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Boundary value problems on Riemannian symmetric spaces of the noncompact type

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BOUNDARY VALUE PROBLEMS ON RIEMANNIAN SYMMETRIC SPACES OF THE NONCOMPACT TYPE

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Abstract. We characterize the image of the Poisson transform on each boundary component of a Riemannian symmetric space of the noncompact type by a system of differential equations. The system corresponds to a generator system of a two sided ideals of an universal enveloping algebra, which are explicitly given by analogues of minimal polynomials of matrices.

1. Introduction

The classical Poisson integral of a function on the unit circle in the complex plane gives a harmonic function on the unit disk. More generally, each eigenfunction of the Laplace-Beltrami operator on the Poincaré disk can be represented by a generalized Poisson integral of a hyperfunction on the unit circle.

The notion of the Poisson integral is generalized to a Riemannian symmetric space $X = G/K$ of the noncompact type, where G is a connected real reductive Lie group and K its maximal compact subgroup. The so called "Helgason conjecture" states that each joint eigenfunction of the invariant differential operators on X has a Poisson integral representation by a hyperfunction on the Furstenberg boundary G/P of X, where P is a minimal parabolic subgroup of G. Helgason proved the conjecture for the Poincaré disk. Kashiwara *et al.* [K–] prove it generally by using the theory of hyperfunctions and the system of differential equations with regular singularities and their boundary value problem due to Kashiwara and Oshima [KO].

The Poisson transform is an intertwining operator from the spherical principal series representation to the eigenspace representation. For generic parameter λ of the principal series representation, the Poisson transform $\mathcal{P}_\lambda$ gives an isomorphism of the representations. The principal series representation is realized on the space of the sections of a homogeneous line bundle over G/P, whose parameter is λ. If $\lambda = \rho$, then the line bundle is trivial and the representation is realized on the space of functions on G/P. Then the image of $\mathcal{P}_\rho$ consists of the harmonic functions, that is the functions which are annihilated by the invariant differential operators on the symmetric space that kill the constant functions. We call this the "harmonic case".

It is natural to pose the problem of characterizing the image of $\mathcal{P}_\lambda$ when the map is not bijective. An interesting case corresponds to the problem of characterizing the image of the Poisson transform from another boundary component of X in one of Satake compactifications of X (cf. [Sa], [O2]). Each boundary component is of the form G/P_Ξ, where P_Ξ is a parabolic subgroup of G. The Furstenberg boundary is the maximal among the boundary components.

For a classical Hermitian symmetric space of tube type, Hua [Hua] studied the Poisson integrals of functions on the Shilov boundary, which are generalization of the classical Poisson integrals on the unit disk. The Poisson integrals are harmonic functions, and moreover, they are annihilated by second order differential operators, which are called the Hua operators. Korányi and Malliavin [KM], and Johnson [J1] showed that the Hua operators characterize the Poisson integrals of hyperfunctions on the Shilov boundary of the Siegel upper half planes. Johnson and Korányi [JK] constructed the Hua operators for a Hermitian symmetric space of tube type in general and proved that they characterize the Poisson integrals of hyperfunctions on the Shilov boundary. The second author [Sn2] generalized the result to non-harmonic cases. In [Sn4], he also constructed a system of differential equations that characterizes the image of the Poisson transform from a certain kind of boundary component of a Hermitian symmetric space.

For a Hermitian symmetric space of non-tube type, Berline and Vergne [BV] defined generalized Hua operators, which are third order differential operators, and proved that these operators with invariant differential operators characterize the Poisson integrals of hyperfunctions on the Shilov boundary, in the harmonic case. Koufany and Zhang [KZ] generalize the result to non-harmonic cases. For $G = U(p, q)$ these authors also showed that second order operators characterize the image of the Poisson transform from the Shilov boundary, even for non-tube cases, that is the case of $p > q$.

Johnson constructed a system of differential equations that characterizes the image of the Poisson transform from each boundary component for $G = SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ in [J2], and for general G in [J3], in the harmonic case.

The first author [O1] proposed a method to study boundary value problems for various boundaries of X. He constructed a system of differential operators corresponding to the boundary $GL(n, \mathbb{R})/P_{n-1,1}$ of $GL(n, \mathbb{R})/O(n)$, where $P_{n-1,1}$ is the maximal parabolic subgroup of $GL(n, \mathbb{R})$ corresponding to the partition $(n - 1, 1)$. To prove that the differential equations indeed characterize the image of the Poisson transform, he used the method of calculating differential equations for boundary values on the Fusrtenberg boundary, which are called "induced equations". All of the above mentioned works on the problem of characterizing the image of the Poisson transform from a boundary component by a system of differential equations use essentially the method of calculating induced equations.

On the other hand, recently the first author ([O4, O5, O6] and [OO] with Oda), studied two sided ideals of a universal enveloping algebra of a complex reductive Lie algebra, which are annihilators of generalized Verma modules, and apply them to boundary value problems for various boundaries of a symmetric space. In this paper, we use two sided ideals constructed explicitly in [O6, OO] to characterize the image of the Poisson transform from a boundary component of a symmetric space, giving several examples including previously known cases. We also study the case of homogeneous line bundle on a Hermitian symmetric space. Since the differential operators come from a two sided ideal of the universal enveloping algebra of the complexification of the Lie algebra of G, the proof that they characterize the image of the Poisson transform is fairly easy. Indeed we do not need to calculate induced equations on the Furstenberg boundary. For the harmonic case, our operators are different from those constructed by Johnson [J2, J3] and more explicit.

This paper is organized as follows. In §2 we review on representations realized on a symmetric space and give basic results on the Poisson transforms on various boundaries.

In §3 we review on minimal polynomials on complex reductive Lie algebras, which give a generator system of the annihilator of a generalized Verma module after [O6, OO] and show that the corresponding differential operators on a Riemannian symmetric space characterize the image of the Poisson transform from a boundary component of the symmetric space.

In §4, we give examples when G is $U(p, q)$, $Sp(n, \mathbb{R})$ or $GL(n, \mathbb{R})$. In particular, for $G = U(p, q)$ or $Sp(n, \mathbb{R})$ and G/P_Ξ the Shilov boundary of X, our operators for the trivial line bundle over $X = G/K$ coincide with the previously known "Hua operators" mentioned above.

2. Representations on symmetric spaces

In this section we review on representations realized on Riemannian symmetric spaces of the noncompact type and their characterizations by differential equations. The statements in this section are known results or at least a reformulation or an easy consequence of known facts (cf. [He2], [He1], [K–], [Ko], [KR], [O1], [Sn1] etc.) and this section can be read without referring to other sections.

Let G be a connected real connected semisimple Lie group, possibly with infinite center. Let K be a maximal compact subgroup of G modulo the center of G, θ be the corresponding Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Fix a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$. Let $\Sigma(\mathfrak{a}_\mathfrak{p})$ be the set of the roots defined by the pair $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ and fix a positive system $\Sigma(\mathfrak{a}_\mathfrak{p})^+$. We denote its Weyl group by $W(\mathfrak{a}_\mathfrak{p})$ and the fundamental system by $\Psi(\mathfrak{a}_\mathfrak{p})$, and the half of the sum of the positive roots counting their multiplicities is denoted by ρ. Let $G = KAN$ be the Iwasawa decomposition of G with $\mathrm{Lie}(A) = \mathfrak{a}_\mathfrak{p}$ and N corresponding to $\Sigma(\mathfrak{a}_\mathfrak{p})^+$. Then $P = MAN$ is a minimal parabolic subgroup of G. Here M is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in K. We denote by $\mathfrak{k}$, $\mathfrak{m}$ and $\mathfrak{n}$ the Lie algebras of K, M and N, respectively.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$, which we identify with the algebra of left invariant differential operators on G. In general, for a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we denote by $U(\mathfrak{l})$ the universal enveloping algebra of the complexification $\mathfrak{l}_\mathbb{C}$ of $\mathfrak{l}$. Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}_\mathbb{C}$. Then the map sym of symmetrization of $S(\mathfrak{g})$ to $U(\mathfrak{g})$ defines a K-linear bijection. By the Killing form on $\mathfrak{g}_\mathbb{C}$ we identify the space $\mathcal{O}(\mathfrak{p}_\mathbb{C})$ of polynomial functions on the complexification $\mathfrak{p}_\mathbb{C}$ of $\mathfrak{p}$ with the symmetric algebra of $\mathfrak{p}_\mathbb{C}$. Let $\mathcal{O}(\mathfrak{p})^K$ be the space of all K-invariant polynomials on $\mathfrak{p}_\mathbb{C}$ and $\mathcal{H}$ be the space of all K-harmonic polynomials on $\mathfrak{p}_\mathbb{C}$. Then we have the following K-linear bijection

$$\begin{aligned} \mathcal{H} \otimes \mathcal{O}(\mathfrak{p})^K \otimes U(\mathfrak{k}) &\xrightarrow{\sim} U(\mathfrak{g}) \\ h \otimes p \otimes k &\mapsto \mathrm{sym}(h) \otimes \mathrm{sym}(p) \otimes k \end{aligned} \tag{2.1}$$

because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and the K-linear bijection

$$\mathcal{H} \otimes \mathcal{O}(\mathfrak{p})^K \ni h \otimes p \mapsto hp \in \mathcal{O}(\mathfrak{p})$$

studied by [KR].

We denote by $\mathcal{A}(G)$ and $\mathcal{B}(G)$ the space of real analytic functions and that of hyperfunctions on G, respectively. Then they are left G-modules by $(\pi_g f)(x) = f(g^{-1}x)$ for $g \in G$ and the functions f in the spaces. We write $\mathcal{A}(G)_K$ for the space of all the K-finite elements of $\mathcal{A}(G)$.

By the decomposition

$$U(\mathfrak{g}) = \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}_\mathfrak{p}) \oplus U(\mathfrak{a}_\mathfrak{p}) \oplus U(\mathfrak{g})\mathfrak{k} \tag{2.2}$$

coming from the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$, we define $D_{\mathfrak{a}_\mathfrak{p}} \in U(\mathfrak{a}_\mathfrak{p})$ for $D \in U(\mathfrak{g})$ so that $D - D_{\mathfrak{a}_\mathfrak{p}} \in \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}_\mathfrak{p}) + U(\mathfrak{g})\mathfrak{k}$ and put $\gamma(D) = e^{-\rho} \circ D_{\mathfrak{a}_\mathfrak{p}} \circ e^{\rho}$. Here e^ρ is the function on A defined by $e^\rho(a) = a^\rho$.

Note that the kernel of the restriction of γ to the space of all the K-invariants $U(\mathfrak{g})^K$ of $U(\mathfrak{g})$ equals $U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$ and the restriction defines the Harish-Chandra isomorphism

$$\bar{\gamma} : \mathbb{D}(G/K) \simeq U(\mathfrak{g})^K / (U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}) \to U(\mathfrak{a}_\mathfrak{p})^W \tag{2.3}$$

onto the space $U(\mathfrak{a}_\mathfrak{p})^W$ of all the $W(\mathfrak{a}_\mathfrak{p})$-invariants in $U(\mathfrak{a}_\mathfrak{p})$. Here $\mathbb{D}(G/K)$ is the algebra of invariant differential operators on G/K. Note also that $\mathrm{sym}(O(\mathfrak{p})^K)$ is isomorphic to $\mathbb{D}(G/K)$ through (2.2) and (2.3) as K-modules.

Identifying $U(\mathfrak{a}_\mathfrak{p})$ with the space of polynomial functions on the complex dual $\mathfrak{a}_\mathbb{C}^*$ of $\mathfrak{a}_\mathfrak{p}$, we put

$$\gamma_\lambda(D) = \gamma(D)(\lambda) \in \mathbb{C} \tag{2.4}$$

for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $D \in U(\mathfrak{g})$. Now we define

$$J_\lambda = U(\mathfrak{g})\mathfrak{k} + \sum_{p\in\mathcal{O}(\mathfrak{p})^K} U(\mathfrak{g})(\mathrm{sym}(p) - \gamma_\lambda(\mathrm{sym}(p))) \tag{2.5}$$

and

$$\mathcal{A}(G/K;\mathcal{M}_\lambda) = \{u \in \mathcal{A}(G); Du = 0 \quad \text{for } D \in J_\lambda\} \tag{2.6}$$

and put $\mathcal{A}(G/K;\mathcal{M}_\lambda)_K = \mathcal{A}(G)_K \cap \mathcal{A}(G/K;\mathcal{M}_\lambda)$. Here $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ is naturally a subspace of the space $\mathcal{A}(G/K)$ of real analytic functions on G/K because the function in $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ is right K-invariant. Now we can state our basic theorem.

Theorem 2.1. *Fix $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and define a bilinear form*

$$\begin{aligned} \mathcal{H}\otimes\mathcal{A}(G/K;\mathcal{M}_\lambda) &\to \mathbb{C} \\ (h,u) &\mapsto \langle h,u\rangle = (\mathrm{sym}(h)u)(e) \end{aligned} \tag{2.7}$$

and for a subspace V of $\mathcal{A}(G/K;\mathcal{M}_\lambda)$, put

$$H(V) = \{h \in \mathcal{H}; \langle h,u\rangle = 0 \quad \textit{for } u \in V\}. \tag{2.8}$$

i) *The bilinear form $\langle\ ,\ \rangle$ is K-invariant and non-degenerate.*
ii) *If V is a subspace of $\mathcal{A}(G/K;\mathcal{M}_\lambda)_K$, then*

$$V = \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda)_K; \langle h,u\rangle = 0 \quad \textit{for } h \in H(V)\}.$$

iii) *There are natural bijections between the following sets of modules.*

$$\begin{aligned} \mathfrak{V}(\lambda) &= \{V \subset \mathcal{A}(G/K;\mathcal{M}_\lambda); V \textit{ is a close subspace of } C^\infty(G) \textit{ and } G\textit{-invariant}\}, \\ \mathfrak{V}(\lambda)_K &= \{V_K \subset \mathcal{A}(G/K;\mathcal{M}_\lambda)_K; V_K \textit{ is a } \mathfrak{g}\textit{-invariant subspace}\}, \\ \mathfrak{J}(\lambda) &= \{J \supset J_\lambda; J \textit{ is a left ideal of } U(\mathfrak{g})\}. \end{aligned}$$

Here the bijections are given by

$$\mathfrak{V}(\lambda) \ni V \mapsto V \cap \mathcal{A}(G)_K \in \mathfrak{V}(\lambda)_K, \tag{2.9}$$

$$\mathfrak{V}(\lambda)_K \ni V_K \mapsto J_\lambda + \sum_{p\in H(V_K)} U(\mathfrak{g})\,\mathrm{sym}(p) \in \mathfrak{J}(\lambda) \tag{2.10}$$

$$\mathfrak{J}(\lambda) \ni J \mapsto \{u \in \mathcal{A}(G); Du = 0 \textit{ for } D \in J\} \in \mathfrak{V}(\lambda) \tag{2.11}$$

Before the proof of this theorem we review on the Poisson transform. The G-module

$$\begin{aligned} \mathcal{B}(G/P;\mathcal{L}_\lambda) = \{f \in \mathcal{B}(G)\,;\, & f(gman) = a^{\lambda-\rho}f(g) \\ & \text{for } (g,m,a,n) \in G\times M\times A\times N\} \end{aligned} \tag{2.12}$$

is the space of hyperfunction sections of spherical principal series of G parametrized by $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Put $\mathcal{A}(G/P;\mathcal{L}_\lambda) = \mathcal{B}(G/P;\mathcal{L}_\lambda) \cap \mathcal{A}(G)$. Define the K-fixed element $K\times A\times N \ni (k,a,n) \mapsto \mathbf{1}_\lambda(kan) = a^{\lambda-\rho}$ of $\mathcal{A}(G/P;\mathcal{L}_\lambda)$ and put $P_\lambda(g) = \mathbf{1}_{-\lambda}(g^{-1})$. By the G-invariant bilinear form

$$\begin{aligned} \mathcal{B}(G/P;\mathcal{L}_\lambda)\times\mathcal{A}(G/P;\mathcal{L}_{-\lambda}) &\to \mathbb{C} \\ (\phi,f) &\mapsto \langle\phi,f\rangle_\lambda = \int_K \phi(k)f(k)dk \end{aligned} \tag{2.13}$$

with the normalized Haar measure dk on K, we define the Poisson transform

$$\begin{aligned} \mathcal{P}_\lambda : \mathcal{B}(G/P;\mathcal{L}_\lambda) &\to \mathcal{B}(G) \\ \phi &\mapsto \mathcal{P}_\lambda\phi(g) = \langle \pi_{g^{-1}}\phi, \mathbf{1}_{-\lambda}\rangle_\lambda = \int_K \phi(gk)dk \\ &= \int_K \phi(k)P_\lambda(k^{-1}g)dk. \end{aligned} \tag{2.14}$$

Then it is known that the image of $\mathcal{P}_\lambda$ is contained in $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ because $DP_\lambda = \gamma_\lambda(D)P_\lambda$ for $D \in U(\mathfrak{g})^K$. (If the center Z of G is infinite, integrations over K in the definitions of pairing $\langle\,\cdot\,,\cdot\,\rangle_\lambda$ and the Poisson transform should be understood to be normalized integral over K/Z. But we write K for simplicity.)

For $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})$ and $w \in W(\mathfrak{a}_\mathfrak{p})$, we put

$$\begin{aligned} \Sigma(\mathfrak{a}_\mathfrak{p})^+_o &= \{\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})^+;\, \tfrac{\alpha}{2} \notin \Sigma(\mathfrak{a}_\mathfrak{p})\}, \\ e_\alpha(\lambda) &= \left\{\Gamma\left(\tfrac{\lambda_\alpha}{4} + \tfrac{m_\alpha}{4} + \tfrac{1}{2}\right)\Gamma\left(\tfrac{\lambda_\alpha}{4} + \tfrac{m_\alpha}{4} + \tfrac{m_{2\alpha}}{2}\right)\right\}^{-1}, \\ e(\lambda) &= \prod_{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+_o} e_\alpha(\lambda), \\ c(\lambda) &= Ce(\lambda)\prod_{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+_o} 2^{-\frac{\lambda_\alpha}{2}}\Gamma\left(\tfrac{\lambda_\alpha}{2}\right), \end{aligned} \tag{2.15}$$

where m_α is the multiplicity of the root α, $\lambda_\alpha = 2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ and C is a constant determined by $c(\rho) = 1$.

The following theorem is the main result in [K–].

Theorem 2.2. *Let* $\lambda \in \mathfrak{a}^*_\mathbb{C}$.

i) $\mathcal{P}_\lambda$ *gives a topological* G*-isomorphism of* $\mathcal{A}(G/P;\mathcal{L}_\lambda)$ *onto* $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ *if and only if* $e(\lambda) \neq 0$.

ii) *Let* w *be an element of* $W(\mathfrak{a}_\mathfrak{p})$ *which satisfies*

$$\operatorname{Re}\langle w\lambda,\alpha\rangle \geq 0 \quad \textit{for all } \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})^+, \tag{2.16}$$

then $\mathcal{P}_{w\lambda}$ *gives a topological* G*-isomorphism*

$$\mathcal{P}_{w\lambda} : \mathcal{B}(G/P;\mathcal{L}_{w\lambda}) \xrightarrow{\sim} \mathcal{A}(G/K;\mathcal{M}_\lambda). \tag{2.17}$$

Remark 2.3. i) The equivalence of the injectivity of $\mathcal{P}_\lambda$ and the condition $e(\lambda) \neq 0$ is proved in [He3].

ii) Suppose $e(\lambda) \neq 0$. Let $\mathcal{D}'(G)$ and $C^\infty(G)$ denote the space of distributions and that of C^∞-functions on G, respectively. Then

$$\begin{aligned} &\mathcal{P}_\lambda(\mathcal{B}(G/P;\mathcal{L}_\lambda) \cap \mathcal{D}'(G)) \\ &\quad = \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda)\,;\text{ there exist } C \text{ and } k \text{ with } |u(g)| \leq C\exp k|g|\}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\mathcal{P}_\lambda(\mathcal{B}(G/P;\mathcal{L}_\lambda) \cap C^\infty(G)) \\ &\quad = \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda)\,;\text{ there exist } k \text{ such that for any } D \in U(\mathfrak{k}) \\ &\qquad\quad \text{we can choose } C_D > 0 \text{ with } |\pi_D u(g)| \leq C_D \exp k|g|\}. \end{aligned} \tag{2.19}$$

Here $U(\mathfrak{k})$ is the universal enveloping algebra of the complexification of $\mathfrak{k}$ and $|g| = \langle H,H\rangle^{\frac{1}{2}}$ with the Killing form $\langle\ ,\ \rangle$ if $g \in K\exp HK$ with $H \in \mathfrak{a}_\mathfrak{p}$. Note that $U(\mathfrak{k})$ in (2.19) may be replaced by $U(\mathfrak{g})$.

In fact, (2.18) is given in [OS1, Corollary 5.5]. Suppose $u = \mathcal{P}_\lambda f$. Since P_λ is contained in the set (2.18), the $U(\mathfrak{g})$-equivariance and the last expression in (2.14) implies that the left hand side of (2.19) is contained in the right hand side of (2.19). Note that the inverse of $\mathcal{P}_\lambda$ is the map of taking boundary values. We can see from the definition that the order of distribution of the boundary value f of u

is estimated by k in (2.18) (cf. [OS1, the proof of Lemma 2.19] or [O3]). If u is contained in the right hand side of (2.19), the order of $\pi_D f$ is uniformly bounded for all $D \in U(\mathfrak{k})$ and hence $f|_K \in C^\infty(K)$. A different proof can be found in [BS].

Proof of Theorem 2.1. Let $X \in \mathfrak{g}$, $k \in K$ and $u \in \mathcal{A}(G/K)$. Then

$$\begin{aligned}(X\pi_k u)(e) &= \tfrac{d}{dt}u\big(k^{-1}\exp tX\big)\big|_{t=0} = \tfrac{d}{dt}u\big((\exp t\,\mathrm{Ad}(k^{-1})X)k^{-1}\big)\big|_{t=0}\\ &= (\mathrm{Ad}(k)^{-1}Xu)(e)\end{aligned}$$

and therefore the bilinear from $\langle\ ,\ \rangle$ in Theorem 2.1 is K-invariant.

Let $\hat{K}$ be the set of equivalence classes of irreducible representations of K. For δ, $\tau \in \hat{K}$ we denote by $\mathcal{A}(G/K;\mathcal{M}_\lambda)_\delta$ and $\mathcal{H}_\tau$ the δ isotopic components of $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ and τ isotopic components of $\mathcal{H}$, respectively. In general, for a K-module U we denote by U_δ the subspace of K-isotopic components δ. Then the K-equivariant map

$$\mathcal{A}(G/K;\mathcal{M}_\lambda)_\delta \ni u \mapsto (\mathcal{H}_\tau \ni h \mapsto \langle h,u\rangle \in \mathbb{C}) \in \mathcal{H}_{\tau^*}$$

is identically zero if $\delta \neq \tau^*$ by Schur's lemma, where τ^* is the dual of τ.

Suppose $u \in \mathcal{A}(G/K;\mathcal{M}_\lambda)_\delta$ satisfies $\langle h,u\rangle = 0$ for any $h \in \mathcal{H}_{\delta^*}$. Then $\langle h,u\rangle = 0$ for any $h \in \mathcal{H}$ and therefore it follows from (2.1), (2.5) and (2.6) that $(Du)(e) = 0$ for all $D \in U(\mathfrak{g})$. Hence $u = 0$ because u is real analytic. On the other hand, since $\mathcal{H}$ and $\mathcal{A}(G/K;\mathcal{M}_\lambda)_K$ are isomorphic to $\mathrm{Ind}_M^K\mathbf{1}$ (cf. [KR] and Theorem 2.2) $\dim\mathcal{A}(G/K;\mathcal{M}_\lambda)_\delta = \dim\mathcal{H}_{\delta^*}$ and hence we can conclude that $\langle\ ,\ \rangle$ defines a non-degenerate bilinear form on $\mathcal{A}(G/K;\mathcal{M}_\lambda)_\delta \times \mathcal{H}_{\delta^*}$ and we have i) and ii). Here we remark that the results follows from the weaker relation $\dim\mathcal{A}(G/K;\mathcal{M}_\lambda)_\delta \geq \dim\mathcal{H}_{\delta^*}$.

First note that the map (2.9) is a bijection of $\mathfrak{V}(\lambda)$ onto $\mathfrak{V}(\lambda)_K$ whose inverse is the map of taking the closure in $C^\infty(G)$. The map is still bijective even if it is restricted to the spaces killed by a left ideal J of $U(\mathfrak{g})$. Moreover remark that for $X \in \mathfrak{g}$, $D \in U(\mathfrak{g})$ and $u \in \mathcal{A}(G/K)$ we have $(D\pi_X u)(e) = -(XDu)(e)$.

Let $V_K \in \mathfrak{V}(\lambda)_K$. Then $(D\,\mathrm{sym}(h)u)(e) = 0$ for $D \in U(\mathfrak{g})$, $h \in H(V_K)$ and $u \in V_K$ because of the above remark. Note that for a left ideal J of $U(\mathfrak{g})$ and a function u in $\mathcal{A}(G)$, the condition $Du = 0$ for all $D \in J$ is equivalent to the condition $(Du)(e) = 0$ for all $D \in J$. Hence we have

$$\begin{aligned}V_K &= \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda)_K; \langle h,u\rangle = 0 \quad \text{for } h \in H(V_K)\}\\ &= \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda)_K; (D\,\mathrm{sym}(h)u)(e) = 0 \quad \text{for } h \in H(V_K) \text{ and } D \in U(\mathfrak{g})\}\\ &= \{u \in \mathcal{A}(G)_K; Du = 0 \quad \text{for } D \in J_\lambda + \sum_{h\in H(V_K)} U(\mathfrak{g})\,\mathrm{sym}(h)\}.\end{aligned}$$

Let J be a left ideal of $U(\mathfrak{g})$ containing J_λ. Then (2.1) and (2.5) show that

$$\begin{aligned}J &= J_\lambda \oplus \{\mathrm{sym}(h); h \in \mathcal{H}_J\} \quad \text{with}\\ \mathcal{H}_J &= \mathrm{sym}^{-1}(J) \cap \mathcal{H}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}&\{u \in \mathcal{A}(G)_K\,; Du = 0 \quad \text{for} \quad D \in J\}\\ &\qquad= \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda)_K; \mathrm{sym}(h)u = 0 \quad \text{for} \quad h \in \mathcal{H}_J\}.\end{aligned}$$

Hence the map of $\mathfrak{J}(\lambda)$ to $\mathfrak{V}(\lambda)_K$ is injective and we have the Theorem 2.1. □

Theorem 2.4. i) *The map*

$$\mathcal{H} \ni h \ \mapsto \pi_{\mathrm{sym}(h)}\mathbf{1}_\lambda \in \mathcal{A}(G/P;\mathcal{L}_\lambda)_K \tag{2.21}$$

is K-linear. It is bijective if and only if $e(-\lambda) \neq 0$.

ii)

$$\gamma_\lambda(D) = \pi_D(\mathbf{1}_{-\lambda})(e) \quad \textit{for} \;\; D \in U(\mathfrak{g}). \tag{2.22}$$

iii) *Putting*

$$\mathcal{H}_\lambda = \{h \in \mathcal{H}; \gamma_\lambda(\mathrm{sym}(\mathrm{Ad}(k)h)) = 0 \quad \textit{for all } k \in K\} \tag{2.23}$$

and

$$\bar{J}_\lambda = J_\lambda + \sum_{h \in \mathcal{H}_\lambda} U(\mathfrak{g})\,\mathrm{sym}(h), \tag{2.24}$$

we have

$$\mathrm{Im}\,\mathcal{P}_\lambda = \{u \in \mathcal{A}(G);\; Du = 0 \quad \textit{for } D \in \bar{J}_\lambda\}. \tag{2.25}$$

Proof. Since $\mathbf{1}_\lambda$ is K-invariant, the map (2.21) is K-equivariant. Moreover for $h \in \mathcal{H}$, the condition $\pi_{\mathrm{sym}(h)}\mathbf{1}_\lambda = 0$ is equivalent to $(\pi_{\mathrm{sym}(\mathrm{Ad}(k)h)}\mathbf{1}_\lambda)(e) = 0$ for $k \in K$ because $(\pi_{\mathrm{sym}(h)}\mathbf{1}_\lambda)(kan) = (\pi_{\mathrm{sym}(h)}\mathbf{1}_\lambda)(k)a^{\rho-\lambda}$. On the other hand, (2.22) follows from the definition of γ_λ and $\mathbf{1}_{-\lambda}$.

Let $h \in \mathcal{H}_\lambda$. Then $\pi_{\mathrm{sym}(h)}\mathbf{1}_{-\lambda} = 0$ and therefore $\mathrm{sym}(h)P_\lambda = 0$ and hence it is clear from (2.14) that $\mathrm{Im}\,\mathcal{P}_\lambda \subset \{u \in \mathcal{A}(G); Du = 0 \quad \text{for} \;\; D \in \bar{J}_\lambda\}$.

Since $\pi_D\mathbf{1}_\lambda \in \mathbb{C}\mathbf{1}_\lambda$ for $D \in U(\mathfrak{g})^K$, (2.1) shows

$$U(\mathfrak{g})\mathbf{1}_\lambda = \{\pi_{\mathrm{sym}(h)}\mathbf{1}_\lambda; h \in \mathcal{H}\},$$

which is the Harish-Chandra module of the minimal closed G-invariant subspace of $\mathcal{A}(G/P;\mathcal{L}_\lambda)$ containing $\mathbf{1}_\lambda$. For $\phi \in \mathcal{A}(G/P;\mathcal{L}_\lambda)_\delta$, we have $\mathcal{P}_\lambda\phi(g) = \langle \phi, \pi_g\mathbf{1}_{-\lambda}\rangle_\lambda$ and therefore the condition $\mathcal{P}_\lambda\phi = 0$ is equivalent to $\langle \phi, \pi_{\mathrm{sym}(h)}\mathbf{1}_\lambda\rangle_\lambda = 0$ for all $h \in \mathcal{H}_{\delta^*}$. Hence $[\mathrm{Ker}\,\mathcal{P}_\lambda : \delta] = [\mathcal{H}_\lambda : \delta^*]$ and Theorem 2.2 shows $[\mathrm{Im}\,\mathcal{P}_\lambda : \delta] = [\mathcal{A}(G/K;\mathcal{M}_\lambda) : \delta] - [\mathcal{H}_\lambda : \delta^*]$, which means $\dim(\mathrm{Im}\,\mathcal{P}_\lambda)_\delta = \dim\{u \in \mathcal{A}(G);\; Du = 0 \quad \text{for } D \in \bar{J}_\lambda\}_\delta$ and furthermore (2.25) owing to Theorem 2.1 and Remark 2.3 i).

If $e(-\lambda) \neq 0$, the bijectivity of (2.21) follows from Theorem 2.2 because $\mathcal{H}_\lambda = \{0\}$ by the argument above. But it follows directly from the result in [Ko] (cf. [He3]) that $\mathbf{1}_\lambda$ is cyclic in $\mathcal{A}(G/K;\mathcal{L}_\lambda)$ if and only if $e(-\lambda) \neq 0$. □

Remark 2.5 (cf. [Du]). When $\mathfrak{g}$ is a complex semisimple Lie algebra, J_λ is identified with the annihilator of the Verma module of $\mathfrak{g}$ parametrized by λ and the Poisson transform gives a bijection between the two sided ideals of the universal enveloping algebra of $\mathfrak{g}$ containing J_λ which are identified with $\mathfrak{J}(\lambda)$ and the closed G-invariant subspaces of class 1 principal series of G parametrized by λ.

For a subset Ξ of $\Psi(\mathfrak{a}_\mathfrak{p})$, let W_Ξ be a subgroup of $W(\mathfrak{a}_\mathfrak{p})$ generated by reflections with respect to the elements of Ξ and put $P_\Xi = PW_\Xi P$. Let $P_\Xi = M_\Xi A_\Xi N_\Xi$ be the Langlands decomposition of P_Ξ with $A_\Xi \subset A$. For an element μ of the complex dual $\mathfrak{a}^*_{\Xi,\mathbb{C}}$ of the Lie algebra $\mathfrak{a}_\Xi$ of A_Ξ, the space of hyperfunction sections of spherical degenerate series is defined by

$$\begin{aligned}\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu}) = \{f \in \mathcal{B}(G); f(gman) = a^{\mu-\rho}f(g) \\ \text{for } (g,m,a,n) \in G \times M_\Xi \times A_\Xi \times N_\Xi\}.\end{aligned} \tag{2.26}$$

Then as in the case of the minimal parabolic subgroup, we can define Poisson transform

$$\begin{aligned}\mathcal{P}_{\Xi,\mu} : \mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu}) &\to \mathcal{B}(G) \\ \phi &\mapsto (\mathcal{P}_{\Xi,\mu}\phi)(g) = \langle \pi_{g^{-1}}\phi, \mathbf{1}_{\Xi,-\mu}\rangle_{\Xi,\mu} \\ &\qquad = \int_K \phi(gk)dk = \int_K \phi(k)P_{\Xi,\mu}(k^{-1}g)dk.\end{aligned} \tag{2.27}$$

Here $\langle\ ,\ \rangle_{\Xi,\mu}$ is the bilinear form of $\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu})\times\mathcal{A}(G/P_\Xi;\mathcal{L}_{\Xi,-\mu})$ defined on the integral over K, $\mathbf{1}_{\Xi,\mu}(kman)=a^{\mu-\rho}$ for $(k,m,a,n)\in K\times M_\Xi\times A_\Xi\times N_\Xi$ and $P_{\Xi,\mu}(g)=\mathbf{1}_{\Xi,-\mu}(g^{-1})$.

Now we remark

Lemma 2.6. *We have naturally*

$$\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu})\subset\mathcal{B}(G/P;\mathcal{L}_{\mu+\rho(\Xi)}),\tag{2.28}$$

$$\operatorname{Im}\mathcal{P}_{\Xi,\mu}=\operatorname{Im}\mathcal{P}_{\mu-\rho(\Xi)}.\tag{2.29}$$

Here we identify $\mathfrak{a}_\mathfrak{p}$ with its dual by the Killing form and regard $\mu\in\mathfrak{a}^*_{\Xi,\mathbb{C}}$ as an element of $\mathfrak{a}^*_{\mathbb{C}}$ with value zero on $\mathfrak{a}_\Xi^\perp$, and define $\rho_\Xi=\rho|_{\mathfrak{a}_\Xi}$ and $\rho(\Xi)=\rho-\rho_\Xi$.

Proof. The inclusion (2.28) is clear from (2.12) and (2.26), which implies $\mathbf{1}_{\Xi,-\mu}=\mathbf{1}_{-(\mu-\rho(\Xi))}$ because $\mathcal{A}(G/P_\Xi,\mathcal{L}_{\Xi,-\mu})\subset\mathcal{A}(G/P;\mathcal{L}_{-\mu+\rho(\Xi)})$. Since $P_{\Xi,\mu}$ is left M_Ξ-invariant and

$$\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu})|_K=\mathcal{B}(K/M_\Xi\cap K)\text{ and }\mathcal{B}(G/P;\mathcal{L}_{\mu-\rho(\Xi)})|_K=\mathcal{B}(K/M),$$

we have (2.29) from (2.14) and (2.17). □

Corollary 2.7. i) *$\mathcal{P}_{\Xi,\mu}$ is injective if $e(\mu+\rho(\Xi))\neq 0$. In particular, the Poisson transform $\mathcal{P}_{\rho_\Xi}:\mathcal{B}(G/P_\Xi)\to\mathcal{A}(G/K;\mathcal{M}_\rho)$ is injective.*

ii) *If $e(-\mu+\rho(\Xi))e(\mu+\rho(\Xi))\neq 0$, then $\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu})$ is irreducible.*

Proof. The claim i) is a direct consequence of Theorem 2.2 i) and Lemma 2.6.

The K-invariant bilinear form $\langle\ ,\ \rangle_{\Xi,\mu}$ and (2.27) show that the following statements are equivalent:

(2.30) $\mathcal{P}_{\Xi,\mu}$ is injective.

(2.31) $\mathbf{1}_{\Xi,-\mu}$ is cyclic in $\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,-\mu})$.

(2.32) Any non-zero closed G-invariant subspace of $\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu})$ contains $\mathbf{1}_{\Xi,\mu}$.

Hence ii) is clear. □

Remark 2.8. i) The calculation of $\mathcal{H}_\lambda$ in (2.23) is equivalent to the determination of the kernel of $P^\gamma(\lambda)$ defined by [Ko, §4].

ii) Under the notation in Lemma 2.6

$$[\mathcal{H}_{\mu+\rho(\Xi)}:\delta]\geq[\operatorname{Ind}_M^K\mathbf{1}:\delta]-[\operatorname{Ind}_{M_\Xi(K)}^K\mathbf{1}:\delta]\quad\text{for }\delta\in\hat{K}\tag{2.33}$$

and the equality holds if and only if $\mathcal{P}_{\Xi,\mu}$ is injective.

Most of statements in this section can be generalized to line bundles or vector bundles over G/K. We will give necessary modifications when we consider homogeneous line bundles over a Hermitian symmetric space G/K. For simplicity we suppose G is simple and $\mathfrak{k}$ have a non-trivial center. Let K' be the analytic subgroup of G with the Lie algebra $\mathfrak{k}'=[\mathfrak{k},\mathfrak{k}]$ and Y be the central element of $\mathfrak{k}$ normalized so that $\exp tY\in K'_R$ if and only if $\ell\in\mathbb{Z}$, where K' is denoted by K'_R when G is a real form of a simply connected complex Lie group (cf. [Sn1]). Let $\chi_\ell:K\to\mathbb{C}$ be the one dimensional representation of K defined by $\chi_\ell(k)=1$ if $k\in K'$ and $\chi_\ell(\exp tY)=\exp\sqrt{-1}\ell t$. Then we can define the space of real analytic sections of a homogeneous line bundle $\mathbf{E}_\ell$ over G/K associated to the representation χ_ℓ of K:

$$\mathcal{A}(G/K;\mathbf{E}_\ell)=\{u\in\mathcal{A}(G);u(gk)=\chi_\ell(k)^{-1}u(g)\text{ for }k\in K\}.\tag{2.34}$$

Let $\mathbb{D}(\mathbf{E}_\ell)$ be the algebra of invariant differential operators acting on sections of $\mathbf{E}_\ell$. Defining $\gamma_\ell(D) \in U(\mathfrak{a})$ for $D \in U(\mathfrak{a})$ so that

$$e^\rho \circ \gamma_\ell(D) \circ e^{-\rho} \in \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}_\mathfrak{p}) \oplus \textstyle\sum_{X \in \mathfrak{k}} U(\mathfrak{g})(X + \chi_\ell(X)), \tag{2.35}$$

as in the case of γ, we have the Harish-Chandra isomorphism

$$\bar{\gamma}_\ell : \mathbb{D}(\mathbf{E}_\ell) \simeq U(\mathfrak{g})^K / (U(\mathfrak{g})^K \cap \textstyle\sum_{X \in \mathfrak{k}} U(\mathfrak{g})(X + \chi_\ell(X))) \to U(\mathfrak{a}_\mathfrak{p})^W \tag{2.36}$$

onto $U(\mathfrak{a}_\mathfrak{p})^W$. Hence we put

$$J^\ell_\lambda = \textstyle\sum_{X \in \mathfrak{k}} U(\mathfrak{g})(X + \chi_\ell(X)) + \sum_{p \in \mathcal{O}(\mathfrak{p})^K} U(\mathfrak{g})(\mathrm{sym}(p) - \gamma^\ell_\lambda(\mathrm{sym}(p))) \tag{2.37}$$

and

$$\mathcal{A}(G/K; \mathcal{M}^\ell_\lambda) = \{u \in \mathcal{A}(G); Du = 0 \quad \text{for } D \in J^\ell_\lambda\}. \tag{2.38}$$

Theorem 2.9. *Replacing* $\mathcal{M}_\lambda$ *by* $\mathcal{M}^\ell_\lambda$*, the statements* i), ii) *and* iii) *in* Theorem 2.1 *are valid if the K-invariantness of* $\langle\ ,\ \rangle$ *is modified by*

$$\langle \mathrm{Ad}(k)h, \chi_{-\ell}(k)\pi_k u\rangle = \langle h, p\rangle \quad \text{for } k \in K, h \in \mathcal{H} \text{ and } u \in \mathcal{A}(G/K; \mathcal{M}^\ell_\lambda). \tag{2.39}$$

Proof. Recalling the proof of Theorem 2.1, we have only to consider

$$\mathcal{A}(G/K; \mathcal{M}^\ell_\lambda)_{\delta \otimes \chi_\ell} \times \mathcal{H}_{\delta^*} \ni (u, p) \mapsto \langle p, u\rangle \in \mathbb{C} \tag{2.40}$$

because of the K-invariance (2.39). Hence if we have $\dim A(G/K; \mathcal{M}^\ell_\lambda)_{\delta\otimes\chi_\ell} \geq \dim \mathcal{H}_{\delta^*}$, the same argument as in the proof of Theorem 2.1 shows Theorem 2.9.

On the other hand, the proof of [Sn1, Lemma 8.6] says that

$$[A(G/K; \mathcal{M}^\ell_\lambda), \delta \otimes \chi_\ell] \geq [\mathrm{Ind}^K_M \chi_\ell|_M, \delta \otimes \chi_\ell].$$

Tensoring $\chi_{-\ell}$ to the right hand side, we have $[\mathrm{Ind}^K_M \chi_\ell|_M, \delta\otimes\chi_\ell] = [\mathrm{Ind}^K_M \mathbf{1}, \delta]$, which is also equal to $[\mathcal{H}, \delta^*]$. □

Put $\{|\alpha|; \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})\} = \{c_1, \ldots, c_N\}$ with $c_1 > \cdots > c_N$. Then $N = 1$ or 2 or 3 and we fix $\beta_\nu \in \Sigma(\mathfrak{a}_\mathfrak{p})^+$ with $|\beta_\nu| = c_\nu$. Moreover we put

$$\begin{aligned}\mathcal{B}(G/P; \mathcal{L}^\ell_\lambda) = \{f \in \mathcal{B}(G); f(gman) = \chi_\ell(m)^{-1} a^{\lambda-\rho} f(g) \\ \text{for } (g, m, a, n) \in G \times M \times A \times N\}\end{aligned} \tag{2.41}$$

and

$$\begin{aligned} e_\alpha(\lambda, \ell) &= \left\{\Gamma\left(\tfrac{\lambda_\alpha}{2} + \tfrac{m_{\frac{\alpha}{2}}}{4} + \tfrac{1+\ell}{2}\right)\Gamma\left(\tfrac{\lambda_\alpha}{2} + \tfrac{m_{\frac{\alpha}{2}}}{4} + \tfrac{1-\ell}{2}\right)\right\}^{-1},\\ e(\lambda, \ell) &= \prod_{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+,\ |\alpha|=|\beta_1|} e_\alpha(\lambda, \ell) \times \prod_{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+,\ |\alpha|=|\beta_2|} e_\alpha(\lambda),\\ c(\lambda, \ell) &= Ce(\lambda, \ell) \prod_{k=1}^{2} \prod_{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+,\ |\alpha|=|\beta_k|} 2^{-\frac{\lambda_\alpha}{k}}\Gamma\left(\tfrac{\lambda_\alpha}{k}\right),\end{aligned} \tag{2.42}$$

where $C \in \mathbb{R}$ is determined by $c(\rho, 0) = 1$, $e_\alpha(\lambda)$ is given in (2.15) and $m_{\frac{\alpha}{2}}$ may be 0. Then the main result in [Sn1] says

Theorem 2.10. i) *The Poisson transform*

$$\mathcal{P}^\ell_\lambda : \mathcal{B}(G/P; \mathcal{L}^\ell_\lambda) \to \mathcal{B}(G)$$

$$\phi \mapsto (\mathcal{P}^\ell_\lambda \phi)(g) = \int_K \phi(gk)\chi_\ell(k)dk = \int_K \phi(k)\mathbf{1}_{-\lambda,-\ell}(g^{-1}k)dk$$

is a G-homomorphism and $\mathrm{Im}\,\mathcal{P}^\ell_\lambda \subset \mathcal{A}(G/K; \mathcal{M}^\ell_\lambda)$*. Here the function* $\mathbf{1}_{-\lambda,-\ell} \in \mathcal{A}(G/P; \mathcal{L}^{-\ell}_{-\lambda})$ *is defined by* $\mathbf{1}_{-\lambda,-\ell}(kan) = \chi_\ell(k)a^{-\lambda-\rho}$ *for* $(k, a, n) \in K \times A \times N$.

ii) *If*

$$-2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\notin\{1,2,3,\dots\}\quad \text{for } \alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}})^+ \tag{2.43}$$

and $e(\lambda,\ell)\neq 0$, *then* $\mathcal{P}^\ell_\lambda$ *is a topological* G*-isomorphism of* $\mathcal{B}(G/P;\mathcal{L}^\ell_\lambda)$ *onto* $\mathcal{A}(G/K;\mathcal{M}^\ell_\lambda)$.

iii) *Suppose* $\operatorname{Re}\langle\lambda,\alpha\rangle>0$ *for* $\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}})^+$. *Then the following statements are equivalent:*

$$c(\lambda,\ell)\neq 0. \tag{2.44}$$

$$\mathcal{P}^\ell_\lambda \ \textit{is injective.} \tag{2.45}$$

$$\operatorname{Im}\mathcal{P}^\ell_\lambda=\mathcal{A}(G/K;\mathcal{M}^\ell_\lambda). \tag{2.46}$$

Notice that Theorem 2.2 (ii) does not hold in the case of non-trivial line bundles.

Now we consider the degenerate series. Let $M^o_{\Xi,s}$ denote the semisimple part of M_Ξ, namely $M^o_{\Xi,s}$ is the analytic subgroup of M_Ξ with the Lie algebra $[\mathfrak{m}_\Xi,\mathfrak{m}_\Xi]$. Suppose

$$\chi_\ell|_{M^o_{\Xi,s}\cap K}=\mathbf{1}. \tag{2.47}$$

Let $Z(M_\Xi)$ be the center of M_Ξ. Then $Z(M_\Xi)\subset M$ and for $\mu\in\mathfrak{a}^*_\Xi$ we can define a one-dimensional representation $\tau_{\ell,\mu}$ of P_Ξ by

$$\tau_{\Xi,\ell,\mu}(yman)=\chi_\ell(m)a^{\mu-\rho}\quad\text{for } (y,m,a,n)\in M^o_{\Xi,s}\times M\times A_\Xi\times N_\Xi. \tag{2.48}$$

Put

$$\begin{aligned}\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})&=\{f\in\mathcal{B}(G);f(gp)=\tau_{\Xi,\ell,\mu}(p)f(g)\quad\text{for } p\in P_\Xi\},\\ \mathcal{A}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})&=\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})\cap\mathcal{A}(G).\end{aligned} \tag{2.49}$$

and define an element $\mathbf{1}_{\Xi,\ell,\mu}\in\mathcal{A}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})$ by

$$\mathbf{1}_{\Xi,\ell,\mu}(kan)=\chi_{-\ell}(k)a^{\mu-\rho}\quad\text{for } (k,a,n)\in K\times A\times N. \tag{2.50}$$

Then by the G-invariant bilinear form

(2.51)
$$\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})\times\mathcal{A}(G/P_\Xi;\mathcal{L}^{-\ell}_{\Xi,-\mu})\ni(\phi,f)\mapsto\langle\phi,f\rangle_{\Xi,\ell,\mu}=\int_K\phi(k)f(k)dk\in\mathbb{C}$$

we can define the Poisson transform

$$\begin{aligned}\mathcal{P}^\ell_{\Xi,\mu}:\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})&\to\mathcal{A}(G/K;\mathcal{M}^\ell_{\mu+\rho(\Xi)})\\ \phi&\mapsto(P^\ell_{\Xi,\mu}\phi)(g)=\langle\pi_{g^{-1}}\phi,\mathbf{1}_{\Xi,-\mu,-\ell}\rangle_{\Xi,\ell,\mu}\\ &\qquad=\langle\phi,\pi_g\mathbf{1}_{\Xi,-\mu,-\ell}\rangle_{\Xi,\ell,\mu}.\end{aligned} \tag{2.52}$$

We note the following lemma which is similarly proved as Lemma 2.6.

Lemma 2.11.

$$\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})\subset\mathcal{B}(G/P;\mathcal{L}^\ell_{\mu+\rho(\Xi)}), \tag{2.53}$$

$$\operatorname{Im}\mathcal{P}^\ell_{\Xi,\mu}=\operatorname{Im}\mathcal{P}^\ell_{\mu-\rho(\Xi)}. \tag{2.54}$$

Lastly in this section we examine the space of harmonic functions on G/K:

$$\mathcal{H}(G/K):=\mathcal{A}(G/K;\mathcal{M}_\rho).$$

Let X_Ξ be the Satake compactification of G/K where the boundary G/P_Ξ appears. For $a\in A$ we denote by $a\to\infty$ if $\alpha(\log a)\to\infty$ for any $\alpha\in\Psi(\mathfrak{a}_{\mathfrak{p}})$. Then for any $k\in K$ the point $kaK\in G/K\subset X_\Xi$ converges to a point in $G/P_\Xi\subset X_\Xi$. The Poisson transform $\mathcal{P}_\rho$ defines a bijective homomorphism of $\mathcal{B}(G/P)$ onto $\mathcal{H}(G/K)$. Let $\mathcal{C}(G/P_\Xi)$ be the space of continuous functions on G/P_Ξ. Note that $\mathcal{C}(G/P_\Xi)\simeq\mathcal{C}(K/M_\Xi)\subset\mathcal{C}(G/P)\simeq C(K/M)$.

Proposition 2.12. *Let $\mathcal{F}$ be $\mathcal{C}$ or $\mathcal{C}^m$ or $\mathcal{C}^\infty$ or $\mathcal{D}'$ or $\mathcal{B}$. Note that $\mathcal{F}(G/P_\Xi) \simeq \mathcal{F}(K/K\cap M_\Xi) \subset \mathcal{F}(G/P) \simeq \mathcal{F}(K/M)$. Then we have*

$$\begin{aligned} &\mathcal{P}_{\Xi,\rho}\mathcal{F}(G/P_\Xi)\\ &= \{u \in \mathcal{H}(G/K)\,;\, u(ka) \text{ uniformly converges to a continuous function}\\ &\qquad \text{on } K/K\cap M_\Xi \text{ in the strong topology of } \mathcal{F}(K) \text{ when } a\to\infty\}. \end{aligned} \tag{2.55}$$

This is shown as follows. Suppose u is a function in the above left hand side. Then the boundary value βu of u equals $\lim_{a\to\infty} u(ka)$ (cf. for example, [OS1] and [BOS]) and $\mathcal{P}_\rho\beta u = u$. The assumption implies $\beta u \in \mathcal{F}(G/P_\Xi)$ and hence $\mathcal{P}_\rho\beta u = \mathcal{P}_{\Xi,\rho}\beta u$. Moreover the Poisson transform of the function $f \in \mathcal{F}(G/P_\Xi) \subset \mathcal{F}(G/P)$ has a limit $\lim_{a\to\infty}(\mathcal{P}_\rho u)(ka) = u(ka)$ in the strong topology of $\mathcal{F}(K)$.

In particular if $u \in \mathcal{H}(G/K)$ can be continuously extended to the boundary G/P_Ξ in X_Ξ, u satisfies many differential equations corresponding to Ξ because it is in the image of $\mathcal{P}_{\Xi,\rho}$.

These statements can be extended for general eigenspaces $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ by using weighted boundary values given in [BOS, Theorem 3.2].

3. Construction of the Hua type operators

3.1. **Two sided ideals.** We want to study a good generator system characterizing the image of the Poisson transform $\mathcal{P}^\ell_{\Xi,\mu}$ given by (2.52).

The image of the Poisson transform $\mathcal{P}_\lambda$ given by (2.14) is characterized as a simultaneous eigenspace of the invariant differential operators $\mathbb{D}(G/K)$ on the symmetric space G/K when the Poisson transform is injective. The image is also a simultaneous eigenspace of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ with eigenvalues corresponding to the infinitesimal character and in most cases the system of the equations on G/K defined by the generators of $Z(\mathfrak{g})$ is equals to that defined by $\mathbb{D}(G/K)$. In fact, this holds if the image of $Z(\mathfrak{g}) \subset U(\mathfrak{g})^K$ under the identification (2.3) generates $\mathbb{D}(G/K)$, which is valid when G is of classical type. Moreover even when the image of $Z(\mathfrak{g})$ doesn't generates $\mathbb{D}(G/K)$, the system of the equations defined by the generators of $Z(\mathfrak{g})$ characterize the image of the Poisson transform (2.14) for generic parameter λ, which follows from [He4] or [Oc].

Hence we can expect that the system of differential equations characterizing the image of the Poisson transform $\mathcal{P}^\ell_{\Xi,\mu}$ is given by a two sided ideal of $U(\mathfrak{g})$ at least when the parameter μ is generic. In another word it is expected to coincide with the system defined by a certain left ideal of $U(\mathfrak{g})$ studied in the previous section. Note that if $\mathcal{P}^\ell_{\Xi,\mu}$ is injective, the two sided ideal should kill the preimage $\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})$. Hence the system obtained by the operators killing $\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})$ is expected to be the desired one. The annihilator of $\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu})$ corresponds to that of a generalized Verma module, which will be explained.

Let π denote the left regular representation of G on $\mathcal{F}(G)$ defined by

$$(\pi(g)\varphi)(x) = \varphi(g^{-1}x) \quad (x \in G,\, \varphi \in \mathcal{F}(G)).$$

Here $\mathcal{F}$ denotes one of function spaces such as $\mathcal{A}$ (real analytic functions), C^∞ (smooth functions), $\mathcal{D}'$ (distributions), $\mathcal{B}$ (hyperfunctions). The corresponding representation of $\mathfrak{g}$ is denoted by π. That is

$$(\pi(X)\varphi)(x) = \tfrac{d}{dt}\varphi(e^{-tX}x)\big|_{t=0} \quad (\varphi \in C^\infty(G),\, X \in \mathfrak{g},\, x \in G).$$

For an element $X \in \mathfrak{g}$, let L_X denote the differential operator on G defined by

$$(L_X\varphi)(x) = \tfrac{d}{dt}\varphi(xe^{tX})\big|_{t=0} \quad (\varphi \in C^\infty(G),\, X \in \mathfrak{g},\, x \in G).$$

The universal enveloping algebra $U(\mathfrak{g})$ is identified with the algebra of left G-invariant differential operators on G by the correspondence $X \mapsto L_X$. Under this identification we will write $X\varphi = L_X\varphi$ and denote by $\operatorname{ord} D$ the order of the corresponding differential operator for $D \in U(\mathfrak{g})$.

Let $\mathfrak{a}$ be the Cartan subalgebra of the complexification of $\mathfrak{g}$ containing $\mathfrak{a}_\mathfrak{p}$ and let $\Sigma(\mathfrak{a})^+$ be a compatible positive system of the complexification attached to the Cartan subalgebra $\mathfrak{a}$ and let $\mathfrak{b}$ be the corresponding Borel subalgebra. Denoting the fundamental system of $\Sigma(\mathfrak{a})^+$ by $\Psi(\mathfrak{a})$, we have $\Psi(\mathfrak{a}_\mathfrak{p}) = \{\alpha|_{\mathfrak{a}_\mathfrak{p}}\,;\,\alpha \in \Psi(\mathfrak{a})\} \setminus \{0\}$. For a subset $\Xi \subset \Psi(\mathfrak{a}_\mathfrak{p})$ we define a subset

$$\Theta = \{\alpha \in \Psi(\mathfrak{a})\,;\,\alpha|_{\mathfrak{a}_\mathfrak{p}} \in \Xi \cup \{0\}\} \subset \Psi(\mathfrak{a})$$

and let denote by $\mathfrak{p}_\Theta$, $\mathfrak{g}_\Theta$, $\mathfrak{n}_\Theta$ and $\mathfrak{p}_0$ the complexifications of $\mathfrak{p}_\Xi$, $\mathfrak{m}_\Xi + \mathfrak{a}_\Xi$, $\mathfrak{n}_\Xi$ and the Lie algebra of P, respectively. Note that Θ corresponds to a fundamental system of the root system of $\mathfrak{g}_\Theta$. Let λ denote the character of $\mathfrak{p}_\Theta$ defined by

$$\tau_{\Xi.\ell,\mu}(e^X) = e^{\lambda(X)} \quad (X \in \mathfrak{p}_\Xi). \tag{3.1}$$

Let $\mathfrak{a}_\Theta$ be the center of $\mathfrak{g}_\Theta$. Then λ is identified with an element of the dual $\mathfrak{a}_\Theta^*$ of $\mathfrak{a}_\Theta$. Define left ideals

$$\begin{aligned} J_\Theta(\lambda) &= \sum_{X \in \mathfrak{p}_\Theta} U(\mathfrak{g})(X - \lambda(X)), \\ J_0(\lambda) &= \sum_{X \in \mathfrak{p}_0} U(\mathfrak{g})(X - \lambda(X)), \\ J(\lambda) &= \sum_{X \in \mathfrak{b}} U(\mathfrak{g})(X - \lambda(X)) \end{aligned}$$

of $U(\mathfrak{g})$. Then we can see easily that

$$J_\Theta(\lambda) = \{D \in U(\mathfrak{g})\,;\, Df = 0 \quad (\forall\, f \in \mathcal{F}(G/P_\Xi; \mathcal{L}^\ell_{\Xi,\mu})\}.$$

Let a denote the anti-automorphism of $U(\mathfrak{g})$ defined by $a(X) = -X$, $a(XY) = YX$ for X, $Y \in \mathfrak{g}$.

Proposition 3.1. *Assume that $I_\Theta(\lambda)$ is a two sided ideal of $U(\mathfrak{g})$ that satisfies*

$$J_\Theta(\lambda) = I_\Theta(\lambda) + J_0(\lambda). \tag{3.2}$$

Then

$$\mathcal{F}(G/P_\Xi; \mathcal{L}^\ell_{\Xi,\mu}) = \{f \in \mathcal{F}(G/P; \mathcal{L}^\ell_{\mu+\rho(\Xi)})\,;\, \pi(a(D))f = 0 \quad (\forall\, D \in I_\Theta(\lambda))\}$$

Proof. Since

$$(\pi(a(D))f)(g) = ((\mathrm{Ad}(g^{-1})D)f)(g) \quad (\forall\, g \in G)$$

and $I_\Theta(\lambda)$ is a two sided ideal of $U(\mathfrak{g})$, the proposition follows. □

The above proposition shows that the two sided ideal $I_\Theta(\lambda)$, which satisfies (3.2) characterizes $\mathcal{F}(G/P_\Xi; \mathcal{L}^\ell_{\Xi,\mu})$ in $\mathcal{F}(G/P; \mathcal{L}^\ell_{\mu+\rho(\Xi)})$ as a $U(\mathfrak{g})$-submodule. Notice that the condition

$$J_\Theta(\lambda) = I_\Theta(\lambda) + J(\lambda) \tag{GAP}$$

studied in [O4, O5, O6, OO] implies (3.2).

Theorem 3.2. *Suppose that the Poisson transform*

$$\mathcal{P}^\ell_{\mu+\rho(\Xi)} : \mathcal{B}(G/P; \mathcal{L}^\ell_{\mu+\rho(\Xi)}) \to \mathcal{A}(G/K; \mathcal{M}^\ell_{\mu+\rho(\Xi)})$$

is bijective and assume that (3.2) *holds for a two sided ideal $I_\Theta(\lambda)$ of $\mathfrak{g}$. Then the image of the Poisson transform of $\mathcal{B}(G/P_\Xi; \mathcal{L}^\ell_{\Xi,\mu})$ is characterized by the system $\mathcal{M}^\ell_{\mu+\rho(\Xi)}$ together with the system defined by $I_\Theta(\lambda)$.*

Proof. Since the Poisson transform and its inverse map (boundary value map) are both G-equivariant, Proposition 3.1 implies the theorem. $\square$

It is clear that there exists a two sided ideal $I_\Theta(\lambda)$ satisfying (GAP) if and only if

$$J_\Theta(\lambda) = \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr) + J(\lambda). \tag{3.3}$$

Here $M_\Theta(\lambda)$ is the generalized Verma module $U(\mathfrak{g})/J_\Theta(\lambda)$ and

$$\begin{aligned}\operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr) &:= \{D \in U(\mathfrak{g})\,;\, DM_\Theta(\lambda) = 0\}\\ &= \bigcap_{g\in G} \operatorname{Ad}(g)J_\Theta(\lambda)\\ &= \{D \in U(\mathfrak{g})\,;\, \pi(a(D))\mathcal{B}(G/P_\Xi;\mathcal{L}^\ell_{\Xi,\mu}) = 0\}.\end{aligned}$$

The condition (3.3) is satisfied at least λ is dominant and regular, namely,

$$-\frac{\langle\lambda+\tilde\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} \notin \{0,1,2,3,\ldots\} \quad (\forall\alpha\in\Sigma(\mathfrak{a})^+), \tag{3.4}$$

which is a consequence of [OO, Theorem 3.12]. Here $\tilde\rho$ is the half of the sum of the elements of $\Sigma(\alpha)^+$. Hence (3.3) is satisfied for the harmonic case when $\mu = \rho$, $\ell = 0$ and $\lambda = 0$.

When the complexification of $\mathfrak{g}$ equals $\mathfrak{gl}_N$, Oshima [O5] constructs the generator system of $\operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ for any $\Theta \subset \Psi(\mathfrak{a})$ and any character λ of $\mathfrak{p}_\Theta$ through quantizations of elementary divisors and gives necessary and sufficient condition for (3.3) (cf. [OO, Lemma 4.15]), which says that (3.3) is valid at least if λ is regular, namely,

$$\frac{\langle\lambda+\tilde\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} \neq 0 \quad (\forall\alpha\in\Sigma(\mathfrak{a})^+). \tag{3.5}$$

Oshima [O6] constructs a generator system of a two sided ideal $I_\Theta(\lambda)$ when $\mathfrak{g}$ is a real form of finite copies of classical complex Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sp}_n$ or $\mathfrak{o}_n$ and shows that $I_\Theta(\lambda)$ satisfies (GAP) if λ is (strongly) regular, which will be explained in the next subsection.

3.2. **Minimal polynomials.** Oshima [O6] constructed a set of generators of the annihilator of a generalized Verma module of the scalar type for classical reductive Lie algebras. We review on the main result of [O6] and discuss its implication for the Poisson transform on a degenerate series representation.

Let N be a positive integer and let $\mathfrak{gl}_N \simeq \operatorname{End}(\mathbb{C}^N)$ be the general linear Lie algebra. Let $E_{ij} \in M(N,\mathbb{C})$ be the matrix whose (i,j) entry is 1 and the other entries are all zero. We have a triangular decomposition

$$\mathfrak{gl}_N = \bar{\mathfrak{n}}_N + \mathfrak{a}_N + \mathfrak{n}_N,$$

where

$$\mathfrak{a}_N = \sum_{j=1}^N \mathbb{C}E_{ii}, \quad \mathfrak{n}_N = \sum_{1\le j<i\le N} \mathbb{C}E_{ij}, \quad \bar{\mathfrak{n}}_N = \sum_{1\le i<j\le N} \mathbb{C}E_{ij}.$$

Let $\mathfrak{g}$ be one of the classical complex Lie algebras $\mathfrak{gl}_n$, $\mathfrak{o}_{2n}$, $\mathfrak{o}_{2n+1}$ or $\mathfrak{sp}_n$ and put $N = n$, $2n$, $2n+1$, or $2n$, respectively, so that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_N$. Denoting

$$\tilde I_n = \Bigl(\delta_{i,n+1-j}\Bigr)_{\substack{1\le i\le n\\1\le j\le n}} = \begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix} \text{ and } \tilde J_n = \begin{pmatrix} & \tilde I_n\\ -\tilde I_n & \end{pmatrix},$$

we naturally identify

$$\begin{aligned}\mathfrak{o}_n &= \{X \in \mathfrak{gl}_n;\, \sigma_{\mathfrak{o}_n}(X) = X\} \quad \text{with } \sigma_{\mathfrak{o}_n}(X) = -\tilde I_n{}^t X \tilde I_n,\\ \mathfrak{sp}_n &= \{X \in \mathfrak{gl}_{2n};\, \sigma_{\mathfrak{sp}_n}(X) = X\} \quad \text{with } \sigma_{\mathfrak{sp}_n}(X) = -\tilde J_n{}^t X \tilde J_n.\end{aligned} \tag{3.6}$$

Let σ be the involutive automorphism of $\mathfrak{gl}_N$ defined as above so that $\mathfrak{g} = \mathfrak{gl}_N^\sigma := \{X \in \mathfrak{gl}_N\,;\, \sigma(X) = X\}$ (cf. [O6, Definition 3.1]). Put $F_{ij} = E_{ij}$ if $\mathfrak{g} = \mathfrak{gl}_n$ and $F_{ij} = E_{ij} + \sigma(E_{ij})$ with $\mathfrak{g} = \mathfrak{gl}_N^\sigma$ in other cases. Moreover putting $F_i = F_{ii}$ and $\mathbb{F} = (F_{ij})_{1\le i,\,j\le N} \in M(N,\mathfrak{g})$, we have

$$\mathrm{Ad}(g)q(\mathbb{F}) = {}^t g \cdot q(\mathbb{F}) \cdot g^{-1} \quad (\forall g \in G) \tag{3.7}$$

for any polynomial $q(x)$ and the analytic subgroup G of $GL(n,\mathbb{C})$ with the Lie algebra $\mathfrak{g}$.

We have a triangular decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{a} + \mathfrak{n},$$

where $\mathfrak{a} = \mathfrak{g} \cap \mathfrak{a}_N$, $\mathfrak{n} = \mathfrak{g} \cap \mathfrak{n}_N$ and $\bar{\mathfrak{n}} = \mathfrak{g} \cap \bar{\mathfrak{n}}_N$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$. Let $\Theta = \{0 < n_1 < n_2 < \cdots < n_L = n\}$ be a sequence of strictly increasing positive integers ending at n and put $H_\Theta = \sum_{k=1}^L \sum_{i=1}^{n_k} F_i$. Define

$$\begin{cases} \mathfrak{m}_\Theta = \{X \in \mathfrak{g} \,:\, \mathrm{ad}(H_\Theta)X = 0\}, \\ \mathfrak{n}_\Theta = \{X \in \mathfrak{n} \,:\, \langle X, \mathfrak{m}_\Theta\rangle = 0\}, \quad \bar{\mathfrak{n}}_\Theta = \{X \in \bar{\mathfrak{n}} \,:\, \langle X, \mathfrak{m}_\Theta\rangle = 0\} \\ \mathfrak{p}_\Theta = \mathfrak{m}_\Theta + \mathfrak{n}_\Theta. \end{cases}$$

Then $\mathfrak{p}_\Theta$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Put $H_{\bar\Theta} = \sum_{k=1}^{L-1} \sum_{i=1}^{n_k} F_i$ and define $\mathfrak{m}_{\bar\Theta}$, $\mathfrak{n}_{\bar\Theta}$, $\bar{\mathfrak{n}}_{\bar\Theta}$ and $\mathfrak{p}_{\bar\Theta}$ by replacing Θ by $\bar\Theta$ in the above definition.

For $1 \le i \le n$ with $n_{j-1} < i \le n_j$, put $\iota_\Theta(i) = j$. For $\lambda = (\lambda_1, \ldots, \lambda_L) \in \mathbb{C}^L$ define a character of $\mathfrak{p}_\Theta$ by

$$\lambda\big(X + \sum_{i=1}^n C_i F_i\big) = \sum_{i=1}^n C_i \lambda_{\iota_\Theta(i)} \quad \text{for } X \in \mathfrak{n}_\Theta + [\mathfrak{m}_\Theta, \mathfrak{m}_\Theta] \text{ and } C_i \in \mathbb{C}.$$

In this subsection $U(\mathfrak{g})$ denotes the universal enveloping algebra of the complex Lie algebra $\mathfrak{g}$. The generalized Verma module $M_\Theta(\lambda) = U(\mathfrak{g})/J_\Theta(\lambda)$ is a quotient of the Verma module $M(\lambda) = U(\mathfrak{g})/J(\lambda)$. If $\lambda_L = 0$, we similarly define a character of $\mathfrak{p}_{\bar\Theta}$, $J_{\bar\Theta}(\lambda)$ and $M_{\bar\Theta}(\lambda)$.

Define polynomials

$$\begin{cases} q_\Theta(\mathfrak{gl}_n; x, \lambda) = \prod_{j=1}^L (x - \lambda_j - n_{j-1}), \\ q_\Theta(\mathfrak{o}_{2n+1}; x, \lambda) = (x - n) \prod_{j=1}^L (x - \lambda_j - n_{j-1})(x + \lambda_j + n_j - 2n), \\ q_\Theta(\mathfrak{sp}_n; x, \lambda) = \prod_{j=1}^L (x - \lambda_j - n_{j-1})(x + \lambda_j + n_j - 2n - 1), \\ q_\Theta(\mathfrak{o}_{2n}; x, \lambda) = \prod_{j=1}^L (x - \lambda_j - n_{j-1})(x + \lambda_j + n_j - 2n + 1) \end{cases} \tag{3.8}$$

and if $\mathfrak{g} = \mathfrak{sp}_n$, $\mathfrak{o}_{2n+1}$ or $\mathfrak{o}_{2n}$,

$$q_{\bar\Theta}(\mathfrak{g}; x, \lambda) = (x - n_{L-1}) \prod_{j=1}^{L-1} (x - \lambda_j - n_{j-1})(x + \lambda_j + n_j - 2n - \delta_{\mathfrak{g}})$$

with

$$\delta_{\mathfrak{g}} = \begin{cases} 1 & \text{if } \mathfrak{g} = \mathfrak{sp}_n, \\ 0 & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1} \text{ or } \mathfrak{gl}_n, \\ -1 & \text{if } \mathfrak{g} = \mathfrak{so}_{2n}. \end{cases}$$

Define two sided ideals of $U(\mathfrak{g})$

$$
\begin{cases}
I_\Theta(\lambda) = \sum_{i=1}^{N}\sum_{j=1}^{N} U(\mathfrak{g})q_\Theta(\mathfrak{g};\mathbb{F},\lambda)_{ij} + \sum_{j\in J} U(\mathfrak{g})\Big(\Delta_j - \lambda(\Delta_j)\Big),\\
I_{\bar\Theta}(\lambda) = \sum_{i=1}^{N}\sum_{j=1}^{N} U(\mathfrak{g})q_{\bar\Theta}(\mathfrak{g};\mathbb{F},\lambda)_{ij} + \sum_{j\in \bar J} U(\mathfrak{g})\Big(\Delta_j - \lambda(\Delta_j)\Big),
\end{cases}
\tag{3.9}
$$

where $\Delta_1,\ldots,\Delta_n$ are fixed generators of the center $Z(\mathfrak{g})\subset U(\mathfrak{g})$ with

$$
\begin{cases}
\operatorname{ord}\Delta_j = j \quad (1\le j\le n) & \text{if } \mathfrak{g}=\mathfrak{gl}_n,\\
\operatorname{ord}\Delta_j = 2j \quad (1\le j\le n) & \text{if } \mathfrak{g}=\mathfrak{o}_{2n+1} \text{ or } \mathfrak{g}=\mathfrak{sp}_n,\\
\operatorname{ord}\Delta_j = 2j \quad (1\le j< n), \quad \operatorname{ord}\Delta_n = n & \text{if } \mathfrak{g}=\mathfrak{o}_{2n}
\end{cases}
$$

and

$$
\begin{cases}
J=\{1,2,\ldots,L-1\}, \quad N=n & \text{if } \mathfrak{g}=\mathfrak{gl}_n,\\
J=\{1,2,\ldots,L\},\ \bar J=\{1,2,\ldots,L-1\}, \quad N=2n+1 & \text{if } \mathfrak{g}=\mathfrak{o}_{2n+1},\\
J=\bar J=\{1,2,\ldots,L-1\}, \quad N=2n & \text{if } \mathfrak{g}=\mathfrak{sp}_n,\\
J=\bar J=\{1,2,\ldots,L-1\}\cup\{n\}, \quad N=2n & \text{if } \mathfrak{g}=\mathfrak{o}_{2n}.
\end{cases}
\tag{3.10}
$$

Here $\lambda(\Delta_j)\in\mathbb{C}$ are defined so that $\Delta_j-\lambda(\Delta_j)\in\operatorname{Ann}\big(M(\lambda)\big)$. When $\mathfrak{g}=\mathfrak{o}_{2n}$, Δ_j for $j=1,\ldots,n-1$ are fixed and Δ_n is not fixed by a non-trivial outer automorphism of $\mathfrak{o}_{2n}$.

Oshima [O6] studied sufficient conditions on λ such that

$$
J_{\Theta'}(\lambda) = I_{\Theta'}(\lambda) + J(\lambda_\Theta)
\tag{3.11}
$$

with $\Theta'=\Theta$ or $\bar\Theta$. For $\mathfrak{g}=\mathfrak{gl}_n$, a necessary and sufficient condition on λ for (3.11) is given ([O6, Remark 4.5 (i)]). In particular, in the case when $\mathfrak{g}=\mathfrak{gl}_n$, $\mathfrak{o}_{2n+1}$ or $\mathfrak{sp}_n$, (3.11) holds for $\Theta'=\Theta$ if $\lambda|_{\mathfrak{a}}+\tilde\rho$ is regular, that is $\langle\lambda|_{\mathfrak{a}}+\tilde\rho,\alpha\rangle\neq 0$ for any roots $\alpha\in\Sigma(\mathfrak{a})$. When $\mathfrak{g}=\mathfrak{o}_{2n}$, (3.11) holds for $\Theta'=\Theta$ if $\lambda_\Theta|_{\mathfrak{a}}+\tilde\rho$ is strongly regular, that is $\lambda|_{\mathfrak{a}}+\tilde\rho$ is not fixed by the non-trivial outer automorphism of the root system $\Sigma(\mathfrak{a})$. Moreover, for $\mathfrak{g}=\mathfrak{gl}_n$, $\mathfrak{o}_{2n}$, $\mathfrak{o}_{2n+1}$ or $\mathfrak{sp}_n$, (3.11) holds for $\Theta=\bar\Theta$ if λ satisfies the same regularity condition as above and $\lambda_L=0$. See Section 4 of [O6] for details

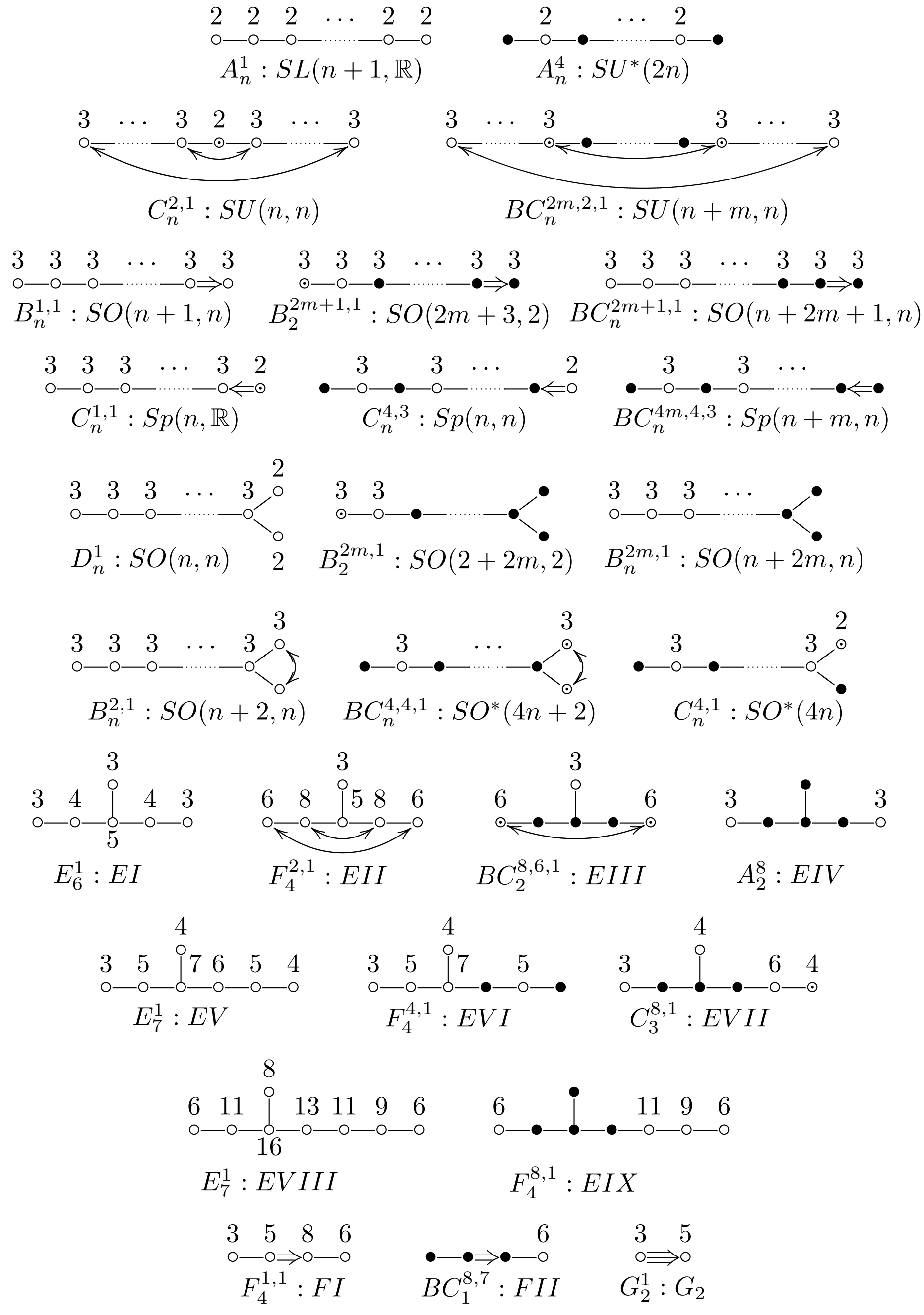
The above construction of the minimal polynomial $q_\Theta(x,\lambda)$ can be extended for any complex reductive Lie algebra $\mathfrak{g}$ by considering a faithful representation $(\pi,\mathbb{C}^N)$ of $\mathfrak{g}$ (see [O6, Section 2]). Oshima and Oda [OO] studied sufficient conditions for the counterpart of (3.11) with $\Theta'=\Theta$ for a general reductive Lie algebra $\mathfrak{g}$ ([OO, Theorem 3.21, Proposition 3.25, Proposition 3.27]). In particular, when $\mathfrak{g}$ is one of the simple exceptional Lie algebras E_6, E_7, E_8, F_4 or G_2, the counterpart of (3.11) with $\Theta'=\Theta$ associated with non-trivial irreducible representation of $\mathfrak{g}$ with minimal degree holds if $\operatorname{Re}\langle\lambda_\Theta+\tilde\rho,\alpha\rangle>0$ for all positive roots $\Sigma(\mathfrak{a})^+$ with respect to a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ (cf. [OO, Remark 4.13]).

Lastly in this section we list the order of the elements of $I_\Theta(\lambda)$ associated with the natural representation or the non-trivial representation with minimal degree according to the condition that $\mathfrak{g}$ is of classical type or exceptional type, respectively, when Θ corresponds to a maximal parabolic subgroup P_Ξ of G as was described in the previous subsection. In the following Satake diagram the number attached to a simple root $\alpha\in\Psi(\mathfrak{a})$ indicates the order of the elements of $I_\Theta(\lambda)$ by the correspondence $\Theta=\Psi(\mathfrak{a}_{\mathfrak{p}})\setminus\{\alpha|_{\mathfrak{a}_{\mathfrak{p}}}\}$. The order is easily seen from (3.8) if $\mathfrak{g}$ is of classical type and it is given in [OO, §4] if $\mathfrak{g}$ is of exceptional type.

The dotted circles correspond to the Shilov boundaries in Hermitian cases.

When $\mathfrak{g}$ is a complex simple Lie algebra, the degree is obtained by the corresponding Dynkin diagram with no arrow and no black circle.

“Degree of minimal polynomials associated to natural/smallest representations”



Remark 3.3. The restricted root system is shown by the notation in [OS1, Appendix] such as $BC_n^{m_1,m_2,m_3}$ and the Lie algebra $\mathfrak{m}_\Xi$ and its complexification for any $\Xi \subset \Psi(\mathfrak{a}_\mathfrak{p})$ can be easily read from the Satake diagram as was explained in [OS2, Appendix B]. Namely, if G is semisimple, the subdiagram corresponding to $\Psi(\Theta) = \{\alpha \in \Psi(\alpha)\,;\, \alpha|_{\mathfrak{a}_\mathfrak{p}} \in \Theta \cup \{0\}\}$ is the Satake diagram of $\mathfrak{m}_\Xi$.

If G is a connected real form of a simply connected semisimple complex Lie group, $M_{\Xi,s}^o$ is a real form of a simply connected complex Lie group and M_Ξ/M_Ξ^o is isomorphic to the direct sum of $(\mathbb{Z}/2\mathbb{Z})^k$ and ℓ copies of $U(1)$. Here M_Ξ^o is the identity component of M_Ξ, ℓ is the number of the arrows pointing roots in

$\Psi(\mathfrak{a}) \setminus \Psi(\Theta)$ and k is the number of the roots in $\Psi(\mathfrak{a}) \setminus \Psi(\Theta)$ which are not pointed by any arrow and are not directly linked by any line attached to any root $\alpha \in \Psi(\mathfrak{a})$ with $\alpha|_{\mathfrak{a}_\mathfrak{p}} = 0$.

4. Examples

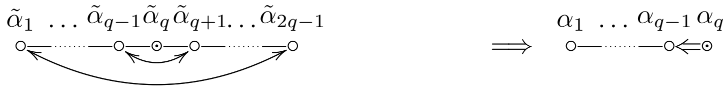
In this section we examine in detail the differential operators on a homogeneous line bundle over a Riemannian symmetric space G/K induced from the two sided ideal given in the preceding section for $G = U(p,q)$, $Sp(n,\mathbb{R})$ and $GL(n,\mathbb{R})$.

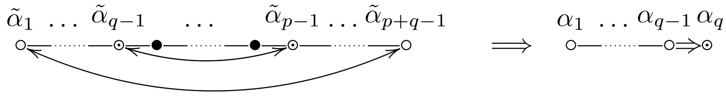
4.1. $U(p,q)$**.** Let σ be the complex linear involution of $\mathfrak{g} = \mathfrak{gl}_{p+q}$ defined by

$$\sigma(X) = I_{p,q} X I_{p,q} \qquad \text{with } \ I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Here $1 \le q \le p$. Then $G = U(p,q) = \{g \in GL(p+q,\mathbb{C})\,;\, g = I_{p,q}\overline{{}^t g^{-1}} I_{p,q}\}$ and $K = U(p,q) \cap U(p+q) = U(p) \times U(q)$.

The corresponding Satake diagram and the Dynkin diagram of the restricted root system are as follows.

$U(p,q) \quad (p = q):$ 

$U(p,q) \quad (p > q):$ 

In this subsection, we restrict ourselves to a parabolic subgroup P_Ξ that satisfies (2.47) for $\ell \neq 0$. This is the case $\Xi \subset \Psi(\mathfrak{a}_\mathfrak{p}) \setminus \{\alpha_q\}$. We fix $L+1$ non-negative integers $0 = n_0 < n_1 < \cdots < n_L = q$ and put $\Xi = \{\alpha_i\,;\, i \in \{1,\ldots,q\} \setminus \{n_1, n_2, \ldots, n_L\}\}$. Then

$$\Theta = \{\tilde\alpha_\nu\,;\, q-1 < \nu < p-1\} \cup \bigcup_{j=1}^{L} \{\tilde\alpha_\nu, \tilde\alpha_{p+q-\nu}\,;\, n_{j-1} < \nu < n_j\}.$$

We have

$$\mathfrak{m}_\Xi + \mathfrak{a}_\Xi \simeq \mathfrak{gl}_{n_1} \oplus \mathfrak{gl}_{n_2 - n_1} \oplus \cdots \oplus \mathfrak{gl}_{n_L - n_{L-1}} \oplus \mathfrak{u}(p-q). \tag{4.1}$$

In particular, if $L = 1$, then $\Xi = \Psi(\mathfrak{a}_\mathfrak{p}) \setminus \{\alpha_q\}$ and G/P_Ξ is the Shilov boundary of G/K.

We examine the system of differential equations characterizing the image of the Poisson transform $\mathcal{P}^\ell_{\Xi,\mu}$ of the space of hyperfunction sections over the boundary G/P_Ξ of G/K. If $p > q$, then the differential operators corresponding to the generators of the two sided ideal of $U(\mathfrak{g})$ given by the minimal polynomial described in Section 3 is of order $2L+1$, but we will show that the image of the Poisson transform can be characterized by operators of order at most $2L$, by reducing the operators in the two sided ideals modulo $\sum_{X \in \mathfrak{k}} U(\mathfrak{g})(X + \chi_\ell(X))$ and taking a K-invariant left ideal (cf. Theorem 4.2, Corollary 4.3). For $L = 1$ these second order differential operators are the Hua operators (cf. Remark 4.5).

Let i, j, k, ℓ, μ and ν are indices which satisfy

$$1 \le i,\ j \le q \quad \text{and} \quad q < k,\ \ell \le p \quad \text{and} \quad 1 \le \mu,\ \nu \le p$$

and put

$$\bar{i} = p + q + 1 - i, \quad \bar{j} = p + q + 1 - j.$$

Put

$$\mathfrak{k}_{\mathbb{C}} = \sum_{\mu,\nu} \mathbb{C}E_{\mu,\nu} + \sum_{i,j} \mathbb{C}E_{\bar{i},\bar{j}}$$

and

$$\mathfrak{a}_{\mathbb{C}} = \sum_{i=1}^{q} \mathbb{C}E_i \quad \text{with} \quad E_i = E_{i,\bar{i}} + E_{\bar{i},i}.$$

Let $e_i \in \mathfrak{a}_{\mathbb{C}}^*$ defined by

$$e_i(E_j) = \delta_{i,j}.$$

Define

$$\begin{aligned}
Y_i &= -E_{i,i} + E_{i,\bar{i}} - E_{\bar{i},i} + E_{\bar{i},\bar{i}}, \\
Y_{i,k} &= E_{i,k} + E_{\bar{i},k}, \qquad Y_{k,i} = E_{k,i} - E_{k,\bar{i}}, \\
Y_{i,j,+} &= E_{i,j} + E_{\bar{i},j} - E_{i,\bar{j}} - E_{\bar{i},\bar{j}} \quad \text{for } i \neq j, \\
Y_{i,j,1} &= E_{i,j} + E_{\bar{i},j} + E_{i,\bar{j}} + E_{\bar{i},\bar{j}} \quad \text{for } i < j, \\
Y_{i,j,2} &= E_{j,i} - E_{\bar{j},i} - E_{j,\bar{i}} + E_{\bar{j},\bar{i}} \quad \text{for } i < j
\end{aligned}$$

and let $\mathfrak{n}_{\mathbb{C}}$ be the nilpotent subalgebra of $\mathfrak{gl}_{p+q}$ spanned by Y_i, $Y_{i,k}$, $Y_{i,j,+}$ with $i \neq j$, $Y_{i,j,1}$ and $Y_{i,j,2}$ with $i < j$. Then $\mathfrak{gl}_{p+q} = \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}$ is the complexification of the Iwasawa decomposition $\mathfrak{u}(p,q) = \mathfrak{k} + \mathfrak{a}_{\mathfrak{p}} + \mathfrak{n}$ of the Lie algebra of $U(p,q)$.

For a polynomial $f(x)$ we will examine $\mathbb{F} \in M\big(p+q, U(\mathfrak{g})\big)$ defined by $\mathbb{F} = f(\mathbb{E})$ with $\mathbb{E} = (E_{i,j}) \in M\big(p+q, U(\mathfrak{g})\big)$. Note that $V_f = \sum_{i,j} \mathbb{C}F_{i,j}$ is a $\mathfrak{g}$-module by the adjoint action of $\mathfrak{g}$ and it is decomposed into 4 $\mathfrak{k}$-submodules.

$$V_f = \bigoplus_{\epsilon_1,\epsilon_2=0,1} V_f^{\epsilon_1,\epsilon_2} \quad \text{with} \quad V_f^{\epsilon_1,\epsilon_2} := \sum_{\substack{p\epsilon_1 < i \leq p+q\epsilon_1 \\ p\epsilon_2 < j \leq p+q\epsilon_2}} \mathbb{C}F_{i,j}. \tag{4.2}$$

We will calculate $\gamma_\ell(F_{i,j})$ (cf. (2.35)) for $\mathbb{F} = (F_{i,j})$ to get $V_f^{\epsilon_1,\epsilon_2}$ killing the image of the Poisson transform $\mathcal{P}_{\Xi,\mu}^{\ell}$. A similar calculation was done in the proof of [O6, Proposition 3.4]. The polynomial $f(x)$ so that V_f characterizes the image of $\mathcal{P}_{\Xi,\mu}^{\ell}$ is given in the preceding section and then the degree of $f(x)$ which is the maximal order of the elements of V_f equals $2L+1$ or $2L$ if $p > q$ or $p = q$, respectively. It happens that V_f doesn't kill the image but $V_f^{\epsilon_1,\epsilon_2}$ does so for suitable $f(x)$ and (ϵ_1, ϵ_2) and then we will get the system of differential equations of order $\leq 2L$ characterizing the image also in the case $p > q$.

Note that for $H \in \mathfrak{a}_{\mathbb{C}}$

$$\begin{aligned}
[H, Y_i] &= 2e_i(H)Y_i, \\
[H, Y_{i,k}] &= e_i(H)Y_{i,k}, & [H, Y_{k,i}] &= e_i(H)Y_{k,i}, \\
[H, Y_{i,j,+}] &= (e_i + e_j)(H)Y_{i,j,+}, \\
[H, Y_{i,j,1}] &= (e_i - e_j)(H)Y_{i,j,1}, & [H, Y_{i,j,2}] &= (e_i - e_j)(H)Y_{i,j,2}.
\end{aligned}$$

Then the root system $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ is of type $BC_q^{2(p-q),2,1}$ and

$$\begin{aligned}
\Psi(\mathfrak{a}_{\mathfrak{p}}) &= \{e_1 - e_2, e_2 - e_3, \dots, e_{q-1} - e_q, e_q\}, \\
\rho &= (p+q-1)e_1 + (p+q-2)e_2 + \cdots + (p-q+1)e_q, \\
E_{i,\bar{i}} &= \frac{1}{2}E_i + \frac{1}{2}Y_i + \frac{1}{2}(E_{i,i} - E_{\bar{i},\bar{i}}), \\
E_{\bar{i},i} &= \frac{1}{2}E_i - \frac{1}{2}Y_i - \frac{1}{2}(E_{i,i} - E_{\bar{i},\bar{i}}), \\
E_{k,\bar{i}} &= -Y_{k,i} + E_{k,i}, \qquad E_{\bar{i},k} = Y_{i,k} - E_{i,k}, \\
E_{i,\bar{j}} &= \frac{1}{2}(Y_{i,j,1} - Y_{i,j,+}) - E_{\bar{i},\bar{j}} \quad \text{for } i < j,
\end{aligned}$$

$$E_{\bar{i},j} = \frac{1}{2}(Y_{i,j,1} + Y_{i,j,+}) - E_{i,j} \quad \text{for } i < j,$$
$$E_{i,\bar{j}} = -\frac{1}{2}(Y_{i,j,+} + Y_{j,i,2}) + E_{i,j} \quad \text{for } i > j,$$
$$E_{\bar{i},j} = \frac{1}{2}(Y_{i,j,+} - Y_{j,i,2}) + E_{\bar{i},\bar{j}} \quad \text{for } i > j.$$

Suppose $F_{a,b} \in U(\mathfrak{g})$ for $1 \leq a, b \leq p+q$ satisfy

$$[E_{i,j}, F_{a,b}] = \delta_{j,a}F_{i,b} - \delta_{i,b}F_{a,j} \quad \text{for } 1 \leq i,\ j,\ a,\ b \leq p+q.$$

Fix $s, t \in \mathbb{C}$ and let $\tau_{s,t}$ be the one dimensional representation of $\mathfrak{k}_{\mathbb{C}}$ with $\tau_{s,t}(E_{\mu,\nu}) = \tau_{s,t}(E_{\bar{i},\bar{j}}) = 0$ if $\mu \neq \nu$ and $i \neq j$ and $\tau_{s,t}(E_{\nu,\nu}) = s$ and $\tau_{s,t}(E_{\bar{i},\bar{i}}) = t$. Note that $\chi_\ell(X) = \tau_{s,t}(X)$ with $\ell = s - t$ for $X \in \mathfrak{k}_{\mathbb{C}}$ with $\operatorname{Trace} X = 0$. Put

$$\tilde{F}_{u,v} = \sum_{w=1}^{p+q} E_{u,w}F_{w,v}.$$

Considering in modulo $\mathfrak{n}_{\mathbb{C}}U(\mathfrak{g}) + \sum_{X\in\mathfrak{k}_{\mathbb{C}}} U(\mathfrak{g})(X - \tau_{s,t}(X))$, we have

$$\begin{aligned}
\tilde{F}_{i,a} &= \sum_{\nu} E_{i,\nu}F_{\nu,a} + \sum_{i>j} E_{i,\bar{j}}F_{\bar{j},a} + E_{i,\bar{i}}F_{\bar{i},a} + \sum_{i<j} E_{i,\bar{j}}F_{\bar{j},a} \\
&\equiv \sum_{\nu}(F_{i,a} - \delta_{ia}F_{\nu,\nu}) + sF_{i,a} + \sum_{i>j} E_{i,j}F_{\bar{j},a} + \frac{1}{2}(E_i + E_{i,i} - E_{\bar{i},\bar{i}})F_{\bar{i},a} \\
&\quad - \sum_{i<j} E_{\bar{i},\bar{j}}F_{\bar{j},a} \\
&\equiv (p+s)F_{i,a} - \delta_{ia}\sum_{\nu} F_{\nu,\nu} - \delta_{ia}\sum_{i>j} F_{\bar{j},j} + \frac{E_i + s - t - 1}{2}F_{\bar{i},a} - \frac{1}{2}\delta_{ia}F_{\bar{i},i} \\
&\quad + \frac{1}{2}\delta_{\bar{i}a}F_{\bar{i},\bar{i}} - \sum_{i<j}(F_{\bar{i},a} - \delta_{\bar{i}a}F_{\bar{j},\bar{j}}), \\
\tilde{F}_{k,a} &= \sum_{\nu} E_{k,\nu}F_{\nu,a} + \sum_{j} E_{k,\bar{j}}F_{\bar{j},a} \\
&\equiv \sum_{\nu}(F_{k,a} - \delta_{ka}F_{\nu,\nu}) + sF_{k,a} + \sum_{j} E_{k,j}F_{\bar{j},a} \\
&= (p+s)F_{k,a} - \delta_{ka}\sum_{\nu} F_{\nu,\nu} - \sum_{j} \delta_{ka}F_{\bar{j},j}, \\
\tilde{F}_{\bar{i},a} &= \sum_{i>j} E_{\bar{i},j}F_{j,a} + E_{\bar{i},i}F_{i,a} + \sum_{i<j} E_{\bar{i},j}F_{j,a} + \sum_{k} E_{\bar{i},k}F_{k,a} + \sum_{j} E_{\bar{i},\bar{j}}F_{\bar{j},a} \\
&\equiv \sum_{i>j} E_{\bar{i},\bar{j}}F_{j,a} + \frac{1}{2}(E_i - E_{i,i} + E_{\bar{i},\bar{i}})F_{i,a} - \sum_{i<j} E_{i,j}F_{j,a} - \sum_{k} E_{i,k}F_{k,a} \\
&\quad + \sum_{j}(F_{\bar{i},a} - \delta_{\bar{i},a}F_{\bar{j},\bar{j}}) + tF_{\bar{i},a} \\
&\equiv tF_{\bar{i},a} - \delta_{\bar{i},a}\sum_{i>j} F_{j,\bar{j}} + \frac{E_i - s + t - 1}{2}F_{i,a} - \frac{1}{2}\delta_{\bar{i},a}F_{i,\bar{i}} + \frac{1}{2}\delta_{i,a}F_{i,i} \\
&\quad - \sum_{i<j}(F_{i,a} - \delta_{i,a}F_{j,j}) - \sum_{k}(F_{i,a} - \delta_{i,a}F_{k,k}) + \sum_{j}(F_{\bar{i},a} - \delta_{\bar{i},a}F_{\bar{j},\bar{j}}).
\end{aligned}$$

Suppose

$$F_{a,b} = 0 \quad \text{if} \quad |a-b| \neq 0,\ p.$$

Then we have

$$\tilde{F}_{a,b} = 0 \quad \text{if} \quad |a-b| \neq 0,\ p$$

and

$$\begin{aligned}
\tilde{F}_{i,i} &= (p+s)F_{i,i} - \sum_{\nu=1}^{p} F_{\nu,\nu} - \sum_{j<i} F_{\bar{j},j} + \left(\frac{E_i+s-t}{2} - q + i - 1\right) F_{\bar{i},i}\\
&= sF_{i,i} + \left(\frac{E_i+s-t}{2} - q\right) F_{\bar{i},i} - \sum_{\nu=1}^{p}(F_{\nu,\nu} - F_{i,i}) - \sum_{j=1}^{i-1}(F_{\bar{j},j} - F_{\bar{i}.i}),\\
\tilde{F}_{i,\bar{i}} &= (p+s)F_{i,\bar{i}} + \left(\frac{E_i+s-t}{2} - q + i\right) F_{\bar{i},\bar{i}} + \sum_{i<j} F_{\bar{j},\bar{j}}\\
&= (p+s)F_{i,\bar{i}} + \frac{E_i+s-t}{2} F_{\bar{i},\bar{i}} + \sum_{j=i+1}^{q}(F_{\bar{j},\bar{j}} - F_{\bar{i},\bar{i}}),\\
\tilde{F}_{k,k} &= (p+s)F_{k,k} - \sum_{\nu=1}^{p} F_{\nu,\nu} - \sum_{j=1}^{q} F_{\bar{j},j}\\
&= sF_{k,k} - \sum_{j=1}^{q} F_{\bar{j},j} - \sum_{\nu=1}^{p}(F_{\nu,\nu} - F_{k,k}),\\
\tilde{F}_{\bar{i},i} &= \left(\frac{E_i-s+t}{2} - p + i\right) F_{i,i} + \sum_{\nu=i+1}^{p} F_{\nu,\nu} + (q+t)F_{\bar{i},i}\\
&= (q+t)F_{\bar{i},i} + \frac{E_i-s+t}{2} F_{i,i} + \sum_{\nu=i+1}^{p}(F_{\nu,\nu} - F_{i,i}),\\
\tilde{F}_{\bar{i},\bar{i}} &= -\sum_{j<i} F_{j,\bar{j}} + \left(\frac{E_i-s+t}{2} - p + i - 1\right) F_{i,\bar{i}} - \sum_{j\neq i} F_{\bar{j},\bar{j}} + (q+t-1)F_{\bar{i},\bar{i}},\\
&= tF_{\bar{i},\bar{i}} + \left(\frac{E_i-s+t}{2} - p\right) F_{i,\bar{i}} - \sum_{j=1}^{q}(F_{\bar{j},\bar{j}} - F_{\bar{i},\bar{i}}) - \sum_{j=1}^{i-1}(F_{j,\bar{j}} - F_{i,\bar{i}}).
\end{aligned}$$

Put

$$F^1_{i,\bar{i}} = \frac{E_i+s-t}{2},\quad F^1_{\bar{i},i} = \frac{E_i-s+t}{2},\quad F^1_{i,i} = F^1_{k,k} = s+\lambda_1 \quad\text{and}\quad F^1_{\bar{i},\bar{i}} = t+\lambda_1.$$

Suppose (u,v) are in $\{(i,i),\ (i,\bar{i}),\ (\bar{i},i),\ (\bar{i},\bar{i}),\ (k,k)\}$ and $F^{m-1}_{u,v}$ are defined. By putting $F_{u,v} = F^{m-1}_{u,v}$, define $\tilde{F}^{m-1}_{u,v} = \tilde{F}_{u,v}$ by the above equations and moreover

$$F^m_{u,v} = \tilde{F}^{m-1}_{u,v} + \lambda_m F^{m-1}_{u,v} \in U(\mathfrak{a}_\mathfrak{p}).$$

Thus we inductively define $F^m_{u,v}$. Note that

$$\begin{aligned}
\bigl(F^m_{a,b}\bigr)_{\substack{1\le a\le p+q\\ 1\le b\le p+q}} &\equiv \prod_{j=1}^{m}\Bigl(\bigl(E_{a,b} + \lambda_j\delta_{a,b}\bigr)_{\substack{1\le a\le p+q\\ 1\le b\le p+q}}\Bigr)\\
F^m_{i,\bar{i}} &= (\lambda_m+p+s)F^{m-1}_{i,\bar{i}} + \frac{E_i+s-t}{2} F^{m-1}_{\bar{i},\bar{i}} + \sum_{j=i+1}^{q}(F^{m-1}_{\bar{j},\bar{j}} - F^{m-1}_{\bar{i},\bar{i}}),\\
F^m_{\bar{i},\bar{i}} &= (t+\lambda_m)F^{m-1}_{\bar{i},\bar{i}} - \sum_{j=1}^{q}(F^{m-1}_{\bar{j},\bar{j}} - F^{m-1}_{\bar{i},\bar{i}})\\
&\quad + \left(\frac{E_i-s+t}{2} - p\right) F^{m-1}_{i,\bar{i}} - \sum_{j=1}^{i-1}(F^{m-1}_{j,\bar{j}} - F^{m-1}_{i,\bar{i}}).
\end{aligned}$$

Putting

$$F^m_{\pm i} = F^m_{\bar{i},\bar{i}} \pm F^m_{i,\bar{i}},$$

we have

$$\begin{aligned}
2F^m_{\pm i} &= (\lambda_m + p + s)(F^{m-1}_{\pm i} - F^{m-1}_{\mp i}) \pm \frac{E_i + s - t}{2}(F^{m-1}_{\pm i} + F^{m-1}_{\mp i}) \\
&\quad \pm \sum_{j=i+1}^{q} (F^{m-1}_{\pm j} - F^{m-1}_{\pm i}) \pm \sum_{j=i+1}^{q} (F^{m-1}_{\mp j} - F^{m-1}_{\mp i}) \\
&\quad + (t + \lambda_m)(F^{m-1}_{\pm i} + F^{m-1}_{\mp i}) - \sum_{j=1}^{q}(F^{m-1}_{\pm j} - F^{m-1}_{\mp i}) - \sum_{j=1}^{q}(F^{m-1}_{\pm j} - F^{m-1}_{\mp i}) \\
&\quad \pm \left(\frac{E_i - s + t}{2} - p\right)(F^{m-1}_{\pm i} - F^{m-1}_{\mp i}) \\
&\quad \mp \sum_{j=1}^{i-1}(F^{m-1}_{\pm j} - F^{m-1}_{\pm i}) \pm \sum_{j=1}^{i-1}(F^{m-1}_{\mp j} - F^{m-1}_{\mp i})
\end{aligned}$$

and

$$F^m_i = \left(\lambda_m + \frac{E_i + s + t}{2}\right) F^{m-1}_i - \sum_{j=1}^{i-1}(F^{m-1}_j - F^{m-1}_i),$$

$$F^m_{-i} = \left(\lambda_m + p - \frac{E_i + s + t}{2}\right) F^{m-1}_{-i} - (p + s - t)F^{m-1}_i - \sum_{j=i+1}^{q} (F^{m-1}_{-j} - F^{m-1}_{-i}).$$

For $0 = n_0 < n_1 < \cdots < n_L = q$ and $(\mu_1, \ldots, \mu_L) \in \mathbb{C}^L$ put

$$E_i = 2\mu_\ell \quad \text{if there exists } \ell \text{ with} \quad n_{\ell-1} < i \le n_\ell$$

and

$$\lambda_k = \begin{cases} -\mu_k - \dfrac{s+t}{2} - n_{k-1} & \text{if} \quad k \le L, \\ \mu_{2L+1-k} - \dfrac{s+t}{2} - p + n_{2L+1-k} & \text{if} \quad L < k \le 2L. \end{cases} \tag{4.3}$$

$$f(x) = \prod_{j=1}^{2L}\bigl(x + \lambda_k\bigr) = \prod_{k=1}^{L}\bigl(x - \mu_k - \tfrac{s+t}{2} - n_{k-1}\bigr)\bigl(x + \mu_k - \tfrac{s+t}{2} - p + n_k\bigr). \tag{4.4}$$

Then for $i > 0$ inductively we can prove

$$F^m_i = 0 \quad \text{if} \quad m \ge L \quad \text{or} \quad i \le n_m$$

and moreover by the induction for $i = q,\, q-1, \ldots, 1$

$$F^m_{i,\bar{i}} = 0 \quad \text{if} \quad m > L \quad \text{and} \quad i > n_{2L-m}.$$

In particular we have $F^{2L}_{\bar{i}\bar{i}} = F^{2L}_{\bar{i},\bar{i}} = 0$ for $i = 1, \ldots, q$ and hence $F^{2L}_{a,b} = 0$ for $a = 1, \ldots, p+q$ and $b = p+1, \ldots, p+q$,

Note that when $p = q$, the same argument as above proves $F^{2L}_{a,b} = 0$ also for $a = 1, \ldots, p+q$ and $b = 1, \ldots, p$.

Lemma 4.1. *Suppose* $M = \Bigl(M_{ij}\Bigr)_{\substack{1\le i\le p+q\\ 1\le j\le p+q}} \in M\bigl(p+q, U(\mathfrak{g})\bigr)$ *satisfies*

$$[E_{ij}, M_{k\ell}] = \delta_{jk}M_{i\ell} - \delta_{\ell i}M_{k\ell}.$$

Put $\tilde{M} = M\Bigl(E_{ij} + \lambda\delta_{ij}\Bigr)_{\substack{1\le i\le p+q\\ 1\le j\le p+q}}$ *and* $\tilde{M}' = \Bigl(E_{ij} + \lambda\delta_{ij}\Bigr)M$*. Then*

$$\tilde{M}'_{a\nu} \equiv 0 \mod \sum_{\substack{1\le b\le p+q\\ p<c\le p+q}} U(\mathfrak{g})M_{bc} \qquad \textit{for} \quad 1 \le a \le p+q,\ p < \nu \le p+q,$$

$$\tilde{M}_{a\nu} \equiv M_{a\nu}\bigl(\lambda + s + q\bigr) \mod \sum_{X\in\mathfrak{k}} U(\mathfrak{g})\bigl(X - \chi_{s,t}(X)\bigr) + \sum_{\substack{1\le b\le p+q\\ p<c\le p+q}} U(\mathfrak{g})M_{bc}$$
$$\text{for} \quad 1 \le a \le p+q,\ 1 \le \nu \le p.$$

Proof. If $1 \le \nu \le p$, then

$$\begin{aligned}
\tilde{M}_{a\nu} &= \sum_{b=1}^{p+q} M_{ab}(E_{b\nu} + \lambda\delta_{b\nu}) \\
&\equiv M_{a\nu}(\lambda + s) + \sum_{b=p+1}^{p+q} M_{ab}E_{b\nu} \mod \sum_{X\in\mathfrak{k}} U(\mathfrak{g})(X - \chi_{s,t}(X)) \\
&\equiv M_{a\nu}(\lambda + s + q) + \sum_{X\in\mathfrak{k}} U(\mathfrak{g})\bigl(X - \chi_{s,t}(X)\bigr) + \sum_{\substack{1\le b\le p+q\\ p<c\le p+q}} U(\mathfrak{g})M_{bc}.
\end{aligned}$$

The former relation is clear. □

Thus we have the following theorem.

Theorem 4.2. *Put* $\mathbb{E} = (E_{i,j})_{\substack{1\le i\le p+q\\ 1\le j\le p+q}} \in M\bigl(p+q, \mathfrak{g}\bigr)$ *and define*

$$\tilde{f}(x) = (x - s - q)\prod_{k=1}^{L}\bigl(x - \mu_k - \tfrac{s+t}{2} - n_{k-1}\bigr)\bigl(x + \mu_k - \tfrac{s+t}{2} - p + n_k\bigr) \tag{4.5}$$

and put

$$I^0_\Xi(\mu, s, t) := \sum_{1\le i\le p+q,\ p<j\le p+q} U(\mathfrak{g})f(\mathbb{E})_{i,j} = U(\mathfrak{g})V_f^{0,1} + U(\mathfrak{g})V_f^{1,1}, \tag{4.6}$$

$$I_\Xi(\mu, s, t) := \sum_{1\le i\le p+q,\ 1\le j\le p+q} U(\mathfrak{g})f(\mathbb{E})_{i,j} = U(\mathfrak{g})V_f, \tag{4.7}$$

$$\tilde{I}_\Xi(\mu, s, t) := \sum_{1\le i\le p+q,\ 1\le j\le p+q} U(\mathfrak{g})\tilde{f}(\mathbb{E})_{i,j} = U(\mathfrak{g})V_{\tilde{f}}. \tag{4.8}$$

Then $I^0_\Xi(\mu, s, t)$ *is a left ideal of* $U(\mathfrak{g})$ *satisfying*

$$\begin{aligned}
D \equiv 0 \mod \mathfrak{n}_{\mathbb{C}}U(\mathfrak{g}) &+ \sum_{X\in\mathfrak{k}} U(\mathfrak{g})\bigl(X - \chi_{s,t}(X)\bigr) \\
&+ \sum_{i=1}^{L}\sum_{\nu=n_0+\cdots+n_{i-1}+1}^{n_0+\cdots+n_i} U(\mathfrak{g})(E_\nu - 2\mu_i) \quad (\forall D \in I^0_\Xi(\mu, s, t))
\end{aligned} \tag{4.9}$$

and $\tilde{I}_\Xi(\mu, s, t)$ *is a two-side ideal of* $U(\mathfrak{g})$ *satisfying*

$$\tilde{I}_\Xi(\mu, s, t) \subset I^0_\Xi(\mu, s, t) + \sum_{X\in\mathfrak{k}} U(\mathfrak{g})\bigl(X - \chi_{s,t}(X)\bigr). \tag{4.10}$$

If $p = q$*, the left ideal* $I^0_\Xi(\mu, s, t)$ *in the claim* (4.9) *may be replaced by the two sided ideal* $I_\Xi(\mu, s, t)$*.*

The polynomial $\tilde{f}(x)$ or $f(x)$ equals the minimal polynomial $q_\Theta(\mathfrak{gl}_{p+q}; x, t)$ given in the last section when $p > q$ or $p = q$, respectively. Hence this theorem and the argument in the preceding section give the following corollary.

Corollary 4.3. *Suppose the infinitesimal character of* $\mathcal{B}(G/P_\Xi; \mathcal{L}^\ell_{\Xi,\mu})$ *is regular and* $c(\mu + \rho(\Xi), \ell) \neq 0$*. Then the image of* $\mathcal{P}^\ell_{\Xi,\mu}$ *is identified with the subspace of* $\mathcal{A}(G)$ *killed by* $I^0_\Xi(\mu, s, t)$ *(resp.* $I_\Xi(\mu, s, t)$*),* $\sum_{X\in\mathfrak{k}} U(\mathfrak{g})(X - \chi_{s,t}(X))$ *and* $\Delta_i - c_i$ *for* $i = 2, \ldots, L-1$ *with* $\Delta_i = \operatorname{Trace}\mathbb{E}^i$ *when* $p > q$ *(resp.* $p = q$*). Here the complex* *parameters μ, s, t of $I^0_\Xi(\mu,s,t)$ or $I_\Xi(\mu,s,t)$ and $c_i\in\mathbb{C}$ are determined according to the parameter μ and ℓ of $\mathcal{P}^\ell_{\Xi,\mu}$,*

Example 4.4 (Shilov boundary)**.** Consider the case when $L=1$. We will write

$$\mathbb{E}=\begin{pmatrix}K_1 & P\\ Q & K_2\end{pmatrix}\in M\big(p+q,U(\mathfrak{g})\big)$$

for simplicity. Here $K_1=(E_{i,j})_{1\le i,\,j\le p}$ etc. Then

$$\begin{aligned}
&\mathbb{E}\equiv\begin{pmatrix}s & P\\ Q & t\end{pmatrix},\\
&K_1P\equiv(p+s)P\mod\sum_{X\in\mathfrak{k}}U(\mathfrak{g})\big(X-\chi_{s,t}(X)\big),\\
&K_2Q\equiv(q+t)Q\mod\sum_{X\in\mathfrak{k}}U(\mathfrak{g})\big(X-\chi_{s,t}(X)\big),\\
&\mathbb{E}^2\equiv\begin{pmatrix}K_1 & P\\ Q & K_2\end{pmatrix}\begin{pmatrix}s & P\\ Q & t\end{pmatrix}=\begin{pmatrix}PQ+sK_1 & (K_1+t)P\\ (K_2+s)Q & QP+tK_2\end{pmatrix}\\
&\quad\equiv\begin{pmatrix}PQ+s^2 & (p+s+t)P\\ (q+s+t)Q & QP+t^2\end{pmatrix},\\
&\big(\mathbb{E}-\lambda-\tfrac{s+t}{2}\big)\big(\mathbb{E}+\lambda-p-\tfrac{s+t}{2}\big)\\
&\quad\equiv\mathbb{E}^2-\big(p+s+t\big)\mathbb{E}-\big(\lambda+\tfrac{s+t}{2}\big)\big(\lambda-p-\tfrac{s+t}{2}\big)\\
&\quad\equiv\begin{pmatrix}PQ+s^2-(p+\frac{s+t}{2})s & 0\\ (q-p)Q & QP+t^2-t(p+\frac{s+t}{2})\end{pmatrix}-\big(\lambda+\tfrac{s+t}{2}\big)\big(\lambda-p-\tfrac{s+t}{2}\big)\\
&\quad\equiv\begin{pmatrix}PQ-s(p+t) & 0\\ (q-p)Q & QP-t(p+s)\end{pmatrix}-\big(\lambda+\tfrac{s+t}{2}\big)\big(\lambda-p-\tfrac{s+t}{2}\big)\\
&\quad=\begin{pmatrix}PQ-(s-t)p & 0\\ (q-p)Q & QP\end{pmatrix}-\big(\lambda+\tfrac{s-t}{2}\big)\big(\lambda-p-\tfrac{s-t}{2}\big).
\end{aligned}$$

Then the system of the equations of the second order characterizing the image of the corresponding Poisson transform equals

$$(QP)_{i,j}u=\delta_{i,j}\big(\lambda+\tfrac{s-t}{2}\big)\big(\lambda-p-\tfrac{s-t}{2}\big)u\quad(1\le i,j\le q).\tag{4.11}$$

Note that the element Trace QP of $U(\mathfrak{g})$ defines a G-invariant differential operator on the homogeneous line bundle $\mathbf{E}_\ell$ over G/K with $\ell=s-t$, which is a constant multiple of the Laplace-Beltrami operator on $\mathbf{E}_\ell$.

Remark 4.5. The second order operators $(QP)_{i,j}$ in Example 4.4 are nothing but the so called Hua operators for $G=U(p,q)$. In the case of the trivial line bundle over G/K, that is the case of $s=t=0$, the fact that they characterize the image of the Poisson transform on the Shilov boundary was proved in [JK] for $p=q$ and $\lambda=p$, [Sn2] for $p=q$ and generic λ, [BV] for $p>q$ and $\lambda=p$, and [KZ] for $p>q$ and generic λ. Our result gives a further generalization to line bundles over G/K. Moreover the differential operators of order $2L$ corresponding to G/P_Ξ in Corollary 4.3 can be considered to be a generalization of the second order Hua operators corresponding to the Shilov boundary.

4.2. $Sp(n,\mathbb{R})$**.** We calculate the system of differential equations characterizing the image of the Poisson transform $\mathcal{P}^\ell_{\Xi,\mu}$ attached to the Shilov boundary of the symmetric space $Sp(n,\mathbb{R})/U(n)$ as in the case of the symmetric space $U(p,q)/U(p)\times U(q)$. Putting

$$\mathbb{F}=\begin{pmatrix}K & P\\ Q & -{}^tK\end{pmatrix}\quad\text{with}\quad\begin{cases}2K_{ij}=E_{ij}-E_{j+n,i+n},\\ 2P_{ij}=E_{i,j+n}+E_{j,i+n},\\ 2Q_{ij}=E_{i+n,j}+E_{j+n,i},\end{cases}$$

we have $\sum_{1\le i,\,j\le 2n}\mathbb{C}F_{ij}\simeq\mathfrak{sp}_n$.

$$
\begin{aligned}
[E_{ij}-E_{j+n,i+n},E_{k,\ell+n}+E_{\ell,k+n}] &= \delta_{jk}E_{i,\ell+n}+\delta_{j\ell}E_{i,k+n}+\delta_{j\ell}E_{k,i+n}+\delta_{jk}E_{\ell,i+n}\\
[K_{ij},P_{k\ell}] &= \tfrac12\delta_{jk}P_{i\ell}+\tfrac12\delta_{j\ell}P_{ik},\\
[K_{ij},Q_{k\ell}] &= -\tfrac12\delta_{ik}Q_{j\ell}-\tfrac12\delta_{i\ell}Q_{jk},\\
\sum_\nu K_{i\nu}P_{\nu j}-\sum_\nu P_{\nu j}K_{i\nu} &= \tfrac{n}{2}P_{ij}+\tfrac12 P_{ij}=\tfrac{n+1}{2}P_{ij},\\
\sum_\nu -K_{\nu i}Q_{\nu j}+\sum_\nu Q_{\nu j}K_{\nu i} &= \tfrac{n}{2}PQ_{ij}+\tfrac12 Q_{ij}=\tfrac{n+1}{2}Q_{ij},\\
\begin{pmatrix}K & P\\ Q & -{}^tK\end{pmatrix}\begin{pmatrix}\ell & P\\ Q & -\ell\end{pmatrix} &= \begin{pmatrix}\ell K+PQ & KP-\ell P\\ \ell Q-{}^tKQ & QP+\ell\,{}^tK\end{pmatrix}\\
&\equiv \begin{pmatrix}PQ+\ell^2 & \frac{n+1}{2}P\\ \frac{n+1}{2}Q & QP+\ell^2\end{pmatrix}\\
\mathbb{E}\big(\mathbb{E}-\tfrac{n+1}{2}\big) &\equiv \begin{pmatrix}PQ+\ell(\ell-\frac{n+1}{2}) & 0\\ 0 & QP+\ell(\ell+\frac{n+1}{2})\end{pmatrix}\\
\big(\mathbb{E}-\lambda\big)\big(\mathbb{E}+\lambda-\tfrac{n+1}{2}\big) &\equiv \begin{pmatrix}PQ-(n+1)\ell & 0\\ 0 & QP\end{pmatrix}\\
&\quad -\big(\lambda+\ell\big)\big(\lambda-\ell-\tfrac{n+1}{2}\big).
\end{aligned}
$$

Hence the system of the differential equations is

$$
\begin{cases}(PQ)_{i,j}u=\delta_{i,j}\big(\lambda-\ell\big)\big(\lambda+\ell-\frac{n+1}{2}\big)u & (1\le i,\,j\le n),\\ (QP)_{i,j}u=\delta_{i,j}\big(\lambda+\ell\big)\big(\lambda-\ell-\frac{n+1}{2}\big)u & (1\le i,\,j\le n).\end{cases}\tag{4.12}
$$

Remark 4.6. The second order operators $(PQ)_{i,j}$ and $(QP)_{i,j}$ are nothing but the so called Hua operators for $G=Sp(n,\mathbb{R})$. The fact that the equations (4.12) characterize the image of the Poisson transform on the Shilov boundary was proved by the second author [Sn3] for generic ℓ and λ. In the case of the trivial line bundle over G/K, that is the case of $\ell=0$, it was proved in [KM] for $n=2$ and $\lambda=\frac32$, [J1] for $\lambda=\frac{n+1}{2}$, and [Se] for generic λ.

4.3. $GL(n,\mathbb{R})$**.** Lastly we give a lemma which helps the calculation of the system of differential operators on the symmetric space $GL(n,\mathbb{R})_+/SO(n,\mathbb{R})$. Here $GL(n,\mathbb{R})_+=\{g\in GL(n,\mathbb{R})\,;\,\det g>0\}$.

Lemma 4.7 ($GL(n,\mathbb{R})$)**.** *Put*

$$
\mathbb{E}=K+P=\Big(K_{ij}+P_{ij}\Big)\quad\textit{with}\ \begin{cases}K_{ij}=\frac12(E_{ij}-E_{ji}),\\ P_{ij}=\frac12(E_{ij}+E_{ji}),\end{cases}
$$

$$
\mathfrak{g}_{\mathbb{C}}=\sum_{i,\,j=1}^n\mathbb{C}E_{ij}\simeq\mathfrak{gl}_n,\quad\textit{and}\quad\mathfrak{k}_{\mathbb{C}}=\sum_{i,\,j=1}^n\mathbb{C}K_{ij}\simeq\mathfrak{o}_n.
$$

Then for $m=0,1,2\ldots$

$$
\begin{aligned}
KP^m &= \tfrac{n}{2}P^m-\tfrac12\operatorname{Trace}(P^m)+\sum_{\nu=1}^n(P^m)_{\nu j}K_{i\nu}\\
&\equiv \tfrac{n}{2}P^m-\tfrac12\operatorname{Trace}(P^m)\mod U(\mathfrak{g})\mathfrak{k},
\end{aligned}\tag{4.13}
$$

$$
(\mathbb{E}-\tfrac{n}{2})P^m\equiv P^{m+1}-\tfrac12\operatorname{Trace}(P^m),\tag{4.14}
$$

$$
P^m\equiv(\mathbb{E}-\tfrac{n}{2})^{m-1}\mathbb{E}+\tfrac12\sum_{k=2}^m(\mathbb{E}-\tfrac{n}{2})^{m-k}\operatorname{Trace}(P^{k-1}),\tag{4.15}
$$

$$
\operatorname{Trace}(P^m)\equiv\operatorname{Trace}\big((\mathbb{E}-\tfrac{n-1}{2})^{m-1}\mathbb{E}\big).\tag{4.16}
$$

Proof. Since

$$\begin{aligned}
[E_{ij}, E_{kl}] &= \delta_{jk}E_{il} - \delta_{li}E_{kj},\\
[E_{ij}, E_{kl} + E_{kl}] &= \delta_{jk}E_{il} - \delta_{li}E_{kj} + \delta_{jl}E_{ik} - \delta_{ki}E_{lj},\\
[E_{ij} - E_{ji}, E_{kl} + E_{kl}] &= \delta_{jk}E_{il} - \delta_{li}E_{kj} + \delta_{jl}E_{ik} - \delta_{ki}E_{lj}\\
&\quad - \delta_{ik}E_{jl} + \delta_{lj}E_{ki} - \delta_{il}E_{jk} + \delta_{kj}E_{li}\\
&= 2(\delta_{jk}P_{il} + \delta_{jl}P_{ik} - \delta_{il}P_{jk} - \delta_{ik}P_{jl})\\
[K_{ij}, P_{kl}] &= \tfrac{1}{2}(\delta_{jk}P_{il} + \delta_{jl}P_{ik} - \delta_{ik}P_{jl} - \delta_{il}P_{jk}),\\
\sum_{\mu,\nu}(P^p)_{\mu\nu}K_{i\mu}(P^q)_{\nu j} - &\sum_{\mu,k}(P^{p+1})_{\mu k}K_{i\mu}(P^{q-1})_{kj}\\
&= \sum_{\mu,\nu,k}(P^p)_{\mu\nu}[K_{i\mu}, P_{\nu k}](P^{q-1})_{kj}\\
&= \tfrac{1}{2}\sum_{\mu,\nu,k}(P^p)_{\mu\nu}(\delta_{\mu\nu}P_{ik} + \delta_{\mu k}P_{i\nu} - \delta_{i\nu}P_{\mu k} - \delta_{ik}P_{\mu\nu})(P^{q-1})_{kj}\\
&= \tfrac{1}{2}(\operatorname{Trace} P^p)(P^q)_{ij} - \tfrac{1}{2}(\operatorname{Trace} P^{p+1})(P^{q-1})_{ij},
\end{aligned}$$

we have (4.13) by the sum of these equations for $p = m - q = 0, \dots, m$. Moreover (4.14) follows from (4.13). Then (4.14) proves (4.15) by the induction on m and (4.16) corresponds to the trace of the matrices in (4.15). □

Remark 4.8. Our study of characterizing the images of Poisson transform for general boundaries of a symmetric space by two sided ideals is originated in [O4] for the boundaries of $GL(n,\mathbb{R})_+/SO(n)$, where generators of the ideals that are different from minimal polynomials are constructed. The ideal spanned by the components of $\gamma\big(f(\mathbb{E})\big)$ for any polynomial $f(x)$ is calculated by [O6, Theorem 4.19] for the symmetric space $GL(n,\mathbb{C})/U(n)$.

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