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Holomorphic curves into the product space of the Riemann spheres

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Contents

1.	Introduction	1
2.	Notation and Preliminaries	3
3.	Meromorphic connection and Blowing-up	4
4.	Proof of the bigness of $\widetilde{D}' + \widetilde{D}''$	9
5.	Proof of the main theorem	11
References		19

1. Introduction

The porpose of this paper is to prove the second main theorem for a holomorphic map from the complex plane $\mathbb{C}$ to the product space of the one-dimentional projective spaces $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $[X_0 : X_1]$ and $[Y_0 : Y_1]$ be the homogenious cordinates in the first and second factors of the product space of the $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let m', n', m'', n'' be positive integers. We define the effective divisors D', D'' on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ by the polynomials $X_0^{m'} Y_0^{n'} - X_1^{m'} Y_1^{n'}$, $X_0^{m''} Y_1^{n''} - X_1^{m''} Y_0^{n''}$. We prove the second main theorem for divisors D' and D''.

The second main theorem for hyperplanes in $\mathbb{P}^n(\mathbb{C})$ is proved by Cartan [1]. The case of non-linear hypersurfaces had been studied by many authors. Although $\mathbb{P}^2(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ are birationally equivallent, the second main theorem for $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ has not been obtained.

Let $i : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the inclusion map where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then Z_k is the compactification of semi-Abelian variety $\mathbb{C}^* \times \mathbb{C}^*$.

In Noguchi, Winkelmann and Yamanoi [5], [6], the second main theorem for a holomorphic map f from $\mathbb{C}$ to a semi-Abelian variety A with D is proved, where D is an effective reduced divisor on A.

Theorem 1 ([6])**.** *Let $f : \mathbb{C} \to A$ be a holomorphic map such that the image of f is Zariski dense in A. There is the compactification of A such that $\overline{A}$ is smooth, equivalent with respect to the A-action, independent of f, and it follows that*

$$T_f(r, [\overline{D}]) \leq N_1(r, f^* D) + \varepsilon T_f(r, [\overline{D}])\|_\varepsilon,$$

for all $\varepsilon > 0$, where $\overline{D}$ is the closure of D in $\overline{A}$.

If $A = \mathbb{C}^* \times \mathbb{C}^*$, and $D = D' + D''$ in Theorem 1, our main theorem deals with the holomorphic curves into $\overline{A}$.

Now we state our main theorem precisely. Let $H_{1,0}$, $H_{1,1}$, $H_{2,0}$ and $H_{2,1}$ be the hyperplanes in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which are defined by the monomials X_0, X_1, Y_0 and Y_1. Put $Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Then there exists the sequence of the blowing-up

$$\begin{gathered} \pi_{1,0} : Z_1 \to Z_0, \\ \pi_{2,1} : Z_2 \to Z_1, \\ \vdots \\ \pi_{k,k-1} : Z_k \to Z_{k-1}. \end{gathered}$$

which satisfies the following condition $(*)$:

Put $\pi_{j,i} = \pi_{i+1,i} \circ \cdots \circ \pi_{j,j-1}$ for $i < j$. Let $\widetilde{D}'$, $\widetilde{D}''$ and $\widetilde{H}_{i,j}$, $1 \le i \le 2, 0 \le j \le 1$ be the proper transform of D', D'', and $H_{i,j}$ under $\pi_{k,0}$. Let E_i, $1 \le i \le k$ be the exceptional divisor of the blowing-up $\pi_{i,i-1}$, and let $\widetilde{E}_i$ be the proper transform of E_i under $\pi_{k,i}$. Then

$(*)$ $\quad \widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{2} \sum_{j=0}^{1} \widetilde{H}_{i,j} + \sum_{i=1}^{k} \widetilde{E}_i \quad$ is simple normal crossing in Z_k.

Our goal is the following theorem.

Theorem 2 (Main Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map. Let $\widetilde{f} : \mathbb{C} \to Z_k$ be the lift of f. Assume that*

$$f(\mathbb{C}) \not\subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \quad | \quad C_0 X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\},$$

for all $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and all $(C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, and assume that there exist no holomorphic functions g_1, g_2 on $\mathbb{C}$ and no $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$f = (\exp g_1, \exp g_2),$$

$$a g_1 + b g_2 = (\text{constant}),$$

on $\mathbb{C}$. Then $\widetilde{D}' + \widetilde{D}''$ is a big divisor on Z_k, and it follows that

$$\begin{aligned} T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \le\ & N_2(r, \widetilde{f}^* \widetilde{D}') + N_2(r, \widetilde{f}^* \widetilde{D}'') \\ & + 2 \sum_{i=1}^{2} \sum_{j=0}^{1} N_1(r, \widetilde{f}^* \widetilde{H}_{i,j}) + 2 \sum_{i=1}^{k} N_1(r, \widetilde{f}^* \widetilde{E}_i) + S_f(r), \end{aligned}$$

where $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$. Here "$\|$" means that the inequality holds for all $r \in (0, +\infty)$ possibly except for subset with finite Lebesgue measure.

In Siu [7], a meromorphic connection is used to prove the second main theorem. In this paper, we also use the meromorphic connection on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Because $\mathbb{C}^* \times \mathbb{C}^*$ is a Lie group, there exists the canonical connection on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. We extend this connection to the meromorphic connection on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This connection does not "vary" under the blowing-up. This meromorphic connection plays an important role in our arguments.

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2. Notation and Preliminaries

We introduce some functions which play an important role in the Nevanlinna theory. Let E be an effective divisor on $\mathbb{C}$. We write $E = \sum m_j P_j$, where $\{P_j\}$ is a set of discrete points in $\mathbb{C}$ and m_j are positive integers. Put $n_k(r, E) = \sum_{|P_j|<r} \min\{k, m_j\}$. We define the counting function of E by

$$N_k(r, E) = \int_1^r \frac{n_k(t, E)}{t} dt.$$

Let X be a complex projective algebraic manifold, and let D be divisors on X. Let $[D]$ be the holomorphic line bundle on X which is defined by the divisor D, and let $\operatorname{supp} D$ be the support of D. Let $f : \mathbb{C} \to X$ be a non-constant holomorphic map. We define the proximity function of D by

$$m_f(r, D) = \int_0^{2\pi} \log \frac{1}{\|\sigma(f(re^{i\theta}))\|} \frac{d\theta}{2\pi},$$

where $\| \cdot \|$ is a Hermitian metric in L. Let $R(L, \| \cdot \|)$ be the curvature form of the metrized line bundle $(L, \| \cdot \|)$ representing the first Chern class. Then we define the characteristic function of L by

$$T_f(r, L) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* R(L, \| \cdot \|),$$

where $\Delta(t) = \{z \in \mathbb{C} \,|\, |z| < t\}$. We set $T_f(r) = T_f(r, L)$ if L is an ample line bundle on X. The equation

$$T_f(r, L) = N(r, f^* D) + m_f(r, D) + O(1)$$

is called the First Main Theorem (cf. Noguchi and Ochiai [4], Chapter V, §2). If $X = \mathbb{P}^1(\mathbb{C})$, f is a meromorphic function on $\mathbb{C}$. Then we have the lemma on logarithmic derivative (cf. Noguchi and Ochiai [4], Chapter VI, §1)

$$\int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \le S_f(r),$$

where $\log^+ r = \max\{0, \log r\}$, and $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$. Here "$\|$" means that the inequality holds for all $r \in (0, +\infty)$ possibly except for a subset with finite Lebesgue measure.

The following lemma is also fundamental in Nevanlinna theory.

Lemma 1. *Let $h(r) > 0$ be a monotone increasing function in $r \ge 1$. Then, for arbitrary $\delta > 0$, we have*

$$\frac{dh(r)}{dr} \le (h(r))^{1+\delta} \|.$$

Proof. See Noguchi-Ochiai [4], Chapter V, §5. □

Let X be a complex projective algebraic manifold, and let Y be the smooth hypersurface of X. Let s be the holomorphic function on an open subset $U \subset X$ such that $Y \cap U = \{x \in U \,|\, s(x) = 0\}$. Let ∇ be a holomorphic connection on U. We write

$$\nabla^{(m)} = \overbrace{\nabla \circ \cdots \circ \nabla}^{m-\text{times}}.$$

Then the following lemma holds (see the proof of Lemma 11.13. of J.-P. Demailly [2]).

Lemma 2. *Let $f : \mathbb{C} \to U$ be a holomorphic function. Assume that Y is totally geodesic with respect to ∇ on U. Then there exist holomorphic functions $h_0, h_1, \cdots, h_m$ on U such that*

$$ds \cdot \nabla_{f'}^{(m)} f'(z) = h_0(f(z)) s \circ f(z) + \sum_{i=1}^{m} h_i(f(z)) \frac{d^i (s \circ f)}{dz^i}(z) + \frac{d^{m+1}(s \circ f)}{dz^{m+1}}(z),$$

for $z \in f^{-1}(U)$.

Let X and $\widetilde{X}$ be n-dimensional complex projective algebraic manifolds. Let $\pi : \widetilde{X} \to X$ be a surjective holomorphic map. Then there exists a proper subvariety S of X such that $\widetilde{X} \setminus \pi^{-1}(S)$ and $X \setminus S$ are locally biholomorphic. Let ∇ be a meromorphic connection on X. Let V be a small neighborhood of $p \in \widetilde{X}$, and let u, v be holomorphic vector fields on a small neighborhood V of p. Then $V \setminus \pi^{-1}(S)$ is locally biholomorphic with $\pi(V) \setminus S$. We define the meromorphic connection $\pi^* \nabla$ on $\widetilde{X} \setminus \pi^{-1}(S)$ by

$$(\pi^* \nabla)_u v|_{V \setminus \pi^{-1}(S)} = (\pi_*|_{V \setminus \pi^{-1} S})^{-1} \, \nabla_{\pi_* u} \pi_* v.$$

Then the meromorphic vector field $(\pi^* \nabla)_u v$ on $V \setminus \pi^{-1} S$ is uniquely extended to the meromorphic vector field $(\pi^* \nabla_u v)$ on V. In this way, we define the meromorphic connection $\pi^* \nabla$ on $\widetilde{X}$.

3. Meromorphic connection and Blowing-up

Let $i : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the inclusion map. Then $\operatorname{supp} i^* D'$ is a subgroup of $\mathbb{C}^* \times \mathbb{C}^*$. Therefore there exists the canonical connection ∇ on $\mathbb{C}^* \times \mathbb{C}^*$ such that $\operatorname{supp} i^* D'$ is totally geodesic with respect to ∇. This connection is extended to the meromorphic connection on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. We also denote this extended connection by ∇. Let $U_{i,j} = \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_i \neq 0, Y_j \neq 0\}$, $0 \leq i, j \leq 1$. Take the canonical local cordinate system (z, w) on $U_{i,j} \simeq \mathbb{C} \times \mathbb{C}$. Then, the meromorphic connection ∇ is written by

$$d + \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix},$$

on $U_{i,j}$. It is easy to see that $\operatorname{supp} i^* D''$ is also totally geodesic with respect to ∇.

Lemma 3. *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map such that $f(\mathbb{C})$ is not contained in the support of $H_{i,j}$, $i = 1, 2$, $j = 0, 1$. Then f satisfies*

$$f' \wedge \nabla_{f'} f' \equiv 0,$$

if and only if f satisfies the following condition (i) or (ii):
(i)

$$f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \,|\, X_0^{r_1} Y_0^{r_2} - C X_1^{r_1} Y_1^{r_2} = 0\},$$

for some $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and some $C \in \mathbb{C} \setminus \{0\}$.

(ii)
There exist holomorphic functions g_1, g_2 on $\mathbb{C}$ and $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

$$f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),$$

$$a g_1 + b g_2 = (\text{constant}),$$

on $\mathbb{C}$.

Proof. Without loss of generality, we may assume that $f(0) \in \mathbb{C}^* \times \mathbb{C}^*$. The holomorphic map

$$\big(\exp(2\pi\sqrt{-1}\,\cdot\,), \exp(2\pi\sqrt{-1}\,\cdot\,)\big) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*,$$

is the univeral covering of $\mathbb{C}^* \times \mathbb{C}^*$. The induced connection on the covering space $\mathbb{C} \times \mathbb{C}$ by ∇ is the flat connection d. We put $f = (f_1, f_2)$ where f_1 and f_2 are meromorphic functions on $\mathbb{C}$. Let

$$h_i = \frac{1}{2\pi\sqrt{-1}} \log f_i, \quad i = 1, 2.$$

Assume that $f' \wedge \nabla_{f'} f' \equiv 0$. Then there exists a meromorphic function h on $\mathbb{C}$ such that

$$\begin{pmatrix} h_1''(z) \\ h_2''(z) \end{pmatrix} = h(z) \begin{pmatrix} h_1'(z) \\ h_2'(z) \end{pmatrix},$$

on $\mathbb{C}$.

This means that

$$h_i'(z) = h_i'(0) \exp H(z), \quad i = 1, 2,$$

in a simple connected neighborhood U of $0 \in \mathbb{C}$. Here

$$H(z) = \int_0^z h(t)dt.$$

If $h_i'(0) = 0$, it follows that h_i is a constant function. So $(h_1'(0), h_2'(0)) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$. It holds that

$$h_i(z) = h_i'(0) \int_0^z \exp H(t)dt + h_i(0), \quad i = 1, 2.$$

It follows that

$$h_2'(0)h_1(z) - h_1'(0)h_2(z) = h_2'(0)h_1(0) - h_1'(0)h_2(0).$$

Conversely, assume that there exist $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

$$ah_1(z) + bh_2(z) = (\text{constant}),$$

on $\mathbb{C}$. Then $ah_1'(z)+bh_2'(z) = 0, ah_1''(z)+bh_2''(z) = 0$. So it follows that $f' \wedge \nabla_{f'} f' \equiv 0$.

Therefore $f' \wedge \nabla_{f'} f' \equiv 0$ if and only if there exist $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

$$a \log f_1(z) + b \log f_2(z) = (\text{constant}),$$

on $\mathbb{C}$.

Assume that

$$a \log f_1(z) + b \log f_2(z) = c, \tag{1}$$

for some $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$, $c \in \mathbb{C}$. Without loss of generality, we may assume that $a = 1$. For $x \in \mathbb{C}$, we put

$$f_i(z) = (z - x)^{r_i} h_i(z), \quad i = 1, 2,$$

where $r_i \in \mathbb{Z}$, $h_i(z)$ is a holomorphic function on an open neighborhood of x such that $h_i(x) \neq 0$. Then, by (1), we have $r_1 + br_2 = 0$. When $r_2 \neq 0$ for some $x \in \mathbb{C}$, it follows that

$$\log(f_1(z))^{r_2} + \log(f_2(z))^{-r_1} = r_2 c.$$

Then it holds that the meromorphic function $(f_1(z))^{r_2}(f_2(z))^{-r_1}$ is a constant function on $\mathbb{C}$. This means that

$$f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid X_1^{r_2} Y_0^{r_1} - C X_0^{r_2} Y_1^{r_1} = 0\},$$

for $r_1 \in \mathbb{Z}$, $r_2 \in \mathbb{Z} \setminus \{0\}$, $C \in \mathbb{C}$. When $r_2 = 0$ for all $x \in \mathbb{C}$, we have $r_1 = 0$ for all $x \in \mathbb{C}$. This means that there exist holomorphic functions g_1, g_2 on $\mathbb{C}$ such that $f_i = \exp g_i$, $i = 1, 2$. Then $g_1 + b g_2 = c$.

Conversely, if f satisfies the condition (i) or (ii). It is easy to see that there exist $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

$$a \log f_1(z) + b \log f_2(z) = (\text{constant}),$$

on $\mathbb{C}$. $\square$

Remark 1. The condition of (b) in Lemma 4 does not mean the algebraical degeneracy of $f(\mathbb{C})$. For example, take

$$f(z) = (\exp z, \exp \sqrt{-1} z) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}).$$

The divisor

$$D' + D'' + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j},$$

is not simple normal crossing at $\{([0:1],[0:1]), ([0:1],[1:0]), ([1:0],[0:1]), ([1:0],[1:0])\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Put $Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $\pi_{1,0} : Z_1 \to Z_0$ be the blowing-up of Z_0 at the center $\{([0:1],[0:1]), ([0:1],[1:0]), ([1:0],[0:1]), ([1:0],[1:0])\}$. Let $D'_1, D''_1, H_{i,j,1}$ be the strict transform of $D', D'', H_{i,j}$ under $\pi_{1,0}$, and let E_1 be the exceptional divisor of $\pi_{1,0}$.

If the divisor

$$D'_1 + D''_1 + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,1} + E_1,$$

is not simple normal crossing in Z_1, we blow up Z_1 at the points where that divisor is not simple normal crossing. We repeat this process for several times. We put the l-th blowing-up $\pi_{l,l-1} : Z_l \to Z_{l-1}$. Let E_l be the exceptional divisor of $\pi_{l,l-1}$. We define

$$\pi_{j,i} = \pi_{i+1,i} \circ \pi_{i+2,i+1} \circ \cdots \circ \pi_{j,j-1},$$

for $i \leq j$ (we define $\pi_{i,i} = \mathrm{Id}$). Let D'_l, D''_l, $H_{i,j,l}$ be the strict transform of D', D'', $H_{i,j}$ under $\pi_{l,0}$, and let $E_{i,l}$, $1 \leq i \leq l$, be the strict transform of E_i under $\pi_{l,i}$.

Then there exists a positive integer k such that

$$D'_k + D''_k + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,k} + \sum_{i=1}^{k} E_{i,k},$$

is simple normal crossing. We put $\widetilde{D}' = D'_k$, $\widetilde{D}'' = D''_k$, $\widetilde{H}_{i,j} = H_{i,j,k}$, and $\widetilde{E}_i = E_{i,k}$.

Example 1. Let D', D'' be the divisor which are defined by the polynomials

$$X_0^2 Y_0 - X_1^2 Y_1, \quad X_0^3 Y_1^2 - X_1^3 Y_0^2.$$

Let $\pi_{1,0} : Z_1 \to Z_0$ be the blowing-up as above. Then $D'_1 + D''_1 + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,1} + E_1$ is not simple normal crossing at four points in Z_1. So $\pi_{2,1} : Z_2 \to Z_1$ is the blowing-up at these four points. We see that $D'_2 + D''_2 + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,2} + \sum_{i=1}^{2} E_{i,2}$

is not simple normal crossing at two points in Z_2. So $\pi_{3,2} : Z_3 \to Z_2$ is the blowing-up at these two points. Then $D'_2 + D''_2 + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,2} + \sum_{i=1}^{2} E_{i,2}$ is normal crossing.

Let E'_i and E''_i be irreducible components of E_i such that $\pi_{i,0}(\operatorname{supp} E'_i) \subset \operatorname{supp} D'$ and $\pi_{i,0}(\operatorname{supp} E''_i) \subset \operatorname{supp} D''$. Then $E_1 = E'_1 + E''_1$, $E_2 = E'_2 + E''_2$ and $E_3 = E''_3$. Let $\widetilde{E}'_i$ and $\widetilde{E}''_i$ be the proper transform of E'_i and E''_i. Then it follows that

$$\pi_{3,0}^* D' = \widetilde{D}' + \widetilde{E}'_1 + 2\widetilde{E}'_2,$$

and

$$\pi_{3,0}^* D' = \widetilde{D}'' + 2\widetilde{E}''_1 + 3\widetilde{E}''_2 + 6\widetilde{E}''_3.$$

Lemma 4. *There exist affine open covering $\{U_s^l\}_{1\le s\le N_l}$ of Z_l, for $0 \le l \le k$, such that every U_s^l satisfies the following five conditions:*
(i)

$$U_s^l \simeq \mathbb{C} \times \mathbb{C}.$$

Take the canonical local coordinate system (z, w) of U_s^l.
(ii)

$$\sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,l}|_{U_s^l} + \sum_{1\le i\le l} E_{i,l}|_{U_s^l} = (z) + (w),$$

on U_s^l.
(iii)

$$D'_l|_{U_s^l} = (z^{p'} - w^{q'}) \quad (\text{or} \quad (1 - z^{p'}w^{q'}) \quad \text{respectively}),$$

on U_s^l, where p' and q' are non-negative integers (p', q' may depend on l and s).
(iv)

$$D''_l|_{U_s^l} = (1 - z^{p''}w^{q''}) \quad (\text{or} \quad (z^{p''} - w^{q''}) \quad \text{respectively}),$$

on U_s^l, where p'' and q'' are non-negative integers (p'', q'' may depend on l and s).
(v)

$$\pi_{l,0}^* \nabla|_{U_s^l} = d + \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix},$$

on U_s^l.

Proof. We take affine open coverings $\{U_s^l\}_{1\le s\le N_l}$ by induction over l. For $l = 0$, we put $\{U_s^0\}_{1\le s\le 4} = \{U_{i,j}\}_{0\le i,j\le 1}$. Here $U_{i,j} = \{[X_0 : Y_0], [X_1 : Y_1] \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_i \neq 0, Y_j \neq 0\}$. Then $\{U_s^0\}_{1\le s\le 4}$ satisfies above five conditions. Assume that we take the affine open covering $\{U_s^{l-1}\}_{1\le s\le N_{l-1}}$ of Z_{l-1} for $l \le k$ which satisfies the above five conditions. Let $U_t^{l-1} \in \{U_s^{l-1}\}_{1\le s\le N_{l-1}}$. Take the canonical local coordinate system (z, w) of $U_t^{l-1} \simeq \mathbb{C} \times \mathbb{C}$.

If $D'_{l-1}|_{U_t^{l-1}} = (z^{p'} - w^{q'})$ for some positive integers p', q'. Then

$$D''_{l-1}|_{U_t^{l-1}} = (1 - z^{p''}w^{q''}),$$

for some non-negative integers p'', q''. The divisor

$$D'_{l-1} + D''_{l-1} + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,l-1} + \sum_{1\le i\le l-1} E_{i,l-1},$$

in Z_{l-1}, is not normal crossing at $(0,0) \in U_t^{l-1}$. Then $(0,0)$ is contained in the center of the blowing-up $\pi_{l,l-1}$. We have

$$\pi_{l,l-1}^{-1}(U_t^{l-1}) = \{((z,w),[W_0 : W_1]) \in U_t^{l-1} \times \mathbb{P}^1(\mathbb{C}) \,|\, zW_1 = wW_0\}.$$

Let $V_i = \{((z,w),[W_0 : W_1]) \in \pi_{l,l-1}^{-1}(U_t^{l-1}) \,|\, W_i \neq 0\}$, $i = 0,1$. Then $\{V_0, V_1\}$ are affine open covering of $\pi_{l,l-1}^{-1}(U_t^{l-1})$. We show that affine open sets V_0 and V_1 satisfy the five conditions of lemma. Let $u = W_1/W_0$ be the holomorphic function on V_0. Then (z,u) is the local cordinate system of V_0. It is easy to verify that V_0 satisfies (i), (ii), (iii) and (iv). Since

$$\pi_{l,l-1}^* z = z, \quad \pi_{l,l-1}^* w = zu,$$

we have

$$\pi_{l,l-1\,*}\left(\frac{\partial}{\partial z}\ \frac{\partial}{\partial u}\right) = \left(\frac{\partial}{\partial z}\ \frac{\partial}{\partial w}\right)\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}.$$

Let Γ be the connection form of $\pi_{l,l-1}^*\nabla|_{V_0}$ with respect to the frame $\partial/\partial z, \partial/\partial u$. Then it follows that

$$\Gamma = \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} d\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} \pi_{l,l-1}^*\begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{u}{z} & \frac{1}{z} \end{pmatrix},$$

we have

$$\begin{aligned}
\Gamma &= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -\frac{u}{z} & \frac{1}{z} \end{pmatrix}\begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dz}{z} - \frac{du}{u} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} -\frac{dz}{z} & 0 \\ \frac{u}{z}\frac{dz}{z} & -\frac{1}{z}\left(\frac{dz}{z} + \frac{du}{u}\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} -\frac{dz}{z} & 0 \\ \frac{u}{z}\frac{dz}{z} - \frac{u}{z}\left(\frac{dz}{z} + \frac{du}{u}\right) & -\frac{dz}{z} - \frac{du}{u} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{du}{u} \end{pmatrix}.
\end{aligned}$$

So V_0 satisfies (v). In the same way, we can show that V_1 satisfies the conditions of the lemma.

If

$$D'_{l-1}|_{U_t^{l-1}} = (1 - z^{p'}w^{q'}), \quad D''_{l-1}|_{U_t^{l-1}} = (z^{p''} - w^{q''}),$$

for some non-negative integers p', q' and some positive integers p'', q''. In the same way as above, we can take the affine open sets in $\pi_{l,l-1}^{-1}(U_t^{l-1})$ which satisfy the five conditions of the lemma.

If

$$D'_{l-1}|_{U_t^{l-1}} = (1 - z^{p'}w^{q'}), \quad D''_{l-1}|_{U_t^{l-1}} = (1 - z^{p''}w^{q''}),$$

for some non-negative integers p', q', p'', q''. Then $U_t^{l-1} \simeq \pi_{l,l-1}^{-1}(U_t^{l-1})$ because U_t^{l-1} does not contain the center of the blowing-up $\pi_{l,l-1}$. By the assumption of induction, the affine open subset $\pi_{l,l-1}^{-1}(U_t^{l-1})$ satisfies the five conditions of the lemma.

This completes the proof. □

4. Proof of the bigness of $\widetilde{D}' + \widetilde{D}''$

In this section, we show that $\widetilde{D}' + \widetilde{D}''$ is big in Z_k. We note that there exists the proof of the bigness for more general cases in Proposition 3.9. of [6].

To prove the bigness of the line bundle $\widetilde{D}' + \widetilde{D}''$, it is sufficient to show the following lemma (cf. Theorem 2.2.16. of R. Lazarsfeld [3]).

Lemma 5. *The divisor $\widetilde{D}' + \widetilde{D}''$ is nef and $(\widetilde{D}' + \widetilde{D}'')^2 > 0$.*

Proof. Because
$$(\widetilde{D}' + \widetilde{D}'')^2 = (\widetilde{D}')^2 + 2\widetilde{D}' \cdot \widetilde{D}'' + (\widetilde{D}'')^2,$$
it is enough to show that $(\widetilde{D}')^2 = (\widetilde{D}'')^2 = 0$ and $\widetilde{D}'$ and $\widetilde{D}''$ are nef. Without loss of generality, we may assume that $m' \le n'$. Let E'_1 be the reduced divisor on Z_1 such that
$$\pi^*_{1,0} D' = D'_1 + m' E'_1.$$
Let F' be the divisor on Z_k such that
$$\pi^*_{k,1} D'_1 = \widetilde{D}' + F'.$$
It follows that
$$\begin{aligned}
(\widetilde{D}')^2 &= (\pi^*_{k,0} D' - F' - m'\pi^*_{k,1} E'_1)^2 \\
&= (\pi^*_{k,0} D')^2 + (F')^2 + m'^2(\pi^*_{k,1} E'_1)^2 - 2\pi^*_{k,0} D' \cdot F' \\
&\quad + 2m' F' \cdot \pi^*_{k,1} E'_1 - 2m'\pi^*_{k,1} E'_1 \cdot \pi^*_{k,0} D' \\
&= 2m'n' + (F')^2 - 2m'^2 - 2(\widetilde{D}' + F' + m'\pi^*_{k,1} E'_1) \cdot F' \\
&\quad + 2m' F' \cdot \pi^*_{k,1} E'_1 - 2m'\pi^*_{k,1} E'_1 \cdot (\pi^*_{k,1} D'_1 + m'\pi^*_{k,1} E'_1) \\
&= 2m'n' - 2m'^2 - (F')^2 - 2\widetilde{D}' \cdot F' - 2m' D'_1 \cdot E'_1 - 2m'^2 (E'_1)^2.
\end{aligned}$$
Because $D'_1 \cdot E'_1 = 2m'$, we have
$$(\widetilde{D}')^2 = 2m'n' - 2m'^2 - (F')^2 - 2\widetilde{D}' \cdot F'. \tag{2}$$
If $m' = n'$, then $\widetilde{D}' = D'_1$, $F' = 0$ and we have $(\widetilde{D}')^2 = 0$.

Now we prove $(\widetilde{D}')^2 = 0$ by the induction over the positive integer $m' + n'$. Let E'_i, $i = 2, \cdots k$ be reduced effective divisors on Z_i such that
$$\operatorname{supp}(\pi^*_{i,i-1} D'_{i-1} - D'_i) = \operatorname{supp} E'_i.$$
Let $\widetilde{E}'_i$ be the strict transform of E'_i under $\pi_{k,i}$. There exist non-negative integers $a_2, a_3, \cdots, a_k$ such that
$$F = a_2 \widetilde{E}'_2 + a_3 \widetilde{E}'_3 + \cdots + a_k \widetilde{E}'_k.$$
Now we take another divisor A' on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is defined by the polynomial
$$X_0^{m'} Y_0^{n'-m'} - X_1^{m'} Y_1^{n'-m'}.$$
There is, as in Section 3, the sequence of the blowing-up
$$\begin{aligned}
\sigma_{1,0} \quad &: \quad W_1 \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \\
&\ \ \vdots \\
\sigma_{k-1,k-2} \quad &: \quad W_{k-1} \to W_{k-2},
\end{aligned}$$

such that the following condition $(**)$ satisfies:

Let S be the reduced divisor such that

$$\operatorname{supp}\left(\sigma_{k-1,0}^{*}\left(A'+\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j}\right)\right)=\operatorname{supp}S,$$

where $\sigma_{k-1,1}=\sigma_{1,0}\circ\cdots\circ\sigma_{k-1,k-2}$. Then
$(**)$ S is normal crossing in W_{k-1}.

Let B'_i be the exceptional divisor of $\sigma_{i,i-1}$, and let $\widetilde{B}'_i$ be the strict transform of B'_i under $\sigma_{i+1,i}\circ\cdots\circ\sigma_{k-1,k-2}$. Let $\widetilde{A}'$ be the strict transform of A' under $\sigma_{1,0}\circ\cdots\circ\sigma_{k-1,k-2}$. It follows that

$$(\sigma_{1,0}\circ\cdots\circ\sigma_{k-1,k-2})^{*}A'=\widetilde{A}'+a_2\widetilde{B}'_1+a_3\widetilde{B}'_2+\cdots+a_k\widetilde{B}'_{k-1},$$

and

$$\widetilde{E}'_i\cdot\widetilde{E}'_j=\widetilde{B}'_{i-1}\cdot\widetilde{B}'_{j-1},\qquad \widetilde{D}'\cdot\widetilde{E}'_i=\widetilde{A}'\cdot\widetilde{B}'_{i-1},$$

for all $2\leq i,j\leq k$. Put $G'=a_2\widetilde{B}'_1+a_3\widetilde{B}'_2+\cdots+a_k\widetilde{B}'_{k-1}$. We have

$$(F')^2=(G')^2,\qquad \widetilde{D}'\cdot F'=\widetilde{A}'\cdot G'. \tag{3}$$

It follows that

$$\begin{aligned}(\widetilde{A}')^2&=(\sigma_{k-1,0}^{*}A'-G')^2\\&=2m'(n'-m')-2\sigma_{k-1,0}^{*}A'\cdot G'+(G')^2\\&=2m'(n'-m')-2(\widetilde{A}'+G')\cdot G'+(G')^2\\&=2m'(n'-m')-(G')^2-2\widetilde{A}'\cdot G'.\end{aligned}$$

By the assumption of the induction, we have

$$(G')^2+2\widetilde{A}'\cdot G'-2m'(n'-m')=0. \tag{4}$$

. By (2), (3) and (4), it follows that

$$(\widetilde{D}')^2=2m'n'-2m'^2-(F')^2-2\widetilde{D}'\cdot F'=2m'(n'-m')-(G')^2-2\widetilde{A}'\cdot G'=0.$$

Then we complete the induction. By the same way, we can show that $(\widetilde{D}'')^2=0$.

Now we show that $\widetilde{D}'$ is nef. Let $m'=dp$, $n'=dq$, where d is the greatest common divisor of m' and n'. Then it follows that

$$X_0^{m'}Y_0^{n'}-X_1^{m'}Y_1^{n'}=\prod_{i=0}^{d-1}\left(X_0^pY_0^q-(\varepsilon_d)^iX_1^pY_1^q\right),$$

where $\varepsilon_d=\exp((2\pi\sqrt{-1})/d)$. Let C_i be the irreducible divisor on $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ which is defined by the polynomial $X_0^pY_0^q-(\varepsilon_d)^iX_1^pY_1^q$, and let $\widetilde{C}_i$ be the strict transform of C_i under $\pi_{k,0}$. By the above arguments, we have $(\widetilde{C}_0)^2=0$. Because $\widetilde{C}_0$ and $\widetilde{C}_i$, $1\leq i\leq d-1$, are linearly equivalent, we have

$$\widetilde{C}_i\cdot\widetilde{D}'=(\widetilde{C}_i)^2=(\widetilde{C}_0)^2=0.$$

Therefore D' is nef. By the same way, we can show that $\widetilde{D}''$ is nef. $\square$

5. Proof of the main theorem

Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the holomorphic map, let $\widetilde{f} : \mathbb{C} \to Z_k$ be the lift of f, and let $\widetilde{\nabla} = \pi_{k,0}^* \nabla$.

$$\begin{tikzcd}
Z_0 & Z_1 \arrow[l, "\pi_{1,0}"'] & Z_2 \arrow[l, "\pi_{2,1}"'] & \cdots \arrow[l, "\pi_{3,2}"'] & Z_k \arrow[l, "\pi_{k,k-1}"'] \\
\mathbb{C} \arrow[u, "f"] \arrow[urrrr, dashed, "\widetilde{f}"']
\end{tikzcd}$$

Let $\widetilde{\sigma}' \in \Gamma(Z_k, [\widetilde{D}'])$, $\widetilde{\sigma}'' \in \Gamma(Z_k, [\widetilde{D}''])$, $\widetilde{h}_{i,j} \in \Gamma(Z_k, [\widetilde{H}_{i,j}])$, $\widetilde{e}_i \in \Gamma(Z_k, \widetilde{E}_i)$ be the holomorphic section such that

$$(\widetilde{\sigma}') = \widetilde{D}', \quad (\widetilde{\sigma}'') = \widetilde{D}'', \quad (\widetilde{h}_{i,j}) = \widetilde{H}_{i,j}, \quad (\widetilde{e}_i) = \widetilde{E}_i.$$

Lemma 6. *Assume that*

$$f(\mathbb{C}) \not\subset \operatorname{supp}\left(D' + D'' + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j} \right),$$

and assume that

$$f' \wedge \nabla_{f'} f' \not\equiv 0.$$

Then it follows that

$$\int_{|z|=r} \log^+ \frac{\|\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}'(z)\|_{\bigwedge^2 TZ_k}}{\|\widetilde{\sigma}'(\widetilde{f})\|_{[\widetilde{D}']} \|\widetilde{\sigma}''(\widetilde{f})\|_{[\widetilde{D}'']} \prod_{i=1}^{2} \prod_{j=0}^{1} \|\widetilde{h}_{i,j}(\widetilde{f})\|_{[\widetilde{H}_{i,j}]} \prod_{i=1}^{k} \|\widetilde{e}_i(\widetilde{f})\|_{[\widetilde{E}_i]}} \frac{d\theta}{2\pi}$$
$$= S_f(r).$$

Proof. For the convinience of the notation, we assume that D' and D'' are irreducible. Put

$$A = \widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{2} \sum_{j=0}^{1} \widetilde{H}_{i,j} + \sum_{i=1}^{k} \widetilde{E}_i,$$

and put

$$\xi(z) = \frac{\|\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}'(z)\|_{\bigwedge^2 TZ_k}}{\|\widetilde{\sigma}'(\widetilde{f})\|_{[\widetilde{D}']} \|\widetilde{\sigma}''(\widetilde{f})\|_{[\widetilde{D}'']} \prod_{i=1}^{2} \prod_{j=0}^{1} \|\widetilde{h}_{i,j}(\widetilde{f})\|_{[\widetilde{H}_{i,j}]} \prod_{i=1}^{k} \|\widetilde{e}_i(\widetilde{f})\|_{[\widetilde{E}_i]}}.$$

Note that A is simple normal crossing in Z_k.

Let

$$x \in \bigcup_{i=1}^{2} \bigcup_{j=0}^{1} \operatorname{supp} \widetilde{H}_{i,j} \cap \bigcup_{i=1}^{k} \operatorname{supp} \widetilde{E}_l.$$

By Lemma 4, there exist an affine open neighborhood U_x of x and local cordinate system z_x, w_x on U_x which satisfies the five conditions of Lemma 4. We put

$$V_x = U_x \setminus \operatorname{supp}(\widetilde{D}' + \widetilde{D}''),$$

and put

$$\widetilde{f}_1 = z_x \circ \widetilde{f}, \quad \widetilde{f}_2 = w_x \circ \widetilde{f},$$

on $\widetilde{f}^{-1}(V_x)$. It follows that

$$\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'}\widetilde{f}' = \left(\widetilde{f}_1'\widetilde{f}_2'' - \widetilde{f}_1''\widetilde{f}_2' + \widetilde{f}_1'\widetilde{f}_2'\frac{\widetilde{f}_1'}{\widetilde{f}_1} - \widetilde{f}_1'\widetilde{f}_2'\frac{\widetilde{f}_2'}{\widetilde{f}_2} \right) \frac{\partial}{\partial z_x} \wedge \frac{\partial}{\partial w_x}.$$

on $\widetilde{f}^{-1}(V_x)$. Then it follows that

$$\xi(z) = \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1}\frac{\widetilde{f}_2''}{\widetilde{f}_2} - \frac{\widetilde{f}_1''}{\widetilde{f}_1}\frac{\widetilde{f}_2'}{\widetilde{f}_2} + \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1}\right)^2 \frac{\widetilde{f}_2'}{\widetilde{f}_2} - \frac{\widetilde{f}_1'}{\widetilde{f}_1}\left(\frac{\widetilde{f}_2'}{\widetilde{f}_2}\right)^2 \right) \Phi_x(f(z)), \tag{5}$$

on $\widetilde{f}^{-1}(V_x)$, where Φ_x is a smooth function on V_x.

Let

$$x \in \operatorname{supp} \widetilde{D}' \cap \left(\bigcup_{i=1}^{2}\bigcup_{j=0}^{1} \operatorname{supp} \widetilde{H}_{i,j} \cup \bigcup_{i=1}^{k} \operatorname{supp} \widetilde{E}_i \right)$$

$$\left(\text{or} \quad x \in \operatorname{supp} \widetilde{D}'' \cap \left(\bigcup_{i=1}^{2}\bigcup_{j=0}^{1} \operatorname{supp} \widetilde{H}_{i,j} \cup \bigcup_{i=1}^{k} \operatorname{supp} \widetilde{E}_i \right), \text{ respectively} \right)$$

By Lemma 4, there exists an affine open neighborhood U_x of x and local cordinate system z_x, w_x on U_x which satisfies the five condition of Lemma 4. Because D' and D'' are irreducible, it follows that

$$D'|_{U_x} = (z_x - 1) \quad (\text{ or } D''|_{U_x} = (z_x - 1), \text{ respectively}),$$

and $z_x(x) = 1, w_x(x) = 0$. We take $z_x' = z_x - 1$. Let V_x be an affine open subset of U_x such that

$$A|_{V_x} = (z_x') + (w_x),$$

and

$$\nabla|_{V_x} = d + \begin{pmatrix} -(dz_x')/(z_x'+1) & 0 \\ 0 & -(dw_x)/w_x \end{pmatrix}.$$

We note that $z_x'(x) + 1 \neq 0$ on $\widetilde{f}^{-1}(V_x)$. We put

$$\widetilde{f}_1 = z_x' \circ \widetilde{f}, \quad \widetilde{f}_2 = w_x \circ \widetilde{f},$$

on $\widetilde{f}^{-1}(V_x)$. It follows that

$$\begin{aligned} \xi(z) = {} & \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1}\frac{\widetilde{f}_2''}{\widetilde{f}_2} - \frac{\widetilde{f}_1''}{\widetilde{f}_1}\frac{\widetilde{f}_2'}{\widetilde{f}_2} - \frac{\widetilde{f}_1'}{\widetilde{f}_1}\left(\frac{\widetilde{f}_2'}{\widetilde{f}_2}\right)^2 \right) \Phi_x(f(z)) \\ & + \widetilde{f}_1'\frac{\widetilde{f}_1'}{\widetilde{f}_1}\frac{\widetilde{f}_2'}{\widetilde{f}_2}\Psi_x(f(z)), \end{aligned} \tag{6}$$

on $\widetilde{f}^{-1}(V_x)$, where Φ_x and Ψ_x are smooth functions on V_x.

Let $x \in \operatorname{supp} \widetilde{D}' \cap \operatorname{supp} \widetilde{D}''$. There exists an affine open neighborhood V_x of x and holomorphic functions z_x, w_x on V_x such that

$$\widetilde{D}'|_{V_x} = (z_x), \quad \widetilde{D}''|_{V_x} = (w_x),$$

$$A|_{V_x} = (z_x) + (w_x),$$

on V_x. It follows that dz_x and dw_x are linearly independent on V_x. We put

$$\widetilde{f}_1 = z_x \circ \widetilde{f}, \quad \widetilde{f}_2 = w_x \circ \widetilde{f}.$$

By Lemma 2, there exist holomorphic functions g_0, g_1, h_0, h_1 on V_x such that

$$dz_x \cdot \nabla_{\widetilde{f}'} \widetilde{f}'(\gamma) = g_0(\widetilde{f}(\gamma))\widetilde{f}_1(\gamma) + g_1(\widetilde{f}(\gamma))\widetilde{f}_1'(\gamma) + \widetilde{f}_1''(\gamma),$$

for all $\gamma \in \widetilde{f}^{-1}(V_x)$, and

$$dw_x \cdot \nabla_{\widetilde{f}'} \widetilde{f}'(\gamma) = h_0(\widetilde{f}(\gamma))\widetilde{f}_2(\gamma) + h_1(\widetilde{f}(\gamma))\widetilde{f}_2'(\gamma) + \widetilde{f}_2''(\gamma),$$

for all $\gamma \in \widetilde{f}^{-1}(V_x)$. It follows that

$$\begin{aligned}&\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}' \\ &= \left[\widetilde{f}_1'\left(h_0(\widetilde{f})\widetilde{f}_2 + h_1(\widetilde{f})\widetilde{f}_2' + \widetilde{f}_2''\right) - \widetilde{f}_2'\left(g_0(\widetilde{f})\widetilde{f}_1 + g_1(\widetilde{f})\widetilde{f}_1' + \widetilde{f}_2''\right)\right] \frac{\partial}{\partial z_x} \wedge \frac{\partial}{\partial w_x}.\end{aligned}$$

Then it follows that

$$\begin{aligned}\xi(z) &= \Phi_{x,1}(\widetilde{f})\frac{\widetilde{f}_1'}{\widetilde{f}_1} + \Phi_{x,2}(\widetilde{f})\frac{\widetilde{f}_2'}{\widetilde{f}_2} \\ &\quad + \Phi_{x,3}(\widetilde{f})\frac{\widetilde{f}_1'}{\widetilde{f}_1}\frac{\widetilde{f}_2'}{\widetilde{f}_2} + \Phi_{x,4}(\widetilde{f})\frac{\widetilde{f}_1'}{\widetilde{f}_1}\frac{\widetilde{f}_2''}{\widetilde{f}_2} + \Phi_{x,5}(\widetilde{f})\frac{\widetilde{f}_1''}{\widetilde{f}_1}\frac{\widetilde{f}_2'}{\widetilde{f}_2},\end{aligned} \tag{7}$$

on $\widetilde{f}^{-1}(V_x)$, where $\Phi_{x,1}, \dots, \Phi_{x,5}$ are smooth functions on V_x.

Let $R = \{x \in Z_k |\ x \text{ is contained in two irreducible components of } A\ \}$. Note that R is a finite subset of Z_k. For $x \in R$, we take affine open subset V_x and holomorphic functions z_x, w_x as above arguments. Then $\{V_x\}_{x\in R}$ is an open covering of Z_k. We take an open covering $\{V_x'\}_{x\in R}$ of Z_k such that $V_x' \subset V_x$ and V_x' is relatively compact in V_x. We take a partition of unity $\{\phi_x\}_{x\in R}$ which is subordinate to the covering $\{V_x'\}_{x\in R}$. Fix $x \in R$. Let $\widetilde{f}_1 = z_x \circ \widetilde{f}$, $\widetilde{f}_2 = w_x \circ \widetilde{f}$ be a holomorphic function on $\widetilde{f}^{-1}(V_x)$. Then $\widetilde{f}_1$ and $\widetilde{f}_2$ are extended to meromorphic functions on $\mathbb{C}$. By (5), (6) and (7), we have

$$\begin{aligned}&\int_{|z|=r} \phi_i(\widetilde{f}(z)) \log^+ \xi(z)\frac{d\theta}{2\pi} \\ &\leq \int_{|z|=r} \Gamma(\widetilde{f}(z))\frac{d\theta}{2\pi} + 4\sum_{i=1}^{2}\int_{|z|=r} \log^+ \frac{|\widetilde{f}_i'(z)|}{|\widetilde{f}_i(z)|}\frac{d\theta}{2\pi} \\ &+ \sum_{i=1}^{2}\int_{|z|=r} \log^+ \frac{|\widetilde{f}_i''(z)|}{|\widetilde{f}_i(z)|}\frac{d\theta}{2\pi} + \int_{|z|=r} \log^+ |\widetilde{f}_1'(z)|\frac{d\theta}{2\pi},\end{aligned}$$

where Γ is a bounded smooth function on Z_k. By using the lemma on logarithmic derivative, it follows that

$$\int_{|\gamma|=r} \log^+ \frac{|\widetilde{f}_i'(\gamma)|}{|\widetilde{f}_i(\gamma)|}\frac{d\theta}{2\pi} \leq S_{\widetilde{f}}(r).$$

It holds that

$$\begin{aligned}\int_{|z|=r} \log^+ |\widetilde{f}_1'(z)|\frac{d\theta}{2\pi} &= \frac{1}{2}\int_{|z|=r} \log^+ |\widetilde{f}_1'(z)|^2\frac{d\theta}{2\pi} \\ &\leq \frac{1}{2}\int_{|z|=r} \log^+ \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{d\theta}{2\pi} + O(1),\end{aligned}$$

where $\|\cdot\|_{TZ_k}$ is a hermitian metric of TZ_k. By Lemma 1 and the concavity of log, we have that

$$\begin{aligned}
&\frac{1}{2}\int_{|z|=r} \log^+ \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{d\theta}{2\pi} \\
&\le \frac{1}{2}\int_{|z|=r} \log\{\|\widetilde{f}'(z)\|^2_{TZ_k}+1\}\frac{d\theta}{2\pi} \\
&\le \frac{1}{2}\log\left(1+\int_{|z|=r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{d\theta}{2\pi}\right)+O(1) \\
&\le \frac{1}{2}\log\left(1+\frac{1}{2\pi r}\frac{d}{dr}\int_{|z|\le r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)+O(1) \\
&\le \frac{1}{2}\log\left(1+\frac{1}{2\pi r}\left(\int_{|z|\le r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{1+\delta}\right)+O(1)\| \\
&= \frac{1}{2}\log\left(1+\frac{r^\delta}{2\pi}\left(\frac{d}{dr}\int_1^r\frac{dt}{t}\int_{|z|\le r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{1+\delta}\right)+O(1)\| \\
&\le \frac{1}{2}\log\left(1+\frac{r^\delta}{2\pi}\left(\int_1^r\frac{dt}{t}\int_{|z|\le r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{(1+\delta)^2}\right)+O(1)\| \\
&\le S_f(r),
\end{aligned}$$

where δ is any positive number.

Because $\sum_{x\in R}\phi_x(\widetilde{f})=1$ on $\mathbb{C}$, it follows that

$$\int_{|z|=r}\log^+\xi(z)\frac{d\theta}{2\pi}=\sum_{x\in R}\int_{|z|=r}\phi_x(\widetilde{f}(z))\log^+\xi(z)\frac{d\theta}{2\pi}\le S_f(r).$$

□

The following lemma is useful.

Lemma 7. *It follows that*

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi^*_{k,0}H_{i,j}=\sum_{i=1}^{2}\sum_{j=0}^{1}\widetilde{H}_{i,j}+\sum_{i=1}^{k}\pi^*_{k,i}E_i+\sum_{i=1}^{k}\widetilde{E}_i.$$

Proof. Let the divisor $H_{i,j,l}$ on Z_l be the strict transform of $H_{i,j}$ under $\pi_{l,0}$, and let $E_{i,l}$, $i\le l$, be the strict transform of E_i under $\pi_{l,i}$, where $E_{l,l}=E_l$.

We show

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi^*_{l,0}H_{i,j}=\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,l}+\sum_{i=1}^{l}\pi^*_{l,i}E_i+\sum_{i=1}^{l}E_{i,l},$$

by induction over l. If $l=1$, we have

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi^*_{1,0}H_{i,j}=\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,1}+2E_i.$$

Therefore the statement of the induction holds for $l=1$. Assume that the statement holds for $l-1$, $1<l\le k$. Let C_i $i=1,2,\dots,r$ be irreducible divisor on Z_{l-1} such

that

$$\operatorname{supp}\left(\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l-1,0}^{*}H_{i,j}\right)=\bigcup_{i=1}^{r}\operatorname{supp}C_i.$$

There exist positive integers $a_1, a_2, \dots, a_r$ such that

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l-1,0}^{*}H_{i,j}=\sum_{i=1}^{r}a_iC_i.$$

By the assumption of the induction, we have

$$\sum_{i=1}^{l}\pi_{l,i}^{*}E_i=\sum_{i=1}^{r}(a_i-1)C_i.$$

Let $x \in Z_{l-1}$ be one of the points of the center of $\pi_{l,l-1}$, and let F_l be the irreducible component of E_l such that $\pi_{l,l-1}(\operatorname{supp}F_l)=x$. Assume that $x \in \operatorname{supp}C_p \cap \operatorname{supp}C_q$ for $1 \le p < q \le r$. Then the coefficients of F_l in $\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l,0}^{*}H_{i,j}$ is a_p+a_q, and the coefficient of F_l in $\sum_{i=1}^{l-1}\pi_{l,i}^{*}E_i$ is a_p+a_q-2. Therefore we have

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l,0}^{*}H_{i,j}-\sum_{i=1}^{l}\pi_{l,i}^{*}E_i=\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,l}+\sum_{i=1}E_{i,l}.$$

This complete the induction, and the lemma follows. $\square$

Proof of the Main Theorem. We put $W_{\widetilde{\nabla}}(\widetilde{f})=\widetilde{f}'\wedge\widetilde{\nabla}_{\widetilde{f}'}\widetilde{f}'$. We denote by $\operatorname{ord}_z g$ the order of zero of g, where g is a holomorphic section of a line bundle on a neighborhood of z. By (5), (6) and (7) in Lemma 6, it follows that

$$\begin{aligned}&\operatorname{ord}_z\left(\widetilde{\sigma}'(\widetilde{f})\widetilde{\sigma}''(\widetilde{f})\prod_{i=1}^{2}\prod_{j=0}^{1}\widetilde{h}_{i,j}(\widetilde{f})\prod_{i=1}^{k}\widetilde{e}_i(\widetilde{f})\right)-\operatorname{ord}_z\left(W_{\widetilde{\nabla}}(\widetilde{f})\right)\\&\le\min\{\operatorname{ord}_z\widetilde{\sigma}'(\widetilde{f}),2\}+\min\{\operatorname{ord}_z\widetilde{\sigma}''(\widetilde{f}),2\}\\&+2\sum_{i=1}^{2}\sum_{j=0}^{1}\min\{\operatorname{ord}_z\widetilde{h}_{i,j}(\widetilde{f}),1\}+2\sum_{i=1}^{k}\min\{\operatorname{ord}_z\widetilde{e}_i(\widetilde{f}),1\}.\end{aligned}$$

Therefore it follows that

$$\begin{aligned}&T_{\widetilde{f}}(r,K_{Z_k})+T_{\widetilde{f}}(r,[\widetilde{D}'+\widetilde{D}''])+\sum_{i=1}^{2}\sum_{j=0}^{1}T_{\widetilde{f}}(r,\widetilde{H}_{i,j})+\sum_{i=1}^{k}T_{\widetilde{f}}(r,\widetilde{E}_i)\\&\le N_2(r,\widetilde{f}^*\widetilde{D}')+N_2(r,\widetilde{f}^*\widetilde{D}'')+2\sum_{i=1}^{2}\sum_{j=0}^{1}N_1(r,\widetilde{f}^*\widetilde{H}_{i,j})\\&\quad+2\sum_{1\le i\le k}N_1(r,\widetilde{f}^*\widetilde{E}_i)+S_f(r),\end{aligned}\tag{8}$$

where K_{Z_k} is the canonical line bundle of Z_k. The canonical line bundle of Z_k is equal to

$$\pi_{k,0}^{*}K_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})}+\pi_{k,1}^{*}E_1+\pi_{k,2}^{*}E_2+\cdots+E_k.$$

By Lemma 7, it follows that

$$-T_f(r, K_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})}) = T_f(r,\mathcal{O}(2,2)) = \sum_{i=1}^{2}\sum_{j=0}^{1} T_{\widetilde{f}}(r, \pi_{k,0}^* H_{i,j}) \tag{9}$$
$$= \sum_{i=1}^{2}\sum_{j=0}^{1} T_{\widetilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \pi_{k,i}^* E_i) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \widetilde{E}_i)$$

By (8), (9), it follows that

$$T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \leq N_2(r, \widetilde{f}^*\widetilde{D}') + N_2(r, \widetilde{f}^*\widetilde{D}'') + 2\sum_{i=1}^{2}\sum_{j=0}^{1} N_1(r, \widetilde{f}^*\widetilde{H}_{i,j}) + 2\sum_{i=1}^{k} N_1(r, \widetilde{f}^*\widetilde{E}_i) + S_f(r).$$

By Lemma 4 and Lemma 5, our main theorem follows. □

Corollary 1. *Let $f : \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant map. Assume that*

$$f(\mathbb{C}) \not\subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \,|\, C_0 X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\},$$

for all $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and all $(C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, and assume that there exist no $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$a \log f_1 + b \log f_2 = (\text{constant}),$$

on $\mathbb{C}$. Then it follows that

$$T_{\widetilde{f}}(r, [\widetilde{D}]) \leq N_2(r, f^*D') + N_2(r, f^*D'') + S_f(r).$$

Proof. Because $N_2(r, \widetilde{f}^*\widetilde{H}_{i,j}) = 0$ and $N_2(r, \widetilde{f}^*\widetilde{E}_i) = 0$, we have the corollary. □

Example 2. Let D', D'' be the divisor which are defined by the polynomials

$$X_0Y_0 - X_1Y_1, \quad X_0Y_1 - X_1Y_0.$$

Then

$$D_1' + D_1'' + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,1} + E_1,$$

is normal crossing in Z_1. Therefore $\widetilde{D}' = D_1'$, $\widetilde{D}'' = D_1''$. Let $E_{(0,0)}, E_{(0,\infty)}, E_{(\infty,0)}, E_{(\infty,\infty)}$ be irreducible components of E_1 such that

$$\pi_{1,0}(\operatorname{supp} E_{(0,0)}) = ([0:1],[0:1]), \quad \pi_{1,0}(\operatorname{supp} E_{(0,\infty)}) = ([0:1],[1:0]),$$
$$\pi_{1,0}(\operatorname{supp} E_{(\infty,0)}) = ([1:0],[0:1]), \quad \pi_{1,0}(\operatorname{supp} E_{(\infty,\infty)}) = ([1:0],[1:0]).$$

Let $f = (f_1, f_2) : \mathbb{C} \to Z_0$ be a non-constant holomorphic map, and let $\widetilde{f} : \mathbb{C} \to Z_1$ be the lift of f. It follows that

$$T_{\widetilde{f}}(r, [\widetilde{D}']) = T_{\widetilde{f}}(r, [\pi_{1,0}^* D']) - T_{\widetilde{f}}(r, [E_{(0,\infty)}]) - T_{\widetilde{f}}(r, [E_{(\infty,0)}]),$$

and

$$T_{\widetilde{f}}(r, [\pi_{1,0}^* D']) = T_f(r, \mathcal{O}(1,1)) = T(r, f_1) + T(r, f_2),$$

where

$$T(r, f_i) = \int_{|z|=r} \log^+ |f_i| \frac{d\theta}{2\pi} + N(r, (f_i)_\infty),$$

for $i = 1, 2$. By the first main theorem, we have

$$T_{\widetilde{f}}(r, E_{(0,\infty)}) = N(r, \widetilde{f}^* E_{(0,\infty)}) + m_{\widetilde{f}}(r, E_{(0,\infty)}),$$

$$T_{\widetilde{f}}(r, E_{(\infty,0)}) = N(r, \widetilde{f}^* E_{(\infty,0)}) + m_{\widetilde{f}}(r, E_{(\infty,0)}).$$

It holds that

$$m_{\widetilde{f}}(r, E_{(0,\infty)}) = \int_{|z|=r} \log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} \frac{d\theta}{2\pi},$$

and

$$m_{\widetilde{f}}(r, E_{(\infty,0)}) = \int_{|z|=r} \log^+ \frac{1}{\sqrt{|f_1^{-1}|^2 + |f_2|^2}} \frac{d\theta}{2\pi}.$$

By these equations, we have

$$\begin{aligned} T_{\widetilde{f}}(r, \widetilde{D}') = {} & N(r, (f_1)_\infty) + N(r, (f_2)_\infty) - N(r, \widetilde{f}^* E_{(0,\infty)}) - N(r, \widetilde{f}^* E_{(\infty,0)}) \\ & + \int_{|z|=r} \left(\log^+ |f_1| + \log^+ |f_2|\right) \frac{d\theta}{2\pi} \\ & - \int_{|z|=r} \left(\log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} + \log^+ \frac{1}{\sqrt{|f_1^{-1}|^2 + |f_2|^2}} \right) \frac{d\theta}{2\pi} \end{aligned}$$

Let $f_1 = P(z), f_2 = \exp z$, where $P(z)$ is a polynomial of degree p on $\mathbb{C}$. Then $T(r, f_1) = p \log r + O(1)$, and $T(r, f_2) = |r| + O(1)$. Because

$$\log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} \leq \log^+ \frac{1}{|f_1|},$$

it follows that

$$m_{\widetilde{f}}(r, E_{(0,\infty)}) \leq T(r, f_1^{-1}) = T(r, f_1) + O(1) = p \log |r| + O(1).$$

So we have

$$m_{\widetilde{f}}(r, E_{(0,\infty)}) = o(r).$$

By the same arguments, we have

$$m_{\widetilde{f}}(r, E_{(\infty,0)}) = o(r).$$

Then it holds that

$$T_{\widetilde{f}}(r, \widetilde{D}') = r + o(r).$$

Let D' and D'' be divisors on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which are defined by the polynomials

$$X_0^m Y_0^n - X_1^m Y_1^n, \quad X_0^n Y_1^m - X_1^n Y_0^m.$$

(i,e,. $m = m' = n''$ and $n = n' = m''$.) We have the following theorem.

Theorem 3. *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be non-constant holomorphic map such Let $\widetilde{f} : \mathbb{C} \to Z_k$ be the lift of f. Assume that*

$$f(\mathbb{C}) \not\subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \,|\, C_0 X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\},$$

for all $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and all $(C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, and assume that there exists no holomorphic functions g_1, g_2 on $\mathbb{C}$ and no $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),$$
$$ag_1 + bg_2 = (\text{constant}),$$

on $\mathbb{C}$. Then it follows that

$$\left(1 - \frac{4}{m+n}\right) T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \leq N_2(r, \widetilde{f}^* \widetilde{D}') + N_2(r, \widetilde{f}^* \widetilde{D}'') + S_f(r).$$

Proof. Let $a_1 = \min\{m, n\}$. It follows that

$$\pi_{1,0}^*(D' + D'') = D_1' + D_1'' + a_1 E_1,$$

on Z_1, where D_1' and D_1'' are proper transform of D' and D'' under $\pi_{1,0}$. Let $a_2 = \min\{\max\{m,n\} - a_1, a_1\} \leq a_1$. It follows that

$$\pi_{2,0}^*(D' + D'') = D_2' + D_2'' + a_2 E_2 + a_1 \pi_{2,1}^* E_1,$$

on Z_2, where D_2' and D_2'' are proper transform of D' and D'' under $\pi_{2,0}$. Repeating this process, there exist positive integers $a_3 \ \cdots, a_k$ such that

$$\pi_{k,0}^*(D' + D'') = \widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{k} a_i \pi_{k,i}^* E_i.$$

Without loss of generality, we may assume that $m \leq n$. Then it holds that $m \geq a_1 \geq a_2 \geq \cdots \geq a_k$. It follows that

$$T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \geq T_{\widetilde{f}}(r, \pi_{k,0}^* \mathcal{O}(m+n, m+n)) - m \sum_{i=1}^{k} T_{\widetilde{f}}(r, \pi_{k,i}^* E_i).$$

By Lemma 7, we have

$$T_{\widetilde{f}}(r, \pi_{k,0}^* \mathcal{O}(2,2)) = \sum_{i=1}^{2} \sum_{j=0}^{1} T_{\widetilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \pi_{k,i}^* E_i) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \widetilde{E}_i)$$

Then we have

$$\begin{aligned}
&T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \\
&\geq \frac{m+n}{2} \left(T_{\widetilde{f}}(r, \pi_{k,0}^* \mathcal{O}(2,2)) - \sum_{i=1}^{k} T_{\widetilde{f}}(r, \pi_{k,i}^* E_i) \right) + \left(\frac{m+n}{2} - m \right) \sum_{i=1}^{k} T_{\widetilde{f}}(r, \pi_{k,i}^* E_i) \\
&\geq \frac{m+n}{2} \left(\sum_{i=1}^{2} \sum_{j=0}^{1} T_{\widetilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \widetilde{E}_i) \right).
\end{aligned}$$

By Theorem 2, it follows that

$$T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \leq N_2(r, \widetilde{f}^* \widetilde{D}') + N_2(r, \widetilde{f}^* \widetilde{D}'')) + \frac{4}{m+n} T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) + S_f(r).$$

Then the theorem follows. □

Corollary 2. *Assume the hypothesis of Theorem 3, and assume that*

$$f(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \operatorname{supp}(D' + D'').$$

If $m + n \geq 5$, then it follows that $f(\mathbb{C}) \subset \operatorname{supp} H_{i,j}$ for $i = 1, 2$ and $j = 0, 1$.

Proof. Assume that $f(\mathbb{C})$ is not contained in the support of $\sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j}$. By Theorem 3, f satisfies the following condition (i) or condition (ii):

(i) $f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\}$,

for some $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and some $C_1 \in \mathbb{C} \setminus \{(0)\}$.

(ii) There exists holomorphic functions g_1, g_2 on $\mathbb{C}$ and $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

$$f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),$$
$$ag_1 + bg_2 = (\text{constant}),$$

on $\mathbb{C}$.

If f satisfies condition (i), without loss of generality, we may assume that $r_1 > 0, r_2 \geq 0$. Assume that $r_2 > 0$. Let R be an irreducible component of $\{X_0^{r_1} Y_0^{r_2} - CX_1^{r_1} Y_1^{r_2} = 0\}$. Then $([0 : 1], [1 : 0]), ([1 : 0], [0 : 1]) \in \operatorname{supp} R \cap \operatorname{supp} D'$, and $\operatorname{supp} R \cap \operatorname{supp} D''$ contains at least one point which is not $([0 : 1], [1 : 0])$ nor $([1 : 0], [0 : 1])$. Therefore the holomorphic map

$$f : \mathbb{C} \to \operatorname{supp} R \setminus \operatorname{supp}(D' + D'')$$

is a constant map.
Assume that $r_2 = 0$. We have

$$f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}(\mathbb{C}) \times \mathbb{P}(\mathbb{C}) \mid X_0^{r_1} - CX_1^{r_1} = 0\}.$$

Let S be an irreducible component of $\{X_0^{r_1} - CX_1^{r_1} = 0\}$. Because $m + n \geq 5$, m or n is more than 2, it follows that $\operatorname{supp} S \cap \operatorname{supp} D'$ or $\operatorname{supp} S \cap \operatorname{supp} D''$ contains at least three points. Then f is a constant map.

If f satisfies condition (ii), it is easy to see that f is a constant map. □

Remark 2. Let $x_{1,0} = ([0 : 1], [1 : 1]), x_{1,1} = ([1 : 0], [1 : 1]), x_{2,0} = ([1 : 1], [0 : 1]), x_{2,1} = ([1 : 1], [1 : 0]) \in Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $W = Z_0 \setminus \operatorname{supp} D' \cup \operatorname{supp} D''$, and let $W^* = W \setminus \{x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}\}$. By Corollary 2, there exist no non-constant holomorphic map from $\mathbb{C}$ to W^*.

Let $i : W^* \to W$ be the inclusion map, and let d_{W^*}, d_W be the Kobayashi pseudo distance of W^*, W (see Noguchi-Ochiai [4]). By Proposition 1.3.14. of [4], we have $i^* d_W = d_{W*}$. Therefore W^* is Brody hyperbolic but not Kobayashi hyperbolic.

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