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Lipschitz stability estimates in inverse source problems for a fractional diffusion equation of half order in time by Carleman estimates

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Abstract

In this article, we consider a fractional diffusion equation of half order in time. We study inverse problems of determining the space-dependent factor in the source term from additional data at a fixed time and interior or boundary data over an appropriate time interval. We establish the global Lipschitz stability estimates in the inverse source problems. Our methods are based on Carleman estimates. Here we prove and use the Carleman estimates for a fractional diffusion equation of half order in time.

1 Introduction

Let $T > 0$, $L > 0$, $\Omega = (0, L)$ and $\partial\Omega = \{0\} \cup \{L\}$. We set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. We consider the following one dimensional fractional diffusion equation of half order in time:

$$(\partial_t^{\frac{1}{2}} - L_2)u(x, t) = g(x, t), \quad (x, t) \in Q \tag{1.1}$$

where $\partial_t^{\frac{1}{2}}$ is the Caputo type fractional derivative defined by

$$\partial_t^{\alpha} u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_t u(x, s)}{(t-s)^{\alpha}} \, ds, \quad (x, t) \in Q$$

for $0 < \alpha < 1$ and

$$L_2 u(x, t) := \partial_x(a(x)\partial_x u(x, t)) + b(x)\partial_x u(x, t) + c(x)u(x, t), \quad (x, t) \in Q. \tag{1.2}$$

Here and Henceforth we use notations: $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$.
We always assume that $a \in C^3(\overline{\Omega})$, $b, c \in C^2(\overline{\Omega})$, and there exists a constant $\mu > 0$ such that

$$\frac{1}{\mu} \le a(x) \le \mu, \quad x \in \Omega.$$

In this article, we consider a fractional diffusion equation known as a macroscopic model for the anomalous diffusion in heterogeneous media. The study of the anomalous diffusion has various applications in existing problems such as environmental problems. For instance, the anomalous diffusion phenomena arise from the diffusion of contaminants underground. From field data, the diffusion in heterogeneous aquifer is unable to be explained by the classical diffusion equation and it is showed that this diffusion is slower than the classical diffusion by Adams and Gelhar [1]. In Hatano and Hantano [12], they discussed the continuous time random walk (CTRW) model as a microscopic model for the anomalous diffusion. A fractional diffusion equation is derived by the CTRW model with an argument similar to that used to obtain the classical diffusion equation from the random walk model (see [24]).

Our main results are composed of two parts. One is the Carleman estimate, the other is the conditional stability estimate in inverse source problems. We establish Carleman estimates for the 1-D half order time fractional diffusion equation (1.1). Here we derive two kinds of Carleman estimates: with interior data (Theorem 2.5) and with boundary data (Theorem 2.6). Then we consider the inverse problems of determining the space–dependent source factor $f(x)$ where $g(x,t) = f(x)R(x,t)$. Using Carleman estimates, we derive the Lipschitz type conditional stability estimates in our inverse source problems by the additional data at an arbitrarily fixed time and two types of observations: interior observations (Theorem 2.7) and boundary observations (Theorem 2.8).

The main techniques in the proof of the Carleman estimates for the fractional diffusion equation (1.1) are the integration by parts and the reduction from (1.1) to the following fourth order parabolic equation:

$$(\partial_t - L_2^2)u(x,t) = \widetilde{g}(x,t), \quad (x,t) \in Q. \tag{1.3}$$

This reduction is obtained in Xu, Cheng and Yamamoto [34]. By the integration by parts, we establish Carleman estimates for the fourth order parabolic equation and then we derive Carleman estimates for (1.1).

To establish the stability estimates in inverse source problems, we use the methodology based on Imanuvilov and Yamamoto [13], which use the global Carleman estimate to derive the global Lipschitz stability estimate in inverse source problems for parabolic equations. Their methodology is originated in the Bukhgeim-Klibanov method using the Carleman estimate to derive the global uniqueness in inverse problems (see [4, 15, 16, 17, 35] and references therein).

As a pioneer work for the Carleman estimates for a fractional diffusion equation of half order in time, we can refer to Xu, Cheng and Yamamoto [34] in 2011. Reducing $\partial_t^{\frac{1}{2}} - \partial_x^2$ to $\partial_t - \partial_x^4$, they established the local Carleman estimate for functions with compact support. Our Carleman estimates are motivated by their result. As a development for the local Carleman estimate, Cheng, Lin and Nakamura [5] treated the multi dimensional case in space. Moreover Lin and Nakamura [20] established the Carleman estimate with any time fractional order $\alpha \in (0,1)$. Related to our Carleman estimates, there are some previous researches for fourth order parabolic equations such as the Kuramoto-Sivashinsky equations. The Carleman estimate with boundary data for $\partial_t + L_4$ was proved in [3, 6, 23] where L_4 is the fourth order differential operator defined later in (2.2). The Carleman estimate with interior data for $\partial_t + \partial_x^4$ was firstly established in [39] and improved in [8, 9, 10, 11].

However the Carleman estimate with interior data for the fourth order parabolic equation with variable coefficients has not derived yet. In this article, we prove the Carleman estimate for $\partial_t - L_4$ with interior data (Lemma 2.3). As its application, we establish the Carleman estimate for the fractional diffusion equation (Theorem 2.5 and Theorem 2.6).

There are a lot of works for inverse source problems for fractional diffusion equations [2, 7, 18, 27, 28, 29, 31, 32, 33, 37, 38]. As theoretical approaches, for instance, Zhang and Xu [38] proved the uniqueness concerning the identification of the space–dependent source term from boundary data $u(0,t)$, $t \in (0,T)$ by the analytic continuation and the Laplace transform. Sakamoto and Yamamoto [28] established the Lipschitz stability in determining the space–dependent source factor from final overdetermining data $u(x,T)$. Liu, Rundell and Yamamoto [22] gave the uniqueness result in identifying the time–dependent source factor by using the strong maximum principle for fractional diffusion equations (see also [21]). Sakamoto and Yamamoto [27], Fujishiro and Kian [7] obtained the Lipschitz stability in the determination of the time–dependent factor from data $u(x_0,t)$ at fixed $x_0 \in \overline{\Omega}$ for any $t \in (0,T)$. Related to the inverse source problems, we also refer to some results for the coefficient inverse problems [14, 19, 25, 26, 30, 36]. Especially, Yamamoto and Zhang [36] investigated the inverse source problem of determining the space–dependent source factor as a step to prove the stability estimate in the coefficient inverse problem. For their inverse source problem, they derived the Hölder stability estimate by data $u(x,t_0)$ at fixed $t_0 \in (0,T)$ (see Lemma 1 in [36]) by using the Carleman estimate established in [34].

As for the conditional stability estimate in the determination of the space–dependent source factor $f(x)$, to the author's knowledge, the Lipschitz stability estimates by using interior or boundary observations have not established yet, although the estimates by final overdetermining data were studied (see e.g., [28, 29]), We will prove the Lipschitz type conditional stability estimates by the additional data at an arbitrarily fixed time and interior/boundary observations (Theorem 2.7/Theorem 2.8).

This article is organized as follows. Section 1 gives this introduction. In Section 2, we describe main results for the Carleman estimates and Lipschitz stability estimate. Section 3 contains proofs of main results. Section 4 is devoted to Appendix.

2 Main results

In this section we present our main results composed of Carleman estimates in subsection 2.1 and Lipschitz stability estimates for inverse source problems in subsection 2.2.

2.1 Carleman estimates for fractional diffusion equations

Firstly we state the reduction from the 1-D half order time fractional diffusion equation (1.1) to the fourth order parabolic equation (1.3).

In general, $\partial_t = \partial_t^{\frac{1}{2}}\partial_t^{\frac{1}{2}}$ does not hold for the Caputo type fractional derivative. Hence we need the following Lemma derived in Xu, Cheng and Yamamoto [34].

Lemma 2.1 (Lemma 2.2 in Xu, Cheng and Yamamoto [34])**.** *Let* $0 < \alpha + \beta \le 1$, $\alpha, \beta > 0$*. If* $\psi \in C[0,T] \cap W^{1,1}(0,T)$ *and* $\psi(0) = \partial_t^{\beta}\psi(0) = 0$*, then we have*

$$\partial_t^{\alpha}\partial_t^{\beta}\psi(t) = \partial_t^{\alpha+\beta}\psi(t), \quad t \in (0,T).$$

Noting that the coefficients of L_2 are independent of t and using the Lemma 2.1, we have the following Lemma.

Lemma 2.2. *If* $u \in C([0,T]; H^4(\Omega)) \cap C^1([0,T]; H^2(\Omega))$ *satisfies*

$$\begin{aligned} &(\partial_t^{\frac{1}{2}} - L_2)u(x,t) = g(x,t), && (x,t) \in Q, \\ &u(x,0) = 0, && x \in \Omega, \end{aligned}$$

then we have the equation (1.3)*:*

$$(\partial_t - L_2^2)u(x,t) = \widetilde{g}(x,t), \quad (x,t) \in Q,$$

where

$$\widetilde{g}(x,t) = (\partial_t^{\frac{1}{2}} + L_2)g(x,t) + \frac{g(x,0)}{\sqrt{\pi t}}, \quad (x,t) \in Q.$$

This Lemma is a modified Lemma 2.3 in [34]. Here we omit the proof of Lemma 2.2. Replacing ∂_x^2 in the proof of Lemma 2.3 in [34] by L_2, we may obtain Lemma 2.2.

Since $\partial_t - L_2^2$ is a fourth order parabolic operator, in general, we consider the following fourth order parabolic equation:

$$(\partial_t - L_4)u(x,t) = G(x,t), \quad (x,t) \in Q \tag{2.1}$$

where

$$\begin{aligned} L_4 u(x,t) := {} & a_0(x,t)\partial_x^4 u(x,t) + a_1(x,t)\partial_x^3 u(x,t) \\ & + a_2(x,t)\partial_x^2 u(x,t) + a_3(x,t)\partial_x u(x,t) + a_4(x,t)u(x,t), \quad (x,t) \in Q. \end{aligned} \tag{2.2}$$

We always assume that $a_0 \in C^1([0,T]; C^2(\overline{\Omega}))$, $a_k \in L^\infty(Q)$ $(k = 1,2,3,4)$ and that there exists a constant $m > 0$ such that

$$\frac{1}{m} \le a_0(x,t) \le m, \quad \text{for } (x,t) \in Q.$$

Remark 1. *Here we remark that coefficients of L_4 depend on x and t. In the proof of Carleman estimates, we are not required to assume that coefficients are independent of t. Carleman estimates for $\partial_t - L_4$ themselves have applications in various problems for fourth order parabolic equations such as Kuramoto-Sivashinsky equations. See [3, 6, 8, 9, 10, 11, 23, 39].*

For our Carleman estiamtes, we introduce weight functions φ_0, α_0 and φ_1, α_1, taking two kinds of distance functions d_0 and d_1. Let ω, ω_0 be an arbitrary fixed sub-domain of Ω such that $\overline{\omega} \subset \Omega$, $\overline{\omega_0} \subset \omega$. We take a distance function $d_0 \in C^4(\overline{\Omega})$ satisfying: there exists a positive constant $\sigma_0 > 0$ such hat

$$d_0(x) > 0, \quad x \in \Omega, \quad d_0(x) = 0, \quad x \in \partial\Omega, \quad |\partial_x d_0(x)| > \sigma_0, \quad x \in \overline{\Omega \setminus \omega_0}.$$

Moreover we choose another distance function $d_1 \in C^4(\overline{\Omega})$ satisfying: there exists a positive constant $\sigma_1 > 0$ such that

$$d_1(x) > 0, \quad x \in \Omega, \quad \partial_x d_1(x) \le -\sigma_1, \quad x \in \overline{\Omega}.$$

We set weight functions

$$\varphi_i(x,t) = \ell(t) e^{\lambda d_i(x)}, \quad \alpha_i(x,t) = \ell(t)\left(e^{\lambda d_i(x)} - e^{2\lambda \|d_i\|_{C(\overline{\Omega})}}\right) \tag{2.3}$$

for $i = 0, 1$, where $\ell(t) = \frac{1}{t(T-t)}$.

Now we ready to state Carleman estimates for (2.1).

Lemma 2.3 (Carleman estimate with interior data for (2.1))**.** *There exists $\lambda_0 > 0$ such that for any $\lambda \ge \lambda_0$, we can choose $s_0(\lambda) > 0$ satisfying: there exists $C = C(s_0, \lambda_0) > 0$ such that*

$$\int_Q \left[\frac{1}{s\varphi_0}\left(|\partial_t u|^2 + |\partial_x^4 u|^2\right) + s\lambda^2\varphi_0|\partial_x^3 u|^2 + s^3\lambda^4\varphi_0^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_0^5|\partial_x u|^2 + s^7\lambda^8\varphi_0^7|u|^2\right] e^{2s\alpha_0}\,dxdt$$
$$\le C\int_Q |(\partial_t - L_4)u|^2 e^{2s\alpha_0}\,dxdt + Ce^{C(\lambda)s}\int_{\omega\times(0,T)} |u|^2\,dxdt$$

for all $s > s_0$ and all $u \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;H^2(\Omega))$ satisfying $u(x,t) = \partial_x u(x,t) = 0$, $(x,t) \in \Sigma$.

Lemma 2.4 (Carleman estimate with boundary data for (2.1))**.** *There exists $\lambda_0 > 0$ such that for any $\lambda \ge \lambda_0$, we can choose $s_0(\lambda) > 0$ satisfying: there exists $C = C(s_0, \lambda_0) > 0$ such that*

$$\int_Q \left[\frac{1}{s\varphi_1}\left(|\partial_t u|^2 + |\partial_x^4 u|^2\right) + s\lambda^2\varphi_1|\partial_x^3 u|^2 + s^3\lambda^4\varphi_1^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_1^5|\partial_x u|^2 + s^7\lambda^8\varphi_1^7|u|^2\right] e^{2s\alpha_1}\,dxdt$$
$$\le C\int_Q |(\partial_t - L_4)u|^2 e^{2s\alpha_1}\,dxdt + Ce^{C(\lambda)s}\int_0^T \left(|\partial_x^3 u(0,t)|^2 + |\partial_x^2 u(0,t)|^2\right) dt$$

for all $s > s_0$ and all $u \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;H^2(\Omega))$ satisfying $u(x,t) = \partial_x u(x,t) = 0$, $(x,t) \in \Sigma$.

Remark 2. *In this article, we do not prove Lemma 2.4. The Carleman estimate with boundary data for $\partial_t + L_4$ is established by Meléndez (Proposition 3 in [23]). By the change of variables $\widetilde{u}(x,t) = u(x, T-t)$, the Carleman estimate for $(\partial_t + L_4)\widetilde{u}$ gives us Lemma 2.4 .*

In this article, we prove Lemma 2.3. Moreover using the proof of Lemma 2.3, we show the following our main Carleman estimates for the fractional diffusion equation of half order in time (1.1).

Theorem 2.5 (Carleman estimate with interior data for (1.1))**.** *Let $g \in L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))$. There exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, we can choose $s_0(\lambda) > 0$ satisfying: there exists $C = C(s_0, \lambda_0) > 0$ such that*

$$\int_Q \left[\frac{1}{s\varphi_0} \left(|\partial_t u|^2 + |\partial_x^4 u|^2 \right) + s\lambda^2\varphi_0|\partial_x^3 u|^2 + s^3\lambda^4\varphi_0^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_0^5|\partial_x u|^2 + s^7\lambda^8\varphi_0^7|u|^2 \right] e^{2s\alpha_0}\, dxdt$$
$$\leq C\int_Q |(\partial_t - L_2^2)u|^2 e^{2s\alpha_0}\, dxdt + Ce^{C(\lambda)s}\int_{\omega\times(0,T)} |u|^2\, dxdt$$

for all $s > s_0$ and all solution $u \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;H^2(\Omega))$ of (1.1), $u(x,0) = 0$, $x \in \Omega$, and $u(x,t) = \partial_x u(x,t) = 0$, $(x,t) \in \Sigma$.

Theorem 2.6 (Carleman estimate with boundary data for (1.1))**.** *Let $g \in L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))$ with $g(x,t) = \partial_x g(x,t) = 0$, $(x,t) \in \Sigma$. There exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, we can choose $s_0(\lambda) > 0$ satisfying: there exists $C = C(s_0, \lambda_0) > 0$ such that*

$$\int_Q \left[\frac{1}{s\varphi_1} \left(|\partial_t u|^2 + |\partial_x^4 u|^2 \right) + s\lambda^2\varphi_1|\partial_x^3 u|^2 + s^3\lambda^4\varphi_1^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_1^5|\partial_x u|^2 + s^7\lambda^8\varphi_1^7|u|^2 \right] e^{2s\alpha_1}\, dxdt$$
$$\leq C\int_Q |(\partial_t - L_2^2)u|^2 e^{2s\alpha_1}\, dxdt + Ce^{C(\lambda)s}\int_0^T \left(|\partial_x u(0,t)|^2 + |\partial_t^{\frac{1}{2}}\partial_x u(0,t)|^2 + |\partial_t\partial_x u(0,t)|^2 \right) dt$$
$$+ Ce^{C(\lambda)s}\int_0^T \left(|\partial_x u(L,t)|^2 + |\partial_t^{\frac{1}{2}}\partial_x u(L,t)|^2 + |\partial_t\partial_x u(L,t)|^2 \right) dt$$

for all $s > s_0$ and all the solution $u \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;H^2(\Omega))$ of (1.1), $u(x,0) = 0$, $x \in \Omega$ and $u(x,t) = 0$, $(x,t) \in \Sigma$.

2.2 Inverse source problems

We present the global Lipschitz stability estimate in inverse source problems by interior observations or boundary observations.

Let $t_0 \in (0,T)$ be arbitrarily fixed. We assume that

$$R \in C^1([0,T];C^2(\overline{\Omega})), \quad \partial_t^{\frac{1}{2}} R \in C^1([0,T];C(\overline{\Omega})), \quad |R(x,t_0)| > 0, \quad x \in \overline{\Omega}. \tag{2.4}$$

Taking $\delta > 0$ such that

$$0 < t_0 - \delta < t_0 < t_0 + \delta < T,$$

we set $Q_\delta = \Omega \times (t_0 - \delta, t_0 + \delta)$, $\Sigma_\delta = \partial\Omega \times (t_0 - \delta, t_0 + \delta)$, $Q_{\omega,\delta} = \omega \times (t_0 - \delta, t_0 + \delta)$ and we set weight functions

$$\varphi_{\delta,i}(x,t) = \ell_\delta(t)e^{\lambda d_i(x)}, \quad \alpha_{\delta,i}(x,t) = \ell_\delta(t)\left(e^{\lambda d_i(x)} - e^{2\lambda\|d_i\|_{C(\overline{\Omega})}}\right)$$

for $i = 0, 1$, where $\ell_\delta(t) = \frac{1}{(t-(t_0-\delta))(t_0+\delta-t)}$.

2.2.1 Interior observations

Let us consider an inverse source problem for the following fractional diffusion equation and initial boundary value conditions:

$$\partial_t^{\frac{1}{2}} u(x,t) - L_2 u(x,t) = f(x)R(x,t), \qquad (x,t) \in Q, \tag{2.5}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{2.6}$$

$$u(x,t) = \partial_x u(x,t) = 0, \qquad (x,t) \in \Sigma. \tag{2.7}$$

Inverse source problem by interior observations: Determine f by interior data $u(x,t)$, $(x,t) \in \omega \times (0,T)$ and data $u(x,t_0)$, $x \in \Omega$ where $\omega \subset \Omega$ is an open sub-domain such that $\overline{\omega} \subset \Omega$.

Now we ready to state the Lipschitz stability estimate in our inverse source problem.

Theorem 2.7. *We assume that* $u, \partial_t u \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;H^2(\Omega))$ *and* u *satisfies* (2.5)–(2.7), *and that* R *satisfies* (2.4). *Then we have*

$$\|f\|_{H^2(\Omega)} \le C\left(\|\partial_t^{\frac{1}{2}} u\|_{L^2(Q_{\omega,\delta})} + \|\partial_t u\|_{L^2(Q_{\omega,\delta})} + \|\partial_t \partial_t^{\frac{1}{2}} u\|_{L^2(Q_{\omega,\delta})}\right) + C\|u(\cdot,t_0)\|_{H^4(\Omega)}$$

for any $f \in H^2(\Omega)$ *satisfying* $f(0) = \partial_x f(0) = 0$.

2.2.2 Boundary observations

We consider the following fractional diffusion equation and initial boundary value conditions:

$$\partial_t^{\frac{1}{2}} u(x,t) - L_2 u(x,t) = f(x)R(x,t), \qquad (x,t) \in Q, \tag{2.8}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{2.9}$$

$$u(x,t) = 0, \qquad (x,t) \in \Sigma. \tag{2.10}$$

Inverse source problem by boundary observations: Determine f by boundary data $\partial_x u(x,t)$, $(x,t) \in \Sigma$ and data $u(x,t_0)$, $x \in \Omega$.

We have the following theorem for our inverse source problem.

Theorem 2.8. *We assume that* $u, \partial_t u \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;H^2(\Omega))$ *and* u *satisfies* (2.8)–(2.10) *and that* R *satisfies* (2.4). *Then we have*

$$\begin{aligned}\|f\|_{H^2(\Omega)} \le C\Bigg(&\|\partial_t \partial_x u(0,\cdot)\|_{L^2(t_0-\delta,t_0+\delta)} + \|\partial_t \partial_t^{\frac{1}{2}} \partial_x u(0,\cdot)\|_{L^2(t_0-\delta,t_0+\delta)} + \|\partial_t^2 \partial_x u(0,\cdot)\|_{L^2(t_0-\delta,t_0+\delta)} \\ &+ \|\partial_t \partial_x u(L,\cdot)\|_{L^2(t_0-\delta,t_0+\delta)} + \|\partial_t \partial_t^{\frac{1}{2}} \partial_x u(L,\cdot)\|_{L^2(t_0-\delta,t_0+\delta)} + \|\partial_t^2 \partial_x u(L,\cdot)\|_{L^2(t_0-\delta,t_0+\delta)}\Bigg) \\ &+ C\|u(\cdot,t_0)\|_{H^4(\Omega)}\end{aligned}$$

for any $f \in H^2(\Omega)$ *satisfying* $f(x) = \partial_x f(x) = 0$, $x \in \partial\Omega$.

Remark 3. *Under the assumptions of Theorem 2.7, we have*

$$\|f\|_{H^2(\Omega)} \le C\|u(\cdot,t_0)\|_{H^4(\Omega)}$$

for any $f \in H^2(\Omega)$ *satisfying* $f(x) = \partial_x f(x) = 0$, $x \in \partial\Omega$.

3 Proofs of Main results

This Section is composed of proofs of Carleman estimates and Lipschitz stability estimate.

3.1 Proofs of Carleman estimates

Here we will prove the Carleman estimate with interior data for fourth order parabolic equation: Lemma 2.3. Then we show the Carleman estimates for the 1-D half order time fractional diffusion equation (1.1): Theorem 2.5 and 2.6.

Before the proof of Carleman estimates, we see properties of weight functions defined in (2.3).

$$\partial_x\varphi_i = \lambda(\partial_x d_i)\varphi_i, \quad \partial_x\alpha_i = \lambda(\partial_x d_i)\varphi_i \quad \text{in } Q$$

and

$$\varphi_i \ge C, \quad \alpha_i < 0, \quad \partial_t\varphi_i \le C\varphi_i^2, \quad |\partial_t\alpha_i| \le C(\lambda)\varphi_i^2 \quad \text{in } Q$$

for $i = 0, 1$. Here and henceforth $C > 0$ denote generic constants which are independent of s, λ and may vary from line to line. Moreover we denote constants depending on λ by $C(\lambda)$.

3.1.1 Proof of Lemma 2.3

It is sufficient to derive the Carleman estimate for $\partial_t u - \partial_x^2(a_0\partial_x^2 u)$. We assume that the Carleman estimate for $\partial_t u - \partial_x^2(a_0\partial_x^2 u)$ is derived:

$$\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_0}\left(|\partial_t u|^2 + |\partial_x^4 u|^2\right) + s\lambda^2\varphi_0|\partial_x^3 u|^2 + s^3\lambda^4\varphi_0^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_0^5|\partial_x u|^2 + s^7\lambda^8\varphi_0^7|u|^2\right] e^{2s\alpha_0}\,dxdt \\
&\le C\int_Q |\partial_t u - \partial_x^2(a_0\partial_x^2 u)|^2 e^{2s\alpha_0}\,dxdt + Ce^{C(\lambda)s}\int_{\omega\times(0,T)} |u|^2\,dxdt. \qquad (3.1)
\end{aligned}$$

Noting that there exists a constant $C > 0$ such that

$$|\partial_t u - \partial_x^2(a_0\partial_x^2 u)|^2 \le |(\partial_t - L_4)u|^2 + C|\partial_x^3 u|^2 + C|\partial_x^2 u|^2 + C|\partial_x u|^2 + C|u|^2,$$

we have

$$\int_Q \left[\frac{1}{s\varphi_0}\left(|\partial_t u|^2 + |\partial_x^4 u|^2\right) + s\lambda^2\varphi_0|\partial_x^3 u|^2 + s^3\lambda^4\varphi_0^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_0^5|\partial_x u|^2 + s^7\lambda^8\varphi_0^7|u|^2\right] e^{2s\alpha_0}\,dxdt$$

$$\le C\int_Q |(\partial_t - L_4)u|^2 e^{2s\alpha_0}\,dxdt + Ce^{C(\lambda)s}\int_{\omega\times(0,T)} |u|^2\,dxdt$$
$$+ C\int_Q \left(|\partial_x^3 u|^2 + |\partial_x^2 u|^2 + |\partial_x u|^2 + |u|^2\right) e^{2s\alpha_0}\,dxdt. \tag{3.2}$$

Taking sufficient large $s > 0$, we may absorb the third term on the right-hand side of (3.2). Then we get the Carleman estimate for $\partial_t - L_4$. Henceforth we prove the estimate (3.1).

Set $z = ue^{s\alpha_0}$. We note that

$$\lim_{t\to 0+0} \partial_x^j z(x,t) = \lim_{t\to T-0} \partial_x^j z(x,t) = 0, \quad x \in \Omega, \quad j = 0, 1, 2, \dots$$

Define P by $Pz := e^{s\alpha_0}(\partial_t(ze^{-s\alpha_0}) - \partial_x^2(a_0\partial_x^2(ze^{-s\alpha_0})))$. Then we decompose

$$Pz = P_1 z + P_2 z - Rz$$

with

$$\begin{aligned}
P_1 z :=& -\partial_x^2(a_0\partial_x^2 z) - 6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0\partial_x^2 z - 12s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0\partial_x z\\
&- 12s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0\partial_x z - 6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)\partial_x z - s^4\lambda^4\varphi_0^4(\partial_x d_0)^4 a_0 z,\\
P_2 z :=& \partial_t z + 4s\lambda\varphi_0(\partial_x d_0)\partial_x(a_0\partial_x^2 z) + 4s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0\partial_x z + 6s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z,\\
Rz :=& s(\partial_t\alpha_0)z - 6s\lambda^2\varphi_0(\partial_x d_0)^2 a_0\partial_x^2 z - 6s\lambda\varphi_0(\partial_x^2 d_0)a_0\partial_x^2 z - 2s\lambda\varphi_0(\partial_x d_0)(\partial_x a_0)\partial_x^2 z\\
&- 4s\lambda^3\varphi_0(\partial_x d_0)^3 a_0\partial_x z - 12s\lambda^2\varphi_0(\partial_x d_0)(\partial_x^2 d_0)a_0\partial_x z - 4s\lambda\varphi_0(\partial_x^3 d_0)a_0\partial_x z\\
&- 6s\lambda^2\varphi_0(\partial_x d_0)^2(\partial_x a_0)\partial_x z - 6s\lambda\varphi_0(\partial_x^2 d_0)(\partial_x a_0)\partial_x z - 2s\lambda\varphi_0(\partial_x d_0)(\partial_x^2 a_0)\partial_x z\\
&+ (7s^2\lambda^4\varphi_0^2 - s\lambda^4\varphi_0)(\partial_x d_0)^4 a_0 z + (-6s^3\lambda^3\varphi_0^3 + 18s^2\lambda^3\varphi_0^2 - 6s\lambda^3\varphi_0)(\partial_x d_0)^2(\partial_x^2 d_0)a_0 z\\
&+ (3s^2\lambda^2\varphi_0^2 - 3s\lambda^2\varphi_0)(\partial_x^2 d_0)^2 a_0 z + (4s^2\lambda^2\varphi_0^2 - 4s\lambda^2\varphi_0)(\partial_x d_0)(\partial_x^3 d_0)a_0 z - s\lambda\varphi_0(\partial_x^4 d_0)a_0 z\\
&+ (-2s^3\lambda^3\varphi_0^3 + 6s^2\lambda^3\varphi_0^2 - 2s\lambda^3\varphi_0)(\partial_x d_0)^3(\partial_x a_0)z + (6s^2\lambda^2\varphi_0^2 - 6s\lambda^2\varphi_0)(\partial_x d_0)(\partial_x^2 d_0)(\partial_x a_0)z\\
&- 2s\lambda\varphi_0(\partial_x^3 d_0)(\partial_x a_0)z + (s^2\lambda^2\varphi_0^2 - s\lambda^2\varphi_0)(\partial_x d_0)^2(\partial_x^2 a_0)z - s\lambda\varphi_0(\partial_x^2 d_0)(\partial_x^2 a_0)z.
\end{aligned}$$

Setting $h := Pz + Rz = P_1 z + P_2 z$ and taking L^2-norm for h, we obtain

$$\|h\|^2_{L^2(Q)} \le 2\|Pz\|^2_{L^2(Q)} + 2\|Rz\|^2_{L^2(Q)}$$

and

$$\|h\|^2_{L^2(Q)} = \|P_1 z + P_2 z\|^2_{L^2(Q)} = \|P_1 z\|^2_{L^2(Q)} + \|P_2 z\|^2_{L^2(Q)} + 2(P_1 z, P_2 z)_{L^2(Q)}.$$

By above inequalities, we have

$$\frac{1}{2}\|P_1 z\|^2_{L^2(Q)} + \frac{1}{2}\|P_2 z\|^2_{L^2(Q)} + (P_1 z, P_2 z)_{L^2(Q)} \le \|Pz\|^2_{L^2(Q)} + \|Rz\|^2_{L^2(Q)}. \tag{3.3}$$

Now we compute each of terms on the left-hand side of (3.3): $\|P_1 z\|^2_{L^2(Q)}$, $\|P_2 z\|^2_{L^2(Q)}$ and $(P_1 z, P_2 z)_{L^2(Q)}$.

$$\|P_1 z\|^2_{L^2(Q)} = \int_Q |P_1 z|^2\,dxdt$$

$$
\begin{aligned}
&\geq \varepsilon \int_Q \frac{1}{s\varphi_0}|P_1 z|^2\,dxdt \\
&\geq \frac{\varepsilon}{2}\int_Q \frac{1}{s\varphi_0}|\partial_x^2(a_0\partial_x^2 z)|^2\,dxdt \\
&\quad - \varepsilon\int_Q \frac{1}{s\varphi_0}\Big| 6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0\partial_x^2 z + 12s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0\partial_x z \\
&\qquad\qquad + 12s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0\partial_x z \\
&\qquad\qquad + 6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)\partial_x z + s^4\lambda^4\varphi_0^4(\partial_x d_0)^4 a_0 z\Big|^2\,dxdt \\
&\geq \frac{\varepsilon}{2}\int_Q \frac{1}{s\varphi_0}|\partial_x^2(a_0\partial_x^2 z)|^2\,dxdt - C\varepsilon\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt \\
&\quad - C\int_Q (s^3\lambda^6\varphi_0^3 + s^3\lambda^4\varphi_0^3)|\partial_x z|^2\,dxdt - C\varepsilon\int_Q s^7\lambda^8\varphi_0^7(\partial_x d_0)^8 a_0^2|z|^2\,dxdt
\end{aligned}
$$

for $\varepsilon \in (0,1]$ and large $s > 0$ such that $s\varphi_0 \geq 1$. The second inequality from the bottom is obtained by $|\alpha+\beta| \geq \frac{1}{2}|\alpha|^2 - |\beta|^2$. So we have

$$
\begin{aligned}
&\frac{\varepsilon}{2}\int_Q \frac{1}{s\varphi_0}|\partial_x^2(a_0\partial_x^2 z)|^2\,dxdt - C\varepsilon\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt - C\varepsilon\int_Q s^7\lambda^8\varphi_0^7(\partial_x d_0)^8 a_0^2|z|^2\,dxdt \\
&\leq \|P_1 z\|^2_{L^2(Q)} + C\int_Q (s^3\lambda^6\varphi_0^3 + s^3\lambda^4\varphi_0^3)|\partial_x z|^2\,dxdt. \qquad (3.4)
\end{aligned}
$$

Similarly we have the following inequality for $\|P_2 z\|^2_{L^2(Q)}$:

$$
\begin{aligned}
&\frac{\varepsilon}{2}\int_Q \frac{1}{s\varphi_0}|\partial_t z|^2\,dxdt - C\varepsilon\int_Q s\lambda^2\varphi_0(\partial_x d_0)^2|\partial_x(a_0\partial_x^2 z)|^2\,dxdt - C\varepsilon\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt \\
&\leq \|P_2 z\|^2_{L^2(Q)} + C\int_Q s^5\lambda^8\varphi_0^5|z|^2\,dxdt \qquad (3.5)
\end{aligned}
$$

for $\varepsilon \in (0,1]$.

Let us estimate $(P_1 z, P_2 z)_{L^2(Q)}$ from below.

$$
\begin{aligned}
&(P_1 z, P_2 z)_{L^2(Q)} \\
&= \big(-\partial_x^2(a_0\partial_x^2 z), \partial_t z\big)_{L^2(Q)} + \big(-\partial_x^2(a_0\partial_x^2 z), 4s\lambda\varphi_0(\partial_x d_0)\partial_x(a_0\partial_x^2 z)\big)_{L^2(Q)} \\
&\quad + \big(-\partial_x^2(a_0\partial_x^2 z), 4s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0\partial_x z\big)_{L^2(Q)} + \big(-\partial_x^2(a_0\partial_x^2 z), 6s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\big)_{L^2(Q)} \\
&\quad + \big(-6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0\partial_x^2 z, \partial_t z\big)_{L^2(Q)} + \big(-6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0\partial_x^2 z, 4s\lambda\varphi_0(\partial_x d_0)\partial_x(a_0\partial_x^2 z)\big)_{L^2(Q)} \\
&\quad + \big(-6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0\partial_x^2 z, 4s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0\partial_x z\big)_{L^2(Q)} \\
&\quad + \big(-6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0\partial_x^2 z, 6s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\big)_{L^2(Q)} \\
&\quad + \big(-12s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0\partial_x z, \partial_t z\big)_{L^2(Q)} + \big(-12s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0\partial_x z, 4s\lambda\varphi_0(\partial_x d_0)\partial_x(a_0\partial_x^2 z)\big)_{L^2(Q)} \\
&\quad + \big(-12s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0\partial_x z, 4s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0\partial_x z\big)_{L^2(Q)} \\
&\quad + \big(-12s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0\partial_x z, 6s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\big)_{L^2(Q)} \\
&\quad + \big(-12s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0\partial_x z, \partial_t z\big)_{L^2(Q)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(-12s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0\partial_x z, 4s\lambda\varphi_0(\partial_x d_0)\partial_x(a_0\partial_x^2 z)\right)_{L^2(Q)} \\
&+ \left(-12s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0\partial_x z, 4s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0\partial_x z\right)_{L^2(Q)} \\
&+ \left(-12s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0\partial_x z, 6s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\right)_{L^2(Q)} \\
&+ \left(-6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)\partial_x z, \partial_t z\right)_{L^2(Q)} \\
&+ \left(-6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)\partial_x z, 4s\lambda\varphi_0(\partial_x d_0)\partial_x(a_0\partial_x^2 z)\right)_{L^2(Q)} \\
&+ \left(-6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)\partial_x z, 4s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0\partial_x z\right)_{L^2(Q)} \\
&+ \left(-6s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)\partial_x z, 6s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\right)_{L^2(Q)} \\
&+ \left(-s^4\lambda^4\varphi_0^4(\partial_x d_0)^4 a_0 z, \partial_t z\right)_{L^2(Q)} + \left(-s^4\lambda^4\varphi_0^4(\partial_x d_0)^4 a_0 z, 4s\lambda\varphi_0(\partial_x d_0)\partial_x(a_0\partial_x^2 z)\right)_{L^2(Q)} \\
&+ \left(-s^4\lambda^4\varphi_0^4(\partial_x d_0)^4 a_0 z, 4s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0\partial_x z\right)_{L^2(Q)} \\
&+ \left(-s^4\lambda^4\varphi_0^4(\partial_x d_0)^4 a_0 z, 6s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\right)_{L^2(Q)} \\
&=: \sum_{k=1}^{24} I_k. \qquad (3.6)
\end{aligned}
$$

We calculate each term $I_1, \ldots, I_{24}$ by the integration by parts and the Schwarz inequality.

$$
\begin{aligned}
I_1 &= -\int_Q (\partial_t z)\partial_x^2(a_0\partial_x^2 z)\,dxdt \\
&= \frac{1}{2}\int_Q (\partial_t a_0)|\partial_x^2 z|^2\,dxdt + \int_0^T \left[a_0(\partial_x\partial_t z)\partial_x^2 z\right]_{x=0}^{L} dt - \int_0^T \left[(\partial_t z)\partial_x(a_0\partial_x^2 z)\right]_{x=0}^{L} dt \\
&\geq -C\int_Q |\partial_x^2 z|^2\,dxdt + \int_0^T \left[a_0(\partial_x\partial_t z)\partial_x^2 z\right]_{x=0}^{L} dt - \int_0^T \left[(\partial_t z)\partial_x(a_0\partial_x^2 z)\right]_{x=0}^{L} dt. \qquad (3.7)
\end{aligned}
$$

Here and henceforth we use the following notation: $\left[F(x)\right]_{x=0}^{L} := F(L) - F(0)$.

$$
\begin{aligned}
I_2 &= -4\int_Q s\lambda\varphi_0(\partial_x d_0)\partial_x(a_0\partial_x^2 z)\partial_x^2(a_0\partial_x^2 z)\,dxdt \\
&= 2\int_Q s\lambda^2\varphi_0(\partial_x d_0)^2|\partial_x(a_0\partial_x^2 z)|^2\,dxdt + 2\int_Q s\lambda\varphi_0(\partial_x^2 d_0)|\partial_x(a_0\partial_x^2 z)|^2\,dxdt \\
&\quad - 2\int_0^T \left[s\lambda\varphi_0(\partial_x d_0)\left|\partial_x(a_0\partial_x^2 z)\right|^2\right]_{x=0}^{L} dt \\
&\geq 2\int_Q s\lambda^2\varphi_0(\partial_x d_0)^2|\partial_x(a_0\partial_x^2 z)|^2\,dxdt - C\int_Q s\lambda\varphi_0|\partial_x(a_0\partial_x^2 z)|^2\,dxdt \\
&\quad - 2\int_0^T \left[s\lambda\varphi_0(\partial_x d_0)\left|\partial_x(a_0\partial_x^2 z)\right|^2\right]_{x=0}^{L} dt. \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= -4\int_Q s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0(\partial_x z)\partial_x^2(a_0\partial_x^2 z)\,dxdt \\
&= -18\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt - 36\int_Q s^3\lambda^5\varphi_0^3(\partial_x d_0)^5 a_0^2(\partial_x z)(\partial_x^2 z)\,dxdt
\end{aligned}
$$

$$
\begin{aligned}
&-12\int_Q s^3\lambda^4\varphi_0^3\partial_x\left[(\partial_x d_0)^4 a_0\right]a_0(\partial_x z)(\partial_x^2 z)\,dxdt+4\int_Q s^3\lambda^3\varphi_0^3\partial_x\{(\partial_x d_0)^3 a_0\}(\partial_x z)\partial_x(a_0\partial_x^2 z)\,dxdt\\
&-6\int_Q s^3\lambda^3\varphi_0^3(\partial_x d_0)^2(\partial_x^2 d_0)a_0^2|\partial_x^2 z|^2\,dxdt+12\int_0^T\left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2(\partial_x z)\partial_x^2 z\right]_{x=0}^L dt\\
&+2\int_0^T\left[s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0^2|\partial_x^2 z|^2\right]_{x=0}^L dt-4\int_0^T\left[s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0(\partial_x z)\partial_x(a_0\partial_x^2 z)\right]_{x=0}^L dt\\
\geq&-18\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt-C\int_Q s\lambda\varphi_0|\partial_x(a_0\partial_x^2 z)|^2\,dxdt\\
&-C\int_Q(s^3\lambda^3\varphi_0^3+s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2\,dxdt-C\int_Q(s^4\lambda^6\varphi_0^4+s^5\lambda^5\varphi_0^5+s^4\lambda^4\varphi_0^4)|\partial_x z|^2\,dxdt\\
&+12\int_0^T\left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2(\partial_x z)\partial_x^2 z\right]_{x=0}^L dt+2\int_0^T\left[s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0^2|\partial_x^2 z|^2\right]_{x=0}^L dt\\
&-4\int_0^T\left[s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0(\partial_x z)\partial_x(a_0\partial_x^2 z)\right]_{x=0}^L dt. \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
I_4=&-6\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\partial_x^2(a_0\partial_x^2 z)\,dxdt\\
=&-6\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt\\
&+18\int_Q s^3\lambda^5\varphi_0^3(\partial_x d_0)^5 a_0 z\partial_x(a_0\partial_x^2 z)\,dxdt+6\int_Q s^3\lambda^4\varphi_0^3\partial_x\left[(\partial_x d_0)^4 a_0\right]z\partial_x(a_0\partial_x^2 z)\,dxdt\\
&-18\int_Q s^3\lambda^5\varphi_0^3(\partial_x d_0)^5 a_0^2(\partial_x z)\partial_x^2 z\,dxdt-6\int_Q s^3\lambda^4\varphi_0^3\partial_x\left[(\partial_x d_0)^4 a_0\right]a_0(\partial_x z)\partial_x^2 z\,dxdt\\
&+6\int_0^T\left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2(\partial_x z)\partial_x^2 z\right]_{x=0}^L dt-6\int_0^T\left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\partial_x(a_0\partial_x^2 z)\right]_{x=0}^L dt\\
\geq&-6\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt-C\int_Q\lambda^2|\partial_x(a_0\partial_x^2 z)|^2\,dxdt-C\int_Q s^2\lambda^4\varphi_0^2|\partial_x^2 z|^2\,dxdt\\
&-C\int_Q(s^4\lambda^6\varphi_0^4+s^4\lambda^4\varphi_0^4)|\partial_x z|^2\,dxdt-C\int_Q(s^6\lambda^8\varphi_0^6+s^6\lambda^6\varphi_0^6)|z|^2\,dxdt\\
&+6\int_0^T\left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2(\partial_x z)\partial_x^2 z\right]_{x=0}^L dt-6\int_0^T\left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\partial_x(a_0\partial_x^2 z)\right]_{x=0}^L dt. \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
I_5=&-6\int_Q s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0(\partial_t z)\partial_x^2 z\,dxdt\\
=&\,12\int_Q s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0(\partial_t z)\partial_x z\,dxdt+12\int_Q s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0(\partial_t z)\partial_x z\,dxdt\\
&+6\int_Q s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)(\partial_t z)\partial_x z\,dxdt-6\int_Q s^2\lambda^2\varphi_0(\partial_t\varphi_0)(\partial_x d_0)^2 a_0|\partial_x z|^2\,dxdt\\
&-3\int_Q s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_t a_0)|\partial_x z|^2\,dxdt-6\int_0^T\left[s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0(\partial_t z)\partial_x z\right]_{x=0}^L dt\\
\geq&\,12\int_Q s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0(\partial_t z)\partial_x z\,dxdt+12\int_Q s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0(\partial_t z)\partial_x z\,dxdt\\
&+6\int_Q s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)(\partial_t z)\partial_x z\,dxdt-C\int_Q(s^2\lambda^2\varphi_0^3+s^2\lambda^2\varphi_0^2)|\partial_x z|^2\,dxdt
\end{aligned}
$$

$$-6\int_0^T \left[s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0(\partial_t z)\partial_x z\right]_{x=0}^{L} dt. \tag{3.11}$$

$$\begin{aligned}
I_6 &= -24\int_Q s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0(\partial_x^2 z)\partial_x(a_0\partial_x^2 z)\,dxdt\\
&= 36\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt + 36\int_Q s^3\lambda^3\varphi_0^3(\partial_x d_0)^2(\partial_x^2 d_0)a_0^2|\partial_x^2 z|^2\,dxdt\\
&\quad - 12\int_0^T \left[s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0^2|\partial_x^2 z|^2\right]_{x=0}^{L} dt\\
&\geq 36\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt - C\int_Q s^3\lambda^3\varphi_0^3|\partial_x^2 z|^2\,dxdt\\
&\quad - 12\int_0^T \left[s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0^2|\partial_x^2 z|^2\right]_{x=0}^{L} dt.
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
I_7 &= -24\int_Q s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2(\partial_x z)\partial_x^2 z\,dxdt\\
&= 60\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt + 12\int_Q s^5\lambda^5\varphi_0^5\partial_x\left[(\partial_x d_0)^5 a_0^2\right]|\partial_x z|^2\,dxdt\\
&\quad - 12\int_0^T \left[s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2|\partial_x z|^2\right]_{x=0}^{L} dt\\
&\geq 60\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt - C\int_Q s^5\lambda^5\varphi_0^5|\partial_x z|^2\,dxdt\\
&\quad - 12\int_0^T \left[s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2|\partial_x z|^2\right]_{x=0}^{L} dt.
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
I_8 &= -36\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2 z\partial_x^2 z\,dxdt\\
&= 180\int_Q s^5\lambda^7\varphi_0^5(\partial_x d_0)^7 a_0^2 z\partial_x z\,dxdt + 36\int_Q s^5\lambda^6\varphi_0^5\partial_x\left[(\partial_x d_0)^6 a_0^2\right] z\partial_x z\,dxdt\\
&\quad + 36\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt - 36\int_0^T \left[s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2 z\partial_x z\right]_{x=0}^{L} dt\\
&\geq 36\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt - C\int_Q (s^4\lambda^6\varphi_0^4 + s^4\lambda^4\varphi_0^4)|\partial_x z|^2\,dxdt\\
&\quad - C\int_Q s^6\lambda^8\varphi_0^6|z|^2\,dxdt - 36\int_0^T \left[s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2 z\partial_x z\right]_{x=0}^{L} dt.
\end{aligned} \tag{3.14}$$

$$I_9 = -12\int_Q s^2\lambda^3\varphi_0^2(\partial_x d_0)^3 a_0(\partial_t z)\partial_x z\,dxdt. \tag{3.15}$$

$$\begin{aligned}
I_{10} &= -48\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0(\partial_x z)\partial_x(a_0\partial_x^2 z)\,dxdt\\
&= 144\int_Q s^3\lambda^5\varphi_0^3(\partial_x d_0)^5 a_0^2(\partial_x z)(\partial_x^2 z)\,dxdt + 48\int_Q s^3\lambda^4\varphi_0^3\partial_x\left[(\partial_x d_0)^4 a_0\right] a_0(\partial_x z)(\partial_x^2 z)\,dxdt\\
&\quad + 48\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt - 48\int_0^T \left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2(\partial_x z)\partial_x^2 z\right]_{x=0}^{L} dt
\end{aligned}$$

$$
\begin{aligned}
&\geq 48\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt - C\int_Q s^2\lambda^4\varphi_0^2|\partial_x^2 z|^2\,dxdt\\
&\quad - C\int_Q (s^4\lambda^6\varphi_0^4 + s^4\lambda^4\varphi_0^4)|\partial_x z|^2\,dxdt - 48\int_0^T \Big[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2(\partial_x z)\partial_x^2 z\Big]_{x=0}^{L}\,dt. \qquad (3.16)
\end{aligned}
$$

$$
I_{11} = -48\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt. \qquad (3.17)
$$

$$
\begin{aligned}
I_{12} &= -72\int_Q s^5\lambda^7\varphi_0^5(\partial_x d_0)^7 a_0^2 z\partial_x z\,dxdt\\
&\geq -C\int_Q s^4\lambda^6\varphi_0^4|\partial_x z|^2\,dxdt - C\int_Q s^6\lambda^8\varphi_0^6|z|^2\,dxdt. \qquad (3.18)
\end{aligned}
$$

$$
I_{13} = -12\int_Q s^2\lambda^2\varphi_0^2(\partial_x d_0)(\partial_x^2 d_0)a_0(\partial_t z)\partial_x z\,dxdt. \qquad (3.19)
$$

$$
\begin{aligned}
I_{14} &= -48\int_Q s^3\lambda^3\varphi_0^3(\partial_x d_0)^2(\partial_x^2 d_0)a_0(\partial_x z)\partial_x(a_0\partial_x^2 z)\,dxdt\\
&\geq -C\int_Q s\lambda\varphi_0|\partial_x(a_0\partial_x^2 z)|^2\,dxdt - C\int_Q s^5\lambda^5\varphi_0^5|\partial_x z|^2\,dxdt. \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
I_{15} &= -48\int_Q s^5\lambda^5\varphi_0^5(\partial_x d_0)^4(\partial_x^2 d_0)a_0^2|\partial_x z|^2\,dxdt\\
&\geq -C\int_Q s^5\lambda^5\varphi_0^5|\partial_x z|^2\,dxdt. \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
I_{16} &= -72\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^5(\partial_x^2 d_0)a_0^2 z\partial_x z\,dxdt\\
&\geq -C\int_Q s^4\lambda^6\varphi_0^4|\partial_x z|^2\,dxdt - C\int_Q s^6\lambda^6\varphi_0^6|z|^2\,dxdt. \qquad (3.22)
\end{aligned}
$$

$$
I_{17} = -6\int_Q s^2\lambda^2\varphi_0^2(\partial_x d_0)^2(\partial_x a_0)(\partial_t z)\partial_x z\,dxdt. \qquad (3.23)
$$

$$
\begin{aligned}
I_{18} &= -24\int_Q s^3\lambda^3\varphi_0^3(\partial_x d_0)^3(\partial_x a_0)(\partial_x z)\partial_x(a_0\partial_x^2 z)\,dxdt\\
&\geq -C\int_Q s\lambda\varphi_0|\partial_x(a_0\partial_x^2 z)|^2\,dxdt - C\int_Q s^5\lambda^5\varphi_0^5|\partial_x z|^2\,dxdt. \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
I_{19} &= -24\int_Q s^5\lambda^5\varphi_0^5(\partial_x d_0)^5(\partial_x a_0)a_0|\partial_x z|^2\,dxdt\\
&\geq -C\int_Q s^5\lambda^5\varphi_0^5|\partial_x z|^2\,dxdt. \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
I_{20} &= -36\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6(\partial_x a_0)a_0 z(\partial_x z)\,dxdt\\
&\geq -C\int_Q s^4\lambda^4\varphi_0^4|\partial_x z|^2\,dxdt - C\int_Q s^6\lambda^8\varphi_0^6|z|^2\,dxdt. \qquad (3.26)
\end{aligned}
$$

$$\begin{aligned}
I_{21} &= -\int_Q s^4\lambda^4\varphi_0^4(\partial_x d_0)^4 a_0(\partial_t z)z\,dxdt \\
&= 2\int_Q s^4\lambda^4\varphi_0^3(\partial_t\varphi_0)(\partial_x d_0)^4 a_0|z|^2\,dxdt + \frac{1}{2}\int_Q s^4\lambda^4\varphi_0^4(\partial_x d_0)^4(\partial_t a_0)|z|^2\,dxdt \\
&\geq -C\int_Q (s^4\lambda^4\varphi_0^5 + s^4\lambda^4\varphi_0^4)|z|^2\,dxdt. \qquad (3.27)
\end{aligned}$$

$$\begin{aligned}
I_{22} &= -4\int_Q s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0 z\partial_x(a_0\partial_x^2 z)\,dxdt \\
&= -30\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt - 100\int_Q s^5\lambda^7\varphi_0^5(\partial_x d_0)^7 a_0^2 z(\partial_x z)\,dxdt \\
&\quad - 20\int_Q s^5\lambda^6\varphi_0^5\partial_x\left[(\partial_x d_0)^6 a_0^2\right] z(\partial_x z)\,dxdt + 4\int_Q s^5\lambda^5\varphi_0^5\partial_x\left[(\partial_x d_0)^5 a_0\right] a_0 z(\partial_x^2 z)\,dxdt \\
&\quad - 2\int_Q s^5\lambda^5\varphi_0^5\partial_x\left[(\partial_x d_0)^5 a_0^2\right]|\partial_x z|^2\,dxdt + 20\int_0^T\left[s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2 z\partial_x z\right]_{x=0}^{L}dt \\
&\quad + 2\int_0^T\left[s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2|\partial_x z|^2\right]_{x=0}^{L}dt - 4\int_0^T\left[s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2 z\partial_x^2 z\right]_{x=0}^{L}dt \\
&\geq -30\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt - C\int_Q s^3\lambda^3\varphi_0^3|\partial_x^2 z|^2\,dxdt \\
&\quad - C\int_Q (s^4\lambda^6\varphi_0^4 + s^5\lambda^5\varphi_0^5)|\partial_x z|^2\,dxdt - C\int_Q (s^6\lambda^8\varphi_0^6 + s^6\lambda^6\varphi_0^6 + s^7\lambda^7\varphi_0^7)|z|^2\,dxdt \\
&\quad + 20\int_0^T\left[s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2 z\partial_x z\right]_{x=0}^{L}dt + 2\int_0^T\left[s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2|\partial_x z|^2\right]_{x=0}^{L}dt \\
&\quad - 4\int_0^T\left[s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2 z\partial_x^2 z\right]_{x=0}^{L}dt. \qquad (3.28)
\end{aligned}$$

$$\begin{aligned}
I_{23} &= -4\int_Q s^7\lambda^7\varphi_0^7(\partial_x d_0)^7 a_0^2 z(\partial_x z)\,dxdt \\
&= 14\int_Q s^7\lambda^8\varphi_0^7(\partial_x d_0)^8 a_0^2|z|^2\,dxdt + 2\int_Q s^7\lambda^7\varphi_0^7\partial_x\left[(\partial_x d_0)^7 a_0^2\right]|z|^2\,dxdt \\
&\quad - 2\int_0^T\left[s^7\lambda^7\varphi_0^7(\partial_x d_0)^7 a_0^2|z|^2\right]_{x=0}^{L}dt \\
&\geq 14\int_Q s^7\lambda^8\varphi_0^7(\partial_x d_0)^8 a_0^2|z|^2\,dxdt - C\int_Q s^7\lambda^7\varphi_0^7|z|^2\,dxdt \\
&\quad - 2\int_0^T\left[s^7\lambda^7\varphi_0^7(\partial_x d_0)^7 a_0^2|z|^2\right]_{x=0}^{L}dt. \qquad (3.29)
\end{aligned}$$

$$I_{24} = -6\int_Q s^7\lambda^8\varphi_0^7(\partial_x d_0)^8 a_0^2|z|^2\,dxdt. \qquad (3.30)$$

By (3.6)–(3.30), we have

$$\begin{aligned}
&2\int_Q s\lambda^2\varphi_0(\partial_x d_0)^2|\partial_x(a_0\partial_x^2 z)|^2\,dxdt + 60\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt \\
&\quad + 18\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt + 8\int_Q s^7\lambda^8\varphi_0^7(\partial_x d_0)^8 a_0^2|z|^2\,dxdt
\end{aligned}$$

$$
\begin{aligned}
&= (P_1 z, P_2 z)_{L^2(Q)} + C\int_Q (s\lambda\varphi_0 + \lambda^2)|\partial_x(a_0\partial_x^2 z)|^2\,dxdt + C\int_Q (s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2 + 1)|\partial_x^2 z|^2\,dxdt \\
&\quad + C\int_Q (s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4 + s^4\lambda^4\varphi_0^4 + s^2\lambda^2\varphi_0^3 + s^2\lambda^2\varphi_0^2)|\partial_x z|^2\,dxdt \\
&\quad + C\int_Q (s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6 + s^6\lambda^6\varphi_0^6 + s^4\lambda^4\varphi_0^5 + s^4\lambda^4\varphi_0^4)|z|^2\,dxdt + B
\end{aligned}
\tag{3.31}
$$

where

$$
\begin{aligned}
B &= 2\int_0^T \left[s^7\lambda^7\varphi_0^7(\partial_x d_0)^7 a_0^2|z|^2\right]_{x=0}^L dt + 16\int_0^T \left[s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2 z\partial_x z\right]_{x=0}^L dt \\
&\quad + 4\int_0^T \left[s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2 z\partial_x^2 z\right]_{x=0}^L dt + 6\int_0^T \left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0 z\partial_x(a_0\partial_x^2 z)\right]_{x=0}^L dt \\
&\quad + 6\int_0^T \left[s^2\lambda^2\varphi_0^2(\partial_x d_0)^2 a_0(\partial_t z)\partial_x z\right]_{x=0}^L dt + \int_0^T \left[(\partial_t z)\partial_x(a_0\partial_x^2 z)\right]_{x=0}^L dt \\
&\quad + 10\int_0^T \left[s^5\lambda^5\varphi_0^5(\partial_x d_0)^5 a_0^2|\partial_x z|^2\right]_{x=0}^L dt + 30\int_0^T \left[s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2(\partial_x z)\partial_x^2 z\right]_{x=0}^L dt \\
&\quad + 4\int_0^T \left[s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0(\partial_x z)\partial_x(a_0\partial_x^2 z)\right]_{x=0}^L dt + 10\int_0^T \left[s^3\lambda^3\varphi_0^3(\partial_x d_0)^3 a_0^2|\partial_x^2 z|^2\right]_{x=0}^L dt \\
&\quad + 2\int_0^T \left[s\lambda\varphi_0(\partial_x d_0)\left|\partial_x(a_0\partial_x^2 z)\right|^2\right]_{x=0}^L dt - \int_0^T \left[a_0(\partial_x\partial_t z)\partial_x^2 z\right]_{x=0}^L dt.
\end{aligned}
$$

Let us calculate $\|Rz\|^2_{L^2(Q)}$.

$$
\begin{aligned}
&\|Rz\|^2_{L^2(Q)} \\
&= \int_Q |Rz|^2\,dxdt \\
&\le C\int_Q (s^2\lambda^4\varphi_0^2 + s^2\lambda^2\varphi_0^2)|\partial_x^2 z|^2\,dxdt + C\int_Q s^2\lambda^6\varphi_0^2|\partial_x z|^2\,dxdt \\
&\quad + C\int_Q (s^4\lambda^8\varphi_0^4 + s^6\lambda^6\varphi_0^6)|z|^2\,dxdt + C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt
\end{aligned}
\tag{3.32}
$$

for large $s > 0$ and $\lambda > 0$. By (3.3), (3.4), (3.5), (3.31) and (3.32), we have

$$
\begin{aligned}
&\frac{\varepsilon}{4}\int_Q \frac{1}{s\varphi_0}|\partial_t z|^2\,dxdt + \frac{\varepsilon}{4}\int_Q \frac{1}{s\varphi_0}|\partial_x^2(a_0\partial_x^2 z)|^2\,dxdt \\
&+ \left(2 - \frac{C\varepsilon}{2}\right)\int_Q s\lambda^2\varphi_0(\partial_x d_0)^2|\partial_x(a_0\partial_x^2 z)|^2\,dxdt + \left(60 - \frac{C\varepsilon}{2}\right)\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt \\
&\quad + \left(18 - \frac{C\varepsilon}{2}\right)\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt + \left(8 - \frac{C\varepsilon}{2}\right)\int_Q s^7\lambda^8\varphi_0^7(\partial_x d_0)^8 a_0^2|z|^2\,dxdt \\
&\le \|Pz\|^2_{L^2(Q)} + C\int_Q (s\lambda\varphi_0 + \lambda^2)|\partial_x(a_0\partial_x^2 z)|^2\,dxdt + C\int_Q (s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2\,dxdt \\
&\quad + C\int_Q \left(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\right)|\partial_x z|^2\,dxdt + C\int_Q \left(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\right)|z|^2\,dxdt \\
&\quad + C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt + B
\end{aligned}
\tag{3.33}
$$

for large $s > 0$ and $\lambda > 0$. Taking sufficient small $\varepsilon \in (0, 1]$ such that each term on the left-hand side of (3.33) is positive, we obtain

$$\begin{aligned}
&\int_Q \frac{1}{s\varphi_0}\left(|\partial_t z|^2+|\partial_x^2(a_0\partial_x^2 z)|^2\right)dxdt+\int_Q s\lambda^2\varphi_0(\partial_x d_0)^2|\partial_x(a_0\partial_x^2 z)|^2\,dxdt\\
&\quad+\int_Q s^3\lambda^4\varphi_0^3(\partial_x d_0)^4 a_0^2|\partial_x^2 z|^2\,dxdt+\int_Q s^5\lambda^6\varphi_0^5(\partial_x d_0)^6 a_0^2|\partial_x z|^2\,dxdt+\int_Q s^7\lambda^8\varphi_0^7(\partial_x d_0)^8 a_0^2|z|^2\,dxdt\\
&\le C\|Pz\|^2_{L^2(Q)}+C\int_Q(s\lambda\varphi_0+\lambda^2)|\partial_x(a_0\partial_x^2 z)|^2\,dxdt+C\int_Q(s^3\lambda^3\varphi_0^3+s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2\,dxdt\\
&\quad+C\int_Q\left(s^5\lambda^5\varphi_0^5+s^4\lambda^6\varphi_0^4\right)|\partial_x z|^2\,dxdt+C\int_Q\left(s^7\lambda^7\varphi_0^7+s^6\lambda^8\varphi_0^6\right)|z|^2\,dxdt\\
&\quad+C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt+CB. \qquad (3.34)
\end{aligned}$$

Noting that $\frac{1}{m}\le a_0$ in Q and $|\partial_x d_0|\ge\sigma_0$ in $\overline{\Omega\setminus\omega_0}$, we have

$$\begin{aligned}
&\int_Q\left[\frac{1}{s\varphi_0}\left(|\partial_t z|^2+|\partial_x^2(a_0\partial_x^2 z)|^2\right)+s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2+s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2+s^5\lambda^6\varphi_0^5|\partial_x z|^2+s^7\lambda^8\varphi_0^7|z|^2\right]dxdt\\
&\le C\|Pz\|^2_{L^2(Q)}+C\int_Q(s\lambda\varphi_0+\lambda^2)|\partial_x(a_0\partial_x^2 z)|^2\,dxdt+C\int_Q(s^3\lambda^3\varphi_0^3+s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2\,dxdt\\
&\quad+C\int_Q\left(s^5\lambda^5\varphi_0^5+s^4\lambda^6\varphi_0^4\right)|\partial_x z|^2\,dxdt+C\int_Q\left(s^7\lambda^7\varphi_0^7+s^6\lambda^8\varphi_0^6\right)|z|^2\,dxdt\\
&\quad+C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt+CB\\
&\quad+C\int_{\omega_0\times(0,T)}\left(s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2+s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2+s^5\lambda^6\varphi_0^5|\partial_x z|^2+s^7\lambda^8\varphi_0^7|z|^2\right)dxdt. \qquad (3.35)
\end{aligned}$$

Estimation of boundary terms. Let us estimate boundary terms B. By the boundary conditions $u=\partial_x u=0$ on Σ, we obtain $z=\partial_t z=\partial_x z=\partial_x\partial_t z=0$ on Σ. So we have

$$\begin{aligned}
B=&\,10\int_0^T s^3\lambda^3\varphi_0^3(L,t)(\partial_x d_0)^3(L)a_0^2(L,t)|\partial_x^2 z(L,t)|^2\,dt\\
&-10\int_0^T s^3\lambda^3\varphi_0^3(0,t)(\partial_x d_0)^3(0)a_0^2(0,t)|\partial_x^2 z(0,t)|^2\,dt\\
&+2\int_0^T s\lambda\varphi_0(L,t)\partial_x d_0(L)|\partial_x a_0(L,t)\partial_x^2 z(L,t)+a_0(L,t)\partial_x^3 z(L,t)|^2\,dt\\
&-2\int_0^T s\lambda\varphi_0(0,t)\partial_x d_0(0)|\partial_x a_0(0,t)\partial_x^2 z(0,t)+a_0(0,t)\partial_x^3 z(0,t)|^2\,dt.
\end{aligned}$$

Noting that $\partial_x d_0(0)\ge 0$ and $\partial_x d_0(L)\le 0$, we see that

$$B\le 0. \qquad (3.36)$$

Estimation of interior terms. Now we replace

$$\int_{\omega_0\times(0,T)}\left(s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2+s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2+s^5\lambda^6\varphi_0^5|\partial_x z|^2+s^7\lambda^8\varphi_0^7|z|^2\right)dxdt \qquad (3.37)$$

on the right-hand side of (3.35) by

$$\int_{\omega\times(0,T)} s^7\lambda^8\varphi_0^7|z|^2\,dxdt.$$

The main techniques are integration by parts and cut off functions arguments. We will estimate (3.37) by three steps. Firstly we estimate the third order term on (3.37) by lower order terms.

Taking an open subset $\omega_1 \subset\subset \omega$ such that $\omega_0 \subset\subset \omega_1$, we consider a cut off function $\rho_1 \in C_0^\infty(\Omega)$ such that $0 \le \rho_1 \le 1$ in Ω, $\operatorname{supp} \rho_1 \subset \omega_1$ and $\rho_1 \equiv 1$ in ω_0. Here and henceforth $A \subset\subset B$ denotes $\overline{A} \subset B$. Then we estimate the following integral

$$-\int_Q \rho_1 s\lambda^2\varphi_0 a_0(\partial_x^2 z)\partial_x^2(a_0\partial_x^2 z)\, dxdt$$

from above and below.

$$-\int_Q \rho_1 s\lambda^2\varphi_0 a_0(\partial_x^2 z)\partial_x^2(a_0\partial_x^2 z)\, dxdt \le \frac{1}{\eta_1}\int_Q \frac{1}{s\varphi_0}|\partial_x^2(a_0\partial_x^2 z)|^2\, dxdt + 4\eta_1\int_{\omega_1\times(0,T)} s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2\, dxdt \tag{3.38}$$

for $\eta_1 > 0$. Here we used $ab = \left(\frac{1}{\sqrt{2\eta}}a\right)(\sqrt{2\eta}b) \le \frac{1}{\eta}a^2 + 4\eta b^2$ for $\eta > 0$. On the other hand

$$\begin{aligned}
&-\int_Q \rho_1 s\lambda^2\varphi_0 a_0(\partial_x^2 z)\partial_x^2(a_0\partial_x^2 z)\, dxdt \\
&= \int_Q (\partial_x\rho_1)s\lambda^2\varphi_0 a_0(\partial_x^2 z)\partial_x(a_0\partial_x^2 z)\, dxdt + \int_Q \rho_1 s\lambda^3\varphi_0(\partial_x d_0)a_0(\partial_x^2 z)\partial_x(a_0\partial_x^2 z)\, dxdt \\
&\quad + \int_Q \rho_1 s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2\, dxdt \\
&\ge \int_{\omega_0\times(0,T)} s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2\, dxdt - C\int_Q \lambda^2|\partial_x(a_0\partial_x^2 z)|^2\, dxdt - C\int_Q (s^2\lambda^4\varphi_0^2 + s^2\lambda^2\varphi_0^2)|\partial_x^2 z|^2\, dxdt.
\end{aligned} \tag{3.39}$$

Combining (3.38) with (3.39), we have

$$\begin{aligned}
&\int_{\omega_0\times(0,T)} s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2\, dxdt \\
&\le \frac{1}{\eta_1}\int_Q \frac{1}{s\varphi_0}|\partial_x^2(a_0\partial_x^2 z)|^2\, dxdt + 4\eta_1\int_{\omega_1\times(0,T)} s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2\, dxdt \\
&\quad + C\int_Q \lambda^2|\partial_x(a_0\partial_x^2 z)|^2\, dxdt + C\int_Q (s^2\lambda^4\varphi_0^2 + s^2\lambda^2\varphi_0^2)|\partial_x^2 z|^2\, dxdt.
\end{aligned} \tag{3.40}$$

By (3.35), (3.36) and (3.40), we obtain

$$\begin{aligned}
&\int_Q \Bigg[\frac{1}{s\varphi_0}|\partial_t z|^2 + \left(1 - \frac{C}{\eta_1}\right)\frac{1}{s\varphi_0}|\partial_x^2(a_0\partial_x^2 z)|^2 + s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2 \\
&\qquad + s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\Bigg]\, dxdt \\
&\le C\|Pz\|_{L^2(Q)}^2 + C\int_Q (s\lambda\varphi_0 + \lambda^2)|\partial_x(a_0\partial_x^2 z)|^2\, dxdt + C\int_Q (s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2\, dxdt \\
&\quad + C\int_Q \left(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\right)|\partial_x z|^2\, dxdt + C\int_Q \left(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\right)|z|^2\, dxdt \\
&\quad + C(\lambda)\int_Q s^2\varphi_0^4|z|^2\, dxdt + C\int_{\omega_1\times(0,T)} \left[(4\eta_1+1)s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\right]\, dxdt
\end{aligned}$$

for large $s > 0$ and $\lambda > 0$. Taking sufficient large $\eta_1 > 0$, we get

$$\int_Q \left[\frac{1}{s\varphi_0} \left(|\partial_t z|^2 + |\partial_x^2(a_0 \partial_x^2 z)|^2 \right) + s\lambda^2\varphi_0 |\partial_x(a_0\partial_x^2 z)|^2 + s^3\lambda^4\varphi_0^3 |\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5 |\partial_x z|^2 + s^7\lambda^8\varphi_0^7 |z|^2 \right] dxdt$$
$$\leq C\|Pz\|_{L^2(Q)}^2 + C\int_Q (s\lambda\varphi_0 + \lambda^2)|\partial_x(a_0\partial_x^2 z)|^2 \, dxdt + C\int_Q (s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2 \, dxdt$$
$$+ C\int_Q \left(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\right)|\partial_x z|^2 \, dxdt + C\int_Q \left(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\right)|z|^2 \, dxdt$$
$$+ C(\lambda)\int_Q s^2\varphi_0^4 |z|^2 \, dxdt + C\int_{\omega_1\times(0,T)} \left(s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\right) dxdt. \tag{3.41}$$

We take an open subset $\omega_2 \subset\subset \omega$ such that $\omega_1 \subset\subset \omega_2$. We choose a cut off function $\rho_2 \in C_0^\infty(\Omega)$ such that $0 \leq \rho_2 \leq 1$ in Ω, $\operatorname{supp}\rho_2 \subset \omega_2$ and $\rho_2 \equiv 1$ in ω_1. Then we estimate the following integral

$$-\int_Q \rho_2 s^3\lambda^4\varphi_0^3(\partial_x z)\partial_x(a_0\partial_x^2 z) \, dxdt$$

from above and below.

$$-\int_Q \rho_2 s^3\lambda^4\varphi_0^3(\partial_x z)\partial_x(a_0\partial_x^2 z) \, dxdt \leq \frac{1}{\eta_2}\int_Q s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2 \, dxdt + 4\eta_2\int_{\omega_2\times(0,T)} s^5\lambda^6\varphi_0^5|\partial_x^2 z|^2 \, dxdt \tag{3.42}$$

for $\eta_2 > 0$. On the other hand

$$-\int_Q \rho_2 s^3\lambda^4\varphi_0^3(\partial_x z)\partial_x(a_0\partial_x^2 z) \, dxdt$$
$$= \int_Q (\partial_x\rho_2)s^3\lambda^4\varphi_0^3 a_0(\partial_x z)(\partial_x^2 z) \, dxdt + 3\int_Q \rho_2 s^3\lambda^5\varphi_0^3(\partial_x d_0)a_0(\partial_x z)(\partial_x^2 z) \, dxdt$$
$$+ \int_Q \rho_2 s^3\lambda^4\varphi_0^3 a_0|\partial_x^2 z|^2 \, dxdt$$
$$\geq \frac{1}{m}\int_{\omega_1\times(0,T)} s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 \, dxdt - C\int_Q s^2\lambda^4\varphi_0^2|\partial_x^2 z|^2 \, dxdt - C\int_Q (s^4\lambda^6\varphi_0^4 + s^4\lambda^4\varphi_0^4)|\partial_x z|^2 \, dxdt. \tag{3.43}$$

Combining (3.42) with (3.43), we have

$$\int_{\omega_1\times(0,T)} s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 \, dxdt$$
$$\leq \frac{m}{\eta_2}\int_Q s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)| \, dxdt + 4m\eta_2\int_{\omega_2\times(0,T)} s^5\lambda^6\varphi_0^5|\partial_x^2 z| \, dxdt$$
$$+ C\int_Q s^2\lambda^4\varphi_0^2|\partial_x^2 z|^2 \, dxdt + C\int_Q (s^4\lambda^6\varphi_0^4 + s^4\lambda^4\varphi_0^4)|\partial_x z|^2 \, dxdt. \tag{3.44}$$

By (3.41) and (3.44), we obtain

$$\int_Q \left[\frac{1}{s\varphi_0}\left(|\partial_t z|^2 + |\partial_x^2(a_0\partial_x^2 z)|^2\right) + \left(1 - \frac{Cm}{\eta_2}\right) s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2 \right.$$
$$\left. + s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2 \right] dxdt$$
$$\leq C\|Pz\|_{L^2(Q)}^2 + C\int_Q (s\lambda\varphi_0 + \lambda^2)|\partial_x(a_0\partial_x^2 z)|^2 \, dxdt + C\int_Q (s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2 \, dxdt$$

$$+ C\int_Q \big(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\big)|\partial_x z|^2\,dxdt + C\int_Q \big(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\big)|z|^2\,dxdt$$
$$+ C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt + C\int_{\omega_2\times(0,T)} \big[(4m\eta_2+1)s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\big]\,dxdt$$

for large $s > 0$ and $\lambda > 0$. Taking sufficient large $\eta_2 > 0$, we get

$$\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_0}\big(|\partial_t z|^2 + |\partial_x^2(a_0\partial_x^2 z)|^2\big) + s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2 + s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\right]dxdt\\
&\le C\|Pz\|^2_{L^2(Q)} + C\int_Q (s\lambda\varphi_0 + \lambda^2)|\partial_x(a_0\partial_x^2 z)|^2\,dxdt + C\int_Q (s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2\,dxdt\\
&\quad + C\int_Q \big(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\big)|\partial_x z|^2\,dxdt + C\int_Q \big(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\big)|z|^2\,dxdt\\
&\quad + C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt + C\int_{\omega_2\times(0,T)} \big(s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\big)\,dxdt.
\end{aligned}\tag{3.45}$$

We carry out a similar argument again. We take a cut-off function $\rho_3 \in C_0^\infty(\Omega)$ such that $0 \le \rho_3 \le 1$ in Ω, $\operatorname{supp}\rho_3 \subset \omega$ and $\rho_3 \equiv 1$ in ω_2. Then we estimate the following integral

$$-\int_Q \rho_3 s^5\lambda^6\varphi_0^5 z\partial_x^2 z\,dxdt$$

from above and below.

$$-\int_Q \rho_3 s^5\lambda^6\varphi_0^5 z\partial_x^2 z\,dxdt \le \frac{1}{\eta_3}\int_Q s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2\,dxdt + 4\eta_3\int_{\omega\times(0,T)} s^7\lambda^8\varphi_0^7|z|^2\,dxdt \tag{3.46}$$

for $\eta_3 > 0$. On the other hand

$$\begin{aligned}
&-\int_Q \rho_3 s^5\lambda^6\varphi_0^5 z\partial_x^2 z\,dxdt\\
&= \int_Q (\partial_x\rho_3)s^5\lambda^6\varphi_0^5 z\partial_x z\,dxdt + 5\int_Q \rho_3 s^5\lambda^7\varphi_0^5(\partial_x d_0)z\partial_x z\,dxdt + \int_Q \rho_3 s^5\lambda^6\varphi_0^5|\partial_x z|^2\,dxdt\\
&\ge \int_{\omega_2\times(0,T)} s^5\lambda^6\varphi_0^5|\partial_x z|^2\,dxdt - C\int_Q s^4\lambda^6\varphi_0^4|\partial_x z|^2\,dxdt - C\int_Q (s^6\lambda^8\varphi_0^6 + s^6\lambda^6\varphi_0^6)|z|^2\,dxdt.
\end{aligned}\tag{3.47}$$

Combining (3.46) with (3.47), we have

$$\begin{aligned}
&\int_{\omega_2\times(0,T)} s^5\lambda^6\varphi_0^5|\partial_x z|^2\,dxdt\\
&\le \frac{1}{\eta_3}\int_Q s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2\,dxdt + 4\eta_3\int_{\omega\times(0,T)} s^7\lambda^8\varphi_0^7|z|^2\,dxdt\\
&\quad + C\int_Q s^4\lambda^6\varphi_0^4|\partial_x z|^2\,dxdt + C\int_Q (s^6\lambda^8\varphi_0^6 + s^6\lambda^6\varphi_0^6)|z|^2\,dxdt.
\end{aligned}\tag{3.48}$$

By (3.45) and (3.48), we obtain

$$\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_0}\big(|\partial_t z|^2 + |\partial_x^2(a_0\partial_x^2 z)|^2\big) + s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2\right.\\
&\qquad \left. + \left(1 - \frac{C}{\eta_3}\right)s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\right]dxdt
\end{aligned}$$

$$
\begin{aligned}
&\le C\|Pz\|^2_{L^2(Q)} + C\int_Q (s\lambda\varphi_0 + \lambda^2)|\partial_x(a_0\partial_x^2 z)|^2\,dxdt + C\int_Q (s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2\,dxdt\\
&\quad + C\int_Q \big(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\big)|\partial_x z|^2\,dxdt + C\int_Q \big(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\big)|z|^2\,dxdt\\
&\quad + C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt + C\int_{\omega\times(0,T)} (4\eta_3+1)s^7\lambda^8\varphi_0^7|z|^2\,dxdt
\end{aligned}
$$

for large $s > 0$ and $\lambda > 0$. Taking sufficient large $\eta_3 > 0$, we get

$$
\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_0}\big(|\partial_t z|^2 + |\partial_x^2(a_0\partial_x^2 z)|^2\big) + s\lambda^2\varphi_0|\partial_x(a_0\partial_x^2 z)|^2 + s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\right] dxdt\\
&\le C\|Pz\|^2_{L^2(Q)} + C\int_Q (s\lambda\varphi_0 + \lambda^2)|\partial_x(a_0\partial_x^2 z)|^2\,dxdt + C\int_Q (s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2)|\partial_x^2 z|^2\,dxdt\\
&\quad + C\int_Q \big(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\big)|\partial_x z|^2\,dxdt + C\int_Q \big(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\big)|z|^2\,dxdt\\
&\quad + C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt + C\int_{\omega\times(0,T)} s^7\lambda^8\varphi_0^7|z|^2\,dxdt. \qquad (3.49)
\end{aligned}
$$

By $\partial_x(a_0\partial_x^2 z) = a_0\partial_x^3 z + (\partial_x a_0)(\partial_x^2 z)$, we get

$$
|\partial_x(a_0\partial_x^2 z)|^2 \ge \frac{1}{2m^2}|\partial_x^3 z|^2 - C|\partial_x^2 z|^2, \quad |\partial_x(a_0\partial_x^2 z)|^2 \le C|\partial_x^3 z|^2 + C|\partial_x^2 z|^2.
$$

Similarly, $\partial_x^2(a_0\partial_x^2 z) = a_0\partial_x^4 z + 2(\partial_x a_0)(\partial_x^3 z) + (\partial_x^2 a_0)(\partial_x^2 z)$ implies that

$$
|\partial_x^2(a_0\partial_x^2 z)|^2 \ge \frac{1}{2m^2}|\partial_x^4 z|^2 - C|\partial_x^3 z|^2 - C|\partial_x^2 z|^2.
$$

By (3.49), we have

$$
\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_0}\big(|\partial_t z|^2 + |\partial_x^4 z|^2\big) + s\lambda^2\varphi_0|\partial_x^3 z|^2 + s^3\lambda^4\varphi_0^3|\partial_x^2 z|^2 + s^5\lambda^6\varphi_0^5|\partial_x z|^2 + s^7\lambda^8\varphi_0^7|z|^2\right] dxdt\\
&\le C\|Pz\|^2_{L^2(Q)} + C\int_Q \big(s\lambda\varphi_0 + \lambda^2\big)|\partial_x^3 z|^2\,dxdt + C\int_Q \big(s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2\big)|\partial_x^2 z|^2\,dxdt\\
&\quad + C\int_Q \big(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\big)|\partial_x z|^2\,dxdt + C\int_Q \big(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\big)|z|^2\,dxdt\\
&\quad + C(\lambda)\int_Q s^2\varphi_0^4|z|^2\,dxdt + C\int_{\omega\times(0,T)} s^7\lambda^8\varphi_0^7|z|^2\,dxdt \qquad (3.50)
\end{aligned}
$$

for large $s > 0$ and $\lambda > 0$. Rewriting (3.50) in terms of u, we get

$$
\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_0}\big(|\partial_t u|^2 + |\partial_x^4 u|^2\big) + s\lambda^2\varphi_0|\partial_x^3 u|^2 + s^3\lambda^4\varphi_0^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_0^5|\partial_x u|^2 + s^7\lambda^8\varphi_0^7|u|^2\right] e^{2s\alpha_0}\,dxdt\\
&\le C\int_Q |\partial_t u - \partial_x^2(a_0\partial_x^2 u)|^2 e^{2s\alpha_0}\,dxdt + C\int_Q \big(s\lambda\varphi_0 + \lambda^2\big)|\partial_x^3 u|^2 e^{2s\alpha_0}\,dxdt\\
&\quad + C\int_Q \big(s^3\lambda^3\varphi_0^3 + s^2\lambda^4\varphi_0^2\big)|\partial_x^2 u|^2 e^{2s\alpha_0}\,dxdt + C\int_Q \big(s^5\lambda^5\varphi_0^5 + s^4\lambda^6\varphi_0^4\big)|\partial_x u|^2 e^{2s\alpha_0}\,dxdt\\
&\quad + C\int_Q \big(s^7\lambda^7\varphi_0^7 + s^6\lambda^8\varphi_0^6\big)|u|^2 e^{2s\alpha_0}\,dxdt\\
&\quad + C(\lambda)\int_Q s^2\varphi_0^4|u|^2 e^{2s\alpha_0}\,dxdt + C\int_{\omega\times(0,T)} s^7\lambda^8\varphi_0^7|u|^2 e^{2s\alpha_0}\,dxdt. \qquad (3.51)
\end{aligned}
$$

for large $s > 0$ and $\lambda > 0$. Taking sufficient large $\lambda > 0$, we have

$$
\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_0} \left(|\partial_t u|^2 + |\partial_x^4 u|^2 \right) + s\lambda^2 \varphi_0 |\partial_x^3 u|^2 + s^3\lambda^4\varphi_0^3 |\partial_x^2 u|^2 + s^5\lambda^6\varphi_0^5 |\partial_x u|^2 + s^7\lambda^8\varphi_0^7 |u|^2 \right] e^{2s\alpha_0}\, dxdt \\
&\leq C' \int_Q |\partial_t u - \partial_x^2(a_0 \partial_x^2 u)|^2 e^{2s\alpha_0}\, dxdt + C' \int_Q \lambda^2 |\partial_x^3 u|^2 e^{2s\alpha_0}\, dxdt + C' \int_Q s^2\lambda^4\varphi_0^2 |\partial_x^2 u|^2 e^{2s\alpha_0}\, dxdt \\
&\quad + C' \int_Q s^4\lambda^6\varphi_0^4 |\partial_x u|^2 e^{2s\alpha_0}\, dxdt + C' \int_Q s^6\lambda^8\varphi_0^6 |u|^2 e^{2s\alpha_0}\, dxdt \\
&\quad + C'(\lambda) \int_Q s^2\varphi_0^4 |u|^2 e^{2s\alpha_0}\, dxdt + C' \int_{\omega\times(0,T)} s^7\lambda^8\varphi_0^7 |u|^2 e^{2s\alpha_0}\, dxdt. \qquad (3.52)
\end{aligned}
$$

Here C' is a constant depending on λ_0. Next we choose sufficient large $s > 0$, we can absorb the second term to the sixth term on the right-hand side of (3.52).

$$
\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_0} \left(|\partial_t u|^2 + |\partial_x^4 u|^2 \right) + s\lambda^2 \varphi_0 |\partial_x^3 u|^2 + s^3\lambda^4\varphi_0^3 |\partial_x^2 u|^2 + s^5\lambda^6\varphi_0^5 |\partial_x u|^2 + s^7\lambda^8\varphi_0^7 |u|^2 \right] e^{2s\alpha_0}\, dxdt \\
&\leq C'' \int_Q |\partial_t u - \partial_x^2(a_0 \partial_x^2 u)|^2 e^{2s\alpha_0}\, dxdt + C'' \int_{\omega\times(0,T)} s^7\lambda^8\varphi_0^7 |u|^2 e^{2s\alpha_0}\, dxdt. \qquad (3.53)
\end{aligned}
$$

Here C'' is a constant depending on $s_0(\lambda)$, λ_0. Thus we complete the proof. $\square$

3.1.2 Proof of Theorem 2.5

Note that $\partial_t - L_2^2$ is a fourth order parabolic operator and coefficients of L_2^2 satisfy the assumption for coefficients of L_4. Hence, by Lemma 2.3, we may obtain the Carleman estimate with interior data for (1.1) : Theorem 2.5. $\square$

3.1.3 Proof of Theorem 2.6

Since a constant $C > 0$ exists such that $|\partial_t u - \partial_x^2(a^2\partial_x^2 u)|^2 \leq C|(\partial_t - L_2^2)u|^2 + C|\partial_x^3 u|^2 + C|\partial_x^2 u|^2 + C|\partial_x u|^2 + C|u|^2$, it is sufficient to prove the Carleman estimate for $\partial_t u - \partial_x^2(a^2\partial_x^2 u)$.

Setting $\widetilde{z} = ue^{s\alpha_1}$, by means of an argument to the proof of Lemma 2.3, we obtain the following inequality as we get (3.34).

$$
\begin{aligned}
&\int_Q \frac{1}{s\varphi_1} \left(|\partial_t \widetilde{z}|^2 + |\partial_x^2(a^2\partial_x^2 \widetilde{z})|^2 \right) dxdt + \int_Q s\lambda^2\varphi_1 (\partial_x d_1)^2 |\partial_x(a^2\partial_x^2\widetilde{z})|^2\, dxdt \\
&\quad + \int_Q s^3\lambda^4\varphi_1^3 (\partial_x d_1)^4 a^4 |\partial_x^2 \widetilde{z}|^2\, dxdt + \int_Q s^5\lambda^6\varphi_1^5 (\partial_x d_1)^6 a^4 |\partial_x \widetilde{z}|^2\, dxdt + \int_Q s^7\lambda^8\varphi_1^7 (\partial_x d_1)^8 a^4 |\widetilde{z}|^2\, dxdt \\
&\leq C \|(\partial_t u - \partial_x^2(a^2\partial_x^2 u))e^{s\alpha}\|_{L^2(Q)}^2 + C \int_Q (s\lambda\varphi_1 + \lambda^2) |\partial_x(a^2\partial_x^2\widetilde{z})|^2\, dxdt \\
&\quad + C \int_Q (s^3\lambda^3\varphi_1^3 + s^2\lambda^4\varphi_1^2) |\partial_x^2\widetilde{z}|^2\, dxdt + C \int_Q \left(s^5\lambda^5\varphi_1^5 + s^4\lambda^6\varphi_1^4 \right) |\partial_x \widetilde{z}|^2\, dxdt \\
&\quad + C \int_Q \left(s^7\lambda^7\varphi_1^7 + s^6\lambda^8\varphi_1^6 \right) |\widetilde{z}|^2\, dxdt + C(\lambda) \int_Q s^2\varphi_1^4 |\widetilde{z}|^2\, dxdt + C\widetilde{B} \qquad (3.54)
\end{aligned}
$$

where

$$\begin{aligned}
\widetilde{B} &= 2\int_0^T \left[s^7\lambda^7\varphi_1^7(\partial_x d_1)^7 a^4|\widetilde{z}|^2 \right]_{x=0}^L dt + 16\int_0^T \left[s^5\lambda^6\varphi_1^5(\partial_x d_1)^6 a^4 \widetilde{z}\partial_x\widetilde{z} \right]_{x=0}^L dt \\
&\quad + 4\int_0^T \left[s^5\lambda^5\varphi_1^5(\partial_x d_1)^5 a^4 \widetilde{z}\partial_x^2\widetilde{z} \right]_{x=0}^L dt + 6\int_0^T \left[s^3\lambda^4\varphi_1^3(\partial_x d_1)^4 a^2 \widetilde{z}\partial_x(a^2\partial_x^2\widetilde{z}) \right]_{x=0}^L dt \\
&\quad + 6\int_0^T \left[s^2\lambda^2\varphi_1^2(\partial_x d_1)^2 a^2 (\partial_t\widetilde{z})\partial_x\widetilde{z} \right]_{x=0}^L dt + \int_0^T \left[(\partial_t\widetilde{z})\partial_x(a^2\partial_x^2\widetilde{z}) \right]_{x=0}^L dt \\
&\quad + 10\int_0^T \left[s^5\lambda^5\varphi_1^5(\partial_x d_1)^5 a^4|\partial_x\widetilde{z}|^2 \right]_{x=0}^L dt + 30\int_0^T \left[s^3\lambda^4\varphi_1^3(\partial_x d_1)^4 a^4 (\partial_x\widetilde{z})\partial_x^2\widetilde{z} \right]_{x=0}^L dt \\
&\quad + 4\int_0^T \left[s^3\lambda^3\varphi_1^3(\partial_x d_1)^3 a^2 (\partial_x\widetilde{z})\partial_x(a^2\partial_x^2\widetilde{z}) \right]_{x=0}^L dt + 10\int_0^T \left[s^3\lambda^3\varphi_1^3(\partial_x d_1)^3 a^4|\partial_x^2\widetilde{z}|^2 \right]_{x=0}^L dt \\
&\quad + 2\int_0^T \left[s\lambda\varphi_1(\partial_x d_1)\left|\partial_x(a^2\partial_x^2\widetilde{z})\right|^2 \right]_{x=0}^L dt - \int_0^T \left[a^2(\partial_x\partial_t\widetilde{z})\partial_x^2\widetilde{z} \right]_{x=0}^L dt.
\end{aligned}$$

Noting that $\frac{1}{\mu} \le a$ and $|\partial_x d_1| \ge \sigma_1$ in $\overline{\Omega}$, we have

$$\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_1}\left(|\partial_t\widetilde{z}|^2 + |\partial_x^2(a^2\partial_x^2\widetilde{z})|^2\right) + s\lambda^2\varphi_1|\partial_x(a^2\partial_x^2\widetilde{z})|^2 + s^3\lambda^4\varphi_1^3|\partial_x^2\widetilde{z}|^2 + s^5\lambda^6\varphi_1^5|\partial_x\widetilde{z}|^2 + s^7\lambda^8\varphi_1^7|\widetilde{z}|^2 \right] dxdt \\
&\le C\|(\partial_t u - \partial_x^2(a^2\partial_x^2 u))e^{s\alpha_1}\|^2_{L^2(Q)} + C\int_Q (s\lambda\varphi_1 + \lambda^2)|\partial_x(a^2\partial_x^2\widetilde{z})|^2\, dxdt \\
&\quad + C\int_Q (s^3\lambda^3\varphi_1^3 + s^2\lambda^4\varphi_1^2)|\partial_x^2\widetilde{z}|^2\, dxdt + C\int_Q \left(s^5\lambda^5\varphi_1^5 + s^4\lambda^6\varphi_1^4\right)|\partial_x\widetilde{z}|^2\, dxdt \\
&\quad + C\int_Q \left(s^7\lambda^7\varphi_1^7 + s^6\lambda^8\varphi_1^6\right)|\widetilde{z}|^2\, dxdt + C(\lambda)\int_Q s^2\varphi_1^4|\widetilde{z}|^2\, dxdt + C\widetilde{B}.
\end{aligned} \tag{3.55}$$

By an argument similar to that used to derive (3.53) from (3.50) in the proof of Lemma 2.3, we have

$$\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_1}\left(|\partial_t u|^2 + |\partial_x^4 u|^2\right) + s\lambda^2\varphi_1|\partial_x^3 u|^2 + s^3\lambda^4\varphi_1^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_1^5|\partial_x u|^2 + s^7\lambda^8\varphi_1^7|u|^2 \right] e^{2s\alpha_1}\, dxdt \\
&\le C''\int_Q |\partial_t u - \partial_x^2(a^2\partial_x^2 u)|^2 e^{2s\alpha_1}\, dxdt + C''\widetilde{B}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\int_Q \left[\frac{1}{s\varphi_1}\left(|\partial_t u|^2 + |\partial_x^4 u|^2\right) + s\lambda^2\varphi_1|\partial_x^3 u|^2 + s^3\lambda^4\varphi_1^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_1^5|\partial_x u|^2 + s^7\lambda^8\varphi_1^7|u|^2 \right] e^{2s\alpha_1}\, dxdt \\
&\le C''\int_Q |(\partial_t u - L_2^2)u|^2 e^{2s\alpha_1}\, dxdt + C''\widetilde{B}.
\end{aligned} \tag{3.56}$$

Estimation of boundary terms. We estimate boundary terms $\widetilde{B}$. By the assumption $u = 0$ on Σ, we have $\widetilde{z} = \partial_t\widetilde{z} = 0$ on Σ. Hence we get

$$\begin{aligned}
\widetilde{B} &= 10\int_0^T \left[s^5\lambda^5\varphi_1^5(\partial_x d_1)^5 a^4|\partial_x\widetilde{z}|^2 \right]_{x=0}^L dt + 30\int_0^T \left[s^3\lambda^4\varphi_1^3(\partial_x d_1)^4 a^4 (\partial_x\widetilde{z})\partial_x^2\widetilde{z} \right]_{x=0}^L dt \\
&\quad + 4\int_0^T \left[s^3\lambda^3\varphi_1^3(\partial_x d_1)^3 a^2 (\partial_x\widetilde{z})\partial_x(a^2\partial_x^2\widetilde{z}) \right]_{x=0}^L dt + 10\int_0^T \left[s^3\lambda^3\varphi_1^3(\partial_x d_1)^3 a^4|\partial_x^2\widetilde{z}|^2 \right]_{x=0}^L dt \\
&\quad + 2\int_0^T \left[s\lambda\varphi_1(\partial_x d_1)\left|\partial_x(a^2\partial_x^2\widetilde{z})\right|^2 \right]_{x=0}^L dt - \int_0^T \left[a^2(\partial_x\partial_t\widetilde{z})\partial_x^2\widetilde{z} \right]_{x=0}^L dt.
\end{aligned}$$

Since $u=0$ on Σ, we have

$$\partial_x \widetilde{z} = (\partial_x u)e^{s\alpha_1} \quad \text{on } \Sigma.$$

Noting that $g=0$ and $\partial_t^{\frac{1}{2}} u = 0$ on Σ, we see that

$$\partial_x^2 \widetilde{z} = -\frac{\partial_x a + b}{a}(\partial_x u)e^{s\alpha_1} + 2s\lambda\varphi_1(\partial_x d_1)(\partial_x u)e^{s\alpha_1} \quad \text{on } \Sigma.$$

Moreover, by $g = \partial_x g = 0$ on Σ, we obtain

$$\begin{aligned}\partial_x^3 \widetilde{z} = {} & \frac{1}{a}(\partial_t^{\frac{1}{2}}\partial_x u)e^{s\alpha_1} \\ & + \left[\frac{1}{a^2}(\partial_x a)(\partial_x a + b) + \frac{1}{a^2}(\partial_x a + b)^2 - \frac{1}{a}(\partial_x^2 a + \partial_x b + c) - \frac{3}{a}s\lambda\varphi_1(\partial_x d_1)(\partial_x a + b)\right. \\ & \qquad \left. + 3s^2\lambda^2\varphi_1^2(\partial_x d_1)^2 + 3s\lambda^2\varphi_1(\partial_x d_1)^2 + 3s\lambda\varphi_1(\partial_x^2 d_1)\right](\partial_x u)e^{s\alpha_1}.\end{aligned}$$

Hence we have

$$\begin{aligned}\widetilde{B} \le {} & Ce^{C(\lambda)s}\int_0^T \left(|\partial_x u(0,t)|^2 + |\partial_t^{\frac{1}{2}}\partial_x u(0,t)|^2 + |\partial_t\partial_x u(0,t)|^2\right) dt \\ & + Ce^{C(\lambda)s}\int_0^T \left(|\partial_x u(L,t)|^2 + |\partial_t^{\frac{1}{2}}\partial_x u(L,t)|^2 + |\partial_t\partial_x u(L,t)|^2\right) dt. \end{aligned} \tag{3.57}$$

By (3.56) and (3.57), we obtain

$$\begin{aligned}& \int_Q \left[\frac{1}{s\varphi_1}\left(|\partial_t u|^2 + |\partial_x^4 u|^2\right) + s\lambda^2\varphi_1|\partial_x^3 u|^2 + s^3\lambda^4\varphi_1^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi_1^5|\partial_x u|^2 + s^7\lambda^8\varphi_1^7|u|^2\right] e^{2s\alpha_1}\, dxdt \\ & \le C''\int_Q |\partial_t u - \partial_x^2(a^2\partial_x^2 u)|^2 e^{2s\alpha_1}\, dxdt + C''e^{C(\lambda)s}\int_0^T \left(|\partial_x u(0,t)|^2 + |\partial_t^{\frac{1}{2}}\partial_x u(0,t)|^2 + |\partial_t\partial_x u(0,t)|^2\right) dt \\ & \quad + C''e^{C(\lambda)s}\int_0^T \left(|\partial_x u(L,t)|^2 + |\partial_t^{\frac{1}{2}}\partial_x u(L,t)|^2 + |\partial_t\partial_x u(L,t)|^2\right) dt.\end{aligned}$$

Thus we complete the proof. $\square$

3.2 Proofs of the Lipschitz stability estimate in inverse source problems

3.2.1 Proof of Theorem 2.7

Proof. By (2.5) and (2.6), we have

$$\partial_t u(x,t) - L_2^2 u(x,t) = \widetilde{g}(x,t), \quad (x,t) \in Q \tag{3.58}$$

where

$$\widetilde{g}(x,t) = f(x)\partial_t^{\frac{1}{2}}R(x,t) + L_2(f(x)R(x,t)) + f(x)\frac{R(x,0)}{\sqrt{\pi t}}, \quad (x,t) \in Q. \tag{3.59}$$

By (3.58) at $t = t_0$, we see that

$$\widetilde{g}(x,t_0) = \partial_t u(x,t_0) - L_2^2 u(x,t_0), \quad x \in \Omega.$$

Hence we have

$$\int_\Omega |\widetilde{g}(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le C\int_\Omega |\partial_t u(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx + C\int_\Omega |L_2^2 u(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx$$
$$\le C\int_\Omega |\partial_t u(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx + Ce^{Cs}\int_\Omega \sum_{j=0}^4 |\partial_x^j u(x,t_0)|^2\,dx. \tag{3.60}$$

First we estimate the following integral

$$\int_\Omega |\partial_t u(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx$$

by the Carleman estimate.

Setting $y = \partial_t u$, (3.58) and (2.7) give us

$$\partial_t y(x,t) - L_2^2 y(x,t) = \partial_t \widetilde{g}(x,t), \qquad (x,t)\in Q, \tag{3.61}$$
$$y(x,t) = \partial_x y(x,t) = 0, \qquad (x,t)\in \Sigma. \tag{3.62}$$

Fix $\lambda > 0$. With an argument similar to that used to prove the Theorem 2.5, we may prove the following Carleman estimate for (3.61) and (3.62):

$$\int_{Q_\delta}\left[\frac{1}{s\varphi_{\delta,0}}\left(|\partial_t y|^2 + |\partial_x^4 y|^2\right) + s\varphi_{\delta,0}|\partial_x^3 y|^2 + s^3\varphi_{\delta,0}^3|\partial_x^2 y|^2 + s^5\varphi_{\delta,0}^5|\partial_x y|^2 + s^7\varphi_{\delta,0}^7|y|^2\right] e^{2s\alpha_{\delta,0}}\,dxdt$$
$$\le C\int_{Q_\delta}|\partial_t \widetilde{g}|^2 e^{2s\alpha_{\delta,0}}\,dxdt + Ce^{Cs}\int_{Q_{\omega,\delta}}|y|^2\,dxdt. \tag{3.63}$$

Noting that $\lim_{t\to t_0-\delta+0} e^{2s\alpha_{\delta,0}(x,t_0)} = 0$, $x\in\overline{\Omega}$, we obtain

$$\int_\Omega |y(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx = \int_{t_0-\delta}^{t_0}\partial_t\left(\int_\Omega |y|^2 e^{2s\alpha_{\delta,0}}\,dx\right)dt$$
$$= \int_\Omega\int_{t_0-\delta}^{t_0}\left[2y\partial_t y + 2s(\partial_t\alpha_{\delta,0})|y|^2\right] e^{2s\alpha_{\delta,0}}\,dtdx$$
$$\le C\int_{Q_\delta}|y||\partial_t y| e^{2s\alpha_{\delta,0}}\,dxdt + C\int_{Q_\delta} s\varphi_{\delta,0}^2|y|^2 e^{2s\alpha_{\delta,0}}\,dxdt. \tag{3.64}$$

In the last inequality, we used $|\partial_t\alpha_{\delta,0}| \le C\varphi_{\delta,0}^2$. By the Schwarz inequality, we get

$$|y||\partial_t y| = \left(\frac{1}{s^2\sqrt{\varphi_{\delta,0}}}|\partial_t y|\right)\left(s^2\sqrt{\varphi_{\delta,0}}|y|\right)$$
$$\le C\frac{1}{s^4\varphi_{\delta,0}}|\partial_t y|^2 + Cs^4\varphi_{\delta,0}|y|^2. \tag{3.65}$$

Combining (3.64) with (3.65), we have

$$\int_\Omega |y(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le C\int_{Q_\delta}\frac{1}{s^4\varphi_{\delta,0}}|\partial_t y|^2 e^{2s\alpha_{\delta,0}}\,dxdt + C\int_{Q_\delta} s^4\varphi_{\delta,0}^2|y|^2 e^{2s\alpha_{\delta,0}}\,dxdt \tag{3.66}$$

for large $s > 0$. By (3.63) and (3.66), we obtain

$$\int_\Omega |y(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le \frac{C}{s^3}\int_{Q_\delta}|\partial_t\widetilde{g}|^2 e^{2s\alpha_{\delta,0}}\,dxdt + Ce^{Cs}\int_{Q_{\omega,\delta}}|y|^2\,dxdt.$$

That is

$$\int_\Omega |\partial_t u(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le \frac{C}{s^3}\int_{Q_\delta} |\partial_t \widetilde{g}|^2 e^{2s\alpha_{\delta,0}}\,dxdt + Ce^{Cs}\int_{Q_{\omega,\delta}} |\partial_t u|^2\,dxdt. \tag{3.67}$$

Let us estimate $|\partial_t \widetilde{g}|^2$. Since

$$\begin{aligned}\partial_t \widetilde{g} &= (\partial_t \partial_t^{\frac{1}{2}} R)f + a(\partial_t R)\partial_x^2 f + [(\partial_x a)\partial_t R + 2a(\partial_t\partial_x R)]\partial_x f + \partial_x(a\partial_t\partial_x R)f \\ &\quad + b(\partial_t R)\partial_x f + b(\partial_t\partial_x R)f + c(\partial_t R)f - \frac{R(x,0)}{\sqrt{\pi}}\cdot\frac{1}{2t\sqrt{t}}f \quad \text{in } Q,\end{aligned}$$

we have

$$|\partial_t \widetilde{g}|^2 \le C\sum_{k=0}^{2} |\partial_x^k f|^2 \quad \text{in } Q_\delta. \tag{3.68}$$

Here we note that C depend on t_0 and δ. Therefore (3.67) and (3.68) yield

$$\int_\Omega |\partial_t u(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le \frac{C}{s^3}\int_{Q_\delta}\sum_{k=0}^{2} |\partial_x^k f|^2 e^{2s\alpha_{\delta,0}}\,dxdt + Ce^{Cs}\int_{Q_{\omega,\delta}} |\partial_t u|^2\,dxdt.$$

Moreover, by $\alpha_{\delta,0}(x,t) \le \alpha_{\delta,0}(x,t_0)$ for $(x,t)\in\overline{Q_\delta}$, we have

$$\int_\Omega |\partial_t u(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le \frac{C}{s^3}\int_{Q_\delta}\sum_{k=0}^{2} |\partial_x^k f|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dxdt + Ce^{Cs}\int_{Q_{\omega,\delta}} |\partial_t u|^2\,dxdt.$$

Together this with (3.60), we obtain

$$\begin{aligned}\int_\Omega |\widetilde{g}(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx &\le \frac{C}{s^3}\int_{Q_\delta}\sum_{k=0}^{2} |\partial_x^k f|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dxdt + Ce^{Cs}\int_{Q_{\omega,\delta}} |\partial_t u|^2\,dxdt \\ &\quad + Ce^{Cs}\int_\Omega \sum_{j=0}^{4} |\partial_x^j u(x,t_0)|^2\,dx. \end{aligned}\tag{3.69}$$

Next we estimate

$$\int_\Omega |\widetilde{g}(x,t_0)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx$$

from below. We remind

$$\begin{aligned}\widetilde{g}(\cdot,t_0) &= aR(\cdot,t_0)\partial_x^2 f + \big((\partial_x a)R(\cdot,t_0) + 2a\partial_x R(\cdot,t_0) + bR(\cdot,t_0)\big)(\partial_x f) \\ &\quad + \left(L_2 R(\cdot,t_0) + \partial_t^{\frac{1}{2}} R(\cdot,t_0) + \frac{R(\cdot,0)}{\sqrt{\pi t_0}}\right) f \quad \text{in } \Omega\end{aligned}$$

is an elliptic equation with respect to f in Ω. Noting that $|aR(\cdot,t_0)| > 0$ in $\overline{\Omega}$ and $f(0) = \partial_x f(0) = 0$, we apply the Carleman estimate for elliptic equations (Lemma 4.1 in Appendix):

$$\begin{aligned}&\int_\Omega \left[\frac{1}{s\varphi_{\delta,0}(x,t_0)}|\partial_x^2 f(x)|^2 + s\varphi_{\delta,0}(x,t_0)|\partial_x f(x)|^2 + s^3\varphi_{\delta,0}^3(x,t_0)|f(x)|^2\right] e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \\ &\le C\int_\Omega |\widetilde{g}(x,t_0)|^2 e^{2s\alpha_{\delta,0(x,t_0)}}\,dx + Ce^{Cs}\int_\omega |f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx.\end{aligned}$$

Hence we have

$$\frac{1}{s}\int_\Omega \sum_{k=0}^{2} |\partial_x^k f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx$$
$$\le C\int_\Omega |\widetilde{g}(x,t_0)|^2 e^{2s\alpha_{\delta,0(x,t_0)}}\,dx + Ce^{Cs}\int_\omega |f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx. \tag{3.70}$$

By (3.69) and (3.70), we obtain

$$\frac{1}{s}\int_\Omega \sum_{k=0}^{2} |\partial_x^k f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le \frac{C}{s^3}\int_{Q_\delta} \sum_{k=0}^{2} |\partial_x^k f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dxdt + Ce^{Cs}\int_{Q_{\omega,\delta}} |\partial_t u|^2\,dxdt$$
$$+ Ce^{Cs}\int_\Omega \sum_{j=0}^{4} |\partial_x^j u(x,t_0)|^2\,dx + Ce^{Cs}\int_\omega |f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx.$$

So we have

$$\left(\frac{1}{s}-\frac{C}{s^3}\right)\int_\Omega \sum_{k=0}^{2} |\partial_x^k f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le Ce^{Cs}\int_{Q_{\omega,\delta}} |\partial_t u|^2\,dxdt + Ce^{Cs}\int_\Omega \sum_{j=0}^{4} |\partial_x^j u(x,t_0)|^2\,dx$$
$$+ Ce^{Cs}\int_\omega |f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx.$$

Taking sufficient large $s>0$, we have

$$\int_\Omega \sum_{k=0}^{2} |\partial_x^k f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le Ce^{Cs}\int_{Q_{\omega,\delta}} |\partial_t u|^2\,dxdt + Ce^{Cs}\int_\Omega \sum_{j=0}^{4} |\partial_x^j u(x,t_0)|^2\,dx$$
$$+ Ce^{Cs}\int_\omega |f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx. \tag{3.71}$$

In the end, let us estimate

$$\int_\omega |f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx.$$

$$\begin{aligned}
\int_\omega |f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx &= \int_\omega \left|\frac{1}{R(x,t_0)}\left[\partial_t^{\frac{1}{2}} u(x,t_0) - L_2 u(x,t_0)\right]\right|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \\
&\le C\int_\omega \left|\partial_t^{\frac{1}{2}} u(x,t_0)\right|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx + Ce^{Cs}\int_\omega \sum_{j=0}^{2} \left|\partial_x^j u(x,t_0)\right|^2\,dx \\
&\le C\int_{t_0-\delta}^{t_0} \partial_t\left(\int_\omega |\partial_t^{\frac{1}{2}} u|^2 e^{2s\alpha_{\delta,0}}\,dx\right)\,dt + Ce^{Cs}\int_\Omega \sum_{j=0}^{2} \left|\partial_x^j u(x,t_0)\right|^2\,dx \\
&\le C\int_\omega \int_{t_0-\delta}^{t_0} \left[2(\partial_t^{\frac{1}{2}} u)(\partial_t\partial_t^{\frac{1}{2}} u) + 2s(\partial_t\alpha_{\delta,0})|\partial_t^{\frac{1}{2}} u|^2\right] e^{2s\alpha_{\delta,0}}\,dtdx \\
&\quad + Ce^{Cs}\int_\Omega \sum_{j=0}^{2} \left|\partial_x^j u(x,t_0)\right|^2\,dx \\
&\le Ce^{Cs}\int_{Q_{\omega,\delta}} \left(|\partial_t^{\frac{1}{2}} u|^2 + |\partial_t\partial_t^{\frac{1}{2}} u|^2\right)\,dxdt + Ce^{Cs}\int_\Omega \sum_{j=0}^{2} \left|\partial_x^j u(x,t_0)\right|^2\,dx.
\end{aligned} \tag{3.72}$$

Hence (3.71) and (3.72) give us

$$\int_\Omega \sum_{k=0}^{2} |\partial_x^k f(x)|^2 e^{2s\alpha_{\delta,0}(x,t_0)}\,dx \le Ce^{Cs}\int_{Q_{\omega,\delta}} \left(|\partial_t^{\frac12} u|^2 + |\partial_t u|^2 + |\partial_t\partial_t^{\frac12} u|^2\right) dxdt$$

$$+ Ce^{Cs}\int_\Omega \sum_{j=0}^{4} |\partial_x^j u(x,t_0)|^2\,dx.$$

Fixing $s > 0$, we may obtain the stability estimate. Thus we complete the proof. □

3.2.2 Proof of Theorem 2.8

We may obtain the following Carleman estimate for (3.61) and $y(x,t) = 0$, $(x,t) \in \Sigma$:

$$\begin{aligned}&\int_{Q_\delta} \left[\frac{1}{s\varphi_{\delta,1}}\left(|\partial_t y|^2 + |\partial_x^4 y|^2\right) + s\varphi_{\delta,1}|\partial_x^3 y|^2 + s^3\varphi_{\delta,1}^3|\partial_x^2 y|^2 + s^5\varphi_{\delta,1}^5|\partial_x y|^2 + s^7\varphi_{\delta,1}^7|y|^2\right] e^{2s\alpha_{\delta,1}}\,dxdt\\
&\le C\int_{Q_\delta} |\partial_t\widetilde{g}|^2 e^{2s\alpha_{\delta,1}}\,dxdt + Ce^{Cs}\int_{t_0-\delta}^{t_0+\delta}\left(|\partial_t\partial_x u(0,t)|^2 + |\partial_t\partial_t^{\frac12}\partial_x u(0,t)|^2 + |\partial_t^2\partial_x u(0,t)|^2\right) dt\\
&\quad + Ce^{Cs}\int_{t_0-\delta}^{t_0+\delta}\left(|\partial_t\partial_x u(L,t)|^2 + |\partial_t\partial_t^{\frac12}\partial_x u(L,t)|^2 + |\partial_t^2\partial_x u(L,t)|^2\right) dt. \qquad (3.73)\end{aligned}$$

Here we note that $\partial_t^{\frac12}\partial_t \neq \partial_t\partial_t^{\frac12}$. In the proof of (3.73), we may estimate boundary terms as the estimation of boundary terms in the proof of Theorem 2.6.

Using the Carleman esitmate (3.73), we perform an argument similar to the proof of Theorem 2.7 and then we obtain the conditional stability estimate in our inverse problem for (2.8)–(2.10). □

4 Appendix

In this appendix, we state the Carleman estimate for second order elliptic equations. We used this estimate in the proof of Lipschitz stability estimate in inverse source problems.

we consider the following elliptic equations:

$$\widetilde{L}_2 v(x) = h(x), \quad x \in \Omega, \qquad (4.1)$$

where

$$\widetilde{L}_2 v(x) = \partial_x(\widetilde{a}(x)\partial_x v(x)) + \widetilde{b}(x)\partial_x v(x) + \widetilde{c}(x)v(x), \quad x \in \Omega.$$

We assume that $\widetilde{a} \in C^1(\overline{\Omega})$, $\widetilde{b}, \widetilde{c} \in L^\infty(\Omega)$ and there exists a constant $\widetilde{\mu} > 0$ such that

$$\frac{1}{\widetilde{\mu}} < \widetilde{a}(x) < \widetilde{\mu}, \quad x \in \Omega. \qquad (4.2)$$

Let $\ell_0 > 0$. We set weight functions

$$\widetilde{\varphi}_i(x) = \ell_0 e^{\lambda d_i(x)}, \quad \widetilde{\alpha}_i(x) = \ell_0\left(e^{\lambda d_i(x)} - e^{2\lambda\|d_i\|_{C(\overline{\Omega})}}\right)$$

for $i = 0, 1$. Here we note that d_0, d_1 are the same functions with ones defined in the section 2.

Lemma 4.1 (Carleman estimate with weight $\widetilde{\varphi}_0$, $\widetilde{\alpha}_0$ for the elliptic equations)**.** *There exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, we can choose $s_0(\lambda) > 0$ satisfying: there exists $C = C(s_0, \lambda_0) > 0$ such that*

$$\begin{aligned}&\int_\Omega \left(\frac{1}{s\widetilde{\varphi}_0} |\partial_x^2 v|^2 + s\lambda^2 \widetilde{\varphi}_0 |\partial_x v|^2 + s^3\lambda^4 \widetilde{\varphi}_0^3 |v|^2 \right) e^{2s\widetilde{\alpha}_0}\, dx \\ &\leq C \int_\Omega |\widetilde{L} v|^2 e^{2s\widetilde{\alpha}_0}\, dx + \int_\omega s^3\lambda^4 \widetilde{\varphi}_0^3 |v|^2 e^{2s\widetilde{\alpha}_0}\, dx\end{aligned}$$

for all $s > s_0$ and all $v \in H^2(\Omega)$ satisfying $v(0) = \partial_x v(0) = 0$.

Lemma 4.2 (Carleman estimate with weight $\widetilde{\varphi}_1$, $\widetilde{\alpha}_1$ for the elliptic equations)**.** *There exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, we can choose $s_0(\lambda) > 0$ satisfying: there exists $C = C(s_0, \lambda_0) > 0$ such that*

$$\begin{aligned}&\int_\Omega \left(\frac{1}{s\widetilde{\varphi}_1} |\partial_x^2 v|^2 + s\lambda^2 \widetilde{\varphi}_1 |\partial_x v|^2 + s^3\lambda^4 \widetilde{\varphi}_1^3 |v|^2 \right) e^{2s\widetilde{\alpha}_1}\, dx \\ &\leq C \int_\Omega |\widetilde{L} v|^2 e^{2s\widetilde{\alpha}_1}\, dx\end{aligned}$$

for all $s > s_0$ and all $v \in H^2(\Omega)$ satisfying $v(0) = \partial_x v(0) = 0$.

We may prove the Lemma 4.1 and 4.2 by a similar direct method to the proof of the Carleman estimate for the second order parabolic equation (see [35] and references therein).

proof of Lemma 4.1 . It is sufficient to prove the Carleman estimate for $\partial_x(\widetilde{a}\partial_x v)$.

Set $w = ve^{s\widetilde{\alpha}_0}$ and define $\widetilde{P}$ by $\widetilde{P}w = e^{s\widetilde{\alpha}_0}[\partial_x(\widetilde{a}\partial_x(we^{-s\widetilde{\alpha}_0}))]$. Then we decompose

$$\widetilde{P}w = \widetilde{P}_1 w + \widetilde{P}_2 w + \widetilde{R}w$$

with

$$\begin{aligned}\widetilde{P}_1 w &= -2s\lambda\widetilde{\varphi}_0(\partial_x d_0)\widetilde{a}\partial_x w, \\ \widetilde{P}_2 w &= \partial_x(\widetilde{a}\partial_x w) + s^2\lambda^2\widetilde{\varphi}_0^2(\partial_x d_0)^2\widetilde{a}w, \\ \widetilde{R}w &= s\lambda^2\widetilde{\varphi}_0(\partial_x d_0)^2\widetilde{a}w + s\lambda\widetilde{\varphi}_0\partial_x(\widetilde{a}\partial_x d_0)w.\end{aligned}$$

Setting $\widetilde{h} = \widetilde{P}w + \widetilde{R}w$ and taking L^2-norm for $\widetilde{h}$, we obtain

$$\|\widetilde{h}\|^2_{L^2(\Omega)} = \|\widetilde{P}_1 w + \widetilde{P}_2 w\|^2_{L^2(\Omega)} = \|\widetilde{P}_1 w\|^2_{L^2(\Omega)} + \|\widetilde{P}_2 w\|^2_{L^2(\Omega)} + 2(\widetilde{P}_1 w, \widetilde{P}_2 w)^2_{L^2(\Omega)}.$$

Moreover

$$\|\widetilde{h}\|^2_{L^2(\Omega)} \leq 2\|\widetilde{P}w\|^2_{L^2(\Omega)} + 2\|\widetilde{R}w\|^2_{L^2(\Omega)}.$$

By above inequalities, we have

$$\frac{1}{2}\|\widetilde{P}_2 w\|^2_{L^2(\Omega)} + (\widetilde{P}_1 w, \widetilde{P}_2 w)^2_{L^2(\Omega)} \leq \|\widetilde{P}w\|^2_{L^2(\Omega)} + \|\widetilde{R}w\|^2_{L^2(\Omega)}. \tag{4.3}$$

Let us estimate $(\widetilde{P}_1 w, \widetilde{P}_2 w)^2_{L^2(\Omega)}$ from below.

$$(\widetilde{P}_1 w, \widetilde{P}_2 w)^2_{L^2(\Omega)} = \big(-2s\lambda\widetilde{\varphi}_0(\partial_x d_0)\widetilde{a}\partial_x w, \partial_x(\widetilde{a}\partial_x w) + s^2\lambda^2\widetilde{\varphi}_0^2(\partial_x d_0)^2\widetilde{a}w\big)$$

$$
\begin{aligned}
&= (-2s\lambda\widetilde{\varphi}_0(\partial_x d_0)\widetilde{a}\partial_x w, \partial_x(\widetilde{a}\partial_x w)) + \left(-2s\lambda\widetilde{\varphi}_0(\partial_x d_0)\widetilde{a}\partial_x w, s^2\lambda^2\widetilde{\varphi}_0^2(\partial_x d_0)^2\widetilde{a}w\right) \\
&=: \widetilde{I}_1 + \widetilde{I}_2. \qquad (4.4)
\end{aligned}
$$

Calculating $\widetilde{I}_1$ and $\widetilde{I}_2$ by integrating by parts and the Schwarz inequality, we get

$$
\begin{aligned}
\widetilde{I}_1 &= -2\int_\Omega s\lambda\widetilde{\varphi}_0(\partial_x d_0)\widetilde{a}(\partial_x w)\partial_x(\widetilde{a}\partial_x w)\,dx \\
&= \int_\Omega s\lambda^2\widetilde{\varphi}_0(\partial_x d_0)^2\widetilde{a}^2|\partial_x w|^2\,dx + \int_\Omega s\lambda\widetilde{\varphi}_0(\partial_x^2 d_0)\widetilde{a}^2|\partial_x w|^2\,dx \\
&\quad - s\lambda\widetilde{\varphi}_0(L)\partial_x d_0(L)\widetilde{a}^2(L)|\partial_x w(L)|^2 + s\lambda\widetilde{\varphi}_0(0)\partial_x d_0(0)\widetilde{a}^2(0)|\partial_x w(0)|^2 \\
&\ge \int_\Omega s\lambda^2\widetilde{\varphi}_0(\partial_x d_0)^2\widetilde{a}^2|\partial_x w|^2\,dx - C\int_\Omega s\lambda\widetilde{\varphi}_0|\partial_x w|^2\,dx \\
&\quad - s\lambda\widetilde{\varphi}_0(L)\partial_x d_0(L)\widetilde{a}^2(L)|\partial_x w(L)|^2 + s\lambda\widetilde{\varphi}_0(0)\partial_x d_0(0)\widetilde{a}^2(0)|\partial_x w(0)|^2. \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{I}_2 &= -2\int_\Omega s^3\lambda^3\widetilde{\varphi}_0^3(\partial_x d_0)^3\widetilde{a}^2 w\partial_x w\,dx \\
&= 3\int_\Omega s^3\lambda^4\widetilde{\varphi}_0^3(\partial_x d_0)^4\widetilde{a}^2|w|^2\,dx + \int_\Omega s^3\lambda^3\widetilde{\varphi}_0^3\partial_x[\widetilde{a}^2(\partial_x d_0)^3]|w|^2\,dx \\
&\quad - s^3\lambda^3\widetilde{\varphi}_0^3(L)(\partial_x d_0)^3(L)\widetilde{a}^2(L)|w(L)|^2 + s^3\lambda^3\widetilde{\varphi}_0^3(0)(\partial_x d_0)^3(0)\widetilde{a}^2(L)|w(0)|^2 \\
&\ge 3\int_\Omega s^3\lambda^4\widetilde{\varphi}_0^3(\partial_x d_0)^4\widetilde{a}^2|w|^2\,dx - C\int_\Omega s^3\lambda^3\widetilde{\varphi}_0^3|w|^2\,dx \\
&\quad - s^3\lambda^3\widetilde{\varphi}_0^3(L)(\partial_x d_0)^3(L)\widetilde{a}^2(L)|w(L)|^2 + s^3\lambda^3\widetilde{\varphi}_0^3(0)(\partial_x d_0)^3(0)\widetilde{a}^2(L)|w(0)|^2. \qquad (4.6)
\end{aligned}
$$

By (4.4)–(4.6), we obtain

$$
\begin{aligned}
&\int_\Omega s\lambda^2\widetilde{\varphi}_0(\partial_x d_0)^2\widetilde{a}^2|\partial_x w|^2\,dx + 3\int_\Omega s^3\lambda^4\widetilde{\varphi}_0^3(\partial_x d_0)^4\widetilde{a}^2|w|^2\,dx \\
&\le (\widetilde{P}_1 w, \widetilde{P}_2 w)^2_{L^2(\Omega)} + C\int_\Omega s\lambda\widetilde{\varphi}_0|\partial_x w|^2\,dx + C\int_\Omega s^3\lambda^3\widetilde{\varphi}_0^3|w|^2\,dx + J \qquad (4.7)
\end{aligned}
$$

where

$$
\begin{aligned}
J &= s\lambda\widetilde{\varphi}_0(L)\partial_x d_0(L)\widetilde{a}^2(L)|\partial_x w(L)|^2 - s\lambda\widetilde{\varphi}_0(0)\partial_x d_0(0)\widetilde{a}^2(0)|\partial_x w(0)|^2 \\
&\quad + s^3\lambda^3\widetilde{\varphi}_0^3(L)(\partial_x d_0)^3(L)\widetilde{a}^2(L)|w(L)|^2 - s^3\lambda^3\widetilde{\varphi}_0^3(0)(\partial_x d_0)^3(0)\widetilde{a}^2(L)|w(0)|^2.
\end{aligned}
$$

Let us estimate $\|\widetilde{P}_2 w\|_{L^2(\Omega)}$ from below.

$$
\begin{aligned}
\|\widetilde{P}_2 w\|^2_{L^2(\Omega)} &= \int_\Omega \left|\partial_x(\widetilde{a}\partial_x w) + s^2\lambda^2\widetilde{\varphi}_0^2(\partial_x d_0)^2\widetilde{a}w\right|^2\,dx \\
&\ge \int_\Omega \frac{1}{s\widetilde{\varphi}_0}\left|\partial_x(\widetilde{a}\partial_x w) + s^2\lambda^2\widetilde{\varphi}_0^2(\partial_x d_0)^2\widetilde{a}w\right|^2\,dx \\
&\ge \frac{1}{2}\int_\Omega \frac{1}{s\widetilde{\varphi}_0}\left|\partial_x(\widetilde{a}\partial_x w)\right|^2\,dx - \int_\Omega s^3\lambda^4\widetilde{\varphi}_0^3(\partial_x d_0)^4\widetilde{a}^2|w|^2\,dx
\end{aligned}
$$

for large $s > 0$ such that $s\widetilde{\varphi}_0 \ge 1$. Hence we have

$$
\frac{1}{2}\int_\Omega \frac{1}{s\widetilde{\varphi}_0}\left|\partial_x(\widetilde{a}\partial_x w)\right|^2\,dx - \int_\Omega s^3\lambda^4\widetilde{\varphi}_0^3(\partial_x d_0)^4\widetilde{a}^2|w|^2\,dx \le \|\widetilde{P_2}w\|^2_{L^2(\Omega)}. \qquad (4.8)
$$

Next we estimate $\|\widetilde{R}w\|_{L^2(\Omega)}$.

$$
\begin{aligned}
\|\widetilde{R}w\|^2_{L^2(\Omega)} &= \int_\Omega \left|s\lambda^2\widetilde{\varphi}_0(\partial_x d_0)^2\widetilde{a}w + s\lambda\widetilde{\varphi}_0\partial_x(\widetilde{a}\partial_x d_0)w\right|^2\,dx\\
&\le C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_0^2 + s^2\lambda^2\widetilde{\varphi}_0^2\right)|w|^2\,dx. \qquad (4.9)
\end{aligned}
$$

By (4.3), (4.7)–(4.9), we get

$$
\begin{aligned}
&\frac{1}{4}\int_\Omega \frac{1}{s\widetilde{\varphi}_0}\left|\partial_x(\widetilde{a}\partial_x w)\right|^2\,dx + \int_\Omega s\lambda^2\widetilde{\varphi}_0(\partial_x d_0)^2\widetilde{a}^2|\partial_x w|^2\,dx + \frac{5}{2}\int_\Omega s^3\lambda^4\widetilde{\varphi}_0^3(\partial_x d_0)^4\widetilde{a}^2|w|^2\,dx\\
&\le \|\widetilde{P}w\|^2_{L^2(\Omega)} + C\int_\Omega s\lambda\widetilde{\varphi}_0|\partial_x w|^2\,dx + C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_0^2 + s^3\lambda^3\widetilde{\varphi}_0^3 + s^2\lambda^2\widetilde{\varphi}_0^2\right)|w|^2\,dx + J. \qquad (4.10)
\end{aligned}
$$

By (4.10), $\frac{1}{\widetilde{\mu}} \le \widetilde{a}$ and $|\partial_x d_0| \ge \sigma_0$ in $\overline{\Omega\setminus\omega_0}$, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{s\widetilde{\varphi}_0}\left|\partial_x(\widetilde{a}\partial_x w)\right|^2\,dx + \int_\Omega s\lambda^2\widetilde{\varphi}_0|\partial_x w|^2\,dx + \int_\Omega s^3\lambda^4\widetilde{\varphi}_0^3|w|^2\,dx\\
&\le C\|\widetilde{P}w\|^2_{L^2(\Omega)} + C\int_{\omega_0} s\lambda^2\widetilde{\varphi}_0|\partial_x w|^2\,dx + C\int_{\omega_0} s^3\lambda^4\widetilde{\varphi}_0^3|w|^2\,dx\\
&\quad + C\int_\Omega s\lambda\widetilde{\varphi}_0|\partial_x w|^2\,dx + C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_0^2 + s^3\lambda^3\widetilde{\varphi}_0^3 + s^2\lambda^2\widetilde{\varphi}_0^2\right)|w|^2\,dx + CJ. \qquad (4.11)
\end{aligned}
$$

Estimation of boundary terms. Since $v(0) = \partial_x v(0) = 0$, we have $w(0) = \partial_x w(0) = 0$.

$$
J = s\lambda\widetilde{\varphi}_0(L)\partial_x d_0(L)\widetilde{a}^2(L)|\partial_x w(L)|^2 + s^3\lambda^3\widetilde{\varphi}_0^3(L)(\partial_x d_0)^3(L)\widetilde{a}^2(L)|w(L)|^2.
$$

Noting that $\partial_x d_0(L) \le 0$, we see that

$$
J \le 0. \qquad (4.12)
$$

Estimation of interior terms. Let us replace

$$
\int_{\omega_0} s\lambda^2\widetilde{\varphi}_0|\partial_x w|^2\,dx
$$

by

$$
\int_{\omega} s^3\lambda^4\widetilde{\varphi}_0^3|w|^2\,dx
$$

on the right-hand side of (4.11).

We consider a cut off function $\widetilde{\rho} \in C_0^\infty(\Omega)$ such that $\widetilde{\rho} \le 1$ in Ω, $\operatorname{supp}\widetilde{\rho} \subset \omega$ and $\widetilde{\rho} \equiv 1$ in ω_0. Then we estimate the following integral

$$
-\int_\Omega \widetilde{\rho}s\lambda^2\widetilde{\varphi}_0 w\partial_x(\widetilde{a}\partial_x w)\,dx
$$

from above and below.

$$
-\int_\Omega \widetilde{\rho}s\lambda^2\widetilde{\varphi}_0 w\partial_x(\widetilde{a}\partial_x w)\,dx \le \frac{1}{\widetilde{\eta}}\int_\Omega \frac{1}{s\widetilde{\varphi}_0}|\partial_x(\widetilde{a}\partial_x w)|^2\,dx + 4\widetilde{\eta}\int_\omega s^3\lambda^4\widetilde{\varphi}_0^3|w|^2\,dx \qquad (4.13)
$$

for $\widetilde{\eta} > 0$. On the other hand, we have

$$
-\int_\Omega \widetilde{\rho}s\lambda^2\widetilde{\varphi}_0 w\partial_x(\widetilde{a}\partial_x w)\,dx
$$

$$
\begin{aligned}
&= \int_\Omega (\partial_x \widetilde{\rho}) s\lambda^2 \widetilde{\varphi}_0 \widetilde{a} w(\partial_x w)\, dx + \int_\Omega \widetilde{\rho} s\lambda^3 \widetilde{\varphi}_0 (\partial_x d_0) \widetilde{a} w (\partial_x w)\, dx + \int_\Omega \widetilde{\rho} s\lambda^2 \widetilde{\varphi}_0 \widetilde{a} |\partial_x w|^2\, dx \\
&\geq \frac{1}{\widetilde{\mu}} \int_{\omega_0} s\lambda^2 \widetilde{\varphi}_0 |\partial_x w|^2\, dx - C\int_\Omega \lambda^2 |\partial_x w|^2\, dx - C\int_\Omega (s^2\lambda^4 \widetilde{\varphi}_0^2 + s^2\lambda^2\widetilde{\varphi}_0^2)|w|^2\, dx. \qquad (4.14)
\end{aligned}
$$

Combining (4.13) with (4.14), we get

$$
\begin{aligned}
\frac{1}{\widetilde{\mu}} \int_{\omega_0} s\lambda^2 \widetilde{\varphi}_0 |\partial_x w|^2\, dx &\leq \frac{1}{\widetilde{\eta}} \int_\Omega \frac{1}{s\widetilde{\varphi}_0} |\partial_x(\widetilde{a}\partial_x w)|^2\, dx + 4\widetilde{\eta} \int_\omega s^3\lambda^4 \widetilde{\varphi}_0^3 |w|^2\, dx \\
&\quad + C\int_\Omega \lambda^2 |\partial_x w|^2\, dx + C\int_\Omega (s^2\lambda^4\widetilde{\varphi}_0^2 + s^2\lambda^2\widetilde{\varphi}_0^2)|w|^2\, dx. \qquad (4.15)
\end{aligned}
$$

By (4.11), (4.12) and (4.15), we obtain

$$
\begin{aligned}
&\left(1 - \frac{C\widetilde{\mu}}{\widetilde{\eta}}\right) \int_\Omega \frac{1}{s\widetilde{\varphi}_0} \left|\partial_x(\widetilde{a}\partial_x w)\right|^2\, dx + \int_\Omega s\lambda^2\widetilde{\varphi}_0 |\partial_x w|^2\, dx + \int_\Omega s^3\lambda^4\widetilde{\varphi}_0^3 |w|^2\, dx \\
&\leq C\|\widetilde{P}w\|^2_{L^2(\Omega)} + C\int_\omega (4\widetilde{\mu}\widetilde{\eta}+1) s^3\lambda^4\widetilde{\varphi}_0^3 |w|^2\, dx \\
&\quad + C\int_\Omega \left(s\lambda\widetilde{\varphi}_0 + \lambda^2\right) |\partial_x w|^2\, dx + C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_0^2 + s^3\lambda^3\widetilde{\varphi}_0^3 + s^2\lambda^2\widetilde{\varphi}_0^2\right)|w|^2\, dx.
\end{aligned}
$$

Taking sufficient large $\widetilde{\eta} > 0$, we get

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{s\widetilde{\varphi}_0} \left|\partial_x(\widetilde{a}\partial_x w)\right|^2 + s\lambda^2\widetilde{\varphi}_0|\partial_x w|^2 + s^3\lambda^4\widetilde{\varphi}_0^3|w|^2\right) dx \\
&\leq C\|\widetilde{P}w\|^2_{L^2(\Omega)} + C\int_\omega s^3\lambda^4\widetilde{\varphi}_0^3|w|^2\, dx \\
&\quad + C\int_\Omega \left(s\lambda\widetilde{\varphi}_0 + \lambda^2\right)|\partial_x w|^2\, dx + C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_0^2 + s^3\lambda^3\widetilde{\varphi}_0^3 + s^2\lambda^2\widetilde{\varphi}_0^2\right)|w|^2\, dx. \qquad (4.16)
\end{aligned}
$$

Since $\partial_x(\widetilde{a}\partial_x w) = \widetilde{a}\partial_x^2 w + (\partial_x\widetilde{a})(\partial_x w)$, we see that

$$
|\partial_x(\widetilde{a}\partial_x w)|^2 \geq \frac{1}{2\widetilde{\mu}^2}|\partial_x^2 w|^2 - C|\partial_x w|^2.
$$

So we have

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{s\widetilde{\varphi}_0}\left|\partial_x^2 w\right|^2 + s\lambda^2\widetilde{\varphi}_0|\partial_x w|^2 + s^3\lambda^4\widetilde{\varphi}_0^3|w|^2\right) dx \\
&\leq C\|\widetilde{P}w\|^2_{L^2(\Omega)} + C\int_\omega s^3\lambda^4\widetilde{\varphi}_0^3|w|^2\, dx \\
&\quad + C\int_\Omega \left(s\lambda\widetilde{\varphi}_0 + \lambda^2 + \frac{1}{s\widetilde{\varphi}_0}\right)|\partial_x w|^2\, dx + C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_0^2 + s^3\lambda^3\widetilde{\varphi}_0^3 + s^2\lambda^2\widetilde{\varphi}_0^2\right)|w|^2\, dx.
\end{aligned}
$$

Rewriting the above inequality in terms of v, we obtain

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{s\widetilde{\varphi}_0}\left|\partial_x^2 v\right|^2 + s\lambda^2\widetilde{\varphi}_0|\partial_x v|^2 + s^3\lambda^4\widetilde{\varphi}_0^3|v|^2\right) e^{2s\widetilde{\alpha}_0}\, dx \\
&\leq C\int_\Omega |\partial_x(\widetilde{a}\partial_x v)|^2 e^{2s\widetilde{\alpha}_0}\, dx + C\int_\omega s^3\lambda^4\widetilde{\varphi}_0^3|v|^2 e^{2s\widetilde{\alpha}_0}\, dx \\
&\quad + C\int_\Omega \left(s\lambda\widetilde{\varphi}_0 + \lambda^2\right)|\partial_x v|^2 e^{2s\widetilde{\alpha}_0}\, dx + C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_0^2 + s^3\lambda^3\widetilde{\varphi}_0^3\right)|v|^2 e^{2s\widetilde{\alpha}_0}\, dx
\end{aligned}
$$

for large $s > 0$ and $\lambda > 0$. Taking sufficient large $\lambda > 0$, we have

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{s\widetilde{\varphi}_0} \left|\partial_x^2 v\right|^2 + s\lambda^2 \widetilde{\varphi}_0 |\partial_x v|^2 + s^3\lambda^4 \widetilde{\varphi}_0^3 |v|^2 \right) e^{2s\widetilde{\alpha}_0}\, dx \\
&\le C' \int_\Omega |\partial_x(\widetilde{a}\partial_x v)|^2 e^{2s\widetilde{\alpha}_0}\, dx + C' \int_\omega s^3\lambda^4 \widetilde{\varphi}_0^3 |v|^2 e^{2s\widetilde{\alpha}_0}\, dx \\
&\quad + C' \int_\Omega \lambda^2 |\partial_x v|^2 e^{2s\widetilde{\alpha}_0}\, dx + C' \int_\Omega s^2\lambda^4 \widetilde{\varphi}_0^2 |v|^2 e^{2s\widetilde{\alpha}_0}\, dx.
\end{aligned} \tag{4.17}$$

Next we choose sufficient large $s > 1$, we may absorb the third term and the fourth term on the left-hand side of (4.17).

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{s\widetilde{\varphi}_0} \left|\partial_x^2 v\right|^2 + s\lambda^2 \widetilde{\varphi}_0 |\partial_x v|^2 + s^3\lambda^4 \widetilde{\varphi}_0^3 |v|^2 \right) e^{2s\widetilde{\alpha}_0}\, dx \\
&\le C'' \int_\Omega |\partial_x(\widetilde{a}\partial_x v)|^2 e^{2s\widetilde{\alpha}_0}\, dx + C'' \int_\omega s^3\lambda^4 \widetilde{\varphi}_0^3 |v|^2 e^{2s\widetilde{\alpha}_0}\, dx.
\end{aligned} \tag{4.18}$$

Thus we complete the proof. □

proof of Lemma 4.2 . Setting $\widetilde{w} = ve^{s\widetilde{\alpha}_1}$ and calculating in a similar way to get (4.10) in the proof of Lemma 4.1.

$$\begin{aligned}
&\frac{1}{4}\int_\Omega \frac{1}{s\widetilde{\varphi}_1} \left|\partial_x(\widetilde{a}\partial_x \widetilde{w})\right|^2\, dx + \int_\Omega s\lambda^2 \widetilde{\varphi}_1 (\partial_x d_1)^2 \widetilde{a}^2 |\partial_x \widetilde{w}|^2\, dx + \frac{5}{2}\int_\Omega s^3\lambda^4 \widetilde{\varphi}_1^3 (\partial_x d_1)^4 \widetilde{a}^2 |\widetilde{w}|^2\, dx \\
&\le \|\widetilde{P}\widetilde{w}\|^2_{L^2(\Omega)} + C\int_\Omega s\lambda\widetilde{\varphi}_1 |\partial_x \widetilde{w}|^2\, dx + C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_1^2 + s^3\lambda^3\widetilde{\varphi}_1^3 + s^2\lambda^2\widetilde{\varphi}_1^2\right)|\widetilde{w}|^2\, dx + \widetilde{J}
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
\widetilde{J} = {}& s\lambda\widetilde{\varphi}_1(L)\partial_x d_1(L)\widetilde{a}^2(L)|\partial_x\widetilde{w}(L)|^2 - s\lambda\widetilde{\varphi}_1(0)\partial_x d_1(0)\widetilde{a}^2(0)|\partial_x\widetilde{w}(0)|^2 \\
&+ s^3\lambda^3\widetilde{\varphi}_1^3(L)(\partial_x d_1)^3(L)\widetilde{a}^2(L)|\widetilde{w}(L)|^2 - s^3\lambda^3\widetilde{\varphi}_1^3(0)(\partial_x d_1)^3(0)\widetilde{a}^2(L)|\widetilde{w}(0)|^2.
\end{aligned}$$

By (4.19), $\frac{1}{\mu} \le \widetilde{a}$ and $|\partial_x d_1| \ge \sigma_1$ in $\overline{\Omega}$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{s\widetilde{\varphi}_1} \left|\partial_x(\widetilde{a}\partial_x \widetilde{w})\right|^2\, dx + \int_\Omega s\lambda^2 \widetilde{\varphi}_1 |\partial_x \widetilde{w}|^2\, dx + \int_\Omega s^3\lambda^4 \widetilde{\varphi}_1^3 |\widetilde{w}|^2\, dx \\
&\le C\|\widetilde{P}\widetilde{w}\|^2_{L^2(\Omega)} + C\int_\Omega s\lambda\widetilde{\varphi}_1 |\partial_x \widetilde{w}|^2\, dx + C\int_\Omega \left(s^2\lambda^4\widetilde{\varphi}_1^2 + s^3\lambda^3\widetilde{\varphi}_1^3 + s^2\lambda^2\widetilde{\varphi}_1^2\right)|\widetilde{w}|^2\, dx + C\widetilde{J}.
\end{aligned} \tag{4.20}$$

By an argument similar to that used to obtain (4.18) from (4.16) in the proof of Lemma 4.1, We rewrite the estimate for $\widetilde{w}$ as the estimate for v. Moreover we absorb the lower order terms with respect to s, λ, φ_1 on the right-hand side of the estimate. Then we obtain

$$\int_\Omega \left(\frac{1}{s\widetilde{\varphi}_1} \left|\partial_x^2 v\right|^2 + s\lambda^2 \widetilde{\varphi}_1 |\partial_x v|^2 + s^3\lambda^4 \widetilde{\varphi}_1^3 |v|^2 \right) e^{2s\widetilde{\alpha}_1}\, dx \le C'' \int_\Omega |\partial_x(\widetilde{a}\partial_x v)|^2 e^{2s\widetilde{\alpha}_1}\, dx + C''\widetilde{J}. \tag{4.21}$$

Estimation of boundarw terms. By $v(0) = \partial_x v(0) = 0$, we have $w(0) = \partial_x w(0) = 0$. Hence we get

$$\widetilde{J} = s\lambda\widetilde{\varphi}_1(L)\partial_x d_1(L)\widetilde{a}^2(L)|\partial_x\widetilde{w}(L)|^2 + s^3\lambda^3\widetilde{\varphi}_1^3(L)(\partial_x d_1)^3(L)\widetilde{a}^2(L)|\widetilde{w}(L)|^2.$$

Noting that $\partial_x d_1(L) \le 0$, we have

$$\widetilde{J} \le 0. \tag{4.22}$$

(4.21) and (4.22) give us the estimate of Lemma 4.2. □

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