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Regularity and singularity of blow-up curve for $\partial_t^2 u - \partial_x^2 u = |\partial_t u|^p$

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We study a blow-up curve for the one dimensional wave equation $\partial_t^2 u - \partial_x^2 u = |\partial_t u|^p$ with $p > 1$. The purpose of this paper is to show that the blow-up curve is a C^1 curve if the initial values are large and smooth enough. To prove the result, we convert the equation into a first order system, and then apply a modification of the method of Caffarelli and Friedman [2]. Moreover, we present some numerical investigations of the blow-up curves. From the numerical results, we were able to confirm that the blow-up curves are smooth if the initial values are large and smooth enough. Moreover, we can predict that the blow-up curves have singular points if the initial values are not large enough even they are smooth enough.

1 Introduction

In this paper, we consider the nonlinear wave equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = |\partial_t u|^p, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where

$$p > 1 \text{ is a constant such that the function } |s|^p \text{ is of class } C^4. \tag{1.2}$$

Here, u is an unknown real-valued function.

Let R^* and T^* be any positive constants, and set

$$B_{R^*} = \{x \mid |x| < R^*\}, \tag{1.3}$$

$$K_-(x_0,t_0) = \{(x,t) \mid |x - x_0| < t_0 - t,\ t > 0\}, \tag{1.4}$$

$$K_{R^*,T^*} = \bigcup_{x \in B_{R^*}} K_-(x, T^*). \tag{1.5}$$

We then consider the following function

$$T(x) = \sup\{t \in (0,T^*) \mid \ |\partial_t u(x,t)| < \infty\} \quad \text{for } x \in B_{R^*}.$$

In this paper, we call the set $\Gamma = \{(x, T(x)) \mid x \in B_{R^*}\}$ the blow-up curve. Below, we identify Γ with T itself. There are two purposes of this paper. First, we demonstrate that T is continuously differentiable for the suitable initial values. Second, we present some numerical examples of the various blow-up curves. From the numerical results, we were able to confirm that the blow-up curves are smooth if the initial values are large and smooth enough. Moreover, we can predict that the blow-up curves have singular points if the initial values are not large enough even they are smooth enough.

We will state some analytical results from previous studies on the blow-up curves for nonlinear wave equations. The majority of previous studies have considered the following nonlinear wave equation:

$$\partial_t^2 u - \partial_x^2 u = F(u), \qquad x \in \mathbb{R},\ t > 0,$$

and corresponding blow-up curve

$$\tilde{T}(x) = \sup\{t \in (0, T^*) \mid |u(x,t)| < \infty\} \quad \text{for } x \in B_{R^*}.$$

We note that the definition of the blow-up curve is different from ours. The pioneering study on this topic was done by Caffarelli and Friedman [1], [2]. They investigated the case with $F(u) = |u|^p$. They demonstrated that $\tilde{T}$ in that case is continuously differentiable under suitable initial conditions. Moreover, Godin [7] showed that the blow-up curve with $F(u) = e^u$ is also continuously differentiable under appropriate initial conditions. It was also shown that the blow-up curve can be C^∞, in the case of $F(u) = e^u$ (see Godin [8]). Furthermore, Uesaka [13] considered the blow-up curve for the system of nonlinear wave equations.

On the other hand, Merle and Zagg [9] showed that there are cases where the blow-up curve has singular points, while the above results concern the smoothness of the blow-up curve.

As mentioned above, several results have been established on the blow-up curve when there are no nonlinear terms involving the derivative of the solution. On the other hand, to the best of our knowledge only one result has been found concerning the blow-up curve with nonlinear terms involving the derivative of solution. Ohta and Takamura [11] considered the nonlinear wave equation

$$\partial_t^2 u - \partial_x^2 u = (\partial_t u)^2 - (\partial_x u)^2, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{1.6}$$

This equation can be transformed into the wave equation $\partial_t^2 v - \partial_x^2 v = 0$ by

$$v(x,t) = \exp\{-u(x,t)\}, \quad u(x,t) = -\log\{v(x,t)\}.$$

Thanks to the linearization of (1.6), we can study the blow-up curve of (1.6).

However, we cannot apply this linearization to (1.1). Therefore, we employ an alternative method, which is to rewrite to (1.1) as a system that does not include the derivative of the solution in nonlinear terms. We basically apply the method introduced by Caffarelli and Friedman [2] to this system. However, we offer an alternative proof of [2] for showing that the blow-up curve of the blow-up limits is an affine function (Section 5). Consequently, our proof is more elementary and easy to read. Our method would be applied to the original equation of [2].

We define ϕ and ψ as

$$\phi = \partial_t u + \partial_x u, \qquad \psi = \partial_t u - \partial_x u.$$

Then, we see that (1.1) is rewritten as

$$\begin{cases} D_-\phi = 2^{-p}|\phi+\psi|^p, & x \in \mathbb{R},\ t > 0, \\ D_+\psi = 2^{-p}|\phi+\psi|^p, & x \in \mathbb{R},\ t > 0, \\ \phi(x,0) = f(x), \quad \psi(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{1.7}$$

where $D_- v = \partial_t v - \partial_x v$, $D_+ v = \partial_t v + \partial_x v$ and $f = u_1 + \partial_x u_0$, $g = u_1 - \partial_x u_0$. (The equivalency of between (1.1) and (1.7) will be described in Remark 1.2.)

Let $(\tilde{\phi}, \tilde{\psi})$ be the solution of

$$\begin{cases} \dfrac{d\tilde{\phi}}{dt} = 2^{-p}|\tilde{\phi} + \tilde{\psi}|^p, & t > 0, \\ \dfrac{d\tilde{\psi}}{dt} = 2^{-p}|\tilde{\phi} + \tilde{\psi}|^p, & t > 0, \\ \tilde{\phi}(0) = \gamma_1, \quad \tilde{\psi}(0) = \gamma_2, \end{cases} \tag{1.8}$$

where γ_1 and γ_2 are some positive constants which will be fixed later. Then, we see that there exists a positive constant T_1 such that

$$\tilde{\phi}(t) + \tilde{\psi}(t) \to \infty \quad \text{as } t \to T_1.$$

We make the following assumptions.

(A1) $f \geq \gamma_1, \quad g \geq \gamma_2 \quad \text{in } B_{R^*+T^*}$.

(A2) $f, g \in C^4(B_{R^*+T^*})$.

(A3) There exists a constant $\varepsilon_0 > 0$ such that

$$2^{-p}(\gamma_1 + \gamma_2)^p \geq (2 + \varepsilon_0) \cdot \max_{x \in B_{R^*+T^*}} \{|f_x(x)| + |g_x(x)|\}.$$

(A4) $T_1 < T^*$.

(A5.1) There exists a constant $\varepsilon_1 > \dfrac{2}{2p-3}$ such that

$$2^{-p}(\gamma_1 + \gamma_2)^p \geq (2 + \varepsilon_1) \cdot \max_{x \in B_{R^*+T^*}} \{|\partial_x f(x)| + |\partial_x g(x)|\}.$$

(We notice that it follows from (1.2) that $p > 3/2$.)

(A5.2) There exists a constant $C^{(2)} > 0$ such that

$$(f + g)^{2p-1} \geq C^{(2)} \cdot \max_{x \in B_{R^*+T^*}} \{|\partial_x^2 f(x)| + |\partial_x^2 g(x)|\}.$$

(A5.3) There exists a constant $C^{(3)} > 0$ such that

$$(f + g)^{3p-2} \geq C^{(3)} \cdot \max_{x \in B_{R^*+T^*}} \{|\partial_x^3 f(x)| + |\partial_x^3 g(x)|\}.$$

We now state the main results of this paper.

Theorem 1.1. *Let R^* and T^* be arbitrary positive numbers. Assume that (A1)-(A5.3) hold true. Then, there exists a unique $C^1(B_{R^*})$ function T such that $0 < T(x) < T^*$ ($x \in B_{R^*}$) and a unique $(C^{3,1}(\Omega))^2$ solution (ϕ, ψ) of (1.7) satisfying*

$$\phi(x, t), \ \psi(x, t) \to \infty \quad \text{as} \quad t \to T(x) \tag{1.9}$$

for any $x \in B_{R^*}$, where $\Omega = \left\{(x, t) \in \mathbb{R}^2 \mid x \in B_{R^*}, \ 0 < t < T(x)\right\}$.

Remark 1.2. The equation (1.1) is equivalent to (1.7). We set

$$u(x, t) = u_0(x) + \frac{1}{2} \int_0^t (\phi + \psi)(x, s) ds.$$

Then, u satisfies (1.1).

Remark 1.3. The assertion (1.9) implies that $\partial_t u(x, t) \to \infty$ as $t \to T(x) \quad (x \in B_{R^*})$.

Next, we will mention numerical analysis of blow-up of nonlinear partial differential equations. There are many previous works of computation of blow-up solutions of various partial differential equations; See, for example, [10], [6], [3], [14], [12], [4] and [5].

We computed blow-up curve using the method of Cho [5] and obtained the various numerical results of blow-up curves. We will show them in Section 7.

The remainder of this paper is organized as follows. In Section 2, we construct a classical solution for (1.7) in the domain Ω. In Section 3, we give the blow-up rates of the solutions of (1.7). Moreover, we show that the blow-up curve is Lipschitz continuous. In the course of Sections 4–6, we prove that the blow-up curve is continuously differentiable. In Section 7, we show some numerical examples of blow-up curves.

2 Existence and regularity of solutions

In this section, we will demonstrate the existence and regularity of the solutions ϕ and ψ of (1.7) by successive approximation. Let us define $\{\phi_n\}$ and $\{\psi_n\}$ by $\phi_0 \equiv \gamma_1$, $\psi_0 \equiv \gamma_2$, and

$$\begin{cases} D_-\phi_{n+1} = 2^{-p}|\phi_n + \psi_n|^p, & (x,t) \in K_{R^*,T^*}, \\ D_+\psi_{n+1} = 2^{-p}|\phi_n + \psi_n|^p, & (x,t) \in K_{R^*,T^*}, \\ \phi_{n+1}(x,0) = f(x), \quad \psi_{n+1}(x,0) = g(x), & x \in B_{R^*+T^*}, \end{cases} \tag{2.1}$$

for $n \in \mathbb{N} \cup \{0\}$. Here, γ_1 and γ_2 are initial values of (1.8). We note that (2.1) can be rewritten as

$$\begin{cases} \phi_{n+1}(x,t) = f(x+t) + \displaystyle\int_0^t 2^{-p}|\phi_n + \psi_n|^p(x+(t-s),s)ds, \\ \psi_{n+1}(x,t) = g(x-t) + \displaystyle\int_0^t 2^{-p}|\phi_n + \psi_n|^p(x-(t-s),s)ds. \end{cases} \tag{2.2}$$

Remark 2.1. Consider a function $F \in C^1(K_{R^*,T^*})$. We note that it follows from (2.1) and (2.2) that $F(x,t) \geq 0$ in K_{R^*,T^*} if

$$F(x,0) \geq 0 \quad \text{in } B_{R^*+T^*}, \quad \text{and} \quad \begin{cases} D_-F(x,t) \geq 0 \\ \quad\text{or} \\ D_+F(x,t) \geq 0 \end{cases} \quad \text{in } K_{R^*,T^*}.$$

2.1 Lemmas

Now, we introduce two important lemmas.

Lemma 2.2. *Assume that (A1) hold. Then, we have*

$$\begin{matrix} \phi_{n+1} \geq \phi_n \geq 0, \\ \psi_{n+1} \geq \psi_n \geq 0, \end{matrix} \quad \text{in } K_{R^*,T^*}, \tag{2.3}$$

for $n \in \mathbb{N} \cup \{0\}$.

Proof. First, it follows from (A1) that

$$\phi_1(x,t) = f(x+t) + \int_0^t 2^{-p}|\phi_0 + \psi_0|^p(x+(t-s),s)ds \geq \gamma_1 = \phi_0(x,t) \geq 0$$

in K_{R^*,T^*}. Similarly, we have that $\psi_1 \geq \psi_0 \geq 0$ in K_{R^*,T^*}.

Next, we assume that

$$\phi_n \geq \phi_{n-1} \geq 0 \quad \text{and} \quad \psi_n \geq \psi_{n-1} \geq 0 \quad \text{in } K_{R^*,T^*}.$$

Then, we have

$$\begin{aligned}
\phi_{n+1}(x,t) &= f(x+t) + \int_0^t 2^{-p}|\phi_n + \psi_n|^p(x+(t-s),s)ds \\
&\geq f(x+t) + \int_0^t 2^{-p}|\phi_{n-1} + \psi_{n-1}|^p(x+(t-s),s)ds \\
&= \phi_n(x,t) \geq 0
\end{aligned}$$

in K_{R^*,T^*}. Similarly, we have that $\psi_{n+1} \geq \psi_n \geq 0 \quad \text{in } K_{R^*,T^*}$. $\square$

Lemma 2.3. *Assume that (A1)–(A3) hold. Then, we have*

$$\begin{aligned}
\partial_t \phi_n &\geq (1+\varepsilon_0)|\partial_x \phi_n|, \\
\partial_t \psi_n &\geq (1+\varepsilon_0)|\partial_x \psi_n|,
\end{aligned} \quad \text{in } K_{R^*,T^*}, \tag{2.4}$$

for $n \in \mathbb{N} \cup \{0\}$.

Proof. Set $\lambda = 1 + \varepsilon_0$, and

$$\begin{aligned}
J_n &= \partial_t \phi_n + \lambda \partial_x \phi_n, \quad \tilde{J}_n = \partial_t \phi_n - \lambda \partial_x \phi_n, \\
L_n &= \partial_t \psi_n + \lambda \partial_x \psi_n, \quad \tilde{L}_n = \partial_t \psi_n - \lambda \partial_x \psi_n,
\end{aligned}$$

for $n \in \mathbb{N} \cup \{0\}$. Then, it suffices to show that J_n, $\tilde{J}_n$, L_n and $\tilde{L}_n$ are nonnegative for $n \in \mathbb{N} \cup \{0\}$, in K_{R^*,T^*}. First, we note that $J_0 = \tilde{J}_0 = L_0 = \tilde{L}_0 = 0$ in K_{R^*,T^*}. We assume that

$$J_n \geq 0, \quad L_n \geq 0 \quad \text{in } K_{R^*,T^*}.$$

Then, it follows from (A3) that

$$\begin{aligned}
J_{n+1}(x,0) &= \partial_t \phi_{n+1}(x,0) + \lambda \partial_x \phi_{n+1}(x,0) \\
&= (1+\lambda)\partial_x \phi_{n+1}(x,0) + 2^{-p}|\phi_n(x,0) + \psi_n(x,0)|^p \\
&\geq (2+\varepsilon_0)\partial_x f(x) + 2^{-p}(\gamma_1+\gamma_2)^p \geq 0 \quad \text{in } B_{R^*+T^*}.
\end{aligned}$$

Furthermore, it follows from Lemma 2.2 that

$$\begin{aligned}
D_- J_{n+1} &= \partial_t(\partial_t \phi_{n+1} + \lambda \partial_x \phi_{n+1}) - \partial_x(\partial_t \phi_{n+1} + \lambda \partial_x \phi_{n+1}) \\
&= \partial_t(\partial_t \phi_{n+1} - \partial_x \phi_{n+1}) + \lambda \partial_x(\partial_t \phi_{n+1} - \partial_x \phi_{n+1}) \\
&= (\partial_t + \lambda \partial_x) 2^{-p}|\phi_n + \psi_n|^p \\
&= (\partial_t + \lambda \partial_x) 2^{-p}(\phi_n + \psi_n)^p \\
&= 2^{-p} p (\phi_n + \psi_n)^{p-1}(J_n + L_n) \geq 0 \quad \text{in } K_{R^*,T^*}.
\end{aligned}$$

Therefore, we obtain $J_{n+1} \geq 0 \quad \text{in } K_{R^*,T^*}$. Similarly, we obtain that $L_{n+1} \geq$ in K_{R^*,T^*}. In the same way of above, we can show that

$$\tilde{J}_{n+1} \geq 0, \quad \tilde{L}_{n+1} \geq 0 \quad \text{in } K_{R^*,T^*}$$

if we assume that $\tilde{J}_n \geq 0$, $\tilde{L}_n \geq 0$ in K_{R^*,T^*}. Therefore, we have obtained that $J_n, \tilde{J}_n, L_n, \tilde{L}_n \geq 0$ for $n \in \mathbb{N} \cup \{0\}$, in K_{R^*,T^*}. This completes the proof. $\square$

2.2 Proof of existence and regularity of ϕ and ψ

Fix $(x,t) \in K_{R^*,T^*}$. Since $\{\phi_n(x,t)\}$ and $\{\psi_n(x,t)\}$ are increasing sequences on n, we have

$$\lim_{n\to\infty} \phi_n(x,t) = \sup_{n\in\mathbb{N}} \phi_n(x,t) \quad \text{and} \quad \lim_{n\to\infty} \psi_n(x,t) = \sup_{n\in\mathbb{N}} \psi_n(x,t). \tag{2.5}$$

We set

$$\phi(x,t) = \sup_{n\in\mathbb{N}} \phi_n(x,t) \quad \text{and} \quad \psi(x,t) = \sup_{n\in\mathbb{N}} \psi_n(x,t).$$

It follows from Lemma 2.3 that ϕ and ψ are monotone increasing on t. Hence, there exists a function $T(x)$ such that

$$T(x) = \sup\{t \in (0,T^*) \mid (\phi+\psi)(x,t) < \infty\} \quad \text{for} \quad x \in B_{R^*}$$

and

$$\lim_{t\uparrow T(x)} (\phi+\psi)(x,t) \to \infty \quad \text{for} \quad x \in B_{R^*}$$

if $T(x) < T^*$. We set $\Omega = \{(x,t) \mid x \in B_{R^*},\ 0 < t < T(x)\}$.

Remark 2.4. We will show that T is actually a blow-up curve of ϕ and ψ in Section 3.

We state the following local existence lemma.

Lemma 2.5. *Assume that (A1)–(A3) hold. Then, (ϕ,ψ) is a unique $(C^{3,1}(\Omega))^2$ solution of* (1.7).

Proof. We set

$$B(t) = \left\{x \in B_{R^*+T^*} \mid |x-\tilde{x}| \le \tilde{t} - t\right\} \quad \text{for} \quad (\tilde{t},\tilde{x}) \in \Omega.$$

(Proof of regularity.)
First, we will show that (ϕ,ψ) is a $(C^{3,1}(\Omega))^2$ solution of (1.7). We split the proof into 2 steps.

(Step 1.) Fix $(\tilde{x},\tilde{t}) \in \Omega$. We will show that there exists a positive constant M_0 such that

$$\|\phi+\psi\|_{L^\infty(B(t)))} \le M_0 \quad \text{for} \quad t \in [0,\tilde{t}] \tag{2.6}$$

by showing a contradiction.

We set

$$Y_x = \left\{x \in B_{R^*} \mid |x-\tilde{x}| \le \tilde{t} - T(x)\right\}$$

and m is the 1-dimensional Lebesgue measure.

We assume that (2.6) does not hold. Then, there exists $t' \in (0,\tilde{t})$ such that there exist a', b' satisfying $a' < b'$ and

$$(a',b') \subset B(t') \quad \text{and} \quad (x',t') \notin \Omega \quad \text{for} \quad x \in (a',b').$$

By the monotonicity of $\phi+\psi$ on t, we have $T(x) \le t'$ for $x \in (a',b')$, which implies $(a',b') \in Y_x$. Hence, we have $m(Y_x) > 0$.

It follows from the monotonicity of $\phi+\psi$ on t that

$$(x,t) \notin \Omega \qquad \text{if} \quad x \in Y_x \quad \text{and} \quad (t = x + \tilde{t} - \tilde{x} \quad \text{or} \quad t = -x + \tilde{t} + \tilde{x}).$$

Moreover, we have $m(Y_{\tilde{t},+}) > 0$ or $m(Y_{\tilde{t},-}) > 0$ if $m(Y_x) > 0$. Here,

$$\begin{aligned} Y_{\tilde{t},-} &= \left\{s \in (0,\tilde{t}) \mid s = -x + \tilde{t} + \tilde{x}, \quad x \in Y_x\right\}, \\ Y_{\tilde{t},+} &= \left\{s \in (0,\tilde{t}) \mid s = x + \tilde{t} - \tilde{x}, \quad x \in Y_x\right\}. \end{aligned}$$

Then, we have

$$\begin{aligned} \infty >&(\phi_{n+1}+\psi_{n+1})(\tilde{x},\tilde{t}) \\ \ge& \int_{Y_{\tilde{t},-}} 2^{-p}|\phi_n+\psi_n|^p(\tilde{x}+\tilde{t}-s,s)ds + \int_{Y_{\tilde{t},+}} 2^{-p}|\phi_n+\psi_n|^p(\tilde{x}-\tilde{t}+s,s)ds \\ &\to \infty, \qquad \text{as} \quad n \to \infty. \end{aligned}$$

It is a contradiction. Therefore, we obtain (2.6).

(Step 2.) We will show $(\phi,\psi)\in(C^{3,1}(\Omega))^2$. Fix $(\tilde{x},\tilde{t})\in\Omega$. It suffices to show

$$\phi,\psi\in C^{3,1}(K_-(\tilde{x},\tilde{t})).$$

By **(Step 1.)**, we have that there exists a positive constant C_0 depending only on $\tilde{t}$ and $\tilde{x}$ such that

$$\|\phi_n+\psi_n\|_{L^\infty(B(t)))}\le C_0 \qquad \text{for} \quad t\in[0,\tilde{t}] \quad \text{and} \quad n\in\mathbb{N}. \tag{2.7}$$

Then, we have

$$\begin{aligned}
&\|\phi_{n+1}(\cdot\,,t)-\phi_n(\cdot\,,t)\|_{L^\infty(B(t))}+\|\psi_{n+1}(\cdot\,,t)-\psi_n(\cdot\,,t)\|_{L^\infty(B(t))}\\
&\le\int_0^t 2^{-p+1}\Big\||\phi_n+\psi_n|^p(\cdot\,,s_1)-|\phi_{n-1}+\psi_{n-1}|^p(\cdot\,,s_1)\Big\|_{L^\infty(B(s_1))}ds_1
\end{aligned}$$

for $t\in[0,\tilde{t}]$ and $n\in\mathbb{N}$. By (2.7), we have that

$$\begin{aligned}
&\|\phi_{n+1}(\cdot\,,t)-\phi_n(\cdot\,,t)\|_{L^\infty(B(t))}+\|\psi_{n+1}(\cdot\,,t)-\psi_n(\cdot\,,t)\|_{L^\infty(B(t))}\\
&\le pC_0^{p-1}\int_0^t\Big(\big\|\phi_n(\cdot\,,s_1)-\phi_{n-1}(\cdot\,,s_1)\big\|_{L^\infty(B(s_1))}\\
&\qquad\qquad+\big\|\psi_n(\cdot\,,s_1)-\psi_{n-1}(\cdot,s_1)\big\|_{L^\infty(B(s_1))}\Big)ds_1\\
&\le\big(pC_0^{p-1}\big)^2\int_0^t\int_0^{s_1}\Big(\big\|\phi_{n-1}(\cdot\,,s_2)-\phi_{n-2}(\cdot\,,s_2)\big\|_{L^\infty(B(s_2))}\\
&\qquad\qquad+\big\|\psi_{n-1}(\cdot\,,s_2)-\psi_{n-2}(\cdot\,,s_2)\big\|_{L^\infty(B(s_2))}\Big)ds_2ds_1.
\end{aligned}$$

for $t\in[0,\tilde{t}]$ and $n\in\mathbb{N}$. Repeating this argument, we obtain that

$$\begin{aligned}
&\|\phi_{n+1}(\cdot\,,t)-\phi_n(\cdot\,,t)\|_{L^\infty(B(t))}+\|\psi_{n+1}(\cdot\,,t)-\psi_n(\cdot\,,t)\|_{L^\infty(B(t))}\\
&\vdots\\
&\le(pC_0^{p-1})^n\int_0^t\int_0^{s_1}\int_0^{s_2}\cdots\int_0^{s_{n-1}}\\
&\quad\Big(\big\|\phi_1(\cdot\,,s_n)-\phi_0(\cdot\,,s_n)\big\|_{L^\infty(B(s_n))}+\big\|\psi_1(\cdot\,,s_n)-\psi_0(\cdot\,,s_n)\big\|_{L^\infty(B(s_n))}\Big)ds_n\ldots ds_2ds_1.\\
&\le 4C_0\frac{(pC_0^{p-1}T)^n}{n!}\to 0 \qquad \text{as } n\to\infty,
\end{aligned}$$

for $t\in[0,\tilde{t}]$. Hence, it follows from (2.5) that

$$\|\phi_n-\phi\|_{L^\infty(K_-(\tilde{x},\tilde{t}))}+\|\psi_n-\psi\|_{L^\infty(K_-(\tilde{x},\tilde{t}))}\to 0 \quad \text{as } n\to\infty.$$

Next, we will show that $\phi,\psi\in W^{1,\infty}(K_-(\tilde{x},\tilde{t}))$. We see that

$$\left\{\begin{array}{l}
D_-D_\theta\phi_{n+1}=D_\theta 2^{-p}(\phi_n+\psi_n)^p=p2^{-p}(\phi_n+\psi_n)^{p-1}(D_\theta\phi_n+D_\theta\psi_n),\\
D_+D_\theta\psi_{n+1}=D_\theta 2^{-p}(\phi_n+\psi_n)^p=p2^{-p}(\phi_n+\psi_n)^{p-1}(D_\theta\phi_n+D_\theta\psi_n),\\
\left\{\begin{array}{l}
D_\theta\phi_1(x,0)=(\cos\theta+\sin\theta)f_x(x)+\sin\theta\cdot 2^{-p}(\gamma_1+\gamma_2)^p,\\
D_\theta\psi_1(x,0)=(\cos\theta-\sin\theta)g_x(x)+\sin\theta\cdot 2^{-p}(\gamma_1+\gamma_2)^p,
\end{array}\right.\\
\left\{\begin{array}{l}
D_\theta\phi_{n+1}(x,0)=(\cos\theta+\sin\theta)\partial_x f(x)+\sin\theta\cdot 2^{-p}(f+g)^p(x),\\
D_\theta\psi_{n+1}(x,0)=(\cos\theta-\sin\theta)\partial_x g(x)+\sin\theta\cdot 2^{-p}(f+g)^p(x),
\end{array}\right. \quad (n\in\mathbb{N})
\end{array}\right.$$

for $n \in \mathbb{N} \cup \{0\}$. Here, $D_\theta v = \sin\theta \partial_t v + \cos\theta \partial_x v$.

We set $W(t) = C_0^p \exp(pC_0^{p-1}t)$. Then, we have

$$W(t) = C_0^p + \int_0^t pC_0^{p-1} W(s) ds.$$

We will show

$$\|D_\theta \phi_n(\cdot\,,t)\|_{L^\infty(B(t))} \leq W(t), \quad \|D_\theta \psi_n(\cdot\,,t)\|_{L^\infty(B(t))} \leq W(t) \tag{2.8}$$

for $t \in [0, \tilde{t}]$ and $n \in \mathbb{N} \cup \{0\}$.

We see

$$D_\theta \phi_0 = D_\theta \psi_0 = 0 \leq W(t)$$

for $t \geq 0$. Assume that (2.8) holds for n. Then, we have

$$\left\| p2^{-p}(\phi_n + \psi_n)^{p-1}(\cdot\,,t)(D_\theta \phi_n + D_\theta \psi_n)(\cdot\,,t) \right\|_{L^\infty(B(t))} \leq pC_0^{p-1} W(t) \tag{2.9}$$

for $t \in [0, \tilde{t}]$. It follows that (A3) that

$$\begin{aligned}
&\|D_\theta \phi_{n+1}(\cdot\,,t)\|_{L^\infty(B(t))} \\
&\leq 2\|\partial_x f\|_{L^\infty(B(0))} + 2^{-p}\|f+g\|^p_{L^\infty(B(0))} \\
&\qquad + \int_0^t \left\| p2^{-p}(\phi_n + \psi_n)^{p-1}(\cdot\,,s)(D_\theta \phi_n + D_\theta \psi_n)(\cdot\,,s) \right\|_{L^\infty(B(s))} ds \\
&\leq C_0^p + \int_0^t pC_0^{p-1} W(s) ds = W(t) \quad \text{for} \quad t \in [0, \tilde{t}].
\end{aligned} \tag{2.10}$$

Similarly, we have that $\|D_\theta \psi_{n+1}(\cdot\,,t)\|_{L^\infty(B(t))} \leq W(t)$ for $t \in [0, \tilde{t}]$. Thus,

$$\|D_\theta \phi_n(\cdot\,,t)\|_{L^\infty(B(t))} \leq W(t), \quad \|D_\theta \psi_n(\cdot\,,t)\|_{L^\infty(B(t))} \leq W(t)$$

for $t \in [0, \tilde{t}]$ and $n \in \mathbb{N} \cup \{0\}$. We set $C_1 = C_0^p \exp(pC_0^{p-1}T)$. Then, we have

$$\|D_\theta \phi_n(\cdot\,,t)\|_{L^\infty(B(t))} \leq C_1 \quad \text{and} \quad \|D_\theta \psi_n(\cdot\,,t)\|_{L^\infty(B(t))} \leq C_1 \tag{2.11}$$

for $t \in [0, \tilde{t}]$ and $n \in \mathbb{N} \cup \{0\}$.

We see that

$$\begin{aligned}
&\|D_\theta \phi_{n+1}(\cdot\,,t) - D_\theta \phi_n(\cdot\,,t)\|_{L^\infty(B(t))} + \|D_\theta \psi_{n+1}(\cdot\,,t) - D_\theta \psi_n(\cdot\,,t)\|_{L^\infty(B(t))} \\
&\leq \int_0^t p2^{-p+1} \Big\| \Big[(\phi_n + \psi_n)^{p-1}(D_\theta \phi_n + D_\theta \psi_n) \\
&\qquad\qquad - (\phi_{n-1} + \psi_{n-1})^{p-1}(D_\theta \phi_{n-1} + D_\theta \psi_{n-1}) \Big](\cdot\,,s) \Big\|_{L^\infty(B(s))} ds
\end{aligned}$$

It follows from (2.7) and (2.11) that

$$\begin{aligned}
&\|D_\theta \phi_{n+1}(\cdot\,,t) - D_\theta \phi_n(\cdot\,,t)\|_{L^\infty(B(t))} + \|D_\theta \psi_{n+1}(\cdot\,,t) - D_\theta \psi_n(\cdot\,,t)\|_{L^\infty(B(t))} \\
&\leq \int_0^t pC_0^{p-1} \Big(\|D_\theta \phi_n(\cdot\,,s_1) - D_\theta \phi_{n-1}(\cdot\,,s_1)\|_{L^\infty(B(s_1))} \\
&\qquad\qquad + \|D_\theta \psi_n(\cdot\,,s_1) - D_\theta \psi_{n-1}(\cdot\,,s_1)\|_{L^\infty(B(s_1))} \Big) ds_1 \\
&\quad + \int_0^t 2p(p-1)C_1 C_0^{p-2} \Big(\|\phi_n(\cdot\,,s_1) - \phi_{n-1}(\cdot\,,s_1)\|_{L^\infty(B(s_1))} \\
&\qquad\qquad + \|\psi_n(\cdot\,,s_1) - \psi_{n-1}(\cdot\,,s_1)\|_{L^\infty(B(s_1))} \Big) ds_1
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(pC_0^{p-1}\right)^2 \int_0^t \int_0^{s_1} \Big(\|D_\theta \phi_{n-1}(\cdot\,, s_2) - D_\theta \phi_{n-2}(\cdot\,, s_2)\|_{L^\infty(B(s_2))} \\
&+ \|D_\theta \psi_{n-1}(\cdot\,, s_2) - D_\theta \psi_{n-2}(\cdot\,, s_2)\|_{L^\infty(B(s_2))}\Big) ds_2 ds_1 \\
&\quad + C_2^2 \int_0^t \int_0^{s_1} \Big(\|\phi_{n-1}(\cdot\,, s_2) - \phi_{n-2}(\cdot\,, s_2)\|_{L^\infty(B(s_2))} \\
&\qquad\qquad + \|\psi_{n-1}(\cdot\,, s_2) - \psi_{n-2}(\cdot\,, s_2)\|_{L^\infty(B(s_2))}\Big) ds_2 ds_1 \\
&\quad + C_2 \int_0^t \Big(\|\phi_{n}(\cdot\,, s_1) - \phi_{n-1}(\cdot\,, s_1)\|_{L^\infty(B(s_2))} \\
&\qquad\qquad + \|\psi_{n}(\cdot\,, s_1) - \psi_{n-1}(\cdot\,, s_1)\|_{L^\infty(B(s_2))}\Big) ds_1 \\
&\vdots \\
&\leq \left(pC_0^{p-1}\right)^n \int_0^t \int_0^{s_1} \int_0^{s_2} \dots \int_0^{s_{n-1}} \\
&\quad \Big(\|D_\theta \phi_1(\cdot\,, s_n) - D_\theta \phi_0(\cdot\,, s_n)\|_{L^\infty(B(s_n))} \\
&\qquad\qquad + \|D_\theta \psi_1(\cdot\,, s_n) - D_\theta \psi_0(\cdot\,, s_n)\|_{L^\infty(B(s_n))}\Big) ds_1 ds_2 \dots ds_n \\
&\quad + \sum_{j=1}^n 4C_0 \frac{T^j}{j!} \cdot \frac{(C_2 T)^{n-j}}{(n-j)!} \\
&\leq 4C_1 \frac{(pC_0^{p-1}T)^n}{n!} + \sum_{j=1}^n 4C_0 \frac{(C_2T)^n}{j!(n-j)!} \to 0 \qquad \text{as } n \to \infty
\end{aligned}
$$

for $t \in [0, \tilde{t}]$. Here, $C_2 = \max\{pC_0^{p-1}, 2p(p-1)C_1C_0^{p-2}\}$. Thus, there exist $\phi_\theta^{(1)}, \psi_\theta^{(1)} \in L^\infty(K_-(\tilde{x}, \tilde{t}))$ such that

$$
\|D_\theta \phi_n - \phi_\theta^{(1)}\|_{L^\infty(K_-(\tilde{x},\tilde{t}))} + \|D_\theta \psi_n - \psi_\theta^{(1)}\|_{L^\infty(K_-(\tilde{x},\tilde{t}))} \to 0 \qquad \text{as } n \to \infty.
$$

Therefore, $(\phi, \psi) \in (W^{1,\infty}(K_-(\tilde{x}, \tilde{t})))^2$. By repeating the same arguments, we obtain that $(\phi, \psi) \in (W^{4,\infty}(K_-(\tilde{x}, \tilde{t})))^2$. That is, we have $(\phi, \psi) \in (C^{3,1}(K_-(\tilde{x}, \tilde{t})))^2$.

(Proof of uniqueness.)
Next, we will show that (ϕ, ψ) is a unique solution of (1.7). We suppose (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are solutions of (1.7) and T_1 and T_2 are corresponding blow-up curves. Let

$$
\Omega_j = \{(x,t) \mid x \in B_{R^*},\ 0 < t < T_j(x)\} \quad \text{for} \quad j = 1, 2.
$$

Take $(\tilde{x}, \tilde{t}) \in \Omega_1 \cap \Omega_2$ arbitrarily. In the same way of proof of **(Step 2.)**, we have

$$
\begin{aligned}
&\sup_{0 \leq t' \leq t} \Big(\|\phi_1(\cdot\,, t') - \phi_2(\cdot\,, t')\|_{L^\infty(B(t'))} + \|\psi_1(\cdot\,, t') - \psi_2(\cdot\,, t')\|_{L^\infty(B(t'))} \Big) \\
&\leq \sup_{0 \leq t' \leq t} \Big(\int_0^{t'} 2^{-p+1} \big\| |\phi_1 + \psi_1|^p(\cdot\,, s) - |\phi_2 + \psi_2|^p(\cdot\,, s) \big\|_{L^\infty(B(s))} ds \Big) \\
&\leq tpC_0^{p-1} \sup_{0 \leq t' \leq t} \Big(\|\phi_1(\cdot\,, t') - \phi_2(\cdot\,, t')\|_{L^\infty(B(t'))} \\
&\qquad\qquad + \|\psi_1(\cdot\,, t') - \psi_2(\cdot\,, t')\|_{L^\infty(B(t'))} \Big)
\end{aligned}
$$

for t satisfying $0 \leq t \leq \tilde{t}$. Thus,

$$
\sup_{0 \leq t' \leq t} \Big(\|\phi_1(\cdot\,, t') - \phi_2(\cdot\,, t')\|_{L^\infty(B(t'))} + \|\psi_1(\cdot\,, t') - \psi_2(\cdot\,, t')\|_{L^\infty(B(t'))} \Big) = 0
$$

if t is small enough. Since C_0 does not depend on t, by repeating this argument, we obtain

$$\sup_{0\le t'\le \tilde t}\Big(\|\phi_1(\cdot\,,t')-\phi_2(\cdot\,,t')\|_{L^\infty(B(t'))}+\|\psi_1(\cdot\,,t')-\psi_2(\cdot\,,t')\|_{L^\infty(B(t'))}\Big)=0.$$

Therefore, we have

$$(\phi_1,\psi_1)=(\phi_2,\psi_2)\quad \text{in}\quad \Omega_1\cap\Omega_2$$

and

$$T_1(x)=T_2(x)\quad \text{for}\quad x\in B_{R^*}.$$

This completes the proof. □

Lemma 2.6. *Assume that (A1)–(A4) hold. Then, we have*

$$T(x)<T^*\quad \textit{for } x\in B_{R^*}.$$

Proof. Let us define $\{\tilde\phi_n\}$ and $\{\tilde\psi_n\}$ by $\tilde\phi_0=\gamma_1$, $\tilde\psi_0=\gamma_2$ and

$$\begin{cases} \dfrac{d}{dt}\tilde\phi_{n+1}=2^{-p}|\tilde\phi_n+\tilde\psi_n|^p, & t>0,\\ \dfrac{d}{dt}\tilde\psi_{n+1}=2^{-p}|\tilde\phi_n+\tilde\psi_n|^p, & t>0,\\ \tilde\phi_{n+1}(0)=\gamma_1,\quad \tilde\psi_{n+1}(0)=\gamma_2. \end{cases}$$

It suffices to show that $\phi_n(x,t)\ge\tilde\phi_n(t)$ and $\psi_n(x,t)\ge\tilde\psi_n(t)$ in K_{R^*,T^*}, for $n\in\mathbb{N}$. First, we see that

$$\begin{aligned}\phi_1(x,t)-\tilde\phi_1(t)&=f(x+t)-\gamma_1+\int_0^t 2^{-p}|\phi_0+\psi_0|^p(x+(t-s),s)ds\\ &\quad-\int_0^t 2^{-p}|\tilde\phi_0+\tilde\psi_0|^p(s)ds\\ &=f(x+t)-\gamma_1\ge 0,\end{aligned}$$

in K_{R^*,T^*}. Similarly, we have that $\psi_1(x,t)\ge\tilde\psi_1(t)$ in K_{R^*,T^*}.

Next, we assume that $\phi_n(x,t)\ge\tilde\phi_n(t)$ and $\psi_n(x,t)\ge\tilde\psi_n(t)$ in K_{R^*,T^*}. Then, we have that

$$\begin{aligned}\phi_{n+1}(x,t)-\tilde\phi_{n+1}(t)&=f(x+t)-\gamma_1+\int_0^t 2^{-p}|\phi_n+\psi_n|^p(x+(t-s),s)ds\\ &\quad-\int_0^t 2^{-p}|\tilde\phi_n+\tilde\psi_n|^p(s)ds\\ &\ge 0,\end{aligned}$$

in K_{R^*,T^*}. Similarly, we obtain that $\psi_{n+1}(x,t)\ge\tilde\psi_{n+1}(t)$ in K_{R^*,T^*}. Therefore, we have

$$\phi_n(x,t)\ge\tilde\phi_n(t),\qquad \psi_n(x,t)\ge\tilde\psi_n(t)\quad \text{in } K_{R^*,T^*}$$

for $n\in\mathbb{N}$.

This completes the proof. □

3 Blow-up rates of solutions and Lipschitz continuity of T

Now, we will show that T is Lipschitz continuous in B_{R^*}. To prove this fact, we first introduce the following proposition.

Proposition 3.1. *Assume that (A1)–(A4) hold. Then, there exist positive constants C_1 and C_2 depending only on p and ε_0 such that*

$$C_1(\phi+\psi)^p \le \partial_t\phi \le C_2(\phi+\psi)^p, \tag{3.1}$$
$$C_1(T(x)-t)^{-q-1} \le \partial_t\phi(x,t) \le C_2(T(x)-t)^{-q-1}, \tag{3.2}$$
$$C_1(\phi+\psi)^p \le \partial_t\psi \le C_2(\phi+\psi)^p, \tag{3.3}$$
$$C_1(T(x)-t)^{-q-1} \le \partial_t\psi(x,t) \le C_2(T(x)-t)^{-q-1}, \tag{3.4}$$
$$C_1(T(x)-t)^{-q} \le (\phi+\psi)(x,t) \le C_2(T(x)-t)^{-q}, \tag{3.5}$$

in Ω. Here, $q = 1/(p-1)$.

Proof. First, we will show that (3.1) holds. We see that

$$\begin{aligned} D_-\partial_t\phi_{n+1} &= \partial_t D_-\phi_{n+1} = \partial_t 2^{-p}|\phi_n+\psi_n|^p = \partial_t 2^{-p}(\phi_n+\psi_n)^p \\ &= 2^{-p}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n+\partial_t\psi_n) \quad \text{in } K_{R^*,T^*}, \end{aligned} \tag{3.6}$$

for $n \in \mathbb{N}\cup\{0\}$. From Lemma 2.3, we obtain that

$$\begin{aligned} D_- 2^{-p}(\phi_n+\psi_n)^p &= 2^{-p}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n - \partial_x\phi_n + \partial_t\psi_n - \partial_x\psi_n) \\ &\le 2^{-p+1}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n+\partial_t\psi_n) \quad \text{in } K_{R^*,T^*}, \end{aligned} \tag{3.7}$$

for $n \in \mathbb{N}\cup\{0\}$. We set $J_{\phi,n+1} = 2\partial_t\phi_{n+1} - 2^{-p}(\phi_n+\psi_n)^p$. Then, by (3.6) and (3.7), we have

$$\begin{aligned} &D_-J_{\phi,n+1} \\ &\ge 2^{-p+1}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n+\partial_t\psi_n) - 2^{-p+1}p(\phi_n+\psi_n)^{p-1}(\partial_t\phi_n+\partial_t\psi_n) \\ &= 0 \quad \text{in } K_{R^*,T^*}, \end{aligned} \tag{3.8}$$

for $n \in \mathbb{N}\cup\{0\}$. It follows from (A3) that

$$\begin{aligned} J_{\phi,n+1}(x,0) &= 2\partial_t\phi_{n+1}(x,0) - 2^{-p}(\phi_n+\psi_n)^p(x,0) \\ &= 2\partial_x\phi_{n+1}(x,0) + 2^{-p}(\phi_n+\psi_n)^p(x,0) \\ &\ge 2f_x + 2^{-p}(\gamma_1+\gamma_2)^p \ge 0 \quad \text{in } B_{R^*+T^*} \end{aligned} \tag{3.9}$$

for $n \in \mathbb{N}\cup\{0\}$. Then, by (3.9) and (3.8), we obtain that $J_{\phi,n} \ge 0$ in K_{R^*,T^*}, for $n \in \mathbb{N}$.

On the other hand, it follows from Lemma 2.3 that

$$\partial_t\phi_{n+1} = \partial_x\phi_{n+1} + 2^{-p}(\phi_n+\psi_n)^p \le \frac{1}{1+\varepsilon_0}\partial_t\phi_{n+1} + 2^{-p}(\phi_n+\psi_n)^p$$

in K_{R^*,T^*}, for $n \in \mathbb{N}\cup\{0\}$. Hence,

$$\partial_t\phi_{n+1} \le \frac{1+\varepsilon_0}{\varepsilon_0}2^{-p}(\phi_n+\psi_n)^p \quad \text{in } K_{R^*,T^*}, \tag{3.10}$$

for $n \in \mathbb{N}\cup\{0\}$. It follows from the fact that $J_{\phi,n} \ge 0$ and (3.10) that

$$2^{-p-1}(\phi_n+\psi_n)^p \le \partial_t\phi_{n+1} \le \frac{1+\varepsilon_0}{\varepsilon_0}\cdot 2^{-p}(\phi_{n+1}+\psi_{n+1})^p \text{ in } K_{R^*,T^*}, \tag{3.11}$$

for $n \in \mathbb{N}\cup\{0\}$, which implies (3.1) holds. Similarly, we can prove that (3.3) holds.

Next, we will show that (3.5) holds. By considering (3.1), we see that

$$\frac{\partial(\phi+\psi)}{\partial t} \le 2^{-p+1}(1+\varepsilon_0)\varepsilon_0^{-1}(\phi+\psi)^p \quad \text{in} \quad \Omega.$$

Thus, we have

$$\frac{\partial t}{\partial(\phi+\psi)} \geq 2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0(\phi+\psi)^{-p} \quad \text{in} \quad \Omega. \tag{3.12}$$

Fix $x_0 \in B_{R^*}$. By (3.12), we have

$$\begin{aligned} T(x_0) - \varepsilon - \tau &\geq \int_{(\phi+\psi)(x_0,\tau)}^{(\phi+\psi)(x_0,T(x_0)-\varepsilon)} 2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0 z^{-p} dz \\ &= \Big[-(p-1)^{-1}2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0 z^{-(p-1)}\Big]_{(\phi+\psi)(x_0,\tau)}^{(\phi+\psi)(x_0,T(x_0)-\varepsilon)}. \end{aligned}$$

for $\tau > 0$ and $\varepsilon > 0$ satisfying $T(x_0) - \varepsilon - \tau > 0$. Hence, by letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} T(x_0) - \tau &\geq \Big[-(p-1)^{-1}2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0 z^{-(p-1)}\Big]_{(\phi+\psi)(x_0,\tau)}^{\infty} \\ &= (p-1)^{-1}2^{p-1}(1+\varepsilon_0)^{-1}\varepsilon_0(\phi+\psi)^{-(p-1)}(x_0,\tau). \end{aligned}$$

Thus, we have that

$$(\phi+\psi)(x_0,\tau) \geq 2\left((p-1)\varepsilon_0^{-1}(1+\varepsilon_0)\right)^{-1/(p-1)}(T(x_0)-\tau)^{-1/(p-1)} \tag{3.13}$$

for $\tau \in [0, T(x_0))$. Similarly, we obtain that

$$(2^p(p-1)^{-1})^{1/(p-1)}(T(x_0)-\tau)^{-1/(p-1)} \geq (\phi+\psi)(x_0,\tau) \tag{3.14}$$

for $\tau \in [0, T(x_0))$. It follows from (3.13) and (3.14) that (3.5) holds. Moreover, it follows from (3.1) and (3.5) that (3.2) holds. Similarly, we have that (3.4) also holds. This completes the proof. □

By combining the above Lemma 2.3 with Proposition 3.1, we obtain that the blow-up curve T is Lipschitz continuous. That is, the following lemma holds.

Lemma 3.2. *Suppose that (A1)–(A4) hold. Then, we have that*

$$|T(x') - T(x'')| \leq \frac{1}{1+\varepsilon_0}|x' - x''| \quad \textit{for} \quad x', x'' \in B_{R^*}. \tag{3.15}$$

Proof. This proof is based on the Implicit Function Theorem. Let $\varepsilon > 0$ be arbitrary. By (3.5), we see that there exists a positive constant C_1 depending p and ε_0 such that

$$C_1\varepsilon^{-q} \leq (\phi+\psi)(x,t) \quad \text{for} \quad x \in B_{R^*} \quad \text{and} \quad t \in [T(x)-\varepsilon, T(x)).$$

Thus, there exists a positive constant M satisfying $M \geq C_1\varepsilon^{-q}$, and a function $E(x)$ ($x \in B_{R^*}$) such that

$$(\phi+\psi)(x, E(x)) = M \quad \text{and} \quad T(x) - E(x) \leq \varepsilon \quad \text{for } x \in B_{R^*}.$$

First, we will demonstrate continuity of E in B_{R^*}. That is, for $x' \in B_{R^*}$, we will show that $t_n \to E(x')$ if $x_n \to x'$, where $t_n = E(x_n)$.

We take an arbitrary converging subsequence $\{t_{nk}\} \subset \{t_n\}$, and denote its limit by η. Following from the definition of E, we have that $(\phi+\psi)(x_{nk}, t_{nk}) = (\phi+\psi)(x_{nk}, E(x_{nk})) = M$. Thus, it follows from continuity of ϕ and ψ that $(\phi+\psi)(x', \eta) = M$. Since $\partial_t(\phi+\psi) > 0$ in Ω, we have that $\eta = E(x')$. Therefore,

$$\liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = E(x').$$

Thus, we have demonstrated the continuity of E at x'.

Next, we will prove Lipschitz continuity of E. We see that there exists a positive constant h' for $x' \in B_{R^*}$ such that

$$\boldsymbol{B}(x', h') \subset \Omega,$$

where $\boldsymbol{B}(x', h') = \left\{(t, x) \mid \sqrt{(x-x')^2 + (t - E(x'))^2} < h'\right\}$. Following from continuity of E, there exits a positive constant h'' such that $0 < h'' \le h'$ satisfying

$$(x_1, E(x_1)), (x_2, E(x_2)) \in \boldsymbol{B}(x', h') \quad \text{for } x_1, x_2 \in (x' - h'', x' + h'').$$

Let $k = E(x_2) - E(x_1)$ and

$$H(\xi) = (\phi + \psi)(x_1 + \xi(x_2 - x_1), t + \xi k),$$

where ξ is a constant satisfying $0 \le \xi \le 1$. Then, we have

$$\begin{aligned} &H(0) = (\phi + \psi)(x_1, t), \\ &H(1) = (\phi + \psi)(x_2, t + k) = (\phi + \psi)(x_2, t + E(x_2) - E(x_1)). \end{aligned}$$

Take t as $t = E(x_1)$. Then, we have $H(0) = H(1) = M$. By Rolle's Theorem, there exists $\xi' \in (0, 1)$ such that

$$\begin{aligned} H'(\xi') =& (x_2 - x_1)\partial_x(\phi + \psi)(x_1 + \xi'(x_2 - x_1), E(x_1) + \xi' k) \\ &+ k\partial_t(\phi + \psi)(x_1 + \xi'(x_2 - x_1), E(x_1) + \xi' k) = 0. \end{aligned} \tag{3.16}$$

Hence, it follows from Lemma 2.3 and (3.16) that

$$\begin{aligned} |E(x_1) - E(x_2)| = |k| &= \left| \frac{-\partial_x(\phi + \psi)(x_1 + \xi'(x_2 - x_1), E(x_1) + \xi' k)}{\partial_t(\phi + \psi)(x_1 + \xi'(x_2 - x_1), E(x_1) + \xi' k)} \right| |x_1 - x_2| \\ &\le \frac{1}{1 + \varepsilon_0} |x_1 - x_2|. \end{aligned}$$

Thus, E is Lipschitz continuous in $(x' - h'', x' + h'')$. Moreover, it follows from the continuity of E that

$$\begin{aligned} \frac{E(x+h) - E(x)}{h} &= \frac{-h\partial_x(\phi + \psi)(x + \xi h, E(x) + \xi(E(x+h) - E(x)))}{h\partial_t(\phi + \psi)(x + \xi h, E(x) + \xi(E(x+h) - E(x)))} \\ &\to \frac{-\partial_x(\phi + \psi)(x, E(x))}{\partial_t(\phi + \psi)(x, E(x))} \quad \text{as } h \to 0 \quad \text{for} \quad x \in B_{R^*}. \end{aligned}$$

Hence, we have that

$$\frac{\partial}{\partial x} E(x) = \frac{-\partial_x(\phi + \psi)(x, E(x))}{\partial_t(\phi + \psi)(x, E(x))} \quad \text{for} \quad x \in B_{R^*}.$$

By continuity of $\partial_x(\phi + \psi)$, $\partial_t(\phi + \psi)$ and E, we see that $E \in C^1(B_{R^*})$. Hence, we have that

$$|E(x') - E(x'')| \le \left(\sup_{x \in B_{R^*}} |E'(x)| \right) |x' - x''| \le \frac{1}{1 + \varepsilon_0} |x' - x''| \tag{3.17}$$

for $x', x'' \in B_{R^*}$. Therefore, E is Lipschitz continuous in B_{R^*}.

Finally, we will prove Lipschitz continuity of T in B_{R^*}. It follows from (3.17) that

$$\begin{aligned} |T(x') - T(x'')| &\le |T(x') - E(x')| + |E(x') - E(x'')| + |E(x'') - T(x'')| \\ &\le 2\varepsilon + \frac{1}{1 + \varepsilon_0} |x' - x''| \quad \text{for} \quad x', x'' \in B_{R^*}. \end{aligned}$$

Since we let $\varepsilon > 0$ take an arbitrary value, this completes the proof. $\square$

By applying Lemma 3.2, we obtain the following results.

Definition 3.3. By $d(x,t)$, we denote the distance from a point (x,t) in Ω to $\Gamma = \{(x, T(x)) \mid x \in B_{R^*}\}$.

Remark 3.4. It follows from Lemma 3.2 that

$$\frac{T(x)-t}{\sqrt{2}} \le d(x,t) \le T(x) - t.$$

By replacing $T(x) - t$ by $d(x,t)$ in Proposition 3.1, we obtain the following Corollary.

Corollary 3.5. *Assume that (A1)–(A4) hold. Then, there exist positive constants C_1 and C_2 depending only on p and ε_0 such that*

$$C_1 d^{-q}(x,t) \le (\phi+\psi)(x,t) \le C_2 d^{-q}(x,t), \tag{3.18}$$
$$C_1 d^{-q-1}(x,t) \le \partial_t \phi(x,t) \le C_2 d^{-q-1}(x,t), \tag{3.19}$$
$$C_1 d^{-q-1}(x,t) \le \partial_t \psi(x,t) \le C_2 d^{-q-1}(x,t), \tag{3.20}$$

where $q = 1/(p-1)$, in Ω.

From Corollary 3.5, we obtain the following lemma, which states that T is the blow-up curve of both ϕ and ψ:

Lemma 3.6. *Assume that (A1)–(A4) hold. Then, there exist positive constants C_1 and C_2 depending on p and ε_0 such that*

$$C_1(T(x)-t)^{-q} \le \phi(x,t) \le C_2(T(x)-t)^{-q}, \tag{3.21}$$
$$C_1(T(x)-t)^{-q} \le \psi(x,t) \le C_2(T(x)-t)^{-q}, \tag{3.22}$$

where $q = 1/(p-1)$, in Ω.

Proof. We will only show that (3.21) holds. By Corollary 3.5 and Lemma 3.2, there exist positive constants c_1 and c_2 depending p and ε_0 such that

$$\begin{aligned}
\phi(x, T(x)-\varepsilon) &= f(x + T(x) - \varepsilon) \\
&\quad + \int_0^{T(x)-\varepsilon} 2^{-p}(\phi+\psi)^p(x + (T(x)-\varepsilon) - s, s)ds \\
&\ge \int_{T(x)-2\varepsilon}^{T(x)-\varepsilon} 2^{-p}(\phi+\psi)^p(x + (T(x)-\varepsilon) - s, s)ds \\
&\ge c_1\varepsilon \inf_{T(x)-2\varepsilon \le s \le T(x)-\varepsilon} d(x + (T(x)-\varepsilon) - s, s)^{-qp} \\
&\ge c_2 \varepsilon \cdot \varepsilon^{-q-1} = c_2\varepsilon^{-q}.
\end{aligned}$$

On the other hand, it follows from Proposition 3.1 that there exists a positive constant C_2 depending only on p and ε_0 such that $\phi(x, T(x)-\varepsilon) \le C_2\varepsilon^{-q}$. This completes the proof. $\square$

4 Blow-up limits of solutions

In the following, we will show that $T \in C^1(B_{R^*})$. In order to achieve this, we will consider limits of the scaled functions T_λ, ϕ_λ, and ψ_λ (we will define these later) and their properties.

4.1 Estimates of blow-up limits

We set D_θ as

$$D_\theta = \cos\theta\partial_x + \sin\theta\partial_t, \quad \text{where } 0 \leq \theta < 2\pi.$$

First, we introduce the following lemma.

Lemma 4.1. *Assume that (A1)–(A5.3) hold. Then, there exist positive constants C_α and C_α^* depending only on p and ε_1 such that*

$$\max\{|D_\theta^\alpha \phi(x,t)|, |D_\theta^\alpha \psi(x,t)|\} \leq C_\alpha(\phi+\psi)^{p+(\alpha-1)/q}(x,t) \tag{4.1}$$
$$\leq C_\alpha^* d(x,t)^{-(pq+(\alpha-1))} \tag{4.2}$$

for $(x,t) \in \Omega$, where $q = 1/(p-1)$ and $\alpha = 0,1,2,3$.

Proof. We can easily obtain that (4.2) holds by Corollary 3.5 if we prove (4.1). So, we will only prove (4.1).

We also obtain that (4.1) holds in the case of $\alpha = 0,1$, by Lemmas 2.2, 2.3 and Proposition 3.1.

First, we will show that (4.1) holds in the case of $\alpha = 2$. It suffices to show that there exists a positive constant C_2 depending only on p and ε_1 such that

$$\begin{aligned}&\max\{|D_\theta^2\phi_n(x,t)|, |D_\theta^2\psi_n(x,t)|\} \\ &\qquad \leq C_2(\phi_n+\psi_n)^{2p-1}(x,t) \quad \text{for } n \in \mathbb{N}\cup\{0\},\end{aligned} \tag{4.3}$$

in K_{R^*,T^*}. We see that $D_\theta\phi_0 = D_\theta\psi_0 = 0$ in K_{R^*,T^*}. Hence, (4.1) holds for $n=0$.

Assume

$$\max\big\{|D_\theta^2\phi_n(x,t)|, |D_\theta^2\psi_n(x,t)|\big\} \leq C_2(\phi_n+\psi_n)^{2p-1}(x,t) \quad \text{in} \quad K_{R^*,T^*}.$$

Then, it follows from (4.1) in the case $\alpha = 1$ and Proposition 3.1 that

$$\begin{aligned}&|D_-(D_\theta^2\phi_{n+1})(x,t)| \\ &= 2^{-p}|D_\theta^2(\phi_n+\psi_n)^p(x,t)| \\ &\leq 2^{-p}p(p-1)(\phi_n+\psi_n)^{p-2}(x,t)(D_\theta\phi_n + D_\theta\psi_n)^2(x,t) \\ &\qquad + 2^{-p}p(\phi_n+\psi_n)^{p-1}(x,t)|(D_\theta^2\phi_n + D_\theta^2\psi_n)(x,t)| \\ &\leq 2^{-p+1}p\Big(2(p-1)C_1^2 + C_2\Big)|(\phi_n+\psi_n)^{3p-2}(x,t)| \quad \text{in } K_{R^*,T^*},\end{aligned} \tag{4.4}$$

where C_α is the constant in the case of $\alpha = 1,2$ of (4.1). Moreover, it follows from Lemma 2.3 and Proposition 3.1 that

$$\begin{aligned}&D_-C_2(\phi_{n+1}+\psi_{n+1})^{2p-1}(x,t) \\ &= C_2(2p-1)(\phi_{n+1}+\psi_{n+1})^{2p-2}(x,t)D_-(\phi_{n+1}+\psi_{n+1})(x,t) \\ &\geq 2^{-p}C_2(2p-1)\Big(1+\frac{\varepsilon_1}{2(1+\varepsilon_1)}\Big)(\phi_n+\psi_n)^{3p-2}(x,t) \quad \text{in } K_{R^*,T^*}.\end{aligned} \tag{4.5}$$

Let

$$M_n(x,t) = C_2(\phi_n+\psi_n)^{2p-1}(x,t) - D_\theta^2\phi_n(x,t).$$

Then, it follows from (A3) and (A5.2) that

$$M_{n+1}(x,0) \geq \Big(C_2 - 4{C^{(2)}}^{-1} - p2^{-2p+3}\Big)(f+g)^{2p-1}(x), \tag{4.6}$$

in $B_{R^*+T^*}$. On the other hand, it follows from (4.4) and (4.5) that

$$\begin{aligned} D_-M_{n+1}(x,t) \geq 2^{-p}C_2\Big\{(2p-1)\Big(1+\frac{\varepsilon_1}{2(1+\varepsilon_1)}\Big)-2p\Big\}(\phi_n+\psi_n)^{3p-2}(x,t) \\ -2^{-p}4p(p-1)C_1^2(\phi_n+\psi_n)^{3p-2}(x,t) \quad \text{in } K_{R^*,T^*}. \end{aligned} \tag{4.7}$$

By (A5.1), we have

$$(2p-1)\Big(1+\frac{\varepsilon_1}{2(1+\varepsilon_1)}\Big)-2p>0.$$

We take C_2 as

$$\begin{aligned} C_2 > \max\Big\{&4C^{(2)^{-1}}+p2^{-2p+3}, \\ &\Big\{(2p-1)\Big(1+\frac{\varepsilon_1}{2(1+\varepsilon_1)}\Big)-2p\Big\}^{-1}4p(p-1)C_1^2\Big\}. \end{aligned}$$

Then, it follows from (4.6) and (4.7) that $M_{n+1}\geq 0$ in K_{R^*,T^*}. Consequently, we obtain that $M_n\geq 0$ in K_{R^*,T^*}, for $n\in\mathbb{N}\cup\{0\}$. That is, there exists a positive constant C_2 depending p and ε_1 such that

$$C_2(\phi_n+\psi_n)^{2p-1}\geq D_\theta^2\phi_n \quad \text{in } K_{R^*,T^*}$$

for $n\in\mathbb{N}\cup\{0\}$. Similarly, we have the following inequality by retaking C_2 if necessary.

$$\begin{cases} C_2(\phi_n+\psi_n)^{2p-1}\geq -D_\theta^2\phi_n, & \\ C_2(\phi_n+\psi_n)^{2p-1}\geq D_\theta^2\psi_n, & \text{in } K_{R^*,T^*}, \\ C_2(\phi_n+\psi_n)^{2p-1}\geq -D_\theta^2\psi_n, & \end{cases}$$

for $n\in\mathbb{N}\cup\{0\}$. This means (4.3) holds. In the same way, we can prove (4.1) in the case of $\alpha=3$. □

Let $x_0\in B_{R^*}$. Then, we introduce the following scaled functions:

$$\phi_\lambda(y,s)=\lambda^q\phi(x_0+\lambda y,T(x_0)+\lambda s), \tag{4.8}$$

$$\psi_\lambda(y,s)=\lambda^q\psi(x_0+\lambda y,T(x_0)+\lambda s), \tag{4.9}$$

where $\lambda>0$ and $q=1/(p-1)$. Any sequences $\{\phi_{\lambda_n}\}$ and $\{\psi_{\lambda_n}\}$ with $\lambda_n\downarrow 0$ are called blow-up sequences (see. [2]). Now, we see that

$$\begin{cases} D_-\phi_\lambda=2^{-p}(\phi_\lambda+\psi_\lambda)^p, \\ D_+\psi_\lambda=2^{-p}(\phi_\lambda+\psi_\lambda)^p \end{cases} \tag{4.10}$$

for $(y,s)\in\Omega_\lambda$, where $\Omega_\lambda=\big\{(y,s)\in\mathbb{R}^2 \mid (x_0+\lambda y,T(x_0)+\lambda s)\in\Omega\big\}$. By $d_\lambda(y,s)$, we denote the distance from a point $(y,s)\in\Omega_\lambda$ to $\Gamma_\lambda=\{(y,s)\mid s=T_\lambda(y)\}$. Here, T_λ is a blow-up curve of ϕ_λ.

Lemma 4.2. *For each fixed $\lambda>0$,*

$$T_\lambda(y)=\frac{T(x_0+\lambda y)-T(x_0)}{\lambda}. \tag{4.11}$$

Proof. By Lemma 3.6, there exist positive constants C_1 and C_2 depending on p and ε_1 such that

$$\begin{aligned} &\lambda^qC_1\left(T(x_0+\lambda y)-(T(x_0)+\lambda s)\right)^{-q} \\ &\quad\leq \lambda^q\phi(x_0+\lambda y,T(x_0)+\lambda s)\leq \lambda^qC_2\left(T(x_0+\lambda y)-(T(x_0)+\lambda s)\right)^{-q}. \end{aligned}$$

We see that

$$\lambda^q\left(T(x_0+\lambda y)-(T(x_0)+\lambda s)\right)^{-q}=\left(\frac{T(x_0+\lambda y)-T(x_0)}{\lambda}-s\right)^{-q}. \tag{4.12}$$

Therefore, we obtain (4.11). □

Similarly, we can show that the blow-up curve of $\psi_\lambda(y,s)$ is $T_\lambda(y)$.

From Proposition 3.1 and Lemmas 2.3, 3.2 and 4.1, there exist positive constants C_1, C_2, $C_{3,\alpha}$, and $C_{4,\alpha}$, depending only on p and ε_1 such that

$$C_1(\phi_\lambda+\psi_\lambda)^p \le \partial_s\phi_\lambda \le C_2(\phi_\lambda+\psi_\lambda)^p, \tag{4.13}$$

$$C_1(\phi_\lambda+\psi_\lambda)^p \le \partial_s\psi_\lambda \le C_2(\phi_\lambda+\psi_\lambda)^p, \tag{4.14}$$

$$C_1(T_\lambda(y)-s)^{-q} \le \phi_\lambda(y,s) \le C_2(T_\lambda(y)-s)^{-q}, \tag{4.15}$$

$$C_1(T_\lambda(y)-s)^{-q} \le \psi_\lambda(y,s) \le C_2(T_\lambda(y)-s)^{-q}, \tag{4.16}$$

$$|\partial_y\phi_\lambda| \le \frac{1}{1+\varepsilon_1}\partial_s\phi_\lambda, \quad |\partial_y\psi_\lambda| \le \frac{1}{1+\varepsilon_1}\partial_s\psi_\lambda, \tag{4.17}$$

$$|T_\lambda(y)-T_\lambda(y')| \le \frac{1}{1+\varepsilon_1}|y-y'| \quad \text{for} \quad y,y' \in \left(\frac{-R-x_0}{\lambda}, \frac{R-x_0}{\lambda}\right), \tag{4.18}$$

$$\frac{T_\lambda(y)-s}{\sqrt{2}} \le d_\lambda(y,s) \le T_\lambda(y)-s, \tag{4.19}$$

$$\begin{aligned}&\max\{|D_\theta^\alpha\phi_\lambda(y,s)|, |D_\theta^\alpha\psi_\lambda(y,s)|\} \\ &\qquad \le C_{3,\alpha}(\phi_\lambda(y,s)+\psi_\lambda(y,s))^{p+(\alpha-1)/q} \le C_{4,\alpha}d_\lambda(y,s)^{-(pq+\alpha-1)}.\end{aligned} \tag{4.20}$$

where $(y,s) \in \Omega_\lambda$. Here $\alpha = 0,1,2,3$.

4.2 Strategy of proof of the differentiability of T

We will consider the limits of the functions T_{λ_n}, ϕ_{λ_n}, and ψ_{λ_n}. It follows from (4.18) that T_{λ_n} is equicontinuous.

We define I_n by a closed interval satisfying

- $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$,
- $\bigcup_{n=1}^\infty . \, I \subset I_{n_0}$.

By (4.18), there exists a positive constant M_1 such that

$$|T_{\lambda_n}(y)| \le M_1 \qquad \text{for} \quad y \in I_1.$$

By the Ascoli and Arzela theorem, there exist a sequence $\{\lambda_n^{(1)}\} \subset \{\lambda_n\}$ and $T_0^{(1)} \in C(I_1)$ such that $T_{\lambda_n^{(1)}}$ converges to $T_0^{(1)}$ uniformly in I_1.

In the same manner as above, we can see that there exist a sequence $\{\lambda_n^{(2)}\} \subset \{\lambda_n^{(1)}\}$ and $T_0^{(2)} \in C(I_2)$ such that $T_{\lambda_n^{(2)}}$ converges to $T_0^{(2)}$ uniformly in I_2. By repeating the same arguments, there exists $T_0 \in C(\mathbb{R})$ such that T_{Λ_n} converges to T_0 locally uniformly in $\mathbb{R}$, where $\Lambda_n = \lambda_n^{(n)}$.

In the remainder of this paper, we will show that $T \in C^1(B_R)$. We demonstrate this proof through the following two steps.

(Step 1.) First (in Section 5), we will show that T_0, which is defined as above, is an affine function. That is, there exists a constant α_{x_0} such that $T_0(y) = \alpha_{x_0}y$ for $y \in \mathbb{R}$.

(Step 2.) Next (in Section 6), we will demonstrate that a contradiction arises if we assume that there exists $x_0 \in B_{R^*}$ such that T is not differentiable at $x_0 \in B_{R^*}$.

We start by assuming that T is not differentiable at $x_0 \in B_{R^*}$. On the other hand, by (Step 1), we have that for all $y \in \mathbb{R}$,

$$\frac{T_{\Lambda_n}(y)}{y} = \frac{T(x_0+\Lambda_n y)-T(x_0)}{\Lambda_n y} \to \alpha_{x_0} \quad \text{as} \quad \Lambda_n \to 0,$$

where $\{\Lambda_n\} \subset \{\lambda_n\}$ is the sequence appeared in (Step 1). This means that there exist $\{\lambda_{n'}\} \subset \{\lambda_n\}$ and $y' \in \mathbb{R}$ such that

$$\limsup_{\lambda_{n'} \to 0} T_{\lambda_{n'}}(y') > \liminf_{\lambda_{n'} \to 0} T_{\lambda_{n'}}(y'). \tag{4.21}$$

On the other hand, there exist $\{\lambda^{(1)}_{n'}\} \subset \{\lambda_{n'}\}$ and $\{\lambda^{(2)}_{n'}\} \subset \{\lambda_{n'}\}$ such that

$$\lim_{\lambda^{(1)}_{n'} \to 0} T_{\lambda^{(1)}_{n'}}(y') = \limsup_{\lambda_{n'} \to 0} T_{\lambda_{n'}}(y'),$$
$$\lim_{\lambda^{(2)}_{n'} \to 0} T_{\lambda^{(2)}_{n'}}(y') = \liminf_{\lambda_{n'} \to 0} T_{\lambda_{n'}}(y').$$

By repeating the above arguments, there exist $\{\lambda^{(1)}_{n'_k}\} \subset \{\lambda^{(1)}_{n'}\}$ and $\{\lambda^{(2)}_{n'_k}\} \subset \{\lambda^{(2)}_{n'}\}$, and corresponding functions $T^{(1)}_0$, $T^{(2)}_0 \in C(\mathbb{R})$, such that

$$T_{\lambda^{(1)}_{n'_k}} \to T^{(1)}_0, \quad T_{\lambda^{(2)}_{n'_k}} \to T^{(2)}_0 \quad \text{locally uniformly in } \mathbb{R}.$$

It follows from **(Step 1)** that there exist constants $\alpha^{(1)}_{x_0}$ and $\alpha^{(2)}_{x_0}$ such that $T^{(1)}_0(y) = \alpha^{(1)}_{x_0} y$ and $T^{(2)}_0(y) = \alpha^{(2)}_{x_0} y$, respectively. By (4.21), we see that $\alpha^{(1)}_{x_0} \neq \alpha^{(2)}_{x_0}$.

In Section 6, we will demonstrate that a contradiction arises if there exist $\alpha^{(1)}_{x_0}$ and $\alpha^{(2)}_{x_0}$ such that $\alpha^{(1)}_{x_0} \neq \alpha^{(2)}_{x_0}$ and

$$T^{(1)}_0(y) = \alpha^{(1)}_{x_0} y, \quad T^{(2)}_0(y) = \alpha^{(2)}_{x_0} y \qquad \text{for } y \in \mathbb{R}.$$

That is, we obtain that T is differentiable in B_{R^*}. Moreover, we can show that a contradiction arises if we assume that the derivative T' is not continuous in B_{R^*}.

In the remainder of this section, we prepare for our proof of **(Step 1.)**. We consider the limits of blow-up sequences ϕ_{λ_n} and ψ_{λ_n}. We set $\Omega_0 = \{(y, s) \mid y \in \mathbb{R},\ s < T_0(y)\}$. Then, we set J_n as a closed subset of Ω_0 satisfying

- $J_n \subset J_{n+1}$ for $n \in \mathbb{N}$,
- $\bigcup_{n=1}^{\infty} J_n = \Omega_0$.

It follows from the Ascoli and Arzela theorem that there exists a subsequence $\{\tilde\lambda_n\} \subset \{\Lambda_n\}$, such that there exist

$$v_\phi, v_\psi, v^{1,\theta}_\phi, v^{1,\theta}_\psi, v^{2,\theta}_\phi, v^{2,\theta}_\psi, v^{3,\theta}_\phi, v^{3,\theta}_\psi \in C(\Omega_0)$$

satisfying

$$\begin{cases} \phi_{\tilde\lambda_n} \to v_\phi, \quad \psi_{\tilde\lambda_n} \to v_\psi, \\ D_\theta \phi_{\tilde\lambda_n} \to v^{1,\theta}_\phi, \quad D_\theta \psi_{\tilde\lambda_n} \to v^{1,\theta}_\psi, \\ D^2_\theta \phi_{\tilde\lambda_n} \to v^{2,\theta}_\phi, \quad D^2_\theta \psi_{\tilde\lambda_n} \to v^{2,\theta}_\psi, \\ D^3_\theta \phi_{\tilde\lambda_n} \to v^{3,\theta}_\phi, \quad D^3_\theta \psi_{\tilde\lambda_n} \to v^{3,\theta}_\psi, \end{cases} \text{locally uniformly in } \Omega_0 \tag{4.22}$$

for $\theta \in [0, 2\pi)$. Thus, we have that $v_\phi, v_\psi \in C^3(\Omega_0)$. The functions v_ϕ and v_ψ are called blow-up limits of ϕ and ψ (see [2]). By (4.10), (4.13)–(4.20), we have that

$$\begin{cases} D_- v_\phi = 2^{-p}(v_\phi + v_\psi)^p, \\ D_+ v_\psi = 2^{-p}(v_\phi + v_\psi)^p, \end{cases} \tag{4.23}$$

and there exist positive constants $C_1, C_2, C_{3,\alpha}$ and $C_{4,\alpha}$, depending only on p and ε_1, such that

$$C_1(v_\phi + v_\psi)^p \le \partial_s v_\phi \le C_2(v_\phi + v_\psi)^p, \tag{4.24}$$

$$C_1(v_\phi + v_\psi)^p \le \partial_s v_\psi \le C_2(v_\phi + v_\psi)^p, \tag{4.25}$$

$$C_1(T_0(y) - s)^{-q} \le v_\phi(y,s) \le C_2(T_0(y) - s)^{-q}, \tag{4.26}$$

$$C_1(T_0(y) - s)^{-q} \le v_\psi(y,s) \le C_2(T_0(y) - s)^{-q}, \tag{4.27}$$

$$|\partial_y v_\phi| \le \frac{1}{1+\varepsilon_1}\partial_s v_\phi, \quad |\partial_y v_\psi| \le \frac{1}{1+\varepsilon_1}\partial_s v_\psi, \tag{4.28}$$

$$|T_0(y) - T_0(y')| \le \frac{1}{1+\varepsilon_1}|y - y'| \quad \text{for} \quad y, y' \in \mathbb{R}, \tag{4.29}$$

$$\frac{T_0(y) - s}{\sqrt{2}} \le d_0(y,s) \le T_0(y) - s, \tag{4.30}$$

$$\begin{aligned} &\max\{|D_\theta^\alpha v_\phi(y,s)|, |D_\theta^\alpha v_\psi(y,s)|\} \\ &\quad \le C_{3,\alpha}(v_\phi(y,s) + v_\psi(y,s))^{p+(\alpha-1)/q} \le C_{4,\alpha} d_0(y,s)^{-(pq+\alpha-1)}, \end{aligned} \tag{4.31}$$

where $(y,s) \in \Omega_0$. Here, $d_0(y,s)$ is the distance from a point $(y,s) \in \Omega_0$ to $\Gamma_0 = \{(y,s) \mid s = T_0(y), \quad y \in \mathbb{R}\}$ and $\alpha = 0, 1, 2, 3$.

4.3 Convexity of blow-up limits

In order to demonstrate that T_0 is an affine function, we will prove the following lemma.

Lemma 4.3. *Assume that (A1)–(A5.3) hold. Then, we have that*

$$D_\theta^2 v_\phi \ge 0, \quad D_\theta^2 v_\psi \ge 0 \quad \textit{in} \quad \Omega_0 \tag{4.32}$$

for $0 \le \theta < 2\pi$.

Proof. We fix a point $(\tilde{y}, \tilde{s}) \in \Omega_0$. Let $\boldsymbol{K}_-(\tilde{y}, \tilde{s}) = \{(y,s) \in \Omega_0 \mid |\tilde{y} - y| < \tilde{s} - s\}$. Then, it suffices to show that $D_\theta^2 v_\phi, D_\theta^2 v_\psi \ge 0$ in $\boldsymbol{K}_-(\tilde{y}, \tilde{s})$.

Let

$$J_\phi = D_\theta^2 v_\phi + \eta \partial_s v_\phi, \qquad J_\psi = D_\theta^2 v_\psi + \eta \partial_s v_\psi,$$

where η is a positive constant.

In what follows, we will show that

$$J_\phi > 0 \quad \text{and} \quad J_\psi > 0 \quad \text{in } \boldsymbol{K}_-(\tilde{y}, \tilde{s}). \tag{4.33}$$

We see that

$$\begin{aligned} D_- J_\phi &= D_+ J_\psi \\ &= 2^{-p} p(p-1)(v_\phi + v_\psi)^{p-2}(D_\theta v_\phi + D_\theta v_\psi)^2 \\ &\qquad + 2^{-p} p(v_\phi + v_\psi)^{p-1}(J_\phi + J_\psi). \end{aligned} \tag{4.34}$$

We consider J_ϕ and J_ψ in $\boldsymbol{K}_-(\tilde{y}, \tilde{s})$. By (4.30), we have

$$\frac{1}{\sqrt{2}}\left(\frac{T_0(y) - s}{|s|}\right) \le \frac{d_0(y,s)}{|s|} \le \frac{T_0(y) - s}{|s|}.$$

Thus, we obtain that

$$\frac{1}{\sqrt{2}} \le \frac{d_0(y,s)}{|s|} \le 1 \quad \text{for} \quad (y,s) \in \boldsymbol{K}_-(\tilde{y}, \tilde{s}), \quad \text{as } s \to -\infty. \tag{4.35}$$

By (4.31), (4.24), (4.26), (4.27) and (4.30), we have that there exist positive constants c_1 and c_2, depending only on p and ε_1, such that

$$\begin{aligned}\max\{|D_\theta^2 v_\phi(y,s)|, |D_\theta^2 v_\psi(y,s)|\} &\le c_1(v_\phi+v_\psi)^p(y,s)(v_\phi+v_\psi)^{1/q}(y,s)\\ &\le c_2\partial_s v_\phi(y,s)d_0(y,s)^{-1}.\end{aligned} \tag{4.36}$$

Hence, it follows from (4.35) and (4.36) that

$$J_\phi = \eta\partial_s v_\phi(1+O(1/|s|)), \quad J_\psi = \eta\partial_s v_\psi(1+O(1/|s|)), \quad \text{as } s\to-\infty \tag{4.37}$$

in $\boldsymbol{K}_-(\tilde{y},\tilde{s})$. Since $\partial_s v_\phi, \partial_s v_\phi > 0 \quad$ in Ω_0, we have that $J_\phi, J_\psi > 0$ in $\boldsymbol{K}_-(\tilde{y},\tilde{s}) \cap \{(y,s) \mid s < -\sigma\}$ if σ is large enough.

We assume that (4.33) does not hold. Then, there exists $(y',s') \in \boldsymbol{K}_-(\tilde{y},\tilde{s})$ such that

$$J_\phi(y',s') = 0 \quad \text{or} \quad J_\psi(y',s') = 0$$

and

$$J_\phi(y,s) > 0 \quad \text{and} \quad J_\psi(y,s) > 0 \quad \text{for } (y,s) \in \boldsymbol{K}_-(\tilde{y},\tilde{s}) \cap \{(y,s) \mid y\in\mathbb{R},\ s<s'\}.$$

We assume $J_\phi(y',s') = 0$. Then, it follows from (4.34) that

$$\begin{aligned}0 &= J_\phi(y',s')\\ &= J_\phi(y'+M, s'-M)\\ &\quad + \int_0^M 2^{-p}p(p-1)(v_\phi+v_\psi)^{p-2}(D_\theta v_\phi + D_\theta v_\psi)^2(y'+M-s,s)ds\\ &\quad + \int_0^M 2^{-p}p(v_\phi+v_\psi)^{p-1}(J_\phi+J_\psi)(y'+M-s,s)ds\\ &> 0 \quad \text{for} \quad M>0.\end{aligned}$$

This is a contradiction. In the same manner as above, we can show that a contradiction arises if we assume that $J_\psi(y',s') = 0$. Therefore, we obtain that (4.33) holds.

By taking $\eta\to 0$, we have

$$D_\theta^2 v_\phi \ge 0 \quad \text{and} \quad D_\theta^2 v_\psi \ge 0 \quad \text{in } \boldsymbol{K}_-(\tilde{y},\tilde{s}).$$

This completes the proof. □

5 Linearity of the blow-up curve of blow-up limits

In this section, we will prove **(Step 1.)** as stated in Section 4.2. In order to prove this, we will consider

$$\begin{cases} D_- V_\phi = 2^{-p}(V_\phi+V_\psi)^p,\\ D_+ V_\psi = 2^{-p}(V_\phi+V_\psi)^p,\end{cases} \tag{5.1}$$

with some constant $\alpha\in\mathbb{R}$ and the corresponding blow-up curve

$$\{(y,s) \mid s = \alpha y, \quad y\in\mathbb{R}\}. \tag{5.2}$$

We know that (5.1)–(5.2) yield the following special solution:

$$(V_{\phi,\alpha}(y,s), V_{\psi,\alpha}(y,s)) = (C_{\phi,\alpha}(\alpha y - s)^{-q}, C_{\psi,\alpha}(\alpha y - s)^{-q}), \tag{5.3}$$

where

$$C_{\phi,\alpha} = (q(1+\alpha)(1-\alpha)^p)^q, \quad C_{\psi,\alpha} = (q(1+\alpha)^p(1-\alpha))^q.$$

In this section, we will prove the following lemma.

Lemma 5.1. *Assume that (A1)–(A5.3) hold. Then, there exists a positive constant $\alpha \in \mathbb{R}$ such that*

$$T_0(y) = \alpha y \quad \textit{for } y \in \mathbb{R}. \tag{5.4}$$

Moreover, the constant α satisfies $-1 < \alpha < 1$ and

$$v_\phi = V_{\phi,\alpha} \quad \textit{and} \quad v_\psi = V_{\psi,\alpha}. \tag{5.5}$$

In order to prove Lemma 5.1, we will first introduce some lemmas.

Lemma 5.2. *Assume that (A1)–(A5.3) hold. Then, T_0 is concave.*

Proof. Let $\varepsilon > 0$ be arbitrary. Then, by (4.26) we see that there exists a positive constant c_1, depending only on p and ε_1, such that

$$c_1\varepsilon^{-q} \le v_\phi(y,s) \quad \text{for} \quad y \in \mathbb{R} \quad \text{and} \quad s \in [T_0(y) - \varepsilon, T_0(y)).$$

Thus, there exist $M \ge c_1\varepsilon^{-q}$ and $E_0(y)$ such that

$$v_\phi(y, E_0(y)) = M \quad \text{and} \quad T_0(y) - E_0(y) \le \varepsilon \quad \text{for} \quad y \in \mathbb{R}.$$

We set $H_M = \{(y,s) \mid \ s \le E_0(y), \quad y \in \mathbb{R}\}$.

We will show that E_0 is concave. It suffices to show that H_M is convex. We assume that H_M is not convex. Then, there exist $(y_1, s_1), (y_2, s_2) \in H_M$ and $\xi' \in (0,1)$ such that $\xi'(y_1,s_1) + (1-\xi')(y_2,s_2) \notin H_M$ and $\xi'(y_1,s_1) + (1-\xi')(y_2,s_2) \in \Omega_0$. We notice that $\partial_s v_\phi > 0$ in Ω_0. Then, we have

$$\begin{aligned} M = \xi' M + (1-\xi')M &\ge \xi' v_\phi(y_1,s_1) + (1-\xi')v_\phi(y_2,s_2) \\ &\ge v_\phi(\xi'(y_1,s_1) + (1-\xi')(y_2,s_2)) \\ &> M. \end{aligned}$$

This is a contradiction. Hence, H_M is convex. Therefore, E_0 is concave. Thus, we have

$$\begin{aligned} &\xi T_0(y) + (1-\xi)T_0(y') \\ &= \xi(T_0(y) - E_0(y)) + (\xi E_0(y) + (1-\xi)E_0(y')) + (1-\xi)(T_0(y') - E_0(y')) \\ &\le \xi(T_0(y) - E_0(y)) + E_0(\xi y + (1-\xi)y') + (1-\xi)(T_0(y') - E_0(y')) \\ &\le \varepsilon + E_0(\xi y + (1-\xi)y') < \varepsilon + T_0(\xi y + (1-\xi)y'), \end{aligned}$$

for $y, y' \in \mathbb{R}$ and $\xi \in (0,1)$. Since we let $\varepsilon > 0$ take an arbitrary value, this completes the proof. □

We set

$$v_{\phi,\lambda}(y,s) = \lambda^q v_\phi(\lambda y, \lambda s), \qquad v_{\psi,\lambda}(y,s) = \lambda^q v_\psi(\lambda y, \lambda s),$$

with $\lambda \to \infty$. Then, we can easily see that the blow-up curve of $v_{\phi,\lambda}$ and $v_{\psi,\lambda}$ is

$$T_{0,\lambda}(y) = \frac{T_0(\lambda y)}{\lambda}.$$

Lemma 5.3. *Assume that (A1)–(A5.3) hold. Then, we have*

$$T_{0,\lambda_n}(y) \to \begin{cases} \alpha y & (y \ge 0) \\ \beta y & (y < 0) \end{cases} \qquad \textit{as} \quad \lambda_n \to \infty$$

where α and β are constants satisfying $-1 < \alpha \le \beta < 1$.

Proof. First, we see that $T_{0,\lambda_n}(0) = 0$.

Next, since T_0 is concave, we see that $\dfrac{T_{0,\lambda_n}(y)}{y} = \dfrac{T_0(\lambda_n y) - T_0(0)}{\lambda_n y}$ is monotone decreasing on n, for $y > 0$. Here, $\{\lambda_n\}$ is a monotone increasing sequence satisfying $\lambda_n \to \infty$. Thus, we have that

$$\lim_{\lambda_n \to \infty} \frac{T_{0,\lambda_n}(y)}{y} = \inf_{\lambda_n} \frac{T_{0,\lambda_n}(y)}{y} = \inf_{\lambda_n} \frac{T_0(\lambda_n y)}{\lambda_n y} \quad \text{for } y > 0.$$

Let $\alpha = \inf_{\lambda_n} \dfrac{T_0(\lambda_n y)}{\lambda_n y}$. Then, we have that

$$T_{0,\lambda_n}(y) \to \alpha y \quad \text{as } \lambda_n \to \infty,$$

for all $y > 0$ and monotone increasing sequences $\{\lambda_n\}$ satisfying $\lambda_n \to \infty$. By (4.29), we have $-1 < \alpha < 1$. We notice that α does not depend on y and λ_n.

Finally, we can prove

$$\lim_{\lambda_n \to \infty} \frac{T_{0,\lambda_n}(y)}{y} = \sup_{\lambda_n} \frac{T_{0,\lambda_n}(y)}{y} = \sup_{\lambda_n} \frac{T_0(\lambda_n y)}{\lambda_n y} \quad \text{for } y < 0,$$

in the same way of above. We set $\beta = \sup_{\lambda_n} \dfrac{T_0(\lambda_n y)}{\lambda_n y}$. We notice that $-1 < \alpha \leq \beta < 1$. Then, it follows that

$$T_{0,\lambda_n}(y) \to \beta y \quad \text{as } \lambda_n \to \infty,$$

for all $y < 0$ and monotone increasing sequences $\{\lambda_n\}$ satisfying $\lambda_n \to \infty$. This completes the proof. $\square$

Now, we set

$$\tilde{T}_0(y) = \begin{cases} \alpha y & (y \geq 0) \\ \beta y & (y < 0) \end{cases} \quad , \quad \tilde{\Omega}_0 = \left\{ (y, s) \in \mathbb{R}^2 \mid s < \tilde{T}_0(y), \quad y \in \mathbb{R} \right\}.$$

Remark 5.4. In the same way of proof of Lemma 5.2, we obtain that $\tilde{T}_0$ is concave. That is, α and β have the same sign.

Lemma 5.5. *Assume that (A1)–(A5.3) hold. Then, we have that $\alpha = \beta$. Here, α and β are constants as defined in Lemma 5.3.*

Proof. There exists a sequence $\{\lambda_n\}$ such that

$$v_{\phi,\lambda_n} \to w_\phi, \quad v_{\psi,\lambda_n} \to w_\psi, \quad \text{as } \lambda_n \to \infty, \quad \text{locally uniformly in } \tilde{\Omega}_0.$$

In the same arguments for Lemma 4.3, we see that $D_\theta^2 w_\phi \geq 0$ and $D_\theta^2 w_\psi \geq 0$ in $\tilde{\Omega}_0$, for $0 \leq \theta < 2\pi$. Thus, $D_\theta w_\phi$ and $D_\theta w_\psi$ are monotone increasing along the direction θ. We also have that it follows from the estimates $|D_\theta w_\phi|$ and $|D_\theta w_\psi|$, corresponding (4.31) that $|D_\theta w_\phi(y, s)|, |D_\theta w_\psi(y, s)| \to 0$ as $\tilde{d}_0(y, s) \to \infty$, where $\tilde{d}_0(y, s)$ is the distance from a point $(y, s) \in \tilde{\Omega}_0$ to $\tilde{\Gamma}_0 = \{(y, \tilde{T}_0(y)) \mid y \in \mathbb{R}\}$. Therefore, $D_\theta w_\phi$ and $D_\theta w_\psi$ do not occur sign changes in $\tilde{\Omega}_0$.

By Remark 5.4, we see that α and β have the same sign.

We assume that $0 < \alpha < \beta$. We set θ_α and θ_β as $\theta_\alpha = \arctan \alpha$ and $\theta_\beta = \arctan \beta$, respectively. Let us assume that $0 \leq \theta_\alpha < \theta_\beta < \pi/2$ without loss of generality.

If we take $\theta \in S$ where $S = \{\theta \in [0, 3\pi/2) \mid \theta_\alpha < \theta < \theta_\beta + \pi\}$, then $D_\theta w_\phi > 0$, since the closer w_ϕ gets to the blow-up curve $s = \beta y$ $(y < 0)$ or $s = \alpha y$ $(y \geq 0)$, the bigger w_ϕ becomes.

We take $\tilde{\theta}$ as $\theta_\alpha < \tilde{\theta} < \theta_\beta$. Then, we have that $D_{\tilde{\theta}} w_\phi > 0$, since $\tilde{\theta} \in S$. On the other hand, $D_{\tilde{\theta}+\pi} w_\phi > 0$, since $\tilde{\theta} + \pi \in S$. This contradicts the fact that

$$D_{\tilde{\theta}} w_\phi = -D_{\tilde{\theta}+\pi} w_\phi \quad \tilde{\Omega}_0.$$

In the same manner, we can prove that a contradiction arises if we assume that $\alpha < \beta < 0$. Therefore, we have that $\alpha = \beta$. This completes the proof. □

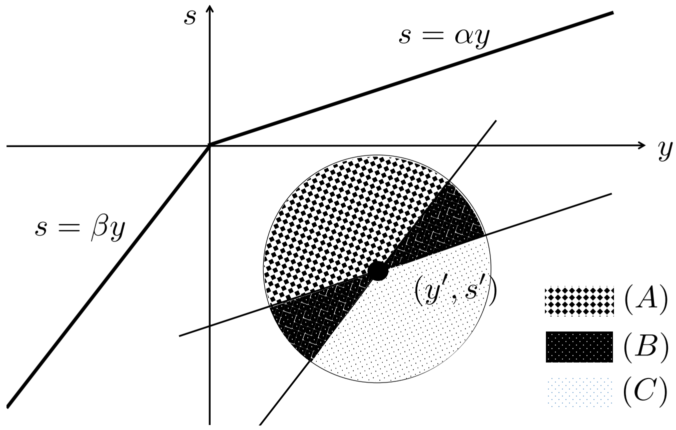


Figure A. The sign of the directional derivative at (y', s').

- (A) and (B) areas: The sign of the directional derivative is positive.
- (C) area : The sign of the directional derivative is negative.
  → If (B) area exists, we can show that a contradiction arises.

Proof of Lemma 5.1. First, we will show that $T_0(y) = \alpha y$. It follows from Lemma 5.5 that

$$\sup_{\lambda_n} \frac{T_0(\lambda_n y)}{\lambda_n y} = \inf_{\lambda_n} \frac{T_0(\lambda_n y)}{\lambda_n y} = \alpha \qquad \text{for } y \in \mathbb{R}.$$

Thus, $T_0(\lambda_n y) = \alpha \lambda_n y$ for $\lambda_n > 0$ and $y \in \mathbb{R}$. Therefore, we obtain that $T_0(y) = \alpha y$ for $y \in \mathbb{R}$.

Next, we will show that $v_\phi = V_{\phi,\alpha}$ and $v_\psi = V_{\psi,\alpha}$. By applying the proof of Lemma 5.5, we obtain that

$$(\alpha \partial_s + \partial_y) v_\phi = 0, \qquad \text{and} \qquad (\alpha \partial_s + \partial_y) v_\psi = 0 \tag{5.6}$$

in Ω_0. By substituting (5.6) for (4.23), we obtain the following system of equations:

$$\begin{cases} (1+\alpha)\partial_s v_\phi = 2^{-p}(v_\phi + v_\psi) \\ (1-\alpha)\partial_s v_\phi = 2^{-p}(v_\phi + v_\psi), \end{cases}$$

with the blow-up curve $T_0(y) = \alpha y$. Therefore, we obtain that $v_\phi = V_{\phi,\alpha}$ and $v_\psi = V_{\psi,\alpha}$ in Ω_0. This completes the proof. □

6 Continuous differentiability of the blow-up curve

In this section, we complete the proof of Theorem 1.1.

First, we will show that T is differentiable in B_{R^*}. We start by assuming that there exists $x_0 \in B_{R^*}$ such that T is not differentiable at $x_0 \in B_{R^*}$. Then, it follows from the arguments of **(Step 2.)** of Section 4.2 that there exist sequences $\{\lambda_n^{(1)}\}, \{\lambda_n^{(2)}\}$ such that there exist constants α_1 and α_2 satisfying

$$\begin{aligned}&\alpha_1, \alpha_2 \in (-1,1), \quad \alpha_1 \neq \alpha_2,\\ &\phi_{\lambda_n^{(j)}} \to V_{\phi,\alpha_j} \quad \text{as} \quad \lambda_n^{(j)} \to 0, \quad \text{locally uniformly in } \Omega_{j,0},\end{aligned}$$

where

$$\Omega_{j,0} = \left\{(y,s) \in \mathbb{R}^2 \mid s < \alpha_j y, \quad y \in \mathbb{R}\right\}$$

for $j = 1, 2$.

Let θ_{α_1} and θ_{α_2} be defined such that $\theta_{\alpha_1} = \arctan \alpha_1$ and $\theta_{\alpha_2} = \arctan \alpha_2$. Let us suppose that

$$0 \leq \theta_{\alpha_j} < \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4} < \theta_{\alpha_j} < \pi \quad (j = 1, 2)$$
$$\text{and}$$
$$\theta_{\alpha_1} < \theta_{\alpha_2}$$

without loss of generality.

We assume that $0 \leq \theta_{\alpha_1} < \theta_{\alpha_2} < \pi/4$. We take $0 < \varepsilon < \pi/2$ as

$$0 < \theta_{\alpha_1} + \varepsilon < \theta_{\alpha_2} - \varepsilon < \frac{\pi}{4}.$$

Then, for $j = 1, 2$, we have that there exist θ_j such that

$$0 < \theta_{\alpha_j} + \varepsilon < \theta_j < \theta_{\alpha_j} + \pi - \varepsilon < \frac{5\pi}{4}.$$

We define

$$S_\varepsilon^{(j)} = \left\{\theta_j \mid \theta_{\alpha_j} + \varepsilon < \theta_j < \theta_{\alpha_j} + \pi - \varepsilon\right\} \quad \text{for} \quad j = 1, 2.$$

We see that there exists $\varepsilon' > 0$ such that

$$D_{\theta'} V_{\phi,\alpha_j} > 2\varepsilon' \quad \text{in} \quad \Omega_{j,0} \cap B_1(0,0),$$

where $B_\rho(y', s') = \left\{(y,s) \in \mathbb{R}^2 \mid \sqrt{(y-y')^2 + (s-s')^2} < \rho\right\}$. Here, ρ is a positive constant.

For $j = 1, 2$, let $(y_j^\pm, s_j^\pm)$ and $(y_j^{\delta_0,\pm}, s_j^{\delta_0,\pm})$ be the intersections of $y^2 + s^2 = 1$ and

$$s = \alpha_{x_j} y \quad \text{and} \quad s = \alpha_{x_j} y - \delta_0,$$

respectively. Here, δ_0 is a positive constant.

We see that there exist $n_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that for $j = 1, 2$,

$$\begin{aligned}&\Omega_{1,0}^{\delta_0} \cap \Omega_{2,0}^{\delta_0} \cap B_1(0,0),\\ &\Omega_{\lambda_{n_0}^{(j)}} \subset \Omega_{j,0}^{-\delta_0}, \quad \Omega_{j,0}^{\delta_0} \subset \Omega_{\lambda_{n_0}^{(j)}}\\ &s_j^{\delta_0,-} < s_j^-,\\ &\text{For } \theta_j \in S_{\varepsilon'}^{(j)}, \quad |D_{\theta_j} \phi_{\lambda_{n_0}^{(j)}} - D_{\theta_j} V_{\phi,\alpha_j}| \leq \varepsilon' \quad \text{in} \quad \Omega_{j,0}^{\delta_0} \cap B_1(0,0).\end{aligned}$$

Here, $\Omega_{j,0}^{\delta_0} = \left\{(y,s) \mid s < \alpha_{x_j} y - \delta_0, \quad y \in \mathbb{R}\right\}$. This means that

$$D_{\theta_j} \phi_{\lambda_{n_0}^{(j)}} > \varepsilon' \quad \text{in} \quad \Omega_{j,0}^{\delta_0} \cap B_1(0,0)$$

for $\theta_j \in S_{\varepsilon'}^{(j)}$ and $j = 1,2$. By (4.13), we can prove

$$D_{\theta_j} \phi_{\lambda_{n_0}^{(j)}} > \varepsilon' \quad \text{in} \quad K_j^{\delta_0} \tag{6.1}$$

where

$$K_j^{\delta_0} = \left\{(y,s) \in \Omega_{\lambda_{n_0}^{(j)}} \cap B_1(0,0) \mid y < \min\{|y_j^{\delta_0,-}|, |y_j^{\delta_0,+}|\}\right\}$$

for $\theta_j \in S_{\varepsilon'}^{(j)}$ and $j = 1,2$. (6.1) means that there exists there exists a positive constant ρ such that

$$0 < \rho \leq 1 \quad \text{and} \quad D_{\theta_j} \phi_{\lambda_{n_0}^{(j)}} > \varepsilon' \quad \text{in} \quad \Omega_{\lambda_{n_0}^{(j)}} \cap B_\rho(0,0) \tag{6.2}$$

for $\theta_j \in S_{\varepsilon'}^{(j)}$ and $j = 1,2$.

Let $\lambda_{n_1} = \min\{\lambda_{n_0}^{(1)}, \lambda_{n_0}^{(2)}\}$. It follows from (6.2) that

$$D_\theta \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_1}\rho}(x_0, T(x_0)).$$

for $\theta \in S_{\varepsilon'}^{(1)} \cup S_{\varepsilon'}^{(2)}$.

In particular,

$$D_{\theta^*} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_1}\rho}(x_0, T(x_0)) \tag{6.3}$$

for $\theta^* \in (\theta_{\alpha_1} + \varepsilon, \theta_{\alpha_2} - \varepsilon)$, since $(\theta_{\alpha_1} + \varepsilon, \theta_{\alpha_2} - \varepsilon) \subset S_\varepsilon^{(1)}$. Moreover, we have

$$D_{\theta^*+\pi} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_1}\rho}(x_0, T(x_0)) \tag{6.4}$$

since $\theta^* + \pi \in (\theta_{\alpha_1} + \pi + \varepsilon, \theta_{\alpha_2} + \pi - \varepsilon) \subset S_\varepsilon^{(2)}$. Then, (6.3) and (6.4) contradict the fact

$$D_{\theta^*} \phi = -D_{\theta^*+\pi} \phi \quad \text{in } \Omega.$$

We can show contradictions in the other cases, that is, in the cases

$$\begin{aligned} &0 \leq \theta_{\alpha_1} < \pi/4, \ 3\pi/4 < \theta_{\alpha_2} < \pi, \\ &3\pi/4 < \theta_{\alpha_1} < \theta_{\alpha_2} < \pi. \end{aligned}$$

Therefore, T is differentiable in B_{R^*}.

Next, we will show that the derivative T' is continuous in B_{R^*}. We start by assuming that there exists $x_0 \in B_{R^*}$ such that T' is discontinuous at $x_0 \in B_{R^*}$. Set $\alpha_{x_0} = T'(x_0)$. Let us suppose that $0 \leq \theta_{\alpha_{x_0}} < \pi/4$ or $3\pi/4 \leq \theta_{\alpha_{x_0}} < 5\pi/4$ without loss of generality.

Since T' is discontinuous at $x_0 \in B_{R^*}$, there exists $0 < \varepsilon' < \pi/2$ such that there exists $\{x_j\} \subset B_{R*}$ satisfying

$$|x_j - x_0| \to 0 \quad \text{as} \quad j \to \infty \quad \text{and} \quad |\theta_{\alpha_{x_j}} - \theta_{\alpha_{x_0}}| > 2\varepsilon' \quad \text{for all } j \in \mathbb{N}. \tag{6.5}$$

By the above argument, there exists $n_0 \in \mathbb{N}$ and $\rho \in \mathbb{R}$ such that

$$D_{\theta_0} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_0}\rho}(x_0, T(x_0))$$

for $\theta_0 \in S_{\varepsilon', x_0} = \{\theta_0 \mid \theta_{\alpha_{x_0}} + \varepsilon' < \theta_0 < \theta_{\alpha_{x_0}} + \pi - \varepsilon'\}$.

Moreover, by the continuity of T and (6.5), there exists $j_0 \in \mathbb{N}$ such that

$$(x_{j_0}, T(x_{j_0})) \in B_{\lambda_{n_0}\rho}(x_0, T(x_0)).$$

We see that there exists $n_{j_0} \in \mathbb{N}$ such that

$$D_{\theta_{j_0}} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_{j_0}} \rho}(x_{j_0}, T(x_{j_0}))$$

for $\theta_{j_0} \in S_{\varepsilon', x_{j_0}} = \{\theta_{j_0} \mid \theta_{\alpha_{x_{j_0}}} + \varepsilon' < \theta_{j_0} < \theta_{\alpha_{x_{j_0}}} + \pi - \varepsilon'\}$.

Then, we have

$$D_\theta \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_0} \rho}(x_0, T(x_0)) \cap B_{\lambda_{n_{j_0}} \rho}(x_{j_0}, T(x_{j_0}))$$

for $\theta \in S_{\varepsilon', x_0} \cup S_{\varepsilon', x_{j_0}}$.

Assume $0 < \theta_{x_0} < \theta_{x_{j_0}} < \pi/4$. By (6.5),

$$\theta_{\alpha_{x_0}} + \varepsilon' < \theta_{\alpha_{x_{j_0}}} - \varepsilon'.$$

Take $\tilde{\theta}$ as $\theta_{\alpha_{x_0}} + \varepsilon' < \tilde{\theta} < \theta_{\alpha_{x_{j_0}}} - \varepsilon'$.

Then,

$$D_{\tilde{\theta}} \phi > 0 \quad \text{and} \quad D_{\tilde{\theta}+\pi} \phi > 0 \quad \text{in} \quad \Omega \cap B_{\lambda_{n_0} \rho}(x_0, T(x_0)) \cap B_{\lambda_{n_{j_0}} \rho}(x_{j_0}, T(x_{j_0})),$$

since $\tilde{\theta}, \tilde{\theta} + \pi \in S_{\varepsilon', x_0} \cup S_{\varepsilon', x_{j_0}}$. This contradicts the fact that

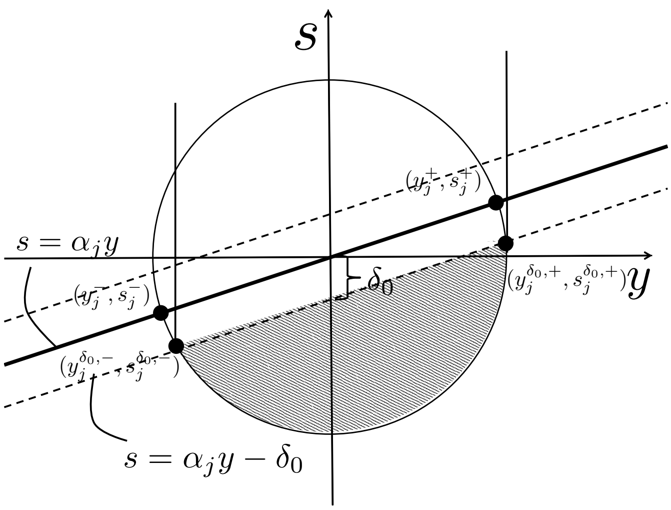
$$D_{\tilde{\theta}+\pi} \phi = -D_{\tilde{\theta}} \phi \quad \text{in} \quad \Omega.$$

In the the other cases, that is, in the cases,

$$\begin{aligned}
&0 \le \theta_{\alpha_{x_0}} < \pi/4,\ 3\pi/4 < \theta_{\alpha_{x_{j_0}}} < \pi, \\
&3\pi/4 < \theta_{\alpha_{x_0}} < \theta_{\alpha_{x_{j_0}}} < \pi, \\
&0 \le \theta_{\alpha_{x_{j_0}}} < \theta_{\alpha_{x_0}} < \pi/4, \\
&0 \le \theta_{\alpha_{x_{j_0}}} < \pi/4,\ 3\pi/4 < \theta_{\alpha_{x_0}} < \pi, \\
&3\pi/4 < \theta_{\alpha_{x_{j_0}}} < \theta_{\alpha_{x_0}} < \pi,
\end{aligned}$$

we can show that contradictions arise in the same way.

This completes the proof.



7 Numerical examples

In this section, we will show some numerical examples of the blow-up curves for (1.7). For simplicity of computation, we consider the equations in a bounded interval $(0,1)$ and pose the periodic boundary condition. We follow the method proposed by Cho [5] for computing the numerical blow-up curve.

For discretization, we employ the finite difference scheme for (1.7). Take a positive integer J and set $x_j = jh$ with $h = 1/J$. As a time variable, we take a positive constant τ as $\tau = h$ and set $t_n = \tau \cdot n$. Then, we consider the following scheme for (1.7):

$$\phi_j^n \approx \phi(x_j, t_n), \qquad \psi_j^n \approx \psi(x_j, t_n) \qquad (1 \le j \le J,\ n \ge 0),$$

$$\begin{cases} \dfrac{\phi_j^{n+1} - \phi_j^n}{\tau} - \dfrac{\phi_{j+1}^n - \phi_j^n}{h} = 2^{-p} \left| \phi_j^n + \psi_j^n \right|^p, \\ \dfrac{\psi_j^{n+1} - \psi_j^n}{\tau} + \dfrac{\psi_j^n - \psi_{j-1}^n}{h} = 2^{-p} \left| \phi_j^n + \psi_j^n \right|^p, \\ \phi_j^0 = f(x_j), \qquad \psi_j^0 = g(x_j), \end{cases}$$
$$(1 \le j \le J,\ n \ge 0),$$

where ϕ_{J+1} and ψ_0^n are set as $\phi_{J+1}^n = \phi_1^n$ and $\psi_0^n = \psi_J^n$.

We define the numerical blow-up curve T_j approximated to $T(x_j)$ by

$$T_j = \tau \cdot n_j(\tau).$$

Here, $n_j(\tau)$ is the smallest positive integer such that

$$\tau \cdot \left(\phi_j^{n_j(\tau)-1} + \psi_j^{n_j(\tau)-1} \right) \ge 1/\mathrm{eps} \quad \text{and} \quad \tau \cdot \left(\phi_j^{n_j(\tau)} + \psi_j^{n_j(\tau)} \right) < 1/\mathrm{eps},$$

where eps > 0 is a stopping criterion given below. We set $\boldsymbol{T} = (T_j)$.

We plot two numerical blow-up curves $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ with two stopping criterion eps1 and eps2, respectively, for several τ in Figure 1–3. We see that $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are almost equal under suitable eps1, eps2 and τ. Therefore, we can regard $\boldsymbol{T}$ is a reasonable approximation of the exact blow-up curve T for (1.7).

First, we examine the shape of blow-up curve T for $p = 2$ and $f(x) = (1 + \sqrt{2.3}) + \frac{1}{2\pi} \sin(2\pi x)$, $g(x) = (1 + \sqrt{2.3}) - \frac{1}{2\pi} \sin(2\pi x)$. In Figure 1, we see that the numerical blow-up curve $\boldsymbol{T}$ converges to a smooth function as $\tau \to 0$. Therefore, we numerically obtain that the blow-up curve T is continuously differentiable if initial values f and g are smooth and large enough. In Figure 2, we also obtain the same result for $p = 3$.

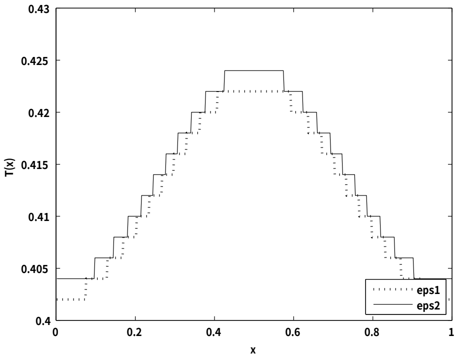
On the other hand, we obtain different results of regularity of the blow-up curve in Figure 3. We see that there is a case where the blow-up curve has the singular points. We notice that all the initial values are smooth in Figures 1–3. However, the initial values f and g occur the sign changes in Figure 3, while the initial values f and g are positive for $x \in (0,1)$ in the case of Figures 1 and 2.

Consequently, we see that we have to impose not only regularity but also largeness on the initial values.

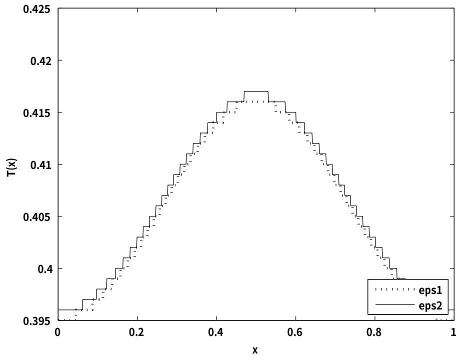
Remark 7.1. Merle and Zagg [9] considered

$$\partial_t^2 u - \partial_x^2 u = u^p.$$

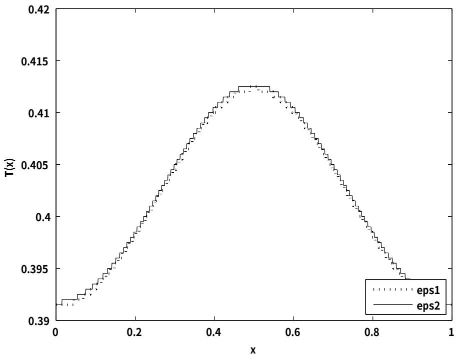
They analytically showed that there are cases where the blow-up curve T has the singular points. However, we do not know the relationship between the our numerical results and the results of [9]



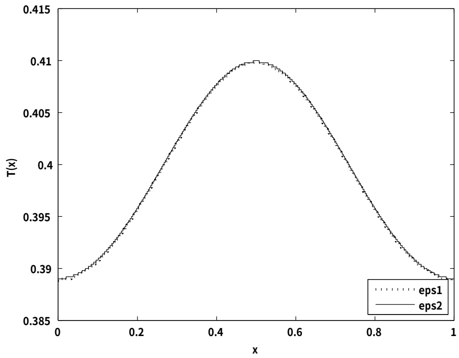
$\tau = h = 1/500.$



$\tau = h = 1/1000.$

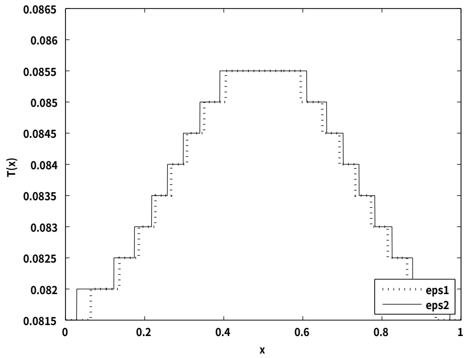


$\tau = h = 1/2000.$

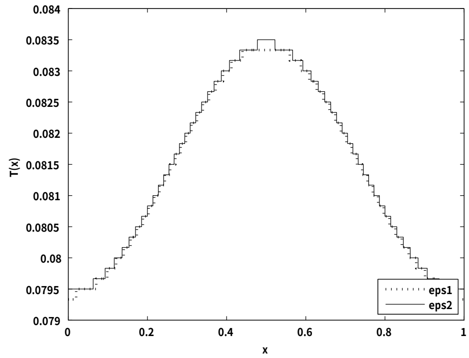


$\tau = h = 1/5000.$

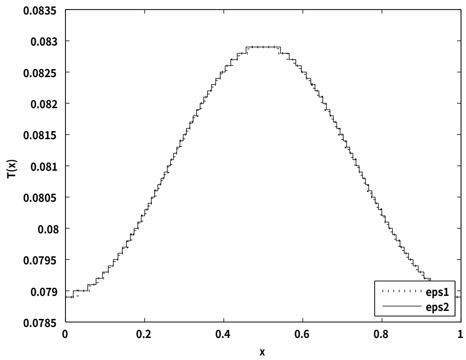
Figure 1: The history of (T_j) for $p = 2$, $f(x) = (1 + \sqrt{2.3}) + \frac{1}{2\pi}\sin(2\pi x)$ and $g(x) = (1 + \sqrt{2.3}) - \frac{1}{2\pi}\sin(2\pi x)$ and stopping criteria $\text{eps1} = 1e - 2$ and $\text{eps2} = 1e - 3$.



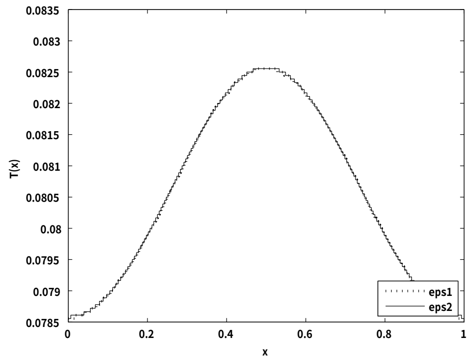
$\tau = h = 1/2000.$



$\tau = h = 1/6000.$

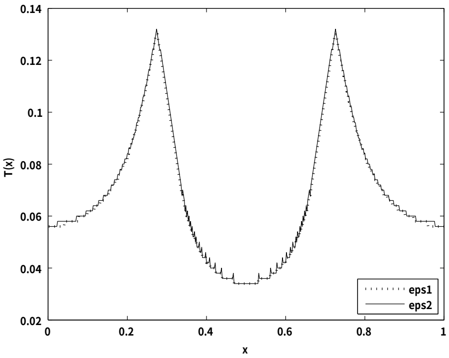


$\tau = h = 1/10000.$

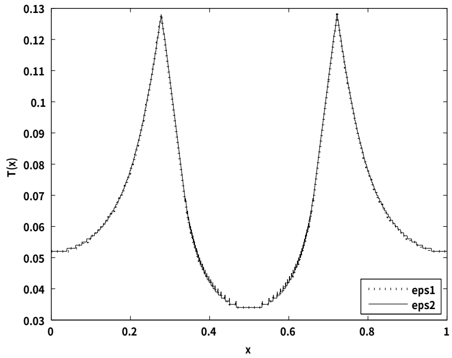


$\tau = h = 1/18000.$

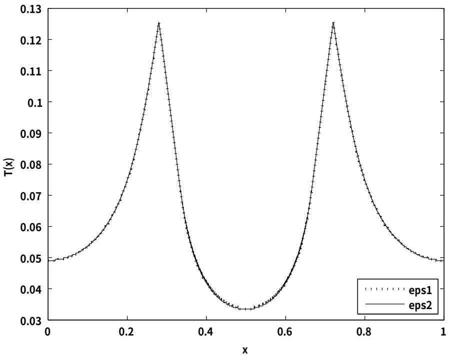
Figure 2: The history of (T_j) for $p = 3$, $f(x) = 2.5 + \frac{1}{2\pi}\sin(2\pi x)$, $g(x) = 2.5 - \frac{1}{2\pi}\sin(2\pi x)$ and stopping criteria eps1 $= 1e-2$ and eps2 $= 1e-3$.



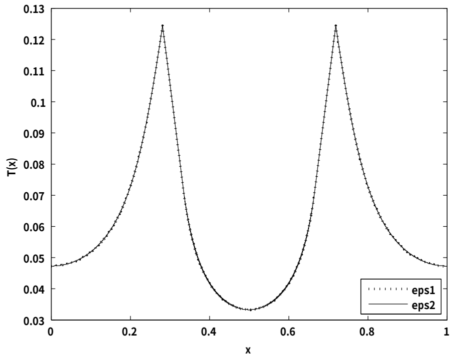
$\tau = h = 1/500.$



$\tau = h = 1/1000.$



$\tau = h = 1/2000.$



$\tau = h = 1/5000.$

Figure 3: The history of (T_j) for $p = 3$, $f(x) = 2 + 10\sin(2\pi x)$, $g(x) = 2 - 10\sin(2\pi x)$ and stopping criteria eps1 $= 1e-2$, and eps2 $= 1e-3$.

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