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Penalty method for stationary Stokes and Navier-Stokes equations with slip boundary condition

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Penalty method for stationary Stokes and Navier-Stokes equations with slip boundary condition

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Abstract The penalty method for solving the stationary Stokes and Navier-Stokes equation with slip boundary condition is considered. For Stokes equations, the optimal error in H^k norm(for any integer $k \geq 0$) is obtain. For Navier-Stokes equations, two penalty equations are proposed, for which the well-posedness and optimal error estimate are investigated. We are also concerned with the finite element approximation to penalty problem using $P1/P1$ and $P1b/P1$ elements. Two implementation methods of penalty term are proposed: the fine-integration and the lower-order-integration schemes. We derive the error estimates and shows the optimal choice of penalty parameter and the finite element discretization parameter(mesh size). The theoretical results are verified by numerical experiments.

Keywords Penalty method · Navier-Stokes · Slip boundary · Error estimates · Finite element method

Mathematics Subject Classification (2000) 35Q30 · 65M60 · 65M15

1 Introduction

Set $\Omega \in \mathbb{R}^d$, $d = 2, 3$, $\partial\Omega = \Gamma \cup C$, $\Gamma \cap C = \emptyset$, see Figure 1.1. Ω is C^3 smooth. We consider the stationary Navier-Stokes problem (NS) with slip boundary

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condition.

$$
\begin{aligned}
& -\nu \Delta u + (u \cdot \nabla) u + \nabla p = f, && \text{in } \Omega, && (1.1a)\\
& \nabla \cdot u = 0, && \text{in } \Omega, && (1.1b)\\
& u_n = 0, \quad \tau_T(u) = 0, && \text{on } \Gamma, && (1.1c)\\
& u = 0 && \text{on } C, && (1.1d)
\end{aligned}
$$

where $\nu > 0$, $u_n = u \cdot n$, n is the unit outer normal vector to Γ, and $\tau_T(u)$ is the tangential component of traction vector on Γ defined in the following.

For velocity u and pressure p, we set the stress tensor,

$$
\sigma(u,p) = (\sigma_{i,j}(u,p)) = -pI + 2\mu \mathcal{E}(u), \tag{1.2a}
$$

$$
\mathcal{E}(u) = (\mathcal{E}_{i,j}(u)) = \left(\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right), \tag{1.2b}
$$

where I denotes the identity. And we introduce the traction vectors:

$$
\tau(u,p) = \sigma(u,p) n, \tag{1.3a}
$$

$$
\tau_n(u,p) = \tau(u,p) \cdot n, \tag{1.3b}
$$

$$
\tau_T(u) = \tau(u,p) - \tau_n(u,p) n. \tag{1.3c}
$$

Slip boundary condition $u_n = 0$ plays important roles in physical fluid models(cf. [14]). To solve Stokes or Navier-Stokes equations with slip boundary condition by finite element method(FEM) is not as easy as the case of no-slip boundary(ex. Dirichlet boundary) problems. The main difficulties lie to find a proper finite element approximation to the slip boundary condition $u_n|_\Gamma = 0$.

Usually, a polygon or polyhedral domain Ω_h is introduced with a triangulation $\mathcal{T}_h$ to approximate the smooth boundary domain Ω in FEM. And it is natural to use the unit outer normal vector n_h to the boundary Γ_h(see Figure 1.2) in numerical method instead of $n(x)$. Some choices of finite element spaces used in the no-slip boundary problem, for example, the $P1/P1$ with stabilization and $P1b/P1$ approximations(cf. [11]), are known to cause variational crime(cf. [2,15]) in the slip boundary case. To briefly explain this problem, we consider the $P1b$ finite element space for velocity

$$
V_h = \{ v_h \in C(\Omega_h)^d \mid v_h|_T \in P_1(T) \oplus B(T), \forall T \in \mathcal{T}_h, v_h = 0 \text{ on } C_h \},
$$

where $P_i(T)$ is the set of polynomials of degree i on T and $B(T)$ stands for the space spanned by bubble function on T. If we set

$$
V_{hn} = \{ v_h \in V_h \mid v_h \cdot n_h = 0 \text{ on } \Gamma_h \},
$$

as the finite element space with slip boundary information. Since n_h is discontinuous on Γ_h, V_{hn} coincides with V_{h0}, where

$$
V_{h0} = \{ v_h \in V_h \mid v_h = 0 \text{ on } \Gamma_h \}.
$$

To tackle this problem, a pioneer work on FEM for Stokes/Navier-Stokes equations with slip boundary condition(slip BC) by Verfürth(cf. [25–27]) established the convergence and error estimates for a special finite element spaces satisfying a coupled *inf-sup condition* given in [26], where the Lagrange multiplier method is employed and the slip BC is enforced in a weak sense.

In [24,23](Tabata, *etc.*), $P1/P1$ element with stabilization is used, and the slip BC is implemented as $v_h(p) \cdot n(p) = 0$ for all p the vertices of Ω_h on Γ(see Figure 1.3). The authors consider the spherical domain in [24,23], where $n(p)$ is easy to calculate; but for general domain, using n_h is more convenient than n. A similar method presented in [10,3] is to introduce a homeomorphism $G_h : \Omega_h \to \Omega$ and consider the $P2/P1$ approximation, then the slip boundary is described as $v_h(p)\cdot n(G_h(p))$(see Figure 1.4), where p is the set of all vertices and midpoint of edges on Γ_h. However, to construct G_h may be very technical for complex domain, and sometimes $P1/P1$ or $P1b/P1$ element spaces are preferred for its less DOFs in computation, especially for 3D simulations.

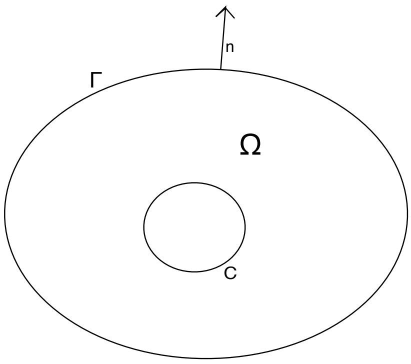

Fig. 1.1 Ω, Γ and C.

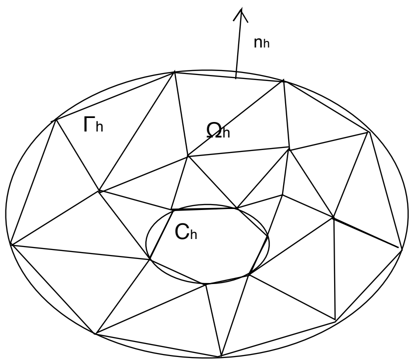

Fig. 1.2 Ω_h, $\partial\Omega_h = \Gamma_h \cup C_h$ and triangulation $\mathcal{T}_h$.

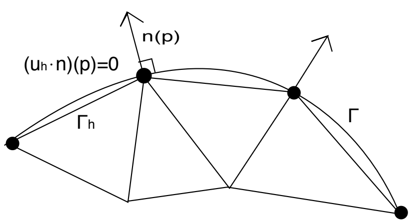

Fig. 1.3

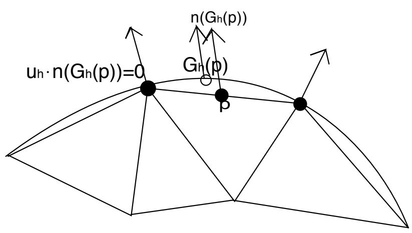

Fig. 1.4

An alternative approach introduced in [10] is to find the local rotational matrix $\mathbf{R}_i$ which rotate the co-ordinate system at nodes i of boundary Γ_h to coincide the tangential and normal directions so that u_n and $u_T = u -$

$u_n n$ become the degree of freedom of node i. This approach requires addition implementation technique.

Instead of searching proper finite element spaces or implement method for slip BC, another popular way is to use the penalty method(cf. [18,6,9,8]), which is easy for computation and widely applied in numerical simulations of fluid motion. The implementation of penalty method can be easily achieved by popular FEM-softwares, such as Freefem++(cf. [13]) and FeniCS(cf. [20]).

The idea of penalty method is based on a Robin-type boundary condition. We state the penalty problem (NS_ϵ):

$$-\nu\Delta u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon + \nabla p_\epsilon = f, \qquad \text{in } \Omega, \tag{1.4a}$$

$$\nabla \cdot u_\epsilon = 0, \qquad \text{in } \Omega, \tag{1.4b}$$

$$\tau_n(u_\epsilon, p_\epsilon) + \frac{1}{\epsilon} u_{\epsilon n} = 0, \quad \tau_T(u_\epsilon) = 0, \qquad \text{on } \Gamma, \tag{1.4c}$$

$$u_\epsilon = 0 \qquad \text{on } C, \tag{1.4d}$$

where $0 < \epsilon \ll 1$ is the penalty parameter, and $u_{\epsilon n} = u_\epsilon \cdot n$. In the variational form described below, the penalty term is $\frac{1}{\epsilon}\int_\Gamma u_{\epsilon n} v_n ds$ with $v \in V = H^1(\Omega)^d$. As $\epsilon \to 0$, $u_{\epsilon n} \to 0$ at least in $L^2(\Gamma)$, which shows $u_{\epsilon n}$ approximate to the slip BC $u_n = 0$.

In fact, the error estimate $\|u - u_\epsilon\|_{H^1(\Omega)} \le C\epsilon$ has been proved for Stokes equation in [8,9] and for Navier-Stokes equations with penalty problem (3.17).

For Stokes equations, in [6], the method of coupled *inf-sup condition* is employed to derive the error estimates. Here, we avoid to use the coupled *inf-sup condition*, and introduce the separation of $p_\epsilon \in L^2(\Omega)$:

$$p_\epsilon = \mathring{p}_\epsilon + k_\epsilon, \quad \mathring{p}_\epsilon \in L_0^2(\Omega), \quad k_\epsilon = \int_\Omega p_\epsilon dx/|\Omega|. \tag{1.5}$$

Then we obtain the error estimates(see Theorem 2.3)

$$\|u - u_\epsilon\|_{H^1(\Omega)} + \| - \tau_n(u,p) - \epsilon^{-1} u_{\epsilon n} + k_\epsilon\|_{H^{-1/2}(\Gamma)} \le C\epsilon.$$

Moreover, for sufficiently smooth domain, we also show that(see Theorem 2.4)

$$\|u_\epsilon\|_{H^k(\Omega)} + \|p_\epsilon\|_{H^{k-1}(\Omega)} \le C\|f\|_{H^{k-2}(\Omega)},$$

and we obtain the error(see Theorem 2.5)

$$\|u - u_\epsilon\|_{H^k(\Omega)} \le C\epsilon,$$

for any integer $k \ge 1$.

For Navier-Stokes equations, we consider the well-posedness of penalty problems (NS_ϵ) and (3.17)(Theorem 3.2,3.3,3.4). The error estimate

$$\|u - u_\epsilon\|_{H^1(\Omega)} \le C\epsilon,$$

is proved for both (NS_ϵ) and (3.17)(see Theorem 4.1,4.2, and Remark 4.1). The penalty scheme (NS_ϵ) may exist "large norm solution" u_ϵ with $\|u_\epsilon\|_{H^1(\Omega)} >$

$C\epsilon^{-1/2}$ for samll ϵ, and we discuss the iteration method for (NS_ϵ) to avoid the "large norm solution"(see Sect. 3.2.1).

For the finite element approximation, [8] consider the $P2/P1$ element homeomorphism G_h introduced before. And in [6], the authors assume the finite element spaces satisfying a discrete version of coupled *inf-sup condition*. The error estimates $\|u - u_h\|_{H^1(\Omega_h)}$ in [8,6] is derived by estimating the error $\|u_\epsilon - u_h\|_{H^1(\Omega_h)}$ and using the inequality $\|u - u_h\|_{H^1(\Omega_h)} \leq \|u_\epsilon - u_h\|_{H^1(\Omega_h)} + \|u - u_\epsilon\|_{H^1(\Omega_h)}$.

Here, we consider the $P1/P1$ element with stabilization and $P1b/P1$ element. We investigate the error estimate $\|u - u_h\|_{H^1(\Omega_h)}$ directly. Most importantly, we proposed two methods to implement the penalty term in finite element method: (1) fine-integration scheme and (2) lower-order integration scheme(see Sect. 5). We derive the error estimates for both two implementation methods(see Theorem 5.2,5.3). And we give some optimal choices of penalty parameter ϵ and mesh size h(see Remark 5.1). For example, when $d = 2$, setting $\epsilon \simeq h^2$, we have the error $O(h)$ for lower-integration scheme. Comparing to the error estimate $O(h^{3/2})$ of $P2/P1$ element with penalty in [8], our error estimate $O(h)$ shows the $P1b/P1$(or $P1/P1$) element also performs well for slip BC penalty method.

As a summary, we prove the error estimates $\|u - u_\epsilon\|_{H^k(\Omega)} \leq C\epsilon(\ k \geq 1)$ for Stokes problem and $\|u - u_\epsilon\|_{H^1(\Omega)} \leq C\epsilon$ for Navier-Stokes problem. Our analysis method is to take the penalty term $\epsilon^{-1}u_{\epsilon n}$ as a Lagrange multiplier, and consider the error of $\| - \tau_n(u,p) - \epsilon^{-1}u_{\epsilon n} + k_\epsilon\|_{H^{-\frac{1}{2}}(\Gamma)}$. We propose two implementation methods of penalty term in FEM for Navier-Stokes equations, and we obtain the error estimates of $\|u - u_h\|_{H^1(\Omega_h)}$.

The paper is organized as follows. In Sect. 2, we consider the penalty method for Stokes equation with slip BC, and derive the error estimates of penalty. Sect. 3 is devoted to the well-posedness of penalty method for Navier-Stokes equations. We give the error estimates of penalty for Navier-Stokes equation in Sect. 4. The finite element approximation is discussed in Sect. 5. The numerical experiments are presented in Sect. 6.

Notations

Throughout this paper, we write $\|\cdot\|_{k,\omega}$ as the norm of Sobolev spaces $H^k(\omega)$, and $\|\cdot\|_{k,p,\omega}$ for $W^{k,p}(\Omega)$(k is omitted if $k = 0$). $(\cdot,\cdot)_\omega$ represents the inner-product of $L^2(\omega)$, and we use $(\cdot,\cdot)$ for the case $\omega = \Omega$. C or C_i represent some constants different case by case, but independent of penalty parameter ϵ and finite element discretization parameter h. C may depend on Ω, d.

2 Penalty method for Stokes equations: error estimates

Let $f \in L^2(\Omega)$. We consider the Stokes equations with slip BC, denoted as (S):

$$-\nu\Delta u + \nabla p = f \qquad \text{in } \Omega, \tag{2.1a}$$
$$\nabla \cdot u = 0 \qquad \text{in } \Omega, \tag{2.1b}$$
$$u_n = 0, \quad \tau_T(u) = 0 \qquad \text{on } \Gamma, \tag{2.1c}$$
$$u = 0 \qquad \text{on } C. \tag{2.1d}$$

Remark 2.1 (cf. [3,4]) For $f \in L^2(\Omega)$, there exists a unique solution $(u,p) \in H^2(\Omega)^d \times H^1(\Omega)$ to (2.1).

Function spaces.

- $V = H^1(\Omega)^d \cap \{v|_C = 0\}, \quad V_n = V \cap \{v | v \cdot n = 0 \text{ on } \Gamma\}$,
- $V_\sigma = V \cap \{v \mid \nabla \cdot v = 0 \text{ in } \Omega\}, \quad V_{\sigma,n} = V_n \cap V_\sigma$,
- $Q = L^2(\Omega), \quad \mathring{Q} = L_0^2(\Omega)$,
- $\Lambda = H^{1/2}(\Gamma), \quad \Lambda^* = H^{-\frac{1}{2}}(\Gamma)$.
- We denote X^* as the dual of Banach space X.

For any $u, v, w \in V$, $p \in Q$, $\eta \in \Lambda$ and $\mu \in \Lambda^*$, we set

$$a(u,v) = 2\nu(\mathcal{E}(u), \mathcal{E}(u)), \tag{2.2a}$$
$$a_1(u,v,w) = \int_\Omega (u \cdot \nabla)v \cdot w dx, \tag{2.2b}$$
$$b(v,p) = -(\nabla \cdot v, p), \tag{2.2c}$$
$$c(\mu,\eta) = \int_\Gamma \mu\eta ds. \tag{2.2d}$$

Some properties of bilinear and trilinear forms.(cf. [11, 5, 26])

- *Coercivity of* a: there exists $\alpha > 0$ such that
$$a(u,u) \geq \alpha \|u\|_{1,\Omega}^2, \quad \forall u \in V. \tag{2.3}$$
- For any $u, v, w \in V$,
$$a(u,v,w) = \int_\Gamma u_n(v \cdot w)ds - a(u,w,v) - \int_\Omega \nabla \cdot u(v \cdot w)dx, \tag{2.4}$$
$$a(u,v,v) = \frac{1}{2}\int_\Gamma u_n|v|^2 ds, \quad \forall u \in V_\sigma. \tag{2.5}$$
- The *inf-sup condition* of b: there exists $\beta > 0$ such that
$$\inf_{p \in L_0^2(\Omega)\setminus\{0\}} \sup_{v \in H_0^1(\Omega)^d\setminus\{0\}} \frac{b(v,p)}{\|v\|_{1,\Omega}\|p\|_\Omega} \geq \beta. \tag{2.6}$$

– The *inf-sup condition* of c: there exists $\gamma_0 > 0$ such that

$$\inf_{\mu\in \Lambda^*\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{c(\mu, v_n)}{\|v\|_{1,\Omega}\|\mu\|_{\Lambda^*}} \geq \gamma_0. \tag{2.7}$$

The variational form of (2.1) reads as: find $(u,p) \in V_n \times \mathring{Q}$ such that,

$$a(u,v) + b(v,p) = (f,v), \quad \forall v \in V_n, \tag{2.8a}$$
$$b(u,q) = 0, \quad \forall q \in \mathring{Q}. \tag{2.8b}$$

Let $0 < \epsilon \ll 1$, the penalty method for (S), denoted as (S_ϵ), reads as:

$$-\Delta u_\epsilon + \nabla p_\epsilon = f \qquad \text{in } \Omega, \tag{2.9a}$$
$$\nabla \cdot u_\epsilon = 0 \qquad \text{in } \Omega, \tag{2.9b}$$
$$\tau_n(u_\epsilon, p_\epsilon) + \frac{1}{\epsilon} u_{\epsilon n} = 0, \quad \tau_T(u_\epsilon) = 0 \qquad \text{on } \Gamma, \tag{2.9c}$$
$$u_\epsilon = 0 \qquad \text{on } C. \tag{2.9d}$$

The variational form of (2.9) reads as: find $(u_\epsilon, p_\epsilon) \in V \times Q$ such that

$$a(u_\epsilon, v) + b(v, p_\epsilon) + \frac{1}{\epsilon} c(u_{\epsilon n}, v_n) = (f, v), \quad \forall v \in V, \tag{2.10a}$$
$$b(u_\epsilon, q) = 0, \quad \forall q \in Q. \tag{2.10b}$$

Remark 2.2 $p_\epsilon \notin \mathring{Q}$. For nonhomogeneous slip boundary condition $u_n = g$ on Γ, we set the penalty term $\frac{1}{\epsilon} c(u_{\epsilon n} - g, v_n)$ in (2.10a), or equivalently, $\tau_n(u_\epsilon, p_\epsilon) + \frac{1}{\epsilon}(u_{\epsilon n} - g) = 0$ in (2.9c).

Theorem 2.1 *Given $f \in V^*$, there exists a unique solution $(u_\epsilon, p_\epsilon) \in V \times Q$ to (2.10), with*

$$\|u_\epsilon\|_{1,\Omega} + \|p_\epsilon\|_\Omega \leq C\|f\|_{V^*}.$$

Proof From the coercivity of a (2.3), we conclude the existence of u_ϵ and $\|u_\epsilon\|_V \leq C\|f\|_{V^*}$. Set $p_\epsilon = \mathring{p}_\epsilon + k_\epsilon$, where $\mathring{p}_\epsilon \in \mathring{Q}$ and $k_\epsilon = \int_\Omega p_\epsilon dx/|\Omega|$. From the *inf-sup condition* of b (2.6), we have(cf. [11]) $\|\mathring{p}_\epsilon\|_\Omega \leq C\|f\|_{V^*}$. To estimate k_ϵ, we choose a trace lifting $v \in V$(cf. [19]) satisfying $v = k_\epsilon n$ on Γ, and $\|v\|_{1,\Omega} \leq Ck_\epsilon$. Substituting this v into (2.10), in view of the fact $\int_\Gamma u_{\epsilon n} ds = 0$, we have

$$|\Gamma| k_\epsilon^2 = k_\epsilon \int_\Gamma v_n dx = -b(v, k_\epsilon) = a(u_\epsilon, v) + b(v, \mathring{p}_\epsilon) - (f, v),$$

which implies

$$|k_\epsilon| \leq C(\|u_\epsilon\|_{1,\Omega} + \|\mathring{p}_\epsilon\|_\Omega + \|f\|_{V^*}) \leq C\|f\|_{V^*}.$$

□

To show the error estimates of penalty method, we introduce the Largrange multipliers $\lambda = -\tau_n(u,p)$ and $\lambda_\epsilon = \frac{1}{\epsilon} u_{\epsilon n}$, then (2.8) and (2.10) are rewritten into the following two equations, respectively:

(1) find $(u, p, \lambda) \in V \times Q \times \Lambda^*$ such that,

$$a(u,v) + b(v,p) + c(\lambda, v_n) = (f,v), \quad \forall v \in V, \tag{2.11a}$$
$$b(u,q) = 0, \quad \forall q \in Q, \tag{2.11b}$$
$$c(u_n, \eta) = 0, \quad \forall \eta \in \Lambda; \tag{2.11c}$$

(2) find $(u_\epsilon, p_\epsilon, \lambda_\epsilon) \in V \times Q \times \Lambda^*$ such that,

$$a(u_\epsilon,v) + b(v,p_\epsilon) + c(\lambda_\epsilon, v_n) = (f,v), \quad \forall v \in V, \tag{2.12a}$$
$$b(u_\epsilon,q) = 0, \quad \forall q \in Q, \tag{2.12b}$$
$$c(u_{\epsilon n}, \eta) = \epsilon c(\lambda_\epsilon, \eta), \quad \forall \eta \in \Lambda. \tag{2.12c}$$

We state the error estimates of penalty method.

Theorem 2.2 *Let (u,p) and (u_ϵ, p_ϵ) be the solutions of (2.1) and (2.9), respectively, then we have*

$$\|u - u_\epsilon\|_{1,\Omega} + \|p - \mathring{p}_\epsilon\|_\Omega + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon\|_\Gamma \le c\sqrt{\epsilon}\|\lambda\|_\Gamma. \tag{2.13}$$

Proof Substituting $v = u - u_\epsilon$ into (2.11a)$-$(2.12a), we have

$$a(u - u_\epsilon, u - u_\epsilon) + c(\lambda - \lambda_\epsilon, u_n - u_{\epsilon n}) = 0. \tag{2.14}$$

Since $u_n = 0$ and $u_{\epsilon n} = \epsilon\lambda_\epsilon$, we have

$$c(\lambda - \lambda_\epsilon, u_n - u_{\epsilon n}) = \epsilon c(\lambda - \lambda_\epsilon, \lambda - \lambda_\epsilon) - \epsilon c(\lambda, \lambda - \lambda_\epsilon). \tag{2.15}$$

From the *coercivity* of a (2.3), (2.14) and (2.15) we obtain

$$\begin{aligned} &\alpha\|u - u_\epsilon\|_{1,\Omega}^2 + \epsilon\|\lambda - \lambda_\epsilon\|_\Gamma^2 \\ \le &\epsilon c(\lambda, \lambda - \lambda_\epsilon) \le \frac{\epsilon}{2}\|\lambda - \lambda_\epsilon\|_\Gamma^2 + \frac{\epsilon}{2}\|\lambda\|_\Gamma^2, \end{aligned}$$

which implies

$$\|u - u_\epsilon\|_{1,\Omega} + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon\|_\Gamma \le c\sqrt{\epsilon}\|\lambda\|_\Gamma. \tag{2.16}$$

From the *inf-sup condition* of b (2.6) and

$$b(p - \mathring{p}_\epsilon, v) = -a(u - u_\epsilon, v), \quad \forall v \in H_0^1(\Omega)^d, \tag{2.17}$$

we have

$$\|p - \mathring{p}_\epsilon\|_\Omega \le C\|u - u_\epsilon\|_{1,\Omega}, \tag{2.18}$$

which gives (2.13). □

Theorem 2.3 *Let (u,p) and (u_ϵ, p_ϵ) be the solutions of (2.1) and (2.9), respectively, then we have*

$$\|u - u_\epsilon\|_{1,\Omega} + \|p - \mathring{p}_\epsilon\|_\Omega + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon + k_\epsilon\|_\Gamma \le C\epsilon(\|\lambda\|_{\frac{1}{2},\Gamma} + 1). \tag{2.19}$$

Proof Subtracting (2.11a) from (2.12a), we have, for any $v \in V$,

$$c(\lambda - \lambda_\epsilon + k_\epsilon, v_n) = -a(u - u_\epsilon, v) - b(v, p - \mathring{p}_\epsilon).$$

In view of the *inf-sup condition* of c (2.7) and (2.18), it yields

$$\|\lambda - \lambda_\epsilon + k_\epsilon\|_{\Lambda^*} \leq C\|u - u_\epsilon\|_{1,\Omega} \tag{2.20}$$

Noticing that $\int_\Gamma u_{\epsilon n} ds = 0$, instead of (2.15), we derive

$$c(\lambda - \lambda_\epsilon, u_n - u_{\epsilon n}) = \epsilon c(\lambda - \lambda_\epsilon + k_\epsilon, \lambda - \lambda_\epsilon + k_\epsilon) - \epsilon c(\lambda + k_\epsilon, \lambda - \lambda_\epsilon + k_\epsilon). \tag{2.21}$$

From the *coercivity* of a (2.3), (2.14) and (2.21), we obtain

$$\begin{aligned} &\alpha\|u - u_\epsilon\|_{1,\Omega}^2 + \epsilon\|\lambda - \lambda_\epsilon + k_\epsilon\|_\Gamma^2 \\ \leq &\epsilon c(\lambda + k_\epsilon, \lambda - \lambda_\epsilon + k_\epsilon) \leq \epsilon\|\lambda + k_\epsilon\|_\Lambda\|\lambda - \lambda_\epsilon + k_\epsilon\|_{\Lambda^*}. \end{aligned} \tag{2.22}$$

From (2.22) and (2.20), we obtain

$$\|u - u_\epsilon\|_{1,\Omega} \leq C\epsilon\|\lambda + k_\epsilon\|_\Lambda,$$

which implies (2.19) because k_ϵ is bounded independent of ϵ(see Theorem 2.1). □

Remark 2.3 From (2.20), we have $\|\lambda - \lambda_\epsilon + k_\epsilon\|_{H^{-1/2}(\Gamma)} \leq C\epsilon$.

In view of

$$\|u_{\epsilon n}\|_{\frac{1}{2},\Gamma} = \|u_{\epsilon n} - u_n\|_{\frac{1}{2},\Gamma} \leq C\|u_\epsilon - u\|_{1,\Omega} \leq C\epsilon,$$

we have

$$\|\tau_n(u_\epsilon, p_\epsilon)\|_{\frac{1}{2},\Gamma} = \|\epsilon^{-1}u_{\epsilon n}\|_{\frac{1}{2},\Gamma} \leq C,$$

which implies

$$\|u_\epsilon\|_{2,\Omega} + \|p_\epsilon\|_{1,\Omega} \leq C.$$

In fact, we have the following regularity result for penalty problem (2.9).

Theorem 2.4 *For arbitrary integer $k \geq 0$, let $\Omega \in C^{k+2}$, $f \in H^k(\Omega)^d$, then there exists a unique solution $(u_\epsilon, p_\epsilon) \in H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$ to (2.9), with*

$$\|u_\epsilon\|_{k+2,\Omega} + \|p_\epsilon\|_{k+1,\Omega} \leq C\|f\|_{k,\Omega}. \tag{2.23}$$

Proof For general domain $\Omega \in C^{k+2}$, the regularity in interior or near C is well known(cf. [7,17]); that is $\|u_\epsilon\|_{k+2,\omega} + \|p_\epsilon\|_{k+1,\omega} \leq C(\omega)\|f\|_{k,\omega}$, where $\omega \subset \Omega$ and $\mathrm{dist}(\overline{w}, \Gamma) \geq \delta > 0$.

For the regularity near Γ, there exists a set of smooth sub-domain in $\mathbb{R}^d$, denoted as $\{U_i\}_{i=1}^m$, satisfying $\Gamma \subset \cup_{i=1}^m U_i$.

We introduce a cut-off function $\theta_i \in C^\infty(\mathbb{R}^d)$ with $\mathrm{supp}\theta_i \subset U_i$, and consider the equations of $(\theta^2 u_\epsilon, \theta^2 p_\epsilon)$ in $U_i \cap \Omega$.

There exists a C^{k+3}-diffeomorphism(cf. [28]) $\mathbf{\Phi}_i : U_i \to Q_R := \mathbb{R}^d_{d,+} \cap \{\tilde{x} \in \mathbb{R}^d, |\ |\tilde{x}| < R\}$, where $\mathbb{R}^d_{d,+} := \{\tilde{x} = (\tilde{x}', \tilde{x}_d) \in \mathbb{R}^d \mid \tilde{x}' \in \mathbb{R}^{d-1}, \tilde{x}_d > 0\}$ is the half-plane, and we also have $\mathbf{\Phi}_i : \Gamma \cap U_i \to \tilde{\Gamma}_i := \{\tilde{x} \mid |\tilde{x}| < R, \tilde{x}_d = 0\}$.

Then we consider the equation of $(\tilde{u}_\epsilon, \tilde{p}_\epsilon) := ((\theta_i^2 u_\epsilon)\circ \mathbf{\Phi}_i, (\theta_i^2 p_\epsilon)\circ \mathbf{\Phi}_i)$ in domain Q_R, to which we apply the famous Agmon-Douglis-Nirenberg' method(cf. [1]) and obtain $\|D_i D_j \tilde{u}_\epsilon\|_\Omega \le C(\|f\|_\Omega + \|u_\epsilon\|_{1,\Omega})$, $i = 1,\dots,d-1$; $j = 1,\dots,d$, where $D_i v = \nabla_{x_i} v$. Hence, we can conclude $\|\tilde{u}_\epsilon\|_{\frac{3}{2},\tilde{\Gamma}_i} \le C\|f\|_{k,\Omega}$, which implies $\|u_{\epsilon n}\|_{\frac{3}{2},\Gamma} \le C\|f\|_\Omega$. Following from well-known regularity result for Stokes equation by Cattabriga [7], it yields $\|u_\epsilon\|_{2,\Omega} + \|p_\epsilon\|_{1,\Omega} \le C\|f\|_\Omega$. For $k \ge 1$, (2.23) can be proved by induction method.

In above, we briefly sketch the strategy of proof. The key point is to consider the equation in the half-plane via some transformations. We refer the readers to [21](Saito, proof of Lemma 4.1) for detailed arguments on those techniques. Here, to make the argument brief, we only prove the case of $k = 0$ and the half-plane domain $\Omega = \mathbb{R}^d_{d,+} := \{x = (x', x_d) \in \mathbb{R}^d \mid x' \in \mathbb{R}^{d-1}, x_d > 0\}$.

Set $D_h^i v = (v(x_1, \cdots, x_i + h, \cdots, x_d) - v(x))/h$, $h > 0$. Substituting $v = D_{-h}^i D_h^i u_\epsilon$ into (2.9), $i = 1,\dots,d-1$, we have, with $\Gamma = \{x \mid x_d = 0\}$,

$$a(u_\epsilon, D_{-h}^i D_h^i u_\epsilon) + b(D_{-h}^i D_h^i u_\epsilon, p_\epsilon) + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n} D_{-h}^i D_h^i u_\epsilon \cdot n ds = (f, D_{-h}^i D_h^i u_\epsilon).$$

Using the fact $(w, D_{-h}^i v) = (D_h^i w, v)$, $\forall w, v \in H^1(\mathbb{R}^d_{d,+})$, we get

$$a(D_h^i u_\epsilon, D_h^i u_\epsilon) + \frac{1}{\epsilon}\int_\Gamma |D_h^i u_{\epsilon n}|^2 ds = (f, D_{-h}^i D_h^i u_\epsilon) \le C\|f\|_\Omega \|D_{-h}^i D_h^i u_\epsilon\|_\Omega.$$

Since $\|D_h^i v\|_\Omega \le C\|\nabla_{x_i} v\|_\Omega$, from the *coercivity* of a (2.3), we have,

$$\|D_h^i u_\epsilon\|_{1,\Omega} + \epsilon^{-1/2}\|D_h^i u_{\epsilon n}\|_\Gamma \le C\|f\|_\Omega, \quad i = 1,\dots,d-1.$$

Let $h \to 0$, and we have

$$\|D_i D_j u_\epsilon\|_\Omega + \epsilon^{-1/2}\|D_i u_{\epsilon n}\| \le C\|f\|_\Omega, \quad i = 1,\dots,d-1;\ j = 1,\dots,d.$$

By trace theorem and $n = (0,\dots,0,1)$, we have

$$\|u_{\epsilon n}\|_{\frac{3}{2},\Gamma} \le C\|f\|_\Omega.$$

And we can conclude $(u_\epsilon, p_\epsilon) \in H^2(\Omega)^d \times H^1(\Omega)$ and (2.23) for $k = 0$(cf. [7]). □

Theorem 2.5 *For any integer $k \ge 0$, $f \in H^k(\Omega)^d$. Let (u,p) and (u_ϵ, p_ϵ) of $H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$ be the solutions of (2.1) and (2.9), respectively, then we have,*

$$\|u - u_\epsilon\|_{k+2,\Omega} + \|p - \mathring{p}_\epsilon\|_{k+1,\Omega} \le C\epsilon\|\lambda\|_{k+\frac{3}{2},\Gamma}. \tag{2.24}$$

Proof Same to Theorem 2.4, to make the argument brief, we only prove the case of $k = 0$($k \ge 1$ follows form induction method) and the half-plane domain $\Omega = \mathbb{R}^d_{d,+}$. Substituting $v = D_{-h}^i D_h(u - u_\epsilon)$, $i = 1,\dots,d-1$, into (2.11a)−(2.12a), we have

$$a(u - u_\epsilon, D_{-h}^i D_h^i(u - u_\epsilon)) + c(\lambda - \lambda_\epsilon + k_\epsilon, D_{-h}^i D_h^i(u - u_\epsilon)\cdot n) = 0,$$

which yields,

$$\begin{aligned}&a(D_h^i(u-u_\epsilon),D_h^i(u-u_\epsilon))+\epsilon c(D_h^i(\lambda-\lambda_\epsilon+k_\epsilon),D_h^i(\lambda-\lambda_\epsilon+k_\epsilon))\\&\quad=\epsilon c(D_h^i(\lambda-\lambda_\epsilon+k_\epsilon),D_h^i(\lambda+k_\epsilon)).\end{aligned}$$

k_ϵ is a constant, so $D_h^i k_\epsilon=0$. Therefore, we have

$$\begin{aligned}&\alpha\|D_h^i(u-u_\epsilon)\|_{1,\Omega}^2+\epsilon\|D_h^i(\lambda-\lambda_\epsilon)\|_\Gamma^2\\&\le C\epsilon\|D_h^i(\lambda-\lambda_\epsilon+k_\epsilon)\|_{-\frac{1}{2},\Gamma}\|D_h^i\lambda\|_{\frac{1}{2},\Gamma}.\end{aligned}\tag{2.25}$$

Via *inf-sup condition* of b, and the equation

$$b(D_h^i(p-\mathring{p}_\epsilon),v)=-a(D_h^i(u-u_\epsilon),v),\quad\forall v\in H_0^1(\mathbb{R}_{d,+}^d),$$

we have $\|D_h^i(p-\mathring{p}_\epsilon)\|_\Omega\le C\|D_h^i(u-u_\epsilon)\|_{1,\Omega}$.

Via *inf-sup condition* of c, and the equation

$$c(D_h^i(\lambda-\lambda_\epsilon+k_\epsilon),v)=-a(D_h^i(u-u_\epsilon),v)-b(D_i(p-\mathring{p}_\epsilon),v),$$

we have

$$\|D_h^i(\lambda-\lambda_\epsilon+k_\epsilon)\|_{-\frac{1}{2},\Gamma}\le C\|D_h^i(u-u_\epsilon)\|_{1,\Omega}.$$

In views of (2.25), we obtain

$$\|D_h^i(u-u_\epsilon)\|_{1,\Omega}\le C\epsilon\|D_h^i\lambda\|_{\frac{1}{2},\Gamma},$$

then letting $h\to 0$, we proved (2.24). □

3 Penalty method for Navier-Stokes equations: well-posedness

3.1 Variational forms of (NS) and (NS_ϵ)

The variational form of (1.1) reads as: find $(u,p)\in V_n\times\mathring{Q}$ such that

$$a(u,v)+a_1(u,u,v)+b(v,p)=(f,v),\quad\forall v\in V_n,\tag{3.1a}$$

$$b(u,q)=0,\quad\forall q\in\mathring{Q}.\tag{3.1b}$$

Remark 3.1 (cf. [11]) For $f=0$, (3.1) admits a unique solution $u=0$. For any $f\in V^*$ and $f\neq 0$, there exists a solution $(u,p)\in V_n\times\mathring{Q}$ for (3.1), with

$$\|u\|_{1,\Omega}\le\|f\|_{V^*}/\alpha,\quad\|p\|_\Omega\le C\|f\|_{V^*}.\tag{3.2}$$

If $\alpha^2>\|f\|_{V^*}$, then the solution is unique.

The variational form of (1.4) reads as: find $(u_\epsilon,p_\epsilon)\in V\times Q$ such that

$$a(u_\epsilon,v)+a_1(u_\epsilon,u_\epsilon,v)+b(v,p_\epsilon)+\frac{1}{\epsilon}\int_\Gamma u_{\epsilon n}v_n ds=(f,v),\quad\forall v\in V,\tag{3.3a}$$

$$b(u_\epsilon,q)=0,\quad\forall q\in Q.\tag{3.3b}$$

3.2 Well-posedness of (NS_ϵ)

For (1.4), it is equivalent to consider the variational form: find $u_\epsilon \in V_\sigma$ such that,

$$a(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n} v_n ds = (f, v), \quad \forall v \in V_\sigma. \tag{3.4}$$

We will use the following Sobolev embedding and trace inequality, which can be found in [4,19]:

$$H^{\frac{1}{2}}(\Gamma) \subset L^4(\Gamma), \quad \|v\|_{L^4(\Gamma)} \le C_1 \|v\|_{\frac{1}{2},\Gamma}, \quad \forall v \in L^4(\Gamma),\ d = 2,3. \tag{3.5}$$

$$\|v\|_{\frac{1}{2},\Gamma} \le C_2 \|v\|_{1,\Omega}, \quad \forall v \in H^1(\Omega). \tag{3.6}$$

From now on, C_1, C_2 represent the constants defined in (3.5) and (3.6).

Theorem 3.1 *Let $f = 0$, u_ϵ is the solution of (3.4):*

(i) $u_\epsilon = 0$*;*
(ii) there may exists $u_\epsilon \neq 0$, but with lower bounds $\|u_\epsilon\|_{1,\Omega} \ge \frac{4}{C_1^2 C_2^2}\sqrt{\frac{\alpha}{\epsilon}}$.

Theorem 3.2 *For $f \neq 0$, we assume ϵ is sufficiently small such that*

$$\epsilon \le \frac{32\alpha^3}{27\|f\|^2_{V_\sigma^*}} C_1^4 C_2^4, \tag{3.7}$$

then we have,

(i) there exists a solution u_ϵ to (3.4), with $\|u_\epsilon\|_{1,\Omega} \le \frac{3}{2\alpha}\|f\|_{V_\sigma^*}$;
(ii) there may exists $u_\epsilon \in V_\sigma \cap \{u \mid \|u\|_{1,\Omega} > \frac{3\|f\|_{V_\sigma^*}}{2\alpha}\}$, but with lower bounds $\|u_\epsilon\|_{1,\Omega} > \frac{\sqrt{8\alpha}}{\sqrt{3\epsilon} C_1^2 C_2^2}$.

Moreover, under the assumption

$$\tilde{\alpha} := \alpha - \frac{3\|f\|_{V_\sigma^*}}{2\alpha} - \frac{1}{2}\sqrt{\frac{3\epsilon}{\alpha}}\|f\|_{V_\sigma^*} C_1^2 C_2^2 > 0, \tag{3.8}$$

there exists a unique solution in $V_\sigma \cap \left\{v \mid \|v\|_{1,\Omega} \le \frac{3}{2\alpha}\|f\|_{V_\sigma^*}\right\}$.

Proof (Proof of Theorem 3.2) We define the function

$$F(v) = a(v, v) + a_1(v, v, v) + \frac{1}{\epsilon}(v_n, v_n) - (f, v), \quad \forall v \in V_\sigma.$$

In view of (2.5), (3.5) and (3.6), we have

$$\begin{aligned} a_1(v, v, v) =& \frac{1}{2}\int_\Gamma v_n |v|^2 ds \le \frac{1}{2}(\|v\|^4_{L^4(\Gamma)} + \|v_n\|^2_{L^2(\Gamma)}) \\ \le& \frac{\epsilon}{8} C_1^4 C_2^4 \|v\|^4_{1,\Omega} + \frac{1}{2\epsilon}\|v_n\|^2_{L^2(\Gamma)}, \quad \forall v \in V_\sigma, \end{aligned}$$

$$F_\epsilon(v) \ge \left(\alpha\|v\|_X - \frac{\epsilon}{8} C_1^4 C_2^4 \|v\|_X^3 - \|f\|_{X_\sigma^*}\right)\|v\|_X + \frac{1}{2\epsilon}\|v_n\|^2_{L^2(\Gamma)}, \quad \forall v \in V_\sigma.$$

Under the assumption (3.7) on ϵ, $F(v) \geq 0$, for any $v \in V_\sigma$ with $\|v\|_{1,\Omega} = \sqrt{\frac{8\alpha}{3\epsilon C_1^4 C_2^4}}$. Following from Browser's fixed point theorem(see [11]. Here, we omit the process of construting the Galerkin's approximate solutions, and we refer the readers to Page 280 of [11] for details), there exists a solution u_ϵ to (3.4), with $\|u_\epsilon\|_{1,\Omega} \leq \sqrt{\frac{8\alpha}{3\epsilon C_1^4 C_2^4}}$. For this u_ϵ, we have $\alpha - \frac{\epsilon}{8} C_1^4 C_2^4 \|u_\epsilon\|_{1,\Omega}^2 \geq \frac{2\alpha}{3}$. Substituting $v = u_\epsilon$ into (3.4), we obtain,

$$\begin{aligned}
\|f\|_{V_\sigma^*} \|u_\epsilon\|_{1,\Omega} \geq (f, u_\epsilon) =& a(u_\epsilon, u_\epsilon) + a_1(u_\epsilon, u_\epsilon, u_\epsilon) + \frac{1}{\epsilon}\|u_{\epsilon n}\|_{L^2(\Gamma)}^2 \\
\geq& \left(\alpha - \frac{\epsilon}{8} C_1^4 C_2^4 \|u_\epsilon\|_{1,\Omega}^2\right) \|u_\epsilon\|_{1,\Omega}^2 + \frac{1}{2\epsilon}\|u_{\epsilon n}\|_{L^2(\Gamma)}^2 \\
\geq& \frac{2\alpha}{3}\|u_\epsilon\|_{1,\Omega}^2 + \frac{1}{2\epsilon}\|u_{\epsilon n}\|_{L^2(\Gamma)}^2,
\end{aligned}$$

which implies

$$\|u_\epsilon\|_{1,\Omega} \leq \frac{3}{2\alpha}\|f\|_{V_\sigma^*}, \quad \|u_{\epsilon n}\|_{L^2(\Gamma)} \leq \sqrt{\frac{3\epsilon}{\alpha}}\|f\|_{V_\sigma^*}. \tag{3.9}$$

Hence, we proved (i)(ii).

We consider the uniqueness of solution. Assume $u_\epsilon^1, u_\epsilon^2$ are two solutions of (3.4), with $\|u_\epsilon^i\|_{1,\Omega} \leq \frac{3}{2\alpha}\|f\|_{V_\sigma^*}$, $i = 1, 2$. Setting $w = u_\epsilon^1 - u_\epsilon^2$, we have

$$a(w, v) + a_1(w, u_\epsilon^1, v) + a_1(u_\epsilon^2, w, v) + \frac{1}{\epsilon}\int_\Gamma w_n v_n ds = 0, \quad \forall v \in V_\sigma. \tag{3.10}$$

Substituting $v = w$ into (3.10), we have

$$\begin{aligned}
0 &= a(w, w) + a_1(w, u_\epsilon^1, w) + \frac{1}{2}\int_\Gamma u_\epsilon^2 \cdot n|w|^2 ds + \frac{1}{\epsilon}\|w_n\|_{L^2(\Gamma)}^2 \\
&\geq \alpha\|w\|_{1,\Omega}^2 - \|w\|_{1,\Omega}^2 \|u_\epsilon^1\|_{1,\Omega} - \frac{1}{2}\|u_\epsilon^2 \cdot n\|_{L^2(\Gamma)} \|w\|_{L^4(\Gamma)}^2 + \frac{1}{\epsilon}\|w_n\|_{L^2(\Gamma)}^2 \\
&\geq \left(\alpha - \frac{3}{2\alpha}\|f\|_{V_\sigma^*} - \frac{1}{2}\|u_{\epsilon n}^2\|_{L^2(\Gamma)} C_1^2 C_2^2\right)\|w\|_{1,\Omega}^2 + \frac{1}{\epsilon}\|w_n\|_{L^2(\Gamma)}^2,
\end{aligned}$$

with (3.9) and the assumption (3.8), we have

$$0 \geq \tilde{\alpha}\|w\|_{1,\Omega}^2 + \frac{1}{\epsilon}\|w_n\|_{L^2(\Gamma)}^2,$$

which gives $w = 0$, and we complete the proof. □

Theorem 3.3 *Let $u_\epsilon \in V_\sigma$ be the solution of (3.4), there exists a unique $p_\epsilon \in Q$ such that, (u_ϵ, p_ϵ) satisfies (3.3) and*

$$\|p_\epsilon\|_\Omega \leq C(\|u_\epsilon\|_{1,\Omega} + \|u_\epsilon\|_{1,\Omega}^2 + \|f\|_{V^*}). \tag{3.11}$$

Proof (Proof of Theorem 3.3) The existence and uniqueness of p_ϵ are obvious. Let $p_\epsilon = \mathring{p}_\epsilon + k_\epsilon$, where $\mathring{p}_\epsilon \in \mathring{Q}$, $k_\epsilon = \int_\Omega p_\epsilon dx/|\Omega|$. We have

$$b(v, \mathring{p}_\epsilon) = -a(u_\epsilon, v) - a_1(u_\epsilon, u_\epsilon, v) + (f, v), \quad \forall v \in H_0^1(\Omega)^d,$$

which implies $\|\mathring{p}_\epsilon\|_\Omega \le C(\|u_\epsilon\|_{1,\Omega} + \|u_\epsilon\|_{1,\Omega}^2 + \|f\|_{V^*})$ by the *inf-sup condition* of b. Let $v \in V$ be the trace lifting defined in Theorem 2.1, where $v_n = k_\epsilon n$ on Γ. Substituting this v into (3.3), we have,

$$k_\epsilon^2 |\Gamma| = a(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + b(v, \mathring{p}_\epsilon) - (f, v),$$

which yields $|k_\epsilon| \le C(\|u_\epsilon\|_{1,\Omega} + \|u_\epsilon\|_{1,\Omega}^2 + \|\mathring{p}_\epsilon\|_\Omega + \|f\|_{V^*})$ by the *inf-sup condition* of c. We proved (3.11). □

3.2.1 Iteration methods

According to (ii) of Theorem 3.1,3.2, there may exist u_ϵ to (3.3) with $\|u_\epsilon\|_{1,\Omega} \ge C\epsilon^{-\frac{1}{2}}$. For sufficiently small ϵ, this *large norm solution* is definitely not the approximation solution to (1.1). We show that under some conditions, the iteration methods for nonlinear problem (3.3) will avoid the *large norm solution*.

First, we take $(u_\epsilon^0, p_\epsilon^0) \in V \times Q$ as the initial value of iteration method, where $(u_\epsilon^0, p_\epsilon^0)$ is the solution to the penalty Stokes problem (2.10), with

$$\|u_\epsilon^0\|_{1,\Omega} \le \frac{\|f\|_{V^*}}{\alpha}, \quad \|u_{\epsilon n}^0\|_{L^2(\Gamma)} \le \sqrt{\epsilon}\|f\|_{V^*}. \tag{3.12}$$

We present two iteration methods in the following.

(i). For $k = 1, 2, \ldots, M_{max}$, find $(u_\epsilon^k, p_\epsilon^k) \in V \times Q$ such that,

$$a(u_\epsilon^k, v) + a_1(u_\epsilon^{k-1}, u_\epsilon^k, v) + b(v, p_\epsilon^k) + \frac{1}{\epsilon\alpha'} u_{\epsilon n}^k v_n ds = (f, v), \quad \forall v \in V, \tag{3.13a}$$

$$b(u_\epsilon^k, q) = 0, \quad \forall q \in Q, \tag{3.13b}$$

$$\text{if } \|u_\epsilon^k - u_\epsilon^{k-1}\|_{1,\Omega} \le \eta_0, \text{ then stop the iteration,} \tag{3.13c}$$

where M_{max} is the maximum iteration number, η_0 is the error of iteration, and $\alpha' := \alpha - \frac{C_1^2 C_2^2 \sqrt{\epsilon}\|f\|_{V^*}}{2} > 0$(with sufficiently small ϵ).

Lemma 3.1 *For sufficiently small ϵ such that $\alpha' := \alpha - \frac{C_1^2 C_2^2 \sqrt{\epsilon}\|f\|_{V^*}}{2} > 0$, we have*

$$\|u_\epsilon^k\|_{1,\Omega} \le \frac{\|f\|_{V^*}}{\alpha'}, \quad \|u_{\epsilon n}^k\|_{L^2(\Gamma)} \le \sqrt{\epsilon}\|f\|_{V^*}, \quad \forall k \ge 1. \tag{3.14}$$

Furthermore, if α is sufficiently large, or $\|f\|_{V^*}$ is very small, such that $(\alpha')^2 > \|f\|_{V^*}$, then $u_\epsilon^k \to u_\epsilon$ in V.

Proof Substituting $v = u_\epsilon^1$ into (3.1) for $k = 1$, with (3.12), and $\alpha' := \alpha - \frac{C_1^2 C_2^2 \sqrt{\epsilon}\|f\|_{V^*}}{2} > 0$, it yields

$$\|u_\epsilon^1\|_{1,\Omega} \le \frac{\|f\|_{V^*}}{\alpha'}, \quad \|u_{\epsilon n}^1\|_{L^2(\Gamma)} \le \sqrt{\epsilon}\|f\|_{V^*}.$$

(3.14) follows from the induction method. (3.14) implies the existence of a subsequence $\{u_\epsilon^m\}_{m\ge 0}$ such that $u_\epsilon^m \to u_\epsilon$ weakly in V as $m \to \infty$.

Next, we show the convergence of $u_\epsilon^k \to u_\epsilon$. Setting $w^k = u_\epsilon^k - u_\epsilon^{k-1}$, we have

$$a(w^{k+1}, v) + a_1(u_\epsilon^k, w^{k+1}, v) + \frac{1}{\alpha'\epsilon}\int_\Gamma w_n^{k+1} v_n ds = -a_1(w^k, u_\epsilon^k, v), \quad \forall v \in V_\sigma.$$

Substituting $v = w^{k+1}$, we obtain

$$\begin{aligned} &a(w^{k+1}, w^{k+1}) + \frac{1}{2}\int_\Gamma u_{\epsilon n}^k |w^{k+1}|^2 ds + \frac{1}{\alpha'\epsilon}\|w_n^{k+1}\|_{L^2(\Gamma)}^2 \\ =& - a_1(w^k, u_\epsilon^k, w^{k+1}) \le \|u_\epsilon^k\|_{1,\Omega}\|w^k\|_{1,\Omega}\|w^{k+1}\|_{1,\Omega}, \end{aligned}$$

which implies

$$\alpha'\|w^{k+1}\|_{1,\Omega} \le \frac{\|f\|_{X^*}}{\alpha'}\|w^k\|_{1,\Omega}.$$

Assume $\frac{\|f\|_{X^*}}{\alpha'^2} \le \gamma < 1$, then $\|w^k\|_{1,\Omega} \to 0$ as $k \to \infty$. □

Remark 3.2 The sufficient condition $\alpha' > \sqrt{\|f\|_{V^*}}$ for convergence of u_ϵ^k is not much different to the assumption of uniqueness of solutions to (NS) and (NS_ϵ)(see Theorem 3.1 and (3.8)).

Instead of the iteration method (3.13), Newton's method is more popular for stationary Navier-Stokes problem due to its fast convergence.

(ii) (Newton's method). For $k = 1, 2, \ldots, M_{max}$, find $(\delta u^k, \delta p^k) \in V \times Q$ such that,

$$\begin{aligned} &a(\delta u^k, v) + a_1(\delta u^k, u_\epsilon^{k-1}, v) + a_1(u_\epsilon^{k-1}, \delta u^k, v) \\ &\quad + b(v, \delta p^k) + \frac{1}{\epsilon}\int_\Gamma \delta u^k \cdot n v_n ds \\ &= (f, v) - a(u_\epsilon^{k-1}, v) - a_1(u_\epsilon^{k-1}, u_\epsilon^{k-1}, v) \\ &\quad - b(v, p_\epsilon^k) - \frac{1}{\epsilon}\int_\Gamma (u_\epsilon^{k-1}\cdot n) v_n ds, \quad \forall v \in V_\sigma, \end{aligned} \tag{3.15a}$$

$$b(\delta u_\epsilon^k, q) = 0, \quad \forall q \in M, \tag{3.15b}$$

$$u_\epsilon^k = u_\epsilon^{k-1} + \delta u^k, \quad p_\epsilon^k = p_\epsilon^{k-1} + \delta p^k, \tag{3.15c}$$

$$\text{if } \|\delta u^k\| \le \eta_0, \text{ then stop the iteration.} \tag{3.15d}$$

Via calculation, we have, for each k,

$$\begin{aligned}&a(\delta u_\epsilon^k, v) + a_1(\delta u_\epsilon^k, u_\epsilon^{k-1}, v) + a_1(u_\epsilon^{k-1}, \delta u^k, v) + \frac{1}{\epsilon}\int_\Gamma \delta u_{\epsilon n}^k v_n ds \\ =&a_1(\delta u^{k-1}, \delta u^{k-1}, v),\end{aligned} \tag{3.16}$$

where $a_1(\delta u^0, \delta u^0, v) := -a_1(u_\epsilon^0, u_\epsilon^0, v)$. Substituting $v = \delta u_\epsilon^k$ into (3.16), we have

$$\begin{aligned}&\underbrace{\left(\alpha - \|u_\epsilon^{k-1}\|_{1,\Omega} - \frac{C_1^2 C_2^2}{2}\|u_{\epsilon n}^{k-1}\|_{L^2(\Gamma)}\right)}_{=:\alpha_k}\|\delta u_\epsilon^k\|_{1,\Omega}^2 + \frac{1}{\epsilon}\|\delta u_{\epsilon n}^k\|_{L^2(\Gamma)}^2 \\ \le&\|\delta u_\epsilon^{k-1}\|_{1,\Omega}^2\|\delta u_\epsilon^k\|_{1,\Omega}.\end{aligned}$$

From (3.12), if α is sufficiently large(or $\|f\|_{V^*}$ is very small) and ϵ is small enough, such that $\alpha_1 > 0$, then we have $\|\delta u_\epsilon^1\|_V \le \frac{1}{\alpha_1}\|u_\epsilon^0 \cdot \nabla u_\epsilon^0\|_\Omega$ and $\|\delta u_{\epsilon n}^1\|_\Gamma \le C\sqrt{\epsilon}$. When $\|\delta u_\epsilon^1\|_V$ is small enough, with induction method, we have $\alpha_k > 0$ and $\|\delta u_\epsilon^k\|_V \le \frac{1}{\alpha_k}\|\delta u_\epsilon^{k-1}\|_V^2$, and it shows the second order convergence of Newton's method *(ii)*. However, the convergence relies on a proper choice(not explicit) of the initial value u_ϵ^0 and ϵ to guarentee the smallness of $\|\delta u_\epsilon^1\|_{1,\Omega}$.

3.3 Penalty method (NS'_ϵ)

Find $(u_\epsilon, p_\epsilon) \in V \times Q$ such that,

$$\begin{aligned}a(u_\epsilon, v) + \frac{1}{2}&[a_1(u_\epsilon, u_\epsilon, v) - a_1(u_\epsilon, v, u_\epsilon)] \\ &+ b(v, p_\epsilon) + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n} v_n ds = (f, v), \quad \forall v \in V,\end{aligned} \tag{3.17a}$$

$$b(u_\epsilon, q) = 0, \quad \forall q \in Q. \tag{3.17b}$$

Remark 3.3 If we replace the homogeneous slip boundary condition $u_n = 0$ of (1.1c) by $u_n = g$ for some given $g \in H^{\frac{1}{2}}(\Gamma)$, then penalty term $\int_\Gamma u_\epsilon v_n ds$ is replaced by $\int_\Gamma (u_\epsilon - g) v_n ds$. For penalty scheme (3.17), $\frac{1}{2}\int_\Gamma g u_\epsilon \cdot v ds$ need to be added to the LHS of (3.17a).

Theorem 3.4 *There exists a solution (u_ϵ, p_ϵ) to (3.17), satisfying*

$$\|u_\epsilon\|_{1,\Omega} \le \|f\|_{V^*}/\alpha, \quad \|p_\epsilon\|_\Omega \le C\|f\|_{V^*}, \quad \|u_{\epsilon n}\|_{L^2(\Gamma)} \le C\sqrt{\epsilon}\|f\|_{V^*}. \tag{3.18}$$

Moreover, assume α is sufficiently large(or $\|f\|_{V^*}$ is small enough), such that $\alpha^2 > \|f\|_{V^*}$, then the solution is unique.

4 Penalty method for Navier-Stokes equations: error estimates

Let $f \in L^2(\Omega)$, we assume there exists a unique solution $(u,p) \in H^2(\Omega) \times H^1(\Omega)$ of (1.1).

Theorem 4.1 *Let u and u_ϵ be the solutions of (1.1) and (3.17), respectively. Assume $\tau_n(u,p) \in L^2(\Gamma)$, and α is sufficiently large(or $\|f\|_\Omega$ is small enough) such that $\alpha^2 > \|f\|_\Omega$, then we have*

$$\|u-u_\epsilon\|_{1,\Omega} + \|p-\mathring{p}_\epsilon\|_\Omega + \sqrt{\epsilon}\|\lambda-\lambda_\epsilon\|_{L^2(\Gamma)} \leq C\sqrt{\epsilon}\|\tau_n(u,p)\|_{L^2(\Omega)}, \tag{4.1}$$

where $p_\epsilon = \mathring{p}_\epsilon + k_\epsilon$, $\mathring{p}_\epsilon \in \mathring{Q}$, and $k_\epsilon = \frac{1}{|\Omega|}\int_\Omega p_\epsilon dx$.

Proof Introducing the Lagrange multiplier $\lambda = -\tau_n(u,p)$ and $\lambda_\epsilon = \frac{1}{\epsilon}u_{\epsilon n}$, we rewrite the variational equations (3.1) and (3.17) into

(1) find $(u,p,\lambda) \in V \times Q \times \Lambda^*$ such that,

$$a(u,v) + a_1(u,u,v) + b(v,p) + c(\lambda, v_n) = (f,v), \quad \forall v \in V, \tag{4.2a}$$
$$b(u,q) = 0, \quad \forall q \in Q, \tag{4.2b}$$
$$c(u_n,\mu) = 0, \quad \forall \mu \in \Lambda; \tag{4.2c}$$

(2) find $(u_\epsilon, p_\epsilon, \lambda_\epsilon) \in V \times Q \times \Lambda^*$ such that,

$$\begin{aligned} a(u_\epsilon, v) + \frac{1}{2}a_1(u_\epsilon,u_\epsilon,v) - \frac{1}{2}a_1(u_\epsilon,v,u_\epsilon) \\ + b(v,p_\epsilon) + c(\lambda_\epsilon, v_n) = (f,v), \quad \forall v \in V, \end{aligned} \tag{4.3a}$$
$$b(u_\epsilon,q) = 0, \quad \forall q \in Q, \tag{4.3b}$$
$$c(u_{\epsilon n},\mu) = \epsilon c(\lambda_\epsilon,\mu), \quad \forall \mu \in \Lambda. \tag{4.3c}$$

Substituting $v = u - u_\epsilon$ into (4.2a)−(4.3a), we have

$$\begin{aligned} a(u-u_\epsilon, u-u_\epsilon) + \frac{1}{4}[a_1(u-u_\epsilon, u+u_\epsilon, u-u_\epsilon) \\ - a_1(u-u_\epsilon, u-u_\epsilon, u+u_\epsilon)] + c(\lambda-\lambda_\epsilon, u_n - u_{\epsilon n}) = 0. \end{aligned}$$

Noticing $u_n = 0$ and $u_{\epsilon n} = \epsilon\lambda_\epsilon$, we derive

$$\begin{aligned} c(\lambda-\lambda_\epsilon, u_n-u_{\epsilon n}) = -\epsilon c(\lambda-\lambda_\epsilon, \lambda_\epsilon) \\ =\epsilon c(\lambda-\lambda_\epsilon, \lambda-\lambda_\epsilon) - \epsilon c(\lambda-\lambda_\epsilon, \lambda). \end{aligned} \tag{4.4}$$

It is proved in Remark 3.1, 3.4, that u and u_ϵ satisfying

$$\|u\|_{1,\Omega}, \|u_\epsilon\|_{1,\Omega} \leq \|f\|_\Omega/\alpha. \tag{4.5}$$

Therefore, we have

$$\begin{aligned} (\alpha - \|f\|_\Omega/\alpha)\|u-u_\epsilon\|^2_{1,\Omega} + \epsilon c(\lambda-\lambda_\epsilon, \lambda-\lambda_\epsilon) \\ \leq \epsilon c(\lambda-\lambda_\epsilon, \lambda) \leq \frac{\epsilon}{2}\|\lambda-\lambda_\epsilon\|^2_{L^2(\Gamma)} + \frac{\epsilon}{2}\|\lambda\|^2_{L^2(\Gamma)}. \end{aligned} \tag{4.6}$$

Under the assumption $\alpha^2 > \|f\|_\Omega$, we obtain,

$$\|u-u_\epsilon\|_{1,\Omega} + \sqrt{\epsilon}\|\lambda-\lambda_\epsilon\|_{L^2(\Gamma)} \leq C\sqrt{\epsilon}\|\lambda\|_{L^2(\Omega)}.$$

Using *inf-sup condition of* b (2.6) and (4.5), we conclude

$$\|p-\mathring{p}_\epsilon\|_\Omega \leq C\|u_\epsilon - u\|_{1,\Omega}. \tag{4.7}$$

The proof is completed. □

Theorem 4.2 *Let $\tau_n(u,p) \in H^{1/2}(\Gamma)$, and with the same assumption of Theorem 4.1, then we have*

$$\|u-u_\epsilon\|_{1,\Omega} + \|p-\mathring{p}_\epsilon\|_\Omega \leq C\epsilon(\|\tau_n(u,p)\|_{H^{1/2}(\Gamma)} + \|f\|_\Omega). \tag{4.8}$$

Proof Instead of using (4.4), we derive

$$\begin{aligned} &c(\lambda-\lambda_\epsilon, u_n - u_{\epsilon n}) = -\epsilon c(\lambda-\lambda_\epsilon+k_\epsilon, \lambda_\epsilon) \\ =&\epsilon c(\lambda-\lambda_\epsilon+k_\epsilon, \lambda-\lambda_\epsilon+k_\epsilon) - \epsilon c(\lambda-\lambda_\epsilon+k_\epsilon, \lambda+k_\epsilon), \end{aligned} \tag{4.9}$$

and obtain

$$\begin{aligned} &(\alpha - \|f\|_\Omega/\alpha)\|u-u_\epsilon\|^2_{1,\Omega} + \epsilon c(\lambda-\lambda_\epsilon+k_\epsilon, \lambda-\lambda_\epsilon+k_\epsilon) \\ \leq&\epsilon c(\lambda-\lambda_\epsilon+k_\epsilon, \lambda+k_\epsilon) \leq \epsilon\|\lambda-\lambda_\epsilon+k_\epsilon\|_{\Lambda^*}\|\lambda+k_\epsilon\|_\Lambda. \end{aligned} \tag{4.10}$$

If we show

$$\|\lambda-\lambda_\epsilon+k_\epsilon\|_{\Lambda^*} \leq C\|u-u_\epsilon\|_{1,\Omega}, \tag{4.11}$$

then with the assumption $\lambda \in H^{1/2}(\Gamma) = \Lambda$, we can derive the error estimate

$$\|u-u_\epsilon\|_{1,\Omega} \leq C\epsilon(\|\lambda\|_\Lambda + k_\epsilon), \tag{4.12}$$

where k_ϵ is bounded independent of ϵ(Theorem 3.4). $\|p-\mathring{p}_\epsilon\|_\Omega \leq C\epsilon$ follows from (4.7) and (4.12). Therefore, we are only left to prove (4.11). Since

$$\begin{aligned} &-c(\lambda-\lambda_\epsilon+k_\epsilon, v_n) \\ =&a(u-u_\epsilon, v) + b(v, p-\mathring{p}_\epsilon) + \frac{1}{2}[a_1(u-u_\epsilon, u, v) \\ &+ a_1(u_\epsilon, u-u_\epsilon, v) + a_1(u_\epsilon-u, v, u_\epsilon) + a_1(u, v, u_\epsilon-u)] \\ \leq&C(1+\|u\|_{1,\Omega}+\|u_\epsilon\|_{1,\Omega})(\|u-u_\epsilon\|_{1,\Omega}+\|p-\mathring{p}_\epsilon\|_\Omega)\|v\|_{1,\Omega}. \end{aligned}$$

From (4.5), (4.7) and the *inf-sup condition* of c (2.7), we obtain (4.11). □

Remark 4.1 In above, we show the error estimates of penalty scheme (3.17). For penalty scheme (3.3), under the assumption that u_ϵ with $\|u_\epsilon\|_{1,\Omega} \leq \frac{3\|f\|_\Omega}{2\alpha}$ and $\alpha^2 > \frac{3\|f\|_\Omega}{2}$, then we can obtain the same error estimates as (4.1) and (4.8).

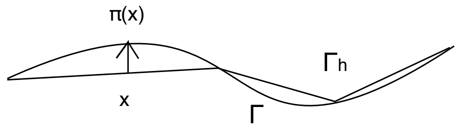

Fig. 5.1 $\pi : \Gamma_h \to \Gamma$.

5 Penalty method for Navier-Stokes equations: FEM

A regular triangulation $\mathcal{T}_h$ is introduced to the smooth domain Ω, where $h = \max_{K\in\mathcal{T}_h} \mathrm{diam}(K)$. $\Omega_h = \cup_{K\in\mathcal{T}_h}\overline{K}$, $\partial\Omega_h = \Gamma_h \cup C_h$, $\Gamma_h \cap C_h = \emptyset$(see Figure 1.2). The boundary mesh $\mathcal{S}_h$ inherited from $\mathcal{T}_h$ is also a regular triangulation of Γ_h in $d-1$ dimension. n_h is the outer unit normal assigned to Γ_h. We assume $C = C_h$ for simplicity. Suppose Γ is C^3 smooth, then we have

(1) $\max_{x\in\Gamma} \mathrm{dist}(x, \Gamma_h) \le Ch^2$.
(2) There exists a continuous bijective mapping(cf. [12,3])

$$\pi : \Gamma_h \to \Gamma; \quad x \mapsto \pi(x).$$

Moreover, for any element S of $\mathcal{S}_h$, we have $\pi, \pi^{-1} \in C^2(S)$ and

$$||D\pi| - 1|, ||D\pi^{-1}| - 1| \le Ch^2, \tag{5.1}$$

where $|D\pi|$ is the Jacobian of transform π, such that $\int_\Gamma v ds = \int_{\Gamma_h} v \circ \pi |D\pi^{-1}| ds$. And we also have(cf. [26])

$$|n_h - n \circ \pi| \le Ch. \tag{5.2}$$

Finite element spaces:

We consider the $P1/P1$ and $P1b/P1$ finite element spaces.

$$V_h = \{v_h \in C(\overline{\Omega_h})^d \mid v_h|_K \in P_1(K),\ K \in \mathcal{T}_h,\ v_h|_{C_h} = 0\},\ \text{for } P1$$
$$V_h = \{v_h \in C(\overline{\Omega_h})^d \mid v_h|_K \in P_1(K) \oplus B(K),\ K \in \mathcal{T}_h,\ v_h|_{C_h} = 0\},\ \text{for } P1b,$$
$$Q_h = \{v_h \in C(\overline{\Omega_h})^d \mid v_h|_K \in P_1(K),\ K \in \mathcal{T}_h\},$$
$$V_{h0} = \{v_h \in V_h \mid v_h = 0 \text{ on } \Gamma_h\}, \quad \mathring{Q}_h = Q_h \cap L_0^2(\Omega_h),$$
$$\Lambda_h = \{v_h \cdot n_h \mid v_h \in V_h\},$$

where $P_l(K)$ is the set of polynomial of order l in K, and $B(K)$ stands for the space spanned by the bubble function on K. We define the following bilinear and trilinear forms:

$$a_h(u_h, v_h) = \int_{\Omega_h} 2\nu\mathcal{E}(u_h)\mathcal{E}(v_h), \quad \forall u_h, v_h \in V_h; \tag{5.3}$$
$$a_{1h}(u_h, v_h, w_h) = \int_{\Omega_h} (u_h \cdot \nabla)v_h w_h dx, \quad \forall u_h, v_h, w_h \in V_h; \tag{5.4}$$
$$b_h(v_h, p_h) = -\int_{\Omega_h} \nabla \cdot v_h p_h dx, \quad \forall v_h \in V_h, p_h \in Q_h, \tag{5.5}$$
$$d_h(p_h, q_h) = \gamma h^2(\nabla p_h, \nabla q_h)_{\Omega_h}, \quad \begin{cases} \gamma = 1 \text{ for } P1/P1, \\ \gamma = 0 \text{ for } P1b/P1. \end{cases} \tag{5.6}$$

Choice of c_h.

(1) *Fine-integration*: For any $\lambda_h, \mu_h \in \Lambda_h$.

$$c_h(\lambda_h, \mu_h) := \int_{\Gamma_h} \lambda_h \mu_h ds. \tag{5.7}$$

$\|\mu_h\|_{c_h} := c_h(\mu_h, \mu_h)^{\frac{1}{2}}$ is equivalent to $\|\mu_h\|_{L^2(\Gamma)}$, for any $\mu_h \in \Lambda_h$.

(2) *Lower-order-integration*: For any $\lambda_h, \mu_h \in \Lambda_h$,

$$c_h(\mu_h, \eta_h) = \sum_{s \in \mathcal{S}_h} |s| \mu_h(m_s) \eta_h(m_s),\ m_s = \begin{cases} \text{midpoint of } s \text{ if } d = 2, \\ \text{barycenter of } s \text{ if } d = 3. \end{cases} \tag{5.8}$$

$\|\mu_h\|_{c_h} = c_h(\mu_h, \mu_h)^{\frac{1}{2}}$ is a semi-norm of Λ_h(there exists $\mu_h \neq 0$ but $c_h(\mu_h, \mu_h) = 0$).

Coercivity and inf-sup conditions.

– *Coercivity* of a_h:

$$a_h(v_h, v_h) \geq \alpha_1 \|v_h\|^2_{1,\Omega_h}, \quad \alpha_1 > 0, \quad \forall v_h \in V_h. \tag{5.9}$$

– *inf-sup condition* of b_h, $\beta_1, \tilde{\beta}_1 > 0$(cf [15,22]):

$$\inf_{p_h \in \mathring{Q}_h \setminus \{0\}} \sup_{v_h \in V_{h0} \setminus \{0\}} \frac{b_h(v_h, p_h)}{\|v_h\|_{1,\Omega} \|p_h\|_{\Omega_h}} \geq \beta_1, \text{ for } P1b/P1. \tag{5.10}$$

$$\sup_{v_h \in V_{h0} \setminus \{0\}} \frac{b_h(v_h, p_h)}{\|v_h\|_{1,\Omega}} \geq \tilde{\beta}_1 \|p_h\|_{\Omega_h} - \gamma C h \|\nabla p_h\|_{\Omega}, \quad \forall p_h \in \mathring{Q}_h, \text{ for } P1/P1. \tag{5.11}$$

– *inf-sup condition* of c_h defined by (5.7):

$$\inf_{\mu_h \in \Lambda_h \setminus \{0\}} \sup_{v_h \in V_h \setminus \{0\}} \frac{\int_{\Gamma_h} v_h \cdot n_h \mu_h}{\|v_h\|_{1,\Omega} \|\mu_h\|_{\Lambda^*}} \geq \gamma_1 > 0. \tag{5.12}$$

Finite element penalty scheme.

The finite element approximation to penalty problem (3.17) reads as: find $(u_h, p_h) \in V_h \times Q_h$ such that,

$$\begin{aligned} a_h(u_h, v_h) &+ \frac{1}{2}[a_{1h}(u_h, u_h, v_h) - a_{1h}(u_h, v, u_h)] \\ &+ b_h(v_h, p_h) + \frac{1}{\epsilon} c_h(u_h \cdot n_h, v_h \cdot n_h) = (\tilde{f}, v_h)_{\Omega_h}, \quad \forall v_h \in X_h, \end{aligned} \tag{5.13a}$$

$$b_h(u_h, q_h) = d_h(p_h, q_h), \quad \forall q_h \in M_h, \tag{5.13b}$$

where $\tilde{f}$ is some extension of f onto $\tilde{\Omega} = \Omega \cup \Omega_h$ with $\|\tilde{f}\|_{\tilde{\Omega}} \leq C \|f\|_{\Omega}$.

In the following we only discuss the $P1b/P1$ element approximation($\gamma = 0$, $b_h(u_h, q_h) = 0$), since the analysis method and results of $P1/P1$ with stabilization($b_h(u_h, q_h) = h^2(\nabla p_h, \nabla q_h)$) and $P1b/P1$ elements are very similar. We refer the readers to [16] for the argument of $P1/P1$ element of Stokes equations.

5.1 Well-posedness and a-priori estimate

Theorem 5.1 *There exists a solution $(u_h, p_h) \in V_h \times Q_h$ to (5.13) with c_h defined by both (5.7) and (5.8), and the solution satisfies*

$$\|u_h\|_{1,\Omega_h} + \|\mathring{p}_h\|_{\Omega_h} + \sqrt{\epsilon}\|u_h \cdot n_h\|_{c_h} \leq C\|\tilde{f}\|_{\Omega h}, \tag{5.14}$$

where $p_h = \mathring{p}_h + k_h$, $\mathring{p}_h \in \mathring{Q}_h$, $k_h = \int_{\Omega_h} p_h dx/|\Omega_h|$, and

$$|k_h| \leq C\left(\|\tilde{f}\|_{\Omega_h} + \|u_h\|_{1,\Omega} + \|u_h\|_{1,\Omega}^2 + \frac{h}{\epsilon}\right). \tag{5.15}$$

Moreover, if $\alpha_1^2 > \|\tilde{f}\|_{\Omega_h}$, then the solution is unique.

Proof The existence and uniqueness of solution (u_h, p_h) and (5.14) follow from the coercivity of a_h, the *inf-sup conditions* of b_h and c_h. Here, we only check the estimate of k_h (5.15). In views of (5.13b) of $\gamma = 0$, we obtain, for c_h defined by both (5.7) and (5.8),

$$c_h(u_h \cdot n_h, 1) = \int_{\Gamma_h} u_h \cdot n_h ds = \sum_{s \in \mathcal{S}_h} |s|(u_h \cdot n_h)(m_s) = -b_h(u_h, 1) = 0. \tag{5.16}$$

Since n_h is discontinuous on Γ_h, we cannot choose the trace lifting $v_h \in V_h$ with $v_h = k_h n_h$ on Γ, as the proof of Theorem 3.3. Let $\{P_i\}_{i=1}^N$ be the set of the vertices of polygon or polyhedral domain Ω_h(nodes of Γ_h), $\Gamma_i = \{s \in \mathcal{S}_h \mid P_i \in \bar{s}\}$(faces/edges contain the vertex P_i), we then define a $v_h \in X_h$ satisfying

$$v_h(P_i) = k_h \frac{1}{\Gamma_i^{\#}} \sum_{s \in \Gamma_i} n_h(s), \quad \|v_h\|_{1,\Omega_h} \leq C k_h,$$

where $\Gamma_i^{\#}$ equals to the number of faces s in Γ_i, and $n_h(s)$ is the value of n_h on s. Since Γ has C^3 smoothness, we have $|v_h - k_h n_h| \leq Ch$ on Γ_h. Then, substituting this v_h into (5.13a), it yields,

$$k_h \int_{\Gamma_h} v_h \cdot n_h = -b_h(v_h, k_h) = a_h(u_h, v_h) + b_h(v_h, \mathring{p}_h) + \frac{1}{\epsilon} c_h(u_h \cdot n_h, v_h \cdot n_h).$$

In view of (5.16), we have

$$\frac{1}{\epsilon} c_h(u_h \cdot n_h, v_h \cdot n_h) = \frac{k_h}{\epsilon} \underbrace{c_h(u_h \cdot n_h, 1)}_{=0} + \frac{1}{\epsilon} c_h(u_h \cdot n_h, (v_h - k_h n_h) \cdot n_h).$$

Therefore, we have

$$\begin{aligned} k_h^2|\Gamma_h| &= k_h \int_{\Gamma_h} k_h n_h \cdot n_h = k_h \int_{\Gamma_h} (k_h n_h - v_h) \cdot n_h \\ &= k_h \int_{\Gamma_h} (k_h n_h - v_h) \cdot n_h + a_h(u_h, v_h) + b_h(v_h, \mathring{p}_h) \\ &\quad + \frac{1}{\epsilon} c_h(u_h \cdot n_h, (v_h - k_h n_h) \cdot n_h), \end{aligned}$$

which implies (5.15) since $|v_h - k_h n_h| \leq Ch$ on Γ_h. □

5.2 Preliminary results

5.2.1 Extension operators and skin domain estimates

We denote the skin domain $\Omega \triangle \Omega_h = (\Omega \backslash \Omega_h) \cup (\Omega_h \backslash \Omega)$, $\tilde{\Omega} := \Omega \cup \Omega_h$.

Lemma 5.1 (cf. [19]) *There exists an extension operator*

$$P \in \mathcal{L}(H^m(\Omega)^d, H^m(\mathbb{R}^d)^d), \quad (0 \leq m \in \mathbb{N}0), \quad v \mapsto Pv =: \tilde{v}$$

such that,

$$\|\tilde{v}\|_{k,\mathbb{R}^d} \leq C_m \|v\|_{k,\Omega}, \quad 0 \leq k \leq m, \quad \forall v \in H^m(\Omega)^d.$$

Moreover, if $\nabla \cdot v = 0$, then we can take the extension $\tilde{v}$ satisfying $\nabla \cdot v = 0$ in $\mathbb{R}^d$.

Lemma 5.2 (cf. [29–31]) *Under the assumption* $\max_{x \in \Gamma}$ *dist*$(x, \Gamma_h) \leq Ch^2$, *we have*

$$\|\tilde{v}\|_{k,\Omega \triangle \Omega_h} \leq Ch \|v\|_{k+1,\Omega}, \quad 0 \leq k \leq m-1, \quad \forall v \in H^m(\Omega)^d.$$

Lemma 5.3 (cf. [29]) *There exists an extension operator $P_h \in \mathcal{L}(V_h, H^1(\tilde{\Omega}))$, such that, $\forall v_h \in V_h$,*

$$\begin{aligned} \|P_h v_h\|_{1,\tilde{\Omega}} &\leq C \|v_h\|_{1,\Omega_h}, \\ \|P_h v_h\|_{k,\Omega \varDelta \Omega_h} &\leq Ch^{\frac{1}{2}} \|v_h\|_{k,K_{\Gamma_h}}, \quad k = 0, 1, \\ \|P_h v_h\|_{\Omega} &\leq Ch \|v_h\|_{1,\Omega_h}, \end{aligned}$$

where $K_{\Gamma_h} := \{K \in \mathcal{T}_h \mid \overline{K} \cap \Gamma_h \neq \emptyset\}$.

5.2.2 Lagrange interpolation and projection operators

We employ the Lagrange interpolation operator I_h and projection operator P_{L^2}(cf. [11,26,27]).

$$I_h : C(\overline{\Omega_h}) \to V_h, \quad v \mapsto I_h v,$$

$$\|v - I_h v\|_{L^p(\Omega_h)} + h \|v - I_h v\|_{W^{1,p}(\Omega_h)} \leq Ch^2 \|v\|_{W^{2,p}(\tilde{\Omega})}, \quad \forall v \in W^{2,p}(\Omega_h).$$

$$\begin{aligned} P_{L^2} : H^1(\Omega_h) &\to V_h, \quad v \mapsto P_{L^2} v, \\ (v - P_{L^2} v, v_h)_{\Omega_h} &= 0, \quad \forall v_h \in V_h, \\ \|v - P_{L^2} v\|_{\Omega_h} &\leq Ch \|v\|_{1,\Omega_h}. \end{aligned}$$

5.2.3 Consistency error estimates

Lemma 5.4 (cf. [16]) *Let $\pi \in C^2(\Gamma_h)$, then we have, for any $v \in H^1(\tilde{\Omega})$,*

(i) $\|v \circ \pi\|_{\Gamma_h} \leq C\|v\|_{\Gamma}$.
(ii) $|\int_\Gamma v ds - \int_{\Gamma_h} v \circ \pi ds| \leq Ch^2\|v\|^2_{\Gamma_h}$.
(iii) $\|v - v \circ \pi\|_{\Gamma_h} \leq Ch\|v\|_{1,\tilde{\Omega}}$.

Proof The proof has been derived in [16]. Here, we present a brief proof for the convenience of readers. (i) is obvious. (ii) follows from the properties of π (5.1),

$$\int_\Gamma v ds - \int_{\Gamma_h} v \circ \pi ds = \int_{\Gamma_h} v \circ \pi(|D\pi^{-1}| - 1) ds \leq Ch^2\|v\|_{\Gamma_h}.$$

(iii) is from [26]((5.1), Verfürth).

Lemma 5.5 (cf. [16]) *Assume $\lambda \in L^2(\Gamma)$(resp. $W^{1,\infty}(\Gamma)$) for c_h defined by (5.7) (resp. (5.8)), and let $\tilde{\lambda} = \lambda \circ \pi$, then we have*

$$|c(v_n, \lambda) - c_h(v \cdot n_h, \tilde{\lambda})| \leq Ch\|v\|_{1,\tilde{\Omega}}, \quad \forall v \in H^1(\tilde{\Omega})^d. \tag{5.17}$$

Proof For c_h defined by (5.7), we have, from (5.2) and (iii) of Lemma 5.4,

$$\begin{aligned}
&|c(v_n, \lambda) - c_h(v \cdot n_h, \tilde{\lambda})| = |c(v_n, \lambda) - \int_{\Gamma_h} v \cdot n_h \tilde{\lambda} ds| \\
\leq &\left| \int_\Gamma v_n \lambda - \int_{\Gamma_h} (v_n \lambda) \circ \pi \right| \\
&+ \left| \int_{\Gamma_h} (v_n \lambda) \circ \pi - v \cdot (n\lambda) \circ \pi + v \cdot (n\lambda) \circ \pi - v \cdot n_h \tilde{\lambda} \right| \\
\leq &Ch\|v\|_{1,\tilde{\Omega}}\|\lambda\|_\Gamma.
\end{aligned}$$

For c_h defined by (5.8), we have

$$\begin{aligned}
&\left| \int_{\Gamma_h} v \cdot n_h \tilde{\lambda} ds - c_h(v \cdot n_h, \tilde{\lambda}) \right| \\
\leq &\sum_{s \in \mathcal{S}_h} \int_s v \cdot n_h |\tilde{\lambda} - \tilde{\lambda}(m_s)| ds \leq Ch\|v\|_{1,\tilde{\Omega}}\|\lambda\|_{W^{1,\infty}(\Gamma)}.
\end{aligned}$$

□

Proposition 5.1 *Let (u, p) and (u_h, p_h) be solutions of (1.1) and (5.13), respectively. Set $\lambda = -\tau_n(u, p)$, $\lambda_h = \frac{1}{\epsilon} u_h \cdot n_h$. We assume $f \in L^2(\Omega)$, and $(u, p) \in H^2(\Omega)^d \times H^1(\Omega)$, and the same assumption of Lemma 5.5. For any $v_h \in V_h$, we set the consistency error*

$$\begin{aligned}
E(v_h) :=& a_h(\tilde{u} - u_h, v_h) + \frac{1}{2}[a_{1h}(\tilde{u} - u_h, \tilde{u}, v_h) + a_{1h}(u_h, \tilde{u} - u_h, v_h) \\
&- a_{1h}(\tilde{u} - u_h, v_h, \tilde{u}) - a_{1h}(\tilde{u}, v_h, \tilde{u} - u_h)] \\
&+ b_h(v_h, \tilde{p} - p_h) + c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h),
\end{aligned}$$

where $(\tilde{u}, \tilde{p})$ is the extension(Lemma 5.1) of (u,p) onto $\tilde{\Omega} = \Omega \cup \Omega_h$. Then, we have

$$|E(v_h)| \leq Ch\|v_h\|_{1,\Omega_h}. \tag{5.18}$$

Proof We denote

$$a_\omega(u,v) := 2\nu(\mathcal{E}(u), \mathcal{E}(v))_\omega, \quad a_{1,\omega}(u,v,w) = (u \cdot \nabla v, w)_\omega,$$

$$b_\omega(v,q) = -(\nabla \cdot v, q)_\omega.$$

From (3.1) and (5.13), we have

$$\begin{aligned} E(v_h) = &- a_{\Omega\setminus\Omega_h}(u, P_h v_h) + a_{\Omega_h\setminus\Omega}(\tilde{u}, v_h) - \frac{1}{2}[a_{1,\Omega\setminus\Omega_h}(u,u,P_h v_h) \\ &- a_{1,\Omega\setminus\Omega_h}(u, P_h v_h, u) - a_{1,\Omega_h\setminus\Omega}(\tilde{u},\tilde{u},v_h) + a_{1,\Omega_h\setminus\Omega}(\tilde{u}, v_h, \tilde{u})] \\ &- b_{\Omega\setminus\Omega_h}(P_h v_h, u) + b_{\Omega_h\setminus\Omega}(v_h, \tilde{u}) + (f, P_h v_h)_{\Omega\setminus\Omega_h} - (\tilde{f}, v_h)_{\Omega_h\setminus\Omega} \\ &- c(P_h v_h \cdot n, \lambda) + c_h(v_h \cdot n_h, \tilde{\lambda}). \end{aligned}$$

(5.18) follows from Lemma 5.2, 5.3 and 5.5. □

5.3 Error estimates: Fine-integration scheme (5.7)

Theorem 5.2 *c_h is defined by (5.7). Let (u,p) and (u_h,p_h) be solutions of (1.1) and (5.13), respectively. Assuming $f \in L^2(\Omega)$, $(u,p) \in H^2(\Omega)^d \times H^1(\Omega)$, and $\alpha_1^2 > \|\tilde{f}\|_{\Omega_h}$, we have*

$$\|\tilde{u} - u_h\|_{1,\Omega_h} + \|\tilde{p} - p_h\|_{\Omega_h} \leq C(\sqrt{h} + \sqrt{\epsilon} + h/\sqrt{\epsilon}). \tag{5.19}$$

Proof Set $v_h = I_h\tilde{u}$. Since $\|\tilde{u} - u_h\|_{1,\Omega_h} \leq \|\tilde{u} - v_h\|_{1,\Omega} + \|u_h - v_h\|_{1,\Omega_h}$ and $\|\tilde{u} - v_h\|_{1,\Omega} \leq Ch\|\tilde{u}\|_{2,\tilde{\Omega}}$, we only need to show the estimate of $\|u_h - v_h\|_{1,\Omega_h}$.

$$\begin{aligned} \alpha_1\|u_h - v_h\|_{1,\Omega_h}^2 &\leq a_h(u_h - v_h, u_h - v_h) \\ &= a_h(v_h - \tilde{u}, v_h - u_h) + a_h(\tilde{u} - u_h, v_h - u_h). \end{aligned} \tag{5.20}$$

$$\begin{aligned} &a_h(\tilde{u} - u_h, v_h - u_h) \\ =&E(v_h - u_h) - \frac{1}{2}[a_{1h}(\tilde{u} - u_h, \tilde{u}, v_h - u_h) + a_{1h}(u_h, \tilde{u} - u_h, v_h - u_h) \\ &- a_{1h}(\tilde{u} - u_h, v_h - u_h, \tilde{u}) - a_{1h}(\tilde{u}, v_h - u_h, \tilde{u} - u_h)] \\ &- b_h(v_h - u_h, \tilde{p} - p_h) - c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h). \end{aligned}$$

In the following, we are aim to prove

$$\begin{aligned} a_h(\tilde{u} - u_h, v_h - u_h) \leq &\frac{\|\tilde{f}\|_{\Omega_h}}{\alpha_1}\|v_h - u_h\|_{1,\Omega_h}^2 + Ch\|v_h - u_h\|_{1,\Omega_h} \\ &- \frac{\epsilon}{4}\|\tilde{\lambda} - \lambda_h\|_{\Gamma_h}^2 + C\frac{h^2}{\epsilon} + \epsilon\|\tilde{\lambda}\|_{\Gamma_h^2} \end{aligned} \tag{5.21}$$

which implies (5.19) under the assumption $\alpha_1^2 > \|\tilde{f}\|_{\Omega_h}$. Firstly, from Proposition 5.1, we have $|E(v_h - u_h)| \le Ch\|v_h - u_h\|_{1,\Omega_h}$. Secondly, Via calculation,

$$\begin{aligned}
&\frac{1}{2}[a_{1h}(\tilde{u} - u_h, \tilde{u}, v_h - u_h) + a_{1h}(u_h, \tilde{u} - u_h, v_h - u_h)\\
&\qquad - a_{1h}(\tilde{u} - u_h, v_h - u_h, \tilde{u}) - a_{1h}(\tilde{u}, v_h - u_h, \tilde{u} - u_h)]\\
\le&\|u_h\|_{1,\Omega_h}\|v_h - u_h\|^2_{1,\Omega_h} + Ch\|v_h - u_h\|_{1,\Omega_h}\|\tilde{u}\|_{2,\tilde{\Omega}}\\
\le&\frac{\|\tilde{f}\|_{\Omega_h}}{\alpha_1}\|v_h - u_h\|^2_{1,\Omega_h} + Ch\|v_h - u_h\|_{1,\Omega_h}.
\end{aligned}$$

Since we can replace p by $p + l$ for any constant l, we set $\tilde{p}$ satisfies $\tilde{p} - p_h \in L^2_0(\Omega_h)$ and $q_h = P_{L^2}\tilde{p}$, $q_h - p_h \in \mathring{Q}_h$. With $b_h(u_h, q_h) = 0$ and $\nabla \cdot \tilde{u} = 0$, we have

$$\begin{aligned}
&- b_h(v_h - u_h, \tilde{p} - p_h)\\
=&b_h(\tilde{u} - v_h, \tilde{p} - q_h) + b_h(\tilde{u} - v_h, q_h - p_h) + b_h(u_h, \tilde{p} - q_h)\\
=&b_h(\tilde{u} - v_h, q_h - p_h) - b_h(v_h - u_h, \tilde{p} - q_h)\\
\le&Ch\|\tilde{u}\|_{2,\tilde{\Omega}}\|q_h - p_h\|_{\Omega_h} + Ch\|\tilde{p}\|_{1,\tilde{\Omega}}\|v_h - u_h\|_{1,\Omega_h}.
\end{aligned}$$

Since $q_h - p_h \in \mathring{Q}_h$, by *inf-sup condition* of b_h, we obtain

$$\|q_h - p_h\|_{\Omega_h} \le Ch(\|\tilde{u}\|_{2,\tilde{\Omega}} + \|\tilde{p}\|_{1,\tilde{\Omega}}) + C\|v_h - u_h\|_{1,\Omega_h}.$$

Therefore, we have $|b_h(v_h - u_h, \tilde{p} - p_h)| \le Ch^2 + Ch\|v_h - u_h\|_{1,\Omega_h}$. We are left to estimate $-c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h)$. In views of $\lambda_h = \frac{1}{\epsilon}u_h \cdot n_h$,

$$\begin{aligned}
&- c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h) = -\epsilon c_h(\tilde{\lambda} - \lambda_h, \tilde{\lambda} - \lambda_h) + \epsilon c_h(\tilde{\lambda}, \tilde{\lambda} - \lambda_h)\\
&\qquad + c_h((\tilde{u} - v_h) \cdot n_h, \tilde{\lambda} - \lambda_h) - c_h(\tilde{u} \cdot n_h, \tilde{\lambda} - \lambda_h)\\
&\le -\epsilon\|\tilde{\lambda} - \lambda_h\|^2_{\Gamma_h} + \epsilon\|\tilde{\lambda}\|^2_{\Gamma_h} + \frac{\epsilon}{4}\|\tilde{\lambda} - \lambda_h\|^2_{\Gamma_h}\\
&\qquad + \frac{1}{\epsilon}\|(\tilde{u} - v_h) \cdot n_h\|^2_{\Gamma_h} + \frac{1}{\epsilon}\|\tilde{u} \cdot n_h\|^2_{\Gamma_h} + \frac{\epsilon}{2}\|\tilde{\lambda} - \lambda_h\|^2_{\Gamma_h}.
\end{aligned} \tag{5.22}$$

Since $\|(\tilde{u} - v_h) \cdot n_h\|_{\Gamma_h} \le C\|\tilde{u} - v_h\|_{1,\tilde{\Omega}} \le Ch\|\tilde{u}\|_{2,\tilde{\Omega}}$ and

$$\|\tilde{u} \cdot n_h\|_{\Gamma_h} \le \|\tilde{u} \cdot (n_h - n \circ \pi) + (\tilde{u} - u \circ \pi)n \circ \pi\|_{\Gamma_h} \le Ch, \quad (u_n = 0 \text{ on } \Gamma)$$

it yields

$$-c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h) \le -\frac{\epsilon}{4}\|\tilde{\lambda} - \lambda_h\|^2_{\Gamma_h} + C\frac{h^2}{\epsilon} + \epsilon\|\tilde{\lambda}\|^2_{\Gamma_h},$$

Combining those inequalities, we proved (5.21). From (5.20), (5.21) and the assumption $\alpha^2 > \|\tilde{f}\|_{\Omega_h}$, we conclude (5.19). $\square$

5.4 Error estimates: Lower-order-integration scheme (5.8)

Lemma 5.6 (cf. [16]) *Let $u \in W^{2,\infty}(\Omega)$ with $u_n|_\Gamma = 0$. For any $s \in \mathcal{S}_h$, $\tilde{u}$ is the extension of u according to Lemma 5.1, then we have*

(i) For $d = 2$, there exists π such that $|n \circ \pi(m_s) - n_h(m_s)| \le Ch^2$; moreover

$$|(I_h\tilde{u} \cdot n_h)(m_s)| \le Ch^2 \|\tilde{u}\|_{W^{2,\infty}(\tilde{\Omega})}.$$

(ii) For $d = 3$, if $\tilde{u} \in W^{2,\infty}(\tilde{\Omega})$ satisfies $\nabla \cdot \tilde{u} = 0$, and $\tilde{u}_n = 0$ on Γ, then we have $|(I_h\tilde{u} \cdot n_h)(m_s)| \le Ch\|\tilde{u}\|_{W^{2,\infty}(\tilde{\Omega})}$.

Proof (i) For $d = 2$, since Γ has C^3 smoothness, there exists $\pi : \Gamma_h \to \Gamma$ satisfying $|n \circ \pi(m_s) - n_h(m_s)| \le Ch^2$ is obvious. In view of $\tilde{u}_n = 0$ on Γ, we have

$$\begin{aligned}
&|(I_h\tilde{u} \cdot n_h)(m_s)| \\
\le& |(I_h\tilde{u} \cdot n_h)(m_s) - I_h\tilde{u}(m_s) \cdot n \circ \pi(m_s)| \\
&+ |I_h\tilde{u}(m_s) \cdot n \circ \pi(m_s) - (\tilde{u}_n) \circ \pi(m_s)| \\
\le& Ch^2 \|\tilde{u}\|_{W^{1,\infty}(\tilde{\Omega})} + Ch^2 \|\tilde{u}\|_{W^{2,\infty}(\tilde{\Omega})}.
\end{aligned}$$

(ii) It follows from (5.2) and the fact $\tilde{u}_n = 0$ on Γ. □

Theorem 5.3 *Let (u, p) and (u_h, p_h) be solutions of (1.1) and (5.13), respectively. We assume $f \in L^2(\Omega)$, $(u, p) \in W^{2,\infty}(\Omega)^d \times W^{1,\infty}(\Omega)$, and $\alpha_1^2 > \|\tilde{f}\|_{\Omega_h}$. We also assume $(\tilde{u}, \tilde{p})$, the extension of (u, p), satisfy the condition of Lemma 5.6, then we have*

$$\|\tilde{u} - u_h\|_{1,\Omega_h} + \|\tilde{p} - p_h\|_{\Omega_h} \le C(h + \sqrt{\epsilon} + h^2/\sqrt{\epsilon}), \quad \text{for } d = 2, \tag{5.23}$$

$$\|\tilde{u} - u_h\|_{1,\Omega_h} + \|\tilde{p} - p_h\|_{\Omega_h} \le C(\sqrt{h} + \sqrt{\epsilon} + h/\sqrt{\epsilon}), \quad \text{for } d = 3. \tag{5.24}$$

Proof In views of the proof of Theorem 5.2, the only difference here is the estimate of $-c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h)$ in (5.22). We have, noticing that $v_h = I_h\tilde{u}$,

$$\begin{aligned}
&- c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h) + \epsilon c_h(\tilde{\lambda} - \lambda_h, \tilde{\lambda} - \lambda_h) \\
=& \epsilon c_h(\tilde{\lambda}, \tilde{\lambda} - \lambda_h) - c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h) \\
\le& -\frac{\epsilon}{2}\|\tilde{\lambda} - \lambda_h\|_{c_h}^2 + C\epsilon\|\tilde{\lambda}\|_{c_h}^2 + C\frac{1}{\epsilon}\|I_h\tilde{u} \cdot n_h\|_{\infty,\Gamma_h}^2.
\end{aligned} \tag{5.25}$$

The error estimates (5.23) and (5.24) follow from Lemma 5.6. □

Remark 5.1 For $d = 2$, from the error estimates (5.19) and (5.23), we conclude the optimal choices of ϵ and h:

(1) Fine-integration scheme: $\epsilon \simeq h$, and the error estimate is $O(\sqrt{h})$;
(2) Lower-order-integration scheme: $\epsilon \simeq h^2$, and the error estimate is $O(h)$.

And we notice that for fine-integration, if $\epsilon \ll h$, then the scheme is not convergence, which is observed by our numerical experiments(Sect. 6). For $d = 3$, we choose $\epsilon \simeq h$, and the error estimate is $O(\sqrt{h})$.

6 Penalty method for Navier-Stokes equations: numerical experiments

Set $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$. We consider the equation (1.1) with exact solution $u = (10x^3y^2, -10x^2y^3)^T$, $p = 10x^2y^2$.

$$\|u\|_\Omega \simeq 1.11, \quad \|u\|_{1,\Omega} \simeq 6.88.$$

Here $\tau_T(u) \neq 0$, therefore we add $\int_\Gamma \tau_T(u)v_T ds$ to the RHS of variational forms (3.3),(3.17), and $\int_{\Gamma_h} \tau_T(u)v_{hT} ds$ to (5.13).

Newton's method is applied to solve the nonlinear equation(see Sect. 3.2.1(ii)). We test two penalty schemes (3.3),(3.17) for both $P1/P1$ with stabilization $h^2(\nabla p_h, \nabla q_h)_{\Omega_h}$ and $P1b/P1$ elements. The error results in our numerical experiments for (3.3) or (3.17) with $P1/P1$ or $P1b/P1$ are all very similar, therefore we only present one of them((3.17) with $P1b/P1$) in the following. But we focus on the difference of error estimates between two implement methods of penalty term(fine-integration scheme (5.7) and lower-order-integration scheme (5.8)), with different choices of ϵ and h($\epsilon \simeq h$ and $\epsilon \simeq h^2$).

From Figure 6.1 and 6.2, the numerical experiments show the H^1 norm error $\|u - u_h\|_{1,\Omega_h}$ is $O(h)$ for both fine and lower-order integration schemes((5.7) and (5.8)). The L^2 norm error $\|u - u_h\|_{\Omega_h}$ seems to be $O(h^2)$ for lower integration scheme with $\epsilon \simeq h^2$. However, the fine-integration fails when $\epsilon \simeq h^2$(or $\epsilon \ll h$), which coincides with our error estimates(Theorem 5.2). An interesting found is that, even for $\epsilon \ll h^2$(we made the numerical tests but do not show here), the lower-order-integration scheme converges well. Hence, the lower-order-integration scheme has well stability for tiny ϵ. (The numerical experiments are implemented with software FeniCS(cf.[20])).

Notice: In Figure 6.2, line $\epsilon \sim h^2, \|\cdot\|_{L^2}$ overlaps with line $y = 2x$; and line $\epsilon \sim h^2, \|\cdot\|_{H^1}$ overlaps with line $\epsilon \sim h, \|\cdot\|_{H^1}$.

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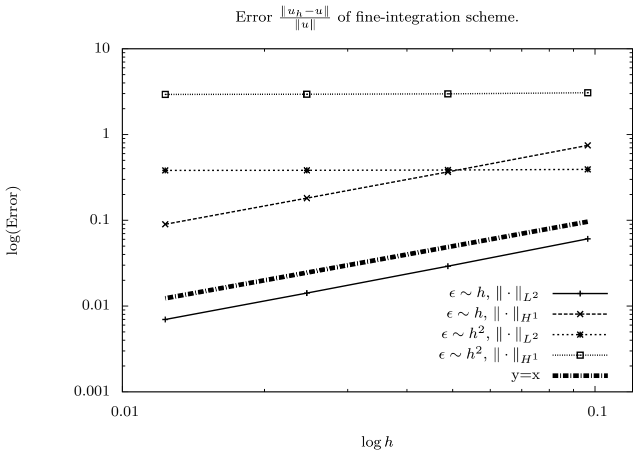

Fig. 6.1 penalty scheme (3.17): fine-integration (5.7)

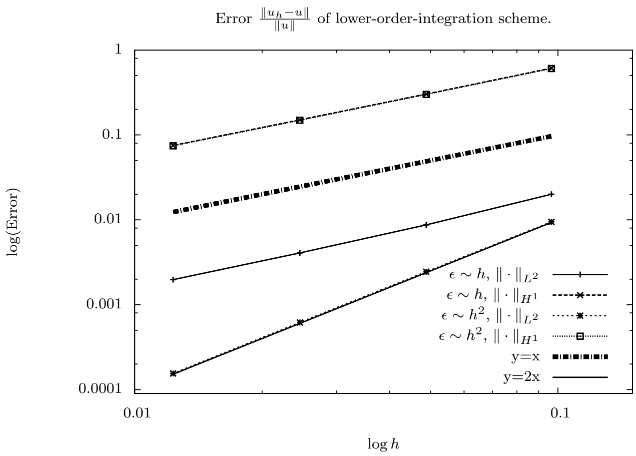

Fig. 6.2 penalty scheme (3.17): lowe-order-integration (5.8)

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