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On Lagrangian embeddings into the complex projective spaces

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ON LAGRANGIAN EMBEDDINGS INTO THE COMPLEX PROJECTIVE SPACES

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Abstract. We prove that for any closed orientable connected 3-manifold L and any Lagrangian immersion of the connected sum $L\#(S^1 \times S^2)$ either into the complex projective 3-space $\mathbb{C}P^3$ or into the product $\mathbb{C}P^1 \times \mathbb{C}P^2$ of the complex projective line and the complex projective plane, there exists a Lagrangian embedding which is homotopic to the initial Lagrangian immersion.

1. Introduction

1.1. **Main result.** A symplectic manifold is an even-dimensional manifold X with a closed non-degenerate 2-form ω. A Lagrangian submanifold L of a symplectic manifold X is a half-dimensional submanifold such that the restriction $\omega\mid_L$ of the symplectic structure is vanishing as a 2-form. The topological classification of Lagrangian submanifolds in a symplectic manifold is an important problem in symplectic topology. Given a Lagrangian immersion of an n-dimensional manifold into a $2n$-dimensional symplectic manifold, it is interesting to know whether it is Lagrangian regularly homotopic to a Lagrangian embedding. In this paper, we show that in some cases all Lagrangian immersions are at least homotopic to Lagrangian embeddings as continuous maps. Our main result is the following.

Theorem 1.1. *Let X be either the complex projective 3-space $\mathbb{C}P^3$ or the product $\mathbb{C}P^1 \times \mathbb{C}P^2$ of the complex projective line and the complex projective plane, where the complex projective space $\mathbb{C}P^n$ is endowed with the Fubini-Study form ω_n, $n = 1, 2, 3$. Then for a closed orientable connected 3-manifold L and a Lagrangian immersion $f\colon L\#(S^1 \times S^2) \to X$, there exists a Lagrangian embedding $L\#(S^1 \times S^2) \to X$ homotopic to f.*

Gromov's h-principle for Lagrangian immersions [4] gives a necessary and sufficient condition for a continuous map f from a 3-manifold L to a 6-dimensional symplectic manifold X to be homotopic to a Lagrangian immersion. In particular, any closed orientable 3-manifold L admits a Lagrangian immersion into a Darboux chart. However, it is not always true that a Lagrangian immersion is homotopic to a Lagrangian embedding. In fact, there are several necessary conditions for a closed 3-manifold L to be a Lagrangian submanifold of the complex projective 3-space $\mathbb{C}P^3$, see Seidel [7] and Biran [1]. For a closed orientable connected 3-manifold L, the connected sum $L\#(S^1 \times S^2)$ satisfies the necessary conditions of Seidel [7]. Indeed, the existence of a Lagrangian embedding of $L\#(S^1 \times S^2)$ into a Darboux chart is proved in [2].

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Theorem 1.2 (Ekholm-Eliashberg-Murphy-Smith [2]). *There exists a Lagrangian embedding $L\#(S^1 \times S^2) \to \mathbb{C}^3$ for any closed orientable connected 3-manifold L, where $\mathbb{C}^3$ is the standard symplectic space.*

Theorem 1.2 was proved by applying the resolving theory of Lagrangian self-intersections by Hamiltonian regular homotopies for certain Lagrangian immersions developed by Eliashberg and Murphy [3]. They constructed a self-transverse Lagrangian immersion $L \to \mathbb{C}^3$ with exactly one double point and resolved the double point by Polterovich's Lagrangian surgery [5].

Remark 1.3. Lagrangian embeddings into a Darboux chart are homotopically trivial. For a closed orientable connected 3-manifold L, if $H^2(L;\mathbb{Z})$ has a non-trivial 4-torsion element then Theorem 1.1 provides a homotopically non-trivial Lagrangian embedding of $L\#(S^1 \times S^2)$ into $\mathbb{C}P^3$, and if $H^2(L;\mathbb{Z})$ has a non-trivial torsion element then Theorem 1.1 provides a homotopically non-trivial Lagrangian embedding of $L\#(S^1 \times S^2)$ into $\mathbb{C}P^1 \times \mathbb{C}P^2$. See Lemmas 3.3 and 3.7 below.

1.2. **Plan of the paper.** In Section 2, we construct a local deformation of Lagrangian immersions. With the help of this local deformation, the arguments in [3] and [2] ensure that Theorems 2.1 and 2.2 hold. Theorems 2.1 and 2.2 are h-principles for self-transverse Lagrangian immersions into 6-dimensional compact symplectic manifolds with the minimal or near-minimal number of double points and with a conical point, respectively. In Sections 3.1 and 3.3, we characterize the homotopy classes of Lagrangian immersions of closed orientable connected 3-manifolds into $\mathbb{C}P^3$ and into $\mathbb{C}P^1 \times \mathbb{C}P^2$, respectively. In Sections 3.2 and 3.4, Theorem 1.1 is proved as an application of Theorems 2.1, 2.2, Lemmas 3.3, 3.7, and Polterovich's Lagrangian surgery [5].

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2. Lagrangian immersions with few double points

We use the definitions introduced in [3] and [2]. Let X be a 6-dimensional oriented manifold, L a 3-manifold, and $f\colon L \to X$ an immersion. We denote by $I(f) \in \mathbb{Z}/2$ and $\mathrm{SI}(f) \in \mathbb{Z}$ the self-intersection number of f and the total number of double points of f, respectively.

To prove Theorem 1.1, we use the following theorems.

Theorem 2.1. *Let (X,ω) be a 6-dimensional simply connected compact symplectic manifold, L a closed connected 3-manifold, and $f_0\colon L \to X$ a Lagrangian immersion. Then there exists a Hamiltonian regular homotopy $f_t\colon L \to X$, $0 \le t \le 1$, from the Lagrangian immersion f_0 to a self-transverse Lagrangian immersion f_1 such that*

$$\mathrm{SI}(f_1) = \begin{cases} 1, & \text{if } I(f_0) = 1; \\ 2, & \text{if } I(f_0) = 0. \end{cases}$$

Theorem 2.2. *Let (X, ω) be a 6-dimensional simply connected compact symplectic manifold, L a connected 3-manifold, and $f_0\colon L \to X$ a Lagrangian immersion with a conical point $p \in L$. Suppose that the Legendrian link of f_0 at p is loose. Then there exists a Hamiltonian regular homotopy $f_t\colon L \to X$, $0 \le t \le 1$, from the Lagrangian immersion f_0 to a self-transverse Lagrangian immersion f_1 with a conical point p such that f_t is the identity in a neighborhood of p and that $\mathrm{SI}(f_1) = |I(f_0)|$.*

Theorems 2.1 and 2.2 can be proved in a way similar to the proof of Theorems 1.1 and 3.7 of [2] for symplectic manifolds of dimension ≥ 8, by using the following lemma instead of Lemma 4.2 of [3] in the proof of Theorem 2.2 of [3].

Lemma 2.3. *Let $A = [0,1] \times S^{n-1} \ni (x, z)$, $n \ge 3$, be the annulus with the coordinates (x, z). Take the dual coordinates (y, u) on the cotangent bundle T^*A so that the canonical Liouville form $\lambda = y\,dx + u\,dz$. Then for any integer $N \ge 10$ there exists a Lagrangian immersion $\Delta\colon A \to T^*A$ with the following properties:*

- $\Delta(A) \subset \left\{|y| \le \frac{12}{N},\ \|u\| \le \frac{12}{N}\right\}$;
- *Δ coincides with the inclusion of the zero section $j_A\colon A \hookrightarrow T^*A$ near ∂A;*
- *there exists a Lagrangian regular homotopy which is the identity near ∂A and connects j_A to Δ in* $\left\{|y| \le \frac{12}{N},\ \|u\| \le \frac{12}{N}\right\}$;
- *for the Δ-image ζ of any path connecting $\{0\} \times S^{n-1}$ to $\{1\} \times S^{n-1}$ in A,* $\int_\zeta \lambda = 1$;
- *the action of any self-intersection point of Δ is* $< \frac{2}{N}$;
- $\mathrm{SI}(\Delta) = 4N^2$.

Proof. We follow the proof of Lemma 4.2 of [3], where Δ was constructed by using the plane curves γ_1, γ_2, and γ_3. We change γ_1 so that $\mathrm{SI}(\Delta) = 4N^2$ as follows.

Consider in $\mathbb{R}^2$ with the coordinates (x, y) the curves $\zeta_k\colon [0,4] \to \mathbb{R}^2$, $k = 1, \dots, N$, defined by

$$\zeta_k(t) = \begin{cases} \left(\frac{1}{12} - \frac{k-1}{N^4}, \left(\frac{6}{N^2} + \frac{2(k-1)}{N^4}\right)t - \frac{k-1}{N^4}\right) & \text{if } 0 \le t \le 1, \\ \left(\left(\frac{1}{6} + \frac{2(k-1)}{N^4}\right)t - \frac{1}{12} - \frac{3(k-1)}{N^4}, \frac{6}{N^2} + \frac{k-1}{N^4}\right) & \text{if } 1 \le t \le 2, \\ \left(\frac{1}{4} + \frac{k-1}{N^4}, -\left(\frac{6}{N^2} + \frac{2k-1}{N^4}\right)t + \frac{18}{N^2} + \frac{5k-3}{N^4}\right) & \text{if } 2 \le t \le 3, \\ \left(-\left(\frac{1}{6} + \frac{2k-1}{N^4}\right)t + \frac{3}{4} + \frac{7k-4}{N^4}, -\frac{k}{N^4}\right) & \text{if } 3 \le t \le 4. \end{cases}$$

Then a product $\eta_N = \zeta_1 \cdot \zeta_2 \cdot \dots \cdot \zeta_N\colon [0,4] \to \mathbb{R}^2$ satisfies

$$\int_{\eta_N} y\,dx = \frac{1}{N} + \frac{1}{6N^2} + \frac{6}{N^4} - \frac{14}{3N^5} - \frac{1}{2N^6} + \frac{1}{6N^7}.$$

FIGURE 1. the curve η_N for $N = 10$

We denote by $T_\varepsilon\colon \mathbb{R}^2 \to \mathbb{R}^2$ the affine map $(x,y) \mapsto (x, y+\varepsilon)$ and let $l_N\colon [0,3] \to \mathbb{R}^2$ be a piecewise linear embedding connecting four points

$$\begin{aligned}
l_N(0) &= \eta_N(4) = \left(\frac{1}{12} - \frac{1}{N^3}, -\frac{1}{N^3}\right),\\
l_N(1) &= \left(\frac{1}{12} - \frac{1}{N^3}, \frac{6}{N^2} + \frac{1}{N^3} - \frac{1}{2N^4}\right),\\
l_N(2) &= \left(\frac{1}{12}, \frac{6}{N^2} + \frac{1}{N^3} - \frac{1}{2N^4}\right), \text{ and}\\
l_N(3) &= T_{\delta_N}(\eta_N(0)) = \left(\frac{1}{12}, \frac{6}{N^2} + \frac{2}{N^3}\right),
\end{aligned}$$

where $\delta_N = \frac{6}{N^2} + \frac{2}{N^3}$. We further let $k_N\colon [0,3] \to \mathbb{R}^2$ be a piecewise linear embedding connecting four points

$$\begin{aligned}
k_N(0) &= T_{(N-1)\delta_N}(\eta_N(4)) = \left(\frac{1}{12} - \frac{1}{N^3}, \frac{6}{N} - \frac{4}{N^2} - \frac{3}{N^3}\right),\\
k_N(1) &= \left(\frac{1}{12} - \frac{1}{N^3}, \frac{6}{N} + \frac{2}{N^2} - \frac{2}{N^3} - \frac{1}{2N^4}\right),\\
k_N(2) &= \left(\frac{1}{4} + \frac{1}{N^3}, \frac{6}{N} + \frac{2}{N^2} - \frac{2}{N^3} - \frac{1}{2N^4}\right), \text{ and}\\
k_N(3) &= \left(\frac{1}{4} + \frac{1}{N^3}, 0\right).
\end{aligned}$$

Then we define a curve $\gamma\colon [0,1] \to \mathbb{R}^2$ by connecting the straight line $\left[0, \frac{1}{12}\right] \times \{0\}$, N-copies $\eta_N, T_{\delta_N}(\eta_N), T_{2\delta_N}(\eta_N), \dots, T_{(N-1)\delta_N}(\eta_N)$ of η_N, $(N-1)$-copies l_N, $T_{\delta_N}(l_N), T_{2\delta_N}(l_N), \dots, T_{(N-2)\delta_N}(l_N)$ of l_N, the curve k_N, and the straight line $\left[\frac{1}{4} + \frac{1}{N^3}, \frac{1}{3}\right] \times \{0\}$. See Figure 2.

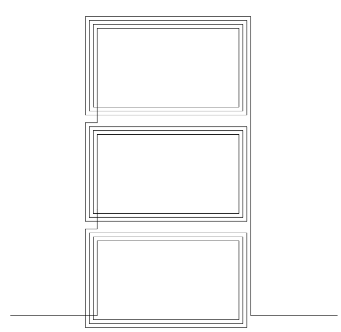

FIGURE 2. the curve γ for $N = 3$

By the construction, the curve γ satisfies the followings:

- $1 - \frac{2}{N} < \int_\gamma y\,dx < 1 + \frac{2}{N}$;
- the action of any self-intersection point of γ is $< \frac{2}{N}$;
- $\mathrm{SI}(\gamma) = N^2$.

Smoothing the corners of γ, we construct an immersed curve γ_1 with transverse self-intersections. We can arrange γ_1 to satisfy the followings:

- $\left| \int_{\gamma_1} y\, dx - 1 \right| < \frac{2}{N}$;
- the action of any self-intersection point of γ_1 is $< \frac{2}{N}$;
- $\mathrm{SI}(\gamma_1) = N^2$;
- the curve γ_1 is contained in the rectangle $\left\{ 0 \leq x \leq \frac{1}{3},\ |y| \leq \frac{7}{N} \right\}$.

We replace the plane curve γ_1 in the proof of Lemma 4.2 of [3] with the above γ_1. Then we define γ_2 and γ_3, and then Δ in a way similar to the proof of Lemma 4.2 of [3]. □

Remark 2.4. Lemma 4.2 of [3] only asserted the construction of such Δ with $\mathrm{SI}(\Delta) \sim N^3$, and hence Theorem 2.2 of [3] was shown for a symplectic manifold which is the negative completion of a compact symplectic manifold and of dimension $2n \geq 8$ in this way.

3. Proof of Theorem 1.1

3.1. **Lagrangian immersions into $\mathbb{C}P^3$.** In view of Gromov's h-principle, the classification of Lagrangian immersions is reduced to a pure algebro-topological problem. In this section, we characterize the homotopy classes of Lagrangian immersions of closed orientable connected 3-manifolds into $\mathbb{C}P^3$.

First the homotopy classes of continuous maps from a 3-manifold L to the complex projective 3-space $\mathbb{C}P^3$ are classified as follows. We denote by $\gamma_n \to \mathbb{C}P^n$ the tautological line bundle and by $c_1(\gamma_n)$ its first Chern class.

Proposition 3.1. *Let L be a 3-manifold. If $n \geq 2$ then the map*

$$[L, \mathbb{C}P^n] \to H^2(L; \mathbb{Z}) : [h] \mapsto -h^* c_1(\gamma_n)$$

is a bijection.

Proof. It follows from the fact that $\mathbb{C}P^\infty$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ and $\mathbb{C}P^n \subset \mathbb{C}P^\infty$ is the $2n$-skeleton. □

We state Gromov's h-principle for Lagrangian immersions.

Theorem 3.2 (Gromov [4])**.** *Let (X, ω) be a $2n$-dimensional symplectic manifold and L an n-dimensional manifold. If $h\colon L \to X$ is a continuous map with $[h^*\omega] = 0$ in $H^2(L; \mathbb{R})$ and $H\colon TL \to TX$ a Lagrangian homomorphism covering h, then there exists a Lagrangian immersion $f\colon L \to X$ which is homotopic to h. Moreover,*

1. *if h is an immersion then one can choose f to be regularly homotopic to h;*
2. *if h is a Lagrangian immersion on a neighborhood of a closed ball in L then one can choose f to be equal to h on the closed ball.*

The above two conditions, $[h^*\omega] = 0$ in $H^2(L; \mathbb{R})$ and the existence of a Lagrangian homomorphism covering h, are simplified in the case where (X, ω) is the complex projective 3-space $\mathbb{C}P^3$ and L is a closed orientable connected 3-manifold.

Lemma 3.3. *Let L be a closed orientable connected 3-manifold and $h\colon L \to \mathbb{C}P^3$ a continuous map. Then the followings are equivalent.*

1. *There exists a Lagrangian immersion $L \to \mathbb{C}P^3$ which is homotopic to h.*
2. *$h^* c_1(\gamma_3)$ is a 4-torsion element in $H^2(L; \mathbb{Z})$.*

Proof. The equality $[\omega_3] = -c_1(\gamma_3) \in H^2(\mathbb{C}P^3;\mathbb{Z})$ and the naturality of coefficient homomorphisms imply that $[h^*\omega_3] = 0$ in $H^2(L;\mathbb{R})$ if and only if $h^*c_1(\gamma_3)$ is a torsion element in $H^2(L;\mathbb{Z})$.

Next, we fix a 3-frame of the tangent bundle TL. Let $P \to \mathbb{C}P^3$ be the principal U(3)-bundle associated to the tangent bundle $T\mathbb{C}P^3$. Then we can identify a Lagrangian homomorphism $H\colon TL \to T\mathbb{C}P^3$ covering h with a map $s\colon L \to P$ which is a lift of h. Thus there exists a Lagrangian homomorphism $H\colon TL \to T\mathbb{C}P^3$ covering h if and only if the principal U(3)-bundle $h^*P \to L$ admits a global section. Since $\dim L = 3$, the obstruction for the existence of a global section $L \to h^*P$ is only the first Chern class $c_1(h^*T\mathbb{C}P^3) = h^*c_1(T\mathbb{C}P^3) = -4h^*c_1(\gamma_3)$. □

Remark 3.4. Using part 1) of Theorem 3.2 and taking the connected sum of Whitney sphere, we can see that for the above pair (h, H) and a number $n \in \mathbb{Z}/2$, there exists a self-transverse Lagrangian immersion $f\colon L \to \mathbb{C}P^3$ which is homotopic to h and satisfies $I(f) = n$.

We state another lemma which is used in the proof of Theorem 1.1 for $\mathbb{C}P^3$. It directly follows from Theorem 2.1 and Lemma 3.3.

Lemma 3.5. *Let L be a closed orientable connected 3-manifold and $h\colon L \to \mathbb{C}P^3$ a continuous map with $4h^*c_1(\gamma_3) = 0$ in $H^2(L;\mathbb{Z})$. Then for an arbitrary Lagrangian immersion $f_0\colon L \to \mathbb{C}P^3$ which is homotopic to h, there exists a Lagrangian regular homotopy $f_t\colon L \to \mathbb{C}P^3$, $0 \le t \le 1$, such that f_1 is self-transverse and*

$$\mathrm{SI}(f_1) = \begin{cases} 1, & \text{if } I(f_0) = 1; \\ 2, & \text{if } I(f_0) = 0. \end{cases}$$

3.2. **Proof of Theorem 1.1 for $\mathbb{C}P^3$.** Let L be a closed orientable connected 3-manifold and $f\colon L\#(S^1 \times S^2) \to \mathbb{C}P^3$ a Lagrangian immersion. Lemma 3.3 provides the equality $4f^*c_1(\gamma_3) = 0$ in $H^2(L\#(S^1\times S^2);\mathbb{Z})$. By the Mayer-Vietoris exact sequence for $L\#(S^1\times S^2) = (L \setminus \mathring{D}^3) \cup (S^1 \times S^2 \setminus \mathring{D}^3)$ where $\mathring{D}^3$ is the interior of a closed 3-disk, there is the isomorphism $H^2(L\#(S^1\times S^2);\mathbb{Z}) \cong H^2(L\setminus \mathring{D}^3;\mathbb{Z}) \oplus H^2(S^1\times S^2 \setminus \mathring{D}^3;\mathbb{Z})$. Since the isomorphism in the Mayer-Vietoris exact sequence is induced by the inclusions and $H^2(S^1 \times S^2 \setminus \mathring{D}^3;\mathbb{Z}) \cong \mathbb{Z}$, the element $f^*c_1(\gamma_3)$ is of the form

$$f^*c_1(\gamma_3) = (h^*c_1(\gamma_3), 0) \in H^2(L \setminus \mathring{D}^3;\mathbb{Z}) \oplus H^2(S^1 \times S^2 \setminus \mathring{D}^3;\mathbb{Z}),$$

where $[h] = \left[f \,\Big|_{L\setminus \mathring{D}^3} \right] \in [L \setminus \mathring{D}^3, \mathbb{C}P^3]$.

In the following, we construct a self-transverse Lagrangian immersion of L into $\mathbb{C}P^3$ with exactly one double point and resolve the double point by Polterovich's Lagrangian surgery [5] to obtain the desired Lagrangian embedding. Since $H^2(L \setminus \mathring{D}^3;\mathbb{Z}) \cong H^2(L;\mathbb{Z})$, we can identify $[L\setminus \mathring{D}^3, \mathbb{C}P^3]$ with $[L, \mathbb{C}P^3]$. Let $[\hat{h}]$ be the element of $[L, \mathbb{C}P^3]$ which is the extension of $[h]$. We note that $4\hat{h}^*c_1(\gamma_3) = 0$ in $H^2(L;\mathbb{Z})$. Applying Lemmas 3.3 and 3.5 to $\hat{h}$, we obtain a self-transverse Lagrangian immersion $f_1\colon L \to \mathbb{C}P^3$ which is homotopic to $\hat{h}$ and satisfies $\mathrm{SI}(f_1) = 1$. Using Polterovich's Lagrangian surgery [5] to resolve the double point of f_1, we obtain a Lagrangian embedding $g\colon L\#(S^1\times S^2)\to \mathbb{C}P^3$. We claim that g is homotopic to f. Indeed, it is enough to show that $g^*c_1(\gamma_3)=f^*c_1(\gamma_3)$, and by the definition of h,

$$\begin{aligned} g^*c_1(\gamma_3) &= \Big(\Big(g\,\Big|_{L\setminus \mathring{D}^3}\Big)^* c_1(\gamma_3), \Big(g\,\Big|_{S^1\times S^2\setminus \mathring{D}^3}\Big)^* c_1(\gamma_3)\Big) \\ &= \Big(\Big(f_1\,\Big|_{L\setminus \mathring{D}^3}\Big)^* c_1(\gamma_3), 0\Big) \\ &= (h^*c_1(\gamma_3), 0) \\ &= f^*c_1(\gamma_3). \end{aligned}$$

The proof of Theorem 1.1 for $\mathbb{C}P^3$ is completed. □

3.3. **Lagrangian immersions into $\mathbb{C}P^1\times\mathbb{C}P^2$.** In this section, we characterize the homotopy classes of Lagrangian immersions of closed orientable connected 3-manifolds into $\mathbb{C}P^1\times\mathbb{C}P^2$.

We need a classification of homotopy classes of continuous maps from a 3-manifold L to the complex projective line $\mathbb{C}P^1$. In [6], Pontrjagin proved the following.

Theorem 3.6 (Pontrjagin [6])**.** *Let K be a 3-dimensional complex. Then there is a bijection*

$$[K,\mathbb{C}P^1]\approx \coprod_{z^2\in H^2(K;\mathbb{Z})} H^3(K;\mathbb{Z})/(2z^2\smile H^1(K;\mathbb{Z})),$$

where $\smile$ denotes the cup product.

We recall the correspondence of the elements in Theorem 3.6 for a closed orientable connected 3-manifold L. For an element $[h]\in[L,\mathbb{C}P^1]$, the cohomology class $z^2\in H^2(L;\mathbb{Z})$ is equal to $-h^*c_1(\gamma_1)$. It represents the primary obstruction for continuous maps from a 3-manifold L to the complex projective line $\mathbb{C}P^1$ to be homotopic. The second obstruction is an element of $H^3(L;\pi_3(\mathbb{C}P^1))\cong H^3(L;\mathbb{Z})$ modulo $2z^2\smile H^1(L;\mathbb{Z})$. For continuous maps $f_1\colon L\to\mathbb{C}P^1$ and $g_1\colon L\to\mathbb{C}P^1$ with $f_1^*c_1(\gamma_1)=g_1^*c_1(\gamma_1)$, the difference between the homotopy classes $[f_1]$ and $[g_1]$ can be realized by the connected sum of an element of $\pi_3(\mathbb{C}P^1)$ since L is connected.

As in Section 3.1, we simplify the two conditions in Theorem 3.2.

Lemma 3.7. *Let L be a closed orientable connected 3-manifold and $h=(h_1,h_2)\colon L\to\mathbb{C}P^1\times\mathbb{C}P^2$ a continuous map. Then the followings are equivalent.*

1. *There exists a Lagrangian immersion $L\to\mathbb{C}P^1\times\mathbb{C}P^2$ which is homotopic to h.*
2. *$h_1^*c_1(\gamma_1)$ and $h_2^*c_1(\gamma_2)$ are torsion elements in $H^2(L;\mathbb{Z})$.*

Proof. Using the equalities $[\omega_n]=-c_1(\gamma_n)$ in $H^2(\mathbb{C}P^n;\mathbb{Z})$ for a positive integer n, $c_1(T\mathbb{C}P^1)=-2c_1(\gamma_1)$, and $c_1(T\mathbb{C}P^2)=-3c_1(\gamma_2)$, the proof can be done in a way similar to the proof of Lemma 3.3. □

Remark 3.8. As with Remark 3.4, the following statement holds. For the above pair (h,H) and a number $n\in\mathbb{Z}/2$, one can choose a self-transverse Lagrangian immersion $f\colon L\to\mathbb{C}P^1\times\mathbb{C}P^2$ which is homotopic to h and satisfies $I(f)=n$.

We state another lemma which is used in the proof of Theorem 1.1 for the product $\mathbb{C}P^1\times\mathbb{C}P^2$. It directly follows from Theorem 2.1 and Lemma 3.7.

Lemma 3.9. *Let $h\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ be a continuous map of a closed orientable connected 3-manifold L. Suppose that $h_1^* c_1(\gamma_1)$ and $h_2^* c_1(\gamma_2)$ are torsion elements in $H^2(L;\mathbb{Z})$. Then for an arbitrary Lagrangian immersion $f_0\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ which is homotopic to h, there exists a Lagrangian regular homotopy $f_t\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$, $0 \le t \le 1$, such that f_1 is self-transverse and*

$$\mathrm{SI}(f_1) = \begin{cases} 1, & \text{if } I(f_0) = 1; \\ 2, & \text{if } I(f_0) = 0. \end{cases}$$

3.4. **Proof of Theorem 1.1 for $\mathbb{C}P^1 \times \mathbb{C}P^2$.** Let L be a closed orientable connected 3-manifold and $f = (f_1, f_2)\colon L\#(S^1 \times S^2) \to \mathbb{C}P^1 \times \mathbb{C}P^2$ a Lagrangian immersion. By Lemma 3.7, the cohomology classes $f_1^* c_1(\gamma_1)$ and $f_2^* c_1(\gamma_2)$ are torsion elements in $H^2(L\#(S^1 \times S^2);\mathbb{Z})$. As in the proof of Theorem 1.1 for $\mathbb{C}P^3$, the cohomology classes $f_j^* c_1(\gamma_j)$ are of the forms

$$f_j^* c_1(\gamma_j) = (h_j^* c_1(\gamma_j), 0) \in H^2(L \setminus \mathring{D}^3;\mathbb{Z}) \oplus H^2(S^1 \times S^2 \setminus \mathring{D}^3;\mathbb{Z}),$$

where $h_j = f_j \big|_{L\setminus \mathring{D}^3}\colon L \setminus \mathring{D}^3 \to \mathbb{C}P^j$ and $j \in \{1,2\}$. We take a continuous map $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ such that $\tilde{h}_j^* c_1(\gamma_j) = h_j^* c_1(\gamma_j)$ via the isomorphism $H^2(L;\mathbb{Z}) \cong H^2(L \setminus \mathring{D}^3;\mathbb{Z})$, $j \in \{1,2\}$, as follows. In view of Proposition 3.1 and the isomorphism $H^2(L;\mathbb{Z}) \cong H^2(L \setminus \mathring{D}^3;\mathbb{Z})$, the cohomology class $h_2^* c_1(\gamma_2) \in H^2(L\setminus \mathring{D}^3;\mathbb{Z})$ determines the unique element in $[L, \mathbb{C}P^2]$. Choosing a representative $\tilde{h}_2$ of the homotopy class, we have $\tilde{h}_2^* c_1(\gamma_2) = h_2^* c_1(\gamma_2)$. Using Theorem 3.6 and the isomorphism $H^2(L;\mathbb{Z}) \cong H^2(L \setminus \mathring{D}^3;\mathbb{Z})$, we can take a continuous map $\tilde{h}_1\colon L \to \mathbb{C}P^1$ with $\tilde{h}_1^* c_1(\gamma_1) = h_1^* c_1(\gamma_1)$ in a similar way. We note that the equality is equivalent to that the maps h_1 and $\tilde{h}_1$ are homotopic on the 2-skeleton of L.

We construct a self-transverse Lagrangian immersion of L into $\mathbb{C}P^1 \times \mathbb{C}P^2$ with exactly one double point. The continuous map $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ satisfies the second condition of Lemma 3.7. Thus there exists a self-transverse Lagrangian immersion $\tilde{f}^0\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ which is homotopic to $\tilde{h}$ and satisfies $I(\tilde{f}^0) = 1$. Applying Lemma 3.9 to $\tilde{f}^0$, we obtain a self-transverse Lagrangian immersion $\tilde{f}^1\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ which is homotopic to $\tilde{h}$ and satisfies $\mathrm{SI}(\tilde{f}^1) = 1$. Moreover, the proof of Theorem 1.1 of [2] shows that there exists a point $p \in \tilde{f}^1(L)$ and a Darboux chart around p, symplectomorphic to the 6-ball B_ε of radius ε with the standard symplectic structure, such that the self-intersection point x of $\tilde{f}^1$ belongs to $B_{\varepsilon/2}$ and $\phi = \tilde{f}^1(L) \cap \partial B_\varepsilon$ is a loose Legendrian sphere in the 5-sphere ∂B_ε with the standard contact structure.

We construct a Lagrangian embedding of $L\#(S^1 \times S^2)$ into $\mathbb{C}P^1 \times \mathbb{C}P^2$ which is homotopic to f. Using Polterovich's Lagrangian surgery [5] to resolve the double point x of $\tilde{f}^1 = (\tilde{f}_1^1, \tilde{f}_2^1)$, we obtain a Lagrangian embedding $g = (g_1, g_2)\colon L\#(S^1 \times S^2) \to \mathbb{C}P^1 \times \mathbb{C}P^2$. Since $g_2^* c_1(\gamma_2) = f_2^* c_1(\gamma_2)$, g_2 is homotopic to f_2. We also have $g_1^* c_1(\gamma_1) = f_1^* c_1(\gamma_1)$. By Theorem 3.6, the difference between the homotopy classes $[g_1]$ and $[f_1]$ in $[L\#(S^1 \times S^2), \mathbb{C}P^1]$ can be realized by the connected sum of an element of $\pi_3(\mathbb{C}P^1)$. Therefore, there exists a continuous map $a\colon S^3 \to \mathbb{C}P^1$ such that $g_1\#a$ is homotopic to f_1. We may assume that the disk in $L\#(S^1 \times S^2)$ which is removed for the connected sum $g_1\#a$ does not intersect $g^{-1}(B_\varepsilon)$. We consider a continuous map $g\#a = (g_1\#a, g_2)\colon L\#(S^1 \times S^2) \to \mathbb{C}P^1 \times \mathbb{C}P^2$. Since $g\#a$ satisfies the assumption of Lemma 3.7, there exists a self-transverse Lagrangian immersion $g^a\colon L\#(S^1 \times S^2) \to \mathbb{C}P^1 \times \mathbb{C}P^2$ such that $I(g^a) = 0$ and g^a is homotopic to $g\#a$ relative to $(g\#a)^{-1}(B_\varepsilon) = g^{-1}(B_\varepsilon)$. Since $\tilde{f}^1 \mid_{g^{-1}(B_\varepsilon)}$ can be glued to $g^a \mid_{L\#(S^1\times S^2)\setminus g^{-1}(B_\varepsilon)}$, we obtain a self-transverse Lagrangian immersion $\tilde{g}^a\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ of $I(\tilde{g}^a) = 1$. In the Darboux chart B_ε, the Lagrangian immersion $\tilde{g}^a$ coincides with $\tilde{f}^1$. Hence, we can replace $\tilde{g}^a(L) \cap B_{\varepsilon/2}$ by the Lagrangian cone over the loose Legendrian knot ϕ. Then we have a Lagrangian immersion $\tilde{g}^0\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ with a conical point q such that the Legendrian link at q is loose and $I(\tilde{g}^0) = 0$. Applying Theorem 2.2 to $\tilde{g}^0$, we obtain a Lagrangian regular homotopy $\tilde{g}^t\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$, $t \in [0,1]$, that is the identity in a neighborhood of the conical point q and that connects $\tilde{g}^0$ to a self-transverse Lagrangian immersion $\tilde{g}^1$ with a conical point q and with $\mathrm{SI}(\tilde{g}^1) = I(\tilde{g}^0) = 0$.

Rescaling $\tilde{g}^a(L) \cap B_{\varepsilon/2}$ and replacing the Lagrangian cone over the loose Legendrian knot ϕ by the rescaled $\tilde{g}^a(L) \cap B_{\varepsilon/2}$, we obtain a self-transverse Lagrangian immersion $\tilde{g}^2\colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ with $\mathrm{SI}(\tilde{g}^2) = 1$. Finally, again resolving the double point x of $\tilde{g}^2$ by Polterovich's Lagrangian surgery [5], we obtain a Lagrangian embedding $g^1\colon L\#(S^1 \times S^2) \to \mathbb{C}P^1 \times \mathbb{C}P^2$ homotopic to g^a relative to a small neighborhood of the point q. In particular, g^1 is homotopic to f. The proof of Theorem 1.1 for $\mathbb{C}P^1 \times \mathbb{C}P^2$ is completed. □

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