. The (un)stable conditionals of the Margulis measure are volumes and scale with h (Lemma 8.6.10). Therefore $h\alpha = \mathscr{L}_X\alpha = \iota_X d\alpha + d\iota_X\alpha = \beta(X)\alpha$ (since $\iota_X\alpha \equiv 0$), so $\beta(X) \equiv h$, hence $\beta = hA + \iota_\eta dA$ (from the proof of Proposition 10.5.13), and

$$h\beta\wedge\alpha = hd\alpha = dh\alpha = d\mathscr{L}_X\alpha = \mathscr{L}_X d\alpha = \mathscr{L}_X\beta\wedge\alpha + \beta\wedge\mathscr{L}_X\alpha = \mathscr{L}_X\beta\wedge\alpha + \beta\wedge h\alpha.$$

Thus $\mathscr{L}_X\beta\wedge\alpha = 0$, hence $\mathscr{L}_X\beta(v) = 0$ for any $v \in \mathbb{R}X \oplus E^-$. Choose $\beta = 0$ on E^+, so $\beta = f(x)A$, hence $\beta = hA$. □

We now apply these invariants to geometric rigidity.

Theorem 10.5.21. *If $GV_0 = c$, $GV_1 = hc$, and $GV_2 = h^2c$ for a negatively curved Riemannian metric on a surface, then the curvature is constant, c is the volume and h the topological entropy.*

This is an immediate consequence of

Proposition 10.5.22. *For the geodesic flow of a negatively curved Riemannian metric on a surface*

$$\frac{GV_0 GV_2}{(GV_1)^2} \geq 1$$

with equality if and only if the curvature is constant.

Proof. Since Lemma 10.5.17 and Proposition 10.5.18 give

$$GV_0 = \int_M A\wedge dA,\ GV_1 = \int_M -uA\wedge dA,\ GV_2 = \int_M u^2 + 3(\mathscr{L}_V u)^2 A\wedge dA,$$

the Cauchy–Schwarz inequality

$$GV_1 = \int_M -uA\wedge dA \leq \Big(\int_M u^2 A\wedge dA\Big)^{\frac12}\Big(\int_M A\wedge dA\Big)^{\frac12} \leq (GV_2)^{\frac12}(GV_0)^{\frac12}$$

allows equality only if $u \equiv$ const,[10] hence $K = -(\dot u + u^2) = -u^2 \equiv$ const. □

Remark 10.5.23. By invoking the Godbillon–Vey invariants, Proposition 10.5.22 implicitly assumes that the invariant foliations are C^2, which by itself is known to imply constant curvature. This necessitates extending the definition to the case of $C^{1+1/2+\epsilon}$ invariant foliations for other applications. However, in both applications below, the invariant foliations are indeed C^2.

[10] and, redundantly, $\mathscr{L}_V u \equiv 0$

Proof of Theorem 10.5.2. The conjugacy F is $C^{k-\epsilon}$ when $\varphi^t \in C^k$ (Theorem 10.2.7), so the invariant foliations are $C^{k-\epsilon}$. A contact Anosov flow is the Reeb flow of a unique contact form. Thus F sends the contact form A for φ^t to that for ψ^t, and likewise for dA and the weak-unstable foliation—which is hence C^2. Thus, the Godbillon–Vey invariants match up, that is, $GV_i^M = GV_i^S$ for $i = 0, 1, 2$, so $\dfrac{GV_0^M GV_2^M}{(GV_1^M)^2} = \dfrac{GV_0^S GV_2^S}{(GV_1^S)^2} = 1$ by Proposition 10.5.22, which implies by Proposition 10.5.22 that M has constant curvature. □

Remark 10.5.24. This theorem is not contingent on defining Godbillon–Vey invariants for lower regularity because the conjugacy sends the smooth maximally isotropic foliation to a C^2 maximally isotropic foliation. The same goes for the next result, which recovers a special case of a rigidity result of Hurder and Katok via a remarkably simple proof.

Proof of Theorem 10.5.1. The C^2 splitting yields a Bott–Kanai connection.

Proposition 10.5.25. *There is a unique φ^t-invariant connection ∇ that parallelizes the geometric structure ($\nabla A = 0$, $\nabla dA = 0$, $\nabla E^\pm \subset E^\pm$) and with $\nabla_{Z^\mp} Z^\pm = p^\pm[Z^\mp, Z^\pm]$ and $\nabla_X Z^\pm = [X, Z^\pm] \pm \gamma Z^\pm$ for any sections $Z^\pm$ of $E^\pm$, where $p^\pm$ is the projection to $E^\pm$ given by the decomposition. If F is a ∇-parallel subbundle of TM then the (rank-1) bundle of volume forms on F has a natural flat connection induced by ∇.*[11]

With $F = E^+$, Proposition 10.5.25 gives a *parallel* unstable volume,[12] which is then holonomy-invariant and hence gives the conditionals of the Bowen–Margulis measure. This establishes the hypothesis of Theorem 10.5.19 (with C^2 splitting), so Theorem 10.5.21 applies. □

[11] See [**43**, Proposition 2.3 & Section 3.2, Lemma 4.1].

[12] See [**43**, Section 4.2]

Part 3

Appendices

CHAPTER 11

Appendix I: Measure-theoretic entropy of maps

1. Lebesgue spaces

To develop subtle notions in ergodic theory such as entropy, it is useful to have a suitable decomposition theory of a measure space, and this is the case for probability spaces adapted to an underlying topological structure. This is a surprisingly mild restriction, and we now develop this notion and some of the resulting properties, following [**90**], which is the definitive exposition of these topics, and with specific references to the locations for the correspondending statements in that book because that is where complete proofs are found. Since these notions are not immediately needed, readers can skip this section and refer back to it as needed.

Definition 11.1.1 (Lebesgue space)**.** A *Lebesgue space* $(X,\mathscr{A},\mu)$ is a set with a probability measure μ on a complete σ-algebra $\mathscr{A}$ that is isomorphic to $([0,1],\overline{\mathscr{B}},\lambda)$, that is, Lebesgue measure on the completion $\mathscr{B}$ of the Borel σ-algebra on the unit interval.

Remark 11.1.2 ([**90**, Lemma 15.2])**.** If $(X,\mathscr{T},\mu)$ is a Lebesgue space, then every complete σ-algebra $\mathscr{A}\subset\mathscr{T}$ is separable.

Being a Lebesgue space is a far less restrictive condition than it seems.

Definition 11.1.3. A *Polish space* is a topological space whose topology is given by a complete separable metric. A *standard Borel space* is a Borel subset of a Polish space (with the completion of the Borel σ-algebra).

Theorem 11.1.4 (Isomorphism Theorem [**90**, Theorem 13.1])**.** *If X is a standard Borel space and μ a nonatomic Borel probability measure on X, then $(X,\overline{\mathscr{B}},\mu)$ is a Lebesgue space.*

Proof. The topology of X has a countable base $\{O_i\}_{i\in\mathbb{N}}$, and

$$\varphi_1\colon X\to\{0,1\}^{\mathbb{N}},\ x\mapsto(\chi_{O_i}(x))_{i\in\mathbb{N}},\quad \varphi_2\{0,1\}^{\mathbb{N}}\to[0,1],\ (a_i)_{i\in\mathbb{N}}\mapsto\sum_i\frac{a_i}{3^i}$$

are injective and Borel, so the Borel injection $\varphi_2 \circ \varphi_1$ is an isomorphism between $(X, \overline{\mathscr{B}}, \mu)$ and $([0,1], \overline{\mathscr{B}}, (\varphi_2 \circ \varphi_1)_*(\mu))$. The latter is isomorphic to $([0,1], \overline{\mathscr{B}}, \lambda)$ via $\varphi(x) := (\varphi_2 \circ \varphi_1)_*(\mu)([0,x))$. □

One important property of Lebesgue spaces is the following.

Theorem 11.1.5 (Measurability Lemma [**90**, Proposition 13.1])**.** *If φ is a measure-preserving measurable map between Lebesgue spaces and A a measurable set with $\varphi(A) \cap \varphi(A^c) = \varnothing$, then $\varphi(A)$ is measurable. In particular, if φ is injective, then it is an isomorphism.*

A crucial notion for entropy theory is particularly well-behaved in the context of Lebesgue spaces.

Definition 11.1.6 (Measurable partition)**.** If $(X, \mathscr{T}, \mu)$ is a measure space, then a partition ξ is a piecewise disjoint cover of X; its elements are called atoms. For $x \in X$ we define $\xi(x)$ by $x \in \xi(x) \in \xi$, and we say that two partitions essentially agree if there is a null set A such that $\xi(x) \smallsetminus A = \eta(x) \smallsetminus A$ for almost all $x \in X$.[1] A partition ξ is said to be *measurable* if it is (essentially[2]) *countably defined*: There is a *basis* for ξ, that is, a countable family of $B_n \in \mathscr{T}$ that separates the elements of ξ, that is, for $C_1 \neq C_2 \in \xi$ there is an n such that $C_1 \subset B_n$ and $C_2 \cap B_n = \varnothing$ or vice versa.

Example 11.1.7. The extreme examples of measurable partitions are the trivial partition $\mathscr{N} := \{X\}$ corresponding to the trivial algebra $\mathscr{A}(\mathscr{N}) = \mathscr{N} := \{\varnothing, X\}$ and the point partition $\mathscr{E} := \{\{x\}\}_{x \in X}$ corresponding to the full algebra $\mathscr{A}(\mathscr{E}) = \mathscr{B}$.

Example 11.1.8. The orbit partition of an irrational rotation is not a measurable partition. Consider $X = S^1 = \mathbb{R}/\mathbb{Z}$, $\alpha \notin \mathbb{Q}$ and the rotation $R_\alpha \colon x \mapsto x + \alpha \pmod 1$. Let $\xi := \{\{R_\alpha^i(x) \mid i \in \mathbb{Z}\}_{x \in S^1}\}$ be the partition of S^1 into the orbits of R_α. Each partition element is countable, hence measurable, but the partition is not. This can be seen by noting that the conditional measures on partition elements are R_α-invariant by uniqueness; since each partition element is a copy of $\mathbb{Z}$ with R_α acting by translation, the conditional measures are translation-invariant probability measures on $\mathbb{Z}$, which is impossible because all integers must have the same measure.[3]

Remark 11.1.9. The reasoning in the preceding example in fact illustrates that the orbit partition of an ergodic transformation always has a trivial factor space.

[1] Equivalently $\xi = \eta \pmod 0$ if for any element $C \in \xi$ of positive measure one can find an element $D \in \eta$ such that $\mu(C \triangle D) = 0$. Here $\triangle$ means symmetric difference: $A \triangle B := (A \cup B) \smallsetminus (A \cap B) = (A \smallsetminus B) \cup (B \smallsetminus A)$..

[2] that is, essentially agrees with a partition with the following property

[3] Alternatively note that by unique ergodicity of R_α the algebra generated by ξ consists of R_α-invariant sets and is hence the trivial algebra $\mathscr{N}$, which engenders the trivial partition, rather than the orbit partition.

The utility of the notion of a Lebesgue space is the correspondence between measurable partitions and various other natural constructs for ergodic theory:

Theorem 11.1.10 (Rokhlin Correspondence [**90**, Theorem 15.1])**.** *If $(X,\mathbb{T},\mu)$ is a Lebesgue space then there are bijections between*

- *measurable partitions of X,*
- *complete σ-algebras in $\mathbb{T}$,*
- *closed subalgebras of $L^0(X,\mathbb{T},\mu)$,*
- *factors of $(X,\mathbb{T},\mu)$ up to isomorphism.*

We now outline the nature of these various bijections.

In a Lebesgue space there is a duality between measurable partitions and complete σ-algebras. This bijection is defined as follows. For a measurable partition ξ the associated complete σ-algebra

$$\mathscr{A}(\xi) := \overline{\{A \in \mathscr{T} \mid A = \bigcup_{x\in A} \xi(x)\}}.$$

is generated by the sets B_n in the definition of ξ being a measurable partition [**90**, Lemma 15.1]. Conversely, given a complete σ-algebra $\mathscr{A} \subset \mathscr{T}$, which is separable and hence generated by countably many sets B_n and null sets, we define the partition $\mathscr{P}(\mathscr{A})$ by

$$\mathscr{P}(\mathscr{A})(x) = \bigcap_{x\in B_n} B_n \smallsetminus \bigcup_{x\notin B_n} B_n;$$

the B_n serve as the sets in the definition of measurability of this partition, and this is well-defined independently of the choice of the B_n. The partition elements are also called the *atoms* of $\mathscr{A}$.

Proposition 11.1.11 ([**90**, Proposition 15.1])**.** *Let $(X,\mathscr{T},\mu)$ be a Lebesgue space. ξ is a measurable partition iff $\mathscr{P}(\mathscr{A}(\xi)) = \xi$. $\mathscr{A} \subset \mathscr{T}$ is a complete σ-algebra iff $\mathscr{A}(\mathscr{P}(\mathscr{A})) = \mathscr{A}$.*

Proof. If ξ is a measurable partition, then $\mathscr{A}(\xi)$ is generated by the B_n in the definition of "measurable partition," and these B_n are then used to define $\mathscr{P}(\mathscr{A}(\xi))$, which is therefore ξ itself. If $\mathscr{A} \subset \mathscr{T}$ is a complete σ-algebra, then it is separable and hence generated by countably many B_n together with null sets; these B_n then define $\mathscr{P}(\mathscr{A})$ and then also $\mathscr{A}(\mathscr{P}(\mathscr{A})) = \mathscr{A}$. □

Measurable partitions are also in an obvious bijective correspondence with factors:

Definition 11.1.12 (Factor [**90**, Definition 15.2])**.** A *factor* of a Lebesgue space $(X,\mathbb{T},\mu)$ is a Lebesgue space $(Y,\mathscr{S},\nu)$ with a measurable map $\pi\colon X \to Y$ such that $\nu = \pi_*\mu$, the *projection.*

In this case $\{\pi^{-1}(\{y\}) \mid y \in Y\}$ is a measurable partition, and conversely, given a measurable partition ξ of $X, \mathbb{T}, \mu)$ one can take its members to define equivalence classes and thus associate with it the factor $(X/\xi, \pi_*\mathbb{T}, \pi_*\mu)$ defined by $\pi(x) := \xi(x)$ and $\pi_*\mathbb{T} := \{A \subset X/\xi \mid \pi^{-1}(A) \in \mathbb{T}\}$. This is a Lebesgue space [**90**, Lemma 15.4], and these two associations are inverses of each other [**90**, Section 15.5].

Finally, the correspondence between σ-algebras in $\mathbb{T}$ and subalgebras of L^0 is: $\mathbb{T} \ni \mathscr{A} \mapsto L^0(X, \mathscr{A}, \mu)$ and $L^0(X, \mathbb{T}, \mu) \ni \mathbf{A} \mapsto \{A \in \mathbb{T} \mid \chi_A \in \mathbf{A}\}$ [**90**, Proposition 15.2].

Definition 11.1.13 (Conditionals)**.** A measurable partition ξ of a Lebesgue space has *conditional measures* $\{\mu_C\}_{C\in\xi}$ such that (C, μ_C) is a Lebesgue space for μ_ξ-almost every $C \in \xi$ and if $A \subset X$ is measurable then $A \cap C$ is μ_C-measurable for almost all $C \in \xi$, the function $C \mapsto \mu_C(A \cap C) =: \mu(A \mid C)$ is measurable on X/ξ and $\mu(A) = \int_\xi \mu_C(A \cap C)\, d\mu_\xi(C)$. This system of conditional measures is unique a.e., that is, if μ_C and μ'_C are conditionals for ξ then $\mu'_C = \mu_C$ for μ_ξ-a.e. $C \in \xi$.

(Whenever $\mu(C) > 0$ this is, of course, the same as in (3.3.3).)

Definition 11.1.14 (Refinement)**.** There is an obvious partial-ordering relation between partitions: $\xi \le \eta$ if and only if for all $D \in \eta$ there exists a $C \in \xi$ such that $D \subset C$. If $\xi \le \eta$ we say that η is a *refinement* of ξ and that ξ is *subordinate* to η.

Remark 11.1.15. This ordering behaves well when passing from partitions to σ-algebras (ordered by inclusion): $\mathscr{A}(\xi) \subset \mathscr{A}(\xi') \Leftrightarrow \xi \le \xi'$.

Definition 11.1.16 (Join)**.** If $\{\xi_i\}_{i\in I}$ is a collection of measurable partitions we define their *join* $\bigvee_{i\in I} \xi_i = \sup_{i\in I} \xi_i$ to be the smallest measurable partition that is a refinement of ξ_i for all $i \in I$; for finite partitions

$$\xi \vee \eta = \{C \cap D \mid C \in \xi, D \in \eta, \mu(C \cap D) > 0\}. \tag{11.1.1}$$

$\bigwedge_{i\in I} \xi_i = \inf_{i\in I} \xi_i$ is the largest partition subordinate to ξ_i for all $i \in I$, and

$$\xi_n \nearrow \xi :\Leftrightarrow \bigvee_{n\in\mathbb{N}} \xi_n = \xi \text{ and } \xi_n \le \xi_{n+1} \text{ for all } n \in \mathbb{N},$$
$$\xi_n \searrow \xi :\Leftrightarrow \bigwedge_{n\in\mathbb{N}} \xi_n = \xi \text{ and } \xi_{n+1} \le \xi_n \text{ for all } n \in \mathbb{N}.$$

Remark 11.1.17. If for σ-algebras $\{\mathscr{A}_i\}_{i\in I}$ we define $\bigvee_{i\in I} \mathscr{A}_i$ to be the smallest σ-algebra that contains all $\mathscr{A}_i$ then

$$\mathscr{A}(\bigvee_{i\in I} \xi_i) = \bigvee_{i\in I} \mathscr{A}(\xi_i) \quad \text{and} \quad \mathscr{A}(\bigwedge_{i\in I} \xi_i) = \bigcap_{i\in I} \mathscr{A}(\xi_i).$$

One can define "$\nearrow$" and "$\searrow$" for σ-algebras analogously to the case of partitions.

An alternative description of this for L^2-functions is given in Example 3.2.15.

2. Entropy and conditional entropy

a. Entropy of a partition. One way of introducing entropy is to consider information about points obtainable from a partition. Given X and a partition ξ, suppose we wish to locate a point $x \in X$. Knowing which element of ξ contains x provides some information; presumably a great deal if this element $\xi(x)$ has small measure—probabilistically speaking, this represents an unlikely event. We therefore wish to define an information function by $I[\xi](x) = \varphi(\mu(\xi(x)))$ with continuous nonnegative φ. A natural choice of φ is determined by the following consideration.

We say that finite partitions ξ and η are *independent* if

$$\mu(C \cap D) = \mu(C) \cdot \mu(D)$$

for all $C \in \xi$, $D \in \eta$. It is natural to wish the information obtained from knowledge about both partitions to be additive in this case, that is, we would like to have

$$I[\xi \vee \eta] = I[\xi] + I[\eta]$$

for independent partitions, where the *joint partition* $\xi \vee \eta$ is the smallest common refinement of ξ and η (Definition 11.1.16). This implies that for two sets C, D with $\mu(C \cap D) = \mu(C) \cdot \mu(D)$ we require

$$\varphi(\mu(C) \cdot \mu(D)) = \varphi(\mu(C \cap D)) = \varphi(\mu(C)) + \varphi(\mu(D)).$$

Up to choice of a factor, this implies that $\varphi = -\log$.

Thus, the entropy of a partition is now defined as the (space) average of the (measurable) *information function*

$$x \mapsto I[\xi](x) = -\log \mu(\xi(x)). \tag{11.2.1}$$

Definition 11.2.1. The *entropy* of a measurable partition ξ is

$$H(\xi) := H_\mu(\xi) := \int_X I[\xi]\, d\mu = \begin{cases} -\sum\limits_{\substack{C\in\xi \\ \mu(C)>0}} \mu(C) \log \mu(C) & \text{if } \mu\Big(\bigcup\limits_{\substack{C\in\xi \\ \mu(C)>0}} C\Big) = 1, \\ \infty & \text{otherwise.} \end{cases} \tag{11.2.2}$$

We denote by $\mathscr{P}_H$ the collection of measurable partitions (mod 0) with finite entropy, and we refer to these as *finite-entropy partitions*.

Finite-entropy partitions are (essentially) finite or countable, and for countable partitions the entropy may be infinite.

In most cases we suppress the dependence of entropy on the measure, but where more than one measure is involved in a discussion we use a subscript. If $f\colon X \to X$ is a measure-preserving transformation, ξ a measurable partition of X, and $f^{-1}(\xi) := \{f^{-1}(C) \mid C \in \xi\}$ then obviously

$$H(f^{-1}(\xi)) = H(\xi). \tag{11.2.3}$$

The definition (11.2.2) illuminates and makes natural the following notion of conditional entropy of a partition with respect to another partition which plays a central role in the entropy theory for measure-preserving transformations.

Definition 11.2.2. Using conditional measures (see (3.3.3) or Definition 11.1.13) define the (measurable) *conditional information function* by

$$I[\xi \mid \eta](x) := -\log \mu(\xi(x) \mid \eta(x)). \tag{11.2.4}$$

We define conditional entropy similarly to (11.2.2): Let ξ, η be two measurable partitions of (X, μ). The *conditional entropy* of ξ with respect to η is

$$H(\xi \mid \eta) := \int_X I[\xi \mid \eta]\, d\mu. \tag{11.2.5}$$

Remark 11.2.3. It may at times be useful in connection with conditional information and entropy to think of ξ as the "numerator" and η as the "denominator" because both expressions are increasing in ξ and decreasing in η.

If $\mathscr{N} := \{X\}$ is the trivial partition then $H(\xi) = H(\xi \mid \mathscr{N})$.

Example 11.2.4. Let $X = [0,1] \times [0,1]$ be the unit square with Lebesgue measure, η the partition into vertical intervals $\{x\} \times [0,1]$, and ξ the partition into vertical intervals $\{x\} \times [0, f(x)]$ and $\{x\} \times (f(x), 1]$, where $f\colon [0,1] \to [0,1]$ is a measurable function. Then

$$H(\xi \mid \eta) = -\int_0^1 [f(x) \log f(x) + (1 - f(x)) \log(1 - f(x))]\, dx.$$

Remark 11.2.5. Alternatively, $H(\xi \mid \eta) = -\sum\limits_{D \in \eta} \mu(D) \sum\limits_{C \in \xi} \mu(C \mid D) \log \mu(C \mid D)$.

For finite or countable measurable partitions note that if we denote by ξ_D the partition of D into the intersections $D \cap C$, $C \in \eta$, such that $\mu(D \cap C) > 0$ then

$$H(\xi \mid \eta) = \sum_{D \in \eta} \mu(D) H_{\mu_D}(\xi_D). \tag{11.2.6}$$

The following proposition summarizes basic properties of entropy and conditional entropy which we use systematically; this includes the behavior relative to the *joint partition* $\xi \vee \eta$ from Definition 11.1.16.

Proposition 11.2.6. *Let $(X, \mathscr{B}, \mu)$ be a probability space and let ξ, η, ζ be finite or countable measurable partitions of X and $\mathscr{N} = \{X\}$. Then:*

(1) $0 \underset{\text{“}=\text{”}\Rightarrow \xi = \mathscr{N}}{\underbrace{\leq}} -\log(\sup_{C \in \xi} \mu(C)) \leq H(\xi) \underset{\text{“}=\text{” if and only if all elements of } \xi \text{ have equal measure}}{\underbrace{\leq}} \log \operatorname{card} \xi$*;*

(2)
- $0 \underset{\text{“}=\text{”}\Leftrightarrow \xi \leq \eta}{\underbrace{\leq}} H(\xi \mid \eta) \underset{\text{“}=\text{”}\Leftrightarrow \xi \text{ and } \eta \text{ are independent}}{\underbrace{\leq}} H(\xi)$*;*
- *If $\zeta \geq \eta$ then $H(\xi \mid \zeta) \leq H(\xi \mid \eta)$.*

(3) *$I[\xi \vee \eta \mid \zeta] = I[\xi \mid \zeta] + I[\eta \mid \xi \vee \zeta]$. Thus,*
   - *$H(\xi \vee \eta \mid \zeta) = H(\xi \mid \zeta) + H(\eta \mid \xi \vee \zeta)$; in particular, $\zeta = \mathscr{N}$ gives*

$$H(\xi \vee \eta) = H(\xi) + H(\eta \mid \xi), \tag{11.2.7}$$

   - *$H(\xi \vee \zeta \mid \zeta) = H(\xi \mid \zeta)$,*
   - *if $\xi \le \eta$, then*

$$H(\eta \mid \zeta) = H(\xi \mid \zeta) + H(\eta \mid \xi \vee \zeta). \tag{11.2.8}$$

(4) *$H(\xi \vee \eta \mid \zeta) \le H(\xi \mid \zeta) + H(\eta \mid \zeta)$; in particular $H(\xi \vee \eta) \le H(\xi) + H(\eta)$.*

(5) *$H(\xi \mid \eta) + H(\eta \mid \zeta) \ge H(\xi \mid \zeta)$.*

(6) *If λ_i are probability measures on X, $a_i \ge 0$, $\sum_{i\in I} a_i = 1$, then for every partition ξ measurable for all λ_i*

$$\sum_{i\in I} a_i H_{\lambda_i}(\xi) \le H_{\sum_{i\in I} a_i\lambda_i}(\xi) \le \sum_{i\in I} a_i H_{\lambda_i}(\xi) + \log \operatorname{card} I.$$

Indeed, the left inequality generalizes to $\int H_{\lambda_\alpha}(\xi)\,d\alpha \le H_{\int \lambda_\alpha\,d\alpha}(\xi)$.

Corollary 11.2.7. *$H(\alpha \vee \beta) = H(\alpha) + H(\beta)$ for independent partitions α and β.*

Proof. Use Proposition 11.2.6(2) and (3). □

Corollary 11.2.8. *For $\xi, \eta \in \mathscr{P}_H$ (Definition 11.2.1) let*

$$d_R(\xi,\eta) := H(\xi \mid \eta) + H(\eta \mid \xi). \tag{11.2.9}$$

Then d_R is a metric on $\mathscr{P}_H$. It is called the Rokhlin metric.

Proof. $d_R(\xi,\eta) \ge 0$ by (2). If $d_R(\xi,\eta) = 0$ then $H(\xi \mid \eta) = H(\eta \mid \xi) = 0$. By (2) $\xi \ge \eta$ and $\eta \ge \xi$. But this immediately implies that $\xi = \eta \pmod 0$. The symmetry of d_R is immediate from (11.2.9). Finally, from (5)

$$\begin{aligned} d_R(\xi,\zeta) &= H(\xi \mid \zeta) + H(\zeta \mid \xi) \\ &\le H(\xi \mid \eta) + H(\eta \mid \zeta) + H(\zeta \mid \eta) + H(\eta \mid \xi) = d_R(\xi,\eta) + d_R(\eta,\zeta). \end{aligned}$$ □

Several of the results in Proposition 11.2.6 are consequences of convexity of the function $x \log x$, so we begin with pertinent convexity lemmas.

Definition 11.2.9. $\phi\colon (a,b) \to \mathbb{R}$ is said to be *convex* if $x, y \in (a,b)$, $\lambda \in [0,1] \Rightarrow$

$$\phi(\lambda x + (1-\lambda y) \le \lambda\phi(x) + (1-\lambda)\phi(y).$$

ϕ is said to be *strictly convex* if equality implies $x = y$ or $\lambda \in \{0,1\}$. Equivalently, the set of points in $\mathbb{R}^2$ above the graph of ϕ is convex, that is, (recursively)

$$\phi\left(\sum \alpha_i x_i\right) \le \sum \alpha_i \phi(x_i) \tag{11.2.10}$$

whenever $x_i \in (a,b)$, $\alpha_i \geq 0$ and $\sum \alpha_i = 1$. If ϕ is strictly convex then equality in (11.2.10) implies that the convex combination is trivial, that is, all x_i such that $\alpha_i \neq 0$ are equal.

Indeed, we have a continuous analog of (11.2.10):

Proposition 11.2.10 (Jensen inequality)**.** *If X is a probability space, $g \in L^1(X)$, and ϕ is convex on $\mathbb{R}$, then $\int_X \phi(g(x))\,dx \geq \phi\big(\int_X g(x)\,dx\big)$.*[4]

Proof. $t \mapsto \frac{\phi(t)-\phi(t_0)}{t-t_0}$ is nondecreasing, so ϕ has one-sided derivatives at each point, and the left derivative never exceeds the right derivative. For m between the left and right derivatives of ϕ at $t_0 = \int_X g$, we then have $\phi(t) \geq m(t-t_0)+\phi(t_0)$ for all t. Set $t = g(x)$ and integrate over X. □

Proposition 11.2.11. *If $\phi'' > 0$ on (a,b), then ϕ is strictly convex.*

Proof. Fix $y > x$ and $\alpha, \beta \in (0,1)$ such that $\alpha+\beta = 1$. By the Mean Value Theorem $\phi(\alpha x+\beta y)-\phi(x) = \phi'(\bar z)\beta(y-x)$ for some $\bar z \in (x, \alpha x+\beta y)$ and $\phi(y)-\phi(\alpha x+\beta y) = \phi'(z)\alpha(y-x)$ for some $z \in (\alpha x+\beta y, y)$ with $\phi'(\bar z) < \phi'(z)$ since $\phi'' > 0$. Then

$$\beta\big(\phi(y)-\phi(\alpha x+\beta y)\big) = \phi'(z)\alpha\beta(y-x) > \phi'(\bar z)\alpha\beta(y-x) = \alpha\big(\phi(\alpha x+\beta y)-\phi(x)\big),$$

hence $\phi(\alpha x+\beta y) < \alpha\phi(x)+\beta\phi(y)$. □

Proposition 11.2.12. *The function $\phi\colon [0,\infty) \to \mathbb{R}$ defined by*

$$\phi(x) := \begin{cases} x\log x & \text{if } x \geq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{11.2.11}$$

is strictly convex.

Proof. $\phi'(x) = 1+\log x$ and $\phi''(x) = 1/x > 0$ for $x \in (0,\infty)$. □

Lemma 11.2.13. *If $b_i \in \mathbb{R}$ and $x_i \geq 0$ for $i = 1,\dots,m$ then*

$$\sum_{i=1}^m x_i(b_i - \log x_i) \leq \sum_{i=1}^m x_i \log\Big(\sum_{j=1}^m e^{b_j}\Big) - \phi\Big(\sum_{i=1}^m x_i\Big) \leq \sum_{i=1}^m x_i \log\Big(\sum_{j=1}^m e^{b_j}\Big) + \frac{1}{e}. \tag{11.2.12}$$

The first inequality is strict unless $x_i = ce^{b_i}$ with c independent of i. In particular, if $b_i = 0$ for all i, this gives

$$-\sum_{i=1}^m x_i \log x_i \leq \sum_{i=1}^m x_i \log m + \frac{1}{e}.$$

[4] For strictly convex ϕ, equality implies that g is constant.

Proof. Let $a_i := e^{b_i}$ for all i, $A := \sum_{i=1}^m a_i$. Strict convexity of $\phi(x) = x\log x$ implies

$$\begin{aligned}\frac{1}{A}\sum_{i=1}^m x_i \log\Big(\frac{x_i}{a_i}\Big) = \sum_{i=1}^m \frac{a_i}{A}\phi\Big(\frac{x_i}{a_i}\Big) \ge \phi\Big(\sum_{i=1}^m \frac{a_i}{A}\frac{x_i}{a_i}\Big) = \phi\Big(\frac{\sum_{i=1}^m x_i}{A}\Big)\\ = \frac{\sum_{i=1}^m x_i}{A}\log\Big(\frac{\sum_{i=1}^m x_i}{A}\Big) = \frac{1}{A}\Big[\phi\Big(\sum_{i=1}^m x_i\Big) - \sum_{i=1}^m x_i \log A\Big]\end{aligned}$$

with equality iff $x_i / e^{b_i} = \text{const}$. Since $\phi(x) \ge -1/e$, this yields the claim. □

Since $\phi(1) = 0$, Lemma 11.2.13 implies

Lemma 11.2.14. *If $\sum_{i=1}^m x_i = 1$ in Lemma 11.2.13, then*

$$\sum_{i=1}^m x_i(b_i - \log x_i) \le \log \sum_{i=1}^m e^{b_i} \quad \textit{with equality if and only if} \quad x_i \sum_{i=1}^m e^{b_i} = e^{b_i}$$

(because $1 = \sum_i x_i = c\sum_i e^{b_i}$). If $b_i = 0$ for all i, this reduces to

$$-\sum_{i=1}^m x_i \log x_i \le \log m \quad \textit{with equality if and only if all } x_i = 1/m.$$

Lemma 11.2.15. *If $x_i, a_i \ge 0, \sum_i a_i = 1$, then*

$$\sum_i x_i \phi(a_i) \le \phi\Big(\sum_i a_i x_i\Big) - \sum_i a_i \phi(x_i) \le 0.$$

Proof. The second inequality is (11.2.10)[5], the first, monotonicity of logarithms:

$$\phi\Big(\sum_i a_i x_i\Big) - \sum_i a_i\phi(x_i) - \sum_i x_i \phi(a_i) = \sum_i \underbrace{a_i x_i}_{\ge 0}\Big[\overbrace{\log\Big(\sum_i a_i x_i\Big) - \log x_i - \log a_i}^{=\log(\sum_i a_i x_i) - \log(x_i a_i) \ge 0 \text{ because log is increasing}}\Big] \ge 0. \quad \square$$

Proof of Proposition 11.2.6. (1) μ is a probability measure, so (11.2.2) implies

$$0 \le -\log(\sup_{C\in\xi} \mu(C)) = \inf I[\xi] \le \int_X I[\xi]\, d\mu = H_\mu(\xi).$$

$H(\xi) \le \log \operatorname{card} \xi$ is vacuous unless $\xi = (C_1, \dots, C_k)$ is finite, in which case Lemma 11.2.14 yields $H(\xi) \le \log k$ with equality if and only if $\mu(C_i) = 1/k$ for all i.

(2) The inequality follows from convexity of ϕ:

$$\begin{aligned} 0 \le H(\xi \mid \eta) &= -\sum_{D\in\eta} \mu(D) \sum_{C\in\xi} \phi(\mu(C \mid D))\\ &= -\sum_{C\in\xi} \underbrace{\sum_{D\in\eta} \mu(D)\phi(\mu(C\mid D))}_{\ge \phi\big(\sum_{D\in\eta}\mu(D)\mu(C|D)\big) = \phi(\mu(C))} \le H(\xi). \qquad (11.2.13)\end{aligned}$$

[5] And hence generalizes as in Proposition 11.2.10.

Now $\phi(x) < 0$ for $0 < x < 1$, so if $H(\xi \mid \eta) = 0$ then for every β with $\mu(D) > 0$ we have $\phi(\mu(C \mid D)) = 0$ for all $C \in \xi$ and consequently $\xi \leq \eta$. If $H(\xi \mid \eta) = H(\xi)$ then we must have equality in (11.2.13) for each term of the summation over α, that is,

$$\phi(\mu(C)) = \phi\Big(\sum_{\substack{D\in\eta\\ \mu(D)>0}} \mu(D)\mu(C \mid D)\Big) = \sum_{\substack{D\in\eta\\ \mu(D)>0}} \mu(D)\phi\big(\mu(C \mid D)\big).$$

By strict convexity of the function ϕ this implies that if $\mu(D) > 0$ and $\mu(C) > 0$ then $\mu(C \mid D) = \mu(C)$, that is, $\mu(C \cap D) = \mu(C) \cdot \mu(D)$.

Applying the inequality $H_{\mu_D}(\xi \mid \zeta) \leq H_{\mu_D}(\xi)$ to the conditional measures μ_D on each element D of the partition η and integrating over that partition we obtain $H(\xi \mid \zeta) = H(\xi \mid \zeta \vee \eta) \leq H(\xi \mid \eta)$.

$$\begin{aligned}(3)\ I[\xi \vee \eta \mid \zeta](x) &= -\log \frac{\mu(\xi(x) \cap \eta(x) \cap \zeta(x))}{\mu(\zeta(x))}\\ &= -\log \frac{\mu(\xi(x) \cap \zeta(x))}{\mu(\zeta(x))} - \log \frac{\mu(\xi(x) \cap \eta(x) \cap \zeta(x))}{\mu(\xi(x) \cap \zeta(x))}\\ &= I[\xi \mid \zeta](x) + I[\eta \mid \xi \vee \zeta](x).\end{aligned}$$

Now integrate with respect to x.

(4) This follows from (3) and the inequality $H(\eta \mid \xi \vee \zeta) \leq H(\eta \mid \zeta)$ which in turn follows from (2) since $\xi \vee \zeta \geq \zeta$.

(5) By (3) and (4) we have $H(\zeta \mid \xi \vee \eta) = H(\xi \vee \zeta \mid \eta) - H(\xi \mid \eta) \leq H(\zeta \mid \eta)$. Using (3) several times we obtain

$$\begin{aligned}H(\zeta \mid \eta) + H(\eta \mid \zeta) &= H(\xi \vee \eta) + H(\eta \vee \zeta) - H(\eta) - H(\zeta)\\ &= H(\xi \vee \eta) + H(\zeta \mid \eta) - H(\zeta)\\ &= H(\xi \vee \eta \vee \zeta) - H(\zeta \mid \xi \vee \eta) + H(\zeta \mid \eta) - H(\zeta)\\ &\geq H(\xi \vee \eta \vee \zeta) - H(\zeta) \geq H(\xi \vee \zeta) - H(\zeta) = H(\xi \mid \zeta).\end{aligned}$$

(6) If $C \in \xi$ then (with $\phi(x) = x \log x$ as in (11.2.11)) Lemma 11.2.15 gives

$$0 \leq -\phi(\sum_i a_i \lambda_i(C)) + \sum_i a_i \phi(\lambda_i(C)) \leq -\sum_i \lambda_i(C)\phi(a_i).$$

Summing over $C \in \xi$ this yields

$$0 \leq H_{\sum_i a_i \lambda_i}(\xi) - \sum_i a_i H_{\lambda_i}(\xi) \leq -\sum \phi(a_i) \leq \log \operatorname{card} I$$

by (1). The continuous generalization of the left inequality is Proposition 11.2.10 implemented in Lemma 11.2.15. □

Remark 11.2.16. The conditional expectation in Corollary 3.2.11 is also defined λ-a.e. uniquely by

$$\varphi_{\mathcal{T}}\restriction_C = \int_C \varphi \, d\lambda_C \text{ for all } C \in \pi(\mathcal{T}),$$

where $\pi(\mathcal{T})$ is as in Definition 11.1.12.[6]

Remark 11.2.17. At times we apply this result to a σ-algebra $\mathcal{T}$ that arises from a partition ξ; in that case we may write $E(\varphi \mid \xi)$ instead of $E(\varphi \mid \mathscr{A}(\xi))$.

With the notation of Definition 11.1.16 we have:

Theorem 11.2.18. *Let ξ be a finite partition of a probability space $(X, \mathscr{B}, \mu)$.*

- *If $\eta_n \nearrow \eta$, then $H(\xi \mid \eta_n) \searrow H(\xi \mid \eta)$ as $n \to \infty$.*
- *If $\eta_n \searrow \eta$, then $H(\xi \mid \eta_n) \nearrow H(\xi \mid \eta)$ as $n \to \infty$.*

Proof. The monotonicity assertions are clear from Proposition 11.2.6.2, and by (11.2.5) the limits are obtained by showing that

$$\int I[\xi \mid \eta_n] \to \int I[\xi \mid \eta].$$

Recall that $I[\xi \mid \eta](x) = -\log \mu(\xi(x) \mid \eta(x))$ and note that

$$\mu(\xi(x) \mid \eta(x)) = \int \chi_{\xi(x)} \, d\mu_{\eta(x)} = E(\chi_{\xi(x)} \mid \eta)(x),$$

where E is the conditional expectation operator from Corollary 3.2.11 (see also Remark 11.2.17 for notation). Thus, for $x \in C \in \xi$ we have

$$\mu(\xi(x) \mid \eta(x)) = E(\chi_C \mid \eta)(x)$$

and hence

$$I[\xi \mid \eta] = -\sum_{C \in \xi} \chi_C \log E(\chi_C \mid \eta).$$

This allows us to write

$$H(\xi \mid \eta) = -\int \sum_{C \in \xi} \chi_C \log E(\chi_C \mid \eta) = -\sum_{C \in \xi} \int \chi_C \log E(\chi_C \mid \eta)$$
$$= -\sum_{C \in \xi} \int E(\chi_C \log E(\chi_C \mid \eta) \mid \eta) = -\sum_{C \in \xi} \int E(\chi_C \mid \eta) \log E(\chi_C \mid \eta).$$

The last step used Proposition 3.2.12.

Thus, we have now observed that it suffices to show

$$-\sum_{C \in \xi} E(\chi_C \mid \eta_n) \log(E(\chi_C \mid \eta_n)) \xrightarrow[n \to \infty]{L^1} -\sum_{C \in \xi} E(\chi_C \mid \eta) \log(E(\chi_C \mid \eta)). \tag{11.2.14}$$

[6] Or rather, $\mathscr{P}(\mathcal{T})$ as in Proposition 11.1.11

To show this, we establish

$$-\sum_{C\in\xi} E(\chi_C \mid \eta_n) \xrightarrow[n\to\infty]{L^2} -\sum_{C\in\xi} E(\chi_C \mid \eta). \tag{11.2.15}$$

This implies that (11.2.15) holds for convergence in measure and hence that (11.2.14) holds for convergence in measure. Since the functions in question are bounded by $e\operatorname{card}\xi < \infty$, we obtain (11.2.14) by the Dominated Convergence Theorem.

To prove (11.2.15) let us note that for $D \in \eta$ one can take $D_n \in \eta_N$ such that $d_\mu(D, D_n) \to 0$ as $n \to \infty$, where $d(A,B) := d_\mu(A,B) := \mu(A \triangle B)$ as in (3.4.5). Then $\chi_{D_n} \in L^2(X, \mathscr{A}(\eta_n), \mu)$, and since $E(\chi_D \mid \eta_n)$ is the orthogonal projection of χ_D to $L^2(X, \mathscr{A}(\eta_n), \mu)$ (Example 3.2.15), we use Proposition 3.2.14 to obtain

$$\|E(\chi_D \mid \eta_n) - \chi_D\|_2^2 \le \|\chi_{D_n} - \chi_D\|_2^2 = \mu(D_n \triangle D) \to 0.$$

Now, $h := E(\chi_C \mid \eta)$ can be L^2-approximated by linear combinations of χ_D with $D \in \mathscr{A}(\eta)$, so the preceding implies that

$$\|E(h \mid \eta_n) - h\|_2^2 \to 0.$$

Since $E(h \mid \eta_n) = E(\chi_C \mid \eta_n)$ (Proposition 3.2.12), we obtain (11.2.15). □

Remark 11.2.19. Using approximations by finite partitions one can show that Theorem 11.2.18 holds with the assumption that ξ has finite entropy instead of the assumption that ξ is finite.

For a measure space (X,μ) and $m \in \mathbb{N}$ consider the space $\mathscr{P}_m$ of all equivalence classes mod 0 of partitions of X into at most m measurable sets. By adding null sets if necessary, we may assume that every partition in $\mathscr{P}_m$ has exactly m elements. For $\xi, \eta \in \mathscr{P}_m$ consider now the set of bijections σ between the elements of ξ and η and set

$$\mathscr{D}(\xi,\eta) := \min_\sigma \sum_{C\in\xi} \mu(C \triangle \sigma(C)) = \min_\sigma \sum_{C\in\xi} d_\mu(C, \sigma(C)). \tag{11.2.16}$$

Obviously $\mathscr{D}$ is a metric. We need the fact that convergence in this metric guarantees convergence in the Rokhlin metric.

Proposition 11.2.20. *For $\epsilon > 0$ there is a $\delta > 0$ such that $\mathscr{D}(\xi,\eta) < \delta \Rightarrow d_R(\xi,\eta) < \epsilon$.*

Remark 11.2.21. In fact, the metrics $\mathscr{D}$ and d_R are equivalent on the space $\mathscr{P}_m$.

Proof. By symmetry it suffices to estimate $H(\eta \mid \xi)$. If $\mathscr{D}(\xi,\eta) = \delta$ write $\xi = (A_1, \dots, A_m)$, $\eta = (B_1, \dots, B_m)$ in such a way that $\sum_{i=1}^m \mu(A_i \triangle B_i) = \delta$. For $i \in$

$\{1,\dots,m\}$ such that $\mu(A_i) > 0$ let $\alpha_i \coloneqq \mu(A_i \smallsetminus B_i)/\mu(A_i)$. Then the contribution of A_i to the expression for $H(\eta \mid \xi)$ in Definition 11.2.2 is

$$\begin{aligned}-\mu(B_i\cap A_i)\log\frac{\mu(B_i\cap A_i)}{\mu(A_i)} &- \sum_{j\neq i}\mu(B_j\cap A_i)\log\frac{\mu(B_j\cap A_i)}{\mu(A_i)}\\ &\le \mu(A_i)[-(1-\alpha_i)\log(1-\alpha_i)-\alpha_i\log\alpha_i+\alpha_i\log(m-1)]\\ &= \mu(A_i)\Big[(1-\alpha_i)\log\frac{1}{1-\alpha_i}+\alpha_i\log\frac{m-1}{\alpha_i}\Big]\le \mu(A_i)\log m.\end{aligned}$$

Here the first inequality follows from Proposition 11.2.6(1) by considering the measure induced on $A_i \smallsetminus B_i = \bigcup_{j\neq i}(A_i\cap B_j)$ and estimating the entropy of η with respect to that measure. The last inequality uses convexity of $-\log x$. Thus

$$H(\eta\mid\xi)\le\sum_{\mu(A_i)\ge\sqrt{\delta}}\mu(A_i)[-(1-\alpha_i)\log(1-\alpha_i)-\alpha_i\log\alpha_i+\alpha_i\log(m-1)]+\sum_{\mu(A_i)<\sqrt{\delta}}\mu(A_i)\log m.$$

The second term does not exceed $m\log m\sqrt{\delta}$. To estimate the first note that

$$\alpha_i\mu(A_i)=\mu(A_i\smallsetminus B_i)=\sum_{j\neq i}\mu(B_j\cap A_i)\le\sum_{j=1}^{m}\mu(A_j\bigtriangleup B_j)=\delta,$$

so for $\mu(A_i)\ge\sqrt{\delta}$ we get $\alpha_i\le\sqrt{\delta}$. Now $\varphi(x)\coloneqq -x\log x-(1-x)\log(1-x)$ is increasing on $(0,1/2)$, so for $\delta<1/4$ the first sum is dominated by $\varphi(\sqrt{\delta})+\sqrt{\delta}\log(m-1)$ and hence $H(\eta\mid\xi)\le\varphi(\sqrt{\delta})+\sqrt{\delta}(m\log m+\log(m-1))$. Since $\varphi(x)\xrightarrow[x\to 0]{}0$, the statement follows. □

b. Entropy of a measure-preserving transformation.

Definition 11.2.22. For a measurable partition ξ and a measure-preserving transformation f we define the *joint partition* as follows. For $I\subset\mathbb{R}$ set

$$\xi_I^f \coloneqq \bigvee_{i\in I\cap\mathbb{Z}} f^i(\xi)$$

and

$$\xi_{-n}^f\coloneqq\xi_{[-n,0)}^f,\qquad \xi_-^f\coloneqq\xi_{(-\infty,0)}^f,\qquad \xi_n^f\coloneqq\xi_{[0,n)}^f,\qquad \xi_+^f\coloneqq\xi_{[0,\infty)}^f,\qquad \xi^f\coloneqq\xi_{\mathbb{Z}}^f.$$

From now on, unless stated otherwise, we assume that all partitions are finite or countable measurable partitions with finite entropy.

Proposition 11.2.23. $\lim_{n\to\infty}\frac{1}{n}H(\xi_{-n}^f)$ *exists (and equals* $\inf_{n\in\mathbb{N}}H(\xi_{-n}^f)/n$*).*

Proof. $H(\xi_{-n-m}^f)\le H(\xi_{-n}^f)+H(\xi_{-m}^f)$ by (11.2.7) and (11.2.3), so the statement follows by the Bowen–Fekete Lemma 4.2.7. □

Definition 11.2.24. $h(f,\xi) \coloneqq h_\mu(f,\xi) \coloneqq \lim_{n\to\infty} H(\xi^f_{-n})/n = \inf_{n\in\mathbb{N}} H_\mu(\xi^f_{-n})/n$ is the *measure-theoretic entropy* of the transformation f *relative* to the partition ξ.

The following proposition gives an alternative proof of existence of the limit $h(f,\xi)$ as well as another expression for it.

Proposition 11.2.25. *$H(\xi \mid \xi^f_{-n}) \searrow h(f,\xi)$ as $n \to \infty$.*

Proof. We first note that

$$H(\xi^f_{-n}) = \sum_{k=0}^{n-1} H(\xi \mid \xi^f_{-k}).$$

For $n = 1$ this is clear since $H(\xi^f_{-n}) = H(\xi^f_{(-n,0]})$, and using (11.2.7) we have

$$H(\xi^f_{-n-1}) = H(\xi^f_{[-n,0]}) = H(\xi \vee \xi^f_{-n}) = H(\xi^f_{-n}) + H(\xi \mid \xi^f_{-n}).$$

By the invariance property (11.2.3), this implies the claim.

Since the partition ξ^f_{-k} in the "denominator" is refined as k increases, by Proposition 11.2.6(2) the sequence $b_n \coloneqq H(\xi \mid \xi^f_{-n})$ of summands is nonincreasing and hence convergent. Thus

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} b_k = \lim_{n\to\infty} \frac{1}{n} H(\xi^f_{-n}) = h_\mu(f,\xi). \qquad \square$$

Corollary 11.2.26. *If $\xi \in \mathscr{P}_H$ then $h(f,\xi) = H(\xi \mid \xi^f_{-})$.*

Proof. Combine Proposition 11.2.25 with Theorem 11.2.18. □

Definition 11.2.27. The *entropy* of f with respect to μ (or the entropy of μ) is

$$h(f) \coloneqq h_\mu(f) \coloneqq \sup\left\{h_\mu(f,\xi) \;\middle|\; \xi \in \mathscr{P}_H\right\}.$$

Obviously entropy is invariant under measure-theoretic isomorphism. We will see soon that this definition is more constructive than it seems; in many cases $h_\mu(f) = h_\mu(f,\xi)$ for an appropriately chosen ξ. (See, for example, Theorem 11.3.7.)

Recalling the definition of the partition entropy through the information function (11.2.1)–(11.2.2) we can interpret the entropy $h_\mu(f,\xi)$ as the average amount of information provided by the knowledge of the "present state" in addition to the knowledge of an arbitrarily long past. Thus, a system with zero entropy can be viewed as strongly deterministic in the sense that an approximate knowledge of the entire past (that is, the past itinerary with respect to a finite partition) precisely determines the future itinerary.

3. Properties of entropy

a. Properties of entropy with respect to a partition. The following proposition summarizes basic properties of the entropy $h(f,\xi)$ as a function of the partition ξ. It prepares the way for subsequent criteria which allow one to calculate the transformation entropy $h(f)$.

Proposition 11.3.1. *Let $f\colon (X,\mu) \to (X,\mu)$ be a measure-preserving transformation of a probability space and $\eta, \xi \in \mathscr{P}_H$. Then:*

(1) $0 \le \overline{\lim}_{n\to\infty} -(1/n)\log(\sup_{C\in\xi^f_{-n}} \mu(C)) \le h(f,\xi) \le H(\xi)$.
(2) $h(f,\xi\vee\eta) \le h(f,\xi) + h(f,\eta)$.
(3) $h(f,\eta) \le h(f,\xi) + H(\eta \mid \xi)$*; in particular if* $\xi \le \eta$ *then* $h(f,\xi) \le h(f,\eta)$.
(4) $|h(f,\xi) - h(f,\eta)| \le H(\xi \mid \eta) + H(\eta \mid \xi)$ *(the* Rokhlin inequality*).*
(5) $h(f, f^{-1}(\xi)) = h(f,\xi) = h(f, f(\xi))$.
(6) $h(f,\xi) = h(f,\xi^f_{-k}) = h(f,\xi^f_{[-k,k]})$ *for* $k \in \mathbb{N}$.
(7) If λ is another probability measure and $p \in [0,1]$ then

$$p h_\mu(f,\xi) + (1-p) h_\lambda(f,\xi) = h_{p\mu+(1-p)\lambda}(f,\xi).$$

Remark 11.3.2. Property (4) means that $h(f,\cdot)$ is a Lipschitz function with Lipschitz constant 1 on $(\mathscr{P}_H, d_R)$ (see (11.2.9)).

Proof. The middle inequality in (1) follows directly from Proposition 11.2.6(1) and the right inequality follows from Proposition 11.2.25 and Proposition 11.2.6(2).

(2) Since $(\xi\vee\eta)^f_{-n} = \xi^f_{-n} \vee \eta^f_{-n}$, this statement follows from (11.2.7) which is a particular case of Proposition 11.2.6(3).

(3) By (11.2.7) $H(\xi^f_{-n}) \le H(\xi^f_{-n} \vee \eta^f_{-n}) = H(\eta^f_{-n}) + H(\xi^f_{-n} \mid \eta^f_{-n})$ and by using Proposition 11.2.6(3) inductively, we obtain

$$\begin{aligned} H(\xi^f_{-n} \mid \eta^f_{-n}) &= H(f^{-1}(\xi) \mid \eta^f_{-n}) + H(f^{-1}(\xi^f_{1-n}) \mid f^{-1}(\xi) \vee \eta^f_{-n}) \\ &\le H(f^{-1}(\xi) \mid f^{-1}(\eta)) + H(f^{-1}(\xi^f_{1-n}) \mid \eta^f_{-n}) \\ &\le H(f^{-1}(\xi) \mid f^{-1}(\eta)) + H(f^{-2}(\xi) \mid f^{-2}(\eta)) + H(f^{-2}(\xi^f_{2-n}) \mid \eta^f_{-n}) \\ &\le \cdots \le n H(\xi \mid \eta). \end{aligned}$$

Property (4) follows directly from (3).

Property (5) follows from the invariance property (11.2.3) since

$$H((f^{-1}(\xi))^f_{-n}) = H(f^{-1}(\xi^f_{-n})) = H(\xi^f_{-n}) = H(f(\xi^f_{-n})) = H((f(\xi))^f_{-n}).$$

(6) $(\xi^f_{-k})^f_{-n} = \xi^f_{[-n-k,-2]}$ and hence

$$h(f,\xi^f_{-k}) = h\Big(f, \xi^f_{[-n-k,-2]}\Big) = \lim_{n\to\infty} \frac{1}{n} H(\xi^f_{1-n-k}) = \lim_{n\to\infty} \frac{1}{n+k-1} H(\xi^f_{1-n-k}) = h(f,\xi).$$

The argument for $f(\xi)$ is similar.

Property (7) follows directly from Proposition 11.2.6(6). □

b. The generator theorem. We can now formulate some criteria for calculating the entropy of a transformation.

Definition 11.3.3. A family $\Xi \subset \mathscr{P}_H$ is said to be *sufficient* with respect to the measure-preserving transformation f if partitions subordinate to partitions of the form $\xi^f_{[-k,k]}$ ($\xi \in \Xi$, $k \in \mathbb{N}$) form a dense subset in $(\mathscr{P}_H, d_R)$ (see (11.2.9)).

Remark 11.3.4. Proposition 11.2.20 allows us to replace the Rokhlin metric by the metric $\mathscr{D}$ from (11.2.16) in this definition. In the case of a nonatomic Borel measure on a compact metric space a more obvious condition that guarantees sufficiency of a family $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ is $\operatorname{diam}(\xi_n) \to 0$, where $\operatorname{diam}(\xi) := \sup_{C\in\xi}(\operatorname{diam}(C))$.

Theorem 11.3.5 (Kolmogorov–Sinai)**.** $h_\mu(f) = \sup_{\xi\in\Xi} h_\mu(f,\xi)$ *for any sufficient family* Ξ *of partitions.*

Proof. For $\eta \in \mathscr{P}_H$ and $\epsilon > 0$ find $\xi \in \Xi$ and $k \in \mathbb{N}$ such that

$$d_R(\eta,\zeta) = H(\eta \mid \zeta) + H(\zeta \mid \eta) < \epsilon$$

for some partition $\zeta \le \xi^f_{[-k,k]}$. Using consecutively Proposition 11.3.1(4), (3), and (6), we obtain

$$h_\mu(f,\eta) \le h_\mu(f,\zeta) + \epsilon \le h_\mu(f,\xi^f_{[-k,k]}) + \epsilon = h_\mu(f,\xi) + \epsilon.$$

Since ϵ is arbitrary, the statement follows. □

Definition 11.3.6. A partition ξ is said to be a *generator* for f if $\Xi = \{\xi\}$ is a sufficient family.

The following corollary is the best-known and simplest-sounding criterion for calculating entropy.

Theorem 11.3.7 (Kolmogorov–Sinai)**.** *If* ξ *is a generator* with finite entropy *for* f *then* $h_\mu(f) = h_\mu(f,\xi)$.

Remark 11.3.8. By Corollary 11.2.26, this can be restated as saying that for a generator we have $h_\mu(f) = H(\xi \mid \xi^f_-)$. This was Kolmogorov's original definition of entropy.

We call a partition ξ a *one-sided generator* or *strong generator* if partitions subordinate to partitions of the form $\xi^f_{[1-k,0]}$ ($k \in \mathbb{N}$) are dense in the metric d_R. Clearly, a one-sided generator is a generator.

Proposition 11.3.9. *If an invertible measure-preserving transformation possesses a one-sided generator* with finite entropy *then* $h_\mu(f) = 0$.

Proof. If ξ is a one-sided generator, then $\xi^f_{(-\infty,0]}$ is the point partition and hence

$$0 = H(f(\xi) \mid \xi^f_{(-\infty,0]}) = H(f(\xi) \mid (f(\xi))^f_-) = h(f, f(\xi))$$

by Corollary 11.2.26. Since f is invertible, this implies $h_\mu(f,\xi) = 0$ by Proposition 11.3.1(5). The claim then follows by Theorem 11.3.7 because ξ is a one-sided generator and hence a generator. □

Note that the existence of countable sufficient families is ensured by separability of d_R (see (11.2.9), Remark 11.2.21). This leads to a slight refinement of Theorem 11.3.5 and a general existence theorem for generators. Suppose $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathscr{P}_H$ is a countable dense family of partitions and define $\xi_n := \bigvee_{i\le n}\zeta_i \in \mathscr{P}_H$. Then this defines an increasing sufficient family, and Theorem 11.3.5 becomes

Proposition 11.3.10. *With these choices,* $h_\mu(f) = \lim_{n\to\infty} h_\mu(f,\xi_n)$.

Proposition 11.3.11. *An ergodic aperiodic transformation has a one-sided generator.*[7]

Proof [**100**, Proposition 9.5]. If $\{\zeta_i\}_{i\in\mathbb{N}} \subset \mathscr{P}_H$ is a countable dense family of partitions, A_i are sets of positive measure, N_i are such that $\mu\big(\bigcup_{0\le j\le N_i} f^{-j}(A_i)\big) > 1-2^{-i}$, then a minimal partition ξ that refines all the $\eta_i := (\zeta_i)^0_{-N_i} \cap A_i$ is a generator because up to Rokhlin distance at most 2^{-i} it is contained in

$$\underbrace{\zeta_i \cap \bigcup_{0\le j\le N_i} f^{-j}(A_i)}_{\zeta_i\cap f^{-j}(A_i)\subset f^{-j}(\eta_i)} \subset (\eta_i)_0^{N_i} \subset (\xi)_0^{N_i}. \qquad \square$$

Remark 11.3.12. Ergodicity is assumed here for (significant) convenience, but is not required for the conclusion. Proposition 11.3.9 indicates that we should not expect a general existence result for one-sided finite-entropy generators; indeed, by Theorem 11.3.7, this can only be the case for systems with finite entropy. In that case, there is even a finite generator for ergodic systems (Krieger's Theorem). Refinements such as this are at the heart of further results such as the Jewett–Krieger Theorem (Remark 3.3.35). At the same time, this also implies that when constructed in this generality, generators can not be expected to encode any geometric information about a dynamical system.

This is an natural moment to make a connection with topological dynamics.

[7] We do not claim finiteness of its entropy!

Proposition 11.3.13. *If f is an expansive homeomorphism and $\epsilon > 0$ an expansivity constant, then for any invariant Borel probability measure a partition with diameter less than ϵ whose boundary is a null set is a generator.*

Proof. Expansivity ensures that the partition refines to the point partition under iteration of f. □

Corollary 11.3.14. *If f is an expansive homeomorphism, then $\mu \mapsto h_\mu(f)$ is upper semicontinuous on $\mathfrak{M}(f)$ with the weak* topology, that is, if $\mu_n \xrightarrow[n\to\infty]{\text{weakly}} \mu$ in $\mathfrak{M}(f)$, then $\lim_{n\to\infty} h_{\mu_n}(f) \le h_\mu(f)$.*

Proof. If $\mu_n \xrightarrow[n\to\infty]{\text{weakly}} \mu$ let ξ be a finite partition with $\operatorname{diam}\xi < \epsilon$ and $\mu\partial\xi = 0$. Then $h_{\mu_n}(f) = h_{\mu_n}(f,\xi) \le \frac{H_{\mu_n}(\xi^f_{-k})}{k} \xrightarrow[n\to\infty]{} \frac{H_\mu(\xi^f_{-k})}{k} \xrightarrow[k\to\infty]{} h_\mu(f)$ (Definition 11.2.24). □

c. Basic properties of entropy. The following proposition is a counterpart for measure-preserving transformations to Proposition 4.2.11 and Proposition 4.2.12.

Proposition 11.3.15. *(1) If $g\colon (Y,\nu) \to (Y,\nu)$ is a factor (see Definition 3.1.1) of $f\colon (X,\mu) \to (X,\mu)$ then $h_\nu(g) \le h_\mu(f)$.*

(2) If A is invariant for f and $\mu(A) > 0$ then

$$h_\mu(f) = \mu(A)h_{\mu_A}(f) + \mu(X \smallsetminus A)h_{\mu_{X\smallsetminus A}}(f).$$

(3) If μ, λ are two invariant probability measures for f then for any $p \in [0,1]$

$$ph_\mu(f) + (1-p)h_\lambda(f) = h_{p\mu+(1-p)\lambda}(f).$$

(4) $h_\mu(f^k) = kh_\mu(f)$ for any $k \in \mathbb{N}$, and $h_\mu(f^{-1}) = h_\mu(f)$, so $h_\mu(f^k) = |k|h_\mu(f)$ for any $k \in \mathbb{Z}$.

(5) $h_{\mu\times\lambda}(f \times g) = h_\mu(f) + h_\lambda(g)$.

Proof. (1) For any measurable partition η of Y, the preimage

$$\pi^{-1}(\eta) = \{\pi^{-1}D \mid D \in \eta\}$$

under the factor map π is a measurable partition of X and by definition $H_\mu(\pi^{-1}\eta) = H_\nu(\eta)$ and $h_\mu(f,\pi^{-1}\eta) = h_\nu(g,\eta)$. Thus

$$\begin{aligned} h_\mu(f) = \sup\{h_\mu(f,\xi) \mid H_\mu(\xi) < \infty\} &\ge \sup\{h_\mu(f,\pi^{-1}(\eta)) \mid H_\mu(\pi^{-1}(\eta)) < \infty\} \\ &= \sup\{h_\nu(g,\eta) \mid H_\nu(\eta) < \infty\} = h_\nu(g). \end{aligned}$$

(2) Let ξ be a measurable partition of X, $H_\mu(\xi) < \infty$, and $\zeta = \{A, X \smallsetminus A\}$. By replacing ξ by $\xi \vee \zeta$ if necessary, we may assume that $\xi \ge \zeta$. Then

$$H_\mu(\xi^f_{-n}) = \mu(A)H_{\mu_A}(\xi^f_{-n}) + \mu(X \smallsetminus A)H_{\mu_{X\smallsetminus A}}(\xi^f_{-n}) - \mu(A)\log\mu(A) - \mu(X \smallsetminus A)\log\mu(X \smallsetminus A)$$

by the definition of the conditional measures μ_A and $\mu_{X\smallsetminus A}$, since A is f-invariant. The two last terms are independent of n and vanish in the limit.

(3) Proposition 11.3.1.7 implies $h_{p\mu+(1-p)\lambda}(f) \le ph_\mu(f) + (1-p)h_\lambda(f)$.

On the other hand, given $C < ph_\mu(f) + (1-p)h_\lambda(f)$ take $C_1 < h_\mu(f)$ and $C_2 < h_\lambda(f)$ such that $pC_1 + (1-p)C_2 > C$ and partitions ξ_1 and ξ_2 such that $h_\mu(f,\xi_1) > C_1$ and $h_\lambda(f,\xi_2) > C_2$. Then Proposition 11.3.1.7 with $\xi := \xi_1 \vee \xi_2$ implies that

$$\begin{aligned} h_{p\mu+(1-p)\lambda}(f,\xi) &= ph_\mu(f,\xi) + (1-p)h_\lambda(f,\xi) \\ &\ge ph_\mu(f,\xi_1) + (1-p)h_\lambda(f,\xi_2) > pC_1 + (1-p)C_2 > C. \end{aligned}$$

Since $C < ph_\mu(f) + (1-p)h_\lambda(f)$ was arbitrary, this implies

$$h_{p\mu+(1-p)\lambda}(f) \ge ph_\mu(f) + (1-p)h_\lambda(f).$$

4. If $k \in \mathbb{N}$ then $\frac{1}{n}H_\mu\Big(\bigvee_{j=0}^{n-1} f^{-kj}\big(\bigvee_{i=0}^{k-1} f^{-i}(\xi)\big)\Big) = \frac{k}{nk}H_\mu\Big(\bigvee_{i=0}^{nk-1} f^{-i}(\xi)\Big)$ and

$$h_\mu\Big(f^k, \bigvee_{i=0}^{k-1} f^{-i}(\xi)\Big) = k\,h_\mu(f,\xi).$$

Furthermore,

$$h_\mu(f,\xi) = h_\mu(f^{-1},\xi)$$

since $\xi_{-n}^f = f^{-n+1}(\xi_{-n}^{f^{-1}})$.

(5) Let ξ, η be measurable partitions of X and Y, correspondingly and $\mathcal{N}_X = \{X\}$ and $\mathcal{N}_Y = \{Y\}$ the trivial partitions. Then $\xi \times \eta = (\xi \times \mathcal{N}_Y) \vee (\mathcal{N}_X \times \eta)$, where $\xi \times \mathcal{N}_Y$ and $\mathcal{N}_X \times \eta$ are independent as partitions of $X \times Y$.

By Corollary 11.2.7

$$H_{\mu\times\lambda}(\xi\times\eta) = H_{\mu\times\lambda}(\xi\times\mathcal{N}_Y) + H_{\mu\times\lambda}(\mathcal{N}_X\times\eta) = H_\mu(\xi) + H_\lambda(\eta).$$

Since $(\xi\times\eta)_{-n}^{f\times g} = \xi_{-n}^f \times \eta_{-n}^g$, this implies $h_{\mu\times\lambda}(f\times g, \xi\times\eta) = h_\mu(f,\xi) + h_\lambda(g,\eta)$ and hence $h_{\mu\times\lambda}(f\times g) \le h_\mu(f) + h_\lambda(g)$. But the family of partitions of $X\times Y$ of the form $\xi\times\eta$ where $H_\mu(\xi) < \infty$ and $H_\lambda(\eta) < \infty$ is sufficient with respect to any measure-preserving transformation of $X \times Y$. Hence $h_{\mu\times\lambda}(f\times g) = h_\mu(f) + h_\lambda(g)$ by Theorem 11.3.5. □

d. Ergodic decomposition of entropy. If $\mu \perp \nu$ in Theorem 4.1.3, the statement can be read as one about an invariant partition of the space into two pieces. That kind of statement holds in much greater generality:

Theorem 11.3.16. *If η is a measurable f-invariant partition by f-invariant sets, then $h(f) = \int_{X/\eta} h(f\restriction_B)\,d\mu_\eta(B)$.*

The proof of this result requires the development of additional properties of entropy for which it is essential to use infinite partitions. The title of the present section reflects an application of Theorem 11.3.16:

Corollary 11.3.17. *If η is the ergodic decomposition of (f,μ) (Theorem 3.3.37), then*

$$h(f) = \int_{X/\eta} h(f\restriction_B)\, d\mu_\eta(B).$$

Remark 11.3.18. Despite this "linearity", the behavior of measure-theoretic entropy as a function of the measure is rather subtle because it is often not continuous (with respect to the weak topology). The coexistence of this "linearity" with discontinuity is related to the fact that even on the set of ergodic measures entropy is not continuous; for example, a weak limit of periodic δ-measures may have positive entropy. This is, in fact, exactly how we obtain measures with large entropy (Proposition 4.3.12).

We now develop the needed further properties of the entropy with respect to a partition.

Lemma 11.3.19. *If $\xi \in \mathscr{P}_H$, η a measurable partition and $\xi \le \eta$ or $\eta \le \xi$ then*

$$\frac{1}{n} H(\xi_n^f \mid \eta_-^f) \to H(\xi \mid \xi_-^f).$$

Remark 11.3.20. By Corollary 11.2.26 one can restate this as $\frac{1}{n} H(\xi_n^f \mid \eta_-^f) \to h(f,\xi)$.

Proof. *Case 1: $\eta \le \xi$.* This does not use $H(\xi) < \infty$. Since $f^{-n}(\eta_-^f \vee \xi_{n-1}^f) \nearrow \xi_-^f$, we have $H(\xi \mid f^{-n}(\eta_-^f \vee \xi_{n-1}^f)) \to H(\xi \mid \xi_-^f)$ by Theorem 11.2.18. Also, Proposition 11.2.6.3 gives

$$\begin{aligned} H(\xi_n^f \mid \eta_-^f) &= H(\xi \mid \eta_-^f) + H(f(\xi) \mid \xi \vee \eta_-^f) + \cdots + H(f^n(\xi) \mid \xi_{n-1}^f \vee \eta_-^f), \\ &= H(\xi \mid \eta_-^f) + H(\xi \mid f^{-1}(\xi \vee \eta_-^f)) + \cdots + H(\xi \mid f^{-n}(\xi_{n-1}^f \vee \eta_-^f)), \end{aligned}$$

which implies the claim.
Case 2: $\xi \le \eta$. On one hand $H(\xi_n^f \mid \eta_-^f)/n \le H(\xi_n^f \mid \xi_-^f)/n \to H(\xi \mid \xi_-^f)$ from the previous case. On the other hand, (11.2.8) gives

$$H(\xi_n^f \mid \eta_-^f) = H(\eta_n^f \mid \eta_-^f) - H(\eta_n^f \mid \xi_n^f \vee \eta_-^f) \ge H(\eta \mid \eta_-^f) - H(\eta_n^f \mid \xi_n^f \vee \xi_-^f),$$

so, using Case 1 with ξ and η interchanged,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}H(\xi_n^f \mid \eta_-^f) &\geq H(\eta \mid \eta_-^f) - \lim_{n\to\infty}\frac{1}{n}H(\eta_n^f \mid \xi_n^f \vee \xi_-^f)\\ &= \lim_{n\to\infty}\Big(\frac{1}{n}H(\eta_n^f \mid \xi_-^f) - \frac{1}{n}H(\eta_n^f \mid \xi_n^f \vee \xi_-^f)\Big)\\ &= \lim_{n\to\infty}\frac{1}{n}H(\xi_n^f \mid \xi_-^f) = H(\xi \mid \xi_-^f),\end{aligned}$$

where the penultimate step again used Proposition 11.2.6.3. □

If ν is a partition, then one would expect $\nu^f_{(-\infty,-n]}$ to essentially "disappear" as $n \to -\infty$. The following statement is a way of making this precise in a specific context.

Lemma 11.3.21. *If $\xi,\eta \in \mathscr{P}_H$ are such that $\xi \leq \eta$ and ν is a measurable partition, then $H(\xi \mid \eta_-^f \vee f^{-n}(\nu_-^f)) \to H(\xi \mid \eta_-^f)$.*

Proof. We first treat the case $\xi = \eta$. By Lemma 11.3.19 applied to $\eta \leq \eta \vee \nu$ and Proposition 11.2.6.3 we have

$$\begin{aligned}H(\eta \mid \eta_-^f) &= \lim_{n\to\infty}\frac{1}{n}H(\eta_n^f \mid \eta_-^f \vee \nu_-^f)\\ &= \lim_{n\to\infty}\frac{1}{n}\Big(H(\eta \mid \eta_-^f \vee \nu_-^f) + \underbrace{H(f(\eta) \mid f(\eta_-^f) \vee \nu_-^f)}_{=H(\eta|\eta_-^f \vee f^{-1}(\nu_-^f))} + \cdots + \underbrace{H(f^n(\eta) \mid f^n(\eta_-^f) \vee \nu_-^f)}_{=H(\eta|\eta_-^f \vee f^{-n}(\nu_-^f))}\Big)\\ &= \lim_{n\to\infty}H(\eta \mid \eta_-^f \vee f^{-n}(\nu_-^f)).\end{aligned}$$

For arbitrary $\xi \leq \eta$, this and (11.2.8) (twice) now give

$$\begin{aligned}\lim_{n\to\infty}H(\xi \mid \eta_-^f \vee f^{-n}(\nu_-^f)) &= \lim_{n\to\infty}\Big(H(\eta \mid \eta_-^f \vee f^{-n}(\nu_-^f)) - H(\eta \mid \xi \vee \eta_-^f \vee f^{-n}(\nu_-^f))\Big)\\ &= H(\eta \mid \eta_-^f) - \underbrace{\lim_{n\to\infty}H(\eta \mid \xi \vee \eta_-^f \vee f^{-n}(\nu_-^f))}_{\leq H(\eta|\xi\vee\eta_-^f)}\\ &= H(\xi \mid \eta_-^f)\\ &\geq H(\xi \mid \eta_-^f \vee f^{-n}(\nu_-^f)).\end{aligned}$$ □

There is also a formula for computing the entropy with respect to the join of two partitions.

Proposition 11.3.22. *If $\xi,\eta \in \mathscr{P}_H$ then $h(f,\xi \vee \eta) = h(f,\eta) + H(\xi \mid \eta^f \vee \xi_-^f)$.*

Proof. $\frac{1}{n}H(\xi_n^f \vee \eta_n^f \mid \xi_-^f \vee \eta_-^f) = \frac{1}{n}H(\eta_n^f \mid \xi_-^f \vee \eta_-^f) + \frac{1}{n}\underbrace{H(\xi_n^f \mid \xi_-^f \vee \eta_-^f \vee \eta_n^f)}_{=\sum_{i=0}^n H(\xi|\xi_-^f \vee \eta_-^f \vee \eta_i^f)} \xrightarrow[n\to\infty]{}$ $H(\eta, \eta_-^f) + H(\xi \mid \xi_-^f \vee \eta^f)$ by Lemma 11.3.19. Now use Corollary 11.2.26. □

Corollary 11.3.23. *If $\eta \in \mathscr{P}_H$ and ξ is fixed by f (that is, consists of f-invariant sets), then $h(f, \xi \vee \eta) = h(f, \eta)$.*

Proof. For $\xi \in \mathscr{P}_H$ this is Proposition 11.3.22 (because the last term there vanishes). To reduce to this case take $\xi_n \nearrow \xi$ with finite entropy (and necessarily fixed); the claim then holds for the ξ_n and this gives Corollary 11.3.23 by Theorem 11.2.18 (Remark 11.2.19). □

Corollary 11.3.23 lends itself to a convex decomposition of entropy as follows.

Proposition 11.3.24. *If $\xi \in \mathscr{P}_H$ and η is a partition fixed by f, then with the notations of* (11.2.6) *and Definition 11.1.6 we have*

$$h(f, \xi) = \int_{X/\eta} h(f\restriction_B, \xi_B)\, d\mu_\eta(B).$$

Proof. Corollary 11.2.26 and (11.2.5) give

$$h(f\restriction_B, \xi_B) = H(\xi_B \mid (\xi_B)_-^{f\restriction_B}) = -\int_B \log \mu(\xi_B(x) \mid (\xi_B)_-^{f\restriction_B}(x))\, d\mu_B$$

so

$$\begin{aligned}\int_{X/\eta} h(f\restriction_B, \xi_B)\, d\mu_\eta(B) &= -\int_{X/\eta}\int_B \log \mu(\xi_B(x) \mid (\xi_B)_-^{f\restriction_B}(x))\, d\mu_B\, d\mu_\eta(B)\\ &= \int_X \log \mu(\xi(x) \mid (\eta \vee \xi_-^f)(x))\, d\mu = H(\xi \mid \eta \vee \xi_-^f),\end{aligned}$$

while Corollary 11.3.23 and Corollary 11.2.26 give

$$\begin{aligned}h(f, \xi) &= h(f, \xi \vee \eta)\\ &= H(\xi \vee \eta \mid (\xi \vee \eta)_-^f)\\ &= H(\xi \vee \eta \mid \eta \vee \xi_-^f)\\ &= H(\xi \mid \eta \vee \xi_-^f) + H(\eta \mid \xi \vee \eta \vee \xi_-^f)\\ &= H(\xi \mid \eta \vee \xi_-^f)\end{aligned}$$

by Proposition 11.2.6.3. □

Proof of Theorem 11.3.16. Let $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathscr{P}_H$ be a countable dense family of partitions and define $\xi_n := \bigvee_{i\le n} \zeta_i \in \mathscr{P}_H$. Then on one hand, Proposition 11.3.24 implies

$$h(f,\xi_n) = \int_{X/\eta} h(f\restriction_B, (\xi_n)_B)\, d\mu_\eta(B).$$

On the other hand, Proposition 11.3.10 implies

$$h(f,\xi_n) \nearrow h(f) \quad \text{and} \quad h(f\restriction_B, (\xi_n)_B) \nearrow h(f\restriction_B).$$

The claim then follows by the Monotone Convergence Theorem. □

e. The Shannon–Macmillan–Breiman Theorem. Definition 11.2.24 suggests that the average measure of elements of ξ^f_{-n} should be about $e^{-nh(f,\xi)}$. For ergodic transformations this turns out to be true in a much stronger sense.

Theorem 11.3.25 (Shannon–Macmillan–Breiman)**.** *If $f\colon X \to X$ is a μ-preserving ergodic transformation and ξ a measurable partition with finite entropy, then*

$$-\frac{1}{n}\log\mu(\xi^f_{-n}(x)) \xrightarrow[n\to\infty]{\text{a.e. and in } L^1} h_\mu(f,\xi),$$

where ξ^f_{-n} is as in Definition 11.2.22.

Remark 11.3.26. To keep the proof simple, we will assume that the partition is finite.

Lemma 11.3.27. *There is an h such that $-\lim\limits_{n\to\infty}\dfrac{1}{n}\log\mu(\xi^f_{-n}(x)) = h$ a.e.*

Proof. We write

$$I_n(x) := I[\xi^f_{-n}](x) = -\log\mu(\xi^f_{-n}(x)).$$

Since

$$f^{-1}\big(\xi^f_{1-n}(f(x))\big) = \Big(\bigvee_{i=2}^{n} f^{-i}(\xi)\Big)(x) \supset \xi^f_{-n}(x)$$

and f is measure-preserving, we have $I_{n-1}(f(x)) \le I_n(x)$. Thus

$$\varliminf_{n\to\infty}\frac{1}{n}I_n(f(x)) \le \varliminf_{n\to\infty}\frac{1}{n}I_n(x).$$

Corollary 3.3.23 implies that there is an $h \in \mathbb{R}$ such that

$$\varliminf_{n\to\infty}\frac{1}{n}I_n(x) \overset{\text{ae}}{=} h. \tag{11.3.1}$$

To establish

$$\lim_{n\to\infty}\frac{1}{n}I_n(x) \overset{\text{ae}}{=} h. \tag{11.3.2}$$

we show that $\overline{\lim}_{n\to\infty} \frac{1}{n} I_n(x) \le h$ a.e. Fix $\epsilon > 0$ and $L > 3$. If

$$\alpha_n := \{x \in X \mid \frac{1}{n} I_n(x) \le h + \epsilon\},$$

then (11.3.1) implies that $\mu\big(\bigcup_{n\ge L} \alpha_n\big) = 1$, so there is an $M \ge L$ such that

$$A := \bigcup_{L\le n\le M} \alpha_n$$

satisfies $\mu(A) > 1 - \epsilon$. The definition of A and α_n yields

$$\forall x \in A\, \exists q \in \mathbb{N}: \quad L \le q \le M \text{ and } \mu(\xi^f_{-n}(x)) \ge e^{-q(h+\epsilon)}. \tag{11.3.3}$$

The Birkhoff Ergodic Theorem shows that $\frac{1}{n}\sum_{i=0}^{n-1} \chi_A \circ F^i \xrightarrow[n\to\infty]{} \mu(A) > 1 - \epsilon$ a.e., hence in measure, so for $\delta > 0$ there is a $B \subset X$ with $\mu(B) > 1 - \delta$ and an $N \in \mathbb{N}$ with

$$\forall x \in B, n > N: \quad \frac{1}{n} \operatorname{card}\{i \mid 0 \le i < n \mid f^i(x) \notin A\} < 2\epsilon. \tag{11.3.4}$$

Claim 11.3.28. *If L is large enough, then $-M_n := \operatorname{card} \xi^f_{-n}\restriction_B \le e^{n(h+2\epsilon(1+\log\operatorname{card}\xi))}$ for all large $n \in \mathbb{N}$.*

M_n can alternatively be described as the number of elements of ξ^f_{-n} that intersect B in a set of positive measure. We prove this claim below.

The claim implies that the "bad" set

$$\{x \in B \mid \mu(\xi^f_{-n}(x)) < e^{-n(h+2\epsilon(2+\log\operatorname{card}\xi))}\}$$

has measure less than $M_n \cdot e^{-n(h+2\epsilon(2+\log\operatorname{card}\xi))} \le e^{-2n\epsilon}$, which is summable. By the Borel–Cantelli Lemma (Theorem 11.3.29) almost every x is in these bad sets for at most finitely many n, so

$$\overline{\lim_{n\to\infty}} \frac{1}{n} I_n(x) \le h + 2\epsilon(2 + \log\operatorname{card}\xi)$$

a.e. in B. Since δ is arbitrary, this holds a.e. on X, and as $\epsilon \to 0$, (11.3.2) follows. □

Here is the measure-theory result we invoked:

Theorem 11.3.29 (Borel–Cantelli)**.** *If $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$ then $\mu(\bigcap_{n\in\mathbb{N}} \bigcup_{i\ge n} A_i) = 0$, that is, almost every point lies in only finitely many A_n.*

Proof. $\mu(\bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n} A_i) \le \mu(\bigcup_{i\ge n} A_i) \le \sum_{i\ge n} \mu(A_i) \to 0$. □

Proof of Claim 11.3.28. Take $C \in \xi^f_{-n}$ with $\mu(C \cap B) > 0$. Then (11.3.3) and (11.3.4) imply that for $x \in C \cap B$ there are pairwise disjoint intervals $[m_k, n_k] \subset [1, n] \subset \mathbb{N}$ such that

(1) $L \le n_k - m_k \le M$,

(2) $\sum_k (n_k - m_k) \geq (1+2\epsilon)n$,

(3) $f^{m_k}(x) \in A$ and $\mu(\xi^f_{n_k-m_k}(f^{m_k}(x))) \geq e^{(m_k-n_k)(h+\epsilon)}$.

To see this, take $m_1 \coloneqq \min\{i \in [1,n] \mid f^i(x) \in A\}$, then $n_1 \coloneqq m_1 + q$, where q is as in (11.3.3) for $f^{m_1}(x)$. Then take $m_2 \coloneqq \min\{i \in (n_1, n] \mid f^i(x) \in A\}$ and so on.

To see how many different such C there can be note that each such C is determined by a sequence of choices of $C_i \in \xi$ for $i \in [1,n]$. In brief, we have

$$\#\text{choices of } C = \#\text{choices of } \{[m_k, n_k]\} \times \#\text{choices on } [m_k, n_k] \times \#\text{choices off } [m_k, n_k].$$

For some i there are $\operatorname{card}\xi$ choices, but (3) provides much better control for the collective choice corresponding to $[m_k, n_k]$. Thus, for a given choice of $\{(m_k, n_k)\}$ this allows at most

$$(\operatorname{card}\xi)^{2\epsilon n} \cdot e^{\sum_k (n_k - m_k)(h+\epsilon)} \leq e^{n(h+\epsilon+2\epsilon\log\operatorname{card}\xi)}$$

different such C.

On the other hand, the number of choices of $\{(m_k, n_k)\}$ can be bounded by noting that $1 \leq k \leq K \coloneqq \lfloor n/L \rfloor$; so we are choosing 2 subsets of $[1,n]$ (the m_k and the n_k) of cardinality at most K. A (generous) upper bound for the possibilities is given by

$$\Big[\sum_{i \leq K} \binom{n}{i}\Big]^2 \leq \Big[K\binom{n}{K}\Big]^2.$$

We bound $K\binom{n}{K}$ using $\binom{n}{K} = \dfrac{n!}{K!(n-K)!}$, the Stirling formula [**258**]

$$n! = \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n e^{\zeta_n} \quad \text{with} \quad \frac{1}{12n+1} < \zeta_n < \frac{1}{12n}, \tag{11.3.5}$$

and writing $K = \ell n$:

$$\begin{aligned} K\binom{n}{K} &= \ell n \frac{n!}{(\ell n)!((1-\ell)n)!} \\ &= \frac{\ell n}{\sqrt{2\pi}} \sqrt{\frac{n}{\ell n (1-\ell) n}} \frac{n^n}{(\ell n)^{\ell n}((1-\ell)n)^{(1-\ell)n}} \underbrace{e^{\zeta_n - \zeta_{\ell n} - \zeta_{(1-\ell)n}}}_{<1 \text{ since } a,b\in\mathbb{N} \Rightarrow \frac{1}{a+b} - \frac{1}{a+1} - \frac{1}{b+1} < 0} \\ &< \sqrt{\frac{\ell n}{2\pi}} \underbrace{\sqrt{\frac{\ell}{1-\ell}}}_{<1 \text{ for } L>1} \big(\ell^\ell (1-\ell)^{1-\ell}\big)^{-n} \\ &< \sqrt{\frac{K}{2\pi}} e^{-[\ell\log\ell + (1-\ell)\log(1-\ell)]n}. \end{aligned}$$

Take L so large that ℓ is small enough for $-[\ell\log\ell+(1-\ell)\log(1-\ell)]\le\epsilon/4$,[8] then take n large enough for $\sqrt{\frac{K}{2\pi}}<\sqrt{\frac{n}{2\pi}}<e^{n\epsilon/4}$to obtain $K\binom{n}{K}\le e^{n\epsilon/2}$, hence

$$M_n\le e^{n(h+\epsilon+2\epsilon\log\operatorname{card}\xi)}\left[K\binom{n}{K}\right]^2\le e^{n(h+\epsilon+2\epsilon\log\operatorname{card}\xi)}e^{\epsilon n}. \qquad \square$$

PROOF OF THE SHANNON–MACMILLAN–BREIMAN THEOREM. To prove convergence in mean, $-\dfrac{1}{n}\log\mu(\xi^f_{-n}(x))\xrightarrow[n\to\infty]{L^1}h$, we use uniform integrability.

Definition 11.3.30. We say that $\mathscr{F}\subset L^1$ is *uniformly integrable* if for $\epsilon>0$ there exists a $\delta>0$ such that $f\in\mathscr{F}$ and $\mu(A)<\delta$ imply $\int|f|\,d\mu<\epsilon$.

Theorem 11.3.31 (Vitali)**.** *If $f_n\to f$ in measure and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable, then $f_n\to f$ in L^1.*

PROOF. $0\le g_n:=|f_n-f|\to0$ in measure and is uniformly integrable. Given $\epsilon>0$ take $\delta>0$ such that $\int_A g_n<\epsilon/2$ for all n whenever $\mu(A)<\delta$ and $N\in\mathbb{N}$ such that $g_n(x)<\epsilon/2$ for all $n\ge N$ and x outside a set A of measure less than δ. Then $\int g_n=\int_A g_n+\int_{X\smallsetminus A}g_n<\epsilon/2+\epsilon/2$ for all $n\ge N$. $\square$

Given $\epsilon>0$, take $\delta\in(0,1/e)$ such that $-\delta\log\delta[1+\operatorname{card}\xi]<\epsilon$. Then

$$\begin{aligned}\mu(A)<\delta\Rightarrow\int_A\frac{1}{n}I_n\,d\mu&=-\frac{1}{n}\sum_{C\in\xi^f_{-n}}\mu(A\cap C)\underbrace{\log\mu(C)}_{\le\log\mu(A\cap C)}\\&\le-\frac{\mu(A)}{n}\sum_{C\in\xi^f_{-n}}\mu(C\mid A)\log(\mu(A)\cdot\mu(C\mid A))\\&=\mu(A)\big[\underbrace{\frac{1}{n}H(\xi^f_{-n}\mid A)}_{\le\frac{1}{n}\log\operatorname{card}\xi^f_{-n}\le\log\operatorname{card}\xi}-\frac{1}{n}\underbrace{\sum_{C\in\xi^f_{-n}}\mu(C\mid A)\log\mu(A)}_{=\log\mu(A)\le-1\text{ since }\mu(A)\le1/e}\big]\\&\le-\mu(A)\log\mu(A)[1+\log\operatorname{card}\xi]<\epsilon,\end{aligned}$$

which is uniform integrability. Therefore (11.3.2) and Theorem 11.3.31 imply

$$\frac{1}{n}I_n(x)\xrightarrow[n\to\infty]{L^1}h=\lim_{n\to\infty}\frac{I_n}{n}=\int\lim_{n\to\infty}\frac{I_n}{n}=\lim_{n\to\infty}\int\frac{I_n}{n}=\lim_{n\to\infty}\frac{1}{n}H(\xi^f_{-n})=h(f,\xi).\quad\square$$

[8] It is interesting to note that $\ell\log\ell+(1-\ell)\log(1-\ell)=-H(\{[0,\ell],[1-\ell,1]\})$.

f. Skew products. We now describe a class of examples to demonstrate that the addition of subexponential complexity does not increase the entropy.

Proposition 11.3.32. *Consider a probability space (Y,μ), an invertible measure-preserving transformation $f\colon Y \to Y$ and the transformation*

$$S\colon X := Y \times S^1 \to X, \quad (y,s) \mapsto (f(y), R_{\phi(y)}(s)),$$

where $\phi\colon Y \to S^1$ is measurable, R_α is the rotation (as in Example 11.1.8), and S^1 carries Lebesgue measure m. Then $h(S) = h(f)$.

Proof. Since f is a factor of S, we have $h(f) \le h(S)$. The main point is the reverse inequality.

We start with some choices and observations.

- Let $\{B_n\}_{n\in\mathbb{N}}$ be a basis for Y,[9]
- $\alpha_m := \bigvee_{n=0}^m \{B_n, Y \smallsetminus B_n\}$,
- $\xi_m := \bigvee_{n=0}^m \{B_n \times S^1, X \smallsetminus (B_n \times S^1)\} \nearrow \xi := \{I_y\}_{y\in Y}$, where $I_y := \{y\} \times S^1$,
- $\beta_m := \{[\frac{i}{m}, \frac{i+1}{m})\}_{i=0}^{m-1}$,
- $\zeta_m := \{Y \times [\frac{i}{m}, \frac{i+1}{m})\}_{i=0}^{m-1}$.

By Theorem 11.2.18 we have

$$H((\zeta_r)_n^S \mid \xi_m) \xrightarrow[m\to\infty]{} H((\zeta_r)_n^S \mid \xi) \le \log rn.$$

The latter inequality is due to the fact that no element of ξ is divided into more than rn pieces by $(\zeta_r)_n^S$. This is the main point: n shows up inside the logarithm.

Thus, we can fix $\epsilon > 0$, and for $r \in \mathbb{N}$ choose $n_r \in \mathbb{N}$ such that $\dfrac{\log rn_r}{n_r} < \dfrac{\epsilon}{2}$, and

$$m_r := 1 + \max\{i \in \mathbb{N} \mid H((\zeta_r)_n^S \mid \xi_i) \ge \log rn + \frac{\epsilon}{2}\} < \infty.$$

Setting

$$M_0 := 1 \quad \text{and} \quad M_r := \max\{M_{r-1}, m_r, r\} \quad \text{for} \quad r \in \mathbb{N}$$

[9] A *basis* $\mathscr{B}$ for a measure space $(X,\mathscr{S},\mu)$ is a countable collection $\mathscr{B} = \{B_i\}_{i\in\mathbb{N}} \subset \mathscr{S}$ whose union is X and for which there is a null set N such that for $x, y \in X \smallsetminus N$ there exists $B \in \mathscr{B}$ such that $x \in B$, $y \notin B$.

gives

$$\begin{aligned}
h(S) \xleftarrow[r\to\infty]{} h(S,\xi_{M_r}\vee\zeta_r) \xleftarrow[k\to\infty]{} \frac{1}{kn_r}\underbrace{H((\xi_{M_r}\vee\zeta_r)^S_{kn_r})}_{=H((\xi_{M_r})^S_{kn_r}\vee(\zeta_r)^S_{kn_r})=\left[H((\xi_{M_r})^S_{kn_r})+H((\zeta_r)^S_{kn_r}|(\xi_{M_r})^S_{kn_r})\right]}\\
&\le \frac{1}{kn_r}\Big[H((\alpha_{M_r})^f_{kn_r})+\underbrace{\sum_{i=0}^{k-1}H(S^{in_r}(\zeta_r)^S_{n_r}\mid(\xi_{M_r})^S_{kn_r})}_{\le kH((\zeta_r)^S_{n_r}|\xi_{M_r})\le k(\log(rn_r)+\frac{\epsilon}{2})}\Big]\\
&\le \frac{1}{kn_r}H((\alpha_{M_r})^f_{kn_r})+\epsilon\\
&\xrightarrow[k\to\infty]{} h(f,\alpha_{M_r})+\epsilon \xrightarrow[r\to\infty]{} h(f)+\epsilon.
\end{aligned}$$

The claim follows since ϵ was arbitrary. □

g. Induced maps.

Definition 11.3.33 (First-return map, section)**.** Let (X,μ) be a measure space, $A\subset X$ measurable with $\mu(A)>0$, $f\colon X\to X$ a measure-preserving transformation, and μ_A the conditional measure on A (see (3.3.3)). For $x\in X$ let $n_A(x):=\min\{n\in\mathbb{N}\mid f^n(x)\in A\}$. Then the μ_A-preserving transformation

$$f_A\colon A\to A,\quad f_A(x):=f^{n_A(x)}(x) \tag{11.3.6}$$

is called the *first-return map* induced by f on the set A. A is called a *section* for f if

$$A_f:=\bigcup_{n=0}^{\infty}f^{-n}A\overset{\text{ae}}{=}X.$$

That f_A preserves μ_A can be seen by considering one level set of n_A at a time. As one would expect, the average return time is $1/\mu(A)$:

Proposition 11.3.34. *Let (X,μ) be a measure space, $A\subset X$ measurable, $\mu(A)>0$, $f\colon X\to X$ measure-preserving. Then $\int_A n_A\,d\mu=\mu(\bigcup_{n=0}^{\infty}f^n(A))>0$.*

Proof. If $A_j:=n_A^{-1}(\{j\})$ then the $f^i(A_j)$ for $0\le i<j\in\mathbb{N}$ are pairwise disjoint, so

$$\mu(\bigcup_{n=0}^{\infty}f^n(A))=\mu(\bigcup_{0\le i<n\in\mathbb{N}}f^i(A_n))=\sum_{0\le i<n\in\mathbb{N}}\mu(f^i(A_n))=\sum_{n\in\mathbb{N}}n\mu(A_n)=\int_A n_A\,d\mu.$$ □

Lemma 11.3.35 (Kac)**.** *If μ is ergodic and $\mu(A)>0$, then $\int n_A\,d\mu_A=1/\mu(A)$.*

Proof. $1=\mu(\bigcup_{n=0}^{\infty}f^n(A))=\int_A n_A\,d\mu=\mu(A)\int n_A\,d\mu_A$. □

The Poincaré Recurrence Theorem guarantees that for any set of positive measure the map induced by a measure-preserving transformation is defined almost everywhere. We next show how the entropy of an induced map is related to that of the map from which it arises. One would expect it to scale by the average return time. This is indeed the case. Specifically, we prove:

Theorem 11.3.36 (Abramov)**.** *If $f\colon X \to X$ is an invertible μ-preserving map and $A \subset X$ is a section (Definition 11.3.33), then*

$$h_\mu(f) = h_{\mu_A}(f_A)\mu(A).$$

Remark 11.3.37. Note that by the Kac Lemma 11.3.35, this does amount to scaling by the return time and that for ergodic f any set A of positive measure is a section.

Proof (Neveu). The central concern of the proof is to understand the relationship between partitions of X and partitions of A. We pay special attention to their joint partitions (Definition 11.2.22) because we will use that by Corollary 11.2.26 we have

$$h_\mu(f,\xi) = H_\mu(\xi \mid \xi_-^f)$$

and a corresponding statement for $h_\mu(f_A)$.

A partition ξ of X naturally defines a partition

$$A \cap \xi \coloneqq \{A \cap C \mid C \in \xi\}$$

whenever $A \subset X$. For a collection A_i of disjoint sets in X and for partitions ξ_i, this gives a partition $\bigcup_i (A_i \cap \xi_i)$ of $\bigcup_i A_i$. The reader is encouraged to verify that if in addition one has a disjoint collection B_j and partitions η_j, then

$$\Big(\bigcup_i (A_i \cap \xi_i)\Big) \vee \Big(\bigcup_j (B_j \cap \eta_j)\Big) = \bigcup_{ij} \big((A_i \cap B_j) \cap (\xi_i \vee \eta_j)\big). \tag{11.3.7}$$

When $\{B_j\}_j = \{X\}$, this reduces to

$$\Big(\bigcup_i (A_i \cap \xi_i)\Big) \vee \eta = \bigcup_i A_i \cap (\xi_i \vee \eta).$$

We decompose the set A in the theorem according to the return times for f^{-1} to A:

$$A_n \coloneqq A \cap f(X \smallsetminus A) \cap \cdots \cap f^{n-1}(X \smallsetminus A) \cap f^n(A) \tag{11.3.8}$$

for $n \in \mathbb{N}$. Given a partition ξ of X, the partition

$$\xi_A \coloneqq \bigcup_{n\in\mathbb{N}} A_n \cap \xi_n^f,$$

of A (where ξ_n^f is as in Definition 11.2.22) is convenient for studying the return map. Inductively, we can express $(\xi_A)_{-l}^{f_A}$ in terms of ξ_{-n}^f as follows:

Claim 11.3.38. $(\xi_A)^{f_A}_{-l} = \bigcup\limits_{n_1,\dots,n_l\in\mathbb{N}} f^{-n_1}(A_{n_1})\cap\cdots\cap f^{-n_1-\cdots-n_l}(A_{n_l})\cap\xi^f_{-n_1-\cdots-n_l}$.

Proof. For $l=1$ this follows from

$$\begin{aligned}(\xi_A)^{f_A}_{-1} = f_A^{-1}(\xi_A) = f_A^{-1}(\bigcup_{n\in\mathbb{N}} A_n\cap\xi^f_n) &= \bigcup_{n\in\mathbb{N}} f_A^{-1}(A_n\cap\xi^f_n)\\ &= \bigcup_{n\in\mathbb{N}} f^{-n}(A_n\cap\xi^f_n) = \bigcup_{n\in\mathbb{N}} f^{-n}(A_n)\cap\xi^f_{-n}.\end{aligned}$$

Using (11.3.7), the step to $l+1$ is accomplished as follows.

$$\begin{aligned}(\xi_A)^{f_A}_{-l-1} &= f_A^{-1}\Big(\underbrace{\xi_A\vee\big((\xi_A)^{f_A}_{-l}\big)}_{=\bigcup_{n\in\mathbb{N}}(A_n\cap\xi^f_n)\vee\big((\xi_A)^{f_A}_{-l}\big)}\Big)\\
&= \bigcup_{n\in\mathbb{N}} f_A^{-1}\Big((A_n\cap\xi^f_n)\vee\big((\xi_A)^{f_A}_{-l}\big)\Big)\\
&= \bigcup_{n\in\mathbb{N}} f^{-n}\Big((A_n\cap\xi^f_n)\vee\big((\xi_A)^{f_A}_{-l}\big)\Big)\\
&= \bigcup_{n\in\mathbb{N}} \Big(\underbrace{f^{-n}(A_n)\cap f^{-n}(\xi^f_n)\vee f^{-n}\big((\xi_A)^{f_A}_{-l}\big)}_{=\big(f^{-n}(A_n)\cap\xi^f_{-n}\big)\vee f^{-n}\Big(\bigcup_{n_1,\dots,n_l\in\mathbb{N}} f^{-n_1}(A_{n_1})\cap\cdots\cap f^{-n_1-\cdots-n_l}(A_{n_l})\cap\xi^f_{-n_1-\cdots-n_l}\Big)}\Big)\\
&= \bigcup_{n,n_1,\dots,n_l\in\mathbb{N}}\Big(f^{-n}(A_n)\cap\cdots\cap f^{-n-n_1-n_2-\cdots-n_l}(A_{n_l})\Big)\cap\Big(\xi^f_{-n}\vee f^{-n}(\xi^f_{-n_1-\cdots-n_l})\Big)\\
&= \bigcup_{n_1,\dots,n_{l+1}\in\mathbb{N}} f^{-n_1}(A_{n_1})\cap\cdots\cap f^{-n_1-\cdots-n_{l+1}}(A_{n_{l+1}})\cap\xi^f_{-n_1-\cdots-n_{l+1}}.\end{aligned}$$ □

The purpose of Claim 11.3.38 is to ascertain the following

Claim 11.3.39. *If $\{f^{-1}(A), X\smallsetminus f^{-1}(A)\}\le\xi$, then (with the notations of Definition 11.2.22)*

$$(\xi_A)^{f_A}_{-} = A\cap\xi^f_{-} \quad \textit{and} \quad (\xi_A)^{f_A} = A\cap\xi^f.$$

Proof. The hypothesis means that $f^{-1}(A)$ (and hence $X\smallsetminus f^{-1}(A)$) is subordinate to ξ, that is, a union of elements of ξ. Using (11.3.8), this implies that

$$f^{-n_1}(A_{n_1})\cap\cdots\cap f^{-n_1-\cdots-n_l}(A_{n_l})$$

is subordinate to $A\cap\xi^f_{-n_1-\cdots-n_l}\le A\cap\xi^f_{-}$. Thus, Claim 11.3.38 implies that

$$(\xi_A)^{f_A}_{-l}\le A\cap\xi^f_{-}.$$

At the same time, Claim 11.3.38 shows that every element of $(\xi_A)^{f_A}_{-l}$ lies in A and in an element of ξ^f_{-l} (since $n_1+\cdots+n_l\ge l$), so

$$A\cap\xi^f_{-l}\le(\xi_A)^{f_A}_{-l}\le A\cap\xi^f_{-}.$$

Since $\xi^f_{-l} \nearrow \xi^f_-$ as $l \to \infty$, this implies $(\xi_A)^{f_A}_- = A \cap \xi^f_-$.

Analogously, $(\xi_A)^{f_A}_+ = A \cap \xi^f_+$, and together these give $(\xi_A)^{f_A} = A \cap \xi^f$. □

To compute the entropy first note that Corollary 11.2.26 and Claim 11.3.39 imply

$$\begin{aligned}\mu(A) h_{\mu_A}(f_A, \xi_A) = \mu(A) H_{\mu_A}(\xi_A \mid (\xi_A)^{f_A}_-) &= \int_A I[\xi_A \mid (\xi_A)^{f_A}_-]\, d\mu \\ &= \int_A I[\xi_A \mid \xi^f_-]\, d\mu, = \sum_{n\in\mathbb{N}} \int_{A_n} I[\xi^f_n \mid \xi^f_-]\, d\mu,\end{aligned}$$

where I is the conditional information function from (11.2.4), and we used the definition of ξ_A. At the same time, Corollary 11.2.26 gives

$$h_\mu(f, \xi) = H_\mu(\xi \mid \xi^f_-) = \int_X I[\xi \mid \xi^f_-]\, d\mu.$$

We thus need to show

Claim 11.3.40.

$$\sum_{n\in\mathbb{N}} \int_{A_n} I[\xi^f_n \mid \xi^f_-]\, d\mu = \int_X I[\xi \mid \xi^f_-]\, d\mu. \tag{11.3.9}$$

Proof. Proposition 11.2.6.3 says that $I[\xi_0 \vee \xi_1 \mid \eta] = I[\xi_0 \mid \eta] + I[\xi_1 \mid \eta \vee \xi_0]$, and recursively, this implies $I[\bigvee_{i=0}^{n-1} \xi_i \mid \eta] = \sum_{i=0}^{n-1} I[\xi_i \mid \eta \vee \bigvee_{j=0}^{i-1} \xi_j]$: For $n = 1$ this is clear, and

$$\begin{aligned}I[\bigvee_{i=0}^{n} \xi_i \mid \eta] = I[\bigvee_{i=0}^{n-1} \xi_i \mid \eta] + I[\xi_n \mid \eta \vee \bigvee_{i=0}^{n-1} \xi_i] \\ = \sum_{i=0}^{n-1} I[\xi_i \mid \eta \vee \bigvee_{j=0}^{i-1} \xi_j] + I[\xi_n \mid \eta \vee \bigvee_{i=0}^{n-1} \xi_i] = \sum_{i=0}^{n} I[\xi_i \mid \eta \vee \bigvee_{j=0}^{i-1} \xi_j].\end{aligned}$$

Thus, on the left-hand side of (11.3.9) we can write

$$I[\xi^f_n \mid \xi^f_-] = \sum_{i=0}^{n-1} I[f^i(\xi) \mid \xi^f_- \vee \bigvee_{j=0}^{i-1} \xi_j] = \sum_{i=0}^{n-1} I[f^i(\xi) \mid f^i\big(\xi^f_-\big)].$$

The conditional information function is “stationary” in that

$$I[f(\xi) \mid f(\eta)] = I[\xi \mid \eta] \circ f.$$

Therefore,

$$\sum_{n\in\mathbb{N}}\int_{A_n} I[\xi_n^f \mid \xi_-^f]\,d\mu, = \sum_{n\in\mathbb{N}}\int_{A_n}\sum_{i=0}^{n-1}\underbrace{I[f^i(\xi)\mid f^i\big(\xi_-^f\big)]}_{=I[\xi|\xi_-^f]\circ f^i}\,d\mu$$

$$= \sum_{n\in\mathbb{N}}\sum_{i=0}^{n-1}\int_{f^{-i}(A_n)} I[\xi \mid \xi_-^f]\,d\mu.$$

A is a section, so $X = \bigcup_{n\in\mathbb{N}}\bigcup_{i=0}^{n-1} f^{-i}(A_n)$, disjointly by definition of A_n. Thus,

$$\sum_{n\in\mathbb{N}}\sum_{i=0}^{n-1}\int_{f^{-i}(A_n)} I[\xi \mid \xi_-^f]\,d\mu = \int_X I[\xi \mid \xi_-^f]\,d\mu. \qquad \square$$

As noted, (11.3.9) implies $\mu(A)h_{\mu_A}(f_A,\xi_A) = h_\mu(f,\xi)$. To obtain Theorem 11.3.36, we pass to suprema over partitions.

Since we used Claim 11.3.39, and hence $\{f^{-1}(A), X\smallsetminus f^{-1}(A)\} \le \xi$, note that for any partition ζ we can choose $\xi := \{f^{-1}(A), X\smallsetminus f^{-1}(A)\}\vee\zeta$, so

$$h_\mu(f) = \sup_{\zeta\in\mathscr{P}_H} h_\mu(f,\zeta) = \sup_{\{f^{-1}(A),X\smallsetminus f^{-1}(A)\}\le\xi\in\mathscr{P}_H} h_\mu(f,\xi),$$

and likewise for the left-hand side:

$$\sup_{\zeta\in\mathscr{P}_H} h_{\mu_A}(f_A,\zeta_A) = \sup_{\{f^{-1}(A),X\smallsetminus f^{-1}(A)\}\le\xi\in\mathscr{P}_H} h_{\mu_A}(f_A,\xi_A).$$

To establish Theorem 11.3.36 it thus suffices, by Theorem 11.3.5, to check that the partitions ζ_A for $\zeta\in\mathscr{P}_H$ form a sufficient family. This is clear because if $\xi\in\mathscr{P}_H$ and η is any finite-entropy partition of A, then $\zeta := \xi\vee(\eta\cup\{X\smallsetminus A\}) \ge \eta$. □

CHAPTER 12

Appendix II: Hyperbolic maps and invariant manifolds

The main purpose of this appendix is to prove the Stable-Manifold Theorem (Theorem 12.5.2). This is done here because it is a discrete-time statement even though its application to time-t maps of flows gives the counterpart for continuous time. Another reason is that the length of the argument here would disrupt the flow of ideas in the main body of the book, and we here have the space to develop the ideas of the proof to further applications.

This main objective is preceded by work that is on one hand preliminary to it and on the other hand of importance in its own right. This is the Banach Contraction-Mapping Principle on one hand, and the study of hyperbolic linear maps in Banach spaces and perturbations of them. The principal application of proof ideas from the Stable-Manifold Theorem is the Inclination Lemma.

Section 12.7 is important beyond the purpose of this book. It builds on the Stable-Manifold Theorem by establishing absolute continuity of the resulting foliations, an essential ingredient for smooth ergodic theory. This result as well is proved for discrete time and applies to time-t maps of flows. The importance beyond this book lies in the fact that we reproduce here a result much more general than needed, which establishes absolute continuity for partially hyperbolic dynamical systems in a rather broad sense. The proof is due to Abdenur and Viana and has not been published elsewhere.

1. The Contraction-Mapping Principle

The Banach Contraction Principle is a fixed-point theorem that is particularly important in hyperbolic dynamics, and we present it here in order to highlight also the dependence of the fixed point on parameters, which is less commonly examined than its mere existence. It is also interesting that this theorem is established by studying a simple dynamical system, the contraction for which it is named.

A map $f\colon X \to X$ is said to be *contracting* if there exists $\lambda < 1$ such that for any $x, y \in X$

$$d(f(x), f(y)) \le \lambda d(x, y). \tag{12.1.1}$$

These maps exhibit both stability of equilibria in the sense of ordinary differential equations and in the sense of persistence under perturbation of the dynamical system. All orbits tend to a fixed point, and changing the contracting map slightly does not move the fixed point much. Some pertinent observations are cast in terms of a regularity notion that extends the one of Lipschitz continuity in a natural way.

Definition 12.1.1 (Lipschitz and Hölder Regularity)**.** Let (X,d), (Y,d) be metric spaces. A map $f\colon X\to Y$ is said to be *Lipschitz (continuous)* if there exists $C>0$ such that $d(x,y)<\epsilon$ implies $d(f(x),f(y))\le C(d(x,y))$, in which case f is said to be C-Lipschitz, and the *Lipschitz constant* $L(f)$ (or $\operatorname{Lip}(f)$) of f is defined by

$$L(f):=\sup_{x\ne y}\frac{d(f(x),f(y))}{d(x,y)}.$$

We say that f is *bi-Lipschitz* if it is Lipschitz and has a Lipschitz inverse.

A map $f\colon X\to Y$ is said to be *Hölder-continuous* with exponent α, or α*-Hölder*, if there exist $C,\epsilon>0$ such that $d(x,y)<\epsilon$ implies $d(f(x),f(y))\le C(d(x,y))^{\alpha}$. A Hölder-continuous map with Hölder-continuous inverse is said to be bi-Hölder.

Remark 12.1.2. This notion is both natural and useful in the context of hyperbolic dynamical systems because it corresponds to saying that if $d(x,y)$ tends to 0 exponentially (as a function of some parameter) then so does $d(f(x),f(y))$.

Proposition 12.1.3 (Contraction-Mapping Principle)**.** *Let X be a complete metric space and $f\colon X\to X$ a contracting map. Then f has a unique fixed point ϕ, and under the action of iterates of f all points converge exponentially to ϕ.*

Indeed, the error at any step can be estimated in terms of the size of the step:

$$d(x,\phi)\le\frac{1}{1-\lambda}d(x,f(x)). \tag{12.1.2}$$

Suppose X,Y are metric spaces, X complete, $f\colon X\times Y\to X$, $\lambda\in(0,1)$ such that $d(f_y(x),f_y(x'))\le\lambda d(x,x')$ for all $x,x'\in X$, $y\in Y$. Denote the fixed point of f_y by ϕ_y. Then

(1) $d(\phi_y,\phi_{y'})\le\frac{1}{1-\lambda}d(f_{y'}(\phi_{y'}),f_y(\phi_{y'}))$.
(2) If f is continuous then so is $y\mapsto\phi_y$.
(3) If $\alpha\in(0,1]$ and $y\mapsto f_y$ is α-Hölder-continuous,[1] *then so is $y\mapsto\phi_y$.*
(4) If X,Y are open subsets of Banach spaces and f is C^r, then so is $y\mapsto\phi_y$, with derivative

$$(1-D^Yf|_{(y,\phi_y)})^{-1}\circ D^Xf|_{(y,\phi_y)},$$

where the superscript denotes the differential in the respective space.

[1] uniformly in x, that is, $\exists\, C\in\mathbb{R}$ such that $d(f_y(x),f_{y'}(x))\le Cd(y,y')^{\alpha}$ for all $x\in X$, $y,y'\in Y$

(5) If $\lambda \in (0,1)$ and

$$d(f_y(x), f_{y'}(x')) \le \lambda \max\{d(x,x'), d(y,y')\}$$

for all $x, x' \in X$, $y, y' \in Y$, then $d(\phi_y, \phi_{y'}) \le \lambda d(y,y')$.

Proof. $\{f^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence because if $m \ge n$ then

$$d(f^m(x), f^n(x)) \le \sum_{k=0}^{m-n-1} \underbrace{d(f^{n+k+1}(x), f^{n+k}(x))}_{\le \lambda^{n+k} d(f(x),x)} \le \frac{\lambda^n}{1-\lambda} d(f(x),x) \xrightarrow[n\to\infty]{} 0. \tag{12.1.3}$$

Then $\phi \coloneqq \lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^{n+1}(x) = \lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x)) = f(\phi)$ exists since X is complete. (12.1.1) implies uniqueness[2], and $m \to \infty$ in (12.1.3) gives

$$d(f^n(x), \phi) \le \frac{\lambda^n}{1-\lambda} d(f(x), x).$$

This proves exponential convergence and for $n = 0$ gives (12.1.2).

(1): Apply (12.1.2) with $x = \phi_{y'} = f_{y'}(\phi_{y'})$.

(2) and (3) follow from (1), and (4) from the Implicit-Function Theorem.

(5): Take $x = \phi_y = f_y(\phi_y)$ and $x' = \phi_{y'} = f_{y'}(\phi_{y'})$ in the assumption and note that the maximum on the right-hand side must be $d(y,y')$. □

Remark 12.1.4. This in particular implies continuous dependence of the fixed point on the contraction when one makes C^1-perturbations.

The robustness of the asymptotic behavior of contractions in Proposition 12.1.3 has a counterpart for hyperbolic maps, even when they are perturbed so as to be nonlinear.

Theorem 12.1.5 (Hyperbolic Fixed-Point Theorem)**.** *If $A\colon E \to E$ is a bounded linear map of a Banach space E and $\mathrm{Id} - A$ is invertible, then a continuous map $F\colon E \to E$ has a unique fixed point ϕ if $\lambda \coloneqq L(F-A)\|(\mathrm{Id}-A)^{-1}\| < 1$. Furthermore, ϕ depends continuously on F, and $\|\phi\| \le \frac{1}{1-\lambda}\|F(0)\|$.*

Remark 12.1.6. $(\mathrm{Id}-A)^{-1}$ is bounded by the Open-Mapping Theorem.

Proof. ϕ is a solution of $(F-A)(x) = x - A(x) = (\mathrm{Id}-A)x$, hence a fixed point of the λ-contraction $(F-A)(\mathrm{Id}-A)^{-1}$. Apply (12.1.2) with $x = 0$. □

This is analogous to the persistence of the fixed point of a contraction under perturbations, but a hyperbolic fixed point is harder to find: The fixed point of a contraction is the limit of the forward orbit of any initial condition. Proposition 12.4.7 shows that this fails for hyperbolic maps except with a lucky starting point.

[2] $f(x) = x$ in (12.1.1) $\Rightarrow y = x$ or $y \ne f(y)$.

2. Generalized eigenspaces

Definition 12.2.1. A vector $v \in \mathbb{C}^n$ (or $\mathbb{R}^n$) is a *generalized eigenvector* of degree p for A if for some $\lambda \in \mathbb{C}$ we have $(A-\lambda I)^p v = 0$ and $(A-\lambda I)^{p-1} v \neq 0$.

Note that generalized eigenvectors of degree 1 are just eigenvectors. Also,

$$(A-\lambda I)^p v = (A-\lambda I)(A-\lambda I)^{p-1} v.$$

So λ is an eigenvalue of A and $(A-\lambda I)^{p-1} v$ is an eigenvector associated with λ.

Proposition 12.2.2. *If v is a generalized eigenvector of degree p for eigenvalue λ, then v, $(A-\lambda I)v, \dots, (A-\lambda I)^{p-1} v$ are linearly independent.*

Proof. Otherwise, there are $c_1, \dots, c_{p-1} \in \mathbb{C}$ with some $c_i \neq 0$ such that

$$c_1 v + c_2 (A-\lambda I) v + \cdots + c_{p-1}(A-\lambda I)^{p-1} v = 0$$

If c_k is the first nonzero coefficient, then $0 \le k \le p-2$ and

$$c_k (A-\lambda I)^k v = \sum_{j=k+1}^{p-1} c_j (A-\lambda I)^j v.$$

Applying $(A-\lambda I)^{p-k-1}$ to each side gives

$$0 \neq c_k (A-\lambda I)^{p-1} v = \sum_{j=k+1}^{p-1} c_j (A-\lambda I)^{p-k-1+j} v = 0,$$

since $p-k-1+j \ge p$ for $j > k$, a contradiction. □

For $A \in \mathcal{M}_n(\mathbb{R})$, let $\mathcal{N}(A) = \ker A$ be the nullspace (or kernel) of A. If λ is an eigenvalue, then

$$\{0\} \subset \mathcal{N}(A-\lambda I) \subseteq \mathcal{N}(A-\lambda I)^2 \subseteq \cdots,$$

while $\dim \mathcal{N}(A-\lambda I)^k \le n$ for k. So there is a smallest $k =: r(\lambda)$ at which the nullspace stabilizes ($\mathcal{N}(A-\lambda I)^k = \mathcal{N}(A-\lambda I)^{k+1}$), and we call $M(\lambda) := \mathcal{N}(A-\lambda I)^{r(\lambda)}$ the *generalized eigenspace* of A for λ.

Lemma 12.2.3. *The following hold.*

(1) $\mathcal{N}(A-\lambda I)^k = M(\lambda)$ *for all* $k \ge r(\lambda)$.
(2) $M(\lambda) = \{v \mid (A-\lambda I)^k v = 0$ *for some* $k \ge 1\}$.
(3) $r(\lambda)$ *is the maximal degree of the generalized eigenvectors for* λ.
(4) $\dim M(\lambda) \ge r(\lambda)$.

Proof. (1): Claim: $\mathcal{N}(A-\lambda I)^k = \mathcal{N}(A-\lambda I)^{k+1} \Rightarrow \mathcal{N}(A-\lambda I)^{k+1} \supseteq \mathcal{N}(A-\lambda I)^{k+2}$: If $v \in \mathcal{N}(A-\lambda I)^{k+2}$, then $(A-\lambda I)^{k+1}(A-\lambda I)v = 0$, so $(A-\lambda I)v \in \mathcal{N}(A-\lambda I)^{k+1} = \mathcal{N}(T-\lambda I)^k$, and$0 = (A-\lambda I)^k (A-\lambda I) v = (A-\lambda I)^{k+1} v$, so $v \in \mathcal{N}(A-\lambda I)^{k+1}$.

(2) is a consequence of (1).

(3) follows from the definition of degree.

(4): Proposition 12.2.2 gives $r(\lambda)$ linearly independent vectors in $M(\lambda)$. □

Now let $R(\lambda)$ be the range of $(A-\lambda I)^{r(\lambda)}$. Then $\dim M(\lambda)+\dim R(\lambda)=n$.

Proposition 12.2.4. *If λ is an eigenvalue of A, then $M(\lambda)$ and $R(\lambda)$ are A-invariant subspaces and $\mathbb{C}^n = M(\lambda)\oplus R(\lambda)$.*

Proof. We will use that $A(A-\lambda I)^k = (A-\lambda I)^k A$. If $v \in M(\lambda)$, then

$$(A-\lambda I)^{r(\lambda)} Av = A(A-\lambda I)^{r(\lambda)} v = A0 = 0,$$

so $Av \in M(\lambda)$, and $M(\lambda)$ is A-invariant.

If $w \in R(\lambda)$, then $w = (A-\lambda I)^{r(\lambda)} v$ for some v, so

$$A(w) = A(A-\lambda I)^{r(\lambda)} v = (A-\lambda I))^{r(\lambda)} Av \in R(\lambda).$$

Since $\dim M(\lambda)+\dim R(\lambda) = n$, we show $M(\lambda)\cap R(\lambda) = \{0\}$ to complete the proof.

If $0 \neq v \in R(\lambda)\cap M(\lambda)$, then there is a $u \in \mathbb{C}^n$ with $0 \neq v = (A-\lambda I)^{r(\lambda)} u$ and $(A-\lambda I)^{r(\lambda)} v = 0$. Thus, u is a generalized eigenvector of degree greater than $r(\lambda)$, contrary to Lemma 12.2.3. □

The eigenvalues of $A \in \mathscr{M}_n(\mathbb{R})$ are the roots of $\det(A-\lambda I) = (-1)^n(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p}$, where $\sum_{i=1}^{p} m_i = n$. The m_i are called the algebraic multiplicities.

Theorem 12.2.5. *Let $\lambda_1,\dots,\lambda_p$ be eigenvalues of A with algebraic multiplicity $m_1,\dots,m_p$. Then $\dim M(\lambda_j) = m_j$ for $1 \le j \le p$ and $\mathbb{C}^n = M(\lambda_1)\oplus\cdots\oplus M(\lambda_p)$.*

Proof. Let λ_j be an eigenvalue of A. Since $\mathbb{C}^n = M(\lambda_j)\oplus R(\lambda_j)$ by Proposition 12.2.4, A can be represented in a basis as

$$T = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_1 is a $\dim(M(\lambda_j))$ square block. Then

$$\det(A_1-\lambda I)\cdot\det(A_2-\lambda I) = \det(A-\lambda I) = (-1)^n \prod_{k=1}^{p} (\lambda-\lambda_k)^{m_k}.$$

Also, $(\lambda-\lambda_j)$ does not divide $\det(A_2-\lambda I)$ since λ_j is not an eigenvalue of A_2.

Then $(\lambda-\lambda_j)^{m_j}$ divides $\det(A_1-\lambda I)$. If there is a $k \neq j$ such that $(\lambda-\lambda_k)$ divides $\det(A_1-\lambda I)$, then $M(\lambda_j)$ has an eigenvector v for some λ_k which is a generalized eigenvector for λ_j, so

$$0 = (A-\lambda_j I)^p v = (A-\lambda_j I)^{p-1} \underbrace{(A-\lambda_j I) v}_{=(\lambda_k-\lambda_j)v} = (\lambda_k-\lambda_j)(A-\lambda_j I)^{p-1} v \neq 0.$$

Thus $(\lambda - \lambda_k)$ does not divide $\det(A_1 - \lambda I)$ for $k \neq j$, and $\det(A_1 - \lambda I) = (-1)^n(\lambda - \lambda_j)^{m_j}$, and $\dim(M(\lambda_j)) = m_j$.

To prove $\mathbb{C}^n = M(\lambda_1) \oplus \cdots \oplus M(\lambda_p)$, suppose to the contrary that $M(\lambda_j) \cap M(\lambda_k) \neq \{0\}$. Since

$$(A - \lambda_k I)(A - \lambda_j I) = (A - \lambda_j)(A - \lambda_k I),$$

$M(\lambda_j)$ is $(A - \lambda_k I)$-invariant and $M(\lambda_k)$ is $(A - \lambda_j I)$-invariant.

Now suppose there exists a nonzero vector $v \in M(\lambda_j) \cap M(\lambda_k)$ where v is of degree q in λ_k. Then $(A - \lambda_k I)^{q-1} v$ is an eigenvector for λ_k and in $M(\lambda_j) \cap M(\lambda_k)$. From previous arguments this is a contradiction.

Suppose there exists $v_1 + \cdots v_p = 0$ such that $v_j \in M(\lambda_j)$ for $1 \leq j \leq p$. Let

$$S_j = (A - \lambda_1 I)^{r(\lambda_1)} \cdots (A - \lambda_{j-1} I)^{r(\lambda_{j-1})} (A - \lambda_{j+1} I)^{r(\lambda_{j+1})} \cdots (A - \lambda_p I)^{r(\lambda_p)}.$$

Then $S_j(v_1 + \cdots v_p) = 0$ if and only if $v_j = 0$. So $v_1 = \cdots = v_p = 0$ and

$$\mathbb{C}^n = M(\lambda_1) \oplus \cdots \oplus M(\lambda_p). \qquad \square$$

The *geometric multiplicity* of λ_j is $\dim \ker(A - \lambda_j I)$. From the previous results the geometric multiplicity is at most the algebraic multiplicity.

3. The spectrum of a linear map

If a linear transformation of a finite-dimensional vector space has no eigenvalues on the unit circle, then the space is the direct sum of an expanding subspace (the sum of the generalized eigenspaces for eigenvalues outside of the unit circle) and a contracting subspace (the sum of the generalized eigenspaces for eigenvalues inside of the unit circle). The purpose of this subsection and the next is to prove the same for transformations of Banach spaces (Theorem 12.4.2).

This involves interesting functional analysis a dynamicist may not otherwise encounter frequently, but the reader may also take the conclusion of Theorem 12.4.2 as a definition of hyperbolicity and skip ahead to Section 12.5.

We now look at a similarly general context that combines contraction and expansion. Here a linear structure helps separate the two, so the natural generality in which this is effective is a Banach space.

It is convenient to consider Banach spaces over the complex numbers. The results we obtain in this context can be used for real Banach spaces E by passing to the complexification $E_{\mathbb{C}}$ (that is, the space $E \otimes \mathbb{C}$ obtained by allowing complex scalars) and then suitably restricting attention to the real part.

$B(z, r)$ denotes the ball of radius r around z in $\mathbb{C}$, and $S(z, r)$ its boundary.

Definition 12.3.1. Let E be a Banach space and $A \colon E \to E$ be a bounded linear map, that is, the norm $\|A\| := \sup_{\|v\|=1} \|Av\|$ of A is finite. The *resolvent set* $R(A)$

of A is the set of $\lambda \in \mathbb{C}$ for which $\lambda\,\mathrm{Id}-A$ has bounded inverse $R_A(\lambda)$, called the *resolvent* of A. We call $\operatorname{sp} A := \mathbb{C} \smallsetminus R(A)$ the *spectrum* of A. The *spectral radius* $r(A)$ of A is defined by $r(A) := \sup\{|\lambda| \mid \lambda \in \operatorname{sp} A\}$.

The *point spectrum* consists of the eigenvalues of A (and $\ker(A-\lambda\,\mathrm{Id})$ is the corresponding eigenspace).

The *continuous spectrum* is $\big\{\lambda \in \operatorname{sp} A \mid A-\lambda\,\mathrm{Id}$ injective, $\overline{(A-\lambda\,\mathrm{Id})(E)} = E\big\}$.

The *residual spectrum* is $\big\{\lambda \in \operatorname{sp} A \mid A-\lambda\,\mathrm{Id}$ is injective & $\overline{(A-\lambda\,\mathrm{Id})(E)} \neq E\big\}$.

Remark 12.3.2. If E is finite-dimensional, then $\operatorname{sp} A$ is the set of eigenvalues: those λ for which $A-\lambda\,\mathrm{Id}$ is not injective—which is also the set of λ for which $A-\lambda\,\mathrm{Id}$ is not surjective. In this case the spectral radius is therefore the largest modulus of an eigenvalue. Invertibility is the only issue in this context because all linear maps between finite-dimensional spaces are bounded. By the Open Mapping Theorem a bounded linear bijection between Banach spaces has bounded inverse, so

$$\operatorname{sp} A = \{\lambda \in \mathbb{C} \mid \lambda\,\mathrm{Id}-A \text{ is not injective}\} \cup \{\lambda \in \mathbb{C} \mid \lambda\,\mathrm{Id}-A \text{ is not surjective}\}.$$

Accordingly, the 3 items in Definition 12.3.1 are a decomposition of $\operatorname{sp} A$.

Lemma 12.3.3. $r(A) \le \|A\|$: $|\lambda|>\|A\| \Rightarrow \lambda \notin \operatorname{sp} A, R_A(\lambda) = \sum\limits_{i=0}^{\infty} \frac{A^i}{\lambda^{i+1}}$ *(Laurent series).*

Proof. $(\lambda\,\mathrm{Id}-A)\displaystyle\sum_{i=0}^{n-1} \frac{A^i}{\lambda^{i+1}} = \sum_{i=0}^{n-1} \frac{A^i}{\lambda^i} - \frac{A^{i+1}}{\lambda^{i+1}} = \mathrm{Id} - \frac{A^n}{\lambda^n} \xrightarrow[n\to\infty]{} \mathrm{Id}$. □

The spectral radius provides an asymptotically sharp bound:

Proposition 12.3.4 (Gelfand Spectral Radius Formula)**.** $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.

Proof. Since $a_n := \log\|A^n\|$ is subadditive, the limit exists by Lemma 4.2.7. By Lemma 12.3.3 the domain of convergence of the Laurent series $\sum_{i=0}^{\infty} A^i/\lambda^{i+1}$ of $R_A(\cdot)$ is $\{|\lambda| > r(A)\}$ while by the root test it is $\{|\lambda| > \lim_{n\to\infty} \|A^n\|^{1/n}\}$. □

Lemma 12.3.5. *If A is a bounded linear operator, then $R(A)$ is the natural domain of analyticity of $R_A(\cdot)$. Thus, $R(A)$ is open, and* $\operatorname{sp} A$ *is compact by Lemma 12.3.3.*

Proof. We show analyticity on $R(A)$ and that $d(\lambda, \operatorname{sp} A) \ge \|(R_A(\lambda)\|^{-1}$ on $R(A)$; this implies openness and $\|R_A(\lambda)\| \xrightarrow[d(\lambda,\operatorname{sp} A)\to 0]{} \infty$, hence the claim.

If $\lambda \in R(A)$ and $|\mu| < \|(R_A(\lambda)\|^{-1}$, then $\|\mu R_A(\lambda)\| < 1$, so $T(\mu) := \sum\limits_{i=0}^{\infty} \mu^i (R_A(\lambda))^{i+1}$ converges,[3] and

$$((\lambda-\mu)\,\mathrm{Id}-A)\,T(\mu) = (\lambda\,\mathrm{Id}-A)\,T(\mu) - \mu T(\mu) = \sum_{i=0}^{\infty} (\mu R_A(\lambda))^i - (\mu R_A(\lambda))^{i+1} = \mathrm{Id},$$

[3] This is the Neumann series for the inverse of $(\lambda-\mu)\,\mathrm{Id}-A = (\lambda\,\mathrm{Id}-A)-\mu\,\mathrm{Id}$

so $\lambda-\mu\in R(A)$ and $R_A(\lambda-\mu)=T(\mu)$ is analytic at $\mu=0$. □

Remark 12.3.6 (Resolvent Equation)**.** For $\mu,\lambda\in R(A)$, multiplying

$$(\mu\,\mathrm{Id}-A)(\lambda\,\mathrm{Id}-A)[R_A(\lambda)-R_A(\mu)]=(\mu\,\mathrm{Id}-A)-(\lambda\,\mathrm{Id}-A)=(\mu-\lambda)\,\mathrm{Id}$$

by $R_A(\lambda)R_A(\mu)$ gives the *resolvent equation*

$$R_A(\lambda)-R_A(\mu)=(\mu-\lambda)R_A(\lambda)R_A(\mu). \tag{12.3.1}$$

Proposition 12.3.7. sp $A\neq\varnothing$ *unless* $E=\{0\}$.

Proof. If sp $A=\varnothing$, then R_A is entire. It is bounded on $\overline{B(0,2\|A\|)}$ by compactness, and $\|R_A(\lambda)\|\le\|A\|^{-1}$ for $|\lambda|>2\|A\|$ because

$$\|(\lambda\,\mathrm{Id}-A)v\|=\|\lambda(\mathrm{Id}-\frac{A}{\lambda})v\|\ge 2\|A\|\cdot\frac{1}{2}\|v\|.$$

Being bounded and entire, R_A is constant by the Liouville Theorem, which implies that Id $=0$, hence $E=\{0\}$. □

The Liouville Theorem applies to this situation since for a bounded linear functional $f\in E^*$, $f\circ R_A$ is an entire bounded scalar function and hence constant.

If A is diagonal, then clearly $\|A\|=r(A)$. The following counterpart to Proposition 5.1.5 is useful for understanding the dynamics of linear maps even if they cannot be diagonalized.

Proposition 12.3.8. *For every* $\delta>0$ *there exists an equivalent norm on* E *with respect to which* $\|A\|<r(A)+\delta$*. This is called an* adapted *or* Lyapunov norm.

Proof. Take n such that $\|A^n\|<(r(A)+\delta)^n$ and $|v|:=\sum_{i=0}^{n-1}\|A^i v\|(r(A)+\delta)^{-i}$. Then

$$\frac{|Av|}{|v|}=\frac{\sum\limits_{i=1}^{n}\|A^i v\|(r(A)+\delta)^{1-i}}{\sum\limits_{i=0}^{n-1}\|A^i v\|(r(A)+\delta)^{-i}}=(r(A)+\delta)\Big[\underbrace{1+\frac{\|A^n v\|(r(A)+\delta)^{-n}-\|v\|}{\sum_{i=0}^{n-1}\|A^i v\|(r(A)+\delta)^{-i}}}_{<1}\Big]. \quad \square$$

Remark 12.3.9. One can conclude from this that for any equivalent norm and for every $\epsilon>0$ there exists C_ϵ such that $\|A^n v\|\le C_\epsilon(r(A)+\epsilon)^n\|v\|$ for any $v\in\mathbb{R}^n$.

Corollary 12.3.10. *If* sp$(A)\subset B(0,1)$*, then there is an equivalent norm on* E *such that* A *is a contraction with respect to the metric generated by that norm.*

Proof. Apply Proposition 12.3.8 with $0<\delta<1-r(A)$ (>0 by compactness). □

The concept of exponential convergence does not depend on a particular choice of an equivalent norm. Thus Proposition 12.1.3 and Corollary 12.3.10 imply

Corollary 12.3.11. *If* $\operatorname{sp}(A) \subset B(0,1)$*, then the positive iterates of every point converge exponentially to the origin. If in addition A is invertible map, then negative iterates of every point go to infinity exponentially.*

4. Hyperbolic linear maps

Next, we consider maps with both contraction and expansion.

Definition 12.4.1. A bounded linear map A of a Banach space E is said to be *hyperbolic* if $\operatorname{sp} A \cap S(0,1) = \varnothing$. It is said to be (ℓ^-,ℓ^+)*-hyperbolic* if $0 < \ell^- < 1 < \ell^+$ and $\operatorname{sp} A \cap \{z \in \mathbb{C} \mid \ell^- \le |z| \le \ell^+\} = \varnothing$.

Theorem 12.4.2. *If E is a Banach space,* $A\colon E \to E$ *continuous linear,* $\gamma := S(0,r) \subset R(A)$*, then there are* $0 < \ell^- < r < \ell^+$ *such that* $\{z \in \mathbb{C} \mid \ell^- \le |z| \le \ell^+\} \subset R(A)$, $\lambda(A) := r(A^-) < \ell^-$ *and* $\mu(A) := 1/r(A^{-1}\upharpoonright_{E^+}) > \ell^+$ *(notation as in 2. below), that is,* $\|(A^-)^n\| \in O(\lambda^n)$ *and* $\|(A^+)^{-n}\| \in O(\mu^{-n})$ *(see Remark 3.2.18). In particular, if A is hyperbolic (*$r = 1$*), then there are* $0 < \ell^- < 1 < \ell^+$ *such that A is* (ℓ^-,ℓ^+)*-hyperbolic.*

If $\gamma \subset \mathbb{C}$ *is a smooth curve bounding a topological disk D and* $\operatorname{sp} A \cap \gamma = \varnothing$*, then there are linear subspaces* E^- *and* E^+ *of E such that*

(1) $E = E^- \oplus E^+$*,*
(2) $AE^- \subset E^-$ *(with equality if* $0 \notin \operatorname{sp} A$*),* $AE^+ = E^+$*; we write* $A^\pm := A\upharpoonright_{E^\pm}$*,*
(3) $\operatorname{sp} A^- = \operatorname{sp}^- A := \operatorname{sp} A \cap D$*,* $\operatorname{sp} A^+ = \operatorname{sp}^+ A := \operatorname{sp} A \smallsetminus D$*.*

Remark 12.4.3. If $\ell^- < 1 < \ell^+$, then these conditions in turn imply that A is hyperbolic, so this is a characterization of hyperbolicity.

If $E^\pm$ are both nontrivial, then the spectrum is contained in 2 annuli. This result readily generalizes to larger numbers of annuli; for instance, if $0 < r_1 < r_2$ and $\operatorname{sp} A \cap S(0,r_i) = \varnothing$, then $\operatorname{sp} A$ lies in the union of 3 annuli; the corresponding subspaces are $E^-_{r_1}$, $E^+_{r_1} \cap E^-_{r_2}$, and $E^+_{r_2}$. Linear maps for which all three subspaces in this decomposition are nontrivial are said to be *partially hyperbolic* if $r_1 < 1 < r_2$.

As in Corollary 12.3.10, there is an *adapted norm* (or *Lyapunov norm*) associated with such (ℓ^-,ℓ^+), that is, a norm $|\cdot|$ equivalent to the given one with

$$\|A^-\| \le \ell^-,\ \|(A^+)^{-1}\| \le 1/\ell^+, \text{ and } |v^- + v^+| = \max(|v^-|,|v^+|) \text{ for } v^\pm \in E^\pm.$$

(Take Lyapunov norms $|\cdot|$ for $A^\pm$ and $|v^- + v^+| := \max(|v^-|,|v^+|)$ for $v^\pm \in E^\pm$.)

Definition 12.4.4. If $\ell^- < 1 < \ell^+$, then E^- is called the *contracting* subspace and E^+ the *expanding* subspace.

Remark 12.4.5. The expanding subspace is not characterized by the fact that vectors in it expand under iterates of the map—all vectors outside the contracting subspace are expanded by a sufficiently large iterate of the map. The characterization of E^+ is given by the description of Remark 12.4.3, namely that preimages contract.

Proof of Theorem 12.4.2. Compactness of $\operatorname{sp} A$ implies the first assertions and the existence of a smooth Jordan curve γ' with γ inside it and $\operatorname{sp} A \setminus D$ outside it.

Claim 12.4.6. $\pi^- := \dfrac{1}{2\pi i}\displaystyle\int_\gamma R_A(\lambda)\,d\lambda = \dfrac{1}{2\pi i}\displaystyle\int_{\gamma'} R_A(\lambda)\,d\lambda$ *is a projection.*

Proof. $\dfrac{1}{2\pi i}\displaystyle\int_c \dfrac{1}{\mu-\lambda}\,d\mu = \begin{cases} 1 & \text{if } \lambda \text{ is inside } c \\ 0 & \text{if } \lambda \text{ is outside } c \end{cases}$ for $c \in \{\gamma,\gamma'\}$, so

$$\pi^-\pi^- = \frac{1}{2\pi i}\int_\gamma R_A(\lambda)\,d\lambda \cdot \frac{1}{2\pi i}\int_{\gamma'} R_A(\mu)\,d\mu = \Big(\frac{1}{2\pi i}\Big)^2 \int_\gamma\int_{\gamma'} \underbrace{R_A(\lambda)R_A(\mu)}_{=\frac{R_A(\lambda)-R_A(\mu)}{\mu-\lambda}\text{ by (12.3.1)}}\,d\mu\,d\lambda$$

$$= \Big(\frac{1}{2\pi i}\Big)^2 \Big[\int_\gamma R_A(\lambda)\underbrace{\int_{\gamma'}\frac{1}{\mu-\lambda}\,d\mu}_{=2\pi i\text{ since }\lambda\in\gamma\text{ inside }\gamma'}\,d\lambda - \int_{\gamma'} R_A(\mu)\underbrace{\int_\gamma \frac{1}{\mu-\lambda}\,d\lambda}_{=0\text{ since }\mu\in\gamma'\text{ outside }\gamma}\,d\mu\Big]$$

$$= \frac{1}{2\pi i}\int_\gamma R_A(\lambda)\,d\lambda = \pi^-. \qquad \square$$

(1): $\pi^+ := \mathrm{Id} - \pi^-$ is then also a projection; take $E^\pm := \pi^\pm(E)$.
(2): $A(E^\pm) = A(\pi^\pm(E)) = \pi^\pm(A(E)) \subset \pi^\pm(E) = E^\pm$ because A commutes with $R_A(\cdot)$ and hence with $\pi^\pm$. $AE^+ = E^+$ because below we show that $0 \notin \operatorname{sp} A^+$.
(3): $E = E^- \oplus E^+$ and $A(E^\pm) \subset E^\pm$ give $\operatorname{sp} A = \operatorname{sp} A^- \oplus A^+ = \operatorname{sp} A^- \cup \operatorname{sp} A^+$, so we show $\operatorname{sp} A^- \subset D$ and $\operatorname{sp} A^+ \cap D = \varnothing$.

$$\underbrace{(\lambda\,\mathrm{Id} - A)}_{=(\mu\,\mathrm{Id}-A)+(\lambda-\mu)\,\mathrm{Id}}\frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda-\mu}R_A(\mu)\,d\mu = \frac{1}{2\pi i}\int_\gamma R_A(\mu) - \frac{\mathrm{Id}}{\mu-\lambda}\,d\mu = \begin{cases} \pi^- & \text{if } \lambda \notin D\cup\gamma, \\ \pi^- - \mathrm{Id} = -\pi^+ & \text{if } \lambda \in D. \end{cases}$$

If $\lambda \notin D \cup \gamma$, restrict to E^- to see that $\lambda\,\mathrm{Id} - A^-$ is invertible, so $\lambda \notin \operatorname{sp} A^-$, and $\operatorname{sp} A^- \subset D$. If $\lambda \in D$, restrict to E^+ to get $\operatorname{sp} A^+ \cap D = \varnothing$, hence (3). □

We now describe the asymptotics of iterates of a hyperbolic linear map.

Proposition 12.4.7. *If E is a Banach space, $A\colon E \to E$ hyperbolic linear, then*

(1) *For every $v \in E^-$, the positive iterates $A^n v$ converge to the origin with exponential speed as $n \to \infty$ and if A is invertible then the negative iterates $A^n v$ go to infinity with exponential speed as $n \to -\infty$.*
(2) *For every $v \in E^+$ the positive iterates of v go to infinity exponentially and if A is invertible then the negative iterates converge exponentially to the origin.*

(3) *For every $v \in E \smallsetminus (E^- \cup E^+)$ the iterates $A^n v$ go to infinity exponentially as $n \to \infty$ and if A is invertible also as $n \to -\infty$.*

Proof. This is mainly a restatement of Theorem 12.4.2 and Remark 12.4.3. If $v \in \mathbb{R}^n \smallsetminus (E^- \cup E^+)$ write $v = v^- + v^+$ where $v^- \in E^- \smallsetminus \{0\}$, $v^+ \in E^+ \smallsetminus \{0\}$ to get

$$\|A^n v\| = \|A^n(v^- + v^+)\| \geq \|A^n v^+\| - \|A^n v^-\| \geq \lambda^n c\|v^+\| - \lambda^{-n} c'\|v^-\| \geq \lambda^n c'',$$

for large positive n, where $\lambda > 1$ and $c, c', c'' > 0$ do not depend on n.

The argument for negative iterates is the same with v^+ and v^- exchanged. □

With the present notations one can recast Theorem 12.1.5 as follows.

Theorem 12.4.8 (Hyperbolic Fixed-Point Theorem II)**.** *If A is a (λ, μ)-hyperbolic bounded linear map of a Banach space and $F\colon E \to E$ is such that $\ell := L(F - A) < \epsilon := \min(1-\lambda, 1-\mu^{-1})$ (see Definition 12.1.1), then F has a unique fixed point $\phi \in E$, and $|\phi| < |F(0)|/(\epsilon - \ell)$, where $|\cdot|$ is an adapted norm. ϕ depends continuously on F.*

This version is more explicit about the closeness assumption in terms of known parameters, but it uses hyperbolicity rather than just $1 \in R(A)$.

Proof. Write $E = E^- \times E^+$, $\pi^\pm\colon E \to E^\pm$, $x \mapsto x^\pm$ for the projections, $F^\pm := \pi^\pm \circ F$ and show that $\bar{F}(x) := \big(F^-(x), x^+ + (A^+)^{-1}(x^+ - F^+(x))\big)$ is a $(1 + \ell - \epsilon)$-contraction. □

Remark 12.4.9. The generality of the present context is motivated by its utility when applied in auxiliary spaces, and the Hyperbolic Fixed-Point Theorem 12.1.5 can be used to prove a variety of results in hyperbolic dynamical systems, including some of our main theorems such as structural stability [**170**, Theorem A]. We immediately show one instance of this: Theorem 12.1.5 can be greatly amplified by applying the very same result in a suitable infinite-dimensional space to show that the dynamics of the almost-linear map f in Theorem 12.1.5 does not only match that of the linear map in that there is a unique fixed point, but that the entire orbit structure of f is the same as that of A.

Theorem 12.4.10. *Let A be a (λ, μ)-hyperbolic bounded linear map of a Banach space and f_1, f_2 Lipschitz-continuous maps with $\Delta f_i := f_i - A$ bounded and*

$$\ell := \max L(\Delta f_i) < \epsilon := \min(1-\lambda, 1-\mu^{-1}, \|A^{-1}\|^{-1}). \tag{12.4.1}$$

Then there is a unique continuous map $h = h_{f_1,f_2}\colon E \to E$ such that $f_1 \circ h = h \circ f_2$ and $\Delta h := h - \mathrm{Id} \in \mathscr{E} := C_b(E, E)$ (bounded continuous maps with the sup norm).

Proof. The f_i are invertible: $f_i(x) = y \Leftrightarrow x = A^{-1}(y - \Delta f_i(x))$, and the right-hand side is an $\ell\|A^{-1}\|$-contraction, so there is a unique such x.

We can thus rewrite the desired conclusion as $f_1 \circ h \circ f_2^{-1} = h$ or

$$(A + \Delta f_1) \circ (\mathrm{Id} + \Delta h) \circ f_2^{-1} = \mathrm{Id} + \Delta h \qquad \text{or}$$

$$\mathscr{F}(\Delta h) \coloneqq \underbrace{A \circ \Delta h \circ f_2^{-1}}_{=:\mathscr{A}(\Delta h) \in \mathscr{E}} + \underbrace{\Delta f_1 \circ (\mathrm{Id} + \Delta h) \circ f_2^{-1} + A \circ f_2^{-1} - \mathrm{Id}}_{=:\Delta\mathscr{F}(\Delta h) \in \mathscr{E}} = \Delta h \in \mathscr{E},$$

a fixed-point problem for $\mathscr{F} = \mathscr{A} + \Delta\mathscr{F}$. $\mathscr{A}$ is hyperbolic: $\mathscr{E} = \mathscr{E}^- \oplus \mathscr{E}^+$, where $\mathscr{E}^\pm \coloneqq C_b(E, E^\pm) = \mathscr{A}(\mathscr{E}^\pm)$, $\|\mathscr{A}^-\| \le \lambda$, and $\|(\mathscr{A}^+)^{-1}\| \le 1/\mu$. Since $L(\Delta\mathscr{F}) \le L(\Delta f_1) < \epsilon$, Theorem 12.1.5 provides the desired unique fixed point $\Delta h \in \mathscr{E}$, and $h \coloneqq \mathrm{Id} + \Delta h$ is the required continuous map. □

This does not quite produce what we promised; for the orbit structures of the maps to be the same, h must be a homeomorphism. This is an easy consequence.

Corollary 12.4.11 (Hartman–Grobman)**.** *Let A be a (λ, μ)-hyperbolic bounded linear map of a Banach space, $f\colon E \to E$ Lipschitz with $\Delta f \coloneqq f - A$ bounded, ϵ as in* (12.4.1)*, and $\ell \coloneqq L(\Delta f) < \epsilon$. Then there is a unique homeomorphism $h\colon E \to E$ depending continuously on f with $h - \mathrm{Id}$ bounded and $h \circ A = f \circ h$.*

Proof. h in Theorem 12.4.10 is a homeomorphism because $f_1 \circ h_{f_1,f_2} = h_{f_1,f_2} \circ f_2$ and (by symmetry) $f_2 \circ h_{f_2,f_1} = h_{f_2,f_1} \circ f_1$, hence

$$f_2 \circ [h_{f_2,f_1} \circ h_{f_1,f_2}] = h_{f_2,f_1} \circ f_1 \circ h_{f_1,f_2} = [h_{f_2,f_1} \circ h_{f_1,f_2}] \circ f_2,$$
$$f_1 \circ [h_{f_1,f_2} \circ h_{f_2,f_1}] = h_{f_1,f_2} \circ f_2 \circ h_{f_2,f_1} = [h_{f_1,f_2} \circ h_{f_2,f_1}] \circ f_1,$$

so uniqueness in Theorem 12.4.10 gives $h_{f_2,f_1} \circ h_{f_1,f_2} = \mathrm{Id} = h_{f_1,f_2} \circ h_{f_2,f_1}$. □

We now describe a localization procedure that connects the global picture in a linear space (such as in Corollary 12.4.11) with local analysis on a manifold.

On a smooth compact manifold M we can choose a Riemannian metric, and then there is an open set $B \subset TM$ such that $0 \in B_x \coloneqq B \cap T_xM$ and $\exp_x\colon B_x \to M$ is an embedding of B_x with $\exp_x(0) = x$.

Theorem 12.4.12. *If f is a C^1-diffeomorphism of M with a compact invariant set Λ, take $\epsilon_0 > 0$ and a C^1-neighborhood U of f such that $g(\exp_x(v)) \in \exp_{f(x)}(B_{f(x)})$ for $g \in U$, $x \in \Lambda$, $\|v\| \le 2\epsilon_0$. If $\rho\colon \mathbb{R} \to [0,1]$ is smooth, $\rho([0,1]) = \{1\}$, $\rho([2,\infty)) = \{0\}$, and $\epsilon < \epsilon_0$ and U are sufficiently small, then the* localization

$$G_x(v) \coloneqq D_xf(v) + \rho(\|v\|/\epsilon)\big(\exp_{f(x)}^{-1} \circ g \circ \exp_x(v) - D_xf(v)\big)$$

of $g \in U$ by is arbitrarily uniformly C^1-close to D_xf.

Proof. Near $v = 0$ this is the choice of ϵ, U; for $\|v\| \ge 2\epsilon$ we have equality. □

Remark 12.4.13. The point is that the continuous map $G\colon T_\Lambda M \to T_\Lambda M := TM_{\restriction_\Lambda}$ defined by $G_{\restriction_{T_xM}} = G_x$ fibers over f, that is, $G_x(T_xM) \subset T_{f(x)}M$, and satisfies

$$\begin{aligned} G(v) &= D_x f(v) && \text{when } \|v\| \geq 2\epsilon \\ \exp_{f(x)} G(v) &= g(\exp_x(v)) && \text{when } \|v\| \leq \epsilon. \end{aligned}$$

Corollary 12.4.11 immediately translates to the following.

Theorem 12.4.14 (Hartman–Grobman Theorem)**.** *Let M be a smooth manifold, $U \subset M$ open, $f\colon U \to M$ continuously differentiable, and $p \in U$ a hyperbolic fixed point of f, that is, the differential $D_p f\colon T_pM \to T_pM$ at p is a hyperbolic linear map. Then there exist neighborhoods U_1, U_2 of p and V_1, V_2 of $0 \in T_pM$ as well as a homeomorphism $h\colon U_1 \cup U_2 \to V_1 \cup V_2$ such that $f = h^{-1} \circ Df_p \circ h$ on U_1, that is, the following diagram commutes:*

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "h"'] & U_2 \arrow[d, "h"] \\ V_1 \arrow[r, "D_p f"] & V_2 \end{tikzcd}$$

5. Admissible manifolds: the Hadamard method

We prove the existence of unstable manifolds by the Hadamard graph transform method. It obtains unstable manifolds as limits of manifolds of an approximately right kind. More specifically, Hadamard's approach is to consider graphs over the unstable subspace and apply the dynamics to these in order to discern successive improvement that leads to an application of the Contraction Mapping Principle. Figure 12.5.2 shows this very idea iconically: The unstable stretch and stable contraction combine to make such graphs "nicer", and they do so in a way that in a suitable norm defines a contraction with a rate that is determined by the contraction and expansion rates. Hadamard's original paper made a point of explaining the core idea well rather than being as strong as possible, and it still makes good reading today [**148**]. The framework in which we present it has the advantage of producing a result that is more general in ways that are essential for some applications. Specifically, admissible (rather than stable or unstable) manifolds are an important product of these arguments.

Recall that for a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ the set of all eigenvalues of A is denoted by $\operatorname{sp}(A)$ (Definition 12.3.1). If A is hyperbolic we define the slowest contraction and expansion rates of A by

$$\lambda(A) := r(A\restriction_{E^-}) = \sup\{|\chi| \mid \chi \in \operatorname{sp}(A), \quad |\chi| < 1\},$$
$$\mu(A) := 1/r(A^{-1}\restriction_{E^+}) = \inf\{|\chi| \mid \chi \in \operatorname{sp}(A), \quad |\chi| > 1\},$$

where the subspaces E^+ and E^- are as in Definition 12.4.4 and Theorem 12.4.2.

By Proposition 12.3.8 for any $\delta > 0$ one can introduce a norm in $\mathbb{R}^n$ such that $\|A\restriction_{E^-}\| < \lambda(A) + \delta$ and $\|A^{-1}\restriction_{E^+}\| < \mu^{-1}(A) + \delta$.

Now we proceed to the local analysis near a general (nonperiodic) orbit. The differentials of the iterates f^k, $k \in \mathbb{Z}$, along such an orbit can not be reduced to the iterates of a single linear map but should be viewed as products of different linear maps. Thus, we can not talk about eigenvalues any more, but rather should define hyperbolicity in terms of expansion and contraction of tangent vectors. We also generalize the situation somewhat by allowing a more general kind of exponential splitting for linear maps into "fast-expanding" or "fast-contracting" directions and the rest. As in the case of a single point one can choose appropriate coordinate systems centered at the points of the reference orbit and express both the nonlinear map and its differential in those coordinates.

Definition 12.5.1. Let $\lambda < \mu$. A sequence of invertible linear maps $L_m \colon \mathbb{R}^n \to \mathbb{R}^n$, $m \in \mathbb{Z}$, is said to admit a (λ, μ)-*splitting* if there exist decompositions $\mathbb{R}^n = E_m^\mu \oplus E_m^\lambda$ such that $L_m E_m^i = E_{m+1}^i$ for $i = \lambda, \mu$ and

$$\|L_m\restriction_{E_m^\lambda}\| \le \lambda, \quad \|L_m^{-1}\restriction_{E_{m+1}^\mu}\| \le \mu^{-1}.$$

We say that $\{L_m\}_{m\in\mathbb{Z}}$ *admits an exponential splitting* or *is partially hyperbolic in the broad sense* if it admits a (λ, μ)-splitting for some λ, μ and $\lambda < 1$, $\dim E_m^\lambda \ge 1$ or $\mu > 1$, $\dim E_m^\mu \ge 1$. We say that $\{L_m\}_{m\in\mathbb{Z}}$ is *hyperbolic* (or *uniformly hyperbolic*) if it admits a (λ, μ)-splitting for some $\lambda < 1 < \mu$. In this case we set $E_m^- := E_m^\lambda$ and $E_m^+ := E_m^\mu$.

By viewing $\mathbb{R}^n$ as a canonical product $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and making a sequence of orthogonal coordinate changes in $\mathbb{R}^n$ one can assume in the previous definition that $E_m^\mu = \mathbb{R}^k \times \{0\}$, $E_m^\lambda = \{0\} \times \mathbb{R}^{n-k}$ for some k, $0 \le k \le n$, and all m.

Thus we have reduced the problem of the local behavior of the iterates of a diffeomorphism near a reference orbit to the study of a sequence of local diffeomorphisms $f_m \colon U_m \to \mathbb{R}^n$, where each U_m is a neighborhood of the origin in $\mathbb{R}^n$ containing a ball of some fixed radius, fixing the origin and such that the sequence of linear maps at the origin $(Df_m)_0$, $m \in \mathbb{Z}$, admits an exponential splitting. Although we are interested only in points whose successive images stay in the neighborhoods, it is convenient to artificially extend our maps from somewhat smaller neighborhoods to the whole space $\mathbb{R}^n$ using Theorem 12.4.12

Here is the stable–unstable manifold theorem in the desired generality.

Theorem 12.5.2 (Hadamard–Perron Theorem)**.** *Let $\lambda < \mu$, $r \ge 1$, and for each $m \in \mathbb{Z}$ let $f_m\colon \mathbb{R}^n \to \mathbb{R}^n$ be a (surjective) C^r diffeomorphism such that for $(x,y) \in \mathbb{R}^k \oplus \mathbb{R}^{n-k}$*

$$f_m(x,y) = (A_m x + \alpha_m(x,y),\ B_m y + \beta_m(x,y))$$

for some linear maps $A_m\colon \mathbb{R}^k \to \mathbb{R}^k$ and $B_m\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ with $\|A_m^{-1}\| \le \mu^{-1}$, $\|B_m\| \le \lambda$ and $\alpha_m(0) = 0$, $\beta_m(0) = 0$.

Then for $0 < \gamma < \min\big(1, \sqrt{\mu/\lambda} - 1\big)$ and

$$0 < \delta < \min\left(\frac{\mu-\lambda}{\gamma+2+1/\gamma},\ \frac{\mu-(1+\gamma)^2\lambda}{(1+\gamma)(\gamma^2+2\gamma+2)}\right)$$

we have: If $\|\alpha_m\|_{C^1} < \delta$ and $\|\beta_m\|_{C^1} < \delta$ for all $m \in \mathbb{Z}$ then there is

(1) a unique family $\{W_m^+\}_{m\in\mathbb{Z}}$ of k-dimensional C^1 manifolds

$$W_m^+ = \{(x, \varphi_m^+(x)) \mid x \in \mathbb{R}^k\} = \operatorname{graph}\varphi_m^+$$

and

(2) a unique family $\{W_m^-\}_{m\in\mathbb{Z}}$ of $(n-k)$-dimensional C^1 manifolds

$$W_m^- = \{(\varphi_m^-(y), y) \mid y \in \mathbb{R}^{n-k}\} = \operatorname{graph}\varphi_m^-,$$

where $\varphi_m^+\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$, $\varphi_m^-\colon \mathbb{R}^{n-k} \to \mathbb{R}^k$, $\sup_{m\in\mathbb{Z}}\|D\varphi_m^\pm\| < \gamma$, and the following properties hold:

(i) $f_m(W_m^-) = W_{m+1}^-$, $f_m(W_m^+) = W_{m+1}^+$.

(ii) $\|f_m(z)\| < \lambda'\|z\|$ for $z \in W_m^-$,
$\|f_{m-1}^{-1}(z)\| < (\mu')^{-1}\|z\|$ for $z \in W_m^+$,
where $\lambda' := (1+\gamma)\big(\lambda + \delta(1+\gamma)\big) < \dfrac{\mu}{1+\gamma} - \delta =: \mu'$.

(iii) Let $\lambda' < \nu < \mu'$. If $\|f_{m+L-1} \circ \cdots \circ f_m(z)\| < C\nu^L\|z\|$ for all $L \ge 0$ and some $C > 0$ then $z \in W_m^-$.
Similarly, if $\|f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(z)\| \le C\nu^{-L}\|z\|$ for all $L \ge 0$ and some $C > 0$ then $z \in W_m^+$.

Finally, if $\mu \ge 1$, then the families $\{W_m^+\}_{m\in\mathbb{Z}}$ consists of C^r manifolds, and if $\lambda \le 1$, then $\{W_m^-\}_{m\in\mathbb{Z}}$ consists of C^r manifolds.

As we intimated before, little of the proof uses the assumption that $\gamma < 1$, giving results about "fast-unstable" manifolds.

The "stationary" special case without dependence of our data on m (corresponding to iterates of a single locally defined map f) may be good to keep in mind on the first reading of the arguments, and it gives

Theorem 12.5.3. *Let p be a hyperbolic fixed point of a local C^r diffeomorphism $f\colon U \to M$, $r \ge 1$. Then there exist C^r embedded discs W_p^+, $W_p^- \subset U$ such that*

- $T_p W_p^{\pm} = E^{\pm}(Df_p)$,
- $f(W_p^-) \subset W_p^-$, *and*
- $f^{-1}(W_p^+) \subset W_p^+$.

There also exist $\lambda < 1 < \mu$ *such that* $\operatorname{sp} Df_p \cap \{z \in \mathbb{C} \mid \lambda \le |z| \le \mu\} = \varnothing$ *and* $C(\delta)$ *such that if* $y \in W_p^-$, $z \in W_p^+$, $m \ge 0$, *then*

$$d(f^m(y), p) < C(\delta)\left(\lambda(Df_p) + \delta\right)^m d(y, p),$$
$$d(f^{-m}(z), p) < C(\delta)\left(\mu^{-1}(Df_p) + \delta\right)^m d(z, p).$$

Furthermore, there exists $\delta_0 > 0$ *such that*

$$\text{if } d(f^m(y), p) \le \delta_0 \text{ for } m \ge 0 \quad \text{then} \quad y \in W_p^-,$$
$$\text{if } d(f^m(z), p) \le \delta_0 \text{ for } m \le 0 \quad \text{then} \quad z \in W_p^+.$$

In fact, there exist a neighborhood $O \subset U$ *of* p *and* C^r *coordinates* $\psi\colon O \to \mathbb{R}^n$ *such that* $\psi(W_p^+ \cap O) \subset \mathbb{R}^k \oplus \{0\}$ *and* $\psi(W_p^- \cap O) \subset \{0\} \oplus \mathbb{R}^{(n-k)}$ *(*adapted coordinates*).*

Remark 12.5.4. The discs W_p^+ and W_p^- are not uniquely defined, but their *germs* are: for any two discs satisfying the assertion of this theorem for W_p^+, their intersection contains a neighborhood of p in each of them. In other words, they are open subsets of a common larger submanifold. The same property holds for W_p^-.

Remark 12.5.5 (Application by localization)**.** The Hadamard method plays a central role in the theory of hyperbolic dynamical systems and applications of Theorem 12.5.2 require *localization* of the context in which it is stated. Fix $r > 0$ and let

$$D_r = \{(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^{n-k} \mid \|x\| \le r, \|y\| \le r\}, \quad W_{m,r}^{\pm} = W_m^{\pm} \cap D_r.$$

If $\lambda' < 1$ (this is true if $\lambda < 1$ and γ and δ are sufficiently small), then by (ii) $f_m(W_{m,r}^-) \subset W_{m+1,r}^-$ and W_m^- is contracted under the action of f_m. Thus in this case $W_{m,r}^-$ is determined by the action of f_m on D_r only. Similar comments apply to W_m^+ if $\mu' > 1$. Thus in these situations we obtain meaningful objects from local data.

If one tries to apply Theorem 12.5.2 via local charts and the previous extension procedure then one obtains meaningful objects (independent of the extensions and determined by local data) *only* in the two cases of the preceding paragraph ($\lambda' < 1$ for W^- or $\mu' > 1$ for W^+).

In particular in the hyperbolic case for sufficiently small γ, δ we have $\lambda' < 1 < \mu'$ and both W_m^+ and $W_{m,r}^+$ are determined locally. In this case W_m^- and W_m^+ are usually called the *stable manifolds* and the *unstable manifolds* at the origin, correspondingly. Furthermore, we can put $v = 1$ in (iii). That shows that stable and unstable manifolds are defined purely topologically, namely,

$$W_m^- = \{z \in \mathbb{R}^n \;|\; \|f_{m+L-1} \circ \cdots \circ f_m z\| \xrightarrow[L\to\infty]{} 0\},$$
$$W_m^+ = \{z \in \mathbb{R}^n \;|\; \|f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1} z\| \xrightarrow[L\to\infty]{} 0\}.$$

In the course of the proof we show that the sequence of differentials $(Df_m)_0$, $m \in \mathbb{Z}$, admits a (λ', μ')-splitting. This immediately implies that $T_0 W_m^\pm = E_m^\pm$.

By considering successive images $p_m = f_{m-1} \circ \cdots \circ f_0(p)$ for $m \geq 0$ and $p_m = f_m^{-1} \circ f_{m+1}^{-1} \circ \cdots \circ f_{-1}^{-1}(p)$ for $m < 0$ of any point $p \in \mathbb{R}^n$ and translating the coordinate systems so that they become centered at p_m, we obtain maps

$$f_m^p(z) = f_m(z + p_m) - p_{m+1}$$

satisfying the hypotheses of the theorem. Thus we can construct manifolds $W_{m,p}^+$ and $W_{m,p}^-$ passing through p and satisfying appropriately modified assertions. In particular (ii) and (iii) imply that if $W_{m,p}^+ \cap W_{m,q}^+ \neq \emptyset$ then $W_{m,p}^+ = W_{m,q}^+$ (similarly for $W_{m,p}^-$). Furthermore, $f_m(W_{m,p}^\pm) = W_{m+1,f_m p}^\pm$. Thus, $\mathbb{R}^n$ splits in two ways into invariant families of manifolds. Naturally, the fields of tangent planes to those manifolds are invariant under the differentials Df_m.

The proof of the Hadamard–Perron Theorem 12.5.2 consists of five steps:

Step 1. Construction of invariant cone families.

Step 2. Construction of invariant sequences of plane fields inside the invariant cone families. Here we also explore other implications of the existence of invariant cones which are used later on a number of occasions.

Step 3. Construction of invariant Lipschitz graphs via an application of the Contraction Mapping Principle to an appropriate operator (graph transform).

Step 4. Verification of differentiability.

Step 5. C^r-smoothness in the hyperbolic case.

In order to distinguish tangent vectors from points in the Euclidean space we usually denote by $(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ a point in $\mathbb{R}^n$ and by $(u, v) \in \mathbb{R}^k \oplus \mathbb{R}^{n-k} \cong T_{(x,y)}\mathbb{R}^n$ a tangent vector at (x, y).

The remainder of this section carries out this proof in these 5 steps.

a. Invariant cones. We begin by defining cone fields.

Definition 12.5.6. If a normed vector bundle E over a metric space Λ decomposes into $E^1 \oplus E^2$, then the *standard horizontal* γ*-cone* field is defined by

$$H_p^\gamma := \{u + v \in E_p^1 \oplus E_p^2 \;|\; \|v\| \leq \gamma \|u\|\}.$$

The *standard vertical γ-cone* is

$$V_p^\gamma := \{u + v \in E_p^1 \oplus E_p^2 \mid \|u\| \le \gamma \|v\|\}.$$

By a *cone field* we mean a map that associates to every point $p \in \mathbb{R}^n$ a cone K_p in $T_p\mathbb{R}^n$. These cone fields are said to be bounded if there is a constant c such that

$$\|u+v\|/c \le \|u\| + \|v\| \le c\|u+v\|$$

for all $p \in \Lambda$, $u \in E_p^1$, $v \in E_p^2$. For a given cone K, the *dual cone* K^* is the closure of the complement of K.

If Λ is an invariant set for a diffeomorphism $f\colon M \to M$, then f naturally acts on cone fields on $E := T_\Lambda M$ by

$$(f_*K)_p := Df_{f^{-1}(p)}(K_{f^{-1}(p)}).$$

We say that a cone family K is (strictly) *invariant* if

$$(f_*K)_p \subset \operatorname{Int} K_p \cup \{0\};$$

we write

$$f_*K \Subset K.$$

Let us look at some examples to clarify the picture involved here. In dimension $n = 2$ all cones look alike. A horizontal cone $|x_2| \le \gamma|x_1|$ is shaded on the left of Figure 12.5.1. Its dual cone is a vertical cone given by $|x_1| \le |x_2|/\gamma$ and. In dimension

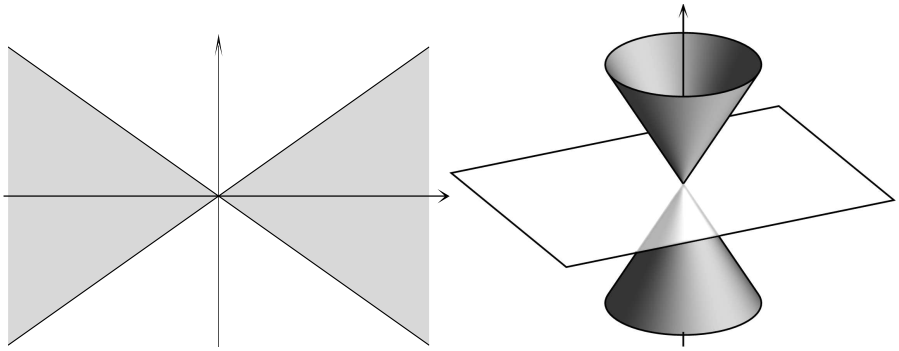


FIGURE 12.5.1. A horizontal cone and a vertical cone

$n = 3$ the following is obviously a cone: Let $u = x_1$, $v = (x_2, x_3)$, $\sqrt{x_2^2 + x_3^2} \le \gamma|x_1|$. So is its dual cone, described by $u = (x_2, x_3)$, $v = x_1$ and $|x_1| \le \sqrt{x_2^2 + x_3^2}/\gamma$. This cone does not look like those designed to hold ice cream. By a *cone family* we mean a sequence of cone fields (Definition 12.5.6). A sequence $f = \{f_m\}_{m\in\mathbb{Z}}$ of diffeomorphisms acts on cone families by

$$(f_* K)_{p,m} = (Df_{m-1})_{f_{m-1}^{-1}(p)}(K_{f_{m-1}^{-1}(p),m-1}).$$

We say that a cone family K is (strictly) *invariant* if

$$(f_* K)_{p,m} \subset \operatorname{Int} K_{p,m} \cup \{0\}.$$

We consider the action of a sequence $f = \{f_m\}_{m\in\mathbb{Z}}$ satisfying the hypotheses of Theorem 12.5.2 on the standard horizontal and vertical cone families which assign to $p \in \mathbb{R}^n$ the cones H_p^γ and V_p^γ for all m, correspondingly.

Lemma 12.5.7. *If* $\delta < \dfrac{\mu-\lambda}{2+\gamma+1/\gamma}$ *then*

$$(Df_m)_p(H_p^\gamma) \subset \operatorname{Int} H_{f_m(p)}^\gamma \text{ and } (Df_m)_p^{-1}(V_{f_m(p)}^\gamma) \subset \operatorname{Int} V_p^\gamma.$$

Proof. If $(u,v) \in H_p^\gamma$, that is, $\|v\| \le \gamma\|u\|$, and

$$(u',v') = (Df_m)_p(u,v) = (A_m u + (D\alpha_m)_p(u,v), B_m v + (D\beta_m)_p(u,v))$$

then

$$\|v'\| = \|B_m v + (D\beta_m)_p(u,v)\| \le \|B_m v\| + \|(D\beta_m)_p(u,v)\| < \lambda\|v\| + \delta\|(u,v)\|.$$

We also have

$$\|u'\| = \|A_m u + (D\alpha_m)_p(u,v)\| \ge \|A_m u\| - \|(D\alpha_m)_p(u,v)\| > \mu\|u\| - \delta\|(u,v)\|$$

and since $\|(u,v)\| \le \|u\| + \|v\| \le (1+\gamma)\|u\|$ we get

$$\|u'\| > \big(\mu - \delta(1+\gamma)\big)\|u\|. \tag{12.5.1}$$

Now $\delta < \dfrac{\mu-\lambda}{2+\gamma+1/\gamma}$, so $\delta(1+\gamma)^2 < \gamma(\mu-\lambda)$, whence

$$\lambda\gamma + \delta(1+\gamma) < \gamma(\mu - \delta(1+\gamma)) \tag{12.5.2}$$

and $\|v'\| < \lambda\|v\| + \delta\|(u,v)\| \le (\lambda\gamma + \delta(1+\gamma))\|u\| < \gamma(\mu-\delta(1+\gamma))\|u\| < \gamma\|u'\|$.

Invariance of vertical γ-cones means that $V_{f_m(p)}^\gamma \subset \operatorname{Int}(Df_m)_p V_p^\gamma$ or, equivalently, $(Df_m)_p(H_p^{1/\gamma}) \subset \operatorname{Int}(H_{f_m(p)}^{1/\gamma})$. But this follows from the observation that the preceding estimates still hold when γ is replaced by $1/\gamma$. □

Let $\tilde{V}^\gamma = f_* V^\gamma$. We now show that vectors in horizontal cones expand and those in vertical cones contract.

Lemma 12.5.8.

$$\|(Df_m)_p(u,v)\| > \left(\frac{\mu}{1+\gamma} - \delta\right)\|(u,v)\| \qquad \textit{for } (u,v) \in H_p^\gamma \textit{ and}$$

$$\|(Df_m)_p(u,v)\| < (1+\gamma)(\lambda+\delta)\|(u,v)\| \qquad \textit{for } (u,v) \in \tilde{V}_p^\gamma.$$

Proof. With the above notation, (12.5.1) implies

$$\|(u',v')\| \ge \|u'\| > \big(\mu - \delta(1+\gamma)\big)\|u\| \ge \frac{\mu-\delta(1+\gamma)}{1+\gamma}\|(u,v)\|$$

for $(u,v) \in H_p^\gamma$. If $(u,v) \in \tilde{V}_p^\gamma$, then $(u',v') \in V_{f_m(p)}^\gamma$, and (12.5.1) yields

$$\underbrace{\|(u',v')\|}_{\le\|u'\|+\|v'\|} \le (1+\gamma)\|v'\| < (1+\gamma)[\lambda\|v\| + \delta\|(u,v)\|] \le (1+\gamma)(\lambda+\delta)\|(u,v)\|. \qquad \square$$

b. Invariant sequences of plane fields. Now we explore the relation between the existence of an invariant sequence of cones and the exponential splitting for a sequence of linear maps. The conclusions of Lemma 12.5.7 and Lemma 12.5.8 applied along each orbit make the results of this step applicable to our setting.

Proposition 12.5.9. *Let $\lambda' < \mu'$ and and $L_m\colon \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ a sequence of invertible linear maps such that there is an $\epsilon > 0$ such that for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ there are $\gamma_m, \gamma'_m > 0$ for which*

(1) $L_m H^{\gamma_m} \subset \operatorname{Int} H^{\gamma_{m+1}}$;
(2) $L_m^{-1} V^{\gamma'_{m+1}} \subset \operatorname{Int} V^{\gamma'_m}$;
(3) $\|L_{m-1} \circ \cdots \circ L_{m-n}(u,v)\| > \epsilon \mu'^n \|(u,v)\|$ for $(u,v) \in H^{\gamma_{m-n}}$;
(4) $\|L_{m-1} \circ \cdots \circ L_{m-n}(u,v)\| < \epsilon^{-1} \lambda'^n \|(u,v)\|$ for $(u,v) \in L_{m-n}^{-1} \circ \cdots \circ L_{m-1}^{-1} V^{\gamma'_m}$.

Then

$$E_m^{\mu'} := \bigcap_{i=0}^{\infty} L_{m-1} \circ L_{m-2} \circ \cdots \circ L_{m-i} H^{\gamma_{m-i}}$$

is a k-dimensional subspace inside H^{γ_m} and

$$E_m^{\lambda'} := \bigcap_{i=0}^{\infty} L_m^{-1} \circ L_{m+1}^{-1} \circ \cdots \circ L_{m+i}^{-1} V^{\gamma'_{m+i+1}}$$

is an $(n-k)$-dimensional subspace inside $V^{\gamma'_m}$.

Proof. Since $\mathbb{R}^k \times \{0\} \subset H^\gamma$ for all γ, condition (1) implies that

$$\begin{aligned} S_j &:= L_{m-1} \circ L_{m-2} \circ \cdots \circ L_{m-j}(\mathbb{R}^k \times \{0\}) \\ &\subset L_{m-1} \circ L_{m-2} \circ \cdots \circ L_{m-j} H^{\gamma_{m-j}} =: T_j. \end{aligned}$$

For each S_j take an ordered orthonormal basis and consider a subsequence such that the sequences of basis elements all converge. Since the intersection of T_j

with the unit sphere is compact it contains the basis consisting of the limits of the basis elements. By the same token any sequence of vectors defined by a fixed set of coefficients converges to a vector in T_j. Hence the span S of the limiting basis belongs to all T_j and thus to the intersection. We need to show that $S = E_m^{\mu'}$.

If $(u, v) \in E_m^{\mu'}$ then, since $S \subset H^{\gamma_m}$ is transverse to $\{0\} \times \mathbb{R}^{n-k}$, we can write $(u, v) = (u, v') + (0, v'')$ with $(u, v') \in S$.

If we let

$$\begin{aligned}(u_j, v_j) &\coloneqq L_{m-j}^{-1} \circ \cdots \circ L_{m-1}^{-1}(u, v),\\ (u'_j, v'_j) &\coloneqq L_{m-j}^{-1} \circ \cdots \circ L_{m-1}^{-1}(u, v'),\\ (u''_j, v''_j) &\coloneqq L_{m-j}^{-1} \circ \cdots \circ L_{m-1}^{-1}(0, v'')\end{aligned}$$

then $(u, v) \in E_m^{\mu'}$ implies that $(u_j, v_j) \in H^{\gamma_{m-j}}$ and by (3) $\|(u_j, v_j)\| \le \epsilon^{-1}(\mu')^{-j}\|(u, v)\|$. By the same token $\|(u'_j, v'_j)\| \le \epsilon^{-1}(\mu')^{-j}\|(u, v')\|$. Thus since $(u''_j, v''_j) \in V^{\gamma'_{m-j}}$ we have by (4) that

$$\begin{aligned}\|v''\| < \epsilon^{-1}(\lambda')^j\|(u''_j, v''_j)\| &\le \epsilon^{-1}(\lambda')^j\left(\|(u_j, v_j)\| + \|(u'_j, v'_j)\|\right)\\ &\le \epsilon^{-2}\left(\frac{\lambda'}{\mu'}\right)^j\left(\|(u, v)\| + \|(u, v')\|\right)\end{aligned}$$

for all $j \in \mathbb{N}$, whence $v'' = 0$ and $(u, v) \in S$.

The argument for $E_m^{\lambda'}$ is similar, using the family $\{L_m^{-1}\}$ instead of L_m. □

Remark 12.5.10. Note that $E_m^{\mu'}$ and $E_m^{\lambda'}$ are unique invariant sequences of subspaces inside the cones H_m^{γ} and $V_m^{\gamma'}$, respectively.

Corollary 12.5.11. *If under the assumptions of Proposition 12.5.9 we have $\lambda' < 1 < \mu'$ then $\{L_m\}$ is a hyperbolic family of linear maps which admits a (λ', μ')-splitting.*

Corollary 12.5.12. *If $\gamma < \sqrt{(\mu/\lambda) - 1}$ and*

$$0 < \delta < \min\left(\frac{\mu - \lambda}{\gamma + \frac{1}{\gamma} + 2}, \frac{\mu - (1+\gamma)^2\lambda}{(2+\gamma)(1+\gamma)}\right) \tag{12.5.3}$$

then

$$(E_p^{\mu})_m = \bigcap_{i=0}^{\infty}((f_*)^i H^{\gamma})_{p,m} = \bigcap_{i=0}^{\infty}\left(f_*(f_*(\ldots f_*(H^{\gamma})\ldots))\right)_{p,m}$$

is a k-dimensional subspace inside H_p^{γ},

$$(Df_m)_p(E_p^{\mu})_m = \left(E_{f_m(p)}^{\mu}\right)_{m+1},$$

and

$$\|(Df_m)_p\xi\| \geq \left(\frac{\mu}{1+\gamma} - \delta\right)\|\xi\|$$

for every $\xi \in (E^\mu_p)_m$.

Similarly $(E^\lambda_p)_m = \bigcap_{i=0}^\infty ((f_*^{-1})^i V^\gamma)_{p,m}$ *is an* $(n-k)$*-dimensional subspace in* V^γ_p,

$$(Df_m)_p(E^\lambda_p)_m = \left(E^\lambda_{f_m(p)}\right)_{m+1},$$

and $\|(Df_m)_p\xi\| \leq (1+\gamma)(\lambda+\delta)\|\xi\|$ *for every* $\xi \in (E^\lambda_p)_m$.

Proof. By Lemma 12.5.7 and Lemma 12.5.8 and (12.5.3) we can apply Proposition 12.5.9 with $\lambda' = (1+\gamma)(\lambda+\delta)$ and $\mu' = \big(\mu/(1+\gamma)-\delta\big)$ since under our assumptions $\lambda' < \mu'$ along each orbit of the sequence $\{f_m\}$. □

Lemma 12.5.13. *For* $m \in \mathbb{Z}$ *the subspaces* $(E^\mu_p)_m$ *and* $(E^\lambda_p)_m$ *are continuous in* p.

Proof. The vectors $v \in (E^\lambda_p)_m$ are characterized by the inequalities

$$\|(Df_{m+j})(Df_{m+j-1})\cdots(Df_m)_p v\| \leq (\lambda')^{j+1}\|v\| \qquad (j \in \mathbb{N})\,. \tag{12.5.4}$$

For a sequence $p_l \to p$ take orthonormal bases $\xi^l_1,\dots\xi^l_k$ of $(E^\lambda_{p_l})_m$ and assume without loss of generality that $\lim_{l\to\infty}\xi^l_i = \xi_i$ $(i = 1,\dots,k)$. Since for any fixed i the vectors ξ^l_i satisfy (12.5.4) for all l we conclude by continuity of all Df_m that ξ_i satisfies (12.5.4) and hence $\xi_i \in (E^\lambda_p)_m$. Since $\dim(E^\lambda_p)_m$ does not depend on p this implies that $\lim_{l\to\infty}(E^\lambda_{p_l})_m = (E^\lambda_p)_m$. □

$(E^\mu_p)_m$ and $(E^\lambda_p)_m$ $(m \in \mathbb{Z})$ are the invariant sequences of plane fields mentioned in the description of the proof.

c. invariant Lipschitz graphs. To get invariant graphs, that is, a family $\{\varphi^+_m\colon \mathbb{R}^k \to \mathbb{R}^{n-k}\}_{m\in\mathbb{Z}}$ of Lipschitz functions such that $f_m(\operatorname{graph}\varphi^+_m) = \operatorname{graph}\varphi^+_{m+1}$ and $\varphi^+_m(0) = 0$ let $C_\gamma(\mathbb{R}^k)$ be the set of functions $\varphi\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$ that are Lipschitz continuous with Lipschitz constant γ. Let $C^0_\gamma(\mathbb{R}^k)$ be the space of $\varphi \in C_\gamma(\mathbb{R}^k)$ such that $\varphi(0) = 0$. The following lemma can be viewed as a nonlinear counterpart of Lemma 12.5.7 and shows that the maps f_m act on the spaces $C_\gamma(\mathbb{R}^k)$ and $C^0_\gamma(\mathbb{R}^k)$:

Lemma 12.5.14. *If* (12.5.3) *holds and* $\varphi \in C_\gamma(\mathbb{R}^k)$ *then* $f_m(\operatorname{graph}\varphi) = \operatorname{graph}\psi$ *for some* $\psi \in C_\gamma(\mathbb{R}^k)$. *The same holds for* $C^0_\gamma(\mathbb{R}^k)$.

Proof. The map $G^m_\varphi\colon \mathbb{R}^k \to \mathbb{R}^k$ given by

$$G^m_\varphi(x) = A_m x + \alpha_m\big(x,\varphi(x)\big) \tag{12.5.5}$$

represents the x-coordinate of f_m acting on graph φ. To show that $f_m(\text{graph }\varphi)$ is

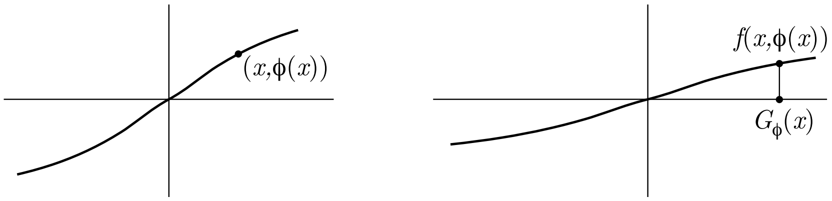


FIGURE 12.5.2. The graph transform

a graph we need to prove that G^m_φ is a bijection. Thus for $x_0 \in \mathbb{R}^k$ we need to find a unique $x \in \mathbb{R}^k$ such that $x_0 = G^m_\varphi(x)$ or equivalently

$$x = F(x) := A_m^{-1} x_0 - A_m^{-1}\big(\alpha_m(x,\varphi(x))\big).$$

$F\colon \mathbb{R}^k \to \mathbb{R}^k$ is a contracting map since

$$\begin{aligned}\|F(x_1) - F(x_2)\| &= \|A_m^{-1}(\alpha_m(x_1,\varphi(x_1)) - \alpha_m(x_2,\varphi(x_2)))\| \\ &\le \mu^{-1}\|\alpha_m\|_{C^1}\cdot(1+\gamma)\|x_1 - x_2\| < \delta\mu^{-1}(1+\gamma)\|x_1 - x_2\|\end{aligned}$$

and $\delta\mu^{-1}(1+\gamma) < 1$ by the second inequality in (12.5.3). Thus by the Contraction Mapping Principle (Proposition 12.1.3) equation F has a unique fixed point, that is, $f_m(\operatorname{graph}\varphi) = \operatorname{graph}\psi$.

Next we show that ψ is γ-Lipschitz continuous. Suppose $\psi(x_1') = y_1'$ and $\psi(x_2') = y_2'$ and take $(x_1, y_1), (x_2, y_2) \in \operatorname{graph}\varphi$ such that for $i = 1, 2$

$$(x_i', y_i') = f_m(x_i, y_i) = \big(A_m x_i + \alpha_m(x_i,\varphi(x_i)),\ B_m\varphi(x_i) + \beta_m(x_i,\varphi(x_i))\big).$$

Then

$$\begin{aligned}\|y_2' - y_1'\| &= \|B_m(\varphi(x_2) - \varphi(x_1)) + \beta_m(x_2,\ \varphi(x_2)) - \beta_m(x_1,\ \varphi(x_1))\| \\ &< \lambda\gamma\|x_2 - x_1\| + \delta(1+\gamma)\|x_2 - x_1\| \\ &= (\lambda\gamma + \delta(1+\gamma))\|x_2 - x_1\|\end{aligned} \tag{12.5.6}$$

and

$$\begin{aligned}\|x_2' - x_1'\| &= \|A_m(x_2 - x_1) + \alpha_m(x_2,\ \varphi(x_2)) - \alpha_m(x_1,\ \varphi(x_1))\| \\ &> \mu\|x_2 - x_1\| - \delta(1+\gamma)\|x_2 - x_1\| \\ &= (\mu - \delta(1+\gamma))\|x_2 - x_1\|.\end{aligned} \tag{12.5.7}$$

Consequently $\|y_2' - y_1'\| \le \dfrac{\lambda\gamma + \delta(1+\gamma)}{\mu - \delta(1+\gamma)}\|x_2' - x_1'\| =: \gamma'\|x_2' - x_1'\|$. But a straightforward calculation shows that the first condition in (12.5.3) is equivalent to $\gamma' < \gamma$. This shows that f_m acts on $C_\gamma(\mathbb{R}^k)$. The same holds for $C^0_\gamma(\mathbb{R}^k)$ since $f_m(0) = 0$. □

Since we eventually want to apply the Contraction Mapping Principle, we introduce a metric on the space $C^0_\gamma(\mathbb{R}^k)$ and show that the action of f_m is a contraction.

Since $\varphi, \psi \in C^0_\gamma(\mathbb{R}^k)$ are Lipschitz continuous with $\varphi(0) = \psi(0) = 0$,

$$d(\varphi,\psi) := \sup_{x\in\mathbb{R}^k \smallsetminus \{0\}} \frac{\|\varphi(x)-\psi(x)\|}{\|x\|}$$

is a well-defined metric. It is easy to check that it is complete.

The next lemma shows that the action of f_m on $C^0_\gamma(\mathbb{R}^k)$ given by

$$f_m(\operatorname{graph}\varphi) = \operatorname{graph}\big((f_m)_*\varphi\big)$$

is a contracting map.

Lemma 12.5.15. *$d\big((f_m)_*\varphi,\ (f_m)_*\psi\big) \le \dfrac{\lambda+\delta(1+\gamma)}{\mu-\delta(1+\gamma)} d(\varphi,\psi)$ for $\varphi,\psi \in C^0_\gamma(\mathbb{R}^k)$.*

Proof. Let $\varphi' = (f_m)_*\varphi$ and $\psi' = (f_m)_*\psi$. Using the map G^m_φ defined by (12.5.5) and the fact that $\psi' \in C^0_\gamma(\mathbb{R}^k)$ we have

$$\begin{aligned}
&\|\varphi'\Big(G^m_\varphi(x)\Big) - \psi'\Big(G^m_\varphi(x)\Big)\| \\
&\qquad\le \|\varphi'\Big(G^m_\varphi(x)\Big) - \psi'\Big(G^m_\psi(x)\Big)\| + \|\psi'\Big(G^m_\psi(x)\Big) - \psi'\Big(G^m_\varphi(x)\Big)\| \\
&\qquad\le \|\big(B_m(\varphi(x)) + \beta_m(x,\varphi(x))\big) - \big(B_m(\psi(x)) + \beta_m(x,\psi(x))\big)\| \\
&\qquad\quad + \gamma\|G^m_\psi(x) - G^m_\varphi(x)\| \\
&\qquad\le \|B_m\big(\varphi(x)-\psi(x)\big)\| + \|\beta_m\big(x,\varphi(x)\big) - \beta_m\big(x,\psi(x)\big)\| \\
&\qquad\quad + \gamma\|\alpha_m\big(x,\psi(x)\big) - \alpha_m\big(x,\varphi(x)\big)\| \\
&\qquad< \lambda\|\varphi(x)-\psi(x)\| + \delta\|\varphi(x)-\psi(x)\| + \gamma\delta\|\varphi(x)-\psi(x)\| \\
&\qquad= \big(\lambda+\delta(1+\gamma)\big)\|\varphi(x)-\psi(x)\|.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|G^m_\varphi(x)\| = \|A_m x + \alpha_m\big(x,\varphi(x)\big)\| &\ge \|A_m x\| - \|\alpha_m\big(x,\varphi(x)\big)\| \\
&\ge \mu\|x\| - \delta(1+\gamma)\|x\| = \big(\mu-\delta(1+\gamma)\big)\|x\|.
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{\|(f_m)_*\varphi\Big(G^m_\varphi(x)\Big) - (f_m)_*\psi\Big(G^m_\varphi(x)\Big)\|}{\|G^m_\varphi(x)\|} &\le \frac{\lambda+\delta(1+\gamma)}{\mu-\delta(1+\gamma)} \cdot \frac{\|\varphi(x)-\psi(x)\|}{\|x\|} \\
&\le \frac{\lambda+\delta(1+\gamma)}{\mu-\delta(1+\gamma)} \cdot d(\varphi,\psi).
\end{aligned}$$

□

Here $\gamma < 1 \Rightarrow \dfrac{\lambda+\delta(1+\gamma)}{\mu-\delta(1+\gamma)} = \gamma^{-1}\dfrac{\lambda\gamma+\delta\gamma(1+\gamma)}{\mu-\delta(1+\gamma)} \le \gamma^{-1}\dfrac{\lambda\gamma+\delta(1+\gamma)}{\mu-\delta(1+\gamma)} < 1$ by (12.5.3) (see (12.5.2)).

We now denote by C^0_γ the space of families $\{\varphi_m\}_{m\in\mathbb{Z}}$ of functions in $C^0_\gamma(\mathbb{R}^k)$. The action of $f = \{f_m\}_{m\in\mathbb{Z}}$ on the space C^0_γ given by

$$f_m(\operatorname{graph}\varphi_m) = \operatorname{graph}\big((f_*\varphi)_{m+1}\big)$$

is called the *graph transform*. Lemma 12.5.15 shows that the graph transform is a contraction with respect to the metric

$$d\big(\{\varphi_m\}_{m\in\mathbb{Z}},\, \{\psi_m\}_{m\in\mathbb{Z}}\big) := \sup_{m\in\mathbb{Z}} d(\varphi_m,\psi_m).$$

Since C^0_γ is complete with this metric, the Contraction Mapping Principle, Proposition 12.1.3, yields a unique fixed point for this action of f, hence an invariant family $\{\varphi^+_m\}$ of graphs, as claimed.

Remark 12.5.16. If $\lambda < 1$ one can show that $\|\varphi^+_m\|_{C^0} < \delta/(1-\lambda)$ by considering only $\varphi \in C^0_\gamma(\mathbb{R}^k)$ bounded by $\delta/(1-\lambda)$ and showing invariance of this condition under f_*. In this case the first estimate in the proof of Lemma 12.5.15 also shows that the graph transform is a contraction with respect to the C^0 topology.

To construct the functions φ^-_m one argues along the same lines. Using the estimates obtained in this step, with γ replaced by $1/\gamma$, one shows that the maps Df^{-1}_m act on families of γ-Lipschitz functions $\varphi\colon \mathbb{R}^{n-k} \to \mathbb{R}^k$ vanishing at the origin, and are contracting.

At this point it is natural to prove (ii) since we use the estimates (12.5.6) and (12.5.7). First replace (x_1, y_1) by $(0,0)$ and (x_2, y_2) by $(x, \varphi^+_m x)$ in (12.5.7). Then

$$\|f_m(x,\varphi^+_m(x))\| \ge \|A_m x + \alpha_m(x,\varphi^+_m(x))\| > (\mu-\delta(1+\gamma))\|x\| \ge \frac{\mu-\delta(1+\gamma)}{1+\gamma}\|(x,\varphi^+_m(x))\|.$$

On the other hand, applying (12.5.6) to $(0,0)$ and $(\varphi^-_m(y), y)$ and using the fact that φ^-_m are γ-Lipschitz yields

$$\begin{aligned}\|f_m(\varphi^-_m(y),y)\| &\le (1+\gamma)\|B_m(y)+\beta_m(\varphi^-_m(y),y)\|\\ &< (1+\gamma)(\lambda\|y\|+\delta(1+\gamma)\|y\|) = (1+\gamma)(\lambda+\delta(1+\gamma))\|(\varphi^-_m(y),y)\|.\end{aligned}$$

d. Differentiability. To prove that the invariant family of functions obtained in the previous step consists of continuously differentiable functions, we introduce the notion of a tangent set for a graph. The results of step 2, the existence of a unique invariant family of continuous plane fields, then imply that the tangent set of each of these graphs is a continuous plane field. But this, by definition, implies that the graphs are graphs of C^1 functions.

Definition 12.5.17. Let $\varphi \in C^0_\gamma(\mathbb{R}^k)$, $x \in \mathbb{R}^k$,

$$\Delta_y\varphi := \frac{(y,\varphi(y)) - (x,\varphi(x))}{\|(y,\varphi(y)) - (x,\varphi(x))\|} \qquad \text{for } y \neq x,$$

$t_x\varphi := \{v \in T_x\mathbb{R}^n \mid \exists \{x_n\}_{n\in\mathbb{N}} \text{ such that } \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \Delta_{x_n}\varphi = v\}$. Then $\tau_x\varphi := \bigcup_{v \in t_x\varphi} \mathbb{R}v$, where $\mathbb{R}v := \{av \mid a \in \mathbb{R}\}$ is the line containing v, is called the *tangent set* of φ at x, and the (disjoint) union $\tau\varphi := \bigcup_{x\in\mathbb{R}^k} \tau_x\varphi$, the *tangent set* of φ.

Note that since for every $v \in \mathbb{R}^k$ one can choose $y = x + tv$ in the definition, $\tau_x\varphi$ projects onto $\mathbb{R}^k$.

As an example consider $\varphi(x) = x\sin(1/x) \in C^0_\gamma(\mathbb{R})$ for which $\tau_0\varphi = \{(x,y) \in \mathbb{R}^2 \mid |y| \leq |x|\} = H^1_0$. Indeed, for $\varphi \in C^0_\gamma(\mathbb{R}^k)$ and $x \in \mathbb{R}^k$ we always have $\tau_x\varphi \subset H^\gamma_x$, since φ has Lipschitz constant γ. Another important observation is that $\varphi \in C^0_\gamma(\mathbb{R}^k)$ is differentiable at x if and only if $\tau_x\varphi$ is a k-dimensional plane.

We can now show that the invariant family $\varphi^+ = \{\varphi^+_m\}_{m\in\mathbb{Z}}$ obtained in step 3 consists of C^1 functions. Associated with φ^+ is the family $\tau\varphi^+ := \{\tau\varphi^+_m\}_{m\in\mathbb{Z}}$ of tangent sets for the functions φ^+_m, $m \in \mathbb{Z}$. Since φ^+ is an invariant family of functions for $f = \{f_m\}_{m\in\mathbb{Z}}$, the associated family $\tau\varphi^+$ of tangent sets is invariant under the action of the differentials Df_m. In step 2 we showed that any such invariant family inside the γ-cones is contained in the unique invariant family E^+_m of continuous plane fields obtained there. Since every tangent set $\tau_p\varphi^+_m$ projects onto $\mathbb{R}^k$, we conclude that $\tau_p\varphi^+_m = (E^+_p)_m$, that is, the φ^+_m are C^1 functions.

Smoothness of φ^-_m is proved likewise. This ends the proof of (i).

It remains to prove (iii). We remarked after the formulation of the theorem that we can construct the manifolds $(W^-_m)_p$ and $(W^+_m)_p$ for any point $p = (x,y)$. We still have $(W^+_m)_p = \text{graph}(\varphi^+_m)_p$ and $(W^-_m)_p = \text{graph}(\varphi^-_m)_p$ for some γ-Lipschitz functions $(\varphi^+_m)_p \colon \mathbb{R}^k \to \mathbb{R}^{n-k}$ and $(\varphi^-_m)_p \colon \mathbb{R}^{n-k} \to \mathbb{R}^k$ and properties analogous to (i) and (ii).

Lemma 12.5.18. *For $p, q \in \mathbb{R}^n$ the intersection $(W^+_m)_p \cap (W^-_m)_q$ is a point.*

Proof. If $z = (x,y) \in (W^+_m)_p \cap (W^-_m)_q$ then $x = (\varphi^-_m)_q(y)$ and $y = (\varphi^+_m)_p(x)$ and hence $x = (\varphi^-_m)_q \circ (\varphi^+_m)_p(x)$. This in turn implies again that $\big(x, (\varphi^+_m)_p(x)\big) \in (W^+_m)_p \cap (W^-_m)_q$. But since we can assume $\gamma < 1$ the map $(\varphi^-_m)_q \circ (\varphi^+_m)_p \colon \mathbb{R}^k \to \mathbb{R}^k$ is a contraction and hence has a unique fixed point. □

Now assume $p \notin (W^-_m)_0$. By Lemma 12.5.18 there is a unique $q \in (W^-_m)_0 \cap (W^+_m)_p$. Using (ii) for $(W^-_m)_0$ and $(W^+_m)_p$ we see that

$$\begin{aligned}
&\|f_{m+L-1}\circ\cdots\circ f_m(p)\| \\
&\qquad \ge \|f_{m+L-1}\circ\cdots\circ f_m(p) - f_{m+L-1}\circ\cdots\circ f_m(q)\| - \|f_{m+L-1}\circ\cdots\circ f_m(q)\| \\
&\qquad \ge (\mu')^L\|p-q\| - (\lambda')^L\|q\| = (\mu')^L\Big(\|p-q\| - \Big(\frac{\lambda'}{\mu'}\Big)^L\|q\|\Big).
\end{aligned}$$

Whenever $\lambda' < \nu < \mu'$ and $C \in \mathbb{R}$ this quantity exceeds $C\cdot\nu^L\|p\|$ for sufficiently large $L \in \mathbb{N}$.

Together with a parallel argument for $(W_m^+)_0$ this proves (iii) and thus also the uniqueness of W_m^+ and W_m^-.

This finishes the proof of the general part of the Hadamard–Perron Theorem.

e. Higher smoothness. To complete the proof of Theorem 12.5.2 we now prove that if $\mu \ge 1$ in Theorem 12.5.2 then $\{W_m^+\}_{m\in\mathbb{Z}}$ consists of manifolds as smooth as the diffeomorphism. Df_m has block form $\begin{pmatrix} A_m^{uu} & A_m^{su} \\ A_m^{us} & A_m^{ss}\end{pmatrix}$ with A_m^{uu} a $k\times k$-matrix with $\|(A_m^{uu})^{-1}\| \le 1/(\mu-\delta)$, A_m^{ss} an $(n-k)\times(n-k)$-matrix with $\|A_m^{ss}\| \le \lambda+\delta$, and $\|A_m^{su}\| < \delta$, $\|A_m^{us}\| < \delta$. By the preceding steps, notably Lemma 6.2.16, we can obtain W_m^+ by taking smooth functions $\varphi_m^0 \in C_\gamma^0(\mathbb{R}^k)$ (such as $\varphi_m^0 = 0$), applying the graph transform repeatedly to obtain families $\{\varphi_m^i\}$ for $i \in \mathbb{N}$, and taking the limit as $i \to \infty$. We plan to show inductively that the $r+1$st derivative of φ_m^i converges as $i\to\infty$, so long as f is C^{r+1}. To that end we note that $D\varphi_m^i$ is the graph of a linear map E_m^i from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$, or, equivalently, the image of the map $\begin{pmatrix} I \\ E_m^i\end{pmatrix}\colon \mathbb{R}^k \to \mathbb{R}^n$. Notice that the image of $D\varphi_m^i$ under Df_m is the image of the linear map

$$\begin{pmatrix} A_m^{uu} & A_m^{su} \\ A_m^{us} & A_m^{ss}\end{pmatrix}\begin{pmatrix} I \\ E_m^i\end{pmatrix} = \begin{pmatrix} A_m^{uu} + A_m^{su}E_m^i \\ A_m^{us} + A_m^{ss}E_m^i\end{pmatrix}.$$

If, referring to (6.2.6), we let $g_m^i \coloneqq (G_{\varphi_{m-1}^i}^{m-1})^{-1}$ then this has to coincide with the image of $\begin{pmatrix} I \\ E_{m+1}^{i+1}\circ(g_{m+1}^i)^{-1}\end{pmatrix}$ which is the same as that of

$$\begin{pmatrix} A_m^{uu} + A_m^{su}E_m^i \\ (E_{m+1}^{i+1}\circ(g_{m+1}^i)^{-1})(A_m^{uu} + A_m^{su}E_m^i)\end{pmatrix},$$

so

$$(E_{m+1}^{i+1}\circ(g_{m+1}^i)^{-1})(A_m^{uu} + A_m^{su}E_m^i) = A_m^{us} + A_m^{ss}E_m^i.$$

Composing with g^i_{m+1} and differentiating r times we get

$$\begin{aligned}D^r E^{i+1}_{m+1}(\alpha^u_{m+1,i+1})^{-1} + E^{i+1}_{m+1}(A^{su}_m \circ g^i_{m+1})(D^r E^i_m \circ g^i_{m+1})(Dg^i_{m+1})^{\otimes r} \\ = (A^{ss}_m \circ g^i_{m+1})(D^r E^i_m \circ g^i_{m+1})(Dg^i_{m+1})^{\otimes r} + \zeta_{m+1,i+1}(\alpha^u_{m+1,i+1})^{-1},\end{aligned}$$

where $\zeta_{m+1,i+1}$ is a polynomial in lower derivatives of E^{i+1}_{m+1} and E^i_m and

$$\alpha^u_{m+1,i+1} \coloneqq [(A^{uu}_m \circ g^i_{m+1}) + (A^{su}_m \circ g^i_{m+1})(E^i_m \circ g^i_{m+1})]^{-1}.$$

Letting $\alpha^s_{m,i} \coloneqq (A^{ss}_{m-1} \circ g^{i-1}_m) - E^i_m(A^{su}_{m-1} \circ g^{i-1}_m)$ this yields

$$\begin{aligned}D^r E^i_m &= \alpha^s_{m,i}(D^r E^{i-1}_{m-1} \circ g^{i-1}_m)(Dg^{i-1}_m)^{\otimes r} \alpha^u_{m,i} + \zeta_{m,i} \\ &= \alpha^s_{m,i}(\alpha^s_{m-1,i-1} \circ g^{i-1}_m) \times \\ &\quad \times (D^r E^{i-2}_{m-2} \circ g^{i-2}_{m-1} \circ g^{i-1}_m)(Dg^{i-2}_{m-1})^{\otimes r}(\alpha^u_{m-1,i-1} \circ g^{i-1}_m)(Dg^{i-1}_m)^{\otimes r} \alpha^u_{m,i} \\ &\quad\quad + \alpha^s_{m,i}(\zeta_{m-1,i-1} \circ g^{i-1}_m)(Dg^{i-1}_m)^{\otimes r} \alpha^u_{m,i} + \zeta_{m,i} \\ &= \dots\end{aligned}$$

Applying this inductively we obtain an expression for $D^r E^i_m$ with a leading term involving $D^r E^0_{m-i}$ between i-fold products

$$\alpha^s_{m,i}(\alpha^s_{m-1,i-1} \circ g^{i-1}_m)(\alpha^s_{m-2,i-2} \circ g^{i-2}_{m-1} \circ g^{i-1}_m) \dots$$

of terms $\alpha^s_{m-l,i-l}$ and

$$\dots (Dg^{i-3}_{m-2})^{\otimes r}(\alpha^u_{m-2,i-2} \circ g^{i-2}_{m-1} \circ g^{i-1}_m)(Dg^{i-2}_{m-1})^{\otimes r}(\alpha^u_{m-1,i-1} \circ g^{i-1}_m)(Dg^{i-1}_m)^{\otimes r} \alpha^u_{m,i}$$

of $\alpha^u_{m-l,i-l}$ and i occurrences of $(Dg^{i-l-1}_{m-l})^{\otimes r}$. This term goes to 0 uniformly as $i \to \infty$: $\|D^r E^0_{m-i}\|$ is uniformly bounded by choice of φ^0_{m-i} and $\|\alpha^s_{m-l,i-l}\| \|\alpha^u_{m-l,i-l}\| < 1$ uniformly by taking small δ. Finally, the assumption $\mu \geq 1$ of this step ensures that the factors $(Dg^{i-l-1}_{m-l})^{\otimes r}$ cause no exponential growth.

The jth of the remaining i summands in the expression for $D^r E^i_m$ similarly consists of $\zeta_{m-j-1,i-j-1}$ between j-fold products of terms $\alpha^s_{m-l,i-l}$ and $\alpha^u_{m-l,i-l}$ as well as j occurrences of $(Dg^{i-l-1}_{m-l})^{\otimes r}$. As before, these terms tend to 0 uniformly as $j \to \infty$ given uniform control of $\zeta_{m-j-1,i-j-1}$. These, however, involve only lower derivatives of E^k_l's which are uniformly bounded by induction assumption, as well as derivatives of order up to order r of coefficients of Df, which are bounded because $f \in C^{r+1}$. Consequently these remaining terms give partial sums of an exponentially convergent series. We already know that lower-order derivatives of E^i_m converge as $i \to \infty$ and thus conclude that the limit of E^i_m is C^r, as desired. □
Note that $(W^+_m)_p$ and $(W^-_m)_p$ for $p \in \mathbb{R}^n$ depend continuously on p: The characterization (iii) of Theorem 12.5.2 yields

Proposition 12.5.19. *If $p_l \to p \in \mathbb{R}^n$ as $l \to \infty$ and $y_l \in (W_m^+)_{p_l}$ for all $l \in \mathbb{N}$ and $y_l \to y \in \mathbb{R}^n$ as $l \to \infty$ then $y \in (W_m^+)_p$.*

Proof. Fix $L \in \mathbb{N}$. Then (ii) of Theorem 12.5.2 implies for $\nu < \mu'$ that

$$\| f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(y_l) - f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(p_l) \| \le \nu^{-L} \| y_l - p_l \|$$

for all $l \in \mathbb{N}$. By continuity of the f_m this implies

$$\| f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(y) - f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(p) \| \le \nu^{-L} \| y - p \|$$

and since L was arbitrary the claim follows by (iii). □

Since on any fixed compact set the assumption that y_l converges is redundant (by passing to a subsequence) this means that $(W_m^+)_{p_l} \to (W_m^+)_p$ when $p_l \to p$. Convergence here is in the pointwise sense of the proposition. Since we know that E_m^+ is continuous, we have continuity of W_m^+ together with its tangent spaces. A similar statement holds for W_m^-.

Another pertinent remark is that we obtain in fact continuous dependence of W^+ and W^- on the family f_m of maps we consider. Since the main ingredient of the proof of the Hadamard–Perron Theorem 12.5.2 was obtaining the invariant manifolds and their tangent bundles as fixed points of a contraction operator associated with the family f_m, we may use Proposition 12.1.3 to infer that the invariant manifolds depend continuously on the diffeomorphisms with respect to the C^1 topology.

Proposition 12.5.20. *The invariant manifolds (with the C^1 topology) obtained in the Hadamard–Perron Theorem 12.5.2 depend continuously on the family f_m if we use the C^1 topology ($\{f_m\}_{m\in\mathbb{N}}, \{g_m\}_{m\in\mathbb{N}}$ are C^1-close if $\sup_m d_{C^1}(f_m, g_m)$ is small).*

Remark 12.5.21. In the hyperbolic case one can use the C^r-topology for invariant manifolds, and one does indeed obtain continuous dependence on the family f_m in the C^r-topology.

Corollary 12.5.22. *If $p_l \to p \in \mathbb{R}^n$ as $l \to \infty$ and $q \in \mathbb{R}^n$ then the sequence y_l given by $(W_m^+)_{p_l} \cap (W_m^-)_q = \{y_l\}$ converges to y given by $\{y\} = (W_m^+)_p \cap (W_m^-)_q$.*

This follows from Proposition 12.5.20 and Lemma 12.5.18 since the y_l are contained in a compact set because the $(W_m^+)_{p_l}$ are Lipschitz graphs.

Since the construction and characterization of $(W_m^+)_p$ and $(W_m^+)_p$ other than $(W_m^+)_0$ and $(W_m^-)_0$ depend on the behavior of points whose orbits do not stay in a neighborhood of the origin, they depend on the extension chosen in the Extension Theorem 12.4.12 and do not represent meaningful objects associated with neighborhoods of a reference orbit on a manifold.

6. The Inclination Lemma and homoclinic tangles

The graph-transform method also yields the *Inclination Lemma*, that successive images of a disk transverse to the stable manifold of a hyperbolic fixed point accumulate (in the C^1 topology) on the unstable manifold of the point.

Theorem 12.6.1 (Inclination Lemma)**.** *Suppose p is a hyperbolic fixed point of a diffeomorphism f and $\mathscr{D}$ is a disk that transversely intersects $W^s(p)$ (and hence has the same dimension as $W^u(p)$). Then the $f^n(\mathscr{D})$ accumulate on $W^u(p)$ in the C^1-topology as $n \to +\infty$. Specifically, for any disk Δ in $W^u(p)$ and any $\epsilon > 0$ there is an $n \in \mathbb{N}$ and a $\mathscr{D}' \subset \mathscr{D}$ such that $d_{C^1}(f^n(\mathscr{D}'), \Delta) < \epsilon$.*

Proof. In order to apply Proposition 12.6.2 below, choose adapted coordinates at p. After possibly conjugating these by f^k for some $k \in \mathbb{N}$ these will contain Δ. Now replace $\mathscr{D}$ by $f^m(\mathscr{D})$ for $m \in \mathbb{N}$ such that (using that $\mathscr{D}$ is C^1 and after possibly shrinking $\mathscr{D}$), $\mathscr{D}$ is in the adapted coordinate system and the hypotheses of Proposition 12.6.2 hold. The conclusion of Proposition 12.6.2 and the Lipschitz-convergence of the graph transform then imply the claim because the Lipschitz topology on C^1 submanifolds induces the C^1-topology. □

To state the main lemma it is convenient to use adapted coordinates as in Theorem 12.5.3 and to let $\pi_1 \colon \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^k$ be the projection to the first coordinate.

Proposition 12.6.2. *Under the hypotheses of Theorem 12.5.3 consider C^r adapted coordinates on a neighborhood O of a hyperbolic fixed point p of $f \colon U \to M$. Given $\epsilon, K, \eta > 0$ there exists an $N \in \mathbb{N}$ such that if $\mathscr{D}$ is a C^1 disk containing $q \in W_p^- \cap O$ with all tangent spaces in horizontal K-cones and such that $\pi_1(\mathscr{D})$ contains an η-ball around $0 \in \mathbb{R}^k \oplus \{0\}$ and $n \geq N$ then $\pi_1(f^n(\mathscr{D})) = W_p^+ \cap O$ and $T_z f^n(\mathscr{D})$ is contained in a horizontal ϵ-cone for every $z \in f^n(\mathscr{D})$.*

Proof. Since $\mathbb{R}^k \oplus \{0\}$ and $\{0\} \oplus \mathbb{R}^{n-k}$ are f-invariant and f is C^1 the differential of f at points $(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ takes the form

$$Df_{(x,y)} = \begin{pmatrix} A_z^{uu} & A_z^{su} \\ A_z^{us} & A_z^{ss} \end{pmatrix},$$

where

$$\begin{aligned} A_z^{uu} &\in M_{k,k}, & \|(A_z^{uu})^{-1}\| &\leq \frac{1}{\mu - \delta}, \\ A_z^{ss} &\in M_{n-k,n-k}, & \|A_z^{ss}\| &< \lambda + \delta, \\ A_z^{us} &\in M_{n-k,k}, & \|A_z^{us}\| &\in o(\|y\|), \\ A_z^{su} &\in M_{k,n-k}, & A_z^{su} &\in o(\|x\|). \end{aligned}$$

Here $\lambda < 1 < \mu$ are the contraction and expansion rates as before, and we used the notation from Remark 3.2.18. δ can be taken arbitrarily small by possibly shrinking the size of the neighborhood O (and replacing $\mathscr{D}$ by its image under an iterate f^n, so that $\mathscr{D}$ intersects the local stable leaf of p in a point in O). Similarly to the proof of smoothness of stable and unstable manifolds it is convenient now to consider planes in horizontal γ-cones as graphs of linear maps whose operator norm (denoted by $\|\cdot\|$) is bounded by γ. After possibly shrinking $\mathscr{D}$ we assume that $\mathscr{D}\cap(\{0\}\oplus\mathbb{R}^{n-k})=\{z\}$ is a single point. Then our first step consists of showing that $T_{z_n}f^n(\mathscr{D})$ is contained in a horizontal $\epsilon/2$-cone for some $n\in\mathbb{N}$, where $z_i = f^i(z)$. To that end consider a linear map $E_z\colon\mathbb{R}^k\to\mathbb{R}^{n-k}$ with $\|E\|\le K$. Its graph is parameterized as the image of the linear map

$$\begin{pmatrix} I \\ E_z \end{pmatrix}\colon \mathbb{R}^k\to\mathbb{R}^k\oplus\mathbb{R}^{n-k},$$

where $I\colon\mathbb{R}^k\to\mathbb{R}^k$ is the identity. The image of the graph under Df_z is then the image of the linear map

$$Df_z\circ\begin{pmatrix} I \\ E_z \end{pmatrix}\colon \mathbb{R}^k\to\mathbb{R}^k\oplus\mathbb{R}^{n-k}.$$

In our coordinates this composition is obtained via the matrix product

$$\begin{pmatrix} A_z^{uu} & A_z^{su} \\ A_z^{us} & A_z^{ss} \end{pmatrix}\begin{pmatrix} I \\ E_z \end{pmatrix}=\begin{pmatrix} A_z^{uu}+A_z^{su}E_z \\ A_z^{us}+A_z^{ss}E_z \end{pmatrix} \tag{12.6.1}$$

with $A_z^{su}=0$ in this case. $A_z^{uu}\colon\mathbb{R}^k\to\mathbb{R}^k$ is nonsingular, so the image of $\begin{pmatrix} A_z^{uu} \\ A_z^{us}+A_z^{ss}E_z \end{pmatrix}$ is that of $\begin{pmatrix} A_z^{uu} \\ A_z^{us}+A_z^{ss}E_z \end{pmatrix}\circ(A_z^{uu})^{-1}=\begin{pmatrix} I \\ A_z^{us}A_z^{uu-1}+A_z^{ss}E_zA_z^{uu-1} \end{pmatrix}$. In other words, $Df_z(T_z\mathscr{D})$ is the graph of the linear map

$$E_{z_1}=A_z^{us}(A_z^{uu})^{-1}+A_z^{ss}E_z(A_z^{uu})^{-1}.$$

Note that $\|E_{z_1}\|\le\dfrac{\|A_{z_0}^{us}\|}{\mu-\delta}+\dfrac{\lambda+\delta}{\mu-\delta}\|E_{z_0}\|$ and inductively

$$\|E_{z_n}\|\le\sum_{i=0}^{n-1}\frac{(\lambda+\delta)^{n-i-1}}{(\mu-\delta)^{n-i}}\|A_{z_i}^{us}\|+\frac{(\lambda+\delta)^n}{(\mu-\delta)^n}\|E_{z_0}\|.$$

Since $\|A_{z_i}^{us}\|\in o(\|y_i\|)$, where $z_i=(x_i,y_i)$, there exists $N\in\mathbb{N}$ such that for $n>N$ both $\sum_{i=N}^{n-1}\dfrac{(\lambda+\delta)^{n-i-1}}{(\mu-\delta)^{n-i}}\|A_{z_i}^{us}\|<\epsilon/6$ and $\dfrac{(\lambda+\delta)^n}{(\mu-\delta)^n}\|E_{z_0}\|<\epsilon/6$. If furthermore $N'\in\mathbb{N}$ is such that $\sum_{i=0}^{N-1}\dfrac{(\lambda+\delta)^{N'+N-i-1}}{(\mu-\delta)^{N'+N-i}}\|A_{z_i}^{us}\|<\epsilon/6$ then for $n\ge N+N'=:N_0$ we have $\|E_{z_n}\|<\epsilon/2$.

After possibly increasing N_0 we may assume that $\|A^{us}_{(x,y)}\| < (1-\lambda-\delta)\epsilon$ whenever $\|(x,y)\| \le \|z_{N_O}\|$. Consequently, by possibly shrinking $\mathscr{D}$, we may assume that all tangent planes to $f^{N_0}(\mathscr{D})$ lie in horizontal ϵ-cones and that $\|A^{us}_z\| < (1-\lambda-\delta)\epsilon$ for $z \in \bigcup_{i \ge N_0} f^i(\mathscr{D})$. If ϵ is sufficiently small then $\|(A^{uu}_z + A^{su}_z E)^{-1}\| < 1$ whenever $\|E\| < \epsilon$. With this choice of parameters the action of f preserves horizontal ϵ-cones because if $\|E_{z_i}\| < \epsilon$ then (12.6.1) gives

$$\begin{aligned}\|E_{z_{i+1}}\| \le \|(A^{us}_{z_i} + A^{ss}_{z_i}E_{z_i})(A^{uu}_{z_i} + A^{su}_{z_i}E_{z_i})^{-1}\| \le \|A^{us}_{z_i} + A^{ss}_{z_i}E_{z_i}\|\,\|(A^{uu}_{z_i} + A^{su}_{z_i}E_{z_i})^{-1}\| \\ < \|A^{us}_{z_i}\| + \|A^{ss}_{z_i}E_{z_i}\| < (1-\lambda-\delta)\epsilon + (\lambda+\delta)\epsilon = \epsilon.\end{aligned}$$

Finally, note that the proof of Lemma 12.5.14 shows that $f^n(\mathscr{D})$ covers $W^n_{\text{loc}}(p)$ under the projection π_1 whenever n is sufficiently large. □

Remark 12.6.3. It is useful to note (for example, by setting $\mu - \delta = 1$ in the calculations) that expansion in the unstable direction is not used in the proof of Proposition 12.6.2; it is needed in Theorem 12.6.1 to assert that arbitrarily large disks are limits of $\mathscr{D}$ under the dynamics. This allows us to invoke Proposition 12.6.2 for time-t maps of flows by including the flow direction with the unstable one.

We next study horseshoes and a generic mechanism that gives rise to them.

By a rectangle in $\mathbb{R}^n$ we mean a set of the form $\Delta = D_1 \times D_2 \subset \mathbb{R}^k \oplus \mathbb{R}^l = \mathbb{R}^n$, where D_1 and D_2 are disks. We denote by $\pi_1\colon \mathbb{R}^n \to \mathbb{R}^k$ and $\pi_2\colon \mathbb{R}^n \to \mathbb{R}^l$ the canonical projections. As in Section 12.5 we refer to the $\mathbb{R}^k$-direction as "horizontal" and the $\mathbb{R}^l$-direction as "vertical".

Definition 12.6.4. Suppose $\Delta \subset U \subset \mathbb{R}^n$ is a rectangle and $f\colon U \to \mathbb{R}^n$ a diffeomorphism. A connected component $C' = fC$ of $\Delta \cap f(\Delta)$ is said to be *full* (for f) if

(1) $\pi_2(C) = D_2$, and
(2) for any $z \in C$, $\pi_1\restriction_{f(C\cap(D_1\times\pi_2(z)))}$ is a bijection onto D_1.

Geometrically, condition (2) means that the image of every horizontal fiber in C meets Δ and "traverses" Δ completely.

Definition 12.6.5. If $U \subset \mathbb{R}^n$ is open then a rectangle $\Delta = D_1 \times D_2 \subset U \subset \mathbb{R}^k \oplus \mathbb{R}^l = \mathbb{R}^n$ is called a *horseshoe* for a diffeomorphism $f\colon U \to \mathbb{R}^n$ if $\Delta \cap f(\Delta)$ contains at least two full components Δ_0 and Δ_1 such that for $\Delta' = \Delta_0 \cup \Delta_1$

(1) $\pi_2(\Delta') \subset \operatorname{int} D_2$, $\quad \pi_1(f^{-1}(\Delta')) \subset \operatorname{int} D_1$,
(2) $D(f\restriction_{f^{-1}(\Delta')})$ preserves and expands a horizontal cone family on $f^{-1}(\Delta')$,
(3) $D(f^{-1}\restriction_{\Delta'})$ preserves and expands a vertical cone family on Δ'.

Conditions (2) and (3) imply by (the discrete-time version of) Proposition 5.1.7 that $\Lambda := \bigcap_{n\in\mathbb{Z}} f^{-n}(\Delta')$ is a hyperbolic set for f with "almost horizontal" expanding and "almost vertical" contracting directions.

We can now establish a connection between transverse homoclinic points and the existence of horseshoes. Figures 6.3.1 and 6.3.2 on page 308 illustrate this.

Theorem 12.6.6 (Birkhoff–Smale: homoclinic tangles produce horseshoes)**.** *Let M be a smooth manifold, $U \subset M$ open, $f\colon U \to M$ an embedding, and $p \in U$ a hyperbolic fixed point with a transverse homoclinic point q. Then in an arbitrarily small neighborhood of p there exists a horseshoe for some iterate of f. Furthermore, the hyperbolic invariant set in this horseshoe contains an iterate of q.*

Proof. Via adapted coordinates on a neighborhood $\mathcal{O}$ we may assume that the hyperbolic fixed point is at the origin and that $W^u_{\mathrm{loc}}(0) := \mathbb{C}(W^u(0) \cap \mathcal{O}, 0) \subset \mathbb{R}^k \oplus \{0\}$ (as in Definition 1.6.13) and $W^s_{\mathrm{loc}}(0) := \mathbb{C}(W^s(0) \cap \mathcal{O}, 0) \subset \{0\} \oplus \mathbb{R}^l$, where $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$. Let D_1, D_2 be small disks around 0 in $W^u_{\mathrm{loc}}(0)$ and $W^u_{\mathrm{loc}}(0)$, respectively, and $B := D_1 \times D_2$.

Take N_0 minimal such that $q' := f^{-N_0}(q) \in \operatorname{Int} D_1$; since q' is transverse homoclinic we can take $\delta > 0$ sufficiently small so that $D_1 \times \{x\}$ is transverse to $W^s_{\mathrm{loc}}(q') := \mathbb{C}\left(W^s(p) \cap \Delta, q'\right) \subset W^s(p)$ for $x \in \delta D_2 := \{\delta z \mid z \in D_2\}$, where $\Delta := D_1 \times \delta D_2$. By the Inclination Lemma, Theorem 12.6.1, we can choose $\delta > 0$ and $N_1 \in \mathbb{N}$ such that if $\mathcal{D}_z := \mathbb{C}\left(f^{N_1}(D_1 \times \{z\}) \cap B, f^{N_1}(D_1 \times \{z\} \cap W^s_{\mathrm{loc}}(q'))\right)$ for $z \in \delta D_2$, then $T_x\mathcal{D}_z$ is in a horizontal ϵ-cone for $x \in \mathcal{D}_z$, and $\pi_1 \mathcal{D}_z = D_1$.

This shows that $\Delta_1 := \bigcup_{z \in \delta D_2} \mathcal{D}_z$ is a full component (Definition 12.6.5) of $\Delta \cap f^{N_1}(\Delta)$. We have in fact shown that in a natural sense this component can be taken arbitrarily close to horizontal. Together with $\Delta_0 := \mathbb{C}\left(\Delta \cap f^{N_1}(\Delta), 0\right)$ which is obviously a full component, we thus have verified (1) of Definition 12.6.5. It remains to prove the required hyperbolicity. Conditions (2) and (3) of Definition 12.6.5 are easy to check for points $x \in f^{-N_1}(\Delta_0)$ since $f^i(x) \in \Delta$ for $i = 1, 2, \dots, N_1$. Consider $f^{-N_1}(\Delta_1)$. Since $f^{N_1}(q')$ is a transverse homoclinic point we can use the decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$ to write $Df^{N_1}(q') = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$ with E nonsingular. The same holds for all $x \in f^{-N_1}(\Delta_1)$ by our choice of δ. If these differentials do not satisfy (2) and (3), replace q' by $q'' = f^{-m}(q')$ and N_1 by $N_2 = N_1 + m + n$ for some $n, m \in \mathbb{N}$ to be specified later. Then

$$Df^{N_2}(q'') = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} \begin{pmatrix} A'_m & B'_m \\ C'_m & D'_m \end{pmatrix}.$$

Since E is nonsingular there exists a $\gamma_0 \in \mathbb{R}$ such that the horizontal γ_0-cone is mapped into the horizontal γ_1-cone with $\gamma_1 < \infty$ by $\begin{pmatrix} E & F \\ G & H \end{pmatrix}$. For $\gamma \in \mathbb{R}_+$ take $m \in \mathbb{N}$ such that the horizontal γ-cone is mapped into the horizontal γ_0-cone by $\begin{pmatrix} A'_m & B'_m \\ C'_m & D'_m \end{pmatrix}$ and $n \in \mathbb{N}$ such that the horizontal γ_1-cone is mapped into the horizontal γ-cone by $\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$. Thus $Df^{N_2}(q'')$ preserves horizontal γ-cones. Enlarging n, m further, if necessary, shows that $Df^{N_2}(q'')$ expands vectors in γ-cones. Since these estimates can be made uniformly on $f^{-N_2}(\Delta_1)$ and even better estimates hold on $f^{-N_2}(\Delta_0)$, we obtain (2) and (3) of Definition 12.6.5. □

7. Absolute continuity

The central argument with which the ergodic theory of hyperbolic dynamical systems started is the Hopf argument, and this argument relies on using the Fubini Theorem, that is, absolute continuity of the invariant foliations. This section establishes absolute continuity of these foliations in a discrete-time setting that is general enough to apply directly to time-1 maps of flows. It is, in fact, more general than that, covering partially hyperbolic diffeomorphisms (Definition 5.5.2) and hence also time-1 maps of partially hyperbolic flows. For our purpose, viewing time-1 maps of Anosov flows as partially hyperbolic diffeomorphisms with one-dimensional center direction shows that the proof of ergodicity also establishes ergodicity of an invariant volume for an Anosov flow. We note that elsewhere, we bypassed the issue of absolute continuity by establishing a weaker property (the Volume Lemma, Proposition 8.4.3) and using the theory of equilibrium states.

Before embarking on the proof, we present here the *original* Katok example of a foliation that is not absolutely continuous, as written down by Keith Burns.

Let A be the hyperbolic automorphism of the torus $\mathbb{T}^2$ defined by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

There is a family $\{f_t \mid t \in [0,1]\}$ of diffeomorphisms preserving the area m and satisfying the following conditions:

(1) f_t is a small perturbation of A for every $t \in [0,1]$;
(2) f_t depends smoothly on t;
(3) $l'(t) \neq 0$, where $l(t)$ is the larger eigenvalue of the derivative of f_t at its fixed point.

The diffeomorphisms f_t are all Anosov, conjugate to A, and ergodic with respect to m. For any s and t in $[0,1]$, the maps f_s and f_t are conjugate via a unique homeomorphism h_{st} close to the identity, i.e., $f_t = h_{st} \circ f_s \circ h_{st}^{-1}$. The homeomorphism h_{st} is Hölder continuous. Let m_{st} be the pushforward of m by h_{st}. Then m_{st} is an ergodic invariant measure for f_t. Using the condition on $l(t)$ and the following lemma, we see that $m \neq m_{st}$ unless $s = t$.

Lemma 12.7.1 (de la Llave [**205**]). *Suppose $f, g\colon \mathbb{T}^2 \to \mathbb{T}^2$ are smooth area-preserving Anosov diffeomorphisms that are conjugate via an area-preserving homeomorphism*

h. Let p be a periodic point for f with least period k. Then $Df^k(p)$ and $Dg^k(h(p))$ have the same eigenvalues up to sign.

Proof. Let λ and λ' be the eigenvalues of $Df^k(p)$ and $Dg^k(h(p))$, respectively, inside the unit circle. Since f and g preserve area, the other eigenvalues of $Df^k(p)$ and $Dg^k(h(p))$ are $1/\lambda$ and $1/\lambda'$ respectively. Choose $x \in W^u_{loc}(p;f) \setminus \{p\}$ and $y \in W^s_{loc}(p;f) \setminus \{p\}$. Let R_n be the smallest "rectangle" with boundary in $W^s_{loc}(p;f)$, $W^s_{loc}(f^{-kn}(x);f)$, $W^u_{loc}(p;f)$, and $W^u_{loc}(f^{kn}(y);f)$, and R'_n the smallest "rectangle" with boundary in $W^s_{loc}(h(p);g)$, $W^s_{loc}(h(f^{-kn}(x));g)$, $W^u_{loc}(h(p);g)$, and $W^u_{loc}(h(f^{kn}(y));g)$. Then

$$\lim_{n\to\infty}\frac{\operatorname{area}(R_{n+1})}{\operatorname{area}(R_n)} = \lambda^{2k} \quad \text{and} \quad \lim_{n\to\infty}\frac{\operatorname{area}(R'_{n+1})}{\operatorname{area}(R_n)} = \lambda'^{2k}.$$

On the other hand, the conjugacy h takes R_n to R'_n for any n. Since h is area-preserving, it follows that $\lambda = \pm\lambda'$. □

A point is *generic* with respect to an invariant measure if the forward and backward Birkhoff averages of any continuous function are defined at the point and are equal to its integral with respect to the measure. If x is generic for f_s with respect to m, then $h_{st}(x)$ is generic for f_t with respect to m_{st} and hence is not generic for f_t with respect to m, unless $s = t$. (To see this, note that the Birkhoff averages of a continuous function φ along the f_t-orbit of $h_{st}(x)$ are the same as the Birkhoff averages of $\varphi \circ h_{st}$ along the f_s orbit of x.)

Now consider the diffeomorphism $F\colon \mathbb{T}^2 \times [0,1] \to \mathbb{T}^2 \times [0,1]$ given by $F(x,t) = (f_t(x), t)$. We have just observed that for any $x \in \mathbb{T}^2$ the set $H(x) = \{(h_{0t}(x), t) \mid t \in [0,1]\}$ contains at most one element of the set $\mathscr{G}$ of points $(y,t) \in \mathbb{T}^2 \times [0,1]$ such that y is generic for f_t with respect to m.

Now, F is a small perturbation of $A \times id_{[0,1]}$ and thus partially hyperbolic. It follows from Theorem 12.7.2 that F has a center foliation whose leaves are small perturbations of the intervals $\{x\} \times [0,1]$ for $x \in \mathbb{T}^2$:

Theorem 12.7.2 (Hirsch, Pugh, Shub [**159**, Theorem 7.5]). *Assume that the central distribution E^c for f is integrable, that the corresponding foliation W^c is smooth and that g is a C^q diffeomorphism sufficiently close to f in the C^1-topology. Then g is partially hyperbolic with integrable central distribution E^c_g.*

Since F maps the tori $\mathbb{T}^2 \times \{t\}$ into themselves, it is easily seen that the leaves of W^c_F are ℓ-normally hyperbolic for any ℓ, and hence are C^∞ by Theorem 12.7.3. On the other hand, for each $x \in \mathbb{T}^2$, the leaf of W^c_F that passes through $(x,0) \in \mathbb{T}^2 \times [0,1]$ is $H(x)$.

Theorem 12.7.3. *Let $f\colon M \to M$ be a partially hyperbolic embedding with integrable central distribution. Then $W^c(x) \in C^\ell$ for every $x \in M$ and ℓ such that the leaves of W^c_F are ℓ-normally hyperbolic.*

The set $\mathcal{G}$ of generic points for F has full measure with respect to m in each torus $\mathbb{T}^2 \times \{t\}$ and hence has full Lebesgue measure in $\mathbb{T}^2 \times [0,1]$, but, as observed above, it intersects each center leaf in at most one point.

To construct an analogous example on $\mathbb{T}^2 \times S^1$ use two periodic points simultaneously instead of the one fixed point. The example here is constructed in such a way that $l(t) = l(s) \Rightarrow t = s$. For a continuous parametrization using $t \in S^1$ this won't work, but starting from the map A^2 instead, which has several fixed points, we use perturbations for which the largest eigenvalues $l_1(t)$ and $l_2(t)$ at two fixed points $x_1(t)$ and $x_2(t)$ satisfy $l_1(t) = l_1(s)$ and $l_2(t) = l_2(s) \Rightarrow t = s \pmod 1$. For example, make $l_1'(t) > 0$ on $(0,1/2)$, $l_1'(t) < 0$ on $(1/2,1)$ and $l_2'(t) = 0$ on $(0,1/2)$, $l_1'(t) > 0$ on $(1/2,3/4)$, $l_2'(t) < 0$ on $(3/4,1)$.

Now we get to work on the proof of absolute continuity. With the conorm from Definition 5.5.1, we define:

Definition 12.7.4. An embedding f is said to be *relatively partially hyperbolic* on Λ if there exists a Riemannian metric called a *Lyapunov metric* in an open neighborhood U of Λ for which there are continuous *functions*

$$0 < \lambda < \zeta \le \xi < \lambda \text{ with } \lambda < 1 < \mu$$

in Λ and a pairwise orthogonal invariant splitting (5.5.2) such that if $x \in \Lambda$ then

$$\|d_x f \restriction E^s(x)\| \le \lambda < \zeta \le [\![d_x f \restriction E^c(x)]\!] \le \|d_x f \restriction E^c(x)\| \le \xi < \mu \le [\![d_x f \restriction E^u(x)]\!].$$

We introduce useful terminology and conventions for our local analysis. For $x \in M$ and $n \in \mathbb{Z}$ let $x_n := f^n(x)$, and for $\varphi\colon M \to (0,\infty)$ and $n \in \mathbb{N}$ define $\varphi_0 := 1$,

$$\varphi_n := \varphi \cdot \varphi \circ f \cdot \varphi \circ f^2 \cdots \varphi \circ f^{n-1} \quad \text{and} \quad \varphi_{-n} := [\varphi \circ f^{-n} \cdot \varphi \circ f^{1-n} \cdots \varphi \circ f^{-1}]^{-1}.$$

We furthermore choose $R > 0$ small enough that every Riemannian R-ball $B(p,R)$ lies in a foliation box (Definition 8.1.16) and if $q, q' \in B(p,R)$ and $q \in W^s_{\text{loc}}(q')$ then $d(f(q), f(q')) \le \lambda(p) d(q,q')$ (where λ is as in Definition 12.7.4). Inductively, this implies that if $q_j, q_j' \in B(p_j, R)$ for $j = 0, \dots, n-1$ and $q \in W^s_{\text{loc}}(q')$ then $d(q_n, q_n') \le \lambda_n d(q,q')$ where $\lambda_n = \prod_{i=0}^{n-1} \lambda(p_i)$.

Definition 8.1.18 describes the property we will verify for the stable and unstable foliations of a partially hyperbolic dynamical system.

Theorem 12.7.5 (Transverse absolute continuity)**.** *The stable and unstable foliations of a $C^{1+\alpha}$ partially hyperbolic diffeomorphism are transversely absolutely continuous with bounded Jacobians (Definition 8.1.18).*

With Proposition 8.1.21 this implies:

Theorem 12.7.6 (Absolute continuity)**.** *The stable and unstable foliations of a $C^{1+\alpha}$ partially hyperbolic diffeomorphism are absolutely continuous with bounded Jacobians (Definition 8.1.19).*

These results cover our needs because we apply them to time-t maps of hyperbolic flows, where one can take $\zeta = \xi = 1$ in Definition 12.7.4. Indeed, these statements go far beyond the applications in this book. They, and their proof are included here in this form to put in the published record the argument by Abdenur and Viana with which one obtains this important ingredient for the study of partially hyperbolic diffeomorphisms [**236**].

Let us explain the strategy before embarking on the proof of Theorem 12.7.5. It is remarkably simple even though the full proof is rather long. To begin with, it suffices to prove this for stable holonomies, passing to the inverse then gives absolute continuity of the unstable foliation.

First, consider what happens to the volume of a large disk mapped from one transversal to another along *short* stable leaves: Any difference between the image of the disk and a disk in the second transversal resides in a small neighborhood of the boundary, so the error in volume is a small percentage. If we are concerned about small sets being mapped by the holonomy anong not-so-short stable leaves, we push the whole picture forward by the dynamics to make the sets large and the stable leaves short. This reduces the situation to the previous one, except that we have to explain that these images are sufficiently "disk-like" and that the distortions of the set under the dynamics applied to one of the transversals versus the other one are not so large as to make the reduction useless. This uses distortion control along exponentially close orbits in ways we have seen before. The proof we give here is classical, but in the details follows an unpublished manuscript of Abdenur and Viana.

Proof of Theorem 12.7.5 (Abdenur–Viana). Referring to Definition 8.1.18 with $\mathscr{F}$ the stable foliation, we will show that

$$m_{\tau_1}(A)/C \le m_{\tau_2}(h_{\tau_1,\tau_2}(A)) \le C m_{\tau_1}(A) \tag{12.7.1}$$

whenever A is a *disk* in τ_1. To see that this implies the same conclusion for any measurable set $E \subset \tau_1$ cover it by disks A_i with $\sum m_{\tau_1}(A_i) \le m_{\tau_1}(E) + \epsilon$ to get

$$m_{\tau_2}(h_{\tau_1,\tau_2}(E)) \le \sum m_{\tau_2}(h_{\tau_1,\tau_2}(A_i)) \le C \sum m_{\tau_1}(A_i) \le C(m_{\tau_1}(E) + \epsilon),$$

where ϵ can be arbitrarily small. The reverse inequality follows from the same argument applied to h_{τ_2,τ_1}. Indeed, this symmetry shows that we only need to establish $m_{\tau_2}(h_{\tau_1,\tau_2}(A)) \le C m_{\tau_1}(A)$ for disks.

Partial hyperbolicity provides $m \in \mathbb{N}$ and $\theta \in (0,1)$ (a measure of the gap between stable and center-unstable behavior) for which

$$\alpha_m(x) := \|Df^m{\restriction_{E^s(x)}}\| < \theta^2 \min(1, \beta_m(x)) \text{ for all } x \in M, \tag{12.7.2}$$

where $\beta_m(x) := \|Df^{-m}{\restriction_{E^{cu}(f^m(x))}}\|^{-1}$ (in the hyperbolic case we take $\beta_m(x) \equiv 1$). Using a Lyapunov metric or by passing to f^m we may (and will) take $m = 1$ here.

It is useful to get a quantitative measure of closeness of $f^n(\tau_1)$ and E^{cu} that improves exponentially in n. (This is reminiscent of the Inclination Lemma, Theorem 12.6.1.) Writing $d(E_1, E_2) := \max_{v \in E_2,\ \|v\|=1} d(v, E_1)$ we have:

Lemma 12.7.7. *There is a $K_1 > 0$ such that $d(T_{f^n(x)} f^n(\tau_1), E^{cu}(f^n(x))) \le K_1 \theta^{2n}$ for $n \in \mathbb{N}$ and $x \in \tau_1$. Similarly for τ_2.*

Proof. For a vector $v_n \in T_{f^n(x)} f^n(\tau_1)$ write $v_n = Df^n(x)v$ with $v \in T_x\tau_1$ and decompose $v = v^s + v^{cu}$ with $v^i \in E^i(x)$. Transversality of τ_1 and E^s gives a $K_1 > 0$ that depends only on τ_1 such that $\|v^s\| \le K_1 \|v^{cu}\|$. Now, for $i \le n$, (12.7.2) gives

$$\left\| \frac{Df^i(x)v}{\|Df^i(x)v^{cu}\|} - \frac{Df^i(x)v^{cu}}{\|Df^i(x)v^{cu}\|} \right\| = \frac{\|Df^i(x)v^s\|}{\|Df^i(x)v^{cu}\|} \le \theta^{2i} \frac{\|v^s\|}{\|v^{cu}\|} \le K_1 \theta^{2i}. \quad \square \tag{12.7.3}$$

We now refine the explanation of our proof strategy a little. While we will indeed apply the partially hyperbolic diffeomorphism f repeatedly to A and $h_{\tau_1,\tau_2}(A)$, the resulting sets are highly distorted, and instead of trying to control their sizes directly, we will instead cover $f^n(A)$ with disks $B_n(x) := B(r(n,x), f^n(x))$ of radius $r(n,x)$ chosen large relative to the distance between $f^n(\tau_1)$ and $f^n(\tau_2)$ but small with respect to the "thinnest" direction of $f^n(A)$. Actually, more to the point, $r(n,x)$ will be chosen small enough for the Jacobian of f^n to be close enough to constant on $f^{-n}(B_n(x))$ (Lemma 12.7.14) and to also agree across the holonomy gap at time n (Proposition 12.7.15). This amounts to choosing it in the gap between the contracting rates in the stable direction and the rates in the center direction as follows.

Recalling (12.7.2), we note that by continuity of α_1, β_1 in $\alpha_1(x) < \theta^2 \min(1, \beta_1(x))$ we can choose $\delta > 0$ such that $a < \theta^2 \min(1, b)$, where $a(x) := \sup\{\alpha_1(y) \mid d(x,y) < \delta\}$ and $b(x) := \sup\{\beta_1(y) \mid d(x,y) < \delta\}$.

Now,

$$\mu(n,x) := \prod_{i=0}^{n-1} a(f^i(x))$$

is an upper bound for the stable contraction along *any* orbit segment that stays within δ of that of x for the first n steps, and

$$\sigma(n,x) := \prod_{i=0}^{n-1} b(f^i(x))$$

is a corresponding lower bound for center-unstable behavior; the preceding estimates imply

$$\mu(n,x) < \theta^{2n}\min(1,\sigma(n,x))$$

for all x and $n \in \mathbb{N}$. With $c := a/\theta$, the desired radius of balls in $f^n(\tau_1)$ is

$$r(n,x) := \prod_{i=0}^{n-1} c(f^i(x)).$$

As advertized, it satisfies

$$\mu(n,x) \le \theta^n r(n,x) \le \theta^{2n}\min(1,\sigma(n,x)). \tag{12.7.4}$$

Having chosen these radii, we now show that it is not only the last points of an orbit segment that are exponentially close, but the whole segment.

Lemma 12.7.8. *If $f^n(\xi) \in B_n(x)$ then $d(f^i(x), f^i(\xi)) \le 3\theta^n$ when $-\frac{\log 2+\log K_1}{2\log\theta} \le p \le i \le n$.*

Proof. We prove this by induction downwards from $i = n$, in which case the conclusion is the definition of $B_n(x)$. More precisely, there is a piecewise smooth curve γ_n in $f^n(\tau_1)$ from $f^n(x)$ to $f^n(\xi)$ with length less than $r(n,x)$, and we show that the length of $\gamma_i := f^{i-n}(\gamma_n)$ is less than $3\theta^n$ for $i \le n$.

Decomposing the tangent vector as $\dot\gamma_i = \dot\gamma_i^s + \dot\gamma_i^{cu} \in E^s \oplus E^{cu}$, Lemma 12.7.7 (or (12.7.3)) gives $\|\dot\gamma_i^s\|/\|\dot\gamma_i^{cu}\| \le K_1\theta^{2i}$, so for $i \ge p \in \mathbb{N}$ such that $K_1\theta^{2p} \le 1/2$ we obtain

$$\|\dot\gamma_i\| \le \frac{3}{2}\|\dot\gamma_i^{cu}\| \quad \text{and} \quad \|\dot\gamma_i\| \ge \frac{1}{2}\|\dot\gamma_i^{cu}\|.$$

For purposes of induction suppose now that the claim is known for $i+1,\dots,n$. To show that the length of $\gamma_i := f^{i-n}(\gamma_n)$ is less than $3\theta^n$ note first that by assumption it is bounded above by

$$\|Df^{-1}\|\ell(\gamma_{i+1}) \le \|Df^{-1}\|3\theta^n,$$

and assume n has been chosen large enough for the right-hand side to be less than δ. This implies that γ_j lies in a δ-ball around $f^j(x)$ for $i \le j \le n$, and we can use the definition of σ:

$$\frac{2}{3}\|\dot\gamma_i\| \le \|\dot\gamma_i^{cu}\| = \|dF^{i-n}\dot\gamma_n^{cu}\| \le \frac{\|\dot\gamma_n^{cu}\|}{\sigma(n-i,f^i(x))} \le 2\frac{\|\dot\gamma_n\|}{\sigma(n-i,f^i(x))},$$

so

$$\ell(\gamma_i) \le \frac{3}{2}\cdot 2\frac{\ell(\gamma_n)}{\sigma(n-i,f^i(x))} \le 3\frac{r(n,x)}{\sigma(n-i,f^i(x))} = 3\frac{r(i,x)r(n-i,f^i(x))}{\sigma(n-i,f^i(x))}.$$

(12.7.4) now implies the claim: $r(i,x)r(n-i,f^i(x)) \le \theta^i\theta^{n-i}\sigma(n-i,f^i(x))$. □

Having studied the dynamics on transversals, we now start to look at the way the 2 transversals become closer under repeated application of f. The first statement sounds obvious, but takes some care to establish.

Lemma 12.7.9. *There is a $K_2 > 0$ such that $d_s(y, h_n(y)) \le K_2\mu(n,x)$ for $n \in \mathbb{N}$ and $y \in B_n(x)$, where*

$$h_n := h_{f^n(\tau_1), f^n(\tau_2)}$$

and d_s denotes distance within a stable leaf.

Proof. Let $C_1 \ge \sup\{d_s(\xi, h_0(\xi) \mid \xi \in \tau_1,\ h_0(\xi) \in \tau_2\}$ and write $\xi = f^{-n}(y)$. The choice of θ then implies

$$d_s(f^i(\xi), f^i(h_0(\xi))) \le C_1 \sup \|Df\restriction_{E^s}\} \le C_1\theta^{2i}$$

for all $i \in \mathbb{N}$. While this is an exponential estimate, the point for now is merely that for $p \in \mathbb{N}$ such that $C_1\theta^{2i} < \delta/2$ this implies

$$d(f^i(\xi), f^i(h_0(\xi))) \le d_s(f^i(\xi), f^i(h_0(\xi))) \le C_1 \sup \|Df\restriction_{E^s}\} \le \delta/2$$

for all $i \ge p$. At the same time, taking p as in Lemma 12.7.8 gives $d(f^i(x), f^i(\xi)) \le 3\theta^n$ for $p \le i \le n$, where the right-hand side is less than $\delta/2$ if n is chosen large enough. Combining these, we find that $p \le i \le n$ implies

$$d(f^i(x), f^i(h_0(\xi))) < \delta \quad and \quad d(f^i(x), f^i(\xi)) < \delta.$$

This allows us to bring in the definition of μ:

$$\begin{aligned} d_s(y, h_n(y)) \le d(f^n(\xi), f^n(h_0(\xi))) &\le \mu(n-p, f^p(x)) d_s(f^p(\xi), f^p(h_0(\xi))) \\ &= \frac{\mu(n,x)}{\mu(p,x)} d_s(f^p(\xi), f^p(h_0(\xi))) < \frac{\delta}{\inf_x \mu(p,x)} \mu(n,x). \quad \square \end{aligned}$$

These preparations will let us show that $B_n(x)$ and $h_n(B_n(x))$ are graphs of maps from E^{cu} to E^s and that these 2 maps are C^1 close exponentially in n. This will make it possible to compare volumes and is the content of the next 2 lemmas.

Lemma 12.7.10. *There is a disk $D_1 \subset E^{cu}(f^n(x))$ and a C^1-map $g_1\colon D_1 \to E^s(f^n(x))$ such that $B_n(x) = \operatorname{graph}(g_1)$. Likewise for $h_n(B_n(x))$.*

Proof. Consider the balls $B^{cu}(y,\rho)$ and $B^s(y,\rho)$ around 0 in $E^{cu}(y)$ and $E^s(y)$, respectively, and choose ρ such that the exponential map $\exp_y\colon T_yM \to M$ is an embedding of $B^{TM}(y,\rho) := B^{cu}(y,\rho) \times B^s(y,\rho)$. Then $B_n(f^{-n}(y)) \subset \exp_y(B^{TM}(y,\rho))$ for large enough n, so we can consider $B_n(x)$ as a subset of $T_{f^n(x)}M$. By Lemma 12.7.7 $f^n(\tau_1)$, hence $B_n(x)$, is nearly tangent to E^{cu}, hence transverse to E^s. So each $z \in B_n(x)$ corresponds to a unique $(z^{cu}, z^s) \in E^{cu}(f^n(x)) \times E^s(f^n(x))$ with z^{cu}-values in a disk $D_1 \subset E^{cu}(f^n(x)) \ni 0$ and defines a map $g\colon D_1 \to E^s(f^n(x))$, $z^{cu} \mapsto z^s$. Smoothness of the leaves of the foliations implies that g_1 is C^1. $\square$

Lemma 12.7.11. *There are $K_3 > 0$, $\alpha \in (0,1)$ with $\|Dg_1\| \le K_3\theta^{\alpha n}$. Likewise for g_2.*

Proof. Write $E^{cu}(y) = \text{graph}(\xi_y)$ for y near $f^n(x)$ with $\xi_y\colon E^{cu}(f^n(x)) \to E^s(f^n(x))$ satisfying $\|\xi_y\| \le C(df^n(x), y)^\alpha$ by Hölder continuity of E^{cu} (Theorem 7.4.1). Since $d_{f^n(\tau_1)}(f^n(x), y) \le r(n,x) \le \theta^n$, allowing for slight distortion under exp gives

$$d(f^n(x), y) \le 2d_{f^n(\tau_1)}(f^n(x), y) \le 2\theta^n,$$

hence $\|\xi_y\| \le C2^\alpha\theta^{\alpha n}$.

Meanwhile, Lemma 12.7.7 gives $d(T_yB_n(x), E^{cu}(y)) \le K_1\theta^{2n}$, which implies

$$T_yB_n(x) = \text{graph}(\zeta_y), \qquad \zeta_y\colon E^{cu}(f^n(y)) \to E^s(f^n(y))$$

for large enough n, with $\|\zeta_y - \xi_y\| \le Ld(T_yB_n(x), E^{cu}(y))$ for some L, so

$$\|\zeta_y\| \le \|\xi_y\| + \|\zeta_y - \xi_y\| \le C2^\alpha\theta^{\alpha n} + LK_1\theta^{2n} \le (C2^\alpha + LK_1)\theta^{\alpha n}. \qquad \square$$

We have now achieved in precise terms the first step of the proof strategy: in forward time these balls are close to each other in the C^1-topology, indeed exponentially so. As a result of this, any difference in volume amounts to an exponentially small percentage error concentrated around the edges:

Lemma 12.7.12. *For some $\alpha' \in (0,\alpha)$ and $K_4 > 0$, D_1 contains the ball around zero of radius $r(n,x)(1 - K_4\theta^{\alpha' n})$ and is contained in the ball around zero of radius $r(n,x)(1 + K_4\theta^{\alpha' n})$. Likewise for D_2.*

Proof. This follows from the small-angle property $d(T_yB_n(x), E^{cu}(f^n(x))) \le K_1\theta^{2n}$ we just proved for $y \in B_n(x)$ because D_1 is the projection along E^s of $B_n(x)$ to $E^{cu}(f^n(x))$, hence coincides with the $r(n,x)$-ball around 0 up to a factor $e^{\theta^{\alpha' n}}$. $\square$

This allows us to conclude, as planned, that the volume of $B_n(x)$ is changed arbitrarily little under the holonomy for large enough n.

Lemma 12.7.13. $\sup_{x\in\tau_1}\left|\dfrac{m_{f^n(\tau_1)}(B_n(x))}{m_{f^n(\tau_2)}(h_n(B_n(x)))} - 1\right| \xrightarrow[n\to\infty]{} 0.$

Proof. By Lemma 12.7.12 D_1 and D_2 contain the projection to $E^{cu}(f^n(x))$ of the ball $A(n,x)$ in $f^n(\tau_1)$ of radius $r(n,x)(1 - K_4\theta^{\alpha' n})$ around 0 and lie in the projection of the ball $C(n,x)$ of radius $r(n,x)(1 + K_4\theta^{\alpha' n})$. Lemma 12.7.11 implies

$$\max_{x\in\tau_1}\left|\frac{m_{f^n(\tau_1)}(A(n,x))}{m_{f^n(\tau_1)}(C(n,x))} - 1\right| \xrightarrow[n\to\infty]{} 0 \tag{12.7.5}$$

and

$$\max_{x\in\tau_1}\left|\frac{m_{f^n(\tau_2)}(h_n(A(n,x)))}{m_{f^n(\tau_2)}(h_n(C(n,x)))} - 1\right| \xrightarrow[n\to\infty]{} 0.$$

Writing $P(u, g_k(u)) \coloneqq u$ for $k = 1,2$ and large enough $n \in \mathbb{N}$ gives

$$P(A(n,x)) \subset P(h_n(B_n(x))) \subset P(C(n,x)) \tag{12.7.6}$$

while for $k = 1,2$ and a disk $D \subset f^n(\tau_k)$ the definition of $m_{f^n(\tau_k)}$ gives

$$m_{f^n(\tau_k)}(D) = \iint_{P(D)} \sqrt{\det g^k_{ij}}\, du_1 \ldots du_{\dim E^{cu}}.$$

The coefficients $g^k_{ij} = \delta_{ij} + \frac{\partial g_k}{\partial u_i}\frac{\partial g_k}{\partial u_j}$ of the inner product only involve first derivatives of g_k, so Lemma 12.7.11 and (12.7.6) imply that for $\epsilon > 0$ there is an $n \in \mathbb{N}$ with

$$\begin{aligned} m_{f^n(\tau_1)}(A(n,x)) &= \iint_{P(A(n,x))} \sqrt{\det g^k_{ij}}\, du_1 \ldots du_{\dim E^{cu}} \\ &\le \iint_{P(h_n(B_n(x)))} \sqrt{\det g^k_{ij}}\, du_1 \ldots du_{\dim E^{cu}} + \epsilon m(P(h_n(B_n(x)))) \\ &= (1+\epsilon) m(P(h_n(B_n(x)))), \end{aligned}$$

where m is the standard measure in $T_{f^n(x)}B_n(x)$. Now (12.7.5) lets us replace the left-hand side by $m_{f^n(\tau_1)}(B_n(x))$ and on the right-hand side $m(P(h_n(B_n(x))))$ is arbitrarily close to $m_{f^n(\tau_2)}(h_n(B_n(x)))$.

Similar arguments bound $m_{f^n(\tau_2)}(h_n(B_n(x)))$ in terms of $m_{f^n(\tau_1)}(C(n,x))$. □

As promised in the proof outline, we have now shown that the volume of $B_n(x)$ is essentially preserved by the holonomy. From here it is downhill. We first show that pulling this result back by Df^{-n} involves on either side a distortion that is essentially constant, and the next step is to check that these constants are close enough to each other.

Lemma 12.7.14. $\sup\limits_{n\in\mathbb{N}, z_1, z_2 \in B_n(x)} \left| \log\det Df^{-n}(z_1)\restriction_{T_{z_1}B_n(x)} - \log\det Df^{-n}(z_2)\restriction_{T_{z_2}B_n(x)} \right| < \infty.$ *Likewise for* $\log\det Df^{-n}(z_k)\restriction_{T_{z_k}h_n(B_n(x))}$.

Proof. Note that $|\log\det Df^{-1}(z)\restriction_{A_1} - \log\det Df^{-1}(z)\restriction_{A_2}| \le C_1 d(A_1, A_2)$ for nearby $\dim E^{cu}$-dimensional subspaces A_k and that Hölder continuity of E^{cu} implies the same for $\log\det Df^{-1}\upharpoonright_{E^{cu}}$. For $z_1, z_2 \in f^{-n(B_n(x))}$, Lemma 12.7.7 thus implies

$$\begin{aligned}
\Big|\log\frac{\det Df^{-n}(z_1)\upharpoonright_{T_{z_1}B_n(x)}}{\det Df^{-n}(z_2)\upharpoonright_{T_{z_2}B_n(x)}}\Big| &\le \underbrace{\Big|\log\frac{\det Df^{-n}(z_1)\upharpoonright_{T_{z_1}B_n(x)}}{\det Df^{-n}(z_1)\upharpoonright_{E^{cu}(z_1)}}\Big|}_{\le C_1\sum_{i=0}^{n-1} d(T_{f^{-i}(z_1)}f^{-i}(B_n(x)),E^{cu}(f^{-i}(z_1)))} \\
&+\underbrace{\Big|\log\frac{\det Df^{-n}(z_1)\upharpoonright_{E^{cu}(z_1)}}{\det Df^{-n}(z_2)\upharpoonright_{E^{cu}(z_2)}}\Big|}_{\le C_2\sum_{i=0}^{n-1} d(f^{-i}(z_1),f^{-i}(z_2))^{\alpha}}+\underbrace{\Big|\log\frac{\det Df^{-n}(z_2)\upharpoonright_{E^{cu}(z_2)}}{\det Df^{-n}(z_2)\upharpoonright_{T_{z_2}B_n(x)}}\Big|}_{\le C_1\sum_{i=0}^{n-1} d(T_{f^{-i}(z_2)}f^{-i}(B_n(x)),E^{cu}(f^{-i}(z_2)))} \\
&\le (C_2+2C_1K_1)\sum_{i=0}^{n-1}\theta^{\alpha(n-i)}d(z_1,z_2) < \frac{C_2+2C_1K_1}{1-\theta^{\alpha}}d(z_1,z_2).
\end{aligned}$$

Likewise for $\log\det Df^{-n}(z_k)\upharpoonright_{T_{z_k}h_n(B_n(x))}$. □

A similar argument now establishes our advertized goal:

Proposition 12.7.15. $\displaystyle\sup_n \log\frac{m_{\tau_1}(f^{-n}(B_n(x)))}{m_{\tau_2}(h_0(f^{-n}(B_n(x))))} < \infty.$

Proof. We write the numerator and denominator as

$$\begin{aligned}
m_{\tau_1}(f^{-n}(B_n(x))) &= \int_{B_n(x)} |\det Df^{-n}(z)\upharpoonright_{T_zB_n(x)}|\, dm_{f^m(\tau_1)}(z), \\
m_{\tau_2}(h_0(f^{-n}(B_n(x)))) &= \int_{h_n(B_n(x))} |\det Df^{-n}(z)\upharpoonright_{T_{h_n(z)}h_n(B_n(x))}|\, dm_{f^m(\tau_2)}(z)
\end{aligned}$$

and note that the integrals are over sets of comparable measures $m_{f^m(\tau_1)}(B_n(x))$ and $m_{f^m(\tau_2)}(h_n(B_n(x)))$, so it suffices to show that the ratio of the integrands is bounded. To that end note first that by the last lemma, we can replace the integrands by their value at any z_1 and $h_n(z_1)$, respectively, up to a uniformly bounded factor. This leaves us to bound

$$\Big|\log\frac{\det Df^{-n}(z_1)\restriction_{T_{z_1}B_n(x)}}{\det Df^{-n}(z_1)\restriction_{T_{h_n(z_1)}h_n(B_n(x))}}\Big| \le \underbrace{\Big|\log\frac{\det Df^{-n}(z_1)\restriction_{T_{z_1}B_n(x)}}{\det Df^{-n}(z_1)\restriction_{E^{cu}(z_1)}}\Big|}_{\le C_1\sum_{i=0}^{n-1} d(T_{f^{-i}(z_1)}f^{-i}(B_n(x)),E^{cu}(f^{-i}(z_1)))}$$

$$+\underbrace{\Big|\log\frac{\det Df^{-n}(z_1)\restriction_{E^{cu}(z_1)}}{\det Df^{-n}(h_n(z_1))\restriction_{E^{cu}(h_n(z_1))}}\Big|}_{\le C_2\sum_{i=0}^{n-1} d(f^{-i}(z_1),f^{-i}(h_n(z_1)))^{\alpha}}+\underbrace{\Big|\log\frac{\det Df^{-n}(h_n(z_1))\restriction_{E^{cu}(h_n(z_1))}}{\det Df^{-n}(z_1)\restriction_{T_{h_n(z_1)}h_n(B_n(x))}}\Big|}_{\le C_1\sum_{i=0}^{n-1} d(T_{f^{-i}(h_n(z_1))}f^{-i}(h_n(B_n(x))),E^{cu}(f^{-i}(h_n(z_1))))}$$

$$\le (C_2+2C_1K_1)\sum_{i=0}^{n-1}\theta^{\alpha(n-i)}d(z_1,h_n(z_1)) < \frac{C_2+2C_1K_1}{1-\theta^{\alpha}} < \infty.$$

This uniformly controls the distortion effected by h_0 on $f^{-n}(B_n(x))$. □

Proposition 12.7.15 says that pullbacks of suitably chosen balls behave well under the holonomy. As promised, a simple covering argument now establishes (12.7.1) and hence completes the proof of Theorem 12.7.5.

With A as in (12.7.1) choose $\epsilon > 0$ such that $A_\epsilon := \{y \in A \mid d_{\tau_1}(y, \tau_1 \smallsetminus A) > \epsilon\}$ satisfies

$$\Big|\frac{m_{\tau_1}(A_\epsilon)}{m_{\tau_1}(A)} - 1\Big| < \frac{1}{2} \quad\text{and}\quad \Big|\frac{m_{\tau_1}h_0((A_\epsilon))}{m_{\tau_1}h_0((A))} - 1\Big| < \frac{1}{2}.$$

The cover $\mathscr{B}_n := \{B_n(x) \mid x \in A_\epsilon\}$ of A_ϵ satisfies $\bigcup\mathscr{B}_n \subset f^n(A)$ for large n since the radius of $B_n(x)$ is $r(n,x) \le \theta^n\sigma(n,x) < \text{const.}\theta^n d(\partial f^n(A), \partial f^n(A_\epsilon))$ by (12.7.4). The Besicovich Covering Theorem provides a countable subcover $\mathscr{C}_n \subset \mathscr{B}_n$ of A_ϵ such that no point of A_ϵ is contained in more than ℓ elements of $\mathscr{C}_n$, where ℓ depends only on the dimension of A_ϵ (this is where we use that A is a disk). Thus

$$\frac{1}{2\ell}m_{\tau_1}(A) \le \frac{1}{\ell}m_{\tau_1}(A_\epsilon) \le \sum_{B\in\mathscr{C}_n} m_{\tau_1}(f^{-n}(B)) \le \ell m_{\tau_1}(A),$$

and likewise,

$$\frac{1}{2\ell}m_{\tau_2}(h_0(A)) \le \frac{1}{\ell}m_{\tau_2}(h_0(A_\epsilon)) \le \sum_{B\in\mathscr{C}_n} m_{\tau_2}(h_0(f^{-n}(B))) \le \ell m_{\tau_2}(h_0(A)),$$

so Theorem 12.7.5 follows by applying Proposition 12.7.15 to each $B \in \mathscr{C}_n$. □

Hints and answers to the exercises

Exercise 1.1. Show and then use that $\mathbb{T} := \{t \mid \varphi^t(x) = x\} \ni 0$ is closed (continuity), $\mathbb{T} + \mathbb{T} = \mathbb{T}$, and (1)$\Leftrightarrow$ $\mathbb{T} = \{0\}$, (2)$\Leftrightarrow$ 0 is an accumulation point of $\mathbb{T}$, (3)$\Leftrightarrow$ $\mathbb{T} = \mathbb{R}$.

Exercise 1.2. $\dfrac{dH}{dt} = \dfrac{dH}{x}\dfrac{dx}{dt} + \dfrac{dH}{dv}\dfrac{dv}{dt} = g(x) \cdot v + v \cdot (-g(x)) \equiv 0.$

Exercise 1.7. $y \in W^s(\{x\}) \Leftrightarrow \varnothing \neq \omega(y) \subset \{x\} \Leftrightarrow \omega(y) = \{x\} \Leftrightarrow \varphi^t(y) \to x \Leftrightarrow y \in W^s(x)$.

Exercise 1.9. As usual, reflexivity (take $h = \mathrm{Id}$) and symmetry (replace h by h^{-1}) are easy. Transitivity: compose 2 conjugacies and check that this is as required.

Exercise 1.11. See [**220**, p. 59].

Exercise 1.12. Same steps as for Exercise 1.9.

Exercise 1.13. Use a circle around the attracting fixed point as a fundamental domain to define the conjugacy analogously to the linear case for all orbits that tend to this equilibrium; extend by continuity to the orbits ending on the saddle.

Exercise 1.14. A fixed point is a constant sequence, and sequences asymptotic to it are those which are eventually (on the left or the right, respectively) constant.

Exercise 1.15. A periodic point is a periodic sequence, so like in Exercise 1.14, sequences asymptotic to it are those which are eventually (on the left or the right, respectively) periodic.

Exercise 1.22. All other points have empty first prolongational limit sets.

Exercise 1.25. Example 1.4.14 is a counterexample.

Exercise 1.28. This can be shown from the definitions or from Theorem 1.5.41 (L induces a continuous injection into $\mathbb{R}$, which is a homeomorphism onto its image by invariance of domain), or follows from the next exercise.

Exercise 1.29. Check that the Lyapunov function from Theorem 1.5.41 is well-defined modulo "$\sim$" and defines a continuous bijection $\mathscr{R}(\Phi)/\sim\ \to L(\mathscr{R}(\Phi))$, which is the ternary Cantor set or a finite subset. This is then a homeomorphism by compactness (invariance of domain).

Exercise 2.2. This need not be computational, but these computations suffice: $E\wedge dE(P,V,Q)=\cos^2\theta(\cos^2\theta+\sin^2\theta)+\sin^2\theta(\cos^2\theta+\sin^2\theta)\equiv 1$, $E(P)=1$, $E(Q)=0$, $E(V)=0$, $Z\in\{Q,V\}\Rightarrow dE(P,Z)=\underbrace{\mathscr{L}_P\overbrace{E(Z)}^{\equiv 0}}_{=0}-\underbrace{\mathscr{L}_Z\overbrace{E(P)}^{\equiv 1}}_{=0}-\underbrace{E(\overbrace{[P,Z]}^{\in -\{V,Q\}})}_{=0}=0$, $\zeta^\pm:=Q\pm V\Rightarrow$ $[P,\zeta^\pm]=-\cos^2\theta[H,X]\mp\cos\theta[V,X]-\sin^2\theta[H,X]\pm\sin\theta[H,V]=\mp\zeta^\pm$.

Exercise 2.9. $\theta=(1/2)\sum_{i=1}^n(p_i\,dq_i-q_i\,dp_i)$, $v=-(-q,p)$.

Exercise 2.13. Decompose into blocks $\begin{pmatrix}\lambda & 0\\ 0 & 1/\lambda\end{pmatrix}$ for real λ, rotations R_α for $\lambda=e^{i\alpha}$, and $\begin{pmatrix}\rho R_\alpha & 0\\ 0 & \rho^{-1}R_{-\alpha}\end{pmatrix}$ for $\lambda=\rho e^{i\alpha}$.

Exercise 2.14. ω^n is a volume and exterior multiplication on forms induces a multiplicative structure on cohomology, hence the second cohomology of ω is nonzero.

Exercise 2.15. Use the previous exercise.

Exercise 2.16. Use the Moser “homotopy trick” in the proof of the Darboux Theorem 2.6.11.

Exercise 2.17. Use rotational symmetry (this is an instance of the Noether Theorem). The integral obtained is angular momentum. Independence can be seen by studying how the integral depends on momenta.

Exercise 2.19. To show that $\omega(v,w)$ depends only on the projection of v and w use that the projections are along the flow, hence invariant, and $\omega(X,X_H)=0$ for every $X\in TM_c$.

Exercise 2.20. For $n=2$ a geodesic is an oriented great circle and hence identified with an oriented plane which in turn is defined by a unit vector (a positive normal). The space of these is S^2. By rotational symmetry the volume is the standard one. Alternatively take a single great circle together with the unit tangent vectors pointing into one complementary hemisphere as a transversal and compactify by the two tangent directions to again get a sphere.

Exercise 3.1. Although length is preserved in Example 1.1.5, there is no invariant Borel probability measure (any open interval has infinitely many disjoint images of equal measure, which must be 0 for finite total measure). Example 1.1.7 only has the point mass at 0; by similar reasons no open interval in $\mathbb{R}\smallsetminus\{0\}$ has positive measure. As noted, Example 1.3.5 is conjugate to Example 1.1.5 hence has no invariant Borel probability measure. From before, no invariant Borel probability measure has positive measure in the interior of Example 1.3.6, which leaves the Dirac masses at the ends and their convex combinations. Likewise, Example 1.3.9 has only the Dirac mass at the fixed point. More generally, *any* probability measure on the fixed-point set in Example 1.3.11 is an invariant probability measure, and these are it. By like arguments, only the circle of fixed points in Example 1.4.14 supports invariant Borel probability measures, and any Borel probability measure on this circle is an invariant Borel probability measure. In Figure 1.4.1 again only the 2 fixed points support invariant probability measures, so $\mathfrak{M}(\Phi$ consists of convex combinations of 2 Dirac masses, that is, an interval. Likewise for Figure 1.5.4 but with 3 points (so $\mathfrak{M}(\Phi$ is now a triangle). In Figure 1.5.11 all interior points are wandering, so the invariant Borel probability measures are supported on the boundary, and any Borel probability measure on the boundary is invariant. Figure 1.1.4 is the most complex. All points are fixed or periodic orbits and hence support a Dirac measure. Standard area is also invariant (this is the Hamiltonian nature of the pendulum), as is area multiplied by any constant of motion as a density, and we can expect a multitude of other invariant measures. However, the aforementioned Dirac measures are the only *ergodic* ones, so $\mathfrak{M}(\Phi)$ is their closed convex hull.

Exercise 3.2. To check Proposition 1.6.9(3), suppose $\varnothing \neq U, V \subset \operatorname{supp}\mu$ are open, hence have positive measure. By ergodicity, $\mathcal{U} := \varphi^{\mathbb{R}}(U)$ has full measure, hence intersects V.

Exercise 3.4. Use the argument for (4)⇒(1) in Proposition 1.6.9: if $U_1, U_2, \dots$ is a base for the topology of $\operatorname{supp}\mu$, then $E_i := \{x \in \operatorname{supp}\mu \mid \varphi^{\mathbb{R}}(x) \cap U_i \neq \varnothing\}$ has full measure by the Birkhoff Ergodic Theorem applied to χ_{U_i} and so then has the desired set $\bigcap_{i \in \mathbb{N}} E_i$.

Exercise 3.5. Example 3.3.15 provides the essential insight since a circle rotation is a factor of a suspension: The exceptional set is $\mathbb{Q}$. The quickest proof is to invoke Remark 3.3.5 and Example 3.3.15, the most satisfying one would be to consider the actionof the time-p/q-map on $X \times ([0, 1/2q) + \mathbb{Z}/q)$, say.

Exercise 5.1. Check that S is sufficiently large if

$$\int_S^{S+t} \underline{\lambda}^{-2s} \left(\|D\varphi^s v\|_{\varphi^s(x)}\right)^2 ds < \int_0^t \underline{\lambda}^{-2s} \left(\|D\varphi^s v\|_{\varphi^s(x)}\right)^2 ds$$

for all $t > 0$. To see that there is such an S use that there are $c, \overline{\lambda} > 0$ such that $\|D\varphi^s v\|_{\varphi^s(x)} \geq c\overline{\lambda}^s \|x\|_x$ for all $s \geq 0$.

Exercise 5.2. The rectangle Δ is an isolating neighborhood.

Exercise 5.4. Use the Hartman–Grobman Theorem.

Exercise 5.6. Calculate the volume of a sphere by integrating the volume element generated by orthonormal Jacobi fields.

Exercise 5.7. Use the previous exercise.

Exercise 5.8. Show that $\exp_x \colon T_x\widetilde{M} \to \widetilde{M}$ is a diffeomorphism.

Exercise 5.9. Use the previous exercise.

Exercise 6.3. Apply Exercise 5.2 and Theorem 6.2.7 or check directly (noting that here and generally for suspensions $t(x,y) \equiv 0$ in Proposition 6.2.2).

Exercise 6.4. Apply Exercise 5.3 and Theorem 6.2.7 or check directly.

Exercise 6.5. This means that the geodesic flow has periodic points with incommensurate periods, which follows from Theorem 6.2.12 and Remark 2.4.6.

Exercise 6.6. This is essentially a restatement of Theorem 6.2.12(3) (which holds by Remark 2.4.6): Each neighborhood of v in $S\Sigma$ contains vectors that generate closed geodesics of incommensurate lengths, so at least one of those lengths is incommensurate with that of γ_v. This is the choice of v_i

Exercise 6.7. Apply Proposition 6.2.18 to the geodesic flow.

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