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Abstract. We study the behavior of a variational solution to the Neumann type problem of the heat equation on a moving thin domain $\Omega_\varepsilon(t)$ that converges to an evolving surface $\Gamma(t)$ as the width of $\Omega_\varepsilon(t)$ goes to zero. We show that, under suitable assumptions, the average in the normal direction of $\Gamma(t)$ of a variational solution to the heat equation converges weakly in a function space on $\Gamma(t)$ as the width of $\Omega_\varepsilon(t)$ goes to zero, and that the limit is a unique variational solution to a limit equation on $\Gamma(t)$, which is a new type of linear diffusion equation involving the mean curvature and the normal velocity of $\Gamma(t)$. We also estimate the difference between variational solutions to the heat equation on $\Omega_\varepsilon(t)$ and the limit equation on $\Gamma(t)$.

1. Introduction

For $t \in [0,T]$, $T > 0$, let $\Omega_\varepsilon(t)$ be a moving thin domain in $\mathbb{R}^n$, $n \geq 2$, with width of order $\varepsilon > 0$ that converges to an evolving closed hypersurface $\Gamma(t)$ as $\varepsilon \to 0$. We consider the Neumann type problem of the heat equation of the form

$$(H_\varepsilon) \qquad \begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = 0 & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon} u^\varepsilon + V_\varepsilon^N u^\varepsilon = 0 & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0). \end{cases}$$

Here $Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \Omega_\varepsilon(t) \times \{t\}$, $\partial_\ell Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \partial\Omega_\varepsilon(t) \times \{t\}$, and ν_ε, V_ε^N are the unit outward normal vector field of $\partial\Omega_\varepsilon(t)$ and the outer normal velocity of $\partial\Omega_\varepsilon(t)$, respectively. The term $V_\varepsilon^N u^\varepsilon$ in the boundary condition is added so that the total amount of heat $\int_{\Omega_\varepsilon(t)} u^\varepsilon \, dx$ is conserved, see the beginning of Section 3. Also, if u^ε denotes the concentration of some chemicals, the boundary condition says that chemicals do not move and flux is just caused by the motion of the boundary.

We are interested in the behavior of a solution u^ε to (H_ε) as $\varepsilon \to 0$. Our goal is to characterize its limit as well as its convergence. Let us explain the simplest case when $\Omega_\varepsilon(t)$ is the set of all points in $\mathbb{R}^n$ with distance less than ε from $\Gamma(t)$ so that the width of $\Omega_\varepsilon(t)$ is 2ε. Let ν be the unit outward normal vector field of $\Gamma(t)$ and $V_\Gamma = V_\Gamma^N \nu + V_\Gamma^T$ be the velocity of $\Gamma(t)$, where V_Γ^N and V_Γ^T are the outer normal velocity of $\Gamma(t)$ and a given tangential velocity. Then our main result formally implies that, under suitable assumptions on the initial data u_0^ε of (H_ε), the limit v is a solution to

$$\partial_t v + V_\Gamma^N \nu \cdot \nabla v - V_\Gamma^N H v - \Delta_{\Gamma(t)} v = 0 \quad \text{on} \quad S_T. \tag{1.1}$$

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Here $S_T := \bigcup_{t\in(0,T)} \Gamma(t)\times\{t\}$ and ∇ is the usual gradient in $\mathbb{R}^n$. Also, $H := -\mathrm{div}_{\Gamma(t)}\nu$ and $\Delta_{\Gamma(t)} := \mathrm{div}_{\Gamma(t)}\nabla_{\Gamma(t)}$ are the mean curvature of $\Gamma(t)$ and the Laplace-Beltrami operator on $\Gamma(t)$, where $\mathrm{div}_{\Gamma(t)}$ and $\nabla_{\Gamma(t)}$ are the surface divergence operator and the tangential gradient on $\Gamma(t)$, respectively (see Section 2 for their definitions). We will give a heuristic derivation of the limit equation (1.1) in the appendix. The equation (1.1) is equivalent to

$$\partial^\bullet v + (\mathrm{div}_{\Gamma(t)} V_\Gamma)v - \Delta_{\Gamma(t)} v - \mathrm{div}_{\Gamma(t)}(vV_\Gamma^T) = 0 \quad \text{on} \quad S_T, \tag{1.2}$$

which we will actually derive in Section 6. Here $\partial^\bullet v = \partial_t v + V_\Gamma^N \nu\cdot\nabla v + V_\Gamma^T\cdot\nabla_{\Gamma(t)} v$ denotes the material derivative of v. Note that the equation (1.1) is independent of the tangential velocity V_Γ^T. In other words, the evolution of the limit v is not affected by advection along $\Gamma(t)$. Such a phenomenon does not occur in an advection-diffusion equation widely studied in recent years [2–8, 18, 27]:

$$\partial^\bullet v + (\mathrm{div}_{\Gamma(t)} V_\Gamma)v - \Delta_{\Gamma(t)} v = 0 \quad \text{on} \quad S_T. \tag{1.3}$$

This equation is derived from the conservation law such that, for an arbitrary portion $\mathcal{M}(t)$ of $\Gamma(t)$,

$$\frac{d}{dt}\int_{\mathcal{M}(t)} v\, d\mathcal{H}^{n-1} = -\int_{\partial\mathcal{M}(t)} q\cdot\mu\, d\mathcal{H}^{n-2}$$

holds, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure for $k\in\mathbb{N}$, μ is the co-normal to the boundary $\partial\mathcal{M}(t)$, and q is the surface flux, see [3, Section 3] and [4, Section 3.1] for details.

Partial differential equations on thin domains are studied over the years [11–15, 19–23, 25, 26], and many researchers deal with a nonmoving thin domain of the form

$$\Omega_\varepsilon = \{(x', x_n)\in\mathbb{R}^{n-1}\times\mathbb{R} \mid x'\in\omega,\ \varepsilon g_0(x') < x_n < \varepsilon g_1(x')\}, \quad \varepsilon > 0, \tag{1.4}$$

where ω is a domain in $\mathbb{R}^{n-1}$ and g_0, g_1 are functions on ω. In their pioneering works [11, 12], Hale and Raugel compared the dynamics of reaction-diffusion equations and damped wave equations on Ω_ε of the form (1.4) (with $g_0 = 0$ and slightly modified g_1) and that of corresponding limit equations on ω by the scaling argument. They transformed the equations on Ω_ε into scaled equations on a fixed reference domain $\Omega_0 = \omega\times(0,1)$ by the change of variables, and formally derived the limit equations on ω by letting $\varepsilon\to 0$ in the scaled equations on Ω_0 and omitting divergent terms. Then they compared the dynamics of the scaled equations on Ω_0 and that of the limit equations on ω by analyzing weighted bilinear forms that appear in variational formulations of the scaled equations and the limit equations. Their scaling argument is applicable to more general thin domains such as a thin L-shaped domain [13] and a moving thin domain of the form (1.4) where $g_0 = 0$ and g_1 depends on time [22]. Prizzi and Rybakowski [20] generalized the scaling argument in [11, 12] to study reaction-diffusion equations on a (nonmoving) thin domain with holes around a lower dimensional domain. The generalized scaling argument in [20] is also valid for a (nonmoving) thin domain with holes around a lower dimensional manifold [19, 21]. We refer to [23] and references therein for other examples of thin domains.

In contrast to the above papers, the limit hypersurface $\Gamma(t)$ of our thin domain $\Omega_\varepsilon(t)$ evolves. Such a situation has been considered only in the paper [7], which deals with a diffuse interface model for the advection-diffusion equation (1.3). See also [8] for numerical computations of the advection-diffusion equation (1.3) based on the diffuse interface model. In [7], however, the limit equation (1.3) on the evolving surface is given and the equation on the moving thin domain involves a weight function that vanishes on the boundary of the domain. Therefore, there is no literature on initial-boundary value problems of partial differential equations on moving thin domains around evolving surfaces whose limit equations are unknown in advance, even in the case of the heat equation.

The difficulty caused by the evolution of the hypersurface $\Gamma(t)$ is in transforming equations on $\Omega_\varepsilon(t)$ and $\Gamma(t)$ into equations on fixed (in time and width) domain and hypersurface. In particular, transformations of differential operators on $\Gamma(t)$ into those on a fixed hypersurface is so complicated that we can hardly find a limit equation on the fixed hypersurface and convert it into an equation on $\Gamma(t)$, see [6] for the actual transformations of differential operators.

To avoid this difficulty, we employ another method that does not require transformations of $\Omega_\varepsilon(t)$ and $\Gamma(t)$. Let us explain our idea of derivation of a limit equation on $\Gamma(t)$. We start from a variational formulation of (H_ε) (see (3.2)) that consists of integrals over the noncylindrical domain $Q_{\varepsilon,T}$ of a variational solution u^ε to (H_ε) and a test function defined on $Q_{\varepsilon,T}$. In this variational formulation, we take a test function independent of the normal direction of $\Gamma(t)$ and apply the co-area formula (see (5.4)) and a weighted average operator M_ε (see Definition 5.1) to get a variational formulation (with some residual terms) of the average $M_\varepsilon u^\varepsilon$ (see (6.1)) that consists of integrals over the space-time manifold S_T of $M_\varepsilon u^\varepsilon$ and a test function defined on S_T. Then we obtain a variational formulation of a limit equation on $\Gamma(t)$ (see (6.13)) by omitting the residual terms in the variational formulation of $M_\varepsilon u^\varepsilon$. Moreover, we prove that $M_\varepsilon u^\varepsilon$ converges weakly in a function space on S_T as $\varepsilon \to 0$ and that the limit is a unique variational solution to the limit equation (see Theorem 6.10), and estimate the $L^2(Q_{\varepsilon,T})$-norm of the difference between variational solutions to (H_ε) and the limit equation (see Theorem 6.13). These weak convergence result and estimate indicate that our limit equation on $\Gamma(t)$ derived as above is indeed the "limit" of (H_ε),

In our derivation of a limit equation, Lemma 5.7 and Lemma 5.14 play an important role. In Lemma 5.7 we approximate an H^1-bilinear form on $\Omega_\varepsilon(t)$ for each $t \in [0,T]$ by that on $\Gamma(t)$ with the tangential gradient of the average $M_\varepsilon u$ of a function u on $\Omega_\varepsilon(t)$. The proof of Lemma 5.7 is based on simple representations of the gradient in $\mathbb{R}^n$ and the tangential gradient on $\Gamma(t)$ under a special local coordinate system for each fixed point on $\Gamma(t)$. On the other hand, Lemma 5.14 gives an integral formula that formally represents a relation between the weak time derivative of a function u on $Q_{\varepsilon,T}$ and the weak material derivative of its average $M_\varepsilon u$ (in fact, we do not explicitly deal with the time derivative of u). Lemma 5.14 essentially follows from Lemma 5.12, which gives a relation between the time derivative and the material derivative of functions defined on S_T.

Average operators in the thin direction were originally introduced by Hale and Raugel [11, 12], but they took the average of functions on the scaled domain $\Omega_0 = \omega \times (0,1)$. Average operators on actual thin domains Ω_ε appears in the study of the Navier-Stokes equations on three dimensional thin domains [14, 15, 25, 26]. Temam and Ziane [25, 26] first employed them to study the global existence of strong solutions to the Navier-Stokes equations for large initial data and external forces and the behavior of solutions as $\varepsilon \to 0$ when Ω_ε is a three dimensional thin product domain $\Omega_\varepsilon = \omega \times (0,\varepsilon)$ with a bounded domain ω in $\mathbb{R}^2$ and a thin spherical domain

$\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid a < |x| < (1+\varepsilon)a\}$ with a constant $a > 0$. In [14, 15], average operators were employed to study the dynamics of the Navier-Stokes equations on Ω_ε of the form (1.4). In particular, the authors of [15] compared the dynamics of the Navier-Stokes equations with that of limit equations by estimating the difference of the average of solutions to the Navier-Stokes equations and solutions to the limit equations.

We point out that our weighted average operator given in Definition 5.1 is a generalization of average operators given in [14, 15, 25] and that its weight function is different from that of an average operator given in [26]. In fact, the weight function of our average operator is a Jacobian that appears when we change variables of integrals over a tubular neighborhood of $\Gamma(t)$ in terms of the normal coordinate system around $\Gamma(t)$. Our choice of the weighted function enables us to transform easily a bilinear form on a function space on $Q_{\varepsilon,T}$ including the weak time derivative of a function u on $Q_{\varepsilon,T}$ into a bilinear form on a function space on S_T including the weak material derivative of the average $M_\varepsilon u$, see Lemma 5.14. We also note that, contrary to our case, Kublik, Tanushev, and Tsai [16] employed the same Jacobian and co-area formula to transform integrals over boundaries of domains into those over their tubular neighborhoods. Based on this transformation, they proposed a new approach to numerical computations of boundary integrals without explicit parametrizations of boundaries and a simple formulation for constructing boundary integral methods to solve Poisson's equations. Their method of the numerical computations of boundary integrals is also applicable to integrals over nonclosed manifolds of higher codimension, such as curves in $\mathbb{R}^3$ with different endpoints, see [17] for details.

Finally we mention variational formulations of partial differential equations on evolving surfaces. There are several kinds of variational frameworks for equations on evolving surfaces, mainly the advection-diffusion equation (1.3), see [3, 18, 27] for example. In addition, Alphonse, Elliott, and Stinner [1, 2] proposed an abstract variational setting with evolving Hilbert spaces and applied it to some equations on moving domains and evolving surfaces. Among these variational frameworks, we adopt the one introduced by Olshanskii, Reusken, and Xu [18]. Their variational formulation is imposed on function spaces on S_T, which is suitable for our calculation of bilinear forms on function spaces on S_T and $Q_{\varepsilon,T}$ performed in Section 5 and Section 6.

This paper is organized as follows. In Section 2 we introduce notations related to the evolving surface $\Gamma(t)$ and define the moving thin domain $\Omega_\varepsilon(t)$. In Section 3 we define a variational solution to (H_ε) and prove its existence and uniqueness. We also derive an energy estimate of a variational solution to (H_ε) with a constant independent of ε. In Section 4 we define and investigate function spaces on S_T introduced in [18]. In Section 5 we define the weighted average operator M_ε and establish estimates and formulas related to M_ε. In Section 6, we derive a limit equation on $\Gamma(t)$ of the form (1.2) via its variational formulation and prove our main theorems (Theorem 6.10 and Theorem 6.13). In the appendix, we give a heuristic derivation of the limit equation (1.1) when $\Omega_\varepsilon(t)$ is the set of all points in $\mathbb{R}^n$ with distance less than ε from $\Gamma(t)$.

2. Evolving surfaces and moving thin domains

For each $t \in [0,T]$, let $\Gamma(t)$ be a closed (that is, compact and without boundary), connected and oriented smooth hypersurface in $\mathbb{R}^n$. We set $\Gamma_0 := \Gamma(0)$ and define a space-time manifold $S_T \subset \mathbb{R}^{n+1}$ as $S_T := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. We assume that each point y on $\Gamma(t)$ evolves with velocity $V_\Gamma(y,t)$, which is not necessarily normal to $\Gamma(t)$, and the velocity field $V_\Gamma \colon \overline{S_T} \to \mathbb{R}^n$ is smooth. Let $\Phi(\cdot,t)\colon \Gamma_0 \to \Gamma(t)$ be a flow map of V_Γ, that is, $\Phi(\cdot,t)$ is a diffeomorphism from Γ_0 onto $\Gamma(t)$ for each $t \in [0,T]$ and satisfies

$$\Phi(Y,0) = Y, \quad \frac{\partial \Phi}{\partial t}(Y,t) = V_\Gamma(\Phi(Y,t),t) \quad \text{for all} \quad Y \in \Gamma_0,\, t \in [0,T].$$

We assume that Φ and its inverse Φ^{-1} are smooth on $\Gamma_0 \times [0,T]$ and $\overline{S_T}$, respectively. Due to this assumption, $\overline{S_T}$ is a compact smooth manifold in $\mathbb{R}^{n+1}$.

Let $\nu : \overline{S_T} \to \mathbb{R}^n$ be the unit outward normal vector field of $\Gamma(t)$. The velocity V_Γ is decomposed into $V_\Gamma = V_\Gamma^N \nu + V_\Gamma^T$, where $V_\Gamma^N : \overline{S_T} \to \mathbb{R}$ is the outer normal velocity and $V_\Gamma^T : \overline{S_T} \to \mathbb{R}^n$ is a tangential velocity field. Note that to describe the geometric motion of $\Gamma(t)$ it is sufficient to prescribe the normal velocity. However, to describe a limit equation on $\Gamma(t)$ we will derive in Section 6, we also need to consider a tangential velocity, which represents advection along $\Gamma(t)$.

For each $t \in [0,T]$, let $d(\cdot,t)$ be the signed distance function from $\Gamma(t)$ that increases the direction of the normal vector $\nu(\cdot,t)$. By the smoothness (in space and time) and compactness of $\Gamma(t)$, there is an open set $N(t)$ in $\mathbb{R}^n$ of the form $N(t) = \{x \in \mathbb{R}^n \mid -\delta < d(x,t) < \delta\}$ for each $t \in [0,T]$, where $\delta > 0$ is a constant independent of t, that satisfies the following conditions.

- The signed distance function d is smooth on $\overline{N_T}$, where $N_T \subset \mathbb{R}^{n+1}$ is a noncylindrical domain given by $N_T := \bigcup_{t\in(0,T)} N(t) \times \{t\}$.
- For each $(x,t) \in \overline{N_T}$, there is a unique point $p(x,t) \in \Gamma(t)$ such that

$$x = p(x,t) + d(x,t)\nu(p(x,t),t), \quad \nabla d(x,t) = \nu(p(x,t),t).$$

The set $N(t)$ is called a tubular neighborhood of $\Gamma(t)$. Based on the above equality, we extend the normal vector ν to $\overline{N_T}$ by setting $\nu(x,t) := \nabla d(x,t)$ for $(x,t) \in \overline{N_T}$. Then, by the smoothness of d, the extended normal vector ν and the projection mapping p are smooth on $\overline{N_T}$. Also, the normal velocity V_Γ^N of $\Gamma(t)$ is given by $V_\Gamma^N = -\partial_t d$ on $\overline{S_T}$.

Next, we give definitions of differential operators on evolving surfaces. For a function v and a vector field F on S_T, we define the tangential gradient of v and the surface divergence of F as

$$\begin{aligned}\nabla_{\Gamma(t)} v(y,t) &:= [I_n - \nu(y,t) \otimes \nu(y,t)]\nabla \overline{v}(y,t),\\ \mathrm{div}_{\Gamma(t)} F(y,t) &:= \mathrm{trace}[\{I_n - \nu(y,t)\otimes\nu(y,t)\}\nabla \overline{F}(y,t)]\end{aligned}$$

for $(y,t) \in S_T$. Here I_n is the identity matrix of size n and $\nu \otimes \nu := (\nu_i \nu_j)_{i,j}$ is the tensor product of ν. Also, $\overline{v}$ and $\overline{F}$ are the constant extensions of v and F to the normal direction of $\Gamma(t)$ given by

$$\overline{v}(x,t) := v(p(x,t),t), \quad \overline{F}(x,t) := F(p(x,t),t), \quad (x,t) \in N_T.$$

By definition, $\nu \cdot \nabla_{\Gamma(t)} v = 0$ holds. Hereafter we use the same notations for functions and vector fields on $\Gamma(t)$ with each fixed $t \in [0,T]$.

Finally, we define a moving thin domain. Let g_0 and g_1 be smooth functions on $\overline{S_T}$. We assume that there is a constant $c>0$ such that

$$g(y,t) := g_1(y,t) - g_0(y,t) \geq c \quad \text{for all} \quad (y,t) \in \overline{S_T}. \tag{2.1}$$

Then we define a moving thin domain $\Omega_\varepsilon(t) \subset \mathbb{R}^n$ as

$$\Omega_\varepsilon(t) := \{y + \rho\nu(y,t) \mid y \in \Gamma(t),\, \varepsilon g_0(y,t) < \rho < \varepsilon g_1(y,t)\}, \quad t \in [0,T],\, \varepsilon > 0$$

and a space-time noncylindrical domain $Q_{\varepsilon,T} \subset \mathbb{R}^{n+1}$ as $Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \Omega_\varepsilon(t) \times \{t\}$. Note that $\Omega_\varepsilon(t)$ does not necessarily include $\Gamma(t)$, since we do not assume that g_0 is negative and g_1 is positive. Since g_0 and g_1 are smooth and thus bounded on the compact manifold $\overline{S_T}$, there is a positive number ε_0 such that $\overline{\Omega_\varepsilon(t)} \subset N(t)$ for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T]$. Hereafter we assume that $\varepsilon \in (0,\varepsilon_0)$.

3. Heat equation on moving thin domains

In this section, we consider the initial-boundary problem (H_ε) of the heat equation on the moving thin domain $\Omega_\varepsilon(t)$. First we show that the boundary condition of (H_ε) yields the conservation of heat. Suppose that u^ε satisfies the heat equation in $Q_{\varepsilon,T}$. Then, by the Reynolds transport theorem and Green's formula (see [9, Appendix C]), we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx &= \int_{\Omega_\varepsilon(t)} \partial_t u^\varepsilon\,dx + \int_{\partial\Omega_\varepsilon(t)} V_\varepsilon^N u^\varepsilon\,d\mathcal{H}^{n-1} \\ &= \int_{\Omega_\varepsilon(t)} \Delta u^\varepsilon\,dx + \int_{\partial\Omega_\varepsilon(t)} V_\varepsilon^N u^\varepsilon\,d\mathcal{H}^{n-1} = \int_{\partial\Omega_\varepsilon(t)} (\partial_{\nu_\varepsilon} u^\varepsilon + V_\varepsilon^N u^\varepsilon)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Hence if u^ε additionally satisfies the boundary condition of (H_ε), then we have $\frac{d}{dt}\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx = 0$ for all $t\in(0,T)$, that is, the total amount of heat $\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx$ is conserved.

Next, we give a definition of a variational solution to (H_ε). For each $\varepsilon>0$, we define a function space $L^2_{H^1(\varepsilon)}$ on $Q_{\varepsilon,T}$ and an inner product on $L^2_{H^1(\varepsilon)}$ as

$$\begin{aligned} L^2_{H^1(\varepsilon)} &:= \{u \in L^2(Q_{\varepsilon,T}) \mid \nabla u \in L^2(Q_{\varepsilon,T})\}, \\ (u_1,u_2)_{L^2_{H^1(\varepsilon)}} &:= \int_0^T \int_{\Omega_\varepsilon(t)} (u_1u_2 + \nabla u_1\cdot\nabla u_2)\,dx\,dt. \end{aligned} \tag{3.1}$$

The space $L^2_{H^1(\varepsilon)}$ is a Hilbert space endowed with the above inner product. Let $\|\cdot\|_{L^2_{H^1(\varepsilon)}}$ denote the norm of $L^2_{H^1(\varepsilon)}$ induced by the inner product $(\cdot,\cdot)_{L^2_{H^1(\varepsilon)}}$.

Definition 3.1. Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$. A function $u \in L^2_{H^1(\varepsilon)}$ is said to be a *variational solution* to the initial-boundary value problem (H_ε) if it satisfies

$$\int_0^T \int_{\Omega_\varepsilon(t)} (-u^\varepsilon \partial_t w + \nabla u^\varepsilon \cdot \nabla w)\,dx\,dt - \int_{\Omega_\varepsilon(0)} u_0^\varepsilon w(0)\,dx = 0 \tag{3.2}$$

for all $w \in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T)=0$ in $\Omega_\varepsilon(T)$.

The variational formulation (3.2) is derived as follows. Suppose that u^ε is a classical solution to (H_ε). We multiply both sides of the heat equation in $Q_{\varepsilon,T}$ by an arbitrary function $w \in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T) = 0$ in $\Omega_\varepsilon(T)$ and integrate them over $Q_{\varepsilon,T}$ to get

$$\int_0^T \int_{\Omega_\varepsilon(t)} (\partial_t u^\varepsilon - \Delta u^\varepsilon) w \, dx \, dt = 0.$$

We calculate the left-hand side of the above equality. By the Reynolds transport theorem and the conditions $u^\varepsilon(0) = u_0^\varepsilon$ in $\Omega_\varepsilon(0)$ and $w(T) = 0$ in $\Omega_\varepsilon(T)$, we have

$$\int_0^T \int_{\Omega_\varepsilon(t)} (\partial_t u^\varepsilon) w \, dx \, dt = \\ -\int_0^T \int_{\Omega_\varepsilon(t)} u^\varepsilon \partial_t w \, dx \, dt - \int_0^T \int_{\partial\Omega_\varepsilon(t)} V_\varepsilon^N u^\varepsilon w \, d\mathcal{H}^{n-1} \, dt - \int_{\Omega_\varepsilon(0)} u_0^\varepsilon w(0) \, dx.$$

On the other hand, by integration by parts,

$$-\int_{\Omega_\varepsilon(t)} (\Delta u^\varepsilon) w \, dx \, dt = \int_{\Omega_\varepsilon(t)} \nabla u^\varepsilon \cdot \nabla w \, dx - \int_{\partial\Omega_\varepsilon(t)} (\partial_{\nu_\varepsilon} u^\varepsilon) w \, d\mathcal{H}^{n-1}.$$

Hence it follows that

$$\int_0^T \int_{\Omega_\varepsilon(t)} (-u^\varepsilon \partial_t w + \nabla u^\varepsilon \cdot \nabla w) \, dx \, dt - \int_0^T \int_{\partial\Omega_\varepsilon(t)} (\partial_{\nu_\varepsilon} u^\varepsilon + V_\varepsilon^N u^\varepsilon) w \, d\mathcal{H}^{n-1} \, dt \\ - \int_{\Omega_\varepsilon(0)} u_0^\varepsilon w(0) \, dx = 0$$

and we obtain (3.2) by applying the boundary condition of (H_ε) to the second term of the left-hand side in the above equality.

Our goal in this section is to obtain a unique variational solution to (H_ε) that satisfies an energy estimate with a constant independent of ε. To this end, we transform (3.2) into a variational formulation of some equation on a fixed (in time) domain $\Omega_\varepsilon(0)$ with the aid of a suitable diffeomorphism between $\Omega_\varepsilon(0)$ and $\Omega_\varepsilon(t)$.

Lemma 3.2. *For each $t \in [0, T]$, there exists a diffeomorphism $\Psi_\varepsilon(\cdot, t) \colon \Omega_\varepsilon(0) \to \Omega_\varepsilon(t)$ with its inverse $\Psi_\varepsilon^{-1}(\cdot, t) \colon \Omega_\varepsilon(t) \to \Omega_\varepsilon(0)$ such that Ψ_ε and Ψ_ε^{-1} are smooth on $\overline{\Omega_\varepsilon(0)} \times [0, T]$ and $\overline{Q_{\varepsilon,T}}$, respectively, and $\Psi_\varepsilon(\cdot, 0)$ is the identity mapping on $\Omega_\varepsilon(0)$. Moreover, there exists a constant $c > 0$ independent of ε such that*

$$|\partial_X^\alpha \partial_t^k \Psi_\varepsilon(X, t)| \leq c, \quad |\partial_x^\alpha \partial_t^k \Psi_\varepsilon^{-1}(x, t)| \leq c \tag{3.3}$$

for all $(X, t) \in \Omega_\varepsilon(0) \times (0, T)$, $(x, t) \in Q_{\varepsilon,T}$, and $|\alpha| + k \leq 2$, $k = 0, 1, 2$.

Proof. We observe that for each $X \in \Omega_\varepsilon(0)$ there is a unique $\theta \in (0, 1)$ such that

$$X = p(X, 0) + \varepsilon\{(1 - \theta) g_0(p(X, 0), 0) + \theta g_1(p(X, 0), 0)\} \nu(p(X, 0), 0), \tag{3.4}$$

that is, X divides the line segment $A_0 A_1$ internally in the ratio $\theta \colon 1 - \theta$, where

$$A_i := p(X, 0) + \varepsilon g_i(p(X, 0), 0) \nu(p(X, 0), 0), \quad i = 0, 1.$$

Then, it is natural to define $\Psi_\varepsilon(X, t) \in \Omega_\varepsilon(t)$ as

$$\Psi_\varepsilon(X, t) := \Phi(p(X, 0), t) \\ + \varepsilon\{(1 - \theta) g_0(\Phi(p(X, 0), t), t) + \theta g_1(\Phi(p(X, 0), t), t)\} \nu(\Phi(p(X, 0), t), t), \tag{3.5}$$

that is, $\Psi_\varepsilon(X,t)$ divides the line segment B_0B_1 internally in the ratio $\theta\colon 1-\theta$, where

$$B_i := \Phi(p(X,0),t) + \varepsilon g_i(\Phi(p(X,0),t),t)\nu(\Phi(p(X,0),t),t), \quad i=0,1.$$

To eliminate θ in (3.5), we take the inner product of both sides of (3.4) and $\nu(p(X,0),0)$. Then

$$\{X-p(X,0)\}\cdot\nu(p(X,0),0) = \varepsilon\{(1-\theta)g_0(p(X,0),0)+\theta g_1(p(X,0),0)\}.$$

Since $\{X-p(X,0)\}\cdot\nu(p(X,0),0)=d(X,0)$ and $g_1-g_0=g>0$, it follows that

$$\theta = \frac{d(X,0)-\varepsilon g_0(p(X,0),0)}{\varepsilon g(p(X,0),0)}.$$

Hence, by substituting this for θ in (3.5), we obtain

$$\Psi_\varepsilon(X,t) = \Phi(p(X,0),t) + \{d(X,0)\phi_1(X,t)+\varepsilon\phi_2(X,t)\}\nu(\Phi(p(X,0),t),t) \tag{3.6}$$

for $X\in\Omega_\varepsilon(0)$ and $t\in[0,T]$, where

$$\begin{aligned} \phi_1(X,t) &:= \frac{g(\Phi(p(X,0),t),t)}{g(p(X,0),0)}, \\ \phi_2(X,t) &:= g_0(\Phi(p(X,0),t),t)-\phi_1(X,t)g_0(p(X,0),0). \end{aligned}$$

Similarly we define a mapping Ψ_ε^{-1} as

$$\begin{aligned} &\Psi_\varepsilon^{-1}(x,t) := \\ &\qquad \Phi^{-1}(p(x,t),t) + \{d(x,t)\phi_3(x,t)+\varepsilon\phi_4(x,t)\}\nu(\Phi^{-1}(p(x,t),t),0) \end{aligned} \tag{3.7}$$

for $(x,t)\in Q_{\varepsilon,T}$, where

$$\begin{aligned} \phi_3(x,t) &:= \frac{g(\Phi^{-1}(p(x,t),t),0)}{g(p(x,t),t)}, \\ \phi_4(x,t) &:= g_0(\Phi^{-1}(p(x,t),t),0)-\phi_3(x,t)g_0(p(x,t),t). \end{aligned}$$

By definition, $\Psi_\varepsilon(\cdot,t)\colon\Omega_\varepsilon(0)\to\Omega_\varepsilon(t)$ is a bijection with its inverse $\Psi_\varepsilon^{-1}\colon\Omega_\varepsilon(t)\to\Omega_\varepsilon(0)$ for each $t\in[0,T]$. Also, since $\Phi(\cdot,0)$ is the identity mapping on Γ_0, we have $\phi_1(X,0)=1$, $\phi_2(X,0)=0$ and thus

$$\Psi_\varepsilon(X,0) = p(X,0)+d(X,0)\nu(p(X,0),0) = X \quad \text{for all} \quad X\in\Omega_\varepsilon(0),$$

that is, $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$. Due to the smoothness of Φ, Φ^{-1}, d, p, g_0, and g_1, the right-hand sides of (3.6) and (3.7) are smooth on the compact sets $\overline{N(0)}\times[0,T]$ and $\overline{N_T}$, respectively, and thus bounded independently of ε along their derivatives. From this fact and the inclusion $\overline{\Omega_\varepsilon(t)}\subset N(t)$ for each $t\in[0,T]$, it follows that Ψ_ε and Ψ_ε^{-1} are smooth on $\overline{\Omega_\varepsilon(0)}\times[0,T]$ and $\overline{Q_{\varepsilon,T}}$, respectively, and that the inequality (3.2) holds with a constant $c>0$ independent of ε. In particular, $\Psi_\varepsilon(\cdot,t)\colon\Omega_\varepsilon(0)\to\Omega_\varepsilon(t)$ is a diffeomorpfism for each $t\in[0,T]$. $\Box$

Let Ψ_ε and Ψ_ε^{-1} be mappings given by Lemma 3.2. In (3.2), we set

$$U^\varepsilon(X,t) := u^\varepsilon(\Psi_\varepsilon(X,t),t), \quad W(X,t) := w(\Psi_\varepsilon(X,t),t)$$

for $(X,t) \in \Omega_\varepsilon(0)\times(0,T)$. Then, by the change of variables $x = \Psi_\varepsilon(X,t)$, we transform (3.2) into

$$\int_0^T \{-(U^\varepsilon(t), J^\varepsilon(t)\partial_t W(t))_{L^2} + (A^\varepsilon(t)\nabla U^\varepsilon(t) - U^\varepsilon(t)B^\varepsilon(t), \nabla W(t))_{L^2}\}\,dt - (u_0^\varepsilon, W(0))_{L^2} = 0. \tag{3.8}$$

Here $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\Omega_\varepsilon(0))$ and

$$J^\varepsilon(X,t) := |\det \nabla\Psi_\varepsilon(X,t)| \in \mathbb{R},$$
$$A^\varepsilon(X,t) := J^\varepsilon(X,t)\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T \in \mathbb{R}^{n\times n},$$
$$B^\varepsilon(X,t) := J^\varepsilon(X,t)\partial_t\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t) \in \mathbb{R}^n$$

for $(X,t)\in\Omega_\varepsilon(0)\times(0,T)$, where

$$\nabla\Psi_\varepsilon^{-1} := \begin{pmatrix} \partial_1(\Psi_\varepsilon^{-1})_1 & \dots & \partial_n(\Psi_\varepsilon^{-1})_1 \\ \vdots & \ddots & \vdots \\ \partial_1(\Psi_\varepsilon^{-1})_n & \dots & \partial_n(\Psi_\varepsilon^{-1})_n \end{pmatrix} \quad \text{for} \quad \Psi_\varepsilon^{-1} = \begin{pmatrix} (\Psi_\varepsilon^{-1})_1 \\ \vdots \\ (\Psi_\varepsilon^{-1})_n \end{pmatrix}$$

and $[\nabla\Psi_\varepsilon^{-1}]^T$ denotes the transpose matrix of $\nabla\Psi_\varepsilon^{-1}$. Since $w(T)=0$ in $\Omega_\varepsilon(T)$ and $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$, we have $W(T)=0$ and $J^\varepsilon(0)=1$ in $\Omega_\varepsilon(0)$. Thus, by integration by parts with respect to t, we further transform (3.8) into

$$\int_0^T \{{}_{(H^1)'}\langle\partial_t U^\varepsilon(t), J^\varepsilon(t)W(t)\rangle_{H^1} + (U^\varepsilon(t), W(t)\partial_t J^\varepsilon(t))_{L^2} + (A^\varepsilon(t)\nabla U^\varepsilon(t) - U^\varepsilon(t)B^\varepsilon(t), \nabla W(t))_{L^2}\}\,dt = 0. \tag{3.9}$$

Here ${}_{(H^1)'}\langle\cdot,\cdot\rangle_{H^1}$ is the duality product between $H^1(\Omega_\varepsilon(0))$ and its dual space $(H^1(\Omega_\varepsilon(0)))'$.

Theorem 3.3. *For every $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$, there exists a unique function*

$$U^\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon(0))) \cap L^2(0,T;H^1(\Omega_\varepsilon(0)))$$
$$\text{with} \quad \partial_t U^\varepsilon \in L^2(0,T;(H^1(\Omega_\varepsilon(0)))')$$

that satisfies (3.9) for all $W\in L^2(0,T;H^1(\Omega_\varepsilon(0)))$ and $U^\varepsilon(0)=u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$. Moreover, there exists a constant $c>0$ independent of u_0^ε, U^ε, and ε such that

$$\sup_{t\in(0,T)} \|U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))} + \int_0^T \|\nabla U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}\,dt \le c\|u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon(0))}. \tag{3.10}$$

Proof. For $i,j=1,\dots,n$, let A^ε_{ij} be the (i,j)-entry of A^ε and B^ε_i be the i-th component of B^ε. Suppose that there is a positive constant C independent of ε such that

$$C^{-1} \le J^\varepsilon(X,t) \le C, \tag{3.11}$$
$$|\nabla J^\varepsilon(X,t)| \le C, \quad |\partial_t J^\varepsilon(X,t)| \le C, \quad |A^\varepsilon_{ij}(X,t)| \le C, \quad |B^\varepsilon_i(X,t)| \le C, \tag{3.12}$$
$$A^\varepsilon(X,t)\zeta\cdot\zeta \ge C|\zeta|^2 \tag{3.13}$$

for all $(X,t)\in\Omega_\varepsilon(0)\times(0,T)$, $\zeta\in\mathbb{R}^n$, and $i,j=1,\dots,n$. Then the theorem is proved by a standard Galerkin method and Gronwall argument, see [9, Section 7.1] for details. In particular, the constant c in (3.10) depends only on the above C and thus it is independent of ε.

Let us prove the inequalities (3.11), (3.12), and (3.13). The inequality (3.12) and the right-hand inequality of (3.11) immediately follow from (3.3). From the identity $\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)\nabla\Psi_\varepsilon(X,t)=I_n$, it follows that

$$|\det\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)|J^\varepsilon(X,t)=1,\quad [\nabla\Psi_\varepsilon(X,t)]^T[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T=I_n$$

for all $(X,t)\in\Omega_\varepsilon(0)\times(0,T)$. The first equality yields the left-hand inequality of (3.11) because $|\det\nabla\Psi_\varepsilon^{-1}|$ is bounded on $Q_{\varepsilon,T}$ independently of ε by (3.3). Moreover, the above equality and (3.3) imply that, for all $(X,t)\in\Omega_\varepsilon(0)\times(0,T)$ and $\zeta\in\mathbb{R}^n$,

$$\begin{aligned}|\zeta|^2&=|[\nabla\Psi_\varepsilon(X,t)]^T[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T\zeta|^2\\&\leq c|[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T\zeta|^2\\&=c\{\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T\zeta\}\cdot\zeta\\&=c|\det\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)|A^\varepsilon(X,t)\zeta\cdot\zeta\leq cA^\varepsilon(X,t)\zeta\cdot\zeta\end{aligned}$$

with a constant $c>0$ independent of ε. Thus (3.13) follows. □

Now we can show the existence and uniqueness of a variational solution to (H_ε) and its energy estimate with a constant independent of ε.

Theorem 3.4. *For every $u_0^\varepsilon\in L^2(\Omega_\varepsilon(0))$, there exists a unique variational solution u^ε to (H_ε). Moreover, u^ε satisfies that $u^\varepsilon(0)=u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$ and*

$$\sup_{t\in(0,T)}\|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon(t))}^2+\int_0^T\|\nabla u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon(t))}^2\,dt\leq c\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}^2 \tag{3.14}$$

with a constant $c>0$ independent of u_0^ε, u^ε, and ε.

Proof. For each $u_0^\varepsilon\in L^2(\Omega_\varepsilon(0))$, let U^ε be a unique function given by Theorem 3.3 and we set

$$u^\varepsilon(x,t):=U^\varepsilon(\Psi_\varepsilon^{-1}(x,t),t),\quad (x,t)\in Q_{\varepsilon,T}.$$

Since $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$ by Lemma 3.2 and $U^\varepsilon(0)=u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$ by Theorem 3.3, we have $u^\varepsilon(0)=u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$. Let us show that u^ε satisfies (3.2) for all $w\in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T)=0$ in $\Omega_\varepsilon(T)$. Since Ψ_ε is smooth on $\overline{\Omega_\varepsilon(0)}\times[0,T]$, a function

$$W(X,t):=w(\Psi_\varepsilon(X,t),t),\quad (X,t)\in\overline{\Omega_\varepsilon(0)}\times[0,T]$$

is in $C^1(\overline{\Omega_\varepsilon(0)}\times[0,T])$ and satisfies $W(T)=0$ in $\Omega_\varepsilon(0)$. Hence we can substitute it for W in (3.9) and integrate by parts with respect to t to get (3.8). By changing variables $X=\Psi_\varepsilon^{-1}(x,t)$ in (3.8), we obtain (3.2).

Next we prove the energy estimate (3.14). By the change of variables $x=\Psi_\varepsilon(X,t)$ we have

$$\int_{\Omega_\varepsilon(t)}|u^\varepsilon(x,t)|^2\,dx=\int_{\Omega_\varepsilon(0)}|U^\varepsilon(X,t)|^2|\det\nabla\Psi_\varepsilon(X,t)|\,dX$$

and

$$\int_{\Omega_\varepsilon(t)}|\nabla u^\varepsilon(x,t)|^2\,dx=\\ \int_{\Omega_\varepsilon(0)}|[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T\nabla U^\varepsilon(X,t)|^2|\det\nabla\Psi_\varepsilon(X,t)|\,dX$$

for all $t \in [0,T]$. Hence the inequality (3.3) yields

$$\|u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} \leq c\|U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}, \quad \|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} \leq c\|\nabla U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}$$

with a constant $c > 0$ independent of ε. By this inequality and (3.10), we obtain (3.14) and thus $u \in L^2_{H^1(\varepsilon)}$. Hence u^ε is a variational solution to (H_ε).

Finally, the uniqueness of a variational solution to (H_ε) follows from that of a function given by Theorem 3.3. The proof is complete. □

Remark 3.5. Let u^ε be a unique variational solution to (H_ε) with initial data $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$. Then, it immediately follows from (3.14) that

$$\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}} \leq c\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}, \tag{3.15}$$

where $c > 0$ is a constant independent of u_0^ε, u^ε, and ε. We will use this inequality in Section 6.

4. Basic function spaces on evolving surfaces

In this section, we define and investigate function spaces on the space-time manifold S_T introduced by Olshanskii, Reusken and Xu [18]. These spaces will give an appropriate variational formulation of a limit equation on $\Gamma(t)$ we will derive in Section 6. All results in this section are originally obtained in [18] when $n = 3$. We shall extend them for general $n \geq 2$.

For each fixed $T > 0$, we define a function space H_T and an inner product on H_T as

$$H_T := \{v \in L^2(S_T) \mid \nabla_{\Gamma(t)} v \in L^2(S_T)\}, \tag{4.1}$$

$$(v_1, v_2)_{H_T} := \int_0^T \int_{\Gamma(t)} \{v_1(y,t)v_2(y,t) + \nabla_{\Gamma(t)} v_1(y,t) \cdot \nabla_{\Gamma(t)} v_2(y,t)\}\, d\mathcal{H}^{n-1}(y)\, dt.$$

This inner product induces the norm $\|\cdot\|_{H_T}$ that is equivalent to the one induced by the inner product $\int_{S_T} \{v_1(\sigma)v_2(\sigma) + \nabla_{\Gamma(t)} v_1(\sigma) \cdot \nabla_{\Gamma(t)} v_2(\sigma)\}\, d\mathcal{H}^n(\sigma)$, since the identity

$$\int_0^T \int_{\Gamma(t)} f(y,t)\, d\mathcal{H}^{n-1}(y)\, dt = \int_{S_T} f(\sigma)\{1 + (V_\Gamma \cdot \nu)^2\}^{-1/2}\, d\mathcal{H}^n(\sigma)$$

holds and V_Γ is bounded on S_T. If $T_1 < T_2$, then H_{T_2} is continuously embedded into H_{T_1} just by restricting elements of H_{T_2} on S_{T_1}.

Next we define an auxiliary space. Let $H(\Gamma_0) := \{V \in L^2(\Gamma_0) \mid \nabla_{\Gamma_0} V \in L^2(\Gamma_0)\}$ with the inner product $(V_1, V_2)_{H^1(\Gamma_0)} := \int_{\Gamma_0} (V_1 V_2 + \nabla_{\Gamma_0} V_1 \cdot \nabla_{\Gamma_0} V_2)\, d\mathcal{H}^{n-1}$, where ∇_{Γ_0} is the tangential gradient on Γ_0. Then we define a Hilbert space $\widehat{H}_T$ as

$$\widehat{H}_T := L^2(0,T; H^1(\Gamma_0)), \quad (V_1, V_2)_{\widehat{H}_T} := \int_0^T (V_1(t), V_2(t))_{H^1(\Gamma_0)}\, dt$$

and let $\|\cdot\|_{\widehat{H}_T}$ denote the norm of $\widehat{H}_T$ induced by the inner product $(\cdot,\cdot)_{\widehat{H}_T}$.

For a function V on $\Gamma_0 \times (0,T)$, we define a function $v = LV$ on S_T as

$$v(y,t) := V(\Phi^{-1}(y,t), t), \quad (y,t) \in S_T. \tag{4.2}$$

Also, for a function v on S_T, we define a function $V = L^{-1}v$ on $\Gamma_0 \times (0,T)$ as

$$V(Y,t) := v(\Phi(Y,t), t), \quad (Y,t) \in \Gamma_0 \times (0,T).$$

Clearly L and L^{-1} are linear mappings and satisfy $L^{-1}(LV)=V$ and $L(L^{-1}v)=v$ for all functions V on $\Gamma_0\times(0,T)$ and v on S_T.

Lemma 4.1. *The linear mapping L given by* (4.2) *defines a homeomorphism between $\widehat{H}_T$ and H_T.*

Proof. Let V be a function on $\Gamma_0\times(0,T)$ and $v:=LV$. We shall show that there are constants $c_1,c_2>0$ independent of V and v such that

$$c_1\|V(t)\|_{L^2(\Gamma_0)}\leq\|v(t)\|_{L^2(\Gamma(t))}\leq c_2\|V(t)\|_{L^2(\Gamma_0)}, \tag{4.3}$$

$$c_1\|\nabla_{\Gamma_0}V(t)\|_{L^2(\Gamma_0)}\leq\|\nabla_{\Gamma(t)}v(t)\|_{L^2(\Gamma(t))}\leq c_2\|\nabla_{\Gamma_0}V(t)\|_{L^2(\Gamma_0)} \tag{4.4}$$

for all $t\in(0,T)$. These inequalities imply that $c_1'\|V\|_{\widehat{H}_T}\leq\|v\|_{H_T}\leq c_2'\|V\|_{\widehat{H}_T}$ for some constants $c_1',c_2'>0$ independent of V and v, which means that L is a homeomorphism between $\widehat{H}_T$ and H_T.

Since Γ_0 is compact, we can take a finite family $\{U_k\}_{k=1}^N$ of bounded open sets of $\mathbb{R}^{n-1}$ and smooth local parametrizations $\mu_0^k\colon U_k\to\Gamma_0$, $k=1,\dots,N$, such that $\{\mu_0^k(U_k)\}_{k=1}^N$ is an open covering of Γ_0. For each $t\in[0,T]$, we define mappings $\mu_t^k\colon U_k\to\Gamma(t)$ as $\mu_t^k(s):=\Phi(\mu_0^k(s),t)$. Then μ_t^k is a smooth local parametrization of $\Gamma(t)$ for each k and $\{\mu_t^k(U_k)\}_{k=1}^N$ is an open covering of $\Gamma(t)$. Also, let $\{\psi_0^k\}_{k=1}^N$ be a partition of unity of Γ_0 subordinate to the covering $\{\mu_0^k(U_k)\}_{k=1}^N$ and $\{\psi_t^k\}_{k=1}^N$ be a family of functions given by $\psi_t^k:=\psi_0^k\circ\mu_0^k\circ(\mu_t^k)^{-1}$. Then $\{\psi_t^k\}_{k=1}^N$ is a partition of unity of $\Gamma(t)$ subordinate to the covering $\{\mu_t^k(U_k)\}_{k=1}^N$.

Using these local parametrizations and partitions of unity, it is sufficient for (4.3) and (4.4) to show that, for each k, there are constants $c_1^k,c_2^k>0$ such that

$$c_1^k\int_{\mu_0^k(Q_k)}|V(t)|^2\,d\mathcal{H}^{n-1}\leq\int_{\mu_t^k(Q_k)}|v(t)|^2\,d\mathcal{H}^{n-1}\leq c_2^k\int_{\mu_0^k(Q_k)}|V(t)|^2\,d\mathcal{H}^{n-1} \tag{4.5}$$

and

$$\begin{aligned}c_1^k\int_{\mu_0^k(Q_k)}|\nabla_{\Gamma_0}V(t)|^2\,d\mathcal{H}^{n-1}&\leq\int_{\mu_t^k(Q_k)}|\nabla_{\Gamma(t)}v(t)|^2\,d\mathcal{H}^{n-1}\\&\leq c_2^k\int_{\mu_0^k(Q_k)}|\nabla_{\Gamma_0}V(t)|^2\,d\mathcal{H}^{n-1}\end{aligned} \tag{4.6}$$

for all $t\in(0,T)$ and all functions V supported in $\mu_0^k(Q_k)\times(0,T)$. Here $Q_k\subset\mathbb{R}^{n-1}$ is a compact subset of U_k such that $\operatorname{supp}\psi_0^k\subset\mu_0^k(Q_k)$. Note that in this case $\operatorname{supp}\psi_t^k\subset\mu_t^k(Q_k)$ holds for all $t\in[0,T]$ and $v=LV$ is supported in $\bigcup_{t\in(0,T)}\mu_t^k(Q_k)\times\{t\}$.

For simplicity, we fix each $k=1,\dots,N$ and omit it. Let $g_t=(g_{t,ij})_{i,j}$ be a matrix given by

$$g_{t,ij}(s):=\frac{\partial\mu_t}{\partial s_i}(s)\cdot\frac{\partial\mu_t}{\partial s_j}(s),\quad (s,t)\in U\times[0,T],\ i,j=1,\dots,n-1, \tag{4.7}$$

and $g_t^{-1}=(g_t^{ij})_{ij}$ be the inverse matrix of g_t. By the definition of the integral over hypersurfaces,

$$\int_{\mu_0(Q)}|V(Y,t)|^2\,d\mathcal{H}^{n-1}=\int_Q|V(\mu_0(s),t)|^2\sqrt{\det g_0(s)}\,ds,$$

$$\int_{\mu_t(Q)}|v(y,t)|^2\,d\mathcal{H}^{n-1}=\int_Q|v(\mu_t(s),t)|^2\sqrt{\det g_t(s)}\,ds.$$

Since $\sqrt{\det g_t(s)}$ is continuous and does not vanish as a function of (s,t) on the compact set $Q\times[0,T]$, there is a constant $c>0$ such that

$$c^{-1}\leq\sqrt{\det g_t(s)}\leq c\quad\text{for all}\quad(s,t)\in Q\times[0,T].\tag{4.8}$$

Moreover, by the definition of L and μ_t,

$$v(\mu_t(s),t)=V(\Phi^{-1}(\mu_t(s),t),t)=V(\Phi^{-1}(\Phi(\mu_0(s),t),t),t)=V(\mu_0(s),t)\tag{4.9}$$

for all $(s,t)\in U\times[0,T]$. Hence (4.5) follows. Similarly, by (4.8) and the equality

$$\int_{\mu_0(Q)}|\nabla_{\Gamma_0}V(Y,t)|^2\,d\mathcal{H}^{n-1}=\int_Q|\nabla_{\Gamma_0}V(\mu_0(s),t)|^2\sqrt{\det g_0(s)}\,ds,$$
$$\int_{\mu_t(Q)}|\nabla_{\Gamma(t)}v(y,t)|^2\,d\mathcal{H}^{n-1}=\int_Q|\nabla_{\Gamma(t)}v(\mu_t(s),t)|^2\sqrt{\det g_t(s)}\,ds,$$

it is sufficient for (4.6) to show that there are constants $c_1,c_2>0$ such that

$$c_1|\nabla_{\Gamma_0}V(\mu_0(s),t)|^2\leq|\nabla_{\Gamma(t)}v(\mu_t(s),t)|^2\leq c_2|\nabla_{\Gamma_0}V(\mu_0(s),t)|^2\tag{4.10}$$

for all $(s,t)\in Q\times[0,T]$. The tangential gradients $\nabla_{\Gamma_0}V$ and $\nabla_{\Gamma(t)}v$ are locally expressed as (see [5, Section 2.1 and Section 2.2] for example)

$$\nabla_{\Gamma_0}V(\mu_0(s),t)=\sum_{i,j=1}^{n-1}g_0^{ij}(s)\frac{\partial}{\partial s_j}(V(\mu_0(s),t))\frac{\partial\mu_0}{\partial s_i}(s),$$
$$\nabla_{\Gamma(t)}v(\mu_t(s),t)=\sum_{i,j=1}^{n-1}g_t^{ij}(s)\frac{\partial}{\partial s_j}(v(\mu_t(s),t))\frac{\partial\mu_t}{\partial s_i}(s)$$

for $(s,t)\in U\times[0,T]$ and their Euclidean norms are

$$|\nabla_{\Gamma_0}V(\mu_0(s),t)|^2=\sum_{i,j=1}^{n-1}g_0^{ij}(s)\frac{\partial}{\partial s_i}(V(\mu_0(s),t))\frac{\partial}{\partial s_j}(V(\mu_0(s),t)),$$
$$|\nabla_{\Gamma(t)}v(\mu_t(s),t)|^2=\sum_{i,j=1}^{n-1}g_t^{ij}(s)\frac{\partial}{\partial s_i}(v(\mu_t(s),t))\frac{\partial}{\partial s_j}(v(\mu_t(s),t)).$$

Then, by (4.9), it is sufficient for (4.10) to show that there are constants $c_1,c_2>0$ such that

$$c_1g_0^{-1}(s)a\cdot a\leq g_t^{-1}(s)a\cdot a\leq c_2g_0^{-1}(s)a\cdot a\tag{4.11}$$

for all $(s,t,a)\in Q\times[0,T]\times\mathbb{R}^{n-1}$. To this end, we consider a real-valued function

$$F(s,t,a):=g_t^{-1}(s)a\cdot a,\quad(s,t,a)\in Q\times[0,T]\times\mathbb{R}^{n-1}.$$

It is continuous on $Q\times[0,T]\times\mathbb{R}^{n-1}$ and satisfies $F(s,t,a)=b\cdot g_t(s)b=|B(s,t,b)|^2$, where $b=(b_1,\dots,b_{n-1}):=g_t^{-1}(s)a$ and $B:=\sum_{i=1}^{n-1}b_i\partial\mu_t/\partial s_i(s)$. Since $b\neq0$ for $a\neq0$, it follows that $B\neq0$ and thus $F(s,t,a)\neq0$ for $a\neq0$. In particular, F is continuous and does not vanish on the compact set $Q\times[0,T]\times S^{n-2}$, where S^{n-2} is the unit sphere of $\mathbb{R}^{n-1}$. Hence there is a constant $c>0$ such that $c^{-1}\leq F(s,t,a)\leq c$ for all $(s,t,a)\in Q\times[0,T]\times S^{n-2}$ and thus

$$c^{-1}|a|^2\leq g_t^{-1}(s)a\cdot a\leq c|a|^2\quad\text{for all}\quad(s,t,a)\in Q\times[0,T]\times\mathbb{R}^{n-1}.$$

This inequality yields (4.11) and thus (4.10) follows. Hence we obtain the inequality (4.6) and the proof is complete. $\square$

Let $C_0^1(S_T)$ be the space of all functions in $C^1(S_T)$ with compact support in S_T. That is, each function in $C_0^1(S_T)$ vanishes near $t=0$ and $t=T$.

Lemma 4.2. *The space H_T is a Hilbert space and $C_0^1(S_T)$ is dense in H_T.*

Proof. Since $\widehat{H}_T$ is a Hilbert space, Lemma 4.1 implies that H_T is a Hilbert space. Also, since $C_0^1(\Gamma_0\times(0,T))$ is dense in $\widehat{H}_T$ (see [18, Lemma 3.1]) and $C_0^1(S_T)$ includes $L[C_0^1(\Gamma_0\times(0,T))]$, Lemma 4.1 again implies that $C_0^1(S_T)$ is dense in H_T. $\square$

The space H_T is continuously embedded into $L^2(S_T)$. Moreover, H_T is dense in $L^2(S_T)$ since it includes a dense subspace $C_0^1(S_T)$ of $L^2(S_T)$. Hence we have continuous and dense embeddings $H_T \hookrightarrow L^2(S_T) \hookrightarrow H_T'$, where H_T' is the dual space of H_T.

For $v\in C^1(S_T)$, we define its (strong) material derivative $\partial^\bullet v$ as

$$\partial^\bullet v(\Phi(Y,t),t) := \frac{d}{dt}(v(\Phi(Y,t),t)), \quad (Y,t)\in\Gamma_0\times(0,T). \tag{4.12}$$

From the Leibniz formula (see [3, Lemma 2.2])

$$\frac{d}{dt}\int_{\Gamma(t)} v\,d\mathcal{H}^{n-1} = \int_{\Gamma(t)}(\partial^\bullet v + v\,\mathrm{div}_{\Gamma(t)}V_\Gamma)\,d\mathcal{H}^{n-1}, \quad v\in C^1(S_T),$$

we have the integration by parts identity

$$\begin{aligned}\int_0^T\int_{\Gamma(t)}&(v_2\partial^\bullet v_1 + v_1\partial^\bullet v_2 + v_1v_2\,\mathrm{div}_{\Gamma(t)}V_\Gamma)\,d\mathcal{H}^{n-1}\,dt \\ &= \int_{\Gamma(T)} v_1(T)v_2(T)\,d\mathcal{H}^{n-1} - \int_{\Gamma(0)} v_1(0)v_2(0)\,d\mathcal{H}^{n-1}\end{aligned} \tag{4.13}$$

for all $v_1,v_2\in C^1(S_T)$. Based on this identity, we define the weak material derivative of $v\in H_T$ as a functional $\partial^\bullet v$ such that

$$\langle\partial^\bullet v,\psi\rangle_T := -\int_0^T\int_{\Gamma(t)}(v\,\partial^\bullet\psi + v\psi\,\mathrm{div}_{\Gamma(t)}V_\Gamma)\,d\mathcal{H}^{n-1}\,dt, \quad \varphi\in C_0^1(S_T). \tag{4.14}$$

If $v\in C^1(S_T)$, then its weak material derivative agrees with the strong one. We set

$$\|\partial^\bullet v\|_{H_T} := \sup_{\varphi\in C_0^1(S_T)\setminus\{0\}}\frac{\langle\partial^\bullet v,\varphi\rangle_T}{\|\varphi\|_{H_T}}, \quad v\in H_T.$$

If $\|\partial^\bullet v\|_{H_T}$ is finite for some $v\in H_T$, then $\partial^\bullet v$ can be extended to a bounded linear functional on H_T because $C_0^1(S_T)$ is dense in H_T (see Lemma 4.2). In this case, we say that $\partial^\bullet v$ is in H_T' and we define a function space W_T and its norm as

$$W_T := \{v\in H_T \mid \partial^\bullet v\in H_T'\}, \quad \|v\|_{W_T} := \left(\|v\|_{H_T}^2 + \|\partial^\bullet v\|_{H_T'}^2\right)^{1/2}. \tag{4.15}$$

For $T_1<T_2$, the space W_{T_2} is continuously embedded into W_{T_1} since $C_0^1(S_{T_1})\subset C_0^1(S_{T_2})$ and H_{T_2} is continuously embedded into H_{T_1}.

To investigate the property of W_T, we define an auxiliary Hilbert space and its norm as

$$\widehat{W}_T := \{V\in\widehat{H}_T \mid \partial_tV\in\widehat{H}_T'\}, \quad \|V\|_{\widehat{W}_T} := \left(\|V\|_{\widehat{H}_T}^2 + \|\partial_tV\|_{\widehat{H}_T'}^2\right)^{1/2}.$$

Here $\widehat{H}_T'$ is the dual space of $\widehat{H}_T$ and $\partial_t V$ is the weak time derivative of $V \in \widehat{H}_T$ defined as

$$[\partial_t V, \Psi]_T := -\int_0^T \int_{\Gamma_0} V\, \partial_t \Psi \, d\mathcal{H}^{n-1}\, dt, \quad \Psi \in C_0^1(\Gamma_0 \times (0,T)),$$

and we say $\partial_t V \in \widehat{H}_T'$ if $\|\partial_t V\|_{\widehat{H}_T'} := \sup_{\Psi \in C_0^1(\Gamma_0\times(0,T))\setminus\{0\}} [\partial_t V, \Psi]_T / \|\Psi\|_{\widehat{H}_T}$ is finite.

Lemma 4.3. *The linear mapping L given by* (4.2) *defines a homeomorphism between $\widehat{W}_T$ and W_T.*

Proof. (1) As in the proof of Lemma 3.3 in [18], we need certain integral transformation formulas to prove the lemma. Let $U \subset \mathbb{R}^{n-1}$ be an open set and $\mu_0 \colon U \to \Gamma_0$ be a smooth local parametrization of Γ_0. Then, as in the proof of Lemma 4.1, a mapping $\mu_t \colon U \to \Gamma(t)$ given by $\mu_t(s) := \Phi(\mu_0(s), t)$ is a smooth local parametrization of $\Gamma(t)$ for each $t \in [0,T]$. We define functions Λ and λ as

$$\Lambda(\mu_0(s), t) := \sqrt{\frac{\det g_t(s)}{\det g_0(s)}}, \quad \lambda(\mu_t(s), t) := \sqrt{\frac{\det g_0(s)}{\det g_t(s)}}, \quad (s,t) \in U \times [0,T],$$

where the matrix $g_t = (g_{t,ij})_{ij}$ is given by (4.7). We can show that $\Lambda(\mu_0(s),t)$ and $\lambda(\mu_t(s),t) = \lambda(\Phi(\mu_0(s),t),t)$ are independent of the choice of a local parametrization μ_0. From this fact and the smoothness assumption on Φ, the functions Λ and λ are well-defined and smooth on the compact manifolds $\Gamma_0 \times [0,T]$ and $\overline{S_T}$, respectively. In particular, they are bounded on $\Gamma_0 \times [0,T]$ and $\overline{S_T}$ along with their derivatives.

Using these functions, the local parametrizations $\mu_t^k \colon U_k \to \Gamma(t)$ and the partition of unity $\{\psi_t^k\}_{k=1}^N$ of $\Gamma(t)$ subordinate to the covering $\{\mu_t^k(U_k)\}_{k=1}^N$ given in the proof in Lemma 4.1, we obtain integral transformation formulas

$$\int_{\Gamma(t)} v(y,t)\, d\mathcal{H}^{n-1} = \int_{\Gamma_0} V(Y,t)\Lambda(Y,t)\, d\mathcal{H}^{n-1}, \tag{4.16}$$

$$\int_{\Gamma_0} V(Y,t)\, d\mathcal{H}^{n-1} = \int_{\Gamma(t)} v(y,t)\lambda(y,t)\, d\mathcal{H}^{n-1} \tag{4.17}$$

for all functions V on $\Gamma_0 \times (0,T)$ and all $t \in (0,T)$, where $v := LV$.

(2) Now let us prove the lemma. Let $V \in \widehat{W}_T$ and $v := LV$. Then Lemma 4.1 implies that $v \in H_T$ and $\|v\|_{H_T} \leq c\|V\|_{\widehat{H}_T}$. For every $\psi \in C_0^1(S_T)$, we have $\Psi := L^{-1}\psi \in C_0^1(\Gamma_0 \times (0,T))$ and $\partial^\bullet \psi(y,t) = \partial_t \Psi(Y,t)$ for $y = \Phi(Y,t)$. Thus (4.16) yields

$$\begin{aligned} \langle \partial^\bullet v, \psi \rangle_T &= -\int_0^T \int_{\Gamma(t)} (v\, \partial^\bullet \psi + v\psi \operatorname{div}_{\Gamma(t)} V_\Gamma)\, d\mathcal{H}^{n-1}\, dt \\ &= -\int_0^T \int_{\Gamma_0} (V\, \partial_t \Psi + V\Psi F)\Lambda\, d\mathcal{H}^{n-1}\, dt, \end{aligned}$$

where $F := L^{-1}(\operatorname{div}_{\Gamma(t)} V_\Gamma) \in C^\infty(\Gamma_0 \times [0,T])$. Since $\Psi\Lambda \in C_0^1(\Gamma_0 \times (0,T))$,

$$-\int_0^T \int_{\Gamma_0} V\Lambda\, \partial_t \Psi \, d\mathcal{H}^{n-1}\, dt = [\partial_t V, \Psi\Lambda]_T + \int_0^T \int_{\Gamma_0} V\Psi\, \partial_t \Lambda \, d\mathcal{H}^{n-1}\, dt$$

by the definition of the weak time derivative $\partial_t V$. Thus it follows that

$$\begin{aligned}|\langle\partial^\bullet v,\psi\rangle_T| &= \left|[\partial_t V,\Psi\Lambda]_T+\int_0^T\int_{\Gamma_0}(V\Psi\,\partial_t\Lambda-V\Psi\Lambda F)\,d\mathcal{H}^{n-1}\,dt\right| \\ &\leq c(\|\partial_t V\|_{\widehat{H}_T'}\|\Psi\Lambda\|_{\widehat{H}_T}+\|V\|_{\widehat{H}_T}\|\Psi\|_{\widehat{H}_T})\leq c\|V\|_{\widehat{W}_T}\|\psi\|_{H_T}\end{aligned}$$

with a constant $c>0$ independent of V and ψ, because F and Λ are bounded on $\Gamma_0\times(0,T)$ along with their derivatives. Hence we have $v=LV\in W_T$ and $\|v\|_{W_T}\leq c\|V\|_{\widehat{W}_T}$ for all $V\in\widehat{W}_T$.

Similarly, by (4.17) and the smoothness of λ on $\overline{S_T}$ we can show that $V:=L^{-1}v$ is in $\widehat{W}_T$ and $\|V\|_{\widehat{W}_T}\leq c\|v\|_{W_T}$ holds for all $v\in W_T$. Hence L is a homeomorphism between $\widehat{W}_T$ and W_T. □

Lemma 4.4. *The space W_T is a Hilbert space and $C^1(S_T)$ is dense in W_T. Moreover, the trace operator $v\mapsto v(t)$ from $C^1(S_T)$ into $L^2(\Gamma(t))$ for each $t\in[0,T]$ can be extended to a bounded linear operator from W_T to $L^2(\Gamma(t))$ and there exists a constant $c>0$ such that*

$$\max_{t\in[0,T]}\|v(t)\|_{L^2(\Gamma(t))}\leq c\|v\|_{W_T}$$

for all $v\in W_T$.

Proof. Since $\widehat{W}_T$ is a Hilbert space, Lemma 4.3 implies that W_T is a Hilbert space. For the rest of the proof, see [18, Theorem 3.6]. □

Finally we show an integration by parts formula which we will use in Section 6.

Lemma 4.5. *For all $v_1,v_2\in W_T$, we have*

$$\begin{aligned}\langle\partial^\bullet v_1,v_2\rangle_T+\langle\partial^\bullet v_2,v_1\rangle_T+\int_0^T\int_{\Gamma(t)}v_1v_2\,\mathrm{div}_{\Gamma(t)}V_\Gamma\,d\mathcal{H}^{n-1}\,dt \\ =\int_{\Gamma(T)}v_1(T)v_2(T)\,d\mathcal{H}^{n-1}-\int_{\Gamma_0}v_1(0)v_2(0)\,d\mathcal{H}^{n-1}.\end{aligned}\tag{4.18}$$

Note that, by Lemma 4.4, $v_i(0)$ and $v_i(T)$, $i=1,2$, are well-defined as functions in $L^2(\Gamma_0)$ and $L^2(\Gamma(T))$, respectively.

Proof. For $v\in C^1(S_T)$, its weak material derivative agrees with the strong one. Thus we have

$$\langle\partial^\bullet v,\psi\rangle_T=\int_0^T\int_{\Gamma(t)}(\partial^\bullet v)\psi\,d\mathcal{H}^{n-1}\,dt,\quad \varphi\in C_0^1(S_T).$$

Moreover, since $C_0^1(S_T)$ is dense in H_T (see Lemma 4.2), the above equality holds for all $\psi\in H_T$ and thus (4.18) follows from (4.13) when $v_1,v_2\in C^1(S_T)$. Since $C^1(S_T)$ is dense in W_T (see Lemma 4.3), the density argument shows that (4.18) holds for general $v_1,v_2\in W_T$. □

5. Average operator

5.1. **Definition and basic properties of the average operator.** For $(y,t) \in \overline{S_T}$, let $\kappa_1(y,t),\dots,\kappa_{n-1}(y,t)$ be the principal curvatures of $\Gamma(t)$ at y (see [10, Section 14.6]). Since $\overline{S_T}$ is smooth and compact, $\kappa_1,\dots,\kappa_{n-1}$ are smooth and bounded along their derivatives on $\overline{S_T}$. We define a function J on $\overline{S_T}\times(-\delta,\delta)$ as

$$J(y,t,\rho) := \prod_{i=1}^{n-1}\{1-\rho\kappa_i(y,t)\}, \quad (y,t)\in\overline{S_T},\ \rho\in(-\delta,\delta).$$

Here $\delta>0$ is a half of the width of the tubular neighborhood $N(t)$ of $\Gamma(t)$, which is independent of $t\in[0,T]$ (see Section 2). Since $\kappa_1,\dots,\kappa_{n-1}$ are smooth on $\overline{S_T}$, the function J is smooth on $\overline{S_T}\times(-\delta,\delta)$. By taking δ sufficiently small, we may assume that there is a constant $c>0$ such that

$$c^{-1} \le 1-\rho\kappa_i(y,t) \le c \quad \text{for all} \quad (y,t)\in\overline{S_T},\ \rho\in(-\delta,\delta),\ i=1,\dots,n-1, \tag{5.1}$$

and thus

$$c^{-1} \le J(y,t,\rho) \le c \quad \text{for all} \quad (y,t)\in\overline{S_T},\ \rho\in(-\delta,\delta). \tag{5.2}$$

Moreover, since $J(y,t,\rho)$ is of the form

$$J(y,t,\rho) = 1+\rho\sum_{i=1}^{n-1}\kappa_i(y,t)+\rho^2P(\kappa_1(y,t),\dots,\kappa_{n-1}(y,t),\rho),$$

where $P(z)$ is a polynomial in $z=(z_1,\dots,z_n)\in\mathbb{R}^n$, it follows that

$$|1-J(y,t,\rho)| \le c\varepsilon, \quad |\nabla_{\Gamma(t)}J(y,t,\rho)| \le c\varepsilon, \quad \left|\frac{\partial J}{\partial\rho}(y,t,\rho)\right| \le c \tag{5.3}$$

for all $(y,t)\in\overline{S_T}$ and $\rho\in(\varepsilon g_0(y,t),\varepsilon g_1(y,t))$ with a constant $c>0$ independent of ε. Using the function J, we have the co-area formula of the form (see [10, Section 14.6])

$$\int_{\Omega_\varepsilon(t)} u(x)\,dx = \int_{\Gamma(t)}\int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y+\rho\nu(y,t))J(y,t,\rho)\,d\rho\,d\mathcal{H}^{n-1} \tag{5.4}$$

for a function u on $\Omega_\varepsilon(t)$ with each fixed $t\in[0,T]$. Based on this formula, we define a weighted average operator M_ε as follows.

Definition 5.1. For a function u on $Q_{\varepsilon,T}$, we define its weighted average $M_\varepsilon u$ as

$$M_\varepsilon u(y,t) := \frac{1}{\varepsilon g(y,t)}\int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y+\rho\nu(y,t),t)J(y,t,\rho)\,d\rho, \quad (y,t)\in S_T. \tag{5.5}$$

We use the same notation $M_\varepsilon u$ for the average of a function u on $\Omega_\varepsilon(t)$ with each fixed $t\in[0,T]$.

For simplicity, we identify a function u on $Q_{\varepsilon,T}$ with a function $\underline{u}$ given by

$$\underline{u}(y,t,\rho) := u(y+\rho\nu(y,t),t), \quad (y,t)\in S_T,\ \rho\in(\varepsilon g_0(y,t),\varepsilon g_1(y,t))$$

and use similar identification for functions on $\Omega_\varepsilon(t)$ with each fixed $t\in[0,T]$. Moreover, we omit variables unless we need to specify them. For example, the co-area formula (5.4) is referred to as

$$\int_{\Omega_\varepsilon(t)} u\,dx = \int_{\Gamma(t)}\int_{\varepsilon g_0}^{\varepsilon g_1} uJ\,d\rho\,d\mathcal{H}^{n-1}.$$

Throughout the rest of this subsection and the next subsection, we fix an arbitrary $t \in [0,T]$ and omit it. For example, we refer to $\Gamma(t)$ as Γ. Also, let c denote a general constant independent of t.

Lemma 5.2. *If $v \in L^2(\Gamma)$, then its extension $\overline{v}$ to the normal direction of Γ is in $L^2(\Omega_\varepsilon)$ and*

$$\|\overline{v}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{1/2}\|v\|_{L^2(\Gamma)} \tag{5.6}$$

with a constant $c>0$ independent of ε.

Proof. By the co-area formula (5.4) and (5.2),

$$\begin{aligned}\|\overline{v}\|_{L^2(\Omega_\varepsilon)} &= \left(\int_\Gamma\int_{\varepsilon g_1}^{\varepsilon g_0} |v|^2 J\,d\rho\,d\mathcal{H}^{n-1}\right)^{1/2} \\ &\leq c\left(\int_\Gamma \varepsilon g|v|^2\,d\mathcal{H}^{n-1}\right)^{1/2} \leq c\varepsilon^{1/2}\|v\|_{L^2(\Gamma)}.\end{aligned}$$

□

Lemma 5.3. *If $u \in L^2(\Omega_\varepsilon)$, then $M_\varepsilon u \in L^2(\Gamma)$ and*

$$\|M_\varepsilon u\|_{L^2(\Gamma)} \leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)} \tag{5.7}$$

with a constant $c>0$ independent of ε.

Proof. By Hölder's inequality, (5.2), (2.1), and the co-area formula (5.4),

$$\begin{aligned}\int_\Gamma |M_\varepsilon u|^2\,d\mathcal{H}^{n-1} &\leq \int_\Gamma (\varepsilon g)^{-2}\left(\int_{\varepsilon g_0}^{\varepsilon g_1} J\,d\rho\right)\left(\int_{\varepsilon g_0}^{\varepsilon g_1} |u|^2 J\,d\rho\right)d\mathcal{H}^{n-1} \\ &\leq c\int_\Gamma (\varepsilon g)^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1} |u|^2 J\,d\rho\,d\mathcal{H}^{n-1} \leq c\varepsilon^{-1}\int_{\Omega_\varepsilon} |u|^2\,dx.\end{aligned}$$

Thus (5.7) follows. □

Lemma 5.4. *There exists a constant $c>0$ independent of ε such that*

$$\left\|u - \overline{M_\varepsilon u}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{H^1(\Omega_\varepsilon)} \tag{5.8}$$

for all $u \in H^1(\Omega_\varepsilon)$.

Proof. For $y \in \Gamma$ and $\rho \in (\varepsilon g_0(y), \varepsilon g_1(y))$, we set

$$\begin{aligned} I_1(y,\rho) &= \frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \{u(y,\rho) - u(y,r)\}\,dr, \\ I_2(y) &= \frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} u(y,r)\{1 - J(y,r)\}\,dr.\end{aligned}$$

Then we have $u(y,\rho) - M_\varepsilon u(y) = I_1(y,\rho) + I_2(y)$. Let us estimate I_1 and I_2. Since

$$\begin{aligned}|u(y,\rho) - u(y,r)| &= \left|\int_r^\rho \frac{d}{d\eta}(u(y+\eta\nu(y)))\,d\eta\right| \\ &= \left|\int_r^\rho \nu(y)\cdot\nabla u(y+\eta\nu(y))\,d\eta\right| \leq \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\nabla u(y,\eta)|\,d\eta\end{aligned}$$

for all $\rho, r \in (\varepsilon g_0(y), \varepsilon g_1(y))$ and the right-hand side of the above inequality is independent of r,

$$|I_1(y,\rho)| \leq \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\nabla u(y,\eta)|\,d\eta.$$

On the other hand, by (2.1) and (5.3) we have

$$|I_2(y)| \leq c\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |u(y,r)|\,dr.$$

These inequalities and Hölder's inequality yield

$$\begin{aligned}|u(y,\rho) - M_\varepsilon u(y)| &\leq |I_1(y,\rho)| + |I_2(y)| \leq c\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u(y,r)| + |\nabla u(y,r)|)\,dr \\ &\leq c\left(\varepsilon g(y)\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u(y,r)|^2 + |\nabla u(y,r)|^2)\,dr\right)^{1/2}.\end{aligned}$$

Here the last term is independent of ρ. Hence by the co-area formula (5.4) and (5.2) we obtain

$$\begin{aligned}\left\|u - \overline{M_\varepsilon u}\right\|_{L^2(\Omega_\varepsilon)}^2 &= \int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |u(y,\rho) - M_\varepsilon u(y)|^2 J(y,\rho)\,d\rho\,d\mathcal{H}^{n-1} \\ &\leq c\int_\Gamma \{\varepsilon g(y)\}^2\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u(y,r)|^2 + |\nabla u(y,r)|^2)\,dr\,d\mathcal{H}^{n-1} \\ &\leq c\varepsilon^2\int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u(y,r)|^2 + |\nabla u(y,r)|^2) J(y,r)\,d\rho\,d\mathcal{H}^{n-1} \\ &= c\varepsilon^2\|u\|_{H^1(\Omega_\varepsilon)}^2.\end{aligned}$$

Thus (5.8) follows. □

5.2. **Tangential gradients of the average operator.** Let $\mu\colon U \to \Gamma$ be a local parametrization of Γ with an open set U in $\mathbb{R}^{n-1}$. We set

$$g(s) = (g_{ij}(s))_{i,j} := \left(\frac{\partial \mu}{\partial s_i}(s)\cdot\frac{\partial \mu}{\partial s_j}(s)\right)_{i,j}, \quad s \in U.$$

Then, the tangential gradient of a function v on Γ is locally expressed as

$$\nabla_\Gamma v(y) = \sum_{i,j=1}^{n-1} g^{ij}(s)\frac{\partial \widetilde{v}}{\partial s_j}(s)\frac{\partial \mu}{\partial s_i}(s), \quad y = \mu(s) \in \mu(U), \tag{5.9}$$

where $\widetilde{v}(s) := v(\mu(s))$ and $g^{-1} = (g^{ij})_{i,j}$ denotes the inverse matrix of g. Also, we define a mapping $M\colon U \times (-\delta,\delta) \to N$ as $M(s,\rho) := \mu(s) + \rho\nu(\mu(s))$ for $(s,\rho) \in U \times (-\delta,\delta)$ and set

$$G(s,\rho) = (G_{ij}(s,\rho))_{i,j} := \left(\frac{\partial M}{\partial s_i}(s,\rho)\cdot\frac{\partial M}{\partial s_j}(s,\rho)\right)_{i,j}, \quad (s,\rho) \in U \times (-\delta,\delta),$$

where $s_n := \rho$. Then the gradient (in $\mathbb{R}^n$) of a function u on N is locally expressed as

$$(5.10)\quad \nabla u(x) = \sum_{i,j=1}^{n} G^{ij}(s,\rho)\frac{\partial \widetilde{u}}{\partial s_j}(s,\rho)\frac{\partial M}{\partial s_i}(s,\rho), \qquad x = M(s,\rho) \in M(U\times(-\delta,\delta)),$$

where $\widetilde{u}(s,\rho) := u(M(s,\rho))$ and $G^{-1} = (G^{ij})_{i,j}$ is the inverse matrix of G.

Let v be a function on Γ and $\overline{v}$ be its extension to the normal direction of Γ. Then, their local representations $\widetilde{v} := v\circ\mu$ and $\widetilde{\overline{v}} := \overline{v}\circ M$ satisfy

$$\widetilde{\overline{v}}(s,\rho) = \overline{v}(p(M(s,\rho))) = v(\mu(s)) = \widetilde{v}(s), \quad (s,\rho)\in U\times(-\delta,\delta).$$

Hereafter we use this fact without mention.

Lemma 5.5. *If $v \in H^1(\Gamma)$, then $\overline{v} \in H^1(\Omega_\varepsilon)$ and*

$$(5.11)\quad \|\nabla\overline{v}\|_{L^2(\Omega_\varepsilon)} \le c\varepsilon^{1/2}\|\nabla_\Gamma v\|_{L^2(\Gamma)}, \quad \left\|\nabla\overline{v} - \overline{\nabla_\Gamma v}\right\|_{L^2(\Omega_\varepsilon)} \le c\varepsilon^{3/2}\|\nabla_\Gamma v\|_{L^2(\Gamma)}$$

with a constant $c > 0$ independent of ε.

Proof. Suppose that for all $y \in \Gamma$ and $\rho \in (\varepsilon g_0(y), \varepsilon g_1(y))$ we have

$$(5.12)\quad |\nabla\overline{v}(y,\rho)| \le c|\nabla_\Gamma v(y)|, \quad |\nabla\overline{v}(y,\rho) - \nabla_\Gamma v(y)| \le c\varepsilon|\nabla_\Gamma v(y)|.$$

Then (5.11) follows from the co-area formula (5.4) and (5.2), (5.12) as in the proof of Lemma 5.2. Let us show (5.12). We fix each $y\in\Gamma$ and set $\kappa_i := \kappa_i(y)$ for $i = 1,\dots,n-1$. By a translation and rotation of coordinates, we may assume $y = 0$ and take a local parametrization $\mu\colon U \to \Gamma$ of the form $\mu(s) = (s, f(s))$, where U is an open set in $\mathbb{R}^{n-1}$ containing the origin and $f\colon U \to \mathbb{R}$ is a smooth function satisfying

$$(5.13)\quad f(0) = 0, \quad \nabla' f(0) = 0, \quad (\nabla')^2 f(0) = \mathrm{diag}[\kappa_1,\dots,\kappa_{n-1}],$$

see [10, Section 14.6]. Here $\nabla' f$ is the gradient of f with respect to $s\in\mathbb{R}^{n-1}$ and $(\nabla')^2 f$ is the Hessian matrix of f. In this case, we have $\nu(y) = \nu(\mu(0)) = e_n$ and

$$(5.14)\quad \frac{\partial\mu}{\partial s_i}(0) = e_i, \quad \frac{\partial}{\partial s_i}\Big(\nu(\mu(s))\Big)\Big|_{s=0} = -\kappa_i e_i, \quad i = 1,\dots,n-1,$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. Moreover, the above equality yields

$$(5.15)\quad \frac{\partial M}{\partial s_i}(0,\rho) = (1-\rho\kappa_i)e_i, \quad i = 1,\dots,n-1, \quad \frac{\partial M}{\partial \rho}(0,\rho) = \nu(\mu(0)) = e_n.$$

Hence we have $g(0) = I_n$, $G(0,\rho) = \mathrm{diag}[1-\rho\kappa_1,\dots,1-\rho\kappa_{n-1},1]$, and

$$(5.16)\quad g^{-1}(0) = I_n, \quad G^{-1}(0,\rho) = \mathrm{diag}[(1-\rho\kappa_1)^{-1},\dots,(1-\rho\kappa_{n-1})^{-1},1].$$

Applying (5.14), (5.15) and (5.16) to (5.9) and (5.10) with $u = \overline{v}$, we obtain

$$\nabla_\Gamma v(y) = \sum_{i=1}^{n-1}\frac{\partial\widetilde{v}}{\partial s_i}(0)e_i, \quad \nabla\overline{v}(y,\rho) = \sum_{i-1}^{n-1}(1-\rho\kappa_i)^{-1}\frac{\partial\widetilde{v}}{\partial s_i}(0)e_i$$

and thus (5.1) implies that

$$|\nabla\overline{v}(y,\rho)|^2 = \sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-2}\left(\frac{\partial\widetilde{v}}{\partial s_i}(0)\right)^2 \le c\sum_{i=1}^{n-1}\left(\frac{\partial\widetilde{v}}{\partial s_i}(0)\right)^2 = |\nabla_\Gamma v(y)|^2$$

with a constant $c>0$ independent of y, ρ, and ε. Moreover, by (5.1) we have

$$\begin{aligned}|(1-\rho\kappa_i)^{-1}-1| &= \left|\int_0^\rho \frac{d}{dr}\left((1-r\kappa_i)^{-1}\right)dr\right| \\ &\leq \varepsilon|\kappa_i|\max_{k=0,1}|g_k(y)|\sup_{r\in(-\delta,\delta)}|1-r\kappa_i|^{-2}\leq c\varepsilon\end{aligned}$$

for all $\rho\in(\varepsilon g_0(y),\varepsilon g_1(y))$ and $i=1,\dots,n-1$ with a constant $c>0$ independent of y, ε and thus

$$|\nabla\overline{\varphi}(y,\rho)-\nabla_\Gamma\varphi(y)|^2=\sum_{i=1}^{n-1}\{(1-\rho\kappa_i)^{-1}-1\}^2\left(\frac{\partial\widetilde{v}}{\partial s_i}(0)\right)^2\leq c\varepsilon^2|\nabla_\Gamma\varphi(y)|^2.$$

Hence (5.12) follows and the proof is complete. □

Let u be a function on Ω_ε and $M_\varepsilon u$ be its wighted average. For an open set $U\subset\mathbb{R}^{n-1}$ and a local parametrization $\mu\colon U\to\Gamma$, the local representation of $M_\varepsilon u$ is given by

$$\widetilde{M_\varepsilon u}(s)=\frac{1}{\varepsilon\widetilde{g}(s)}\int_{\varepsilon\widetilde{g}_0(s)}^{\varepsilon\widetilde{g}_1(s)}\widetilde{u}(s,\rho)\widetilde{J}(s,\rho)\,d\rho,\quad s\in U, \tag{5.17}$$

where $\widetilde{M_\varepsilon u}(s)=M_\varepsilon u(\mu(s))$, $\widetilde{u}(s,\rho)=u(M(s,\rho))$, and

$$\widetilde{J}(s,\rho):=J(\mu(s),\rho)=\prod_{i=1}^{n-1}\{1-\rho\kappa_i(\mu(s))\}. \tag{5.18}$$

Let us calculate the derivatives of $\widetilde{M_\varepsilon u}$.

Lemma 5.6. *Let $u\in H^1(\Omega_\varepsilon)$. Then*

$$\begin{aligned}\frac{\partial\widetilde{M_\varepsilon u}}{\partial s_i}(s) &= \frac{1}{\varepsilon\widetilde{g}(s)}\int_{\varepsilon\widetilde{g}_0(s)}^{\varepsilon\widetilde{g}_1(s)}\left\{\frac{\partial\widetilde{u}}{\partial s_i}(s,\rho)\widetilde{J}(s,\rho)+\widetilde{u}(s,\rho)\frac{\partial\widetilde{J}}{\partial s_i}(s,\rho)\right\}d\rho \\ &\quad+\frac{1}{\varepsilon\widetilde{g}(s)}\int_{\varepsilon\widetilde{g}_0(s)}^{\varepsilon\widetilde{g}_1(s)}\left\{\frac{\partial\widetilde{u}}{\partial\rho}(s,\rho)\widetilde{J}(s,\rho)+\widetilde{u}(s,\rho)\frac{\partial\widetilde{J}}{\partial\rho}(s,\rho)\right\}\chi_i(s,\rho)\,d\rho\end{aligned} \tag{5.19}$$

for all $s\in U$ and $i=1,\dots,n-1$, where

$$\chi_i(s,\rho):=\frac{1}{\widetilde{g}(s)}\left\{(\rho-\varepsilon\widetilde{g}_0(s))\frac{\partial\widetilde{g}_1}{\partial s_i}(s)+(\varepsilon\widetilde{g}_1(s)-\rho)\frac{\partial\widetilde{g}_0}{\partial s_i}(s)\right\}. \tag{5.20}$$

Proof. For simplicity, we set $\partial_i=\partial/\partial s_i$ and $\partial_\rho=\partial/\partial\rho$. For each $i=1,\dots,n-1$, we differentiate both sides of (5.17) with respect to s_i to get

$$\partial_i\widetilde{M_\varepsilon u}=\frac{I}{\varepsilon\widetilde{g}}-\frac{\partial_i\widetilde{g}}{\varepsilon(\widetilde{g})^2}\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{u}\widetilde{J}\,d\rho+\frac{1}{\varepsilon\widetilde{g}}\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\{(\partial_i\widetilde{u})\widetilde{J}+\widetilde{u}(\partial_i\widetilde{J})\}\,d\rho, \tag{5.21}$$

where $I=I(s)$ is given by

$$I(s):=\varepsilon\partial_i\widetilde{g}_1(s)\widetilde{u}(s,\varepsilon\widetilde{g}_1(s))\widetilde{J}(s,\varepsilon\widetilde{g}_1(s))-\varepsilon\partial_i\widetilde{g}_0(s)\widetilde{u}(s,\varepsilon\widetilde{g}_0(s))\widetilde{J}(s,\varepsilon\widetilde{g}_0(s)).$$

Since $I=\left[\widetilde{u}(\rho)\widetilde{J}(\rho)\chi_i(\rho)\right]_{\rho=\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}=\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\partial_\rho(\widetilde{u}\widetilde{J}\chi_i)\,d\rho$ and $\partial_\rho\chi_i=\partial_i\widetilde{g}/\widetilde{g}$, we have

$$\frac{I}{\varepsilon\widetilde{g}}=\frac{\partial_i\widetilde{g}}{\varepsilon(\widetilde{g})^2}\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{u}\widetilde{J}\,d\rho+\frac{1}{\varepsilon\widetilde{g}}\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\{(\partial_\rho\widetilde{u})\widetilde{J}+\widetilde{u}(\partial_\rho\widetilde{J})\}\chi_i\,d\rho. \tag{5.22}$$

Substituting (5.22) for (5.21), we obtain (5.19). □

Using the above lemma, we approximate an H^1-bilinear form on Ω_ε by that on Γ with the tangential gradient of the average operator.

Lemma 5.7. *For $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, let*

$$I_\varepsilon^1(u,\varphi) := \int_{\Omega_\varepsilon} \nabla u \cdot \nabla\overline{\varphi}\,dx - \varepsilon\int_\Gamma g\nabla_\Gamma M_\varepsilon u \cdot \nabla_\Gamma\varphi\,d\mathcal{H}^{n-1}. \tag{5.23}$$

Then, there exists a constant $c > 0$ independent of u, φ and ε such that

$$|I_\varepsilon^1(u,\varphi)| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma\varphi\|_{L^2(\Gamma)}. \tag{5.24}$$

Remark 5.8. The bilinear form $I_\varepsilon^1(u,\varphi)$ is well-defined for $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, since $\overline{\varphi} \in H^1(\Omega_\varepsilon)$ by Lemma 5.5 and $M_\varepsilon u$ is smooth on Γ and thus in $H^1(\Gamma)$ by the compactness of Γ. We will observe later that $I_\varepsilon^1(u,\varphi)$ is well-defined and (5.24) holds for all $u \in H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, see Remark 5.10.

Proof. As in the proof of Lemma 5.6, we set $\partial_i = \partial/\partial s_i$ and $\partial_\rho = \partial/\partial\rho$. For $y \in \Gamma$, we set

$$I(y) := \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \nabla u(y,\rho)\cdot\nabla\overline{\varphi}(y,\rho)J(y,\rho)\,d\rho - \varepsilon g(y)\nabla_\Gamma M_\varepsilon u(y)\cdot\nabla_\Gamma\varphi(y).$$

Suppose that there is a constant $c > 0$ independent of ε such that

$$|I(y)| \leq c\varepsilon|\nabla_\Gamma\varphi(y)|\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u(y,\rho)| + |\nabla u(y,\rho)|)\,d\rho \tag{5.25}$$

for all $y \in \Gamma$. Then, by the co-area formula (5.4), (5.25), Hölder's inequality, and (5.2) we have

$$\begin{aligned}
|I_\varepsilon^1(u,\varphi)| &= \left|\int_\Gamma I(y)\,d\mathcal{H}^{n-1}\right| \leq c\varepsilon\int_\Gamma |\nabla_\Gamma\varphi|\int_{\varepsilon g_0}^{\varepsilon g_1}(|u| + |\nabla u|)\,d\rho\,d\mathcal{H}^{n-1} \\
&\leq c\varepsilon\left(\int_\Gamma |\nabla_\Gamma\varphi|^2\,d\mathcal{H}^{n-1}\right)^{1/2}\left\{\int_\Gamma\left(\int_{\varepsilon g_0}^{\varepsilon g_1}(|u| + |\nabla u|)\,d\rho\right)^2 d\mathcal{H}^{n-1}\right\}^{1/2} \\
&\leq c\varepsilon\|\nabla_\Gamma\varphi\|_{L^2(\Gamma)}\left(\int_\Gamma \varepsilon g\int_{\varepsilon g_0}^{\varepsilon g_1}(|u|^2 + |\nabla u|^2)J\,d\rho\,d\mathcal{H}^{n-1}\right)^{1/2} \\
&\leq c\varepsilon^{3/2}\|\nabla_\Gamma\varphi\|_{L^2(\Gamma)}\|u\|_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

Thus (5.24) holds. Let us prove (5.25). As in the proof of Lemma 5.5, we fix each $y \in \Gamma$. By a translation and rotation of coordinates, we may assume $y = 0$ and take a local parametrization $\mu\colon U \to \Gamma$ of the form $\mu(s) = (s, f(s))$ with an open set $U \subset \mathbb{R}^{n-1}$ containing the origin and a smooth function $f\colon U \to \mathbb{R}$ satisfying (5.13). Then, by (5.14), (5.15), and (5.16) we have

$$\nabla u(y,\rho) = \sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-1}\partial_i\widetilde{u}(0,\rho)e_i + \partial_\rho\widetilde{u}(0,\rho)e_n,$$

$$\nabla_\Gamma M_\varepsilon u(y) = \sum_{i=1}^{n-1}\partial_i\widetilde{M_\varepsilon u}(0)e_i,$$

$$\nabla\overline{\varphi}(y,\rho) = \sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-1}\partial_i\widetilde{\varphi}(0)e_i, \quad \nabla_\Gamma\varphi(y) = \sum_{i=1}^{n-1}\partial_i\widetilde{\varphi}(0)e_i,$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$ and $\kappa_i := \kappa_i(y)$, $i = 1, \dots, n-1$. Hereafter we omit the variables ρ and $s = 0$ unless we need to specify them. The above equality yields

$$\nabla u(y,\rho) \cdot \nabla \overline{\varphi}(y,\rho) = \sum_{i=1}^{n-1} (1-\rho\kappa_i)^{-2} \partial_i \widetilde{u}\, \partial_i \widetilde{\varphi}, \tag{5.26}$$

$$\varepsilon g(y) \nabla_\Gamma M_\varepsilon u(y) \cdot \nabla_\Gamma \varphi(y) = \sum_{i=1}^{n-1} \varepsilon \widetilde{g} \Big(\partial_i \widetilde{M_\varepsilon u}\Big) \partial_i \widetilde{\varphi}.$$

Moreover, (5.19) implies that

$$\varepsilon \widetilde{g} \Big(\partial_i \widetilde{M_\varepsilon u}\Big) = \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \{(\partial_i \widetilde{u})\widetilde{J} + \widetilde{u}(\partial_i \widetilde{J}) + (\partial_\rho \widetilde{u})\widetilde{J}\chi_i + \widetilde{u}(\partial_\rho \widetilde{J})\chi_i\}\, d\rho,$$

where χ_i is given by (5.20), and thus

$$\begin{aligned}\varepsilon g(y) \nabla_\Gamma M_\varepsilon u(y) \cdot \nabla_\Gamma \varphi(y) &= \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \widetilde{J} \sum_{i=1}^{n-1} \partial_i \widetilde{u}\, \partial_i \widetilde{\varphi}\, d\rho \\ &\quad + \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \widetilde{u} \sum_{i=1}^{n-1} \partial_i \widetilde{J}\, \partial_i \widetilde{\varphi}\, d\rho + \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \{(\partial_\rho \widetilde{u})\widetilde{J} + \widetilde{u}(\partial_\rho \widetilde{J})\} \sum_{i=1}^{n-1} \chi_i\, \partial_i \widetilde{\varphi}\, d\rho.\end{aligned}$$

From this equality and (5.26), we obtain $I(y) = I_1 + I_2 + I_3$ with

$$I_1 = \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \widetilde{J} \sum_{i=1}^{n-1} \{(1-\rho\kappa_i)^{-2} - 1\} \partial_i \widetilde{u}\, \partial_i \widetilde{\varphi}\, d\rho,$$

$$I_2 = \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \widetilde{u} \sum_{i=1}^{n-1} \partial_i \widetilde{J}\, \partial_i \widetilde{\varphi}\, d\rho, \quad I_3 = \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \{(\partial_\rho \widetilde{u})\widetilde{J} + \widetilde{u}(\partial_\rho \widetilde{J})\} \sum_{i=1}^{n-1} \chi_i\, \partial_i \widetilde{\varphi}\, d\rho.$$

Let us estimate these integrals. By the definition of $\widetilde{J}$ (see (5.18)), we have

$$\nabla_\Gamma J(y,\rho) = \sum_{i=1}^{n-1} \partial_i \widetilde{J}(0,\rho) e_i, \quad \sum_{i=1}^{n-1} \partial_i \widetilde{J}(0,\rho) \partial_i \widetilde{\varphi}(0) = \nabla_\Gamma J(y,\rho) \cdot \nabla_\Gamma \varphi(y).$$

Hence I_2 is of the form

$$I_2 = \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} u(y,\rho) \nabla_\Gamma J(y,\rho) \cdot \nabla_\Gamma \varphi(y)\, d\rho$$

and by applying (5.3) to the right-hand side we obtain

$$|I_2| \le c\varepsilon |\nabla_\Gamma \varphi(y)| \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |u(y,\rho)|\, d\rho. \tag{5.27}$$

Next we estimate I_3. By the definition of $\widetilde{u}$, $\widetilde{J}$, and χ_i (see (5.18) and (5.20)),

$$\partial_\rho \widetilde{u}(0,\rho) = \nu(y) \cdot \nabla u(y,\rho), \quad \partial_\rho \widetilde{J}(0,\rho) = \partial_\rho J(y,\rho),$$

$$\sum_{i=1}^{n-1} \chi_i(0,\rho) \partial_i \widetilde{\varphi}(0) = \frac{1}{g(y)} \{(\rho - \varepsilon g_0(y)) \nabla_\Gamma g_1(y) + (\varepsilon g_1(y) - \rho) \nabla_\Gamma g_0(y)\} \cdot \nabla_\Gamma \varphi(y).$$

Hence I_3 is of the form

$$I_3 = \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \chi_\varepsilon(y,\rho)\cdot\nabla_\Gamma\varphi(y)\{\nu(y)\cdot\nabla u(y,\rho)J(y,\rho)+u(y,\rho)\partial_\rho J(y,\rho)\}\,d\rho,$$

where $\chi_\varepsilon(y,\rho) := \{(\rho-\varepsilon g_0(y))\nabla_\Gamma g_1(y)+(\varepsilon g_1(y)-\rho)\nabla_\Gamma g_0(y)\}/g(y)$. Since $\nabla_\Gamma g_0$, $\nabla_\Gamma g_1$ are bounded and $g_1 - g_0 = g$, there is a constant $c>0$ independent of y and ε such that

$$|\chi_\varepsilon(y,\rho)| \le \frac{|\nabla_\Gamma g_0(y)|+|\nabla_\Gamma g_1(y)|}{g(y)}\{(\rho-\varepsilon g_0(y))+(\varepsilon g_1(y)-\rho)\} \le c\varepsilon$$

for all $\rho\in(\varepsilon g_0(y),\varepsilon g_1(y))$. From this inequality and (5.2), (5.3), we obtain

$$|I_3| \le c\varepsilon|\nabla_\Gamma\varphi(y)|\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}(|u(y,\rho)|+|\nabla u(y,\rho)|)\,d\rho. \tag{5.28}$$

Let us estimate I_1. For all $\rho\in(\varepsilon g_0(y),\varepsilon g_1(y))$ and $i=1,\dots,n-1$, we have

$$\begin{aligned}|(1-\rho\kappa_i)^{-2}-1| &= \left|\int_0^\rho \frac{d}{dr}((1-r\kappa_i)^{-2})\,dr\right| \\ &\le 2\varepsilon|\kappa_i|\max_{k=0,1}|g_k(y)|\sup_{r\in(-\delta,\delta)}|1-r\kappa_i|^{-3} \le c\varepsilon\end{aligned}$$

with a constant $c>0$ independent of y and ε. Here the last inequality follows from (5.1). This inequality together with Hölder's inequality and (5.1) implies that

$$\begin{aligned}\left|\sum_{i=1}^{n-1}\{(1-\rho\kappa_i)^{-2}-1\}\partial_i\tilde{u}\,\partial_i\tilde{\varphi}\right| &\le c\varepsilon\left(\sum_{i=1}^{n-1}(\partial_i\tilde{u})^2\right)^{1/2}\left(\sum_{i=1}^{n-1}(\partial_i\tilde{\varphi})^2\right)^{1/2} \\ &\le c\varepsilon\left(\sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-2}(\partial_i\tilde{u})^2\right)^{1/2}\left(\sum_{i=1}^{n-1}(\partial_i\tilde{\varphi})^2\right)^{1/2} \le c\varepsilon|\nabla u(y,\rho)||\nabla_\Gamma\varphi(y)|\end{aligned}$$

with some constant $c>0$ independent of y, ρ, and ε. Hence we obtain

$$\begin{aligned}|I_1| &\le c\varepsilon|\nabla_\Gamma\varphi(y)|\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}|\nabla u(y,\rho)|J\,d\rho \\ &\le c\varepsilon|\nabla_\Gamma\varphi(y)|\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}|\nabla u(y,\rho)|\,d\rho,\end{aligned} \tag{5.29}$$

where we used (5.2) in the second inequality. Finally, the inequality (5.25) follows from (5.27), (5.28), and (5.29). The proof is complete. □

The above lemma gives an estimate for the $L^2(\Gamma)$-norm of $\nabla_\Gamma M_\varepsilon u$.

Lemma 5.9. *If $u\in H^1(\Omega_\varepsilon)$, then $M_\varepsilon u\in H^1(\Gamma)$ and*

$$\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)} \le c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)} \tag{5.30}$$

with a constant $c>0$ independent of ε.

Proof. First, we show (5.30) for all $u\in C^\infty(\Omega_\varepsilon)\cap H^1(\Omega_\varepsilon)$. For such u, its average $M_\varepsilon u$ is smooth on Γ and thus in $H^1(\Gamma)$ by the compactness of Γ. We substitute

$M_\varepsilon u$ for φ in (5.23), (5.24) to get

$$\int_\Gamma g|\nabla_\Gamma M_\varepsilon u|^2\,d\mathcal{H}^{n-1} = \varepsilon^{-1}\left(\int_{\Omega_\varepsilon} \nabla u\cdot\nabla\overline{M_\varepsilon u}\,dx - I_\varepsilon^1(u, M_\varepsilon u)\right),$$
$$|I_\varepsilon^1(u, M_\varepsilon u)| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}.$$

Hence, by (2.1), Hölder's inequality and (5.11) we obtain

$$\begin{aligned}
\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}^2 &\leq \int_\Gamma g|\nabla_\Gamma M_\varepsilon u|^2\,d\mathcal{H}^{n-1}\\
&\leq \varepsilon^{-1}\left(\|\nabla u\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\overline{M_\varepsilon u}\right\|_{L^2(\Omega_\varepsilon)} + |I_\varepsilon^1(u, M_\varepsilon u)|\right)\\
&\leq c\varepsilon^{-1}(\varepsilon^{1/2}+\varepsilon^{3/2})\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}\\
&\leq c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}
\end{aligned}$$

and thus (5.30) follows when $u\in C^\infty(\Omega_\varepsilon)\cap H^1(\Omega_\varepsilon)$. Since Ω_ε is bounded, $C^\infty(\Omega_\varepsilon)\cap H^1(\Omega_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$, see [9, Section 5.3.2] for the proof. Hence the density argument together with Lemma 5.3 yields that $M_\varepsilon u\in H^1(\Gamma)$ and (5.30) holds for all $u\in H^1(\Omega_\varepsilon)$. □

Remark 5.10. By Lemma 5.5 and Lemma 5.9, the bilinear form $I_\varepsilon^1(u,\varphi)$ given by (5.23) is well-defined for all $u\in H^1(\Omega_\varepsilon)$ and $\varphi\in H^1(\Gamma)$. Moreover, since $C^\infty(\Omega_\varepsilon)\cap H^1(\Omega_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$, the density argument implies that (5.24) also holds for all $u\in H^1(\Omega_\varepsilon)$ and $\varphi\in H^1(\Gamma)$.

5.3. **Material derivatives of the average operator.** Now let us return to the evolving surface $\Gamma(t)$. Recall the function spaces $L^2_{H^1(\varepsilon)}$ and H_T given by (3.1) and (4.1), respectively. By the results in the previous two subsections, we can show that the average operator M_ε defines a bounded linear operator from $L_{H^1(\varepsilon)}$ into H_T.

Lemma 5.11. *If $u\in L^2_{H^1(\varepsilon)}$, then $M_\varepsilon u\in H_T$ and*

$$\|M_\varepsilon u\|_{H_T}\leq c\varepsilon^{-1/2}\|u\|_{L^2_{H^1(\varepsilon)}} \tag{5.31}$$

with a constant $c>0$ independent of ε.

Proof. Let $u\in L^2_{H^1(\varepsilon)}$. By (5.7) and (5.30), there is a constant independent of ε such that

$$\|M_\varepsilon u(t)\|_{L^2(\Gamma(t))}\leq c\varepsilon^{-1/2}\|u(t)\|_{L^2(\Omega_\varepsilon(t))},$$
$$\|\nabla_{\Gamma(t)}M_\varepsilon u(t)\|_{L^2(\Gamma(t))}\leq c\varepsilon^{-1/2}\|u(t)\|_{H^1(\Omega_\varepsilon(t))}$$

for all $t\in(0,T)$. Hence (5.31) follows. □

By the above lemma, we can consider the weak material derivative of $M_\varepsilon u\in H_T$ for $u\in L^2_{H^1(\varepsilon)}$.

Lemma 5.12. *Let $\varphi\in C^1(S_T)$ and $\overline{\varphi}$ be its extension to the normal direction of $\Gamma(t)$. Then*

$$\partial^\bullet\varphi(p(x,t),t) = \partial_t\overline{\varphi}(x,t) + \{V_\Gamma(p(x,t),t)+a(x,t)\}\cdot\nabla\overline{\varphi}(x,t) \tag{5.32}$$

holds for all $(x,t)\in N_T$ *with a vector field* $a\colon N_T\to\mathbb{R}^n$ *given by*

$$a(x,t) := d(x,t)\{\partial_t\nu(p(x,t),t)+\nabla\nu(p(x,t),t)V_\Gamma(p(x,t),t)\}. \tag{5.33}$$

Here $\nabla\nu := (\partial\nu_i/\partial x_j)_{i,j}$ *is the gradient matrix of* ν.

Proof. We define a mapping Ψ as

$$\Psi(X,t) := \Phi(p(X,0),t)+d(X,0)\nu(\Phi(p(X,0),t),t) \tag{5.34}$$

for $(X,t)\in N(0)\times(0,T)$. For each $t\in[0,T]$, since every $x\in N(t)$ is represented as $x=y+\rho\nu(y,t)$ with unique $y=p(x,t)\in\Gamma(t)$ and $\rho=d(x,t)\in(-\delta,\delta)$, where $\delta>0$ is a constant independent of t, the mapping $\Psi(\cdot,t)\colon N(0)\to N(t)$ is a bijection whose inverse $\Phi^{-1}(\cdot,t)\colon N(t)\to N(0)$ is given by

$$\Psi^{-1}(x,t) := \Phi^{-1}(p(x,t),t)+d(x,t)\nu(\Phi^{-1}(p(x,t),t),0), \quad (x,t)\in N_T. \tag{5.35}$$

Let $\varphi\in C^1(S_T)$. Then it follows from (5.34) and the definition of $\overline{\varphi}$ that

$$\overline{\varphi}(\Psi(X,t),t)=\varphi(p(\Psi(X,t),t),t)=\varphi(\Phi(p(X,0),t),t), \quad (X,t)\in N(0)\times(0,T).$$

We differentiate both sides of the above equality with respect to t. By the definition of the material derivative of $\varphi\in C^1(S_T)$ (see (4.12)), we have

$$\frac{d}{dt}(\varphi(\Phi(p(X,0),t),t)) = \partial^\bullet\varphi(\Phi(p(X,0),t),t).$$

On the other hand, from (5.34) we have $\partial_t\Psi(X,t)=\partial_t\Phi(p(X,0),t)+A(X,t)$, where

$$A(X,t) := d(X,0)\{\partial_t\nu(\Phi(p(X,0),t),t)+\nabla\nu(\Phi(p(X,0),t),t)\partial_t\Phi(p(X,0),t)\}.$$

Hence we obtain

$$\partial^\bullet\varphi(\Phi(p(X,0),t),t) = \partial_t\overline{\varphi}(\Psi(X,t),t)+\{\partial_t\Phi(p(X,0),t)+A(X,t)\}\cdot\nabla\overline{\varphi}(\Psi(X,t),t).$$

Moreover, since $\Phi(p(X,0),t)=p(\Psi(X,t),t)$ for all $(X,t)\in N(0)\times(0,T)$ by (5.34), and $\partial_t\Phi(Y,t)=V_\Gamma(\Phi(Y,t),t)$ for all $(Y,t)\in\Gamma_0\times(0,T)$, it follows that

$$\partial^\bullet\varphi(p(\Psi(X,t),t),t) = \partial_t\overline{\varphi}(\Psi(X,t),t)+\{V_\Gamma(p(\Psi(X,t),t),t)+A(X,t)\}\cdot\nabla\overline{\varphi}(\Psi(X,t),t)$$

for all $(X,t)\in N(0)\times(0,T)$. The above equality implies that

$$\partial^\bullet\varphi(p(x,t),t)=\partial_t\overline{\varphi}(x,t)+\{V_\Gamma(p(x,t),t)+A(\Psi^{-1}(x,t),t)\}\cdot\nabla\overline{\varphi}(x,t)$$

for all $(x,t)\in N_T$, since $\Psi(\cdot,t)\colon N(0)\to N(t)$ is a bijection for each $t\in[0,T]$. Here the vector field $A(\Psi^{-1}(x,t),t)$ is of the form

$$A(\Psi^{-1}(x,t),t)=d(\Psi^{-1}(x,t),0)\{\partial_t\nu(p(x,t),t)+\nabla\nu(p(x,t),t)V_\Gamma(p(x,t),t)\}.$$

This is exactly the vector field $a(x,t)$ given by (5.33) since $d(\Psi^{-1}(x,t),0)=d(x,t)$ holds by (5.35). Hence (5.32) follows. □

Remark 5.13. Let $\varphi\in C^1(S_T)$. Since $p(y,t)=y$ and $d(y,t)=0$ for all $(y,t)\in S_T$, we have

$$\partial^\bullet\varphi=\partial_t\overline{\varphi}+V_\Gamma\cdot\nabla\overline{\varphi}=\partial_t\overline{\varphi}+V_\Gamma^N\nu\cdot\nabla\overline{\varphi}+V_\Gamma^T\cdot\nabla_{\Gamma(t)}\overline{\varphi} \quad \text{on} \quad S_T$$

by Lemma 5.12. Here the last equality follows from the fact that V_Γ^T is tangent to $\Gamma(t)$. Based on this equality, the material derivative operator acting on functions on $\Gamma(t)$ is formally represented as $\partial^\bullet=\partial_t+V_\Gamma\cdot\nabla=\partial_t+V_\Gamma^N\nu\cdot\nabla+V_\Gamma^T\cdot\nabla_{\Gamma(t)}$.

Using the above lemma, we derive an integral formula related to the weak time derivative of a function $u \in L^2_{H^1(\varepsilon)}$ and the weak material derivative of its average $M_\varepsilon u \in H_T$.

Lemma 5.14. *Let $u \in L^2_{H^1(\varepsilon)}$ and $\varphi \in C^1_0(S_T)$. Then we have*

$$\int_0^T \int_{\Omega_\varepsilon(t)} u\, \partial_t \overline{\varphi}\, dx\, dt = - \varepsilon \langle \partial^\bullet M_\varepsilon u, g\varphi \rangle_T - \varepsilon \int_0^T \int_{\Gamma(t)} (\partial^\bullet g + g\, \mathrm{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u)\varphi\, d\mathcal{H}^{n-1}\, dt - \varepsilon \int_0^T \int_{\Gamma(t)} g (M_\varepsilon u) V_\Gamma^T \cdot \nabla_{\Gamma(t)} \varphi\, d\mathcal{H}^{n-1}\, dt + I^2_\varepsilon(u, \varphi; T). \tag{5.36}$$

Here $I^2_\varepsilon(u,\varphi;T)$ is a residual term that satisfies

$$|I^2_\varepsilon(u, \varphi; T)| \le c\varepsilon^{3/2} \int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))} \|\nabla_{\Gamma(t)} \varphi(t)\|_{L^2(\Gamma(t))}\, dt \tag{5.37}$$

with a constant $c > 0$ independent of u, φ, and ε.

Note that the tangential velocity V_Γ^T appears instead of the total velocity V_Γ in the third term of the right-hand side of (5.36), see Remark 5.15 below.

Proof. By (5.32), we have $\overline{\partial^\bullet \varphi} = \partial_t \overline{\varphi} + (\overline{V_\Gamma} + a) \cdot \nabla \overline{\varphi}$ on N_T, where a is the vector field on N_T given by (5.33). Hence if we set

$$I^2_\varepsilon(u, \varphi; T) := - \int_0^T \int_{\Omega_\varepsilon(t)} u \left\{ a \cdot \nabla \overline{\varphi} + \overline{V_\Gamma} \cdot \left(\nabla \overline{\varphi} - \overline{\nabla_{\Gamma(t)} \varphi} \right) \right\} dx\, dt,$$

then we have

$$\int_0^T \int_{\Omega_\varepsilon(t)} u\, \partial_t \overline{\varphi}\, dx\, dt = \int_0^T \int_{\Omega_\varepsilon(t)} u \left(\overline{\partial^\bullet \varphi} - \overline{V_\Gamma} \cdot \overline{\nabla_{\Gamma(t)} \varphi} \right) dx\, dt + I^2_\varepsilon(u, \varphi; T). \tag{5.38}$$

Let us compute the first term of the right-hand side of (5.38). From the co-area formula (5.4) and the definition of the weighted average $M_\varepsilon u$ (see (5.5)),

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon(t)} u(x,t) \overline{\partial^\bullet \varphi}(x,t)\, dx\, dt \\ &\qquad = \int_0^T \int_{\Gamma(t)} \int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y,t,\rho) \partial^\bullet \varphi(y,t) J(y,t,\rho)\, d\rho\, d\mathcal{H}^{n-1}\, dt \\ &\qquad = \varepsilon \int_0^T \int_{\Gamma(t)} g(y,t) M_\varepsilon u(y,t) \partial^\bullet \varphi(y,t)\, d\mathcal{H}^{n-1}\, dt. \end{aligned}$$

On the other hand, by the definition of the weak material derivative of $M_\varepsilon u \in H_T$ (see (4.14)),

$$\begin{aligned}\langle \partial^\bullet M_\varepsilon u, g\varphi\rangle_T &= -\int_0^T\int_{\Gamma(t)} \{(M_\varepsilon u)\partial^\bullet(g\varphi) + (M_\varepsilon u)g\varphi\,\mathrm{div}_{\Gamma(t)}V_\Gamma\}\,d\mathcal{H}^{n-1}\,dt \\ &= -\int_0^T\int_{\Gamma(t)} \{(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)(M_\varepsilon u)\varphi + g(M_\varepsilon u)\partial^\bullet\varphi\}\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

Thus it follows that

$$\begin{aligned}(5.39)\quad \int_0^T\int_{\Omega_\varepsilon(t)} u\,\overline{\partial^\bullet\varphi}\,dx\,dt &= \\ -\varepsilon\langle \partial^\bullet M_\varepsilon u, g\varphi\rangle_T &- \varepsilon\int_0^T\int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)(M_\varepsilon u)\varphi\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

Next, since V_Γ is of the form $V_\Gamma = V_\Gamma^N\nu + V_\Gamma^T$ and $\nabla_{\Gamma(t)}\varphi$ is tangent to $\Gamma(t)$, we have $V_\Gamma\cdot\nabla_{\Gamma(t)}\varphi = V_\Gamma^T\cdot\nabla_{\Gamma(t)}\varphi$ on S_T. This equality together with the co-area formula (5.4) yields

$$\begin{aligned}&\int_{\Omega_\varepsilon(t)} u(x,t)\overline{V_\Gamma}(x,t)\cdot\overline{\nabla_{\Gamma(t)}\varphi}(x,t)\,dx \\ &\qquad= \int_{\Gamma(t)}\int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y,t,\rho)V_\Gamma(y,t)\cdot\nabla_{\Gamma(t)}\varphi(y,t)J(y,t,\rho)\,d\rho\,d\mathcal{H}^{n-1} \\ &\qquad= \varepsilon\int_{\Gamma(t)} g(y,t)M_\varepsilon u(y,t)V_\Gamma^T(y,t)\cdot\nabla_{\Gamma(t)}\varphi(y,t)\,d\mathcal{H}^{n-1}\end{aligned}$$

for all $t\in(0,T)$ and thus

$$(5.40)\quad \int_0^T\int_{\Omega_\varepsilon(t)} u\Big(\overline{V_\Gamma}\cdot\overline{\nabla_{\Gamma(t)}\varphi}\Big)\,dx\,dt = \varepsilon\int_0^T\int_{\Gamma(t)} g(M_\varepsilon u)V_\Gamma^T\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt.$$

Substituting (5.39) and (5.40) for (5.38), we obtain the equality (5.36).

Let us show the inequality (5.37). In (5.33), the first-order derivatives of ν are bounded on N_T and V_Γ is bounded on S_T. Hence there is a constant $c>0$ independent of ε such that

$$|a(x,t)| \le c|d(x,t)| \le c\varepsilon \max_{i=1,2}\sup_{(y,t)\in\overline{S_T}}|g_i(y,t)| \le c\varepsilon$$

for all $(x,t)\in Q_{\varepsilon,T}$. By this inequality, Hölder's inequality, and (5.11) we obtain

$$\begin{aligned}&|I_\varepsilon^2(u,\varphi;T)| \\ &\le c\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\left(\varepsilon\|\nabla\overline{\varphi}(t)\|_{L^2(\Omega_\varepsilon(t))} + \left\|\nabla\overline{\varphi}(t) - \overline{\nabla_{\Gamma(t)}\varphi(t)}\right\|_{L^2(\Omega_\varepsilon(t))}\right)dt \\ &\le c\varepsilon^{3/2}\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt.\end{aligned}$$

Thus (5.37) holds. $\square$

Remark 5.15. Let $\Gamma\subset\mathbb{R}^n$ be a closed, compact, connected, and oriented smooth hypersurface. Then, since $\partial\Gamma=\emptyset$, the integral formula (see [24, Section 7.2])

$$\int_\Gamma \mathrm{div}_\Gamma V\,d\mathcal{H}^{n-1}=-\int_\Gamma (V\cdot\nu)H\,d\mathcal{H}^{n-1}$$

holds for smooth vector fields $V\colon\Gamma\to\mathbb{R}^n$. Here ν is the unit outward normal vector of Γ and $H:=-\mathrm{div}_\Gamma\nu$ is the mean curvature of Γ. This formula yields the equality

$$\int_\Gamma V\cdot\nabla_\Gamma\varphi\,d\mathcal{H}^{n-1}=-\int_\Gamma\{\mathrm{div}_\Gamma V+(V\cdot\nu)H\}\varphi\,d\mathcal{H}^{n-1}$$

for smooth functions φ on Γ. In this equality we decompose $V=V^N\nu+V^T$ into the normal component $V^N:=V\cdot\nu$ and the tangential component $V^T:=V-(V\cdot\nu)\nu$. Then, since $\nu\cdot\nabla_\Gamma\varphi=0$ and

$$\mathrm{div}_\Gamma(V^N\nu)=\nabla_\Gamma V^N\cdot\nu+V^N\mathrm{div}_\Gamma\nu=0+V^N\cdot(-H)=-(V\cdot\nu)H,$$

we obtain a usual integration by parts formula

$$\int_\Gamma V^T\cdot\nabla_\Gamma\varphi\,d\mathcal{H}^{n-1}=-\int_\Gamma\varphi\,\mathrm{div}_\Gamma V^T\,d\mathcal{H}^{n-1},\tag{5.41}$$

which we will use to recover a limit equation on $\Gamma(t)$ from its variational formulation. This is the reason to use the tangential velocity V_Γ^T in (5.36) instead of the total velocity V_Γ of $\Gamma(t)$.

6. Convergence and characterization of the limit

6.1. **Variational formulations of the average of solutions to the heat equation.**

Let us return to the initial-boundary value problem (H_ε) of the heat equation. By Theorem 3.4, for every $u_0^\varepsilon\in L^2(\Omega_\varepsilon(0))$ there is a unique variational solution $u^\varepsilon\in L^2_{H^1(\varepsilon)}$ to (H_ε).

Let M_ε be the average operator defined in Definition 5.1. Our goal in this subsection is to derive a variational formulation of $M_\varepsilon u^\varepsilon$.

Lemma 6.1. *Let $u_0^\varepsilon\in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon\in L^2_{H^1(\varepsilon)}$ be a unique variational solution to (H_ε) given by Theorem 3.4. Then, $M_\varepsilon u^\varepsilon\in H_T$ and it satisfies*

$$\begin{aligned}\langle\partial^\bullet M_\varepsilon u^\varepsilon,g\varphi\rangle_T&+\int_0^T\int_{\Gamma(t)}(\partial^\bullet g+g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)(M_\varepsilon u^\varepsilon)\varphi\,d\mathcal{H}^{n-1}\,dt\\&+\int_0^T\int_{\Gamma(t)}g\{\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon+(M_\varepsilon u^\varepsilon)V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt=I_\varepsilon(u^\varepsilon,\varphi;T)\end{aligned}\tag{6.1}$$

for all $\varphi\in C_0^1(S_T)$. Here $I_\varepsilon(u^\varepsilon,\varphi;T)$ is a residual term that satisfies

$$|I_\varepsilon(u^\varepsilon,\varphi;T)|\le c\varepsilon^{1/2}\int_0^T\|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt\tag{6.2}$$

with a constant $c>0$ independent of u^ε, φ, and ε.

Proof. Since $u^\varepsilon\in L^2_{H^1(\varepsilon)}$, we have $M_\varepsilon u^\varepsilon\in H_T$ by Lemma 5.11. For each $\varphi\in C_0^1(S_T)$, its extension $\overline{\varphi}$ is in $C^1(\overline{Q_{\varepsilon,T}})$ and satisfies $\overline{\varphi}(0)=0$ in $\Omega_\varepsilon(0)$ and $\overline{\varphi}(T)=0$

in $\Omega_\varepsilon(T)$. Thus, by substituting $\overline{\varphi}$ for w in the variational formulation (3.2) we obtain

$$\int_0^T\int_{\Omega_\varepsilon(t)}(-u^\varepsilon\partial_t\overline{\varphi}+\nabla u^\varepsilon\cdot\nabla\overline{\varphi})\,dx\,dt=0. \tag{6.3}$$

Moreover, from Lemma 5.7 and Lemma 5.14 we have

$$\begin{aligned}\int_0^T\int_{\Omega_\varepsilon(t)}\nabla u^\varepsilon\cdot\nabla\overline{\varphi}\,dx\,dt&= \\ &\varepsilon\int_0^T\int_{\Gamma(t)}g\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt+I_\varepsilon^1(u^\varepsilon,\varphi;T)\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\int_0^T\int_{\Omega_\varepsilon(t)}u^\varepsilon\partial_t\overline{\varphi}\,dx\,dt&= \\ -\varepsilon\langle\partial^\bullet M_\varepsilon u^\varepsilon,g\varphi\rangle_T&-\varepsilon\int_0^T\int_{\Gamma(t)}(\partial^\bullet g+g\operatorname{div}_{\Gamma(t)}V_\Gamma)(M_\varepsilon u^\varepsilon)\varphi\,d\mathcal{H}^{n-1}\,dt \\ &-\varepsilon\int_0^T\int_{\Gamma(t)}g(M_\varepsilon u^\varepsilon)V_\Gamma^T\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt+I_\varepsilon^2(u^\varepsilon,\varphi;T),\end{aligned} \tag{6.5}$$

where $I_\varepsilon^1(u^\varepsilon,\varphi;T)$ and $I_\varepsilon^2(u^\varepsilon,\varphi;T)$ satisfy

$$|I_\varepsilon^k(u^\varepsilon,\varphi;T)|\le c\varepsilon^{3/2}\int_0^T\|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt,\quad k=1,2, \tag{6.6}$$

with some constant $c>0$ independent of ε. Hence, by substituting (6.4) and (6.5) for (6.3) and dividing both sides by ε, we obtain (6.1) with the residual term

$$I_\varepsilon(u^\varepsilon,\varphi;T):=\varepsilon^{-1}\bigl\{I_\varepsilon^2(u^\varepsilon,\varphi;T)-I_\varepsilon^1(u^\varepsilon,\varphi;T)\bigr\},$$

which satisfies (6.2) because $I_\varepsilon^1(u^\varepsilon,\varphi;T)$ and $I_\varepsilon^2(u^\varepsilon,\varphi;T)$ satisfy (6.6). $\square$

6.2. **Estimates for the average $M_\varepsilon u^\varepsilon$ in the space W_T.** In this subsection, we estimate $M_\varepsilon u^\varepsilon$ in the Hilbert space W_T given by (4.15).

Lemma 6.2. *Let $u_0^\varepsilon\in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon\in L^2_{H^1(\varepsilon)}$ be a unique variational solution to (H_ε) given by Theorem 3.4. Then, $M_\varepsilon u^\varepsilon\in W_T$ and there exists a constant $c>0$ independent of u^ε and ε such that*

$$\|\partial^\bullet M_\varepsilon u^\varepsilon\|_{H_T'}\le c\Bigl(\|M_\varepsilon u^\varepsilon\|_{H_T}+\varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\Bigr). \tag{6.7}$$

Proof. Let φ be an arbitrary function in $C_0^1(S_T)$. By substituting $g^{-1}\varphi\in C_0^1(S_T)$ for φ in (6.1), we obtain $\langle\partial^\bullet M_\varepsilon u^\varepsilon,\varphi\rangle_T=I(u^\varepsilon,\varphi)+I_\varepsilon(u^\varepsilon,g^{-1}\varphi;T)$, where

$$\begin{aligned}I(u^\varepsilon,\varphi):=\int_0^T\int_{\Gamma(t)}&\{g^{-1}(V_\Gamma^T\cdot\nabla_{\Gamma(t)}g-\partial^\bullet g)-\operatorname{div}_{\Gamma(t)}V_\Gamma\}(M_\varepsilon u^\varepsilon)\varphi\,d\mathcal{H}^{n-1}\,dt \\ &-\int_0^T\int_{\Gamma(t)}\{\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon+(M_\varepsilon u^\varepsilon)V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt \\ &+\int_0^T\int_{\Gamma(t)}g^{-1}(\nabla_{\Gamma(t)}g\cdot\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon)\varphi\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

Since g and V_Γ are smooth on $\overline{S_T}$, they are bounded on S_T along with their derivatives. Moreover, g^{-1} and V_Γ^T are bounded on S_T. Thus we have

$$|I(u^\varepsilon,\varphi)| \leq c\|M_\varepsilon u^\varepsilon\|_{H_T}\|\varphi\|_{H_T}$$

with a constant $c>0$ independent of u^ε, φ, and ε. Also, by (6.2),

$$\begin{aligned}
&|I_\varepsilon(u^\varepsilon,g^{-1}\varphi;T)| \\
&\leq c\varepsilon^{1/2}\int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}(g^{-1}\varphi)(t)\|_{L^2(\Gamma(t))}\,dt \\
&\leq c\varepsilon^{1/2}\int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\Bigl(\|\varphi(t)\|_{L^2(\Gamma(t))}+\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\Bigr)\,dt \\
&\leq c\varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\|\varphi\|_{H_T}
\end{aligned}$$

with some $c>0$ independent of u^ε, φ, and ε. Hence we obtain

$$\begin{aligned}
|\langle\partial^\bullet M_\varepsilon u^\varepsilon,\varphi\rangle_T| &\leq |I(u^\varepsilon,\varphi)|+|I_\varepsilon(u^\varepsilon,g^{-1}\varphi;T)| \\
&\leq c\Bigl(\|M_\varepsilon u^\varepsilon\|_{H_T}+\varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\Bigr)\|\varphi\|_{H_T}
\end{aligned}$$

for all $\varphi\in C_0^1(S_T)$, which implies that $M_\varepsilon u^\varepsilon\in W_T$ and the inequality (6.7). □

Remark 6.3. Since $M_\varepsilon u^\varepsilon\in W_T$ and $C_0^1(S_T)$ is dense in H_T (see Lemma 4.2), the equality (6.1) also holds for all $\varphi\in H_T$. On the other hand, since W_{T_1} is continuously embedded into W_{T_2} when $T_1>T_2$, we have $M_\varepsilon u^\varepsilon\in W_\tau$ for each $\tau\in[0,T]$. Hence (6.1) and (6.2) with T replaced by each $\tau\in[0,T]$ are also valid for all $\varphi\in H_\tau$.

Lemma 6.4. *Let u_0^ε, u^ε be as in Lemma 6.2. Then there exists a constant $c>0$ independent of u_0^ε, u^ε, and ε such that the energy estimate*

$$\begin{aligned}
(6.8)\quad \|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))}+\int_0^\tau\|\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt& \\
\leq c\Bigl(\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)}+\varepsilon\|u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon(0))}\Bigr)&
\end{aligned}$$

holds for all $\tau\in[0,T]$.

Proof. As we mentioned in Remark 6.3, the equality (6.1) holds with T replaced by each $\tau\in[0,T]$. Hence, by substituting $g^{-1}M_\varepsilon u^\varepsilon\in H_\tau$ for φ in (6.1) with T replaced by τ, we obtain

$$\begin{aligned}
&\langle\partial^\bullet M_\varepsilon u^\varepsilon,M_\varepsilon u^\varepsilon\rangle_\tau+\int_0^\tau\|\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \\
&\quad+\int_0^\tau\int_{\Gamma(t)}\{g^{-1}(\partial^\bullet g-V_\Gamma^T\cdot\nabla_{\Gamma(t)}g)+\mathrm{div}_{\Gamma(t)}V_\Gamma\}|M_\varepsilon u^\varepsilon|^2\,d\mathcal{H}^{n-1}\,dt \\
&\quad\quad+\int_0^\tau\int_{\Gamma(t)}M_\varepsilon u^\varepsilon(V_\Gamma^T-g^{-1}\nabla_{\Gamma(t)}g)\cdot\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon\,d\mathcal{H}^{n-1}\,dt \\
&\qquad\qquad=I_\varepsilon(u^\varepsilon,g^{-1}M_\varepsilon u^\varepsilon;\tau).
\end{aligned}$$

Moreover, from (4.18) with T replaced by τ,

$$\langle \partial^\bullet M_\varepsilon u^\varepsilon, M_\varepsilon u^\varepsilon\rangle_\tau = -\frac{1}{2}\int_0^\tau \int_{\Gamma(t)} |M_\varepsilon u^\varepsilon|^2 \operatorname{div}_{\Gamma(t)} V_\Gamma \, d\mathcal{H}^{n-1}\, dt + \frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} - \frac{1}{2}\|M_\varepsilon u^\varepsilon(0)\|^2_{L^2(\Gamma_0)}.$$

Applying this equality and the relation $u^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$ (see Theorem 3.4) to the above equality, we have

$$\begin{aligned} \frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\, dt \\ &= \frac{1}{2}\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + I_1(\tau) + I_2(\tau) + I_\varepsilon(u^\varepsilon, g^{-1}M_\varepsilon u^\varepsilon; \tau), \end{aligned} \tag{6.9}$$

where

$$I_1(\tau) := -\frac{1}{2}\int_0^\tau \int_{\Gamma(t)} \{2g^{-1}(\partial^\bullet g - V_\Gamma^T \cdot \nabla_{\Gamma(t)} g) + \operatorname{div}_{\Gamma(t)} V_\Gamma\} |M_\varepsilon u^\varepsilon|^2 \, d\mathcal{H}^{n-1}\, dt,$$

$$I_2(\tau) := -\int_0^\tau \int_{\Gamma(t)} M_\varepsilon u^\varepsilon (V_\Gamma^T - g^{-1}\nabla_{\Gamma(t)} g)\cdot \nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon \, d\mathcal{H}^{n-1}\, dt.$$

Since g and V_Γ are smooth on $\overline{S_T}$, they are bounded on S_T along with their derivatives. Also, g^{-1} and V_Γ^T are bounded on S_T. Thus it follows that

$$\begin{aligned} |I_1(\tau)| &\leq c\int_0^\tau \|M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\, dt, \\ |I_2(\tau)| &\leq c\int_0^\tau \|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))} \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}\, dt. \end{aligned} \tag{6.10}$$

On the other hand, the inequality (6.2) with T replaced by τ yields

$$\begin{aligned} &|I_\varepsilon(u^\varepsilon, g^{-1}M_\varepsilon u^\varepsilon;\tau)| \\ &\leq c\varepsilon^{1/2}\int_0^\tau \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))} \|\nabla_{\Gamma(t)}(g^{-1}M_\varepsilon u^\varepsilon)(t)\|_{L^2(\Gamma(t))}\, dt \\ &\leq c\varepsilon^{1/2}\int_0^\tau \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))} \Big(\|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))} + \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}\Big)\, dt. \end{aligned} \tag{6.11}$$

Thus, by applying (6.10) and (6.11) to (6.9), we obtain

$$\begin{aligned} \frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\, dt \\ &\leq \frac{1}{2}\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + \frac{1}{2}\int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\, dt \\ &\qquad + c\int_0^\tau \Big(\|M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))} + \varepsilon\|u^\varepsilon(t)\|^2_{H^1(\Omega_\varepsilon(t))}\Big)\, dt. \end{aligned}$$

We multiply both sides by two and subtract $\int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\, dt$ to get

$$\begin{aligned} \|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\, dt \\ &\leq \|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + c\int_0^\tau \Big(\|M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))} + \varepsilon\|u^\varepsilon(t)\|^2_{H^1(\Omega_\varepsilon(t))}\Big)\, dt. \end{aligned}$$

Hence Gronwall's inequality implies

$$\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(t))} + \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \\ \leq c\Big(\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + \varepsilon\|u^\varepsilon\|^2_{L^2_{H^1(\varepsilon)}}\Big)$$

for all $\tau \in [0,T]$, and we obtain (6.8) by applying (3.15) to the second term of the right-hand side of the above inequality. □

Lemma 6.5. *Let u_0^ε, u^ε be as in Lemma 6.2. Then there exists a constant $c > 0$ independent of u_0^ε, u^ε, and ε such that*

$$\|M_\varepsilon u^\varepsilon\|_{W_T} \leq c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big). \tag{6.12}$$

Proof. It follows from (6.8) that

$$\|M_\varepsilon u^\varepsilon\|_{H_T} \leq c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big).$$

Moreover, by applying this inequality and (3.15) to (6.7) we have

$$\begin{aligned}\|\partial^\bullet M_\varepsilon u^\varepsilon\|_{H_T'} &\leq c\Big(\|M_\varepsilon u^\varepsilon\|_{H_T} + \varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\Big) \\ &\leq c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big).\end{aligned}$$

Thus we obtain (6.12). □

6.3. **Limit equation on evolving surfaces and weak convergence of $M_\varepsilon u^\varepsilon$.** Assume that $I_\varepsilon(u^\varepsilon,\varphi;T) = 0$ holds for all $\varphi \in C_0^1(S_T)$ and $v = M_\varepsilon u^\varepsilon$ is independent of ε in the variational formulation (6.1). Then, v satisfies

$$\langle \partial^\bullet v, g\varphi\rangle_T + \int_0^T\int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v\,\varphi\,d\mathcal{H}^{n-1}\,dt \\ + \int_0^T\int_{\Gamma(t)} g(\nabla_{\Gamma(t)}v + vV_\Gamma^T)\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt = 0 \tag{6.13}$$

for all $\varphi \in C_0^1(S_T)$. In addition we assume that v is sufficiently smooth. Since vector fields gvV_Γ^T and $g\nabla_{\Gamma(t)}v$ are tangent to $\Gamma(t)$ for each $t \in [0,T]$, we can apply the integration by parts formula (5.41) to obtain

$$\langle \partial^\bullet v, g\varphi\rangle_T \\ + \int_0^T\int_{\Gamma(t)} \Big\{(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v - \mathrm{div}_{\Gamma(t)}\big[g(\nabla_{\Gamma(t)}v + vV_\Gamma^T)\big]\Big\}\varphi\,d\mathcal{H}^{n-1}\,dt \\ = 0.$$

Since this equality holds for all $\varphi \in C_0^1(S_T)$, we conclude that v satisfies

$$\partial^\bullet(gv) + (g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v - \mathrm{div}_{\Gamma(t)}\big[g(\nabla_{\Gamma(t)}v + vV_\Gamma^T)\big] = 0 \quad \text{on} \quad S_T.$$

This is the limit equation of (H_ε). To justify the above argument, we employ a variational framework introduced by Olshanskii, Reusken, and Xu [18].

Definition 6.6. Let $v_0 \in L^2(\Gamma_0)$. A function $v \in W_T$ is said to be a *variational solution* to the initial value problem

$$(H_0) \qquad \begin{cases} \partial^\bullet(gv) + (g\,\mathrm{div}_{\Gamma(t)} V_\Gamma)v - \mathrm{div}_{\Gamma(t)}\left[g(\nabla_{\Gamma(t)} v + vV_\Gamma^T)\right] = 0 & \text{on} \quad S_T, \\ v(0) = v_0 & \text{on} \quad \Gamma_0, \end{cases}$$

if v satisfies (6.13) for all $\varphi \in H_T$ and $v(0) = v_0$ in $L^2(\Gamma_0)$.

Remark 6.7. By Lemma 4.4, the trace operator $v \mapsto v(t)$ is well-defined as a bounded linear operator from W_T into $L^2(\Gamma(t))$ for every $t \in [0,T]$. Thus the condition $v(0) = v_0$ in $L^2(\Gamma_0)$ makes sense.

Remark 6.8. Suppose that $v \in W_T$ is a variational solution to (H_0). Then, we have $v \in W_\tau$ for each $\tau \in [0,T]$, since W_τ is continuously embedded into W_T. Moreover, by taking test functions φ from $C_0^1(S_\tau)$ we observe that v is a variational solution to (H_0) with T replaced by τ.

We first prove the uniqueness of a variational solution to the initial value problem (H_0).

Lemma 6.9. *For each $v_0 \in L^2(\Gamma_0)$, there is at most one variational solution to (H_0).*

Proof. Since (H_0) is linear, it is sufficient to show that if $v \in W_T$ is a variational solution to (H_0) with zero initial data, then $v = 0$.

Let v be a variational solution to (H_0) with $v(0) = 0$ in $L^2(\Gamma_0)$. For each $\tau \in [0,T]$, we substitute $g^{-1}v \in H_\tau$ for φ in (6.13) with T replaced by τ and compute as in the proof of Lemma 6.4 (replace $M_\varepsilon u^\varepsilon$ by v and omit $I_\varepsilon(u^\varepsilon, \varphi; \tau)$). Then we have

$$\|v(\tau)\|_{L^2(\Gamma(\tau))}^2 + \int_0^\tau \|\nabla_{\Gamma(t)} v(t)\|_{L^2(\Gamma(t))}^2\,dt \leq \|v(0)\|_{L^2(\Gamma_0)}^2 + c\int_0^\tau \|v(t)\|_{L^2(\Gamma(t))}^2\,dt.$$

Since $v(0) = 0$ in $L^2(\Gamma_0)$, the above inequality yields

$$\|v(\tau)\|_{L^2(\Gamma(t))}^2 \leq \int_0^\tau \|v(t)\|_{L^2(\Gamma(t))}^2\,dt.$$

Hence by Gronwall's inequality we obtain $v(\tau) = 0$ for all $\tau \in [0,T]$. □

Now let us show that $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges weakly in W_T and that the limit is a unique variational solution to the initial value problem (H_0).

Theorem 6.10. *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be a unique variational solution to (H_ε) given by Theorem 3.4. Suppose that the following two conditions are satisfied.*

(a) *There exist constants $c > 0$ and $\gamma \in (0, 1/2)$ such that $\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} \leq c\varepsilon^{-\gamma}$ for all $\varepsilon > 0$.*
(b) *There exists $v_0 \in L^2(\Gamma_0)$ such that $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ converges weakly to v_0 in $L^2(\Gamma_0)$ as $\varepsilon \to 0$.*

Then $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges weakly in W_T as $\varepsilon \to 0$. Moreover, the weak limit $v \in W_T$ of $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ is a unique variational solution to (H_0) with initial data v_0.

Proof. (1) By the condition (b), $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ is bounded in $L^2(\Gamma_0)$. From this fact, the inequality (6.12), and the condition (a) it follows that

$$\|M_\varepsilon u^\varepsilon\|_{W_T} \leq c\big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big) \leq c(1+\varepsilon^{-\gamma+1/2}) \leq c \tag{6.14}$$

with some constant $c>0$ independent of ε. Here the last inequality follows from the condition $\gamma\in(0,1/2)$. Hence $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ is bounded in the Hilbert space W_T and there exist $v\in W_T$ and a sequence $\{\varepsilon_n\}_n$ of positive numbers with $\lim_{n\to\infty}\varepsilon_n=0$ such that $\{M_{\varepsilon_n}u^{\varepsilon_n}\}_n$ converges weakly to v in W_T as $n\to\infty$.

Let us show that v is a unique variational solution to (H_0) with initial data v_0. First we show that v satisfies the variational formulation (6.13) for all $\varphi\in H_T$. To this end, we return to the variational formulation (6.1) of $M_{\varepsilon_n}u^{\varepsilon_n}$:

$$\begin{aligned}\langle\partial^\bullet M_{\varepsilon_n}u^{\varepsilon_n}, g\varphi\rangle_T &+ \int_0^T\int_{\Gamma(t)}(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)(M_{\varepsilon_n}u^{\varepsilon_n})\varphi\,d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^T\int_{\Gamma(t)} g\{\nabla_{\Gamma(t)}M_{\varepsilon_n}u^{\varepsilon_n} + (M_{\varepsilon_n}u^{\varepsilon_n})V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt \\ &= I_{\varepsilon_n}(u^{\varepsilon_n},\varphi;T).\end{aligned} \tag{6.15}$$

Let $n\to\infty$ in (6.15). Since $\{M_{\varepsilon_n}u^{\varepsilon_n}\}_n$ converges weakly to v in W_T as $n\to\infty$ and g, V_Γ are bounded on S_T along with their derivatives, the left-hand side of (6.15) converges to

$$\begin{aligned}\langle\partial^\bullet v, g\varphi\rangle_T &+ \int_0^T\int_{\Gamma(t)}(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v\varphi\,d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^T\int_{\Gamma(t)} g(\nabla_{\Gamma(t)}v + vV_\Gamma^T)\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

On the other hand, it follows from (6.2) and (3.15) that

$$\begin{aligned}|I_{\varepsilon_n}(u^{\varepsilon_n},\varphi;T)| &\leq c\varepsilon_n^{1/2}\int_0^T\|u^{\varepsilon_n}(t)\|_{H^1(\Omega_{\varepsilon_n}(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt \\ &\leq c\varepsilon_n^{1/2}\|u^{\varepsilon_n}\|_{L^2_{H^1(\varepsilon_n)}}\|\varphi\|_{H_T} \leq c\varepsilon_n^{1/2}\|u_0^{\varepsilon_n}\|_{L^2(\Omega_{\varepsilon_n}(0))}\|\varphi\|_{H_T}\end{aligned}$$

with a constant $c>0$ independent of ε_n. This inequality and the condition (a) imply that

$$|I_{\varepsilon_n}(u^{\varepsilon_n},\varphi;T)| \leq c\varepsilon_n^{-\gamma+1/2}\|\varphi\|_{H_T}\to 0, \quad \text{as} \quad n\to\infty, \tag{6.16}$$

since $\gamma\in(0,1/2)$ and c is independent of ε_n. Hence v satisfies (6.13) for all $\varphi\in H_T$.

Next we show that $v(0)=v_0$ in $L^2(\Gamma_0)$. We take a function $\eta\in C^\infty([0,T])$ such that $\eta(0)=1$ and $\eta(T)=0$. For each $\varphi_0\in C^\infty(\Gamma_0)$, we set $\varphi(y,t):=\varphi_0(\Phi^{-1}(y,t))\eta(t)$ for $(y,t)\in\overline{S_T}$. Due to the smoothness of Φ^{-1}, the function φ is smooth on $\overline{S_T}$ and thus $\varphi\in W_T$. Moreover, φ satisfies $\varphi(0)=\varphi_0$ on Γ_0 and $\varphi(T)=0$ on $\Gamma(T)$. Substituting $g^{-1}\varphi$ for φ in (6.13) and (6.15), we have

$$\begin{aligned}\langle\partial^\bullet v, \varphi\rangle_T &+ \int_0^T\int_{\Gamma(t)}(g^{-1}\partial^\bullet g + \mathrm{div}_{\Gamma(t)}V_\Gamma)v\,\varphi\,d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^T\int_{\Gamma(t)} g(\nabla_{\Gamma(t)}v + vV_\Gamma^T)\cdot\nabla_{\Gamma(t)}(g^{-1}\varphi)\,d\mathcal{H}^{n-1}\,dt = 0\end{aligned}$$

and

$$\langle \partial^\bullet M_{\varepsilon_n} u^{\varepsilon_n}, \varphi\rangle_T + \int_0^T \int_{\Gamma(t)} (g^{-1}\partial^\bullet g + \mathrm{div}_{\Gamma(t)} V_\Gamma)(M_{\varepsilon_n} u^{\varepsilon_n})\varphi \, d\mathcal{H}^{n-1}\, dt$$
$$+ \int_0^T \int_{\Gamma(t)} g\{\nabla_{\Gamma(t)} M_{\varepsilon_n} u^{\varepsilon_n} + (M_{\varepsilon_n} u^{\varepsilon_n}) V_\Gamma^T\} \cdot \nabla_{\Gamma(t)}(g^{-1}\varphi)\, d\mathcal{H}^{n-1}\, dt$$
$$= I_{\varepsilon_n}(u^{\varepsilon_n}, g^{-1}\varphi; T).$$

Since φ, v, and $M_{\varepsilon_n} u^{\varepsilon_n}$ are in W_T, we can apply the identity (4.18) to get

$$\langle \partial^\bullet v, \varphi\rangle_T = -\langle \partial^\bullet \varphi, v\rangle_T - \int_{\Gamma_0} v(0)\varphi_0 \, d\mathcal{H}^{n-1} - \int_0^T \int_{\Gamma(t)} v\varphi \, \mathrm{div}_{\Gamma(t)} V_\Gamma \, d\mathcal{H}^{n-1}\, dt$$

and the same identity for $M_{\varepsilon_n} u^{\varepsilon_n}$. Here we used the conditions $\varphi(0) = \varphi_0$ on Γ_0 and $\varphi(T) = 0$ on $\Gamma(T)$. Thus we have

$$(6.17)\quad -\langle \partial^\bullet \varphi, v\rangle_T + \int_0^T \int_{\Gamma(t)} g^{-1}(\partial^\bullet g) v\varphi \, d\mathcal{H}^{n-1}\, dt$$
$$+ \int_0^T \int_{\Gamma(t)} g(\nabla_{\Gamma(t)} v + vV_\Gamma^T) \cdot \nabla_{\Gamma(t)}(g^{-1}\varphi)\, d\mathcal{H}^{n-1}\, dt = \int_{\Gamma_0} v(0)\varphi_0 \, d\mathcal{H}^{n-1}$$

and

$$(6.18)\quad -\langle \partial^\bullet \varphi, M_{\varepsilon_n} u^{\varepsilon_n}\rangle_T + \int_0^T \int_{\Gamma(t)} g^{-1}(\partial^\bullet g)(M_{\varepsilon_n} u^{\varepsilon_n})\varphi \, d\mathcal{H}^{n-1}\, dt$$
$$+ \int_0^T \int_{\Gamma(t)} g\{\nabla_{\Gamma(t)} M_{\varepsilon_n} u^{\varepsilon_n} + (M_{\varepsilon_n} u^{\varepsilon_n}) V_\Gamma^T\} \cdot \nabla_{\Gamma(t)}(g^{-1}\varphi)\, d\mathcal{H}^{n-1}\, dt$$
$$= \int_{\Gamma_0} (M_{\varepsilon_n} u_0^{\varepsilon_n})\varphi_0 \, d\mathcal{H}^{n-1} + I_{\varepsilon_n}(u^{\varepsilon_n}, g^{-1}\varphi; T).$$

Let $n \to \infty$ in (6.18). Since $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ converges weakly to v_0 in $L^2(\Gamma_0)$ as $\varepsilon \to 0$,

$$\lim_{n\to\infty} \int_{\Gamma_0} M_{\varepsilon_n} u_0^{\varepsilon_n} \, \varphi_0 \, d\mathcal{H}^{n-1} = \int_{\Gamma_0} v_0 \, \varphi_0 \, d\mathcal{H}^{n-1}.$$

Moreover, since $\{M_{\varepsilon_n} u^{\varepsilon_n}\}_n$ converges weakly to v in W_T as $n \to \infty$ and (6.16) holds with φ replaced by $g^{-1}\varphi$, both sides of (6.18) converge to

$$(6.19)\quad -\langle \partial^\bullet \varphi, v\rangle_T + \int_0^T \int_{\Gamma(t)} g^{-1}(\partial^\bullet g) v \, \varphi \, d\mathcal{H}^{n-1}\, dt$$
$$+ \int_0^T \int_{\Gamma(t)} g(\nabla_{\Gamma(t)} v + vV_\Gamma^T) \cdot \nabla_{\Gamma(t)}(g^{-1}\varphi)\, d\mathcal{H}^{n-1}\, dt = \int_{\Gamma_0} v_0 \, \varphi_0 \, d\mathcal{H}^{n-1}.$$

Comparing (6.17) and (6.19), we obtain

$$\int_{\Gamma_0} v(0)\varphi_0 \, d\mathcal{H}^{n-1} = \int_{\Gamma_0} v_0 \, \varphi_0 \, d\mathcal{H}^{n-1} \quad \text{for all} \quad \varphi_0 \in C^\infty(\Gamma_0).$$

Since $C^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, it follows that $v(0) = v_0$ in $L^2(\Gamma_0)$. Hence v is a unique variational solution to (H_0) with initial data v_0. Here the uniqueness follows from Lemma 6.9.

(2) Let us prove that $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ itself converges weakly to v in W_T as $\varepsilon \to 0$, that is, the equality $\lim_{\varepsilon\to 0} f(M_\varepsilon u^\varepsilon) = f(v)$ holds for every bounded linear functional f

on W_T. To this end, we assume to the contrary that there are some bounded linear functional f on W_T, sequence $\{\varepsilon_n\}_n$ of positive numbers with $\lim_{n\to\infty}\varepsilon_n = 0$, and positive constant c such that

$$|f(M_{\varepsilon_n}u^{\varepsilon_n}) - f(v)| \geq c \quad \text{for all} \quad n \in \mathbb{N}. \tag{6.20}$$

Since $\{M_{\varepsilon_n}u^{\varepsilon_n}\}_n$ is bounded in W_T by (6.14), there is a subsequence of $\{M_{\varepsilon_n}u^{\varepsilon_n}\}_n$, which is again referred to as $\{M_{\varepsilon_n}u^{\varepsilon_n}\}_n$, such that it converges weakly to some $v^* \in W_T$ as $n \to \infty$. However, as in the first part of the proof, we can show that v^* is a variational solution to (H_0) with initial data v_0. Thus Lemma 6.9 implies that $v^* = v$ and it follows that $\lim_{n\to\infty} f(M_{\varepsilon_n}u^{\varepsilon_n}) = f(v^*) = f(v)$, which contradicts with (6.20). Hence the assumption is false and we conclude that $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ itself converges weakly to v in W_T as $\varepsilon \to 0$. The proof is complete. □

Corollary 6.11. *For every $v_0 \in L^2(\Gamma_0)$, there exists a unique variational solution to (H_0).*

Proof. For each $\varepsilon > 0$, we define a function u_0^ε on $\Omega_\varepsilon(0)$ as

$$u_0^\varepsilon(X) := \frac{v_0(p(X,0))}{J(p(X,0),0,d(X,0))}, \quad X \in \Omega_\varepsilon(0).$$

Clearly $M_\varepsilon u_0^\varepsilon = v_0$ holds on Γ_0. Moreover, by the co-area formula (5.4) and (5.2) we have

$$\begin{aligned}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} &= \left(\int_{\Gamma_0}\int_{\varepsilon g_0(Y,0)}^{\varepsilon g_1(Y,0)} |v_0(Y)|^2 J(Y,0,\rho)^{-1}\,d\rho\,d\mathcal{H}^{n-1}\right)^{1/2} \\ &\leq c\left(\int_{\Gamma_0} \varepsilon g(Y,0)|v_0(Y)|^2\,d\mathcal{H}^{n-1}\right)^{1/2} \leq c\varepsilon^{1/2}\|v_0\|_{L^2(\Gamma_0)}\end{aligned}$$

with a constant $c > 0$ independent of ε. Hence $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and u_0^ε, v_0 satisfy the conditions (a) and (b) of Theorem 6.10. Thus the corollary follows from Theorem 3.4 and Theorem 6.10. □

Remark 6.12. Let $H := -\mathrm{div}_{\Gamma(t)}\nu$ be the mean curvature of $\Gamma(t)$. Since the material derivative operator is formally of the form $\partial^\bullet = \partial_t + V_\Gamma^N \nu\cdot\nabla + V_\Gamma^T\cdot\nabla_{\Gamma(t)}$ and the equality $\mathrm{div}_{\Gamma(t)}(V_\Gamma^N\nu) = -V_\Gamma^N H$ holds (see Remark 5.13 and Remark 5.15), the limit equation (H_0) is formally equivalent to

$$\partial_t(gv) + V_\Gamma^N\nu\cdot\nabla(gv) - V_\Gamma^N Hgv - \mathrm{div}_{\Gamma(t)}(g\nabla_{\Gamma(t)}v) = 0 \quad \text{on} \quad S_T.$$

This equation depends on $V_\Gamma^N\nu$ and $V_\Gamma^N H$, which represent the geometric motion of $\Gamma(t)$. On the other hand, it is independent of the tangential velocity V_Γ^T, which represents advection along $\Gamma(t)$. Hence, as we mentioned in Section 1, the evolution of the limit v given by Theorem 6.10 is not affected by advection along $\Gamma(t)$, but the geometric motion of $\Gamma(t)$.

6.4. **Estimates for the difference between solutions to the heat equation and the limit equation.** Let us estimate the difference between variational solutions to (H_ε) and (H_0). For a function v on S_T, let $\overline{v}$ be its extension to the normal direction of $\Gamma(t)$. Also, for a function u on $Q_{\varepsilon,T}$, we set

$$\|u\|_{L^2(Q_{\varepsilon,T})} := \left(\int_0^T\int_{\Omega_\varepsilon(t)} |u|^2\,dx\,dt\right)^{1/2}.$$

Theorem 6.13. *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be a unique variational solution to (H_ε). Also, let $v_0 \in L^2(\Gamma_0)$ and $v \in W_T$ be a unique variational solution to (H_0). Then, there exists a constant $c > 0$ independent of u_0^ε, u^ε, v_0, v, and ε such that*

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leq c\big(\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon^{3/2}\|v_0\|_{L^2(\Gamma_0)}\big). \tag{6.21}$$

In particular, if $\lim_{\varepsilon\to 0}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} = 0$, then $\lim_{\varepsilon\to 0}\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} = 0$.

We first estimate the difference between $M_\varepsilon u^\varepsilon$ and v in the space W_T.

Lemma 6.14. *Let u_0^ε, u^ε, v_0, v be as in Theorem 6.13. Then, there exists a constant $c > 0$ independent of u_0^ε, u^ε, v_0, v, and ε such that*

$$\|M_\varepsilon u^\varepsilon - v\|_{W_T} \leq c\big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}\big). \tag{6.22}$$

In particular, if $\lim_{\varepsilon\to 0}\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} = 0$ and $\lim_{\varepsilon\to 0}\varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} = 0$, then $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges strongly to v in W_T.

Proof. For each $\tau \in [0,T]$, we subtract both sides of (6.13) with T replaced by τ from those of (6.1). Then we have

$$\begin{aligned}\langle \partial^\bullet(M_\varepsilon u^\varepsilon - v), g\varphi\rangle_\tau &+ \int_0^\tau\int_{\Gamma(t)}(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)(M_\varepsilon u^\varepsilon - v)\varphi\,d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^\tau\int_{\Gamma(t)} g\{\nabla_{\Gamma(t)}(M_\varepsilon u^\varepsilon - v) + (M_\varepsilon u^\varepsilon - v)V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt \\ &= I_\varepsilon(u^\varepsilon,\varphi;\tau)\end{aligned}$$

for all $\varphi \in H_\tau$. Hence, by calculating as in the proof of Lemma 6.2, Lemma 6.4 and Lemma 6.5 (replace $M_\varepsilon u^\varepsilon$ by $M_\varepsilon u^\varepsilon - v$), we obtain (6.22). $\Box$

Proof of Theorem 6.13. (1) First we show the inequality

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leq c\varepsilon^{1/2}\big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\big). \tag{6.23}$$

To this end, we use the triangle inequality

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leq \left\|u^\varepsilon - \overline{M_\varepsilon u^\varepsilon}\right\|_{L^2(Q_{\varepsilon,T})} + \left\|\overline{M_\varepsilon u^\varepsilon} - \overline{v}\right\|_{L^2(Q_{\varepsilon,T})}$$

and estimate the right-hand side of the above inequality. By (5.8) and (3.15), we have

$$\left\|u^\varepsilon - \overline{M_\varepsilon u^\varepsilon}\right\|_{L^2(Q_{\varepsilon,T})} \leq c\varepsilon\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}} \leq c\varepsilon\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}$$

with a constant $c > 0$ independent of ε. On the other hand, by (5.6) and (6.22),

$$\begin{aligned}\left\|\overline{M_\varepsilon u^\varepsilon} - \overline{v}\right\|_{L^2(Q_{\varepsilon,T})} &\leq c\varepsilon^{1/2}\|M_\varepsilon u^\varepsilon - v\|_{H_T} \\ &\leq c\varepsilon^{1/2}\big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}\big).\end{aligned}$$

Hence (6.23) follows.

(2) Next we estimate the right-hand side of (6.23). Hereafter we identify u_0^ε with a function

$$\underline{u_0^\varepsilon}(Y,\rho) := u_0^\varepsilon(Y + \rho\nu(Y,0)), \quad Y \in \Gamma_0,\ \rho \in (\varepsilon g_0(Y,0), \varepsilon g_1(Y,0)),$$

and omit the variables Y, ρ, and $t=0$ unless we need to specify them. We set

$$I_1 := \frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1}(u_0^\varepsilon - v_0)J\,d\rho, \quad I_2 := \frac{v_0}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1}(J-1)\,d\rho.$$

Then $M_\varepsilon u_0^\varepsilon - v_0 = I_1 + I_2$ holds on Γ_0. By Hölder's inequality and (2.1), (5.2), we have

$$|I_1|^2 \le \frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1}|u_0^\varepsilon - v_0|^2 J^2\,d\rho \le c\varepsilon^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1}|u_0^\varepsilon - v_0|^2 J\,d\rho.$$

On the other hand, (5.3) yields $|I_2| \le c\varepsilon|v_0|$. Hence

$$\begin{aligned}\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)}^2 &\le c\int_{\Gamma_0}(|I_1|^2 + |I_2|^2)\,d\mathcal{H}^{n-1} \\ &\le c\int_{\Gamma_0}\left(\varepsilon^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1}|u_0^\varepsilon - v_0|^2 J\,d\rho + \varepsilon^2|v_0|\right)d\mathcal{H}^{n-1} \\ &= c\big(\varepsilon^{-1}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))}^2 + \varepsilon^2\|v_0\|_{L^2(\Gamma_0)}^2\big).\end{aligned}$$

Here we used the co-area formula (5.4) in the last equality. The above inequality is equivalent to

$$\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} \le c\big(\varepsilon^{-1/2}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon\|v_0\|_{L^2(\Gamma_0)}\big). \tag{6.24}$$

Moreover, by the triangle inequality and (5.6),

$$\begin{aligned}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} &\le \|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \|\overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} \\ &\le \|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + c\varepsilon^{1/2}\|v_0\|_{L^2(\Gamma_0)}.\end{aligned} \tag{6.25}$$

Finally, by applying (6.24) and (6.25) to (6.23), we obtain

$$\begin{aligned}\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} &\le c\varepsilon^{1/2}\left(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\right) \\ &\le c\varepsilon^{1/2}\left((\varepsilon^{-1/2} + \varepsilon^{1/2})\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon\|v_0\|_{L^2(\Gamma_0)}\right) \\ &\le c\left(\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon^{3/2}\|v_0\|_{L^2(\Gamma_0)}\right)\end{aligned}$$

with a constant $c>0$ independent of ε. Hence (6.21) holds. □

7. Appendix: Heuristic derivation of the limit equation

In this section, we give a heuristic derivation of the limit equation (1.1) from (H_ε) when $\Omega_\varepsilon(t)$ is of the form $\Omega_\varepsilon(t) = \{x \in \mathbb{R}^n \mid -\varepsilon < d(x,t) < \varepsilon\}$. In this case, the unit outward normal vector field ν_ε of $\partial\Omega_\varepsilon(t)$ and the outer normal velocity V_ε^N of $\partial\Omega_\varepsilon(t)$ are of the form

$$\nu_\varepsilon(x,t) = \pm\nu(p(x,t),t), \quad V_\varepsilon^N(x,t) = \pm V_\Gamma^N(p(x,t),t), \quad (x,t) \in \partial_\ell Q_{\varepsilon,T},$$

according to $d(x,t) = \pm\varepsilon$ (double-sign corresponds). Thus we start from the heat equation

$$\partial_t u^\varepsilon(x,t) - \Delta u^\varepsilon(x,t) = 0, \quad (x,t) \in Q_{\varepsilon,T}$$

with the boundary condition

$$\nu(p(x,t),t)\cdot\nabla u^\varepsilon(x,t) + V_\Gamma^N(p(x,t),t)u^\varepsilon(x,t) = 0, \quad (x,t) \in \partial_\ell Q_{\varepsilon,T}. \tag{7.1}$$

To derive the limit equation, we assume that

(1) The signed distance $d(x,t)$ of $x \in \Omega_\varepsilon(t)$ is negligible ($d(x,t) \approx 0$), although the quantity $\varepsilon^{-1}d(x,t)$ is not negligible.
(2) The relation $V_\Gamma^N(p(x,t),t) \approx -\partial_t d(x,t)$ holds for all $(x,t) \in Q_{\varepsilon,T}$.
(3) The boundary condition (7.1) also holds in the noncylindrical domain $Q_{\varepsilon,T}$.

These assumptions come from the smallness of the width 2ε of $\Omega_\varepsilon(t)$. Taking the third assumption into account, we consider the two equations

$$\partial_t u^\varepsilon(x,t) - \Delta u^\varepsilon(x,t) = 0, \tag{7.2}$$

$$\nu(p(x,t),t) \cdot \nabla u^\varepsilon(x,t) + V_\Gamma^N(p(x,t),t)u^\varepsilon(x,t) = 0 \tag{7.3}$$

for $(x,t) \in Q_{\varepsilon,T}$. Recall that each $x \in \Omega_\varepsilon(t)$ is represented as

$$x = p(x,t) + d(x,t)\nu(p(x,t),t), \quad \nabla d(x,t) = \nu(x,t) = \nu(p(x,t),t).$$

First, we consider the gradient of the projection $p(x,t)$ onto $\Gamma(t)$ given by

$$\nabla p = \begin{pmatrix} \partial_1 p_1 & \dots & \partial_n p_1 \\ \vdots & \ddots & \vdots \\ \partial_1 p_n & \dots & \partial_n p_n \end{pmatrix} \quad \text{for} \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}.$$

By differentiating both sides of $x = p(x,t) + d(x,t)\nu(x,t)$ and using $\nabla d(x,t) = \nu(x,t)$, we have

$$I_n = \nabla p(x,t) + \nu(x,t) \otimes \nu(x,t) + d(x,t)\nabla\nu(x,t).$$

According to the assumption (1), the above equality reads

$$\nabla p(x,t) \approx I_n - \nu(x,t) \otimes \nu(x,t) = I_n - \nu(p(x,t),t) \otimes \nu(p(x,t),t). \tag{7.4}$$

Now let us define a function $v \colon S_T \times (-1,1) \to \mathbb{R}$ as

$$v(y,t,r) := u^\varepsilon(y + \varepsilon r\nu(y,t),t), \quad (y,t) \in S_T,\ r \in (-1,1).$$

Then u^ε is represented by v as

$$u^\varepsilon(x,t) = v(p(x,t),t,\varepsilon^{-1}d(x,t)), \quad (x,t) \in Q_{\varepsilon,T}. \tag{7.5}$$

From the chain rule of differentiation, we have

$$\begin{aligned} \nabla u^\varepsilon(x,t) = [\nabla p(x,t)]^T \nabla v(p(x,t),t,\varepsilon^{-1}d(x,t)) \\ + \varepsilon^{-1}\partial_r v(p(x,t),t,\varepsilon^{-1}d(x,t))\nabla d(x,t). \end{aligned}$$

By (7.4) and $\nabla d(x,t) = \nu(p(x,t),t)$, this equality reads

$$\begin{aligned} \nabla u^\varepsilon(x,t) \approx \nabla_{\Gamma(t)} v(p(x,t),t,\varepsilon^{-1}d(x,t)) \\ + \varepsilon^{-1}\partial_r v(p(x,t),t,\varepsilon^{-1}d(x,t))\nu(p(x,t),t). \end{aligned} \tag{7.6}$$

Here we abused the definition of the tangential gradient of functions on S_T. Applying (7.6) to (7.3) and observing that $\nu(p(x,t),t) \cdot \nabla_{\Gamma(t)} v(p(x,t),t,\varepsilon^{-1}d(x,t)) = 0$, we obtain

$$\varepsilon^{-1}\partial_r v(p(x,t),t,\varepsilon^{-1}d(x,t)) \approx -V_\Gamma^N(p(x,t),t)v(p(x,t),t,\varepsilon^{-1}d(x,t)) \tag{7.7}$$

and thus (7.6) becomes

$$\begin{aligned} \nabla u^\varepsilon(x,t) \approx \nabla_{\Gamma(t)} v(p(x,t),t,\varepsilon^{-1}d(x,t)) \\ - V_\Gamma^N(p(x,t),t)v(p(x,t),t,\varepsilon^{-1}d(x,t))\nu(p(x,t),t). \end{aligned} \tag{7.8}$$

Next we compute $\Delta u^\varepsilon = \operatorname{div}\nabla u^\varepsilon$. For a vector field F on $\Omega_\varepsilon(t)$ with each fixed $t\in[0,T]$,

$$\begin{aligned}\operatorname{div}F(x) &= \operatorname{trace}[\nabla F(x)]\\ &= \operatorname{trace}[\{I_n-\nu(x,t)\otimes\nu(x,t)\}\nabla F(x)]+\operatorname{trace}[\nu(x,t)\otimes\nu(x,t)\nabla F(x)]\\ &= \operatorname{div}_{\Gamma(t)}F(x)+\nu(x,t)\cdot\partial_\nu F(x)\end{aligned}$$

holds since $\nu\otimes\nu$ is a projection matrix onto the ν-direction. Hence we have

$$\begin{aligned}\operatorname{div}\bigl(\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\bigr) &= \operatorname{div}_{\Gamma(t)}\bigl(\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\bigr)\\ &\qquad+\nu(x,t)\cdot\partial_\nu\bigl(\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\bigr).\end{aligned}$$

Moreover, since $p(x+h\nu(x,t),t)=p(x,t)$ and $d(x+h\nu(x,t),t)=d(x,t)+h$ for sufficiently small $h\in\mathbb{R}$, it follows that

$$\begin{aligned}&\nabla_{\Gamma(t)}v(p(x+h\nu(x,t),t),t,\varepsilon^{-1}d(x+h\nu(x,t),t)) =\\ &\qquad\qquad\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t)+\varepsilon^{-1}h)\end{aligned}$$

and thus

$$\partial_\nu\bigl(\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\bigr)=\varepsilon^{-1}\partial_r\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))$$

by the formula $\partial_\nu f(x)=\lim_{h\to0}\{f(x+h\nu(x,t))-f(x)\}/h$ for functions f on $\Omega_\varepsilon(t)$ with fixed $t\in[0,T]$. Hence we obtain

$$\begin{aligned}&\operatorname{div}\bigl(\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\bigr) =\\ &\quad\operatorname{div}_{\Gamma(t)}\bigl(\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\bigr)+\varepsilon^{-1}\partial_r\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t)).\end{aligned}$$

Similarly we have

$$\begin{aligned}&\operatorname{div}\bigl(V_\Gamma^N(p(x,t),t)v(p(x,t),t,\varepsilon^{-1}d(x,t))\nu(p(x,t),t)\bigr)\\ &\quad=\operatorname{div}_{\Gamma(t)}\bigl(V_\Gamma^N(p(x,t),t)v(p(x,t),t,\varepsilon^{-1}d(x,t))\nu(p(x,t),t)\bigr)\\ &\qquad\quad+\nu(x,t)\cdot\{\varepsilon^{-1}V_\Gamma^N(p(x,t),t)\partial_r v(p(x,t),t,\varepsilon^{-1}d(x,t))\nu(p(x,t),t)\}\\ &\quad\approx\operatorname{div}_{\Gamma(t)}\bigl(V_\Gamma^N(p(x,t),t)v(p(x,t),t,\varepsilon^{-1}d(x,t))\nu(p(x,t),t)\bigr)\\ &\qquad\quad-\{V_\Gamma^N(p(x,t),t)\}^2\,v(p(x,t),t,\varepsilon^{-1}d(x,t)).\end{aligned}$$

Here the last approximation follows from $\nu(x,t)=\nu(p(x,t),t)$ and (7.7). Hence, by (7.8),

$$\begin{aligned}\Delta u^\varepsilon(x,t) &\approx \operatorname{div}_{\Gamma(t)}\bigl(\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\bigr)\\ &\quad+\varepsilon^{-1}\partial_r\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\\ &\quad-\operatorname{div}_{\Gamma(t)}\bigl(V_\Gamma^N(p(x,t),t)v(p(x,t),t,\varepsilon^{-1}d(x,t))\nu(p(x,t),t)\bigr)\\ &\qquad\qquad+\{V_\Gamma^N(p(x,t),t)\}^2\,v(p(x,t),t,\varepsilon^{-1}d(x,t)).\end{aligned}\tag{7.9}$$

On the other hand, we differentiate both sides of (7.5) with respect to t to get

$$\begin{aligned}\partial_t u^\varepsilon(x,t) &= \partial_t p(x,t)\cdot\nabla v(p(x,t),t,\varepsilon^{-1}d(x,t))\\ &\qquad+\partial_t v(p(x,t),t,\varepsilon^{-1}d(x,t))+\varepsilon^{-1}\partial_t d(x,t)\partial_r v(p(x,t),t,\varepsilon^{-1}d(x,t)).\end{aligned}$$

To this equality we apply (7.7) and

$$\partial_t p(x,t)=-\partial_t d(x,t)\nu(x,t)-d(x,t)\partial_t\nu(x,t)\approx V_\Gamma^N(p(x,t),t)\nu(p(x,t),t),$$

where the last approximation follows from the assumptions (1), (2), and $\nu(x,t) = \nu(p(x,t),t)$. Then we have

$$\begin{aligned}(7.10)\quad \partial_t u^\varepsilon(x,t) \approx{}& V_\Gamma^N(p(x,t),t)\nu(p(x,t),t)\cdot\nabla v(p(x,t),t,\varepsilon^{-1}d(x,t))\\ &+\partial_t v(p(x,t),t,\varepsilon^{-1}d(x,t)) + \{V_\Gamma^N(p(x,t),t)\}^2\, v(p(x,t),t,\varepsilon^{-1}d(x,t)).\end{aligned}$$

Substituting (7.9) and (7.10) for the equation (7.2), we obtain

$$\begin{aligned}&\partial_t v(p(x,t),t,\varepsilon^{-1}d(x,t)) + V_\Gamma^N(p(x,t),t)\nu(p(x,t),t)\cdot\nabla v(p(x,t),t,\varepsilon^{-1}d(x,t))\\ &+\mathrm{div}_{\Gamma(t)}\big(V_\Gamma^N(p(x,t),t)v(p(x,t),t,\varepsilon^{-1}d(x,t))\nu(p(x,t),t)\big)\\ &-\mathrm{div}_{\Gamma(t)}\big(\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))\big)-\varepsilon^{-1}\partial_r\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t))=0.\end{aligned}$$

Now let us make an additional assumption: the function $v(y,t,r)$ is independent of the variable r. Then, the above equation reads

$$\begin{aligned}\partial_t v(y,t) + V_\Gamma^N(y,t)\nu(y,t)\cdot\nabla v(y,t) + \mathrm{div}_{\Gamma(t)}\big(V_\Gamma^N(y,t)v(y,t)\nu(y,t)\big)&\\ -\mathrm{div}_{\Gamma(t)}\big(\nabla_{\Gamma(t)}v(y,t)\big)&=0\end{aligned}$$

with $y = p(x,t)\in\Gamma(t)$. Finally we observe that

$$\mathrm{div}_{\Gamma(t)}(V_\Gamma^N v\nu) = \nabla_{\Gamma(t)}(V_\Gamma^N v)\cdot\nu + V_\Gamma^N v\,\mathrm{div}_{\Gamma(t)}\nu = 0 + V_\Gamma^N v\cdot(-H) = -V_\Gamma^N Hv,$$

where $H = -\mathrm{div}_{\Gamma(t)}\nu$ is the mean curvature of $\Gamma(t)$, to obtain

$$\partial_t v(y,t) + V_\Gamma^N(y,t)\nu(y,t)\cdot\nabla v(y,t) - V_\Gamma^N(y,t)H(y,t)v(y,t) - \Delta_{\Gamma(t)}v(y,t) = 0$$

for $(y,t)\in S_T$. This is the limit equation (1.1) we mentioned in Section 1.

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