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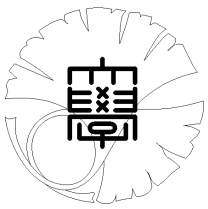

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On cell problems for Hamilton-Jacobi equations with non-coercive Hamiltonians and its application to homogenization problems

Nao Hamamuki[*], Atsushi Nakayasu[†], Tokinaga Namba[†]

Abstract

We study a cell problem arising in homogenization for a Hamilton-Jacobi equation whose Hamiltonian is not coercive. We introduce a generalized notion of effective Hamiltonians by approximating the equation and characterize the solvability of the cell problem in terms of the generalized effective Hamiltonian. Under some sufficient conditions, the result is applied to the associated homogenization problem. We also show that homogenization for non-coercive equations fails in general.

Keywords: Cell Problem, Homogenization, Hamilton-Jacobi Equation, Non-coercive Hamiltonian, Viscosity Solution, Faceted Crystal Growth, Generalized Effective Hamiltonian, Solvability Set

2010 MSC: 35F21, 35D40, 35B27, 49L25, 74N05

1 Introduction

We consider a Hamilton-Jacobi equation of the form

$$H(x, Du(x) + P) = a \quad \text{in } \mathbf{T}^N \tag{CP}$$

and study a problem to find, for a given $P \in \mathbf{R}^N$, a pair of a function $u\colon \mathbf{T}^N \to \mathbf{R}$ and a constant $a \in \mathbf{R}$ such that u is a Lipschitz continuous viscosity solution of (CP). Here, $\mathbf{T}^N := \mathbf{R}^N/\mathbf{Z}^N$ and a function u on $\mathbf{T}^N$ is regarded as a function defined on $\mathbf{R}^N$ with $\mathbf{Z}^N$-periodicity, i.e., $u(x+z) = u(x)$ for all $x \in \mathbf{R}^N$ and $z \in \mathbf{Z}^N$. Moreover, Du denotes the gradient, i.e., $Du = (\partial u/\partial x_1, \cdots, \partial u/\partial x_N)$. This kind of problem is called a *cell problem* in the theory of homogenization. The constant a satisfying (CP) is called a *critical value* if it is uniquely determined.

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In this paper, we assume that the Hamiltonian $H : \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ in (CP) is given by

$$H(x,p) = \sigma(x)m(|p|), \tag{1.1}$$

where σ and m satisfy

(H1) $\sigma\colon \mathbf{T}^N \to (0,\infty)$ is a continuous function,

(H2) $m\colon [0,\infty) \to (0,1)$ is a Lipschitz continuous function,

(H3) m is strictly increasing and $m(r) \to 1$ as $r \to \infty$.

Due to the boundedness of m, our cell problem does not necessarily admit a solution (u,c), and the solvability depends on $P \in \mathbf{R}^N$. One of goals in this paper is to characterize the set of $P \in \mathbf{R}^N$ such that the cell problem admits a solution. The other goal is to apply the result to the associated homogenization problem.

A result for existence of a solution of cell problems for Hamilton-Jacobi equations was first established by Lions, Papanicolaou and Varadhan [19] under the assumption that the Hamiltonian is *coercive*, i.e.,

$$\lim_{r\to\infty} \inf\{H(x,p) \mid x \in \mathbf{T}^N, p \in \mathbf{R}^N,\ |p| \geq r\} = +\infty. \tag{1.2}$$

Their method begins with considering the following approximate equation with a parameter $\delta > 0$:

$$\delta u_\delta(x) + H(x, Du_\delta(x) + P) = 0 \quad \text{in } \mathbf{T}^N. \tag{1.3}$$

By a standard argument of viscosity solutions, it turns out that there exists a unique solution u_δ and that a family of functions $\{\delta u_\delta\}_{\delta>0}$ is uniformly bounded. Thus, (formally) $\{Du_\delta\}_{\delta>0}$ is uniformly bounded thanks to the coercivity. Therefore, by taking a subsequence if necessary, δu_δ and $u_\delta - \min u_\delta$ uniformly converge to a constant $-c$ and a function u as $\delta \to 0$, respectively. A stability argument of viscosity solutions shows that u and c solve (CP). For more details, see [19] and [13]. We point out that the paper [13] also studies second order uniformly elliptic equations by using a similar argument.

Unfortunately, our Hamiltonian (1.1) is not coercive because of the boundedness of the function m. When a Hamiltonian is not coercive, the method of [19] becomes very delicate. Cardaliaguet [8] shows, in fact, that δu_δ may not converge to a constant; this result does not cover our setting. We also refer the reader to [2] as a related work to [8]. Homogenization results with non-coercive Hamiltonians can be seen in [4, 5, 6, 9, 10, 17, 21]. Hamiltonians with some partial coercivity is studied in [4], and [5] treats equations with u/ε-term. The papers [6, 21, 17] are concerned with homogenization on spaces with a (sub-Riemannian) geometrical condition. The authors of [9] study moving interfaces with a sign changing driving force term while [10] considers G-equations being possibly non-coercive. Homogenization for degenerate second order equations has been developed by [1, 7]. Our Hamiltonian (1.1) has not been treated yet in the context of homogenization.

We now present our main results and briefly explain our approach for the non-coercive Hamilton-Jacobi equation (CP). Let us consider an approximate equation of the form

$$(\mathrm{CP}_n) \qquad H_n(x, Du_n(x) + P) = \bar{H}_n(P) \quad \text{in } \mathbf{T}^N$$

for each $n \in \mathbf{N}$. Here $\{H_n\}_{n\in\mathbf{N}}$ is a family of coercive Hamiltonians which approximate H. For the detailed assumptions, see (A1)–(A4) in Section 3. By the coercivity of H_n, the result of [19] ensures that, for each $n \in \mathbf{N}$, the approximate equation has a solution $(u_n, \bar{H}_n(P))$ for every $P \in \mathbf{R}^N$. The function $\bar{H}_n(\cdot)$ is called an *effective Hamiltonian*, which appears in a limit equation in homogenization problems (see [19]). Our first main result is that, for each $P \in \mathbf{R}^N$, there exists a limit $\bar{H}_\infty(P)$ of $\bar{H}_n(P)$ as $n \to \infty$ and its value is independent of approximation (Theorem 3.1). In this paper we call $\bar{H}_\infty(\cdot)$ a *generalized effective Hamiltonian*, which is defined on the whole of $\mathbf{R}^N$ even if (CP) is not solvable for some $P \in \mathbf{R}^N$. We now define the *solvability set* $\mathcal{D}$ as the set of $P \in \mathbf{R}^N$ such that (CP) admits a solution. Our second main result is a characterization of $\mathcal{D}$ in terms of the generalized effective Hamiltonian. We prove that $\mathcal{D} = \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < \underline{\sigma}\}$, where $\underline{\sigma} := \min_{x\in\mathbf{T}^N} \sigma(x)$, and that $\bar{H}_\infty(P)$ is equal to the critical value of (CP) (Theorem 3.2). In the one-dimensional case, it turns out that $\mathcal{D}$ has a more explicit representation (Proposition 4.1).

We next present our homogenization results. Let u^ε be a viscosity solution of

$$(\mathrm{HJ}_\varepsilon) \qquad \begin{cases} u^\varepsilon_t(x,t) + H\left(\dfrac{x}{\varepsilon}, Du^\varepsilon(x,t)\right) = 0 & \text{in } \mathbf{R}^N \times (0,T), \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \mathbf{R}^N. \end{cases}$$

Here, $\varepsilon > 0$ is a parameter and $u_0 : \mathbf{R}^N \to \mathbf{R}$ is a bounded and Lipschitz continuous initial datum. In our homogenization result (Theorem 5.1) we assume either

$$(1) \quad \mathcal{D} = \mathbf{R}^N \quad \text{or} \quad (2) \quad m(\mathrm{Lip}[u_0]) < \underline{\sigma}/\overline{\sigma},$$

where $\overline{\sigma} := \max_{x\in\mathbf{T}^N} \sigma(x)$ and $\mathrm{Lip}[u_0]$ stands for the Lipschitz constant of u_0. Then, we prove that u^ε converges to the solution u of the following problem locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$:

$$(\mathrm{HJ}) \qquad \begin{cases} u_t(x,t) + \bar{H}_\infty(Du(x,t)) = 0 & \text{in } \mathbf{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N. \end{cases}$$

The assumption (1) guarantees that the cell problem is solvable for every $P \in \mathbf{R}^N$. The proof is given by the half-relaxed limit method and the perturbed test function method provided by Evans [13]. The assumption (2) is a sufficient condition that $\{u^\varepsilon\}_{\varepsilon>0}$ is equi-Lipschitz continuous. Since the cell problem may not have a solution for some $P \in \mathbf{R}^N$, we are not able to apply the perturbed test function method directly. We prove the homogenization result by reducing the original equation (CP) to the approximate equation (CP_n) with a coercive Hamiltonian by using the equi-Lipschitz continuity of $\{u^\varepsilon\}_{\varepsilon>0}$. We also show that,

under the condition $\overline{\sigma}m(0) > \underline{\sigma}$, the solutions u^ε do not converge to any function locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$ (Theorem 5.4).

Our non-coercive Hamiltonian (1.1) is originally derived by Yokoyama, Giga and Rybka [22] to study the morphological stability of a faceted crystal. Two functions σ and m represent the rate of supply of molecules and the dimensionless kinetic coefficient, respectively. In [15] and [16] the authors study the large time behavior of a viscosity solution of such non-coercive Hamilton-Jacobi equations.

We conclude this section with the physical explanation of the above homogenization problem and its result. In this problem, we find an average growth of the faceted crystal with a (microscopic) heterogeneous supply of molecules. As we will mention in Subsection 3.3, the cell problem does not have a solution under the condition $\overline{\sigma}m(0) \geq \underline{\sigma}$. Thus, both the assumptions (1) and (2) imply $\overline{\sigma}m(0) < \underline{\sigma}$. This inequality means that the heterogeneity of the supply of molecules is somewhat small. In this case the growth of the faceted crystal is described by (HJ) in view of Theorem 5.1. We point out that the condition $\overline{\sigma}m(0) < \underline{\sigma}$ also appears in [16] to ensure the large time behaviour in the whole space. On the other hand, if $\overline{\sigma}m(0) > \underline{\sigma}$, i.e., the heterogeneity is somewhat large, then the growth of the faceted crystal becomes complicated since homogenization fails (Theorem 5.4).

This paper is organized as follows. Section 2 is devoted to preparation for the viscosity solutions and the critical values. We study the cell problem in Section 3 and 4. In Section 3, we present main theorems and prove them. We also give a sufficient condition for $\mathcal{D} = \mathbf{R}^N$ and some properties of generalized effective Hamiltonians. In Section 4, we focus on the one-dimensional cell problem and give a more explicit representation of $\mathcal{D}$. Section 5 is concerned with an application to homogenization problems.

2 Preliminaries

Let $\mathrm{Lip}(\mathbf{T}^N)$ denote the set of Lipschitz continuous functions on $\mathbf{T}^N$ and $\overline{B(x,r)}$ denote the closure of an open ball $B(x,r)$ of radius $r > 0$ centered at a point x.

We consider Hamilton-Jacobi equations of the form

$$H(x, Du(x)) = 0 \quad \text{in } \mathbf{T}^N. \tag{2.1}$$

Throughout this paper, we assume that the Hamiltonian $H\colon \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ is continuous. In order to define viscosity solutions of (2.1), we recall notions of super- and subdifferentials. For a continuous function $u : \mathbf{T}^N \to \mathbf{R}$ and $x \in \mathbf{T}^N$, we set

$$D^+u(x) := \left\{ D\phi(x) \;\middle|\; \phi \in C^1(\mathbf{T}^N), \quad \max_{\mathbf{T}^N}(u-\phi) = (u-\phi)(x) \right\},$$
$$D^-u(x) := \left\{ D\phi(x) \;\middle|\; \phi \in C^1(\mathbf{T}^N), \quad \min_{\mathbf{T}^N}(u-\phi) = (u-\phi)(x) \right\}.$$

Definition 2.1. We call $u \in C(\mathbf{T}^N)$ a *viscosity subsolution* (resp. *supersolution*) of (2.1) if

$$H(\hat{x}, p) \leq 0 \quad (\text{resp. } H(\hat{x}, p) \geq 0)$$

for all $\hat{x} \in \mathbf{T}^N$ and $p \in D^+u(\hat{x})$ (resp. $p \in D^-u(\hat{x})$).

We call $u \in C(\mathbf{T}^N)$ a *viscosity solution* of (2.1) if it is a viscosity sub- and supersolution of (2.1).

The term "viscosity" is often omitted in this paper.

A pair of a function $u \in \mathrm{Lip}(\mathbf{T}^N)$ and a constant $a \in \mathbf{R}$ satisfying (CP) is called a solution of (CP). If such a constant a is unique, it is called the *critical value* of (CP). If there exists a critical value of the cell problem for every $P \in \mathbf{R}^N$, then we say that the cell problem is *fully solvable*. When the cell problem is fully solvable, we are able to define a function $\bar{H} : \mathbf{R}^N \to \mathbf{R}$ by setting $\bar{H}(P)$ as the associated critical value. We call the function $\bar{H}$ an *effective Hamiltonian* of H.

Proposition 2.2 (Comparison principle for the cell problem)**.** *Let $P \in \mathbf{R}^N$ and let $a, b \in \mathbf{R}$. If there exist a subsolution $u \in \mathrm{Lip}(\mathbf{T}^N)$ of* (CP) *and a supersolution $v \in \mathrm{Lip}(\mathbf{T}^N)$ of $H(x, Dv(x) + P) = b$ in $\mathbf{T}^N$, then $a \geq b$. In particular, if $(u, c), (v, d) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}$ are solutions of the cell problem* (CP)*, then $c = d$ and moreover*

$$\begin{aligned} c &= \inf\{a \in \mathbf{R} \mid \textit{there exists a subsolution of } (\mathrm{CP})\} \\ &= \sup\{a \in \mathbf{R} \mid \textit{there exists a supersolution of } (\mathrm{CP})\}. \end{aligned}$$

The proof is based on the comparison principle for (1.3) with a small $\delta > 0$; see [19, 13]. Here we do not need an extra continuity assumption on H since u and v are now Lipschitz continuous.

Proof. Since u and v is bounded, we may assume that $u > v$ by adding a positive constant to u if necessary. Suppose by contradiction that $a < b$, i.e.,

$$H(x, Du + P) \leq a < b \leq H(x, Dv + P)$$

in the viscosity sense. We then see that

$$\delta u + H(x, Du + P) \leq \frac{a+b}{2} \leq \delta v + H(x, Dv + P).$$

The comparison principle implies that $u \leq v$, which contradicts to $u > v$. Therefore, $a \geq b$.

This observation implies

$$\begin{aligned} &\inf\{a \in \mathbf{R} \mid \text{there exists a subsolution of } (\mathrm{CP})\} =: \overline{c} \\ &\quad \geq \sup\{a \in \mathbf{R} \mid \text{there exists a supersolution of } (\mathrm{CP})\} =: \underline{c}. \end{aligned}$$

We then see that $c = d = \overline{c} = \underline{c}$ since $\overline{c} \leq c \leq \underline{c}$ and $\overline{c} \leq d \leq \underline{c}$ by the definitions. □

Lemma 2.3 (Estimates of the critical value)**.** *Let $P \in \mathbf{R}^N$ and let $(u, c) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}$ be a solution of the cell problem* (CP)*. Then, we have*

$$\left.\begin{array}{l} \displaystyle\sup_{\phi \in C^1(\mathbf{T}^N)} \inf_{x \in \mathbf{T}^N} H(x, D\phi(x) + P) \\ \displaystyle\sup_{x \in \mathbf{T}^N} \sup_{p \in D^+u(x)} H(x, p + P) \end{array}\right\} \leq c \leq \left\{\begin{array}{l} \displaystyle\inf_{\phi \in C^1(\mathbf{T}^N)} \sup_{x \in \mathbf{T}^N} H(x, D\phi(x) + P), \\ \displaystyle\inf_{x \in \mathbf{T}^N} \inf_{p \in D^-u(x)} H(x, p + P). \end{array}\right.$$

Proof. The inequality $c \le \inf_{x\in\mathbf{T}^N} \inf_{p\in D^- u(x)} H(x, p+P)$ is trivial since it is equivalent to the definition of a viscosity supersolution of (CP). Similarly, the inequality $\sup_{x\in\mathbf{T}^N} \sup_{p\in D^+ u(x)} H(x, p+P) \le c$ holds since it is equivalent to the definition of a viscosity subsolution of (CP).

For a fixed $\phi \in C^1(\mathbf{T}^N)$, since $u - \phi$ is periodic and (Lipschitz) continuous, we have $D\phi(\hat{x}) \in D^- u(\hat{x})$ at a minimum point $\hat{x} \in \mathbf{T}^N$ of $u - \phi$. Thus

$$\sup_{x\in\mathbf{T}^N} H(x, D\phi(x) + P) \ge H(\hat{x}, D\phi(\hat{x}) + P) \ge c,$$

which implies that $c \le \inf_{\phi\in C^1(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} H(x, D\phi(x) + P)$. In a similar way, we see that $\sup_{\phi\in C^1(\mathbf{T}^N)} \inf_{x\in\mathbf{T}^N} H(x, D\phi(x) + P) \le c$ by choosing a maximum point of $u - \phi$. □

Remark 2.4. It is worth to note that if the Hamiltonian $H = H(x, p)$ is convex in p for each $x \in \mathbf{T}^N$ and satisfies the coercivity condition (1.2), then

$$\inf_{\phi\in \mathrm{Lip}(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} \sup_{p\in D^+\phi(x)} H(x, p + P) = \inf_{\phi\in C^1(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} H(x, D\phi(x) + P).$$

In particular, we have well-known formulas

$$\begin{aligned} c &= \inf_{\phi\in C^1(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} H(x, D\phi(x) + P) \\ &= \inf_{\phi\in \mathrm{Lip}(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} \sup_{p\in D^+\phi(x)} H(x, p + P) \\ &= \sup_{x\in\mathbf{T}^N} \sup_{p\in D^+ u(x)} H(x, p + P). \end{aligned}$$

We refer the reader to [11] or [20, Subsection 4.2] for details on such a kind of representation formulas of the critical value.

We investigate the cell problem with a coercive Hamiltonian.

Proposition 2.5 ([19])**.** *Assume* (1.2)*. Then, the cell problem* (CP) *is fully solvable.*

Proposition 2.6 (Properties of the effective Hamiltonian)**.** *Assume* (1.2)*.*

(1) If there exists $L > 0$ such that $|H(x, p) - H(x, q)| \le L|p - q|$ for all $x \in \mathbf{T}^N$, $p, q \in \mathbf{R}^N$, then $\bar{H}$ satisfies $|\bar{H}(P) - \bar{H}(Q)| \le L|P - Q|$ for all $P, Q \in \mathbf{R}^N$.

(2) If $H(x, p) \le H(x, kp)$ for all $x \in \mathbf{T}^N$, $p \in \mathbf{R}^N$ and $k \ge 1$, then $\bar{H}(P) \le \bar{H}(kP)$ for all $P \in \mathbf{R}^N$ and $k \ge 1$.

(3) If $H(x, p) = H(x, -p)$ for all $x \in \mathbf{T}^N$ and $p \in \mathbf{R}^N$, then $\bar{H}(P) = \bar{H}(-P)$ for all $P \in \mathbf{R}^N$.

Proof. (1) Let $(u, \bar{H}(P))$ be a solution of (CP). We observe

$$H(x, Du + Q) - L|P - Q| \le H(x, Du + P) = \bar{H}(P).$$

Thus, u is a subsolution of

$$H(x, Du + Q) = \bar{H}(P) + L|P - Q|$$

By Proposition 2.2, we obtain that $\bar{H}(Q) \le \bar{H}(P) + L|P - Q|$.

(2) Let $(u, \bar{H}(P))$ be a solution of (CP). We then see by the assumption that

$$H(x, D(ku) + kP) \ge H(x, Du + P) = \bar{H}(P),$$

which means that ku is a supersolution of $H(x, Dv + kP) = \bar{H}(P)$. Proposition 2.2 implies $\bar{H}(kP) \ge \bar{H}(P)$.

(3) Let $(u, \bar{H}(P))$ be a solution of (CP). Then, since H is even in the second variable, $(-u, \bar{H}(P))$ is a solution of

$$H(x, Dv - P) = \bar{H}(P) \quad \text{in } \mathbf{T}^N.$$

Thus, we have $\bar{H}(P) = \bar{H}(-P)$. □

3 The cell problem

From now on, we study a Hamiltonian H of the form (1.1) with (H1)–(H3). Define

$$\overline{\sigma} := \sup_{x \in \mathbf{T}^N} \sigma(x), \quad \underline{\sigma} := \inf_{x \in \mathbf{T}^N} \sigma(x), \quad m_0 := m(0).$$

3.1 Main results

For each $n \in \mathbf{N}$ let $H_n : \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ be an approximating Hamiltonian of H such that

(A1) H_n is continuous on $\mathbf{T}^N \times \mathbf{R}^N$,

(A2) H_n satisfies the coercivity condition (1.2),

(A3) $\liminf_{n\to\infty} \inf_{\mathbf{T}^N \times \overline{B(0,R)}} (H - H_n) \ge 0$ for all $R > 0$,

(A4) $\limsup_{n\to\infty} \sup_{\mathbf{T}^N \times \mathbf{R}^N} (H - H_n) \le 0$.

By (A1) and (A2), for each $n \in \mathbf{N}$, the approximation cell problem (CP_n) is fully solvable as noted in Proposition 2.5. Let $\bar{H}_n(P)$ be the critical value of (CP_n) for $P \in \mathbf{R}^N$. We define a solvability set $\mathcal{D}$ by

$$\mathcal{D} := \{P \in \mathbf{R}^N \mid \text{(CP) admits a solution } (u, c) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}\}.$$

We are now in a position to state our main theorems.

Theorem 3.1 (Convergence of $\bar{H}_n$)**.** *There exists a unique function $\bar{H}_\infty : \mathbf{R}^N \to \mathbf{R}$ such that, for any sequence $\{H_n\}_{n\in\mathbf{N}}$ satisfying (A1)–(A4), the following conditions hold:*

$$\liminf_{n\to\infty} \inf_{\overline{B(0,R)}} (\bar{H}_\infty - \bar{H}_n) \geq 0 \quad \textit{for all } R > 0, \tag{3.1}$$

$$\limsup_{n\to\infty} \sup_{\mathbf{R}^N} (\bar{H}_\infty - \bar{H}_n) \leq 0. \tag{3.2}$$

We call the function $\bar{H}_\infty$ a *generalized effective Hamiltonian* of H.

Theorem 3.2 (Characterization of the solvability set)**.** *We have $\mathcal{D} = \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < \underline{\sigma}\}$. Moreover, if $P \in \mathcal{D}$, the critical value of* (CP) *is equal to $\bar{H}_\infty(P)$.*

3.2 The proof of Theorem 3.1

The proof consists of four steps. We first prove in Step 1 that $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ is a convergent sequence for every $P \in \mathbf{R}^N$. Then it is shown in Step 2 that the limit is unique no matter how $\{H_n\}_{n\in\mathbf{N}}$ satisfying (A1)–(A4) is chosen. In Step 3 we prove that the convergence is locally uniform when $\{H_n\}_{n\in\mathbf{N}}$ is monotone, and finally, in Step 4, we derive (3.1) and (3.2) for a general approximation.

1. Fix any $P \in \mathbf{R}^N$ and let $(u_n, \bar{H}_n(P)) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}$ be a solution of (CP_n) for each $n \in \mathbf{N}$. We first show that $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ is bounded from below. Indeed, taking a maximum point $x_n \in \mathbf{T}^N$ of u_n, we have

$$H_n(x_n, P) \leq \bar{H}_n(P).$$

Since H_n uniformly converges to H on $\mathbf{T}^N \times \overline{B(0, |P|)}$, we see that

$$H(x_n, P) - 1 \leq \bar{H}_n(P)$$

for sufficiently large n. Thus

$$\left(\inf_{x\in\mathbf{T}^N} H(x, P)\right) - 1 \leq \bar{H}_n(P),$$

which implies $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ is bounded from below.

Fix $\varepsilon > 0$. By (A4) there exists some $K \in \mathbf{N}$ such that

$$H - \frac{\varepsilon}{2} \leq H_n \quad \text{on } \mathbf{T}^N \times \mathbf{R}^N \tag{3.3}$$

for all $n \geq K$. Fix an arbitrary $n \geq K$. Recall that u_n is a Lipschitz continuous function and set $L_n = |P| + \mathrm{Lip}[u_n]$. Then, it follows from (A3) that there exists some $M \geq n$ such that

$$H_m - \frac{\varepsilon}{2} \leq H \quad \text{on } \mathbf{T}^N \times \overline{B(0, L_n)} \tag{3.4}$$

for all $m \geq M$. Combining (3.3) and (3.4), we see that u_n is a subsolution of

$$H_m(x, Dw + P) = \bar{H}_n(P) + \varepsilon \quad \text{in } \mathbf{T}^N.$$

By Proposition 2.2, we have

$$\bar{H}_m(P) \le \bar{H}_n(P) + \varepsilon \tag{3.5}$$

for all $m \ge M$. This inequality implies that $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ is bounded from above. By taking $\limsup_{m\to\infty}$ and $\liminf_{n\to\infty}$, we have

$$\limsup_{m\to\infty} \bar{H}_m \le \liminf_{n\to\infty} \bar{H}_n + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\bar{H}_n(P)$ converges to some value as $n \to \infty$.

2. We next prove that the limit of $\bar{H}_n(P)$ is independent of a choice of $\{H_n\}_{n\in\mathbf{N}}$ satisfying (A1)–(A4). Let $\{H_n\}_{n\in\mathbf{N}}$ and $\{H'_n\}_{n\in\mathbf{N}}$ be two sequences of Hamiltonians satisfying (A1)–(A4). For each $P \in \mathbf{R}^N$, let $(u_n, \bar{H}_n(P))$ and $(u'_n, \bar{H}'_n(P))$ be, respectively, solutions of (CP_n) and

$$H'_n(x, Du'_n + P) = \bar{H}'_n(P) \quad \text{in } \mathbf{T}^N.$$

Consider a new sequence

$$H_1, H'_1, H_2, H'_2, H_3, H'_3, \cdots .$$

This satisfies (A3) and (A4), so that

$$\bar{H}_1(P), \bar{H}'_1(P), \bar{H}_2(P), \bar{H}'_2(P), \bar{H}_3(P), \bar{H}'_3(P), \cdots$$

has a limit $a \in \mathbf{R}$. Therefore

$$a = \lim_{n\to\infty} \bar{H}_n(P) = \lim_{n\to\infty} \bar{H}'_n(P)$$

since both $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ and $\{\bar{H}'_n(P)\}_{n\in\mathbf{N}}$ are subsequences. We denote this common limit by $\bar{H}_\infty(P)$.

3. Assume that $\{H_n\}_{n\in\mathbf{N}}$ is monotone, i.e., $H_n \ge H_{n'}$ on $\mathbf{T}^N \times \mathbf{R}^N$ for all $n \le n'$. By this monotonicity we see that $\bar{H}_n \ge \bar{H}_{n'}$ if $n \le n'$. Indeed, a solution u_n of (CP_n) is always a subsolution of

$$H_{n'}(x, Du_n + P) = \bar{H}_n(P).$$

Thus Proposition 2.2 yields $\bar{H}_n(P) \ge \bar{H}_{n'}(P)$. Since $\bar{H}_\infty$ is continuous in view of Proposition 3.11 (1), Dini's lemma implies that $\bar{H}_n$ converges to $\bar{H}_\infty$ locally uniformly in $\mathbf{R}^N$ as $n \to \infty$. (For the proof of Proposition 3.11 (1) we only need a pointwise convergence of $\bar{H}_n$ to $\bar{H}_\infty$ and the uniqueness of $\bar{H}_\infty$.)

4. We shall show (3.1) and (3.2) for a general $\{H_n\}_{n\in\mathbf{N}}$. Sending $m \to \infty$ in (3.5) of Step 1, we obtain

$$\bar{H}_\infty(P) \le \bar{H}_n(P) + \varepsilon.$$

This inequality holds for all $\varepsilon > 0$, $n \ge K$ and $P \in \mathbf{R}^N$, where K does not depend on P. Accordingly we have $\sup_{\mathbf{R}^N}(\bar{H}_\infty - \bar{H}_n) \le \varepsilon$, and thus taking $\limsup_{n\to\infty}$ yields (3.2) since $\varepsilon > 0$ is arbitrary.

To prove (3.1) we define $\{H'_n\}_{n\in\mathbf{N}}$ by $H'_n(x,p) := \sup_{m\ge n} H_m(x,p)$. Then $\{H'_n\}_{n\in\mathbf{N}}$ is monotone and $H'_n \ge H_n$ on $\mathbf{T}^N \times \mathbf{R}^N$ for all n. Also, $\{H'_n\}_{n\in\mathbf{N}}$ satisfies (A1)–(A4); it is

easy to see that (A2)–(A4) hold while the continuity condition (A1) is due to Ascoli-Arzelà theorem. More precisely, since $\{H_n\}$ has a uniform convergent limit H on each compact set of $\mathbf{T}^N \times \mathbf{R}^N$, it is equi-continuous on the set, which implies that the supremum H_n' is continuous. From Step 3 it follows that $\bar{H}_n'$ converges to $\bar{H}_\infty$ locally uniformly. Therefore, using $\bar{H}_n' \geq \bar{H}_n$, we observe

$$\liminf_{n\to\infty} \inf_{\overline{B(0,R)}} (\bar{H}_\infty - \bar{H}_n) \geq \liminf_{n\to\infty} \inf_{\overline{B(0,R)}} (\bar{H}_\infty - \bar{H}_n') = 0.$$

The proof is now complete.

Remark 3.3. Theorem 3.1 still holds for more general Hamiltonians which are not necessarily of the form (1.1). Indeed, the above proof works if we require H to satisfy

$$|H(x,p) - H(x,q)| \leq L|p-q| \quad \text{for some } L > 0, \tag{3.6}$$

which is used to guarantee Proposition 3.11 (1).

3.3 The proof of Theorem 3.2

We first prepare

Proposition 3.4. *Let $P \in \mathcal{D}$ and let $c \in \mathbf{R}$ be the critical value of* (CP)*. Then,*

$$\left.\begin{array}{r} \underline{\sigma} m(|P|) \\ \overline{\sigma} m_0 \end{array}\right\} \leq c \leq \overline{\sigma} m(|P|), \quad c < \underline{\sigma}.$$

In particular, we have $\mathcal{D} = \emptyset$ if $\overline{\sigma} m_0 \geq \underline{\sigma}$.

Proof. Taking $\phi \equiv 0$ in Lemma 2.3 implies

$$\begin{aligned} c &\leq \inf_{\phi \in C^1(\mathbf{T}^N)} \sup_{x \in \mathbf{T}^N} H(x, D\phi(x) + P) \leq \sup_{x \in \mathbf{T}^N} H(x,P) = \overline{\sigma} m(|P|), \\ c &\geq \sup_{\phi \in C^1(\mathbf{T}^N)} \inf_{x \in \mathbf{T}^N} H(x, D\phi(x) + P) \geq \inf_{x \in \mathbf{T}^N} H(x,P) = \underline{\sigma} m(|P|). \end{aligned}$$

We next show $c < \underline{\sigma}$. Take a solution $u \in \mathrm{Lip}(\mathbf{T}^N)$ of (CP). For every $x \in A^- := \{x \in \mathbf{T}^N \mid D^- u(x) \neq \emptyset\}$, take $p \in D^- u(x)$. Since $|p| \leq \mathrm{Lip}[u]$, we have

$$\inf_{p \in D^- u(x)} H(x, p+P) \leq H(x, p+P) \leq \sigma(x) m(\mathrm{Lip}[u] + |P|).$$

Therefore, by Lemma 2.3,

$$\begin{aligned} c &\leq \inf_{x \in \mathbf{T}^N} \inf_{p \in D^- u(x)} H(x, p+P) \leq \inf_{x \in A^-} \inf_{p \in D^- u(x)} H(x, p+P) \\ &\leq \inf_{x \in A^-} \sigma(x) m(\mathrm{Lip}[u] + |P|). \end{aligned}$$

According to [3, Lemma 1.8 (d)], the set A^- is dense in $\mathbf{T}^N$. Thus, we obtain $c \leq \underline{\sigma} m(\mathrm{Lip}[u] + |P|) < \underline{\sigma}$. The proof of the inequality $\overline{\sigma} m_0 \leq c$ is easier. □

Let $\widehat{\mathcal{D}} := \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < \underline{\sigma}\}$. We note that $\mathcal{D} = \widehat{\mathcal{D}} = \emptyset$ when $\overline{\sigma} m_0 \geq \underline{\sigma}$. Indeed, Proposition 3.4 implies $\mathcal{D} = \emptyset$ and Proposition 3.11 (4) (shown later) implies $\widehat{\mathcal{D}} = \emptyset$. We may hereafter assume that $\overline{\sigma} m_0 < \underline{\sigma}$.

Let us take an approximating Hamiltonian H_n of the form

$$H_n(x,p) = \sigma(x) M_n(|p|),$$

where $M_n : [0,\infty) \to [m_0,\infty)$ is an approximating function of m such that

(B1) M_n is Lipschitz continuous,

(B2) $M_n(r) \to \infty$ as $r \to \infty$,

(B3) there exists $\alpha_n \in \mathbf{R}$ such that

$$M_n(r) = m(r) \quad \text{for } r \in [0,\alpha_n], \quad M_n(r) > m(r) \quad \text{for } r \in (\alpha_n,\infty),$$

for each $n \in \mathbf{N}$ and $\alpha_n \to \infty$ as $n \to \infty$,

(B4) $M_n(r) \geq M_{n'}(r)$ for all $n' \geq n$ and $r \in [0,\infty)$.

For instance,

$$M_n(r) = \max\{m(r), Lr - n\} \tag{3.7}$$

satisfies (B1)–(B4), where L is the Lipschitz constant of m.

Proof of $\mathcal{D} \supset \widehat{\mathcal{D}}$. We define

$$\mathcal{D}_\ell := \{P \in \mathbf{R}^N \mid \bar{H}_\ell(P) \leq \underline{\sigma} m(\alpha_\ell)\}.$$

If we prove that $\mathcal{D} \supset \mathcal{D}_\ell$ for every $\ell \in \mathbf{N}$, the conclusion follows since

$$\mathcal{D} \supset \bigcup_{\ell=1}^{\infty} \mathcal{D}_\ell = \widehat{\mathcal{D}}.$$

Thus, we shall prove that $\mathcal{D} \supset \mathcal{D}_\ell$ for each $\ell \in \mathbf{N}$.

Fix any $P \in \mathcal{D}_\ell$ and let $(u_n, \bar{H}_n(P))$ be a solution of (CP_n). Note that $\bar{H}_n(P)$ is monotone decreasing with respect to n by (B4) and Proposition 2.2. For each $n \in \mathbf{N}$ such that $n \geq \ell$,

$$M_n(|Du_n + P|) = \frac{\bar{H}_n(P)}{\sigma(x)} \leq \frac{\bar{H}_\ell(P)}{\underline{\sigma}} \leq m(\alpha_\ell) \quad \text{in } \mathbf{T}^N$$

in the viscosity sense. Note that the last inequality follows from $P \in \mathcal{D}_\ell$. Since $m \leq M_n$ on $[0,\infty)$ and m is strictly increasing, we have

$$|Du_n(x)| \leq \alpha_\ell + |P| \quad \text{in } \mathbf{T}^N$$

in the viscosity sense. Thus,

$$\sup_{n\geq \ell} \mathrm{Lip}[u_n] \leq \alpha_\ell + |P| < \infty.$$

Set $v_n(y) := u_n(y) - \min u_n$. Then, $\{v_n\}_{n\in\mathbf{N}}$ is uniformly bounded and equi-Lipschitz continuous in $\mathbf{T}^N$. Thus, by taking a subsequence if necessary, Ascoli-Arzelà theorem implies that v_n uniformly converges to some Lipschitz continuous function u in $\mathbf{T}^N$ as $n \to \infty$. Since M_n converges to m locally uniformly in $[0,\infty)$ by (B3) of M_n, the stability of viscosity solutions (see [12]) implies that $(u, \bar{H}_\infty(P))$ is a solution of (CP), which means that $P \in \mathcal{D}$. We get the desired inclusion $\mathcal{D} \supset \widehat{\mathcal{D}}$.

Proof of $\mathcal{D} \subset \widehat{\mathcal{D}}$. Fix any $P \in \mathcal{D}$ and let $(u,c) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}$ be a solution of (CP). The condition (B3) of M_n implies

$$M_n(r) = m(r) \quad \text{for all } r \leq \mathrm{Lip}[u] + |P|$$

for sufficiently large n. Hence, (u,c) is a solution of

$$\sigma(x) M_n(|Du + P|) = c \quad \text{in } \mathbf{T}^N.$$

Since $\bar{H}_n(P)$ is the critical value of the above problem, we have $\bar{H}_n(P) = c$. Sending $n \to \infty$ yields $\bar{H}_\infty(P) = c$. Since $c < \underline{\sigma}$ by Proposition 3.4, we have $\bar{H}_\infty(P) < \underline{\sigma}$. Thus, $\mathcal{D} \subset \widehat{\mathcal{D}}$. The proof of Theorem 3.2 is complete.

Remark 3.5. By the last part of the proof, we see that for every $R > 0$ there exists $N_R \in \mathbf{N}$ such that $\bar{H}_\infty(P) = \bar{H}_n(P)$ for all $P \in \overline{B(0,R)}$ and $n \geq N_R$. This is thanks to the conditions (B1)–(B4).

Remark 3.6. By Lemma 2.3 it is easily seen that $\bar{H}_\infty(0) = \overline{\sigma} m_0$. Thus Theorem 3.2 implies that $0 \in \mathcal{D}$ if $\overline{\sigma} m_0 < \underline{\sigma}$. Moreover, from the Lipschitz continuity of $\bar{H}_\infty$ (Proposition 3.11 (1)) it follows that $B(0, (\underline{\sigma} - \overline{\sigma} m_0)/\overline{\sigma} L) \subset \mathcal{D}$, where L is the Lipschitz constant of m.

Remark 3.7. A similar proof applies to more general Hamiltonians. Let H be a Hamiltonian satisfying (3.6). We define

$$h(\rho) := \inf_{x\in\mathbf{T}^N} \inf_{|p|\geq\rho} H(x,p), \quad h_\infty := \sup_{\rho\geq 0} h(\rho),$$

and assume

(H4) $\inf_{x\in\mathbf{T}^N} \sup_{|p|\leq\rho} H(x,p) < h_\infty$ for all $\rho \geq 0$.

Then it turns out that $\mathcal{D} = \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < h_\infty\}$. We shall give a sketch of the proof of this generalization.

We first note that the critical value c of (CP) satisfies $c < h_\infty$. This follows from (H4). Also, we see that $\{p \in \mathbf{R}^N \mid H(x,p) \leq \tau \text{ for some } x \in \mathbf{T}^N\}$ is bounded for every $\tau < h_\infty$. Indeed, if there were some sequence $\{(x_j, p_j)\}_{j\in\mathbf{N}}$ such that $|p_j| \to \infty$ as $j \to \infty$, we would have $h(|p_j|) \leq H(x_j, p_j) \leq \tau < h_\infty$, which is a contradiction since $\sup_{j\in\mathbf{N}} h(|p_j|) < h_\infty$.

Define $\widehat{\mathcal{D}} := \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < h_\infty\}$, and take an approximate Hamiltonian H_n as $H_n(x,p) = \max\{H(x,p), |p| - n\}$. To prove $\mathcal{D} \supset \widehat{\mathcal{D}}$ we set $\mathcal{D}_\ell := \{P \in \mathbf{R}^N \mid \bar{H}_\ell(P) \le \tau_\ell\}$, where $\{\tau_\ell\}_{\ell \in \mathbf{N}}$ is a sequence such that $\tau_\ell < h_\infty$ and $\tau_\ell \to h_\infty$ as $\ell \to \infty$. Then $\bigcup_{\ell=1}^\infty \mathcal{D}_\ell = \widehat{\mathcal{D}}$. Fix $\ell \in \mathbf{N}$. For every $P \in \mathcal{D}_\ell$ and $n \ge l$, a solution $(u_n, \bar{H}_n(P))$ of (CP_n) satisfies

$$H(x, Du_n + P) \le H_n(x, Du_n + P) = \bar{H}_n(P) \le \bar{H}_\ell(P) \le \tau_\ell.$$

Since $\tau_\ell < h_\infty$, we have $\sup_{n \ge \ell} \mathrm{Lip}[u_n] < \infty$. Ascoli-Arzelà theorem ensures that $u_n - \min u$ subsequently converges to some u, and thus $(u, \bar{H}_\infty(P))$ solves (CP). The proof of $\mathcal{D} \subset \widehat{\mathcal{D}}$ is easier. Indeed, by the choice of H_n, a solution (u,c) of (CP) is also a solution of (CP_n) for n sufficiently large, and therefore $\bar{H}_\infty(P) = \bar{H}_n(P) = c < h_\infty$.

3.4 A sufficient condition for the fully solvability

Applying the result in Theorem 3.2, we give a sufficient condition which guarantees that (CP) is fully solvable, i.e., $\mathcal{D} = \mathbf{R}^n$.

Theorem 3.8. *Assume $\overline{\sigma} m_0 < \underline{\sigma}$. Let $P \in \mathbf{R}^N$ and assume that there exists $\psi \in C^1(\mathbf{T}^N)$ such that $D\psi = -P$ on $\{\sigma \ne \underline{\sigma}\}$. Then $P \in \mathcal{D}$.*

If there exists such a ψ for every $P \in \mathbf{R}^N$, then (CP) is fully solvable. A simple condition for the existence of ψ will be given after the proof; see Remark 3.9.

Proof. We take H_n as in the proof of Theorem 3.2. By the representation of $\mathcal{D}$ obtained in Theorem 3.2, the proof is completed by showing that $\bar{H}_n(P) < \underline{\sigma}$ for $n \in \mathbf{N}$ sufficiently large. To this end, we use the estimate

$$\bar{H}_n(P) \le \inf_{\phi \in C^1(\mathbf{T}^N)} \sup_{x \in \mathbf{T}^N} H_n(x, D\phi(x) + P)$$

in Lemma 2.3. Choosing $\phi = \psi$, where ψ is the function in our assumption, we see

$$\bar{H}_n(P) \le \sup_{x \in \mathbf{T}^N} H_n(x, D\psi(x) + P). \tag{3.8}$$

On $\{\sigma \ne \underline{\sigma}\}$ we compute

$$H_n(x, D\psi(x) + P) = H_n(x, 0) = \sigma(x) m_0 \le \overline{\sigma} m_0 < \underline{\sigma}.$$

For $x \in \mathbf{T}^N$ such that $\sigma(x) = \underline{\sigma}$, we have

$$H_n(x, D\psi(x) + P) = \underline{\sigma} M_n(|D\psi(x) + P|).$$

We now set $r_0 := \max_{x \in \mathbf{T}^N} |D\psi(x) + P| < \infty$ and choose n large so that $M_n(r) = m(r)$ for all $r \le r_0$. Then

$$H_n(x, D\psi(x) + P) \le \underline{\sigma} m(r_0) < \underline{\sigma}.$$

Consequently, (3.8) implies $\bar{H}_n(P) < \underline{\sigma}$. □

Remark 3.9. If $\overline{\{\sigma \neq \underline{\sigma}\}} \subset (0,1)^n$, then there exists ψ in Theorem 3.8 for every $P \in \mathbf{R}^N$. Indeed, letting $A \subset (0,1)^n$ be an open set such that $\overline{\{\sigma \neq \underline{\sigma}\}} \subset A$ and $\overline{A} \subset (0,1)^n$, we are able to construct a function $\psi \in C^1(\mathbf{T}^N)$ so that $\psi(x) = -\langle P, x\rangle$ for $x \in \{\sigma \neq \underline{\sigma}\}$ and $\psi(x) = 0$ for $x \notin A$.

Remark 3.10. The existence of ψ in Theorem 3.8 is not a necessary condition for $P \in \mathcal{D}$. In Example 4.4 (1), where we consider the one-dimensional case, the cell problem is fully solvable, but there is no such periodic ψ for $P \neq 0$ because σ attains a minimum at one point.

3.5 Properties of the generalized effective Hamiltonian

In this subsection we shall derive some properties of the generalized effective Hamiltonian.

Proposition 3.11 (Properties of the generalized effective Hamiltonian)**.** *We have*

(1) $|\bar{H}_\infty(P) - \bar{H}_\infty(Q)| \le \overline{\sigma} L|P - Q|$ *for all* $P, Q \in \mathbf{R}^N$*, where* L *is the Lipschitz constant of* m*,*

(2) $\bar{H}_\infty(kP) \ge \bar{H}_\infty(P)$ *for all* $P \in \mathbf{R}^N$ *and* $k \ge 1$*,*

(3) $\bar{H}_\infty(P) = \bar{H}_\infty(-P)$ *for all* $P \in \mathbf{R}^N$*,*

(4) $\max\{\underline{\sigma} m(|P|), \overline{\sigma} m_0\} \le \bar{H}_\infty(P) \le \overline{\sigma} m(|P|)$ *for all* $P \in \mathbf{R}^N$*.*

Proof. Take H_n as in the proof of Theorem 3.2, where we set M_n by (3.7). Let $\bar{H}_n$ be the effective Hamiltonian of H_n. We then have

$$|H_n(x,p) - H_n(x,q)| \le \overline{\sigma} L|p - q| \quad \text{for all } x \in \mathbf{T}^N,\ p, q \in \mathbf{R}^N.$$

Hence, Proposition 2.6 (1) shows

$$|\bar{H}_n(P) - \bar{H}_n(Q)| \le \overline{\sigma} L|P - Q| \quad \text{for all } P, Q \in \mathbf{R}^N.$$

Sending $n \to \infty$ yields the conclusion (1).

By a similar argument the properties (2)–(3) are verified from Proposition 2.6 since our coercive Hamiltonians H_n satisfy the assumptions of Proposition 2.6 (2)–(3). The property (4) is a consequence of Lemma 2.3. □

4 One-dimensional cell problem

In this section we investigate the cell problem in one dimension. In this case the solvability set $\mathcal{D}$ has a more explicit representation. We first rewrite (CP) as

$$|u'(x) + P| = f_a(x) \quad \text{in } \mathbf{T}, \tag{4.1}$$

where

$$f_a(x) := m^{-1}\left(\frac{a}{\sigma(x)}\right).$$

Here, $m^{-1} : [m_0, 1) \to [0, \infty)$ is the inverse function of m, and f_a is well-defined as a $[0, \infty)$-valued function if $\overline{\sigma} m_0 \le a < \underline{\sigma}$. We now set $m^{-1}(1) = \infty$. Then, $f_{\underline{\sigma}}$ is a $[0, \infty]$-valued function. Note that $a \mapsto f_a(x)$ is increasing for every $x \in \mathbf{T}$.

The authors of [19] consider

$$|u'(x) + P|^2 - V(x) = a \quad \text{in } \mathbf{T}, \tag{4.2}$$

as an example of the cell problem in one dimension. Here, V is a continuous function on $\mathbf{T}$ such that $\min_{\mathbf{T}} V = 0$. According to [19], for each $P \in \mathbf{R}$, the critical value c of (4.2) is given by

$$c = \begin{cases} 0 & \text{if } |P| \le \int_0^1 \sqrt{V(z)}dz, \\ a \quad \text{such that } |P| = \int_0^1 \sqrt{V(z) + a}dz,\ a \ge 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

As an analogue of this formula, we establish

Proposition 4.1. *(1) If $\overline{\sigma} m_0 \ge \underline{\sigma}$, then $\mathcal{D} = \emptyset$.*

(2) If $\overline{\sigma} m_0 < \underline{\sigma}$, then

$$\mathcal{D} = \begin{cases} (-\int_0^1 f_{\underline{\sigma}}(z)dz, \int_0^1 f_{\underline{\sigma}}(z)dz) & \text{if } f_{\underline{\sigma}} \in L^1(0,1), \\ \mathbf{R} & \text{otherwise.} \end{cases}$$

Moreover, the critical value c is given by

$$c = \begin{cases} \overline{\sigma} m_0 & \text{if } |P| \le \int_0^1 f_{\overline{\sigma} m_0}(z)dz, \\ a \quad \text{such that } |P| = \int_0^1 f_a(z)dz & \text{otherwise.} \end{cases} \tag{4.4}$$

Proof. (1) This is obvious by Proposition 3.4.

(2) We set $\widetilde{\mathcal{D}} = (-\int_0^1 f_{\underline{\sigma}}(z)dz, \int_0^1 f_{\underline{\sigma}}(z)dz)$. When $f_{\underline{\sigma}} \notin L^1(0,1)$, we read $\widetilde{\mathcal{D}} = \mathbf{R}$. We first prove $\mathcal{D} \supset \widetilde{\mathcal{D}}$. To do this, take $P \in \widetilde{\mathcal{D}}$. What we have to do is to find $u \in \mathrm{Lip}(\mathbf{T})$ such that (u, c) is a solution of (4.1), where c is the constant in (4.4).

When $|P| \le \int_0^1 f_{\overline{\sigma} m_0}(z)dz$, we set

$$u(x) = \begin{cases} \displaystyle\int_{x_0}^{x} f_{\overline{\sigma} m_0}(z)dz - Px & \text{for } x \in [x_0, x_1], \\ \displaystyle\int_{x}^{x_0+1} f_{\overline{\sigma} m_0}(z)dz + P(1-x) & \text{for } x \in [x_1, x_0 + 1]. \end{cases}$$

Here, $x_0 \in [0, 1]$ and $x_1 \in [x_0, x_0 + 1]$ are points such that

$$f_{\overline{\sigma} m_0}(x_0) = 0, \quad \int_{x_0}^{x_1} f_{\overline{\sigma} m_0}(z)dz = \int_{x_1}^{x_0+1} f_{\overline{\sigma} m_0}(z)dz + P.$$

We regard u as a function on $\mathbf{T}$ by extending it periodically. Then, it is easy to see that u is a solution of (4.1).

When $|P| \geq \int_0^1 f_{\overline{\sigma} m_0}(z)dz$, for c chosen by (4.4), we set

$$u(x) = \mathrm{sign}(P) \int_0^x f_c(z)dz - Px \quad \text{for } x \in \mathbf{R}.$$

Note that u is a $\mathbf{Z}$-periodic function since, by the definition of c,

$$\mathrm{sign}(P) \int_0^1 f_c(z)dz - P = 0.$$

Then, it is easy to see that u is a solution of (4.1). Therefore, we have obtained $\mathcal{D} \supset \widetilde{\mathcal{D}}$.

We next show the reverse inclusion $\mathcal{D} \subset \widetilde{\mathcal{D}}$. Let $P \in \mathcal{D}$ and take a solution (u, c) of (4.1), then

$$|P| = \left| \int_0^1 (u'(z) + P)dz \right| \leq \int_0^1 |u'(z) + P|dz \leq \int_0^1 f_c(z)dz < \int_0^1 f_{\underline{\sigma}}(z)dz.$$

The first equality follows from the periodicity of u. Thus, $P \in \widetilde{\mathcal{D}}$ and so the proof is complete. □

Remark 4.2. The representation of the critical value (4.4) is also obtained via the formula (4.3) given in [19]. In fact, a is a critical value of (CP) if and only if the critical value c_a of

$$|u'(x) + P| = f_a(x) + c_a \quad \text{in } \mathbf{T}$$

is equal to 0. It is easily seen that the condition $c_a = 0$ yields (4.4).

When σ attains a minimum on some interval $[a, b]$ with $a < b$, it is easily seen that $f_{\underline{\sigma}}$ is not integrable since $f_{\underline{\sigma}} = +\infty$ on $[a, b]$. Consequently, (4.1) is fully solvable by Proposition 4.1. If $\sigma(x) = \underline{\sigma}$ at only one point $x \in \mathbf{T}$, the integrability of $f_{\underline{\sigma}}$ depends on σ and m as the next examples indicate.

Example 4.3. Let us consider (4.1) with

$$m(r) = \frac{1}{2}\frac{r}{1+r} + \frac{1}{2} \quad (r \in [0, \infty)), \quad \sigma(x) = x^\alpha (1-x)^\alpha + \beta \quad (x \in [0, 1]),$$

where $\alpha, \beta > 0$. We note that $\overline{\sigma} m_0 < \underline{\sigma}$ holds when $\beta > 1/4^\alpha$. Since

$$f_{\underline{\sigma}}(x) = \frac{\underline{\sigma}}{2}\frac{1}{\sigma(x) - \underline{\sigma}} - 1 = \frac{\underline{\sigma}}{2}\frac{1}{x^\alpha (1-x)^\alpha} - 1,$$

the integrability of $f_{\underline{\sigma}}$ is determined by the choice of $\alpha > 0$.

Example 4.4. Set

$$\sigma(x) = \begin{cases} x + \frac{3}{2} & (0 \le x < \frac{1}{2}), \\ -x + \frac{5}{2} & (\frac{1}{2} \le x < 1). \end{cases}$$

(1) We let

$$m(r) = \frac{1}{2}\frac{r}{1+r} + \frac{1}{2}.$$

We extend σ periodically to $\mathbf{R}$ and still denote it by σ. Note that $\overline{\sigma} m_0 < \underline{\sigma}$ holds. Since

$$f_{\underline{\sigma}}(x) = \frac{\underline{\sigma}}{2}\frac{1}{\sigma(x) - \underline{\sigma}} - \frac{1}{2},$$

we observe that

$$\int_0^1 f_{\underline{\sigma}}(z)dz = 2\int_0^{1/2} f_{\underline{\sigma}}(z)dz = \underline{\sigma}\int_0^{1/2} \frac{1}{z}dz - \frac{1}{2} = \infty.$$

Thus, $\mathcal{D} = \mathbf{R}$, i.e., the cell problem is fully solvable.

(2) We next study

$$m(r) = \frac{1}{2}\tanh r + \frac{1}{2} \quad (r \in [0, \infty)).$$

Note that $\overline{\sigma} m_0 < \underline{\sigma}$. Since

$$f_{\underline{\sigma}}(x) = \frac{1}{2}\log\left(\frac{\underline{\sigma}}{\sigma(x) - \underline{\sigma}}\right) = \frac{1}{2}\{\log \underline{\sigma} - \log(\sigma(x) - \underline{\sigma})\},$$

we observe that

$$\int_0^1 f_{\underline{\sigma}}(z)dz = 2\int_0^{1/2} f_{\underline{\sigma}}(z)dz = 1 + \frac{1}{2}\log 6.$$

Therefore, $\mathcal{D} = \left(-1 - \frac{1}{2}\log 6, 1 + \frac{1}{2}\log 6\right)$.

Proposition 4.5. *We have*

$$\bar{H}_\infty(P) > \bar{H}_\infty(Q) \quad \textit{for all } P, Q \in \mathcal{D} \textit{ such that } |P| > |Q| \ge \int_0^1 f_{\overline{\sigma} m_0}(z)dz.$$

Proof. By (4.4), we observe

$$\int_0^1 \{f_{\bar{H}_\infty(P)}(z) - f_{\bar{H}_\infty(Q)}(z)\}dz = |P| - |Q| > 0,$$

which implies that $\bar{H}_\infty(P) > \bar{H}_\infty(Q)$. □

5 Application to homogenization problems

We present our homogenization result for the equation (HJ_ε) with the Hamiltonian (1.1) satisfying (H1)–(H3). Here, $u_0 : \mathbf{R}^N \to \mathbf{R}$ is a bounded and Lipschitz continuous initial datum. We remark that there exists a unique bounded solution $u^\varepsilon \in C(\mathbf{R}^N \times [0,T))$ of (HJ_ε). Similarly, there exists a unique bounded solution $u \in C(\mathbf{R}^N \times [0,T))$ of (HJ). Indeed, the comparison principle holds for a viscosity sub- and supersolution (see [12]). This yields uniqueness of solutions. Existence is a consequence of Perron's method (see [18]).

Theorem 5.1 (Homogenization result)**.** *Assume either*

$$(1)\quad \mathcal{D} = \mathbf{R}^N \quad \text{or} \quad (2)\quad m(\mathrm{Lip}[u_0]) < \underline{\sigma}/\overline{\sigma}.$$

Then the solution u^ε of (HJ_ε) *converges to the solution u of* (HJ) *locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$.*

Recall that, for each $P \in \mathcal{D}$, $\bar{H}_\infty(P)$ is the critical value of (CP) from Theorem 3.2. As we mentioned in Introduction the assumption (1) means that the cell problem is fully solvable, so the same argument as [13] works for our equation.

We shall hereafter prove Theorem 5.1 under the assumption (2).

Proposition 5.2 (Regularity of the solution of (HJ_ε))**.** *Assume (2) in Theorem 5.1. Then, the solutions u^ε of* (HJ_ε) *satisfy*

$$|u^\varepsilon(x,t) - u^\varepsilon(x,s)| \le L|t-s|, \quad |u^\varepsilon(x,t) - u^\varepsilon(y,t)| \le K|x-y|$$

for all $x, y \in \mathbf{R}^N, t, s \in [0,T)$ with the constants

$$L := \overline{\sigma} m(\mathrm{Lip}[u_0]) < \infty, \quad K := m^{-1}\left(\frac{\overline{\sigma}}{\underline{\sigma}} m(\mathrm{Lip}[u_0])\right) < \infty.$$

We omit the proof since this proposition is verified by the same argument as in [14, Appendix A]. We point out that [14, Proposition 3.17] holds under the assumption $R_+(m) < \infty$ even if the Hamiltonian does not satisfy the coercivity condition (H_{R+}).

We give two different proofs of Theorem 5.1 under the assumption (2).

Proof I of Theorem 5.1 under the assumption (2). By Proposition 5.2, Ascoli-Arzelà theorem implies that u^ε subsequently converges to some Lipschitz continuous function u locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$.

We prove that u is a supersolution of (HJ). The proof is based on the perturbed test function method (see [13]). Let $(x_0, t_0) \in \mathbf{R}^N \times (0,T)$ and $\phi \in C^1(\mathbf{R}^N \times (0,T))$ such that $u - \phi$ has a strict local minimum at (x_0, t_0). Suppose that

$$\phi_t(x_0,t_0) + \bar{H}_\infty(D\phi(x_0,t_0)) =: -\theta < 0.$$

We take $\{H_n\}_{n\in\mathbf{N}}$ as in the proof of Theorem 3.2 and let $\bar{H}_n$ be the effective Hamiltonian of (CP_n). Since $\bar{H}_n$ converges to $\bar{H}_\infty$, we have

$$\phi_t(x_0,t_0)+\bar{H}_n(D\phi(x_0,t_0))\le -\frac{\theta}{2} \tag{5.1}$$

for sufficiently large n.

Set

$$\phi_n^\varepsilon(x,t):=\phi(x,t)+\varepsilon v_n\left(\frac{x}{\varepsilon}\right),$$

where v_n is a solution of (CP_n). By the same argument as in [13], we see that ϕ_n^ε is a subsolution of

$$w_t(x,t)+\sigma\left(\frac{x}{\varepsilon}\right)M_n(|Dw(x,t)|)=0 \tag{5.2}$$

in $B(x_0,r)\times(t_0-r,t_0+r)$ for sufficiently small $r>0$. On the other hand, by the Lipschitz continuity of u^ε and (B3), we see that it is a solution of (5.2) in $\mathbf{R}^N\times(0,T)$ for sufficiently large n. The comparison principle for $(\mathrm{HJ}_\varepsilon)$ implies a contradiction (see [13]) and so u is a supersolution of (HJ).

Similarly, it is proved that u is a subsolution of (HJ), and therefore, u is a unique solution of (HJ). Consequently, u^ε converges to u locally uniformly in $\mathbf{R}^N\times[0,T)$ as $\varepsilon\to 0$ without taking subsequences. □

Proof II of Theorem 5.1 under the assumption (2). Recall that $\{u^\varepsilon\}_{\varepsilon>0}$ is equi-Lipschitz continuous in view of Proposition 5.2 and therefore subsequently converges to some u locally uniformly in $\mathbf{R}^N\times[0,T)$ as $\varepsilon\to 0$. Take $\{H_n\}_{n\in\mathbf{N}}$ as in the proof of Theorem 3.2. By (B3) and the equi-Lipschitz continuity of $\{u^\varepsilon\}_{\varepsilon>0}$, we have

$$u_t^\varepsilon(x,t)+H_n\left(\frac{x}{\varepsilon},Du^\varepsilon(x,t)\right)=0\quad\text{in }\mathbf{R}^N\times(0,T)$$

for all $n\in\mathbf{N}$ large enough and all $\varepsilon>0$. We now apply the homogenization result for coercive Hamiltonians [13] to see that u^ε converges to the solution w_n of

$$\begin{cases} w_t(x,t)+\bar{H}_n\left(Dw(x,t)\right)=0 & \text{in }\mathbf{R}^N\times(0,T),\\ w(x,0)=u_0(x) & \text{in }\mathbf{R}^N \end{cases}$$

locally uniformly in $\mathbf{R}^N\times[0,T)$. Since u is a limit of a subsequence, it turns out that $w_n\equiv u$. Since $\bar{H}_n$ converges to $\bar{H}_\infty$ locally uniformly, the stability result for viscosity solutions yields the conclusion that u is a viscosity solution of (HJ). □

Remark 5.3. The main difference between two proofs is the order of limits of ε and n. We point out that Proof I does not require the locally uniform convergence of $\bar{H}_n$. However, we need the equi-Lipschitz continuity of $\{u^\varepsilon\}_{\varepsilon>0}$ in both proofs in order to ensure that u^ε is a solution of the approximate equation.

Theorem 5.4 (Non-homogenization result)**.** *Assume that $\overline{\sigma}m_0 > \underline{\sigma}$. Let u^ε be the solutions of* (HJ_ε)*. Then, u^ε does not have a locally uniformly convergent limit in $\mathbf{R}^n \times [0,T)$ as $\varepsilon \to 0$.*

Proof. Set

$$u_-^\varepsilon(x,t) := u_0(x) - \sigma\left(\frac{x}{\varepsilon}\right)t, \quad u_+^\varepsilon(x,t) := u_0(x) - \sigma\left(\frac{x}{\varepsilon}\right)m_0 t$$

for $(x,t) \in \mathbf{R}^N \times [0,T)$. Then, we see that u_-^ε and u_+^ε are a subsolution and a supersolution of (HJ_ε) respectively. By the comparison principle, the solution u^ε satisfies

$$u_-^\varepsilon(x,t) \le u^\varepsilon(x,t) \le u_+^\varepsilon(x,t) \tag{5.3}$$

for all $(x,t) \in \mathbf{R}^N \times [0,T)$ and $\varepsilon > 0$.

Let $\overline{u}$ and $\underline{u}$ be the upper half-relaxed limit and the lower half-relaxed limit of u^ε, respectively. Namely,

$$\begin{aligned}
\overline{u}(x,t) &:= \limsup_{\delta\to 0}\{u^\varepsilon(y,s) \mid (y,s) \in B(x,\delta)\times(t-\delta,t+\delta),\ \varepsilon < \delta\},\\
\underline{u}(x,t) &:= \liminf_{\delta\to 0}\{u^\varepsilon(y,s) \mid (y,s) \in B(x,\delta)\times(t-\delta,t+\delta),\ \varepsilon < \delta\}.
\end{aligned}$$

Moreover, let $\overline{u}_-$ and $\underline{u}_+$ be the upper half-relaxed limit of u_-^ε and the lower half-relaxed limit of u_+^ε, respectively. Then, we have

$$\overline{u}_-(x,t) = u_0(x) - \underline{\sigma}t \quad \text{and} \quad \underline{u}_+(x,t) = u_0(x) - \overline{\sigma}m_0 t.$$

Thus, by (5.3) and the assumption $\overline{\sigma}m_0 > \underline{\sigma}$, we have

$$\underline{u}(x,t) \le \underline{u}_+(x,t) < \overline{u}_-(x,t) \le \overline{u}(x,t) \tag{5.4}$$

for all $(x,t) \in \mathbf{R}^N \times [0,T)$. Therefore, $\overline{u}$ and $\underline{u}$ are different and so we conclude that u^ε does not converge to any functions locally uniformly as $\varepsilon \to 0$. □

Remark 5.5. When $\overline{\sigma}m_0 = \underline{\sigma}$, we do not know whether or not u^ε has a limit as $\varepsilon \to 0$. However, by (5.4), we see that the limit of u^ε should be $u_0(x) - \underline{\sigma}t (= u_0(x) - \overline{\sigma}m_0 t)$ if it exists.

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