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Unilateral problem for the Stokes equations: the well-posedness and finite element approximation

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Unilateral problem for the Stokes equations: the well-posedness and finite element approximation

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We consider the stationary Stokes equations under a unilateral boundary condition of Signorini's type, which is one of artificial boundary conditions in flow problems. The well-posedness is discussed through its variational inequality formulation. We also consider the finite element approximation for a regularized penalty problem. The well-posedness, stability and error estimates are established. The lack of a coupled Babuška and Brezzi's condition makes analysis difficult. We offer a new method of analysis; In particular, our device to treat the pressure seems to be new and of interest. Numerical examples to confirm the validity of our theoretical results are also presented.

Key words: Stokes equations, finite element approximation, unilateral boundary condition

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1 Introduction

We suppose that Ω is a bounded domain in $\mathbb{R}^d$ with $d = 2, 3$ and that the boundary $\partial\Omega$ is composed of three parts S_1, S_2 and Γ. Those S_1, S_2 and Γ are assumed to be smooth but the whole boundary $\partial\Omega$ is not necessarily smooth. One may imagine a branched pipe as illustrated in Fig. 1. The first purpose of this paper is to study the well-posedness of the following unilateral boundary value problem for the Stokes equations

$$
\begin{aligned}
&-\nu\Delta u + \nabla p = f, \quad \nabla\cdot u = 0 && \text{in } \Omega, && \text{(1a)}\\
&u = 0 && \text{on } S_1 \cup S_2, && \text{(1b)}\\
&u_n + g_n \ge 0, && \text{on } \Gamma, && \text{(1c)}\\
&\tau_n(u,p) + \alpha_n \ge 0 && \text{on } \Gamma, && \text{(1d)}\\
&(u_n + g_n)(\tau_n(u,p) + \alpha_n) = 0 && \text{on } \Gamma, && \text{(1e)}\\
&\tau_T(u) + \alpha_T = 0 && \text{on } \Gamma && \text{(1f)}
\end{aligned}
$$

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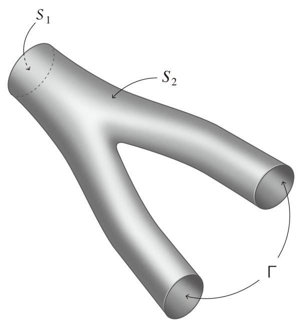

Figure 1: Example of Ω (branched pipe)

for the velocity $u = (u_1, \ldots, u_d)$ and the pressure p with the density $\rho = 1$ and the kinematic viscosity ν of the viscous incompressible fluid under consideration. Therein,

$$\tau(u,p) = \sigma(u,p)n$$

denotes the traction vector on $\partial\Omega$, where n is the outward normal vector to $\partial\Omega$, $\sigma(u,p) = (\sigma_{i,j}(u,p)) = -pI + 2\nu D(u)$ the stress tensor, $D(u) = (D_{i,j}(u)) = \left(\frac{1}{2}\left(\nabla u + \nabla u^{\mathrm{T}}\right)\right)$ the deformation-rate tensor and I the identity matrix. For a vector-valued function v on $\partial\Omega$, its normal and tangential components are denoted, respectively, as

$$v_n = v \cdot n, \quad v_T = v - v_n n.$$

In particular, $\tau_n(u,p) = \tau(u,p) \cdot n$ and $\tau_T(u) = \tau(u,p) - \tau_n(u,p)n$ are normal and tangential traction vectors, respectively. Moreover, f, g and α are prescribed functions. We also consider the finite element approximation for a regularized penalty problem to (1) which is given as

$$-\nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \qquad \text{in } \Omega, \tag{2a}$$

$$u = 0 \qquad \text{on } S_1 \cup S_2, \tag{2b}$$

$$\tau_n(u,p) + \alpha_n = \frac{1}{\varepsilon}\phi_\delta(u_n + g_n) \qquad \text{on } \Gamma, \tag{2c}$$

$$\tau_T(u) + \alpha_T = 0 \qquad \text{on } \Gamma \tag{2d}$$

with $0 < \varepsilon \ll 1$ and $0 < \delta \ll 1$. Therein, $\phi_\delta(s)$ is a C^1 regularization of $[s]_- = \max\{0, -s\}$. We can take, for example,

$$\phi_\delta(s) = \begin{cases} 0 & (s \geq 0) \\ (\sqrt{s^2 + \delta^2} - \delta) & (s < 0). \end{cases} \tag{3}$$

First of all, we mention our motivation for studying (1) and (2). In numerical simulation of real-world flow problems, we often encounter some issues concerning artificial boundary conditions. A typical and important example is the blood flow problem in the large arteries, where the blood is assumed to be a viscous incompressible fluid (cf. [13, 27]). The blood vessel is modeled by a branched pipe as illustrated, for example, by Fig. 1. Then, for $T > 0$, we consider the

Navier-Stokes equations for the velocity $v = (v_1, \dots, v_d)$ and the pressure q,

$$v_t + (v \cdot \nabla)v = \nabla \cdot \sigma(v, q) + f, \quad \nabla \cdot v = 0 \qquad \text{in } \Omega \times (0, T), \tag{4a}$$
$$v = b \qquad \text{on } S_1 \times (0, T), \tag{4b}$$
$$v = 0 \qquad \text{on } S_2 \times (0, T) \tag{4c}$$

with the initial condition $v|_{t=0} = v_0$. We are able to give a velocity profile $b = b(x, t)$ at the *inflow boundary* S_1 and the flow is supposed to be a perfect non-slip on the wall S_2. Then, the blood flow simulation is highly dependent on the choice of artificial boundary conditions posed on the *outflow boundary* Γ.

In a previous paper, Zhou and Saito [29], we discussed an issue of the free-traction condition

$$\tau(v, q) = 0 \quad \text{on} \quad \Gamma, \tag{5}$$

which is one of the common outflow boundary conditions (cf. [17, 18]), and some nonlinear energy-preserving boundary conditions (cf. [4, 5, 6, 7, 8]) from the view-point of energy inequality. Moreover, we proposed a new outflow boundary condition,

$$v_n \geq 0, \quad \tau_n(v, q) \geq 0, \quad v_n \tau_n(v, q) = 0, \quad \tau_T(v) = 0 \quad \text{on } \Gamma. \tag{6}$$

This is an analogy to Signorini's condition in the theory of elasticity (cf. [20]) and is indeed a generalization of the free-traction condition (5). Namely,

$$\text{if } v_n > 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(v, q) = 0 \text{ on } \omega;$$
$$\text{if } v_n = 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(v, q) \geq 0 \text{ on } \omega.$$

An advantage of employing (6) is that (4) admits energy inequality, whereas it is not guaranteed under (5). To describe it more specifically, we take a *reference flow* (g, π) which is the solution of the Stokes system

$$\nabla \cdot \sigma(g, \pi) = 0, \quad \nabla \cdot g = 0 \quad \text{in } \Omega, \tag{7a}$$
$$g = b \text{ on } S_1, \quad g = 0 \text{ on } S_2, \quad g = -g_0(x) \int_{S_1} b \cdot n \; dS_1 \text{ on } \Gamma \tag{7b}$$

for all $t \in [0, T]$, where $g_0 = g_0(x) \in C_0^\infty(\Gamma)^d$ is a prescribed function satisfying

$$\int_\Gamma g_0 \cdot n \; d\Gamma = 1, \quad g_0 \cdot n \geq 0 \;\; \text{on } \Gamma. \tag{8}$$

(The function g is nothing but a lifting function of b.) By using this, we will find (v, q) of the form

$$v = u + g, \qquad q = p + \pi.$$

Assuming (4) admits a smooth solution $(v, q) = (u + g, p + \pi)$ in $0 \leq t \leq T$ and multiplying the both sides of (4a) by u, we have by the integration by parts

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)^d} + \underbrace{2\nu \int_\Omega D_{ij}(u) D_{ij}(u) \; dx + \frac{1}{2}\int_\Gamma v_n |u|^2 \; d\Gamma - \int_\Gamma \tau(v, q) \cdot u \; d\Gamma}_{=I}$$
$$= \int_\Omega [f - g_t - (g \cdot \nabla) g] \cdot u \; dx - \int_\Omega (u \cdot \nabla) g \cdot u \; dx. \tag{9}$$

With the aid of (6), we derive $I \geq 0$ and, consequently,

$$\sup_{t\in[0,T]} \|u\|^2_{L^2(\Omega)^d} + 2\nu \int_0^T D_{ij}(u)D_{ij}(u) \leq C, \tag{10}$$

where C denotes a positive constant depending only on f, u_0, b and T (cf. [29, Theorem 4]). This inequality is of use. It plays a crucial role in the construction of a solution of the Navier-Stokes equations (cf. [29]). Moreover, it is connected with the stability of numerical schemes from the view-point of numerical computation. That is, it is preferred that energy inequality does not spoiled after discretizations (cf. [28]). With (5), we do not know whether $I \geq 0$ or not so that energy inequality cannot be derived even for the continuous case.

The condition (6) is described in terms of inequalities so that it does not fit numerical calculations. However, we can utilize its penalty approximation

$$\tau_n(v,q) = \frac{1}{\varepsilon}[v_n]_-, \quad \tau_T(v) = 0 \quad \text{on } \Gamma, \tag{11}$$

where $0 < \varepsilon \ll 1$ and

$$[s]_\pm = \max\{0, \pm s\}, \quad s = [s]_+ - [s]_- \quad (s \in \mathbb{R}). \tag{12}$$

We also obtain energy inequality with (11) for a sufficiently small ε (cf. [29, Theorem 5]). Moreover, after introducing a C^1 regularization ϕ_δ of $[\cdot]_-$, we can solve (4) with (11) by using, for example, Newton's iteration.

Our final aim is to develop the theory for the initial-boundary value problems for the Navier-Stokes equations (4) with (6) or with (11) from the standpoint both of analysis and numerical computations. As a primary step, we studied the well-posedness of these problems in Ladyzhenskaya's class in [29]. That is, we studied the unique existence of a solution of

$$\begin{aligned}
&u_t + ((u+g)\cdot\nabla)u + (u\cdot\nabla)g - \nabla\cdot\sigma(u,p) = F, \quad \nabla\cdot u = 0 && \text{in } \Omega,\\
&u = 0 && \text{on } S_1 \cup S_2,\\
&u_n + g_n \geq 0, \quad \tau_n(u,p) + \tau_n(g,\pi) \geq 0 && \text{on } \Gamma,\\
&(u_n+g_n)(\tau_n(u,p) + \tau_n(g,\pi)) = 0, \quad \tau_T(u) + \tau_T(g) \geq 0 && \text{on } \Gamma,
\end{aligned}$$

where $F = f - g_t - (g\cdot\nabla)g$.

In the present paper, we concentrate our attention to the discretization of the space variable. Thus, we study the finite element approximation by using model Stokes problems. Consequently, we come to consider Problems (1) and (2).

As a matter of fact, (1) and (2) themselves are not new problems. In a classical monograph, Kikuchi and Oden [20], Chapter 7 is devoted to similar problems. However, their problem contains the traction condition $\tau(u,p) = h$. More precisely, they suppose that S_2 is divided into two parts S_{21}, S_{22} and consider

$$u = 0 \quad \text{on } S_{21}, \quad \tau(u,p) = h \quad \text{on } S_{22} \tag{13}$$

instead of (1b). Then, supposing

$$\overline{\Gamma} \cap \overline{(S_1 \cup S_{21})} = \emptyset, \tag{14}$$

we can prove that there exists a domain constant $C > 0$ satisfying

$$C\left[\|q\|_{L^2(\Omega)} + \|\tau\|_{H^{-1/2}(\Gamma)}\right] \leq \sup_{v\in H^1(\Omega)^d, v|_{S_1\cup S_{21}}=0} \frac{\int_\Omega q(\nabla\cdot v)\,dx + \int_\Gamma \tau v_n\,d\Gamma}{\|v\|_{H^1(\Omega)}} \tag{15}$$

for any $(q, \tau) \in L^2(\Omega) \times H^{-1/2}(\Gamma)$ (cf. [20, Theorem 7.2]). This inequality is usually referred to as the coupled Babuška-Brezzi condition. The well-posedness and error estimates of the corresponding penalty problem (without any regularization) are direct consequences of this result from the general theory (cf. [2]). In contrast, we are interested in establishing a formulation without the traction boundary condition. Unfortunately, if $S_{22} = \emptyset$, (15) is not available and it makes analysis somewhat difficult. Moreover, we do not prefer assuming (14). Consequently, we have to develop a totally new method of analysis in this paper. In particular, we offer a new device to treat the pressure part.

Finite element approximation of another class of unilateral boundary value problems for the Stokes equations, say unilateral problems of *friction type*, are discussed, for example, in [1, 19, 21, 22].

This paper is composed of 7 sections. After having introduced basic definitions and recalled some standard results in Section 2, we state the variational formulation (1) in Section 3. The well-posedness of (1) is also established there. Sections 4 is devoted to the presentation of the finite element approximation for (2). The well-posedness and error estimates are proved in Sections 5 and 6, respectively. Finally, we confirm our results by numerical experiments in Section 7.

2 Preliminary

Geometry We recall that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain and the boundary $\partial\Omega$ is composed of three parts S_1, S_2 and Γ. We deal with the following two cases:

(G1) S_1, S_2 and Γ are smooth surface (curve) and Ω is a Lipschitz domain (cf. Fig. 1);

(G2) S_1, S_2 are polygon (line segment) and Ω is a polyhedral (polygonal) domain.

In what follows, we suppose that Ω is given as (G1) or (G2) unless otherwise stated explicitly. Moreover, we set

$$S = S_1 \cup S_2.$$

Throughout this paper, C denotes various positive constants depending on Ω.

Remark 1. Although we mostly deal with the case illustrated by Fig. 1, our discussion is also valid for the case where $\partial\Omega$ is smooth with $\overline{\Gamma} \cap \overline{S}_2 = \emptyset$, $\overline{S}_2 \cap \overline{S}_1 = \emptyset$, and $\overline{S}_1 \cap \overline{\Gamma} = \emptyset$.

Function spaces and forms We use the standard Lebesgue and Sobolev spaces, for example, $L^2(\Omega)$, $H^1(\Omega)$, $L^2(\Gamma)$, $H^{1/2}(\Gamma)$. (We follow the notation of [25] as for function spaces and their norms.) The abbreviations

$$(v, w) = (v, w)_\Omega = (v, w)_{L^2(\Omega)}, \quad (v, w)_\Gamma = (v, w)_{0,\Gamma} = (v, w)_{L^2(\Gamma)},$$
$$\|v\| = \|v\|_\Omega = \|v\|_{0,\Omega} = \|v\|_{L^2(\Omega)}, \quad \|v\|_1 = \|v\|_{1,\Omega} = \|v\|_{H^1(\Omega)}, \quad \|v\|_\Gamma = \|v\|_{0,\Gamma} = \|v\|_{L^2(\Gamma)}$$

will be employed. Moreover,

$$|v|_m = |v|_{m,\Omega} = |v|_{H^m(\Omega)}, \qquad |v|_{m,\Gamma} = |v|_{H^m(\Gamma)}$$

are the semi-norms of $H^m(\Omega)$, $H^m(\Gamma)$.

For a vector-valued function space, we use the same symbol to denote its norm;

$$\|v\| = \|v\|_{L^2(\Omega)^d} \quad (v \in L^2(\Omega)^d), \qquad \|v\|_1 = \|v\|_{H^1(\Omega)^d} \quad (v \in H^1(\Omega)^d).$$

The basic function spaces of our consideration are

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } S\} \quad \text{and} \quad Q = L^2(\Omega).$$

They are Hilbert spaces equipped with the norms $\|v\|_1$ and $\|q\|$, respectively. We use closed subspaces of V,

$$V^\sigma = \{v \in V \mid \nabla \cdot v = 0 \text{ in } \Omega\}, \quad V_0 = H_0^1(\Omega)^d, \quad V_0^\sigma = \{v \in V_0 \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$

and that of Q,

$$Q_0 = \left\{ q \in Q \mid \int_\Omega q \, dx = 0 \right\}.$$

Convex subsets

$$K = \{v \in V \mid v_n + g_n \geq 0 \text{ on } \Gamma\} \quad \text{and} \quad K^\sigma = \{v \in V^\sigma \mid v_n + g_n \geq 0 \text{ on } \Gamma\}$$

of V and V^σ, respectively, play important roles.

We recall the so-called *Lions-Magenes space* $H_{00}^{1/2}(\Gamma)$. It is defined as (cf. [25, §11.5, Ch. 1])

$$H_{00}^{1/2}(\Gamma) = \{\mu \in H^{1/2}(\Gamma) \mid \rho^{-1/2}\mu \in L^2(\Gamma)\}$$

which is a Hilbert space equipped with the norm

$$\|\mu\|_{H_{00}^{1/2}(\Gamma)} = \left(\|\mu\|_{H^{1/2}(\Gamma)}^2 + \|\rho^{-1/2}\mu\|_\Gamma^2 \right)^{1/2}.$$

Here, $\rho \in C^\infty(\overline{\Gamma})$ denotes any positive function satisfying $\rho|_{\partial\Gamma} = 0$ and, for $x_0 \in \partial\Gamma$,

$$\lim_{x \to x_0} \frac{\rho(x)}{\text{dist }(x, \partial\Gamma)} = d' > 0$$

with some $d' > 0$. Moreover, we know (cf. [25, Theorem 11.7, Ch. 1])

$$H_{00}^{1/2}(\Gamma) = (H_0^1(\Gamma), L^2(\Gamma))_{1/2,2} \quad \text{(algebraly and topologically)}, \tag{16}$$

where the righthand side denotes the real interpolation space between $L^2(\Gamma)$ and $H_0^1(\Gamma)$ with the exponent $1/2$ and $p = 2$. In particular, $H_{00}^{1/2}(\Gamma)$ is *strictly* included in $H^{1/2}(\Gamma)$.

Below we set

$$M = H_{00}^{1/2}(\Gamma), \quad \|\mu\|_{1/2,\Gamma} = \|\mu\|_{H_{00}^{1/2}(\Gamma)}$$

and

$$M_0 = \left\{ \mu \in M \mid \int_\Gamma \mu \, d\Gamma = 0 \right\}.$$

In general, X' denotes the topological dual space of a Banach space X and the norm of X' is defined as

$$\|\varphi\|_{X'} = \sup_{v \in X} \frac{\langle \varphi, v \rangle_{X',X}}{\|v\|_X} = \sup_{v \in X, v \neq 0} \frac{\langle \varphi, v \rangle_{X',X}}{\|v\|_X},$$

where $\langle \cdot, \cdot \rangle_{X',X}$ is the duality pairing between X' and X. For a closed subspace Y of X and $\varphi \in Y'$, we mean by $\|\varphi\|_{Y'}$

$$\|\varphi\|_{Y'} = \sup_{v \in Y} \frac{\langle \varphi, v \rangle_{X',X}}{\|v\|_X}.$$

Set

$$\begin{aligned}\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{V',V} = &\text{ the duality pairing between } V' \text{ and } V,\\ [\cdot,\cdot] = [\cdot,\cdot]_{M',M} = &\text{ the duality pairing between } M' \text{ and } M,\\ [[\cdot,\cdot]] = [[\cdot,\cdot]]_{(M^d)',M^d} = &\text{ the duality pairing between } (M^d)' \text{ and } M^d.\end{aligned}$$

We use the following forms:

$$\begin{aligned} a(u,v) &= 2\nu \int_\Omega D_{i,j}(u)D_{i,j}(v)\ dx && (u,v \in H^1(\Omega)^d);\\ b(p,u) &= -\int_\Omega p(\nabla\cdot u)\ dx && (p\in Q,\ u\in H^1(\Omega)^d).\end{aligned}$$

Trace and lifting operators Let $\mathrm{Tr} = \mathrm{Tr}(\Omega,\Gamma)$ be a trace operator from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$. The meaning of $\mathrm{Tr}(\Omega,S)$ is the same.

Lemma 2. *The trace operator $v\mapsto \mu = \mathrm{Tr}\, v$ is linear and continuous of $V\to M^d$. Conversely, there exists a linear and bounded operator $\mathcal{E}$ of $M^d\to V$, which is called a lifting operator, such that $\mathcal{E}\mu = \mu$ on Γ for all $\mu\in M^d$.*

This result directly follows from [15, Theorem 2.5] and [16, Theorem 1.5.2.3]. A partial result is also reported in [26, Theorems 1.1 and 5.1]. As a consequence of Lemma 2, we obtain a lifting operator $\mathcal{E}_n : M\to V$ such that

$$(\mathcal{E}_n\mu)_n = \mu\ ,(\mathcal{E}_n\mu)_T = 0 \ \text{ on } \Gamma, \qquad \|\mathcal{E}_n\mu\|_1 \le C\|\mu\|_{1/2,\Gamma}$$

for any $\mu\in M$.

Below, we will often write as $v|_\Gamma = \mathrm{Tr}\, v$ if there is no fear of confusion.

Remark 3. In view of Lemma 2 and the standard trace/lifting theorem, the zero extension $\hat\mu$ of $\mu\in M^d$ into $\partial\Omega$;

$$\hat\mu = \begin{cases}\mu & \text{on } \Gamma,\\ 0 & \text{on } \partial\Omega\backslash\Gamma\end{cases}$$

belongs to $H^{1/2}(\partial\Omega)^d$.

Remark 4. Another definition of $H^{1/2}_{00}(\Gamma)$ is given by Baiocchi and Capelo [3, Page 379]. That is,

$$H^{1/2}_{00}(\Gamma) = \{\mathrm{Tr}\, v \mid v\in H^1(\Omega),\ \ \mathrm{Tr}(\Omega,S)v = 0\}$$

which is a Hilbert space equipped with the norm

$$\|\mu\|_{H^{1/2}_{00}(\Gamma)} = \inf\{\|v\|_1 \mid v\in H^1(\Omega),\ \ \mathrm{Tr}(\Omega,S)v=0,\ \ \mathrm{Tr}\, v = \mu\}.$$

Lemmas We collect here several standard results used below.

Lemma 5 (Korn's inequality, [20, Lemma 6.2])**.** *There exists a positive constant C_{K} depending only on Ω such that*

$$a(v,v)\ge C_{\mathrm{K}}\|v\|_1^2 \qquad (v\in V).$$

Lemma 6 ([14, Lemma I.2.1]). *Suppose that $\Phi \in V_0'$ satisfies*

$$\langle \Phi, v\rangle = 0 \qquad (\forall v \in V_0^\sigma).$$

Then, there exists a unique $p \in Q_0$ such that

$$\int_\Omega p\ (\nabla \cdot v)\ dx = \langle \Phi, v\rangle \quad (\forall v \in V_0), \quad \|p\| \le C\|\Phi\|_{V'}.$$

Lemma 7 ([14, Corollary I.2.4]). *For any $q \in Q_0$, there exists $w \in V_0$ such that*

$$\nabla \cdot w = -q \text{ in } \Omega, \quad \|w\|_1 \le C\|q\|.$$

Remark 8. An readily obtainable consequence of Lemma 7 is that there is a positive constant $\gamma > 0$ depending only on Ω satisfying

$$\inf_{q\in Q_0} \sup_{v\in V_0} \frac{b(q,v)}{\|q\|\|v\|_1} \ge \gamma \tag{17}$$

This result is well-known as the inf-sup condition or Babuška and Brezzi's condition.

Lemma 9 ([14, Lemma I.2.2]). *For any $\mu \in H^{1/2}(\partial\Omega)^d$ satisfying $(\mu, n)_{L^2(\partial\Omega)} = 0$, there exists $v \in H^1(\Omega)^d$ such that*

$$\nabla \cdot v = 0 \text{ in } \Omega, \quad v = \mu \text{ on } \partial\Omega, \quad \|v\|_1 \le C\|\zeta\|_{H^{1/2}(\partial\Omega)^d}.$$

Re-definition of traction vectors Let us propose the re-definition of $\tau(u,p)$. If a smooth vector field u and scalar field p satisfy the Stokes equation

$$-\nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega$$

for a given $f \in L^2(\Omega)^d$, they satisfies

$$a(u,v) + b(p,v) + \int_\Gamma \tau(u,p)\cdot v = (f,v) \qquad (\forall v \in V) \tag{18}$$

and

$$a(u,v) + b(p,v) = (f,v) \qquad (\forall v \in V_0). \tag{19}$$

(In (18), $\tau(u,p)$ is understood as a usual function defined on Γ.) Based on those identities, we re-define the traction vector $\tau(u,p)$ for functions $(u,p) \in V^\sigma \times Q$ satisfying (19) as a functional over M^d defined by

$$[[\tau(u,p),\mu]] = a(u,w_\mu) + b(p,w_\mu) - (f,w_\mu) \qquad (\mu \in M^d), \tag{20}$$

where $w_\mu = \mathcal{E}\mu \in V$. Actually, the righthand side of (20) does not depend on the way of extension; Hence, this definition is well-defined. Similarly, we re-define as

$$[[\tau_T(u),\mu]] = a(u,w_\mu) + b(p,w_\mu) - (f,w_\mu) \qquad (\mu \in M^d \text{ with } \mu_n = 0;\ w_\mu = \mathcal{E}\mu) \tag{21}$$

and

$$[\tau_n(u,p),\mu] = a(u,w_\mu) + b(p,w_\mu) - (f,w_\mu) \qquad (\mu \in M;\ w_\mu = \mathcal{E}_n\mu). \tag{22}$$

Then, we deduce an expression

$$[[\tau(u,p),\mu]] = [\tau_n(u,p),\mu_n] + [[\tau_T(u),\mu_T]] \qquad (\mu \in M^d). \tag{23}$$

3 Variational formulations and well-posedness

From now on, we always assume

$$f \in Q^d, \qquad b \in M^d, \qquad \beta \equiv -\int_S b \cdot n \, dS > 0. \tag{24}$$

We take $g \in H^1(\Omega)^d$ satisfying

$$\nabla \cdot g = 0 \text{ in } \Omega, \quad g|_S = b, \quad g|_C = 0, \quad g|_\Gamma = \beta g_0, \tag{25}$$

where g_0 is the function defined as (8). Then, we have

$$g_n \geq 0 \text{ on } \Gamma, \qquad g_n \in M, \qquad \alpha \equiv 2\nu \mathcal{D}(g) n \in (M^d)' \tag{26}$$

Under those assumptions and re-definitions in the previous section, we precisely interpret (1) as follows.

(PDE) Find $(u, p) \in V \times Q$ such that

$$a(u,v) + b(p,v) = (f,v) \qquad (\forall v \in V_0), \tag{27a}$$
$$b(q,u) = 0 \qquad (\forall q \in Q), \tag{27b}$$
$$u_n + g_n \geq 0 \qquad \text{a.e. on } \Gamma, \tag{27c}$$
$$[\tau_n(u,p) + \alpha_n, \mu] \geq 0 \qquad (\forall \mu \in M, \ \mu \geq 0), \tag{27d}$$
$$[\tau_n(u,p) + \alpha_n, u_n + g_n] = 0 \tag{27e}$$
$$[[\tau_T(u) + \alpha_T, \mu]] = 0 \qquad (\forall \mu \in M^d, \ \mu_n = 0). \tag{27f}$$

Obviously, if a solution (u, p) of (PDE) is sufficiently smooth, it solves (1) in the classical sense. (PDE) is equivalent to the following variational inequality.

(VI) Find $(u, p) \in K \times Q$ such that

$$a(u, v-u) + b(p, v-u) \geq (f, v) - [[\alpha, v]] \qquad (\forall v \in K), \tag{28a}$$
$$b(q,u) = 0 \qquad (\forall q \in Q). \tag{28b}$$

In this section, we prove the following two theorems.

Theorem 10. *Problems* (VI) *and* (PDE) *are equivalent.*

Theorem 11. *There exists a unique solution* $(u, p) \in K \times Q$ *of* (VI) *and it holds that*

$$\|u\|_1 + \|p\| \leq C_*, \tag{29}$$

where C_* *denotes a positive constant depending only on* Ω, $\|f\|$, $\|\alpha\|_{(M^d)'}$ *and* $\|g\|_1$.

Remark 12. The boundary condition (27f) is nothing but one of alternatives. One can pose

$$u_T + \alpha'_T = 0 \quad \text{a.e. on } \Gamma \tag{30}$$

with a prescribed α'_T instead of (27f). Actually, the discussion presented below remains true if we re-choose a suitable lifting function g and replace the original V by

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma, \ v_T = 0 \text{ on } \Gamma\}.$$

Proof of Theorem 10. (PDE)⇒(VI). Let $(u, p) \in V \times Q$ be a solution of (PDE). We verify (u, p) is a solution of (VI). First, we have $u \in K$ by (27c) and (27b). By using (20), (23), and (27c)–(27f), we have for any $v \in K$

$$\begin{aligned}
&a(u, v - u) + b(p, v - u) - (f, v) + [[\alpha, v - u]] \\
&= [[\tau(u, p), v - u]] + [[\alpha, v - u]] \\
&= [\tau_n(u, p), v_n - u_n] + \underbrace{[[\tau_T(u) + \alpha_T, v_T - u_T]]}_{=0} + [\alpha_n, v_n - u_n] \\
&= \underbrace{[\tau_n(u, p) + \alpha_n, v_n + g_n]}_{\geq 0} - \underbrace{[\tau_n(u, p) + \alpha_n, u_h + g_n]}_{=0} \geq 0.
\end{aligned}$$

(VI)⇒(PDE). Let $(u, p) \in K \times Q$ be a solution of (VI). We now verify (u, p) actually satisfies (PDE). First, (27b) and (27c) are obvious.

Let $v \in V_0$ be arbitrary. Substituting $v = u \pm v \in K$ into (28a), we have (27a).

We recall $\tau(u, p)$ is defined as (20). Thus, (28a) implies

$$[[\tau(u, p), v - u]] \geq -[[\alpha, v - u]] \qquad (\forall v \in K).$$

Moreover, by (23)

$$[\tau_n(u, p) + \alpha_n, v_n - u_n] + [[\tau_T(u) + \alpha_T, v_T - u_T]] \geq 0 \qquad (\forall v \in K). \tag{31}$$

Let $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n = 0$ on Γ. Substituting $v = u \pm \mathcal{E}\psi \in K$ into (31), we have $[[\tau_T(u) + \alpha_T, \psi]] = 0$. By the density, this implies (27f).

Next, let $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n \geq 0$ and $\psi_T = 0$ on Γ. Substituting $v = u + \mathcal{E}\psi \in K$ into (31), we have $[\tau_n(u, p) + \alpha_n, \psi_n] \geq 0$. By the density, this implies (27d).

Combining (31) and (27f), we have

$$[\tau_n(u, p) + \alpha_n, v_n - u_n] \geq 0 \qquad (\forall v \in K).$$

At this stage, we introduce $w^* \in V$ satisfying

$$w^* = g \text{ on } \Gamma, \quad \|w^*\|_1 \leq C\|g\|_1. \tag{32}$$

Since $g\ |_\Gamma \in M^d$, such w^* really exists in view of the trace theorem. But it does not satisfy the divergence-free condition; Thus, $w^* \notin V^\sigma$. We now have $-w_n^* + g_n = -g_n + g_n \geq 0$ and $2u_n + w_n^* + g_n = 2(u_n + g_n) \geq 0$. Hence, we can choose as $v = -w^*$ and $v = 2u + w^*$ above and obtain (27e). □

Proof of Theorem 11. Since a is a coercive bilinear form in $V^\sigma \times V^\sigma$ by virtue of Lemma 5, we can apply Stampacchia's theorem (cf. [10, Theorem 5.6]) to conclude that there exists a unique $u \in K^\sigma$ satisfying

$$a(u, v - u) \geq (f, v - u) - [[\alpha, v - u]] \qquad (\forall v \in K^\sigma). \tag{33}$$

Taking $v = u \pm \varphi$ with $\varphi \in V_0^\sigma$ in (33), we deduce

$$a(u, \varphi) = (f, \varphi) \qquad (\forall \varphi \in V_0^\sigma). \tag{34}$$

Hence, according to Lemma 6, there exists $\hat{p} \in Q_0$ satisfying

$$-b(\hat{p}, v) = a(u, v) - (f, v) \qquad (\forall v \in V_0). \tag{35}$$

Now we set, for $k \in \mathbb{R}$,

$$p_k = \hat{p} + k \tag{36}$$

and verify, with an appropriate choice of k, that (u, p_k) is a solution of (VI). To this end, it suffices to check that (u, p_k) is a solution of (PDE).

We have by (20) and (33)

$$[\tau_n(u, p_k), v_n - u_n] + [[\tau_T(u), v_T - u_T]] \geq -[[\alpha, v - u]] \qquad (v \in K^\sigma). \tag{37}$$

Let $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n = 0$ on Γ. Then, since $\int_\Gamma \psi_n \, d\Gamma = 0$, there is a function $w \in V$ satisfying $w = \psi$ on Γ, $\nabla \cdot w = 0$ in Ω and $\|w\|_1 \leq C\|\psi\|_{M^d}$. Substituting $v = u \pm w \in K^\sigma$ into (37), we get $[[\tau_T(u) + \alpha_T, \psi_T]] = 0$. By the density, this implies (27f).

Consequently, it follows from (37) that

$$[\tau_n(u, p_k) + \alpha_n, v_n - u_n] \geq 0 \qquad (v \in K^\sigma). \tag{38}$$

We set

$$\gamma = \inf_{\mu \in Y} [\tau_n(u, \hat{p}) + \alpha_n, \mu],$$

where

$$Y = \left\{ \mu \in M \mid \mu \geq 0,\ \mu \not\equiv 0,\ \int_\Gamma \mu \, d\Gamma = 1 \right\}. \tag{39}$$

The, for any $\mu \in M$ with $\mu \geq 0$ and $\mu \not\equiv 0$, we have

$$[\tau_n(u, p_k) + \alpha_n, \mu] = [\tau_n(u, \hat{p}) + \alpha_n, \mu] - k \int_\Gamma \mu \geq \gamma \int_\Gamma \mu - k \int_\Gamma \mu$$

Hence, we deduce (27d) if $k \leq \gamma$.

For the time being, we admit

$$\gamma = \frac{[\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n]}{\beta}, \tag{40}$$

When $u_n + g_n = 0$ on Γ, we obviously have $\gamma = 0$. However, this is impossible since $u_n + g_n \geq 0$ and $\int_\Gamma g_n \, d\Gamma > 0$. We have by (40)

$$[\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n] = \gamma\beta = \gamma \int_\Gamma g_n \, d\Gamma = \gamma \int_\Gamma (u_n + g_n) \, d\Gamma.$$

Therefore, taking

$$k = \gamma,$$

we obtain

$$[\tau_n(u, p_k) + \alpha_n, u_n + g_n] = [\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n] - \gamma \int_\Gamma (u_n + g_n) = 0.$$

Thus, we have verified (27e).

In order to show (40), we use $w^* \in V$ defined as (32) again. From (38) with $k = 0$,

$$[\tau_n(u, \hat{p}) + \alpha_n, v_n + w_n^*] \geq [\tau_n(u, \hat{p}) + \alpha_n, u_n + w_n^*] \qquad (v \in K^\sigma).$$

Since $w^* = g$ on Γ, this is equivalently written as

$$[\tau_n(u, \hat{p}) + \alpha_n, v_n + g_n] \geq [\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n] \qquad (v \in K^\sigma).$$

Moreover, we get

$$\left[\tau_n(u,\hat{p})+\alpha_n, \frac{v_n+g_n}{\beta}\right] \geq \left[\tau_n(u,\hat{p})+\alpha_n, \frac{u_n+g_n}{\beta}\right] \qquad (v\in K^\sigma). \tag{41}$$

Now let $\mu \in Y$ be arbitrary and set $\tilde{\mu} = \beta\mu - g_n \in M$. Since $\int_\Gamma \tilde{\mu}\, d\Gamma = 0$, there exists $\tilde{v} \in V^\sigma$ such that $\tilde{v}_n = \tilde{\mu}$ on Γ according to Remark 3 and Lemma 9. Then, the function $\tilde{v}$ satisfies that $\tilde{v}_n + g_n = \beta\mu \geq 0$ on Γ. Thus, $\tilde{v} \in K^\sigma$. Consequently, we have by (41)

$$\begin{aligned} [\tau_n(u,\hat{p})+\alpha_n, \mu] &= \left[\tau_n(u,\hat{p})+\alpha_n, \frac{\tilde{\mu}+g_n}{\beta}\right] \\ &= \left[\tau_n(u,\hat{p})+\alpha_n, \frac{\tilde{v}_n+g_n}{\beta}\right] \geq \frac{1}{\beta}\left[\tau_n(u,\hat{p})+\alpha_n, u_n+g_n\right], \end{aligned}$$

which implies (40).

It remains to derive (29). First, from (35) and (17), we have

$$\|\hat{p}\| \leq \sup_{v\in V_0} \frac{|(f,v)-a(u,v)|}{\|v\|_1} \leq C(\|f\|+\|u\|). \tag{42}$$

Equation (27e), together with (22), implies

$$a(u,u+g)+b(p,u+g)-(f,u+g)+[\alpha_n, u_n+g_n]=0.$$

Therefore, by virtue of Korn's inequality,

$$C_{\mathrm{K}}\|u+g\|_1^2 \leq C(\|f\|+\|\alpha\|_{(M^d)'}+\|g\|_1)\|u+g\|_1$$

and, consequently,

$$\|u+g\|_1 \leq C_*, \qquad \|u\|_1 \leq C_*. \tag{43}$$

Finally, thanks to the expression (40), we can estimate as

$$|\gamma| \leq \frac{1}{\beta}\|\tau_n(u,\hat{p})+\alpha_n\|_{M'}\|u_n+g_n\|_{1/2,\Gamma} \leq C(\|u\|_1+\|\hat{p}\|)\|u+g\|_1. \tag{44}$$

Combining this with (42) and (43), we obtain (29). □

4 Finite element approximation

Since the problem (PDE) and (VI) are not directly applicable for numerical computation, we propose the penalty approximation for (VI).

As a regularization of $[s]_-(s\in\mathbb{R})$, we introduce a function $\phi_\delta : \mathbb{R}\to\mathbb{R}$ that satisfies

$$\phi_\delta \text{ is a non-increasing } C^1(\mathbb{R}) \text{ function;} \tag{45a}$$

$$|\phi_\delta(s)-[s]_-| \leq C\delta \quad (s\in\mathbb{R}); \tag{45b}$$

$$\phi_\delta(s)=0 \quad (s\geq 0), \qquad 0\leq \phi_\delta(s)\leq -s \quad (s<0); \tag{45c}$$

$$\left|\frac{d}{ds}\phi_\delta(s)\right| \leq C \qquad (s\in\mathbb{R}), \tag{45d}$$

where $\delta\in(0,1]$ is regularized parameter and C's are independent of δ. As mentioned in Introduction, we can take, for example, the function $\phi_\delta(s)$ defined as (3).

For penalty parameter $\varepsilon \in (0,1]$, we consider the following penalty problem;

(PE$_{\varepsilon,\delta}$) Find $(u,p) \in V \times Q$ such that

$$a(u,v) + b(p,v) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_n+g_n)v_n\ d\Gamma = (f,v) - [[\alpha, v]] \qquad (\forall v \in V), \tag{46a}$$

$$b(q,u) = 0 \qquad (\forall q \in Q). \tag{46b}$$

This and the subsequent sections are devoted to the finite element approximation of (PE$_{\varepsilon,\delta}$). In order to avoid unessential difficulties concerning "curved boundary", we consider only the case (G2). Consequently, the unit outer normal vector n to Γ is a constant vector over Γ.

We use the so-called P1 bubble/P1 (P1b/P1) elements for discretization. Let $\{\mathcal{T}_h\}_h$ be a *regular* family of triangulations of Ω. As the granularity parameter, we have employed $h = \max\{h_T \mid T \in \mathcal{T}_h\}$, where h_T denotes the diameter of T. We introduce the following function spaces:

$$\begin{gathered}
V_h = \{v_h \in C^0(\overline{\Omega}) \mid v_h = 0 \text{ on } S,\ v_h|_T \in \mathcal{P}_1^{(d)} \oplus \operatorname{span}\{\varphi_T\}\ (\forall T \in \mathcal{T}_h)\},\\
V_{0h} = V_h \cap H_0^1(\Omega)^d, \qquad V_h^\sigma = \{v_h \in V_h \mid b(q_h,v_h) = 0\ (\forall q_h \in Q_h)\},\\
Q_h = \{q_h \in C^0(\overline{\Omega}) \mid q_h|_T \in \mathcal{P}_1^{(d)}\ (\forall T \in \mathcal{T}_h)\}, \quad Q_{0h} = Q_h \cap Q_0,\\
M_h = \{\mu_h = v_{hn}|_\Gamma \mid v_h \in V_h\}, \quad M_{0h} = \left\{\mu_h \in M_h \mid \int_\Gamma \mu_h\ d\Gamma = 0\right\},
\end{gathered}$$

where $\mathcal{P}_k^{(d)}$ denotes the set of all polynomials in $x_1,\ldots,x_d$ of degree $\le k$, and $\varphi_T = \prod_{i=1}^{d+1}\lambda_{T,i}$, with $\lambda_{T,1},\ldots,\lambda_{T,d+1}$ the barycentric co-ordinates of T.

Let us denote by $\mathcal{S}_h$ the $d-1$ dimensional triangulation of Γ inherited from $\mathcal{T}_h$. We obviously have

$$M_h = \{\mu_h \in C(\overline{\Gamma}) \mid \mu_h|_S \in \mathcal{P}_1^{(d-1)}\ (\forall S \in \mathcal{S}_h),\ \mu_h|_{\partial\Gamma} = 0\} \quad \text{(algebraly)}. \tag{48}$$

Moreover, we introduce a projection operator $\Lambda : Q \to Q_0$ by

$$\Lambda q = q - m(q) \quad \text{with} \quad m(q) = \frac{1}{|\Omega|}\int_\Omega q\ dx \qquad (q \in Q). \tag{49}$$

It is clear that $\|\Lambda q\| \le C\|q\|$ for $q \in Q$ and $\Lambda q_h \in Q_{0h}$ for $q_h \in Q_h$.

Then, the finite element approximation for (PE$_{\varepsilon,\delta,h}$) reads as follows.

(PE$_{\varepsilon,\delta,h}$) Find $(u_h,p_h) \in V_h \times Q_h$ such that

$$a(u_h,v_h) + b(p_h,v_h) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_n+g_n)v_{hn}\ d\Gamma = (f,v_h) - [[\alpha, v_h]] \qquad (\forall v_h \in V_h), \tag{50a}$$

$$b(q_h,u_h) = 0 \qquad (\forall q_h \in Q_h). \tag{50b}$$

Before considering the well-posedness and error estimates, we recall here basic results on the finite element method.

Babuška-Brezzi condition As is well-known, the Babuška-Bezzi condition holds true in $V_{0h} \times Q_{0h}$. That is, there is a constant $\gamma' > 0$, which is independent of h, such that

$$\inf_{q_h \in Q_{0h}} \sup_{v_h \in V_{0h}} \frac{b(q_h,v_h)}{\|v_h\|_1\|q_h\|} \ge \gamma'. \tag{51}$$

Discrete lifting operators and discrete traction vectors The following is a discrete analogue of Lemma 2.

Lemma 13 ([19, Lemma 2.1])**.** (i) *There is a continuous linear operator $\mathcal{E}_h$ from M_h^d to V_h such that $\mathcal{E}_h\mu_h = \mu_h$ on Γ and $\|\mathcal{E}_h\mu_h\|_1 \le C\|\mu_h\|_{1/2,\Gamma}$ for any $\mu_h \in M_h^d$, where C is independent of h.*
(ii) *There is a continuous linear operator $\mathcal{E}_{nh}$ from M_h to V_h such that $(\mathcal{E}_{nh}\mu_h)_n = \mu_h$ and $(\mathcal{E}_{nh}\mu_h)_T = 0$ on Γ and $\|\mathcal{E}_{nh}\mu_h\|_1 \le C\|\mu_h\|_{1/2,\Gamma}$ for any $\mu_h \in M_h$, where C is independent of h.*
(iii) *For $\mu_h \in M_{0h}$, the above $w_h = \mathcal{E}_{nh}\mu_h$ can be chosen in such way that $w_h \in V_h^\sigma$.*

As the continuous case, we define traction vectors $\tau(u_h, p_h) \in (M_h^d)'$, $\tau_T(u_h) \in (M_h^d)'$ and $\tau_n(u_h, p_h) \in M_h'$ for a solution $(u_h, p_h) \in V_h^\sigma \times Q_h$ of

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \qquad (v_h \in V_{0h}) \tag{52}$$

as follows:

$$[[\tau(u_h, p_h), \mu_h]] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h^d,\ w_h = \mathcal{E}_h\mu_h); \tag{53a}$$
$$[\tau_T(u_h), \mu_h] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h^d \text{ with } \mu_{hn} = 0,\ w_h = \mathcal{E}_h\mu_h); \tag{53b}$$
$$[\tau_n(u_h, p_h), \mu_h] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h,\ w_h = \mathcal{E}_{nh}\mu_h) \tag{53c}$$

These definitions do not depend on the way of extensions. In fact, for any μ_h, let $w_h \in V_h$ and $\tilde{w}_h \in V_h$ be both extension of λ_h; $w_{hn} = \tilde{w}_{hn} = \lambda_h$ on Γ. Set $v_h = w_h - \tilde{w}_h$. Then, since $v_h \in V_{0h}$, we deduce, by (52),

$$\begin{aligned} a(u_h, w_h) + b(p_h, w_h) - (f, w_h) - [a(u_h, \tilde{w}_h) + b(p_h, \tilde{w}_h) - (f, \tilde{w}_h)] \\ = a(u_h, v_h) + b(p_h, v_h) - (f, v_h) = 0. \end{aligned}$$

Thus, (53c) is well-defined.

5 Well-posedness of (PE$_{\varepsilon,\delta,h}$)

In this section, we establish the well-posedness of (PE$_{\varepsilon,\delta,h}$). Thus, we shall prove the following two theorems. Recall that C_* denotes a positive constant depending only on Ω, $\|f\|$, $\|g\|_1$ and $\|\alpha\|_{(M^d)'}$.

Theorem 14. *There exists a unique solution $(u_h, p_h) \in V_h \times Q_h$ of (PE$_{\varepsilon,\delta,h}$) , and we have*

$$\|u_h\|_1 + \|\hat{p}_h\| + \left\| \frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n) + k_h \right\|_{M_h'} \le C_*, \tag{54}$$

where $\hat{p}_h = \Lambda p_h$ and $k_h = m(p_h)$.

Theorem 15. *Assume*

(A1) *the family $\{\mathcal{S}_h\}_h$ is of quasi-uniform;*

(A2) *there exists $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| > 0$ which is independent of h, ε, δ and Ω such that $u_{hn}+g_n > 0$ on Γ_1.*

Then, the solution $(u_h, p_h) \in V_h \times Q_h$ of $(\mathrm{PE}_{\varepsilon,\delta,h})$ admits the following estimates:

$$\|u_h\|_1 + \|p_h\| + \left\| \frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n) \right\|_{M_h'} \le C_*; \tag{55a}$$

$$\left\| \frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n) \right\|_{M'} \le C_* \left(1 + \frac{h}{\varepsilon}\right); \tag{55b}$$

$$\frac{1}{\sqrt{\varepsilon}}\|[u_{hn} + g_n]_-\|_\Gamma \le C_* \left(1 + \frac{\delta}{\varepsilon}\right). \tag{55c}$$

Remark 16. Condition (A2) is not restrictive; If β is sufficiently large and h, ε, δ are suitably small, it is natural to suppose this condition.

Remark 17. If $\delta \le c_0\varepsilon$ with some $c_0 > 0$, we have, from (55c), $\|[u_{hn} + g_n]_-\|_\Gamma \to 0$ as $\varepsilon \to 0$.

To prove Theorem 14, we apply the following fundamental result.

Lemma 18 ([24, Theorem 2.1])**.** *Let X be a separable reflexive Banach space and let $T : X \to X'$ be a (possibly nonlinear) operator satisfying the following conditions:*

1. *(boundness) There exist $C, C', m > 0$ s.t. $\|Tu\|_{X'} \le C\|u\|_X^m + C'$ for all $u \in X$;*
2. *(monotonicity) $\langle Tu - Tv, u - v\rangle_{X',X} \ge 0$ for all $u, v \in X$;*
3. *(hemicontinuity) For any $u, v, w \in X$, the function $\lambda \mapsto \langle T(u + \lambda v), w\rangle_{X',X}$ is continuous on $\mathbb{R}$;*
4. *(coerciveness) $\dfrac{\langle Tu, u\rangle_{X',X}}{\|u\|_X} \to \infty$ as $\|u\|_X \to \infty$.*

Then, for any $\varphi \in X'$, there exists $u \in X$ such that $Tu = \varphi$. Furthermore, if T is strictly monotone:

$$\langle Tu - Tv, u - v\rangle_{X',X} > 0 \qquad (\forall u, v \in X, u \ne v),$$

then the solution is unique.

We set $\rho_\delta : V \to V'$ by

$$\langle \rho_\delta(u), v\rangle = -\int_\Gamma \phi_\delta(u_n + g_n)v_n \, d\Gamma \qquad (v \in V).$$

Lemma 19. *ρ_δ is a bounded, monotone and hemicontinuous operator from V to V'.*

Proof. (boundness) By using (45c) and the trace theorem, we have

$$\langle \rho_\delta(u), v\rangle \le \int_\Gamma |u_n + g_n| \cdot |v_n| \, d\Gamma \le (\|u\|_1 + \|g_n\|_\Gamma)\|v\|_1$$

for $u, v \in V$. Hence,

$$\|\rho_\delta(u)\|_{V'} \le \|u\|_1 + \|g_n\|_\Gamma \qquad (u \in V).$$

(monotonicity) Since $-\phi_\delta(s)$ is non-decreasing function, we have

$$\langle \rho_\delta(u) - \rho_\delta(v), u - v\rangle = -\int_\Gamma (\phi_\delta(u_n + g_n) - \phi_\delta(v_n + g_n))(u_n + g_n - (v_n + g_n)) \, d\Gamma \ge 0$$

for $u, v \in V$.

(hemicontinuity) Let $u, v, w \in V$. Then, a real-valued function

$$\langle \rho_\delta(u + \lambda v), w\rangle = -\int_\Gamma \phi_\delta(u_n + \lambda v_n) w_n \, d\Gamma$$

of $\lambda \in \mathbb{R}$ is a continuous function, since the function ϕ_δ is continuous. $\square$

Proof of Theorem 14. It is divided into three steps.
Step 1. First, we prove that there exists a unique $u_h \in V_h^\sigma$ satisfying

$$a(u_h, v_h) + \frac{1}{\varepsilon}\langle \rho_\delta(u_h), v_h\rangle = (f, v_h) - [[\alpha, v_h]] \qquad (\forall v_h \in V_h^\sigma) \tag{56}$$

by using Lemma 18.

To do this, we introduce a nonlinear operator $A_\varepsilon : V_h \to V_h'$ by setting

$$\langle A_\varepsilon u_h, v_h\rangle = a(u_h, v_h) + \frac{1}{\varepsilon}\langle \rho_\delta(u_h), v_h\rangle \qquad (u_h, v_h \in V_h).$$

and verify the conditions of Lemma 18.
(boundness) For $u_h, v_h \in V_h^\sigma$, we have immediately

$$\|A_\varepsilon u_h\|_{(V_h^\sigma)'} \le \left(\|a\| + \frac{1}{\varepsilon}\right)\|u_h\|_1 + \frac{1}{\varepsilon}\|g_n\|_\Gamma \qquad (u_h \in V_h^\sigma).$$

(strictly monotonicity) By virtue of Korn's inequality and monotonicity of ρ_δ,

$$\begin{aligned}\langle A_\varepsilon u_h - A_\varepsilon v_h, u_h - v_h\rangle &= a(u_h - v_h, u_h - v_h) + \frac{1}{\varepsilon}\langle \rho_\delta(u_h) - \rho_\delta(v_h), u_h - v_h\rangle \\ &\ge C_{\mathrm{K}}\|u_h - v_h\|_1^2 > 0\end{aligned}$$

for $u_h, v_h \in V_h^\sigma$, $u_h \ne v_h$.
(hemicontinuity) Let $u_h, v_h, w_h \in V_h^\sigma$. Then, a real-valued function

$$\langle A_\varepsilon(u_h + \lambda v_h), w_h\rangle = a(u_h + \lambda v_h, w_h) + \frac{1}{\varepsilon}\langle \rho_\delta(u_h + \lambda v_h), w_h\rangle$$

of $\lambda \in \mathbb{R}$ is continuous, since $a(\cdot, w_h)$ is continuous and $\rho_\delta(\cdot)$ is hemicontinuous.
(coerciveness) For $u_h \in V_h^\sigma$, we have by (45c)

$$\begin{aligned}\langle \rho_\delta(u_h), u_h\rangle &= -\int_\Gamma \phi_\delta(u_{hn} + g_n) u_{hn} \, d\Gamma \\ &= -\int_\Gamma \phi_\delta(u_{hn} + g_n)\left([u_{hn} + g_n]_+ - [u_{hn} + g_n]_- - [g_n]_+ + [g_n]_-\right) \, d\Gamma \\ &\ge -\int_\Gamma \phi_\delta(u_n + g_n)[g_n]_- \, d\Gamma \\ &\ge -C\left(\|u_h\|_1 + \|g_n\|_\Gamma\right)\|g_n\|_\Gamma.\end{aligned}$$

This gives

$$\begin{aligned}\frac{\langle A_\varepsilon u_h, u_h\rangle}{\|u_h\|_1} &= \frac{a(u_h, u_h)}{\|u_h\|_1} + \frac{1}{\varepsilon}\frac{\langle \rho_\delta(u_h), u_h\rangle}{\|u_h\|_1} \\ &\ge C_{\mathrm{K}}\|u_h\|_1 - \frac{C}{\varepsilon}\frac{\left(\|u_h\|_1 + \|g_n\|_\Gamma\right)}{\|u_h\|_1}\|g_n\|_\Gamma,\end{aligned}$$

and, hence,

$$\frac{\langle A_\varepsilon u_h, u_h\rangle}{\|u_h\|_1} \to \infty \quad \text{as} \quad \|u_h\|_1 \to \infty.$$

As a consequence, we can apply Lemma 18 to conclude that there exists a unique $u_h \in V_h^\sigma$ satisfying $A_\varepsilon u_h = F_h$, where $F_h \in (V_h^\sigma)'$ is defined as $\langle F, v_h\rangle = (f, v_h) - [[\alpha, v_h]]$ for $v_h \in V_h^\sigma$. Thus, we have proved a unique existence of the solution $u_h \in V_h^\sigma$ of (56).

Step 2. We verify the unique existence of $p_h \in Q_h$ such that (u_h, p_h) is a solution of ($\mathrm{PE}_{\varepsilon,\delta,h}$). In view of (51), there exists a unique $\hat{p}_h \in Q_{0h}$ satisfying

$$a(u_h, v_h) + b(\hat{p}_h, v_h) = (f, v_h) \qquad (v_h \in V_{0h}). \tag{57}$$

Now, we find a constant k_h such that (u_h, p_h) is a solution of ($\mathrm{PE}_{\varepsilon,\delta,h}$), where $p_h = \hat{p}_h + k_h$. (We note that, with any $k_h \in \mathbb{R}$, (u_h, p_h) also solves (52).) To do this, we first rewrite (50a) as, by using (53a)–(53c),

$$[\tau_n(u_h, p_h) - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) - \alpha_n, v_{hn}] + [[\tau_T(u_h) - \alpha_T, v_{hT}]] = 0 \quad (v_h \in V_h).$$

Thus, in view of Lemma 13, it suffices to prove the following two equations:

$$[\tau_n(u_h, p_h) - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) - \alpha_n, \mu_h] = 0 \quad (\mu_h \in M_h); \tag{58a}$$

$$[[\tau_T(u_h) - \alpha_T, \mu_h]] = 0 \quad (\mu_h \in M_h^d \text{ with } \mu_{hn} = 0). \tag{58b}$$

Obviously, $v_h = \mathcal{E}_h\mu_h \in V_h$ belongs to V_h^σ for any $\mu_h \in M_h^d$ with $\mu_{hn} = 0$. Hence, (56) and (53b) immediately implies (58b).

On the other hand, combining (56) and (53c), we have

$$[\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \lambda_h] = 0 \qquad (\forall \lambda_h \in M_{0h}). \tag{59}$$

At this stage, let us take

$$\tilde{\mu}_h \in Y_h = \left\{ \mu_h \in M_h \mid \mu_h \ge 0,\ \mu_h \not\equiv 0,\ \int_\Gamma \mu_h \, d\Gamma = 1 \right\}.$$

Then, for any $\mu_h \in M_h$, the function $\mu_h - \kappa_h\tilde{\mu}_h$ belongs to M_{0h}, where $\kappa_h = \int_\Gamma \mu_h \, d\Gamma$.

Therefore, for any $\mu_h \in M_h$,

$$\begin{aligned}
[\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \mu_h] &= [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \mu_h - \kappa_h\tilde{\mu}_h] \\
&\quad + [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \kappa_h\tilde{\mu}_h] \\
&= \kappa_h[\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \tilde{\mu}_h]
\end{aligned} \tag{60}$$

Now, choosing

$$k_h = [\tau_n(u_h, \hat{p}_h) - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \tilde{\mu}_h], \tag{61}$$

we have

$$\begin{aligned}
[\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \tilde{\mu}_h] &= [\tau_n(u_h, \hat{p}_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \tilde{\mu}_h] - k_h \\
&= 0.
\end{aligned}$$

Hence, we get (58a) by (60).

It remains to verify that (61) does not depend on the choice of $\tilde{\mu}_h$. We let $\tilde{\mu}_h, \tilde{\mu}_h' \in Y_h$ with $\tilde{\mu}_h \neq \tilde{\mu}_h'$ and let the corresponding k_h be denoted by $\tilde{k}_h$, $\tilde{k}_h'$, respectively. Then, since $\lambda_h = \mu_h - \mu_h'$ satisfies $\int_\Gamma \lambda_h \, d\Gamma = 0$, we have by (59),

$$\tilde{k}_h - \tilde{k}_h' = [\tau_n(u_h, \hat{p}_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \lambda_h] = 0,$$

which means that k_h defined by (61) is well-defined.

Step 3. Finally, we derive the stability result (54). Substituting $v_h = u_h \in V_h^\sigma$ into (56), we obtain,

$$a(u_h, u_h) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn} + g_n)u_{hn} \, d\Gamma = (f, u_h) - [[\alpha, u_h]]. \tag{62}$$

Noting that, by (45c)

$$\begin{aligned} -\frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn} + g_n)u_{hn} \, d\Gamma &= -\frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn} + g_n)(u_{hn} + g_n) \, d\Gamma + \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn} + g_n)g_n \, d\Gamma \\ &\geq \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn} + g_n)[u_{hn} + g_n]_- \, d\Gamma \geq 0, \end{aligned} \tag{63}$$

we get

$$a(u_h, u_h) \leq (f, u_h) - [[\alpha, u_h]].$$

Hence, by virtue of Korn's inequality,

$$\|u_h\|_1 \leq C(\|f\| + \|\alpha\|_{(M^d)'}). \tag{64}$$

Moreover, according to (51) and (57),

$$\|\hat{p}_h\| \leq \sup_{v_h \in V_{0h}} \frac{b(\hat{p}_h, v_h)}{\|v_h\|_1} = \sup_{v_h \in V_{0h}} \frac{(f, v_h) - a(u_h, v_h)}{\|v_h\|_1} \leq C(\|f\| + \|u_h\|_1). \tag{65}$$

Since (50a) is expressed as

$$\begin{aligned} &\int_\Gamma (\varepsilon^{-1}\phi_\delta(u_{hn} + g_n) + k_h)\mu_h \, d\Gamma \\ &\qquad = a(u_h, v_h) + b(\hat{p}_h, v_h) - (f, v_h) + [[\alpha, v_h]] \qquad (\forall \mu_h \in M_h,\ v_h = \mathcal{E}_{nh}\mu_h \in V_h), \end{aligned}$$

we deduce

$$\left\| \frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n) + k_h \right\|_{M_h'} \leq C(\|u_h\|_1 + \|\hat{p}_h\| + \|f\| + \|\alpha\|_{(M^d)'}). \tag{66}$$

Summing up (64), (65) and (66), we obtain (54). $\square$

We proceed to the proof of Theorem 15. We use the standard Lagrange interpolation operator $i_h : C(\overline{\Gamma}) \to M_h$ defined by

$$i_h\mu(P) = \mu(P) \quad \text{(every node } P \text{ of } \mathcal{S}_h)$$

and the L^2 projection operator $\pi_h : L^2(\Gamma) \to M_h$ defined by

$$\int_\Gamma (\pi_h\mu - \mu)\mu_h \, d\Gamma = 0 \qquad (\mu_h \in M_h). \tag{67}$$

The following results are well-known:

$$\mu \geq 0 \quad \Rightarrow \quad i_h\mu \geq 0, \tag{68a}$$

$$\|i_h\mu - \mu\|_\Gamma + h\|i_h\mu - \mu\|_{1,\Gamma} \leq Ch^2|\mu|_{2,\Gamma} \qquad (\mu \in H^2(\Gamma) \cap H_0^1(\Gamma)), \tag{68b}$$

$$\|\pi_h\mu\|_\Gamma \leq C\|\mu\|_\Gamma \qquad (\mu \in L^2(\Gamma)), \tag{68c}$$

$$\|\pi_h\mu\|_{1,\Gamma} \leq C\|\mu\|_{1,\Gamma} \qquad (\mu \in H_0^1(\Gamma)), \tag{68d}$$

$$\|\pi_h\mu - \mu\|_\Gamma \leq Ch\|\mu\|_{1,\Gamma} \qquad (\mu \in H_0^1(\Gamma)). \tag{68e}$$

In fact, (68a), (68b), (68c), (68e) are standard. On the other hand, (68d) holds true if $\{\mathcal{S}_h\}_h$ is of quasi-uniform (cf. [12, 11, 9]).

Remark 20. According to (68b), $\|i_h\mu\|_{1,\Gamma}$ is bounded by a positive constant depending only on μ if $\mu \in C_0^\infty(\Gamma)$.

Lemma 21. $\|\pi_h\mu - \mu\|_{M'} \leq Ch\|\mu\|_{1/2,\Gamma}$ *for any* $\mu \in M$*.*

Proof. It follows from (68c) that $\|\pi_h\mu - \mu\|_\Gamma \leq C\|\mu\|_\Gamma$. Combining this with (68e), (16) and applying the interpolation theorem (cf. [25, Theorem 5.1, Ch. 1]), we obtain

$$\|\pi_h\mu - \mu\|_\Gamma \leq Ch^{1/2}\|\mu\|_{1/2,\Gamma} \qquad (\mu \in M).$$

We can utilize this in the following way. That is, noting (67),

$$\begin{aligned}\|\pi_h\mu - \mu\|_{M'} &= \sup_{\lambda\in M}\frac{(\pi_h\mu - \mu, \lambda)_\Gamma}{\|\lambda\|_{1/2,\Gamma}} = \sup_{\lambda\in M}\frac{(\pi_h\mu - \mu, \pi_h\lambda - \lambda)_\Gamma}{\|\lambda\|_{1/2,\Gamma}}\\ &\leq \sup_{\lambda\in M}\frac{\|\pi_h\mu - \mu\|_\Gamma\|\pi_h\lambda - \lambda\|_\Gamma}{\|\lambda\|_{1/2,\Gamma}} \leq Ch\|\mu\|_{1/2,\Gamma}.\end{aligned}$$

□

Lemma 22. $\|\phi_\delta(\mu)\|_{1/2,\Gamma} \leq C\|\mu\|_{1/2,\Gamma}$ *for any* $\mu \in M$*.*

Proof. By using (45c) and (45d), we have $\|\phi_\delta(\mu)\|_\Gamma \leq C\|\mu\|_\Gamma$ for $\mu \in Q$ and $\|\phi_\delta(\mu)\|_{1,\Gamma} \leq C\|\mu\|_{1,\Gamma}$ for $\mu \in H_0^1(\Gamma)$. Hence, we can apply the (nonlinear) interpolation theorem (cf. [23, Theorem 3.1]) and (16) to get the desired result. □

Proof of Theorem 15. First, we derive an estimation for k_h. We take $\tilde{\mu} \in C_0^\infty(\Gamma)$ satisfying

$$\tilde{\mu} \geq 0,\ \tilde{\mu} \not\equiv 0 \text{ in } \Gamma, \quad \operatorname{supp}\tilde{\mu} \subset \Gamma_1.$$

Then, setting $\tilde{\mu}_h = i_h\mu \in M_h$, we have

$$\tilde{\mu}_h \geq 0,\ \tilde{\mu} \not\equiv 0 \text{ in } \Gamma, \quad \tilde{\mu}_h = 0 \text{ in } \Gamma\backslash\Gamma_1, \quad \|\tilde{\mu}_h\|_M \leq C, \qquad \left|\int_\Gamma \tilde{\mu}_h\, d\Gamma - \int_\Gamma \tilde{\mu}\, d\Gamma\right| \leq Ch^2, \tag{69}$$

where those C's depend on μ.

Since (A2) gives

$$\phi_\delta(u_{hn} + g_n) = 0 \quad \text{ on } \Gamma_1,$$

we deduce from (50a) and (69)

$$\begin{aligned}k_h\int_\Gamma \tilde{\mu}_h\, d\Gamma &= a(u_h, \tilde{v}_h) + b(\hat{p}_h, \tilde{v}_h) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn} + g_n)\tilde{\mu}_h\, d\Gamma - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]]\\ &= a(u_h, \tilde{v}_h) + b(\hat{p}_h, \tilde{v}_h) - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]],\end{aligned} \tag{70}$$

where $\tilde{v}_h = \mathcal{E}_{nh}\tilde{\mu}_h \in V_h$.

This leads to

$$|k_h| \le C_*, \qquad \|p_h\| \le C_*. \tag{71}$$

Hence, we have proved (55a).

By using (67) and (50a), we can calculate as

$$\begin{aligned}\int_\Gamma \varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n)\mu_h\ d\Gamma &= \int_\Gamma \varepsilon^{-1}\phi_\delta(u_{hn}+g_n)\mu_h\ d\Gamma \\ &= a(u_h,v_h)+b(p_h,v_h)-(f,v_h)+[[\alpha,v_h]] \quad (\forall\mu_h\in M_h,\ v_h=\mathcal{E}_{nh}\mu_h\in V_h).\end{aligned}$$

Hence,

$$\|\varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n)\|_{M_h'} \le (\|u_h\|_1+\|\hat{p}_h\|+\|f\|+\|\alpha\|_{(M^d)'}+\|g\|_1). \tag{72}$$

We write as

$$\begin{aligned}&\sup_{\mu\in M}\frac{(\varepsilon^{-1}\phi_\delta(u_{hn}+g_n),\mu)_\Gamma}{\|\mu\|_{1/2,\Gamma}} \\ &\qquad = \underbrace{\frac{1}{\varepsilon}\sup_{\mu\in M}\frac{(\phi_\delta(u_{hn}+g_n)-\pi_h\phi_\delta(u_{hn}+g_n),\mu)_\Gamma}{\|\mu\|_{1/2,\Gamma}}}_{=I_1}+\underbrace{\sup_{\mu\in M}\frac{(\varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n),\mu)_\Gamma}{\|\mu\|_{1/2,\Gamma}}}_{=I_2}.\end{aligned}$$

By using Lemmas 21 and 22,

$$\begin{aligned}\|\pi_h\phi_\delta(u_{hn}+g_n)-\phi_\delta(u_{hn}+g_n)\|_{M'} &\le Ch\|\phi_\delta(u_{hn}+g_n)\|_{1/2,\Gamma} \\ &\le Ch\|u_{hn}+g_n\|_{1/2,\Gamma} \\ &\le Ch(\|u_h\|_1+\|g\|_1).\end{aligned}$$

Consequently,

$$|I_1| \le C\frac{h}{\varepsilon}(\|u_h\|_1+\|g\|_1).$$

On the other hand, by virtue of (67), (68d) and (72), we have

$$\begin{aligned}I_2 &= \sup_{\mu\in M}\frac{(\varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n),\mu)_\Gamma}{\|\mu\|_{1/2,\Gamma}} \le C\sup_{\mu\in M}\frac{(\varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n),\pi_h\mu)_\Gamma}{\|\pi_h\mu\|_{1/2,\Gamma}} \\ &\le C\sup_{\mu_h\in M_h}\frac{(\varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n),\mu_h)_\Gamma}{\|\mu_h\|_{1/2,\Gamma}}\end{aligned}$$

Therefore, from (72),

$$|I_2| \le C(\|u_h\|_1+\|p_h\|+\|f\|+\|\alpha\|_{(M^d)'}+\|g\|_1)$$

Summing up those estimates, we get (55b).

Finally, using (45b),

$$\begin{aligned}-\frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn}+g_n)u_{hn}\ d\Gamma &\ge \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn}+g_n)[u_{hn}+g_n]_-\ d\Gamma \ge 0 \\ &\ge \frac{1}{\varepsilon}\int_\Gamma ([u_{hn}+g_n]_-^2 - C\delta[u_{hn}+g_n]_-)\ d\Gamma \ge 0.\end{aligned}$$

We apply this to (62) and obtain

$$\frac{1}{\varepsilon}\int_\Gamma [u_{hn}+g_n]_-^2\ d\Gamma \le (f,u_h) - [[\alpha,u_h]] + C\frac{\delta}{\varepsilon}\int_\Gamma [u_{hn}+g_n]_-\ d\Gamma$$
$$\le \|f\|\cdot\|u_h\|_1 + \|\alpha\|_{(M^d)'}\|u_h\|_1 + C\frac{\delta}{\varepsilon}(\|u_h\|_1+\|g\|_1),$$

which implies (55c). □

6 Error estimate

We are now ready to state the error estimates between (PDE) and ($\mathrm{PE}_{\varepsilon,\delta,h}$).

Theorem 23. *Assume that* (A1) *and* (A2) *are satisfied. Let* (u,p) *and* (u_h,p_h) *be solutions of* (PDE) *and* ($\mathrm{PE}_{\varepsilon,\delta,h}$), *respectively, and suppose that* $(u,p)\in H^2(\Omega)^d\times H^1(\Omega)$ *and* $\tau_n(u,p)+\alpha_n\in M$. *Moreover, assume that* $h\le c_1\varepsilon$ *with a constant* $c_1>0$. *Then, we have*

$$\|u-u_h\|_1 + \|\hat{p}-\hat{p}_h\| \le C_{**}(\sqrt{\varepsilon}+\sqrt{\delta}+\sqrt{h}), \tag{73}$$

where $\hat{p}=\Lambda p$, $\hat{p}_h=\Lambda p_h$, C_{**} *denotes a positive constant depending only on* Ω, $|u|_2$, $|p|_1$, $\|\tau_n(u,p)+\alpha_n\|_M$, $\|f\|$, $\|g\|_1$ *and* $\|\alpha\|_{(M^d)'}$. *If, furthermore,*

(A3) *there exists* $\Gamma_0\subset\Gamma$ *with* $|\Gamma_0\cap\Gamma_1|>0$ *such that* $u_n+g_n>0$ *on* Γ_0,

then we have

$$\|u-u_h\|_1 + \|p-p_h\| \le C_{**}(\sqrt{\varepsilon}+\sqrt{\delta}+\sqrt{h}). \tag{74}$$

Remark 24. Since $\int_\Gamma u_n\ d\Gamma = 0$ and $u_n+g_n\ge 0$ on Γ, we may assume that we are given Γ_0 as in (A3).

We use the standard Lagrange interpolation operator $I_h : C(\Omega)^d\to V_h$ and the L^2 projection operator $\Pi_h : Q\to Q_h$. It is well-known that

$$\|v-I_hv\|_1 \le Ch|v|_2 \qquad (v\in[H^2(\Omega)\cap H_0^1(\Omega)]^d), \tag{75a}$$
$$\|q-\Pi_hq\| \le Ch|q|_1 \qquad (q\in H^1(\Omega)). \tag{75b}$$

Proof of Theorem 23. We recall that (27) together with (22) give

$$a(u,v)+b(p,v)-[\tau_n(u,p),v_n] = (f,v)-[[\alpha_T,v_T]] \qquad (v\in V). \tag{76}$$

Hence, errors $u-u_h$ and $p-p_h$ satisfy

$$a(u-u_h,v_h)+b(p-p_h,v_h)-[\tau_n(u,p)+\alpha_n-\varepsilon^{-1}\phi_\delta(u_{hn}+g_n),v_{hn}] = 0 \quad (v_h\in V_h). \tag{77}$$

Setting $\hat{p}=\Lambda p$, $\hat{p}_h=\Lambda p_h$, $k=m(p)$ and $k_h=m(p_h)$, we can write as

$$a(u-u_h,v_h) = \underbrace{-b(\hat{p}-\hat{p}_h,v_h)}_{=J_1(v_h)}$$
$$\underbrace{+[\tau_n(u,p)+\alpha_n-\varepsilon^{-1}\phi_\delta(u_{hn}+g_n)+k-k_h,v_{hn}]}_{=J_2(v_h)} \qquad (v_h\in V_h).$$

In particular, we have

$$b(\Lambda q_h - \hat{p}_h, v_h) = -a(u-u_h, v_h) - b(\hat{p} - \Lambda q_h, v_h) \qquad (v_h \in V_{0h},\ q_h \in Q_h).$$

and, by applying (51),

$$\begin{aligned}
\|\Lambda q_h - \hat{p}_h\| &\le C \sup_{v_h \in V_{0h}} \frac{-a(u-u_h, v_h) - b(\hat{p} - \Lambda q_h, v_h)}{\|v_h\|_1} \\
&\le C(\|u-u_h\|_1 + \|\hat{p} - \Lambda q_h\|) \\
&\le C(\|u-u_h\|_1 + \|p - q_h\|) \qquad (q_h \in Q_h).
\end{aligned} \tag{78}$$

At this stage, we set

$$v_h = I_h u - u_h \in V_h, \qquad q_h = \Pi_h p \in Q_h, \qquad \hat{q}_h = \Lambda q_h \in Q_{0h}. \tag{79}$$

Then,

$$\begin{aligned}
\|\hat{p} - \hat{p}_h\| &\le \|\hat{p} - \hat{q}_h\| + \|\hat{q}_h - \hat{p}_h\| \\
&\le \|p - \Pi_h p\| + C(\|u-u_h\|_1 + \|p - \Pi_h p\|) \\
&\le C_{**} h + C\|u-u_h\|_1.
\end{aligned} \tag{80}$$

By using (75), (78) and $\|\hat{p} - \hat{q}_h\| \le C\|p - q_h\|$, we estimate as

$$\begin{aligned}
|J_1(I_h u - u_h)| &\le |b(\hat{p} - \hat{p}_h, I_h u - u)| + |b(\hat{p} - \hat{q}_h, u - u_h)| + |b(\hat{q} - \hat{p}_h, u - u_h)| \\
&\le \|b\| \cdot \|\hat{p} - \hat{p}_h\| \cdot \|I_h u - u\|_1 + \|b\| \cdot \|\hat{p} - \hat{q}_h\| \cdot \|u - u_h\|_1 + 0 \\
&\le C(C_{**} h + \|u-u_h\|_1) \cdot h|u|_2 + Ch|p|_1 \|u-u_h\|_1 \\
&\le C_{**} h^2 + C_{**} h \|u - u_h\|_1 \\
&\le C_{**} h^2 + C_{**} h(\|u - I_h u_h\|_1 + \|I_h u - u_h\|_1) \\
&\le C_{**} h^2 + C_{**} \|I_h u - u_h\|_1.
\end{aligned}$$

To perform an estimation for J_2, we divide it as

$$\begin{aligned}
J_2(I_h u - u_h) = &\underbrace{[\tau_n(u,p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) + k - k_h, (I_h u)_n - u_n]}_{=J_{21}} \\
&\qquad \underbrace{+[\tau_n(u,p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) + k - k_h, u_n - u_{hn}]}_{=J_{22}}.
\end{aligned}$$

According to stability results (54) and (55b), we deduce

$$\begin{aligned}
|J_{21}| &\le (\|\tau_n(u,\hat{p})\|_{M'} + \|\varepsilon^{-1}\phi_\delta(u_{hn} + g_n)\|_{M'} + \|\alpha_n\|_{M'} + |k_h|)\|(I_h u)_n - u_n\|_{1/2,\Gamma} \\
&\le C_* \|I_h u - u\|_1 \\
&\le C_{**} h.
\end{aligned}$$

Noting

$$\int_\Gamma (u_n - u_{hn})\ d\Gamma = \int_\Omega \nabla \cdot (u - u_h)\ dx = 0$$

and using (27c), (27d), (27e), (45b), (45c) and (54), we can calculate as

$$
\begin{aligned}
J_{22} &= [\tau_n(u,p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), u_n + g_n - (u_{hn} + g_n)] \\
&= -[\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), u_n + g_n] - [\tau_n(u,p) + \alpha_n, u_{hn} + g_n] \\
&\qquad\qquad + [\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), u_{hn} + g_n] \\
&= \underbrace{-[\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), u_n + g_n]}_{\le 0} \underbrace{-[\tau_n(u,p) + \alpha_n, [u_{hn} + g_n]_+]}_{\le 0} \\
&\qquad\qquad + [\tau_n(u,p) + \alpha_n, [u_{hn} + g_n]_-] \underbrace{-[\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), [u_{hn} + g_n]_-]}_{\le 0} \\
&\le [\tau_n(u,p) + \alpha_n, [u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)] + \varepsilon[\tau_n(u,p) + \alpha_n, \varepsilon^{-1}\phi_\delta(u_{hn} + g_n)] \\
&\le \|\tau_n(u,p) + \alpha_n\|_\Gamma \|[u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)\|_\Gamma \\
&\qquad\qquad + \varepsilon\|\tau_n(u,p) + \alpha_n\|_M \|\varepsilon^{-1}\phi_\delta(u_{hn} + g_n)\|_{M'} \\
&\le C_*(\delta + \varepsilon).
\end{aligned}
$$

Summing up those estimates, we obtain

$$
\begin{aligned}
C_{\mathrm{K}}\|I_h u - u_h\|_1^2 &\le a(I_h u - u_h, I_h u - u_h) \\
&= a(I_h u - u, I_h u - u_h) + a(u - u_h, I_h u - u_h) \\
&= a(I_h u - u, I_h u - u_h) + J_1(I_h u - u_h) + J_{21} + J_{22} \\
&\le C_{**}h\|I_h u - u_h\|_1 + C_{**}h^2 + C_{**}h\|I_h u - u_h\|_1 + C_{**}h + C_*(\delta + \varepsilon).
\end{aligned}
$$

Therefore, we deduce

$$
\|I_h u - u_h\|_1 \le C_{**}\left(\sqrt{h} + \sqrt{\varepsilon} + \sqrt{\delta}\right)
$$

and

$$
\|u - u_h\|_1 \le \|u - I_h u\|_1 + \|Iu - u_h\|_1 \le C_{**}\left(\sqrt{h} + \sqrt{\varepsilon} + \sqrt{\delta}\right).
$$

This, together with (80), implies (73).

Finally, we derive an estimation for $|k_h - k|$. As in the proof of Theorem 15 (cf. (69)), We take $\tilde{\mu} \in C_0^\infty(\Gamma)$ satisfying $\tilde{\mu} \ge 0,\ \tilde{\mu} \not\equiv 0$ in Γ and $\operatorname{supp}\tilde{\mu} \subset \Gamma_0 \cap \Gamma_1$. Then, setting $\tilde{\mu}_h = i_h\mu \in M_h$, we have

$$
\tilde{\mu}_h \ge 0,\ \tilde{\mu}_h \not\equiv 0 \text{ in } \Gamma, \quad \tilde{\mu}_h = 0 \text{ in } \Gamma\backslash(\Gamma_0 \cap \Gamma_1), \quad \|\tilde{\mu}_h\|_M \le C, \quad \left|\int_\Gamma \tilde{\mu}_h\ d\Gamma - \int_\Gamma \tilde{\mu}\ d\Gamma\right| \le Ch^2.
$$

Since $u_n + g_n > 0$ on Γ_0, we have $\tau_n(u,p) + \alpha_n = 0$ on Γ_0 in view of (27e). Substituting $\tilde{v}_h = \mathcal{E}_{nh}\tilde{\mu}_h \in V_h \subset V$ into (76) and using (27d), we have

$$
\begin{aligned}
k\int_\Gamma \tilde{\mu}_h\ d\Gamma &= a(u, \tilde{v}_h) + b(\hat{p}, \tilde{v}_h) - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]] - \int_\Gamma (\tau_n(u,p) + \alpha_n)\tilde{\mu}_h\ d\Gamma \\
&= a(u, \tilde{v}_h) + b(\hat{p}, \tilde{v}_h) - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]].
\end{aligned}
$$

This, together with (70), gives

$$
|k_h - k| \le |a(u_h - u, \tilde{v}_h)| + |b(\hat{p}_h - \hat{p}, \tilde{v}_h)| \le C(\|u_h - u\|_1 + \|\hat{p}_h - \hat{p}\|).
$$

Combining those estimates, we completes the proof of (74). □

7 Numerical examples

In this section, we present some results of numerical experiments in order to confirm our theoretical results. We prefer the original setting (4) with (6), (11) to (1) and (2). Thus, we consider a model Stokes problem with a nonlinear Robin condition

$$
\begin{aligned}
&-\nu\Delta v + \nabla q = f, \quad \nabla\cdot v = 0 && \text{in } \Omega, && (81a)\\
&v = b && \text{on } S_1, && (81b)\\
&v = 0 && \text{on } S_2, && (81c)\\
&\tau_n(v,q) = \frac{1}{\varepsilon}\phi_\delta(v_n) && \text{on } \Gamma, && (81d)\\
&v_T = 0 && \text{on } \Gamma, && (81e)
\end{aligned}
$$

where ϕ_δ is the regularized function defined as (3).

Remark 25. As mentioned in Introduction, we are interested in computing v and q in (4). The unknown functions u and p in (1) and (2) are introduced as "perturbations" of those target variables and they make analysis clear. Moreover, the reference flow (g,π) plays an important role in theoretical considerations, whereas it is not obvious that it is always available in actual computations.

Remark 26. In (81), we take $v_T = 0$ instead of $\tau_T(u) = 0$ as a boundary condition for the tangential component of v on Γ. See Remark 12.

The finite element approximation for (81) reads as follows.

($\mathbf{PE}'_{\varepsilon,\delta,h}$) Find $(v_h, q_h) \in W_h \times Q_h$ such that $v_h = i_h b$ on S_1 and

$$
\begin{aligned}
&a(v_h,w_h) + b(q_h,w_h) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(v_n) w_{hn}\, d\Gamma = (f,w_h) && (\forall w_h \in V_h), && (82)\\
&b(r_h,v_h) = 0 && (\forall r_h \in Q_h), && (83)
\end{aligned}
$$

where

$$
W_h = \{v_h \in C^0(\overline{\Omega}) \mid v_h = 0 \text{ on } S_2,\ v_{hT} = 0 \text{ on } \Gamma,\ v_h|_T \in \mathcal{P}_1^{(d)} \oplus \mathrm{span}\{\varphi_T\}\ (\forall T \in \mathcal{T}_h)\}.
$$

First, we deal with a simple example. That is, setting $\Omega = \{(x,y) \mid 0 \le x \le L, -R \le y \le R\}$, $S_1 = \{0\} \times [-R,R]$, and $\Gamma = \{L\} \times [-R,R]$, we impose

$$
b(x,y) = (C_0(R^2 - y^2), 0), \quad f \equiv 0 \tag{84}
$$

with $C_0 > 0$. Then, (81) has the exact solution which is explicitly given as

$$
v(x,y) = \left(C_0(R^2-y^2), 0\right), \qquad q(x,y) = 2\nu C_0 L\left(1 - \frac{x}{L}\right). \tag{85}
$$

This is nothing but the well-known Poiseuille flow.

The details of our computation are as follows. Set $L = 15$, $R = 5$, $\nu = 1/50$, and $C_0 = 5/(\nu L)$. For the triangulation of Ω, we use a uniform mesh composed of $12N^2$ congruent right-angle triangles; The rectangle is divided into $3N \times 2N$ squares. Then, each small square is decomposed into two equal triangles by a diagonal. Consequently, $h = N/\sqrt{2}$. Since we have employed the C^1 regularization ϕ_δ, Newton's method is available for computing the nonlinear equation $(\mathrm{PE}'_{\varepsilon,\delta,h})$. Penalty parameters are chosen as $\varepsilon = \delta = h/20$. Hence, it is ensured by Theorem 23 that

$$
\|v - v_h\|_1 + \|q - q_h\| \le C\sqrt{h}. \tag{86}
$$

In order to verify this, we set

$$E_h^{(1)} = \|v - v_h\|, \quad E_h^{(2)} = \|v - v_h\|_1, \quad E_h^{(3)} = \|q - q_h\|,$$

and observe

$$\rho_h^{(i)} = \frac{\log E_{h'}^{(i)} - \log E_h^{(i)}}{\log h' - \log h} \qquad (i = 1, 2, 3)$$

with $h' \approx h/2$.

The result is reported in Tab. 1; The H^1 error $\rho_h^{(2)}$ for v_h is close to the unity. It is better than our theoretical result (86). However, we have not succeeded in proving those first order convergences at present. We need further study. We could not derive the L^2 error for v_h; From Tab. 1, we observe that the second order convergence actually takes place.

h	$E_h^{(1)}$	$\rho_h^{(1)}$	$E_h^{(2)}$	$\rho_h^{(2)}$	$E_h^{(3)}$	$\rho_h^{(3)}$
1.0743	13.9	–	$1.20 \cdot 10^2$	–	$2.07 \cdot 10^{-1}$	–
0.5371	3.47	2.001	$5.96 \cdot 10^1$	1.010	$6.57 \cdot 10^{-2}$	1.656
0.2685	0.87	2.000	$2.97 \cdot 10^1$	1.003	$2.18 \cdot 10^{-2}$	1.594
0.1342	0.21	2.000	$1.48 \cdot 10^1$	1.001	$7.42 \cdot 10^{-3}$	1.553
0.0665	0.052	2.000	7.17	1.000	$2.56 \cdot 10^{-3}$	1.527

Table 1: Numerical convergence rates of $(\mathrm{PE}'_{\varepsilon,\delta,h})$ for (85).

Finally, we consider a two-dimensional branched pipe as illustrated in Fig. 2. Since this Ω is not a polygon, we approximate it by a polygon Ω_h whose vertices are located on $\partial\Omega$. On S_1, we impose a parabolic inflow similarly to (84). The figure 3 shows the state of a numerical flow velocity v_h.

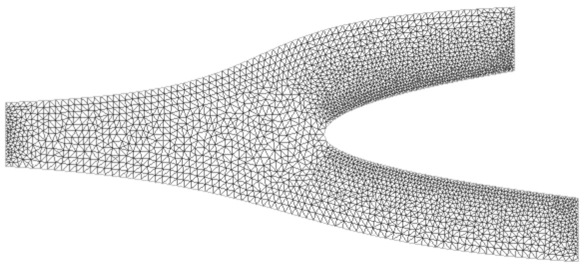

Figure 2: A branched pipe and an example of triangulation.

As before, we observe $\rho_h^{(1)}$, $\rho_h^{(2)}$ and $\rho_h^{(3)}$. Since in this case we cannot obtain the (explicit) exact solution, we make use of numerical solutions with extra fine mesh. Tab. 2 shows the result. We observe that the H^1 error for v_h and the L^2 error for q_h are close to the unity even in the curved domain. Moreover, the L^2 error for v_h is close to 2.

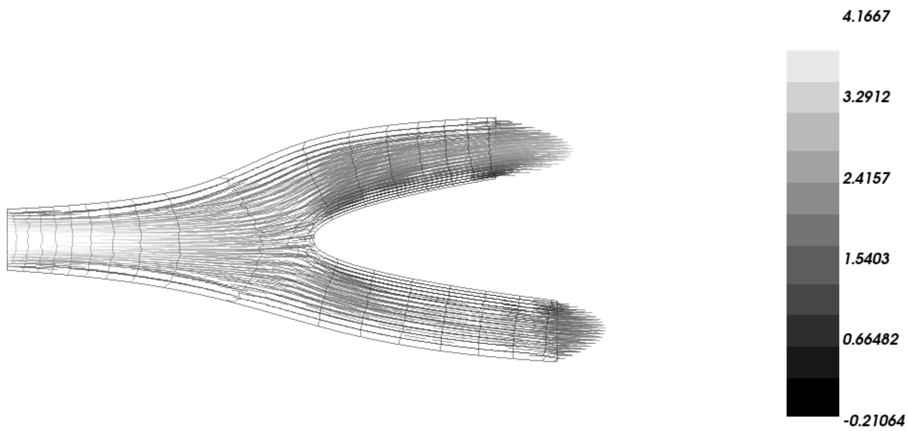

Figure 3: Velocity and pressure field in branched pipe.

h	$E_h^{(1)}$	$\rho_h^{(1)}$	$E_h^{(2)}$	$\rho_h^{(2)}$	$E_h^{(3)}$	$\rho_h^{(3)}$
0.69279	$2.497 \cdot 10^{-1}$	—	5.941	—	$1.786 \cdot 10^{-1}$	—
0.33353	$7.767 \cdot 10^{-2}$	1.552	3.359	0.780	$5.909 \cdot 10^{-2}$	1.513
0.17571	$2.044 \cdot 10^{-2}$	2.083	1.768	1.001	$3.069 \cdot 10^{-2}$	1.022

Table 2: Numerical convergence rates of $(\mathrm{PE}'_{\varepsilon,\delta,h})$ for branched pipe.

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References

[1] M. Ayadi, L. Baffico, M. K. Gdoura, and T. Sassi, *Error estimates for Stokes problem with Tresca friction conditions*, ESAIM Math. Model. Numer. Anal. **48** (2014) 1413–1429.

[2] D. Boffi, F. Brezzi, and M. Fortin, Mixed Finite Element Methods and Applications, Springer, 2013.

[3] C. Baiocchi and A. Capelo, Variational and Quasivariational Inequalities. Applications to Free Boundary Problems, Wiley, 1984.

[4] F. Boyer and P. Fabrie, *Outflow boundary conditions for the incompressible non-homogeneous Navier-Stokes equations*, Discrete Contin. Dyn. Syst. Ser. B, **7** (2007) 219–250.

[5] F. Boyer and P. Fabrie, Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models, Springer, 2013.

[6] C. H. Bruneau and P. Fabrie, *Effective downstream boundary conditions for incompressible Navier-Stokes equations*, Internat. J. Numer. Methods Fluids, **19** (1994) 693–705.

[7] C. H. Bruneau and P. Fabrie, *New efficient boundary conditions for incompressible Navier-Stokes equations: a well-posedness result*, RAIRO Modél. Math. Anal. Numér. **30** (1996) 815–840.

[8] Y. Bazilevs, J. R. Gohean, T. J. R. Hughes, R. D. Moser and Y. Zhang, *Patient-specific isogeometric fluid-structure interaction analysis of thoracic aortic blood flow due to implantation of the Jarvik 2000 left ventricular assist device*, Comput. Methods Appl. Mech. Engrg. **198** (2009) 3534–3550.

[9] M. Boman, *Estimates for the L_2-projection onto continuous finite element spaces in a weighted L_p-norm*, BIT Numer. Math. **46** (2006) 249–260.

[10] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011.

[11] M. Crouzeix, and V. Thomée, *The stability in L_p and W^1_p of the L_2-projection onto finite element function spaces*, Math. Comp. **48** (1987) 521–532.

[12] J. Douglas Jr., T. Dupont, and L. Wahlbin, *The stability in L^q of the L^2-projection into finite element function spaces*, Numer. Math. **23** (1975) 193–197.

[13] L. Formaggia, A. Quarteroni and A. Veneziani (Eds.), Cardiovascular Mathematics, Springer, 2009.

[14] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1986.

[15] P. Grisvard, *Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain*, Numerical Solution of P.D.E's III, (Proc. Third Sympos. (SYNSPADE), Univ. Maryland, College Park, Md., 1975), 207–274, Academic Press, 1976.

[16] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, 1985.

[17] P. M. Gresho and R. L. Sani, Incompressible Flow and the Finite Element Method, Volume 2, Isothermal Laminar Flow, 2000, Wiley.

[18] J. G. Heywood, R. Rannacher and S. Turek, *Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations*, Internat. J. Numer. Methods Fluids, **22** (1996) 325–352.

[19] T. Kashiwabara, *Finite element method for Stokes equations under leak boundary condition of friction type*, SIAM J. Numer. Anal. **51** (2013) 2448–2469.

[20] N. Kikuchi and J. T.Oden, Contact Problems in Elasticity, SIAM, 1988.

[21] Y. Li and R. An, *Semi-discrete stabilized finite element methods for Navier-Stokes equations with nonlinear slip boundary conditions based on regularization procedure*, Numer. Math. **117** (2011) 1–36.

[22] Y. Li and R. An, *Penalty finite element method for Navier-Stokes equations with nonlinear slip boundary conditions*, Internat. J. Numer. Methods Fluids, **69** (2012) 550–576.

[23] J. L. Lions: *Some remarks on variational inequalities*, Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969), 269–282, Univ. Tokyo Press, Tokyo.

[24] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, 1969.

[25] J. L. Lions and E. Magenes, Non-homogeneous Boundary Value Problems and Applications, I, Springer, 1972.

[26] N. Saito and H. Fujita, *Remarks on traces of H^1-functions defined in a domain with corners*, J. Math. Sci. Univ. Tokyo **7** (2000) 325–345.

[27] C. A. Taylor, T. J. Hughes, and C. K. Zarins, *Finite element modeling of blood flow in arteries*, Comput. Methods Appl. Mech. Engrg. **158** (1998) 155–196.

[28] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, AMS, AMS Chelsea Publishing, 2001.

[29] G. Zhou and N. Saito, *The Navier-Stokes equations under a unilateral boundary condition of Signorini's type*, UTMS Preprint Series 2015-1 (`http://www.ms.u-tokyo.ac.jp/preprint/2015/`).

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