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Quiver mutation loops and partition q-series

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Abstract. A quiver mutation loop is a sequence of mutations and vertex relabelings, along which a quiver transforms back to the original form. For a given mutation loop γ, we introduce a quantity called a *partition q-series* $Z(\gamma)$ which takes values in $\mathbb{N}[[q^{1/\Delta}]]$ where Δ is some positive integer. The partition q-series are invariant under pentagon moves. If the quivers are of Dynkin type or square products thereof, they reproduce so-called parafermionic or quasi-particle character formulas of certain modules associated with affine Lie algebras. They enjoy nice modular properties as expected from the conformal field theory point of view.

1. Introduction

Quiver mutations are now ubiquitous in many branch of mathematics and mathematical physics, such as Donaldson-Thomas theory, low dimensional topology, representation theory, quantum field theories. Quiver mutations are now recognized as important tools, along with cluster algebras.

The main purpose of this paper is to introduce quantities called *partition q-series* directly at the level of quiver mutation sequences. The definition requires only combinatorial data of quivers and mutation sequences, and completely independent of the details of the problem. In fact, one motivation is to provide a solid mathematical foundation to extract an essential information of the partition function of a 3-dimensional gauge theory associated with a sequence of quiver mutations which is introduced in [14]. It is hoped that a deeper understanding of the partition q-series will help uncover the hidden combinatorial structure and shed new lights on the mystery of quantization.

A quiver mutation loop is a sequence of mutations and vertex relabelings, along which a quiver transforms back to the original form. For a given mutation loop γ, we associate a quantity called a *partition q-series* $Z(\gamma)$ which takes values in $\mathbb{N}[[q^{1/\Delta}]]$ where Δ is some positive integer. The partition q-series are closely related to the quantum dilogarithms, and satisfy various invariance properties such as pentagon relations. If the quivers are of Dynkin type or square products thereof, they reproduce so-called parafermionic or quasi-particle character formulas of certain modules associated with affine Lie algebras. They enjoy nice modular properties as expected from the conformal field theory point of view.

The paper is organized as follows. In Section 2, we recall the basic definitions of quiver mutations. In Section 3, we introduce the partition q-series $Z(\gamma)$ for the mutation loop γ. Since the definition is a slightly complicated, we supplied a few simple examples. In Section 4, we introduce the notion of "pentagon move" of mutation loops, and show that the partition q-series are invariant under such moves. In the final Sections 5 and 6, we treat the quivers of simply-laced Dynkin type or square products thereof. It is demonstrated that if we choose a special mutation loop, the associated partition q-series are nothing but the "parafermionic character formulas" of certain modules associated with affine Lie algebras. Up to multiplication by appropriate powers of q, they are conjectured to be modular forms with respect to a certain congruence subgroup of $SL(2, \mathbb{Z})$, as expected from the conformal field theory point of view.

The paper [1] propose a relation between four-dimensional gauge theories and parafermion conformal field theories. In particular, they claim that the L^2-trace of the half monodromy is written in terms of characters. It would be interesting to find a precise relation with their work.

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2. Backgrounds

2.1. Quivers and mutations. A *quiver* Q is an oriented graph given by a set of vertices Q_0, a set of arrows Q_1 and two maps "source" $s : Q_1 \to Q_0$ and "target" $t : Q_1 \to Q_0$. A quiver Q is *finite* if the sets Q_0 and Q_1 are finite. Throughout this paper, we will assume all quivers are finite, and an isomorphism $Q_0 \xrightarrow{\sim} \{1, \dots, n\}$, called *labeling*, is fixed.

Let Q be a quiver. A *loop* or 1-*cycle* of Q is an arrow α whose source and target coincide. A 2-*cycle* of Q is a pair of distinct arrows β and γ such that $s(\beta) = t(\gamma)$ and $t(\beta) = s(\gamma)$:

$$\text{loop} \quad \begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd} \qquad \text{2-cycle} \quad \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

In this paper, we treat quivers without 1-loops or 2-cycles. For a quiver Q, Q^{op} denotes the quiver obtained from Q by reversing all arrows.

A *Dynkin quiver* is a quiver Q whose underlying graph $\underline{Q}$, a graph obtained by forgetting the orientation of arrows, is a Dynkin diagram.

A vertex i of a quiver is a *source* (respectively, a *sink*) if there are no arrows α with target i (respectively, with source i). A quiver is *alternating* if each of its vertices is a source or a sink. For an alternating graph Q, the *sign* of the vertex i is defined as $\operatorname{sgn}(i) = 1$ if i is a source and $\operatorname{sgn}(i) = -1$ if i is a sink. Here are some examples of alternating Dynkin quivers:

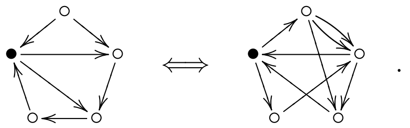
$$A_6 \qquad\qquad D_6 \qquad\qquad E_6$$

$$\begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 & 5 \arrow[l] \arrow[r] & 6 \end{tikzcd} \qquad \begin{tikzcd} & & & & 5 \arrow[dl] \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 & \\ & & & & 6 \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & & 6 & & \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] \arrow[u] & 4 & 5 \arrow[l] \end{tikzcd}$$

For a quiver Q and its vertex k, the *mutated quiver* $\mu_k(Q)$ is defined [3]: it has the same set of vertices as Q; its set of arrows is obtained from that of Q as follows:

1) for each path $i \to j \to k$ of length two, add a new arrow $i \to k$;
2) reverse all arrows with source or target k;
3) remove the arrows in a maximal set of pairwise disjoint 2-cycles.

The following two quivers are obtained from each other by mutating at the black vertex



There is a bijection

$$\left\{\begin{array}{l}\text{the quivers without loops or}\\ \text{2-cycles, } Q_0=\{1,\dots,n\}\end{array}\right\}\longleftrightarrow\left\{\begin{array}{l}\text{the skew-symmetric integer}\\ n\times n\text{-matrices } B=(b_{ij})\end{array}\right\}$$

by

$$b_{ij} = \#\{(i\to j)\in Q_1\} - \#\{(j\to i)\in Q_1\}. \tag{2.1}$$

The above operation of quiver mutation corresponds to matrix mutation defined by Fomin-Zelevinsky [3]. The matrix B' corresponding to $\mu_k(Q)$ is given by [4]

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} + \operatorname{sgn}(b_{ik})\max(b_{ik}b_{kj},0) & \text{otherwise.}\end{cases} \tag{2.2}$$

2.2. **Mutation sequences and mutation loops.** A finite sequence of vertices of Q, $\boldsymbol{m}=(m_1,m_2,\dots,m_T)$ is called *mutation sequence*. This can be regarded as a (discrete) time evolution of quivers:

$$Q(0):=Q, \qquad Q(t):=\mu_{m_t}(Q(t-1)), \qquad (1\le t\le T).$$

$Q(0)$ and $Q(T)$ are called the *initial* and the *final* quiver, respectively.

Suppose further that $Q(0)$ and $Q(T)$ are isomorphic, namely, the composed mutation

$$\mu_{\boldsymbol{m}} := \mu_{m_T}\circ\cdots\circ\mu_{m_2}\circ\mu_{m_1} \tag{2.3}$$

transforms Q into a quiver isomorphic to Q. An isomorphism $\varphi : Q(T)\to Q(0)$ regarded as a bijection on the set of vertices, is called *boundary condition* of the mutation sequence $\boldsymbol{m}$. Using the fixed labeling $Q_0 \xrightarrow{\sim} \{1,\dots,n\}$, we represent φ by an element in the symmetric group S_n. The pair $\gamma=(\boldsymbol{m},\varphi)$ is called a *mutation loop.*

3. Partition q-series and their examples

In this section, we introduce a quantity called partition q-series $Z(\gamma)$ for a mutation loop γ, in the same spirit as partition functions of statistical mechanics. Roughly speaking, $Z(\gamma)$ is defined as a sum of weights over all possible states, while the weights are expressed as a product of local factors. For clarity's sake, sample computations of $Z(\gamma)$ are presented.

3.1. **Definition of partition q-series.** Let $\gamma=(\boldsymbol{m},\varphi)$ be a mutation loop with initial quiver Q. We first introduce a family of *s-variables* $\{s_i\}$ and *k-variables* $\{k_t\}$ as follows.

(i) An "initial" s-variable s_v is attached to each vertex v of the initial quiver Q.
(ii) Every time we mutate at vertex v, we add a "new" s-variable associated with v. We often use s_v, s'_v, s''_v, … to distinguish s-variables attached to the same vertex.
(iii) We associate a k-variable k_t with each mutation at m_t.

(iv) If two vertices are related by a boundary condition, then the corresponding s-variables are identified.

As we will soon see, the s- and k-variables are not considered independent; we impose a linear relation for each mutation step. We also define a *weight* of each mutation as a function of these variables.

Suppose that the quiver $Q(t-1)$ equipped with s-variables $\{s_i\}$ is mutated at vertex $v = m_t$ to give $Q(t)$. Then k- and s-variables are required to satisfy

$$k_t = s_v + s'_v - \sum_{a:(a\to v)} s_a. \tag{3.1}$$

Here, s'_v is the "new" s-variable attached to mutated vertex v, and the sum is over all the arrows of $Q(t-1)$ whose target vertex is v.

The *weight of the mutation* $\mu_{m_t} : Q(t-1) \to Q(t)$ at $v = m_t$ is defined as

$$W(m_t) := \frac{q^{\frac{1}{2}\left(s_v+s'_v-\sum_{a:(a\to v)} s_a\right)\left(s_v+s'_v-\sum_{b:(v\to b)} s_b\right)}}{(q)_{s_v+s'_v-\sum_{a:(a\to v)} s_a}} = \frac{q^{\frac{1}{2}k_t\left(s_v+s'_v-\sum_{b:(v\to b)} s_b\right)}}{(q)_{k_t}}, \tag{3.2}$$

where

$$(x;q)_n := \prod_{k=0}^{n-1}(1-xq^k), \qquad (q)_n := (q;q)_n \tag{3.3}$$

is the q-Pochhammer symbol.

The *weight of the mutation loop* γ is then defined as the product over all mutations,

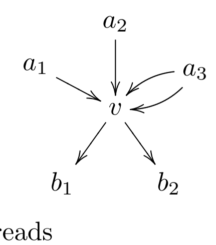
$$W(\gamma) = \prod_{t=1}^{T} W(m_t). \tag{3.4}$$

Clearly $W(\gamma)$ has a structure

$$W(\gamma) = \frac{q^{H(\boldsymbol{s})}}{\prod_{t=1}^{T}(q)_{k_t}} \tag{3.5}$$

where $H(\boldsymbol{s})$ is a quadratic form in s-variables.

For example, suppose we mutate the following quiver at vertex $v = m_t$. (All arrows not incident on v are omitted.)



In this case, the relation (3.1) reads

$$k_t = s_v + s'_v - s_{a_1} - s_{a_2} - 2s_{a_3}$$

and the weight of the mutation is

$$W(m_t) = \frac{q^{\frac{1}{2}(s_v+s'_v-s_{a_1}-s_{a_2}-2s_{a_3})(s_v+s'_v-s_{b_1}-s_{b_2})}}{(q)_{s_v+s'_v-s_{a_1}-s_{a_2}-2s_{a_3}}} = \frac{q^{\frac{1}{2}k_t(s_v+s'_v-s_{b_1}-s_{b_2})}}{(q)_{k_t}}.$$

Note that both the linear relations (3.1) and the weight (3.2) uses only the local information around the mutating vertex.

The relation (3.1) allows us to express each k-variable as a $\mathbb{Z}$-linear combination of s-variables. If these relations are invertible as a whole, namely, if one can express each s-variable as a $\mathbb{Q}$-linear combination of k-variables, then, the mutation loop γ is called *nondegenerate*.[1] Suppose the mutation loop γ is nondegenerate. Then the quadratic form $H(\boldsymbol{s})$ in (3.5) can be expressed as a quadratic form $F(\boldsymbol{k})$ in k-variables:

$$W(\gamma) = \frac{q^{F(\boldsymbol{k})}}{\prod_{t=1}^{T}(q)_{k_t}}. \tag{3.6}$$

Note that there is a positive integer Δ such that $\Delta F(\boldsymbol{k}) \in \mathbb{Z}$ for all $\boldsymbol{k} \in \mathbb{Z}^T$. The mutation loop γ is called *positive*, if $F(\boldsymbol{k}) > 0$ for all $\boldsymbol{k} \in \mathbb{N}^T$, $\boldsymbol{k} \neq 0$. This condition assures finiteness of the set $\{\boldsymbol{k} \in \mathbb{N}^T \mid F(\boldsymbol{k}) = n\}$ for all $n \in \mathbb{Z}$.

Now we are ready to define $Z(\gamma)$. From now on, mutation loops are assumed to be nondegenerate and positive. Let γ be a mutation loop with T mutations. We define its *partition q-series* $Z(Q,\gamma)$ by the multiple sum

$$Z(Q,\gamma) = \sum_{k_1,\dots,k_T=0}^{\infty} W(\gamma) \quad \in \mathbb{N}[[q^{1/\Delta}]]. \tag{3.7}$$

$Z(Q,\gamma)$ is simply written as $Z(\gamma)$ if the initial quiver Q is clear from the context.

Remark 3.1. Occasionally it is convenient to introduce another set of variables, $k^\vee$-variables $\{k_t^\vee\}$ (see e.g. the proof of Theorem 6.1 below). These are "orientation reversed" version of k-variables, and the linear relation now reads

$$k_t^\vee = s_v + s'_v - \sum_{b:(v\to b)} s_b. \tag{3.8}$$

Then, the weight of mutation (3.2) is expressed as

$$W(m_t) = \frac{q^{\frac{1}{2}k_t k_t^\vee}}{(q)_{k_t}}. \tag{3.9}$$

3.2. **Example 1 — A_3 quiver.** We illustrate how to compute the partition q-series using the quiver of type A_3

$$Q = \ 1 \longrightarrow 2 \longleftarrow 3$$

and a mutation loop

$$\gamma = (\boldsymbol{m},\varphi), \qquad \boldsymbol{m} = (2,1,3), \qquad \varphi = \mathrm{id}\,. \tag{3.10}$$

[1]It is not easy to decide whether or not γ is nondegenerate, just looking "local" structure of γ. However, the following remark is in order. For a mutation loop with T mutations, the number of k-variables is T by the rule (iii). The number of independent s-variables is also T; we start with $\#Q_0$ of "initial" s-variables (i), add T "new" ones (ii), but $\#Q_0$ of s-variables are identified via the boundary condition φ (iv). So there is a good chance of γ being nondegenerate.

We label the s- and k-variables as follows:[2]

$$
\begin{tikzcd}
Q(0) \arrow[d, "\mu_2"'] & s_1 \arrow[r] \arrow[d, equal] & s_2 \arrow[d, dotted, "k_2"] & s_3 \arrow[l] \arrow[d, equal] \\
Q(1) \arrow[d, "\mu_1"'] & s_1 \arrow[d, dotted, "k_1"] & s_2' \arrow[l] \arrow[r] \arrow[d, equal] & s_3 \arrow[d, equal] \\
Q(2) \arrow[d, "\mu_3"'] & s_1' \arrow[r] \arrow[d, equal] & s_2' \arrow[r] \arrow[d, equal] & s_3 \arrow[d, dotted, "k_3"] \\
Q(3) \arrow[d, "\mathrm{id}"'] & s_1' \arrow[r] \arrow[d, equal] & s_2' \arrow[d, equal] & s_3' \arrow[l] \arrow[d, equal] \\
Q(0) & s_1 \arrow[r] & s_2 & s_3 \arrow[l]
\end{tikzcd} \tag{3.11}
$$

The relations between k- and s-variables are

$$
k_2 = s_2 + s_2' - s_1 - s_3, \quad k_1 = s_1 + s_1' - s_2', \quad k_3 = s_3 + s_3' - s_2'. \tag{3.12}
$$

Under the boundary conditions $s_i = s_i'$ $(i = 1, 2, 3)$, one can solve (3.12) for s-variables:

$$
\begin{aligned}
&s_1 = s_1' = \frac{1}{4}\left(3k_1 + 2k_2 + k_3\right), \qquad s_2 = s_2' = \frac{1}{2}\left(k_1 + 2k_2 + k_3\right), \\
&s_3 = s_3' = \frac{1}{4}\left(k_1 + 2k_2 + 3k_3\right).
\end{aligned}
$$

So the weight of γ takes the following form:

$$
\begin{aligned}
W(\gamma) &= \frac{q^{\frac{1}{2}(s_2+s_2'-s_1-s_3)(s_2+s_2')}}{(q)_{s_2+s_2'-s_1-s_3}} \frac{q^{\frac{1}{2}(s_1+s_1'-s_2')(s_1+s_1')}}{(q)_{s_1+s_1'-s_2'}} \frac{q^{\frac{1}{2}(s_3+s_3'-s_2')(s_3+s_3')}}{(q)_{s_3+s_3'-s_2'}} \\
&= \frac{q^{\frac{3}{4}k_1^2+k_1k_2+k_2^2+k_2k_3+\frac{3}{4}k_3^2+\frac{1}{2}k_3k_1}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}}.
\end{aligned} \tag{3.13}
$$

Summing over k-variables, we obtain

$$
Z(\gamma) = \sum_{k_1,k_2,k_3=0}^{\infty} \frac{q^{\frac{3}{4}k_1^2+k_1k_2+k_2^2+k_2k_3+\frac{3}{4}k_3^2+\frac{1}{2}k_3k_1}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}} \in \mathbb{N}[[q^{1/4}]]. \tag{3.14}
$$

Exactly the same formula appeared in the study of coset conformal field theories [9]. This is an example of partition q-series which we study more systematically in Section 5. $Z(\gamma)$ can be written as

$$
Z(\gamma) = \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} q^{\frac{3}{4}n^2}, \tag{3.15}
$$

which reveals that $q^{-\frac{1}{24}}Z(\gamma)$ is a modular function for a certain congruence subgroup of $SL_2(\mathbb{Z})$.

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[2] Here the k-variables are indexed by vertex labels instead of mutation order; this is in accordance with the convention of Section 5.

3.3. **Example 2 — pentagon identity.** Let us consider a quiver of type A_2:

$$Q = 1 \longrightarrow 2\ .$$

We take up two mutation loops γ, γ', and compare the associated partition q-series.

The first loop we study is

$$\gamma = (\boldsymbol{m}, \varphi), \qquad \boldsymbol{m} = (1,2), \qquad \varphi = \mathrm{id}\,. \tag{3.16}$$

The s- and k-variables are given as follows:

$$\begin{tikzcd}
Q(0) \arrow[d, "\mu_1"'] & a \arrow[r] \arrow[d, dotted, "k_1"'] & b \arrow[d, equal] \\
Q(1) \arrow[d, "\mu_2"'] & a' \arrow[d, equal] & b \arrow[l] \arrow[d, dotted, "k_2"] \\
Q(2) \arrow[d, "\mathrm{id}"'] & a' \arrow[r] \arrow[d, equal] & b' \arrow[d, equal] \\
Q(0) & a \arrow[r] & b
\end{tikzcd} \tag{3.17}$$

Since there is no incoming arrow on mutating vertices, the relations (3.1) among k- and s-variables are simply

$$k_1 = a + a', \qquad k_2 = b + b'. \tag{3.18}$$

The initial and the new s-variables are identified via boundary condition $\varphi = \mathrm{id}$:

$$a = a', \qquad b = b'. \tag{3.19}$$

So there are two s-variables a, b and two k-variables k_1, k_2. Solving (3.18) and (3.19) for s-variables, we have

$$a = a' = \frac{1}{2}k_1, \qquad b = b' = \frac{1}{2}k_2. \tag{3.20}$$

The weight of mutation loop is thus

$$W(\gamma) = \frac{q^{\frac{1}{2}(a+a')(a+a'-b)}}{(q)_{a+a'}} \frac{q^{\frac{1}{2}(b+b')(b+b'-a')}}{(q)_{b+b'}} = \frac{q^{\frac{1}{2}(k_1^2 - k_1k_2 + k_2^2)}}{(q)_{k_1}(q)_{k_2}}. \tag{3.21}$$

The mutation loop γ is nondegenerate and positive, because the quadratic form $k_1^2 - k_1k_2 + k_2^2$ is positive definite. The partition q-series is, by definition,

$$Z(\gamma) = \sum_{k_1,k_2=0}^{\infty} \frac{q^{\frac{1}{2}(k_1^2 - k_1k_2 + k_2^2)}}{(q)_{k_1}(q)_{k_2}} \in \mathbb{N}[[q^{1/2}]]. \tag{3.22}$$

The second loop we consider is

$$\gamma' = (\boldsymbol{m}', \varphi'), \qquad \boldsymbol{m}' = (2,1,2), \qquad \varphi' = (12), \tag{3.23}$$

where $\varphi' = (12)$ means the transposition of the two vertices. The s- and k-variables are given as follows:

$$\begin{tikzcd}
Q(0) \arrow[d, "\mu_2"'] & a \arrow[r] \arrow[d, equal] & b \arrow[d, dotted, "k_1'"] \\
Q(1) \arrow[d, "\mu_1"'] & a \arrow[d, dotted, "k_2'"] & b' \arrow[l] \arrow[d, equal] \\
Q(2) \arrow[d, "\mu_2"'] & a' \arrow[r] \arrow[d, equal] & b' \arrow[d, dotted, "k_3'"] \\
Q(3) \arrow[d, "(12)"'] & a' \arrow[dr, equal] & b'' \arrow[l] \arrow[dl, equal] \\
Q(0) & a \arrow[r] & b
\end{tikzcd} . \tag{3.24}$$

The relations between k- and s-variables are

$$k_1' = b + b' - a, \qquad k_2' = a + a' - b', \qquad k_3' = b' + b'' - a'. \tag{3.25}$$

The boundary condition implies $a' = b$ and $b'' = a$. Taking this into the account, we can solve (3.25) for s-variables:

$$a = b'' = \frac{k_2'}{2} + \frac{k_3'}{2}, \qquad b = a' = \frac{k_1'}{2} + \frac{k_2'}{2}, \qquad b' = \frac{k_1'}{2} + \frac{k_3'}{2}. \tag{3.26}$$

The weight for the mutation loop γ' is thus

$$\begin{aligned} W(\gamma') &= \frac{q^{\frac{1}{2}(b+b'-a)(b+b')}}{(q)_{b+b'-a}} \frac{q^{\frac{1}{2}(a+a'-b')(a+a')}}{(q)_{a+a'-b'}} \frac{q^{\frac{1}{2}(b'+b''-a')(b'+b'')}}{(q)_{b'+b''-a'}} \\ &= \frac{q^{\frac{1}{2}\left({k_1'}^2+{k_2'}^2+{k_3'}^2+k_1'k_2'+k_2'k_3'+k_3'k_1'\right)}}{(q)_{k_1'}(q)_{k_2'}(q)_{k_3'}}. \end{aligned} \tag{3.27}$$

The partition q-series is now defined as

$$Z(\gamma') = \sum_{k_1',k_2',k_3'=0}^{\infty} \frac{q^{\frac{1}{2}\left({k_1'}^2+{k_2'}^2+{k_3'}^2+k_1'k_2'+k_2'k_3'+k_3'k_1'\right)}}{(q)_{k_1'}(q)_{k_2'}(q)_{k_3'}}. \tag{3.28}$$

It turns out that the partition q-series (3.22) and (3.28) are equal due to the identity (see e.g. [16])

$$\frac{1}{(q)_m(q)_n} = \sum_{\substack{r,s,t\ge 0 \\ m=r+s \\ n=s+t}} \frac{q^{rt}}{(q)_r(q)_s(q)_t}. \tag{3.29}$$

This is no coincidence. In the next section, we state and prove a general result about the conditions on mutation loops, which guarantee the equality of associated partition q-series.

4. Generalized pentagon identity

The main result of this section is Theorem 4.1, saying that as a function of mutation loops, the partition q-series $Z(\gamma)$ is invariant under *pentagon move* of γ, which we define shortly.

4.1. **Pentagon move.** It is convenient to slightly generalize the notion of mutation sequences/loops to keep track of vertex relabeling effect. Let Q be a quiver with vertices $\{1, \cdots, n\}$. A finite sequence $\boldsymbol{f} = (f_1, \ldots, f_r)$ consisting of

(a) mutation μ_i at the vertex i $(1 \le i \le n)$, or
(b) vertex relabeling by an element σ of the symmetric group S_n,

is called a *mutation sequence.* If $\boldsymbol{f}(Q) := f_r(\cdots(f_1(Q))\cdots)$ is isomorphic to Q as a (labeled) quiver, then $\boldsymbol{f}$ is called a *mutation loop.* $\boldsymbol{f}$, $\boldsymbol{f}'$ are considered equivalent if they are related by a series of the following moves (rewriting rules):

- $(\cdots, \sigma_1, \sigma_2, \cdots) \simeq (\cdots, \sigma_2 \circ \sigma_1, \cdots)$,
- $(\cdots, \mu_i, \sigma, \cdots) \simeq (\cdots, \sigma, \mu_{\sigma(i)}, \cdots)$,
- $(\cdots, \mathrm{id}, \cdots) \simeq (\cdots, \cdots)$.

Clearly any mutation loop is equivalent to the form of $(\mu_{m_1}, \ldots, \mu_{m_T}, \varphi)$, $\varphi \in S_n$, i.e. a pair of a mutation sequence and a boundary condition.

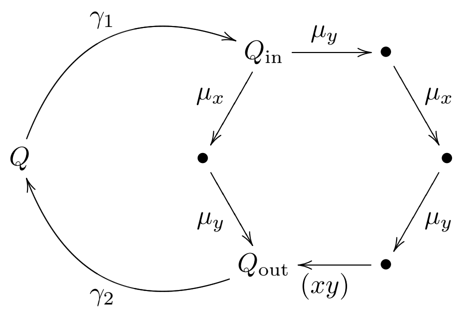

FIGURE 1. Pentagon move. $\gamma = (\gamma_1, \mu_x, \mu_y, \gamma_2)$ and $\gamma' = (\gamma_1, \mu_y, \mu_x, \mu_y, (xy), \gamma_2)$.

A local change of mutation loops of the following type

$$\gamma = (\gamma_1, \mu_x, \mu_y, \gamma_2) \qquad \longleftrightarrow \qquad \gamma' = (\gamma_1, \mu_y, \mu_x, \mu_y, (xy), \gamma_2) \tag{4.1}$$

is called *pentagon move.* Here γ_1, γ_2 are arbitrary mutation sub-sequences, and the vertices x, y are assumed to be connected by a single arrow $x \to y$ in $Q_{\mathrm{in}} := \gamma_1(Q)$. This condition guarantees that $(\mu_x, \mu_y)(Q_{\mathrm{in}})$ and $(\mu_y, \mu_x, \mu_y, (xy))(Q_{\mathrm{in}})$ are isomorphic as labeled quivers: we denote this quiver by Q_{out}. (Figure 1)

The mutation loops of Section 3.3 are the simplest example of those related by a pentagon move. For another example, take $Q = (1 \to 2 \leftarrow 3 \to 4)$. The mutation loop

$$\gamma = (\boldsymbol{m}, \varphi), \qquad \boldsymbol{m} = (4,1,2,3,2,4,1), \qquad \varphi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{smallmatrix}\right)$$

is, via pentagon move, equivalent to

$$\gamma' = (\boldsymbol{m}', \varphi'), \qquad \boldsymbol{m} = (4,2,1,2,3,1,4,2), \qquad \varphi' = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right).$$

Indeed,

$$\begin{aligned} \gamma =& (\mu_4, \underline{\mu_1, \mu_2}, \mu_3, \mu_2, \mu_4, \mu_1, \varphi) \\ \to& (\mu_4, \underline{\mu_2, \mu_1, \mu_2, (12)}, \mu_3, \mu_2, \mu_4, \mu_1, \varphi) \\ \simeq& (\mu_4, \mu_2, \mu_1, \mu_2, \mu_3, \mu_1, \mu_4, \mu_2, (12), \varphi) \\ \simeq& (\mu_4, \mu_2, \mu_1, \mu_2, \mu_3, \mu_1, \mu_4, \mu_2, \varphi') = \gamma'. \end{aligned}$$

4.2. **Generalized pentagon identity.** The main result of this section is the next

Theorem 4.1. *The partition q-series $Z(\gamma)$ is invariant under the pentagon move of the loop γ; that is, for the mutation loops γ, γ' in (4.1), we have*

$$Z(\gamma) = Z(\gamma').$$

The rest of this subsection is devoted to the proof of Theorem 4.1. The key idea is to cut the mutation loop γ at Q_{in} and Q_{out} into two pieces — "internal part" and "external part" (Figure 1), and treat their contribution to the mutation weights separately. We put

$$\begin{aligned} \gamma &= (\gamma_1, \boldsymbol{m}, \gamma_2) \qquad & \boldsymbol{m} &= (\mu_x, \mu_y), \\ \gamma' &= (\gamma_1, \boldsymbol{m}', \gamma_2) \qquad & \boldsymbol{m}' &= (\mu_y, \mu_x, \mu_y, (xy)). \end{aligned} \tag{4.2}$$

The subsequences $\boldsymbol{m}$, $\boldsymbol{m}'$ from Q_{in} to Q_{out} is referred to as "internal"; the rest is considered as "external".

By definition of pentagon move, two mutating vertices x, y are connected by a single arrow in Q_{in} (Figure 2). The vertices x and y can be a source or a target of other arrows in Q_{in}; such arrows are collectively denoted as $a_i \to x$, $b_j \to y$, $x \to c_k$ and $y \to d_l$. Along the mutation paths $\boldsymbol{m}$, $\boldsymbol{m}'$ from Q_{in} to Q_{out}, the quiver will changes as in Figure 3.

$$Q_{\text{in}} = \begin{tikzcd} a_i \arrow[dr] & & & b_j \arrow[dl] \\ & x \arrow[r] \arrow[dl] & y \arrow[dr] & \\ c_k & & & d_l \end{tikzcd}$$

FIGURE 2. The quiver Q_{in}. Only the arrows incident on x or y are shown. Some of the vertices a_i, b_j, c_k, d_l may be missing, duplicated or identified.

Let k_1, k_2 be the k-variables associated with $\boldsymbol{m} = (\mu_x, \mu_y)$, and k_3, k_4, k_5 be those for $\boldsymbol{m}' = (\mu_y, \mu_x, \mu_y, (xy))$. The k-variables on external part are denoted by $\{l_i\}$; they are common to both γ and γ'. For the internal part, the k- and s-variables are related as

$$\begin{aligned} k_1 &= x + x' - \sum a_i, \\ k_2 &= y + y' - \sum a_i - \sum b_j, \\ k_3 &= y + y'' - \sum b_j - x, \\ k_4 &= x + x'' - \sum a_i - y'', \\ k_5 &= y'' + y''' - x''. \end{aligned} \tag{4.3}$$

The relations (4.3) and the identification $y' = x''$, $x' = y'''$ yield the constraint

$$k_3 + k_4 = k_2, \qquad k_4 + k_5 = k_1. \tag{4.4}$$

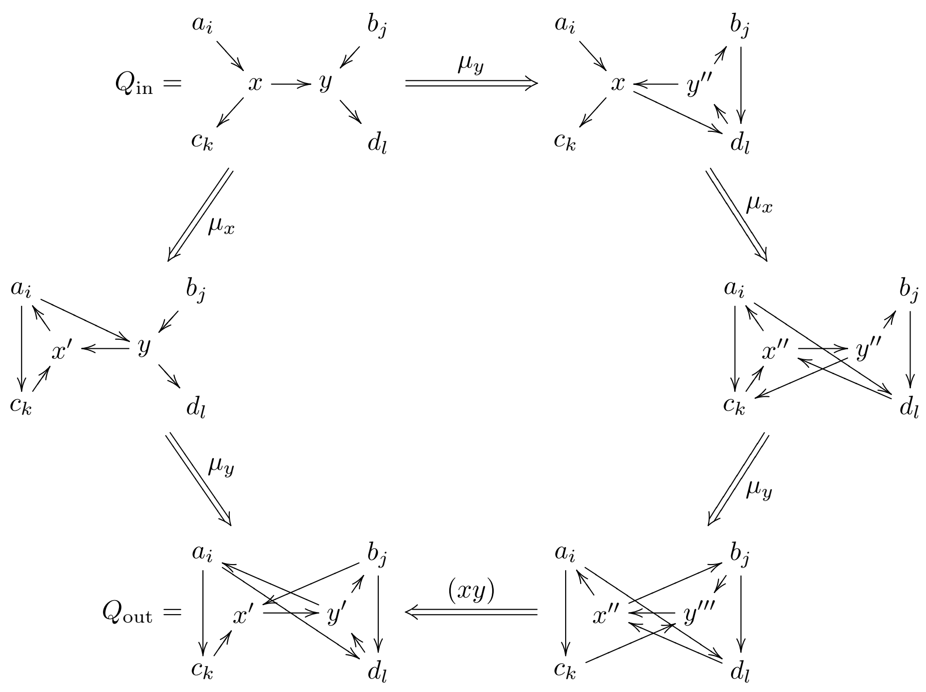

FIGURE 3. Pentagon move and quiver mutations. The vertices of quivers are represented by the corresponding s-variables. $\boldsymbol{m} = (\mu_x, \mu_y)$ and $\boldsymbol{m}' = (\mu_y, \mu_x, \mu_y, (xy))$ represent two internal paths from Q_{in} to Q_{out}. We have $x'' = y'$, $y''' = x'$ via transposition (xy) of vertices.

The partition q-series has the following form:

$$
\begin{aligned}
Z(\gamma) &= \sum_{k_1,k_2,l_i \geq 0} \frac{q^{F_{\mathrm{ext}}(k_1,k_2,l)}}{\prod_i (q)_{l_i}} \times \frac{q^{F_{\mathrm{int}}(k_1,k_2,l)}}{(q)_{k_1}(q)_{k_2}} \\
&= \sum_{l_i \geq 0} \frac{1}{\prod_i (q)_{l_i}} \times \sum_{k_1,k_2 \geq 0} \frac{q^{F_{\mathrm{ext}}(k_1,k_2,l)+F_{\mathrm{int}}(k_1,k_2,l)}}{(q)_{k_1}(q)_{k_2}}, \\
Z(\gamma') &= \sum_{k_3,k_4,k_5,l_i \geq 0} \frac{q^{F'_{\mathrm{ext}}(k_3,k_4,k_5,l)}}{\prod_i (q)_{l_i}} \times \frac{q^{F'_{\mathrm{int}}(k_3,k_4,k_5,l)}}{(q)_{k_3}(q)_{k_4}(q)_{k_5}} \\
&= \sum_{l_i \geq 0} \frac{1}{\prod_i (q)_{l_i}} \times \sum_{k_3,k_4,k_5 \geq 0} \frac{q^{F'_{\mathrm{ext}}(k_3,k_4,k_5,l)+F'_{\mathrm{int}}(k_3,k_4,k_5,l)}}{(q)_{k_3}(q)_{k_4}(q)_{k_5}}.
\end{aligned}
\tag{4.5}
$$

The external part of γ and γ' share the same set of s-variables, so as functions s-variables, $F_{\mathrm{ext}} = F'_{\mathrm{ext}}$. Therefore under the identification (4.4), we have

$$
\left(F_{\mathrm{ext}}(k_1, k_2, l) \,\bigg|_{\substack{k_1 = k_4 + k_5 \\ k_2 = k_3 + k_4}} \right) = F'_{\mathrm{ext}}(k_3, k_4, k_5, l). \tag{4.6}
$$

As for the internal part, we obtain after some computation,

$$\begin{aligned}
F_{\mathrm{int}}(k_1,k_2,l) &= \frac{1}{2}(x+x'-\sum a_i)(x+x'-\sum c_k - y)\\
&\quad + \frac{1}{2}(y+y'-\sum a_i - \sum b_j)(y+y'-x'-\sum d_l)\\
&= \frac{1}{2}\left(k_1^2+k_2^2-k_1k_2+Ak_1+Bk_2\right),
\end{aligned}$$

$$\begin{aligned}
F'_{\mathrm{int}}(k_3,k_4,k_5,l) &= \frac{1}{2}(y+y''-\sum b_j - x)(y+y''-\sum d_l)\\
&\quad + \frac{1}{2}(x+x''-\sum a_i - y'')(x+x''-\sum c_k - \sum d_l)\\
&\quad + \frac{1}{2}(y''+y'''-x'')(y''+y'''-\sum b_j - \sum c_k)\\
&= \frac{1}{2}\big(k_3^2+k_4^2+k_5^2+k_3k_4+k_4k_5+k_3k_5+Bk_3+(A+B)k_4+Ak_5\big),
\end{aligned}$$

where

$$A := \sum a_i - \sum c_k - y, \qquad B := \sum b_j - \sum d_k + x.$$

It is now easy to check that under the relation (4.4),

$$F'_{\mathrm{int}}(k_3,k_4,k_5,l) = \left(F_{\mathrm{int}}(k_1,k_2,l)\,\Big|_{\substack{k_1=k_4+k_5\\ k_2=k_3+k_4}}\right) + k_3k_5. \tag{4.7}$$

By substituting (4.6) and (4.7) into (4.5), we conclude $Z(\gamma)$ and $Z(\gamma')$ are equal thanks to (3.29). This completes the proof of Theorem 4.1.

5. Partition q-series and Characters I — Dynkin case

Let Q be an alternating quiver of Dynkin type A_n, D_n or E_n. Denote by $\boldsymbol{m}_+$, $\boldsymbol{m}_-$ the set of sources, sinks of Q, respectively. We consider the following mutation sequence of length $n = \#Q_0$:

$$\boldsymbol{m} = \boldsymbol{m}_-\boldsymbol{m}_+. \tag{5.1}$$

Here consider $\boldsymbol{m}_\pm$ as sequence of mutations. The ordering within $\boldsymbol{m}_\pm$ does not matter since there are no arrows connecting two sources or two sinks in Q. It is easy to check that

$$\mu_{\boldsymbol{m}_-}(Q) = Q^{op}, \qquad \mu_{\boldsymbol{m}_+}(Q^{op}) = Q.$$

Thus with trivial boundary condition $\varphi = \mathrm{id}$, $\gamma = (\boldsymbol{m}, \mathrm{id})$ makes up a mutation loop. The s-variables s_v and s'_v, before and after the mutation at v, are identified for each $v \in Q_0$. Since every vertex v of Q is mutated exactly once, it is convenient to label the k-variables by vertices, not by mutation order; we use the notation $\boldsymbol{k} = (k_v)_{v\in Q_0}$.

To motivate our main result of this section, Theorem 5.1, we first give an example. Consider an alternating quiver Q of type D_5

$$Q \;=\; \begin{tikzcd}[row sep=small] & & & 4 \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[ur] \arrow[dr] & \\ & & & 5 \end{tikzcd}$$

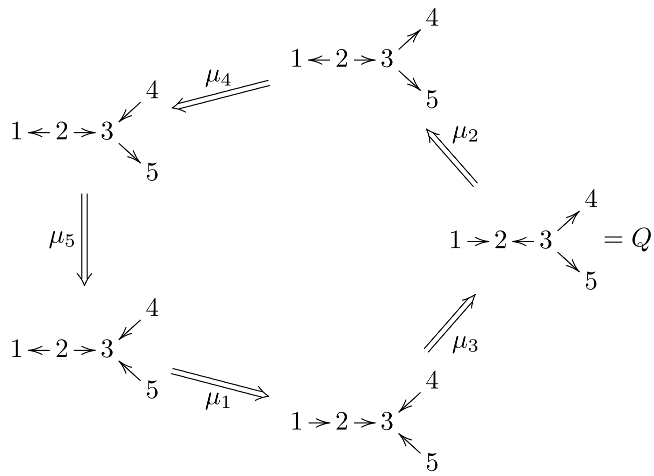

FIGURE 4. Example of type D_5: the mutation loop $\gamma = ((2,4,5,1,3),\mathrm{id})$.

and the mutation sequence

$$\boldsymbol{m}_- = (2,4,5), \quad \boldsymbol{m}_+ = (1,3), \quad \gamma = (\boldsymbol{m}_-\boldsymbol{m}_+,\mathrm{id}) = ((2,4,5,1,3),\mathrm{id}).$$

See Figure 4.

The linear relations between k- and s-variables are

$$\begin{aligned} &k_2 = -s_1 + 2s_2 - s_3, \quad k_4 = 2s_4 - s_3, \quad k_5 = 2s_5 - s_3, \\ &k_1 = 2s_1 - s_2, \quad k_3 = -s_2 + 2s_3 - s_4 - s_5. \end{aligned} \tag{5.2}$$

Recall that $s'_v = s_v$ by the boundary condition. The weight is then expressed as

(5.3)

$$\begin{aligned} W(\gamma) &= \frac{q^{\frac{1}{2}(2s_2-s_1-s_3)2s_2}}{(q)_{2s_2-s_1-s_3}} \frac{q^{\frac{1}{2}(2s_4-s_3)2s_4}}{(q)_{2s_4-s_3}} \frac{q^{\frac{1}{2}(2s_5-s_3)2s_5}}{(q)_{2s_5-s_3}} \frac{q^{\frac{1}{2}(2s_1-s_2)2s_1}}{(q)_{2s_1-s_2}} \frac{q^{\frac{1}{2}(2s_3-s_2-s_4-s_5)2s_3}}{(q)_{2s_3-s_2-s_4-s_5}} \\ &= \frac{q^{2s_1^2-2s_2s_1+2s_2^2+2s_3^2+2s_4^2+2s_5^2-2s_2s_3-2s_3s_4-2s_3s_5}}{(q)_{2s_2-s_1-s_3}(q)_{2s_4-s_3}(q)_{2s_5-s_3}(q)_{2s_1-s_2}(q)_{2s_3-s_2-s_4-s_5}}. \end{aligned}$$

The relation (5.2) is nondegenerate: we can solve (5.2) for s-variables:

$$\begin{cases} s_1 = (2k_1 + 2k_2 + 2k_3 + k_4 + k_5)/2, \\ s_2 = k_1 + 2k_2 + 2k_3 + k_4 + k_5, \\ s_3 = (2k_1 + 4k_2 + 6k_3 + 3k_4 + 3k_5)/2, \\ s_4 = (2k_1 + 4k_2 + 6k_3 + 5k_4 + 3k_5)/4, \\ s_5 = (2k_1 + 4k_2 + 6k_3 + 3k_4 + 5k_5)/4. \end{cases} \tag{5.4}$$

Substituting these into (5.3), we can express $Z(\gamma)$ in terms of k-variables alone:

(5.5)

$$Z(\gamma) = \sum_{k_1,\dots,k_5=0}^{\infty} \frac{q^{k_1^2+2k_2^2+3k_3^2+\frac{5}{4}k_4^2+\frac{5}{4}k_5^2+2k_1k_2+2k_1k_3+k_1k_4+k_1k_5+4k_2k_3+2k_2k_4+2k_2k_5+3k_3k_4+3k_3k_5+\frac{3}{2}k_4k_5}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}(q)_{k_4}(q)_{k_5}}.$$

Let $A[\boldsymbol{x}]$ denote the quadratic form associated with a symmetric $n \times n$ matrix $A = (a_{ij})$:

$$A[\boldsymbol{x}] = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \boldsymbol{x}^T A \boldsymbol{x}, \qquad (\boldsymbol{x} = (x_1, \ldots, x_n)). \tag{5.6}$$

The exponents of q in the summand (5.3) or (5.5) are quadratic form in s- or k-variables; they are neatly expressed as $C[\boldsymbol{s}]$ and $D[\boldsymbol{k}]$, respectively, where

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}, \qquad D = C^{-1} = \frac{1}{4} \begin{pmatrix} 4 & 4 & 4 & 2 & 2 \\ 4 & 8 & 8 & 4 & 4 \\ 4 & 8 & 12 & 6 & 6 \\ 2 & 4 & 6 & 5 & 3 \\ 2 & 4 & 6 & 3 & 5 \end{pmatrix} \tag{5.7}$$

are nothing but the Cartan matrix of type D_5 and its inverse. The linear relations (5.2) and (5.4) are also simply given by

$$\boldsymbol{k} = C\boldsymbol{s}, \qquad \boldsymbol{s} = D\boldsymbol{k}. \tag{5.8}$$

We write the product of q-Pochhammer symbols as

$$(q)_{\boldsymbol{v}} := \prod_{i \in I} (q)_{v_i}, \tag{5.9}$$

where $\boldsymbol{v} = (v_i)_{i \in I}$ is a vector of nonnegative integers. The denominators of the weights are then simply expressed as $(q)_{\boldsymbol{k}}$.

Theorem 5.1. *Let Q be an alternating quiver of simply-laced Dynkin type X_n ($X_n = A_n, D_n, E_n$), and $\gamma = (\boldsymbol{m}_-\boldsymbol{m}_+, \mathrm{id})$ be the mutation loop defined in (5.1). Then the partition q-series $Z(\gamma)$ has a following form:*

$$Z(\gamma) = \sum_{\boldsymbol{k}=(k_1,\ldots,k_n) \in \mathbb{N}^n} \frac{q^{D[\boldsymbol{k}]}}{(q)_{\boldsymbol{k}}}. \tag{5.10}$$

where D is the inverse of the Cartan matrix $C = (c_{ij})$ of type X_n. The relation between k- and s-variables is nondegenerate and is given by $\boldsymbol{k} = C\boldsymbol{s}$.

Proof. First consider the mutation sequence $\boldsymbol{m}_-$ applied on Q. It is important to note that every mutation vertex $a \in \boldsymbol{m}_-$ is a sink of Q. So we have

$$k_a = s_a + s'_a - \sum_{i: i \to a \in Q} s_i = 2s_a - \sum_{i: i \sim a} s_i, \qquad (a \in \boldsymbol{m}_-), \tag{5.11}$$

where $i \sim a$ means that the vertices i and a are adjacent in the underlying Dynkin graph $\underline{Q}$. Here we used the identification $s'_a = s_a$. The weight of the mutation at sink $a \in \boldsymbol{m}_-$ is

$$\frac{q^{\frac{1}{2}(2s_a - \sum_{i:i\sim a} s_i)\cdot(2s_a - 0)}}{(q)_{k_a}} = \frac{q^{\sum_{i=1}^{n} c_{ia} s_i s_a}}{(q)_{k_a}}. \tag{5.12}$$

Next consider the mutation sequence $\boldsymbol{m}_+$ on $\boldsymbol{m}_-(Q) = Q^{op}$. Again, every mutating vertex $b \in \boldsymbol{m}_+$ is a sink of Q^{op}. Therefore

$$k_b = s_b + s'_b - \sum_{i: i \to b \in Q^{op}} s_i = 2s_b - \sum_{i: i \sim b \in \underline{Q}} s_i, \qquad (b \in \boldsymbol{m}_-). \tag{5.13}$$

The weight of the mutation at $b \in \boldsymbol{m}_+$ is given by

$$\frac{q^{\frac{1}{2}(2s_b - \sum_{i:i\sim b} s_i)\cdot(2s_b - 0)}}{(q)_{k_b}} = \frac{q^{\sum_{i=1}^n c_{ib}s_i s_b}}{(q)_{k_b}}. \tag{5.14}$$

Clearly the relations (5.11) and (5.13) are combined into

$$\boldsymbol{k} = C\boldsymbol{s}, \tag{5.15}$$

where C is the Cartan matrix of type $\underline{Q}$.

Collecting (5.12) and (5.14), the mutation weight of γ is expressed as

$$W(\gamma) = \prod_{a\in \boldsymbol{m}_-} \frac{q^{\sum_{i=1}^n c_{ia}s_i s_a}}{(q)_{k_a}} \prod_{b\in \boldsymbol{m}_+} \frac{q^{\sum_{i=1}^n c_{ib}s_i s_b}}{(q)_{k_b}} = \frac{q^{\sum_{i=1}^n c_{ij}s_i s_j}}{\prod_{i=1}^n (q)_{k_i}} = \frac{q^{C[\boldsymbol{s}]}}{(q)_{\boldsymbol{k}}}, \tag{5.16}$$

where we used $c_{ij} = c_{ji}$. Since $\boldsymbol{s} = C^{-1}\boldsymbol{k} = D\boldsymbol{k}$, we have

$$C[\boldsymbol{s}] = C[D\boldsymbol{k}] = \boldsymbol{k}^T(D^T C D)\boldsymbol{k} = \boldsymbol{k}^T D\boldsymbol{k} = D[\boldsymbol{k}].$$

Putting this into (5.16) and summing over $\boldsymbol{k}$, we obtain the desired formula for the partition q-series. $\square$

6. Partition q-series and Characters II — square products

6.1. Products of quivers and their mutations. Let Q, Q' be two quivers without oriented cycles, and $B = (b_{ij}), B' = (b'_{i'j'})$ be the corresponding matrices. The *tensor product* $Q \otimes Q'$ is defined as follows [10] (Figure 5): the set vertices is the product $Q_0 \times Q'_0$, and the associated matrix is given by[3]

$$B(Q \otimes Q') = B(Q) \otimes I_{Q'} + I_Q \otimes B(Q') \tag{6.1}$$

where I_Q, $I_{Q'}$ denotes the identity matrix of size $\#Q_0$, $\#Q'_0$, respectively. In other words, the number of arrows from a vertex (i, i') to a vertex (j, j')

a) is zero if $i \neq j$ and $i' \neq j'$;
b) equals the number of arrows from j to j' if $i = i'$;
c) equals the number of arrows from i to i' if $j = j'$.

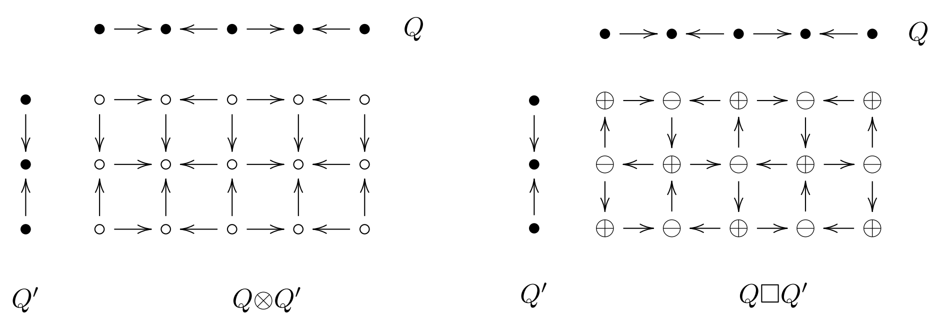

Figure 5. Tensor product and square product of quivers.

[3] There is a natural isomorphism between path algebras $\mathbb{C}(Q \otimes Q') \simeq \mathbb{C}Q \otimes \mathbb{C}Q'$.

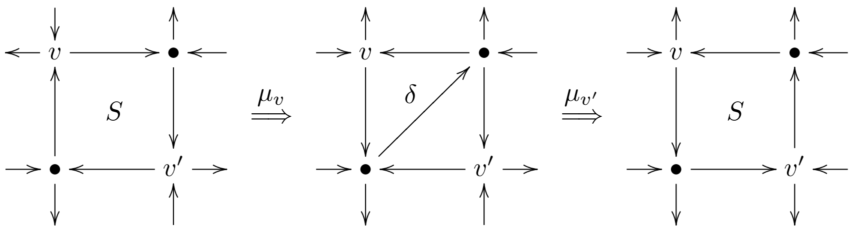

FIGURE 6. Creation and annihilation of a diagonal edge.

Now assume that Q and Q' are alternating, i.e. each vertex is a source or a sink. We define the *square product* $Q\Box Q'$ to be the quiver obtained from $Q\otimes Q'$ by reversing all arrows in the full subquivers of the form $\{i\}\times Q'$ and $Q\times\{i'\}$, where i is a source of Q and i' a sink of Q'.[4] Note that $Q\otimes Q'$ has no oriented cycles, whereas $Q\Box Q'$ is composed of squares with oriented 4-cycle boundaries. It is easy to check that

$$(Q\otimes Q')^{op} = (Q^{op}\otimes Q'^{op}), \qquad (Q^{op}\Box Q') = (Q\Box Q'^{op}) = (Q\Box Q')^{op}. \tag{6.2}$$

In the remainder of this section, we assume Q and Q' are alternating quivers whose underlying graphs are of Dynkin diagram of ADE type. The vertices of $Q\Box Q'$ are partitioned into two subsets: for $\varepsilon=\pm$, we put

$$\boldsymbol{m}_\varepsilon := \left\{ (i,i') \in Q_0\times Q'_0 \mid \operatorname{sgn}(i)\operatorname{sgn}(i') = \varepsilon \right\}. \tag{6.3}$$

In Figure 5, $\boldsymbol{m}_+$ and $\boldsymbol{m}_-$ corresponds to vertices $\oplus$ and $\ominus$, respectively. For each ε, there is no arrows joining two vertices v, v' of $\boldsymbol{m}_\varepsilon$ and thus $\mu_v\circ\mu_{v'}=\mu_{v'}\circ\mu_v$.

The following simple observation will be helpful. Choose a square S in $Q\Box Q'$ and let v, v' be the two vertices of S in the diagonal position (see Figure 6). Suppose we perform two mutations, first at v, and later at v'. By the mutation rule 2), the first mutation creates an arrow δ which is a diagonal of S. But the second mutation at v' eliminates δ by the mutation rule 3). As a result of combined mutation μ_v and $\mu_{v'}$, the diagonal edge δ disappears, and the orientations of all arrows bounding S are reversed. Mutations on vertices other than v, v' can never create or delete δ.

With this observation in mind, it is easy to check that

$$\mu_{\boldsymbol{m}_+}(Q\Box Q') = (Q\Box Q')^{op}, \qquad \mu_{\boldsymbol{m}_-}\left((Q\Box Q')^{op}\right) = Q\Box Q'.$$

Consequently, $\gamma=(\boldsymbol{m}_+\boldsymbol{m}_-,\mathrm{id})$ forms a mutation loop with initial quiver $Q\Box Q'$.

6.2. **Partition q-series.** We now move on to the partition q-series for the mutation loop $\gamma=(\boldsymbol{m}_+\boldsymbol{m}_-,\mathrm{id})$ with initial quiver $Q\Box Q'$.

As in Section 5, every vertex v of $Q\Box Q'$ is mutated exactly once; the s-variables before and after the mutation at v are identified $s_v=s'_v$ by the boundary condition. Both s- and k-variables are thus in one to one correspondence with the vertex set $Q_0\times Q'_0$; let $s_{(i,i')}$, $k_{(i,i')}$ be the s-, k-variable associated with the vertex (i,i'), respectively. It is useful to regard $\boldsymbol{s}=(s_{(i,i')})$ and $\boldsymbol{k}=(k_{(i,i')})$ as column vectors indexed by the set $Q_0\times Q'_0$; we will use lexicographic ordering, if necessary.

The main result of this section is the next

[4] This orientation convention is slightly different from [10].

Theorem 6.1. Let Q, Q' be alternating quivers of type A_n, D_n or E_n with Cartan matrices C_Q, $C_{Q'}$, respectively. Let $\gamma = (\boldsymbol{m}_+\boldsymbol{m}_-, \mathrm{id})$ be the mutation loop described above, with initial quiver $Q\square Q'$. Then the partition q-series $Z(\gamma)$ has a following form:

$$Z(\gamma) = \sum_{\boldsymbol{k}\geq 0} \frac{q^{\frac{1}{2}\left(C_Q\otimes C_{Q'}^{-1}\right)[\boldsymbol{k}]}}{(q)_{\boldsymbol{k}}}. \tag{6.4}$$

The s- and k-variables are related as

$$\boldsymbol{k} = (I_Q \otimes C_{Q'})\boldsymbol{s}, \qquad \boldsymbol{s} = (I_Q \otimes C_{Q'}^{-1})\boldsymbol{k}, \tag{6.5}$$

where I_Q is the identity matrix of size $\#Q_0$.

Remark 6.2. The following remarks are in order.

The partition q-series for the case when Q is type X and Q' is type A_{r-1} is of particular interest. Let $L(r\Lambda_0)$ be the vacuum integrable highest weight module of the untwisted affine Lie algebra of type $X^{(1)}$. The structures of various subquotients of this module, especially explicit description of their basis, are of considerable interest from the viewpoint of mathematical physics, and have been extensively studied [12, 11, 2, 15, 5, 6, 13, 7]. The corresponding characters are often referred to as parafermionic formula or quasi-particle formula. Precisely the same formula as (6.4) appears in the literature (see for example (9) of [11], (0.5) of [6], or (5.40) of [7]). The relation with string functions [8] reveals that when multiplied by a suitable power of q, $q^s Z(\gamma)$ becomes a modular form of some congruence subgroup of $SL_2(\mathbb{Z})$.

Before giving a proof, we illustrate the statement of Theorem 6.1 using an example of $A_3\square A_2$:

$$Q = \begin{array}{ccccc} 1 & \longrightarrow & 3 & \longleftarrow & 5 \\ \uparrow & & \downarrow & & \uparrow \\ 2 & \longleftarrow & 4 & \longrightarrow & 6 \end{array}.$$

Here we enumerate the vertices in the lexicographical order:

$$1 \leftrightarrow (1,1), \quad 2 \leftrightarrow (1,2), \quad 3 \leftrightarrow (2,1), \quad 4 \leftrightarrow (2,2), \quad 5 \leftrightarrow (3,1), \quad 6 \leftrightarrow (3,2).$$

We consider the mutation loop (see Figure 7)

$$\boldsymbol{m}_+ = (1,4,5), \qquad \boldsymbol{m}_- = (2,3,6), \qquad \gamma = ((1,4,5,2,3,6), \mathrm{id}).$$

By the boundary condition, s-variables before and after mutation are identified vertex-wise. The linear relations between k- and s-variables are

$$\begin{aligned} k_1 &= 2s_1 - s_2, \quad & k_4 &= 2s_4 - s_3, \quad & k_5 &= 2s_5 - s_6, \\ k_2 &= 2s_2 - s_1, \quad & k_3 &= 2s_3 - s_4, \quad & k_6 &= 2s_6 - s_5. \end{aligned} \tag{6.6}$$

This may be written as

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix}$$

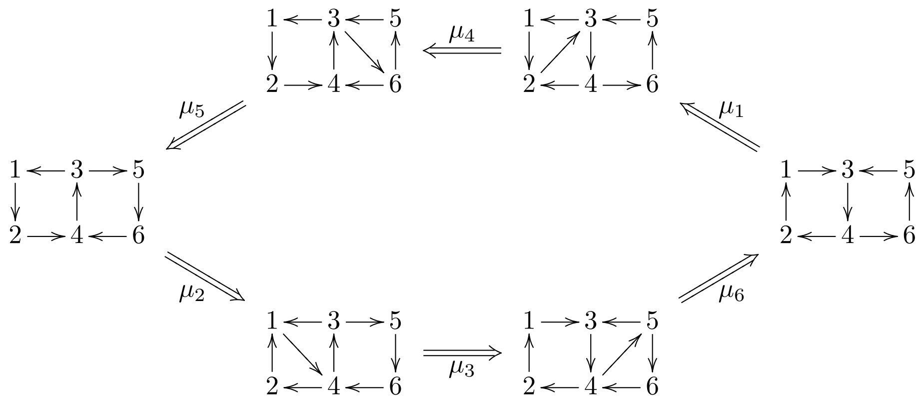

FIGURE 7. $A_3 \square A_2$ and the mutation loop $\gamma = (\boldsymbol{m}_+ \boldsymbol{m}_-, \mathrm{id})$. Here $\boldsymbol{m}_+ = (1, 4, 5)$, $\boldsymbol{m}_- = (2, 3, 6)$.

or more compactly,

$$\boldsymbol{k} = (I_3 \otimes C_{A_2})\, \boldsymbol{s}. \tag{6.7}$$

The weight is given by

$$\begin{aligned} W(\gamma) = &\frac{q^{\frac{1}{2}(2s_1-s_2)(2s_1-s_3)}}{(q)_{2s_1-s_2}} \frac{q^{\frac{1}{2}(2s_4-s_3)(2s_4-s_2-s_6)}}{(q)_{2s_4-s_3}} \frac{q^{\frac{1}{2}(2s_5-s_3)(2s_5-s_6)}}{(q)_{2s_5-s_6}} \\ &\times \frac{q^{\frac{1}{2}(2s_2-s_1)(2s_2-s_4)}}{(q)_{2s_2-s_1}} \frac{q^{\frac{1}{2}(2s_3-s_4)(2s_3-s_1-s_5)}}{(q)_{2s_3-s_4}} \frac{q^{\frac{1}{2}(2s_6-s_4)(2s_6-s_5)}}{(q)_{2s_6-s_5}}. \end{aligned} \tag{6.8}$$

The numerator of (6.8) is of the form $q^{\frac{1}{2}C[\boldsymbol{s}]}$, where $C[\boldsymbol{s}]$ is a quadratic form defined by the positive definite symmetric matrix

$$C = \begin{pmatrix} 4 & -2 & -2 & 1 & 0 & 0 \\ -2 & 4 & 1 & -2 & 0 & 0 \\ -2 & 1 & 4 & -2 & -2 & 1 \\ 1 & -2 & -2 & 4 & 1 & -2 \\ 0 & 0 & -2 & 1 & 4 & -2 \\ 0 & 0 & 1 & -2 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = C_{A_3} \otimes C_{A_2}.$$

We have $\boldsymbol{s} = (I_3 \otimes C_{A_2}^{-1})\boldsymbol{k}$ by inverting the relation (6.7). Substituting this into (6.8), we obtain the partition q-series:

$$Z(\gamma) = \sum_{k_1,k_2,k_3,k_4,k_5,k_6=0}^{\infty} \frac{q^{\frac{1}{2}D[\boldsymbol{k}]}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}(q)_{k_4}(q)_{k_5}(q)_{k_6}}$$

where

$$D = \frac{1}{3}\begin{pmatrix} 4 & 2 & -2 & -1 & 0 & 0 \\ 2 & 4 & -1 & -2 & 0 & 0 \\ -2 & -1 & 4 & 2 & -2 & -1 \\ -1 & -2 & 2 & 4 & -1 & -2 \\ 0 & 0 & -2 & -1 & 4 & 2 \\ 0 & 0 & -1 & -2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \otimes \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = C_{A_3} \otimes (C_{A_2})^{-1}.$$

Proof. of Theorem 6.1:

First consider the sequence of mutations $\boldsymbol{m}_+$ applied to $Q\square Q'$. Pick a vertex $v = (i, i') \in \boldsymbol{m}_+$ (marked with $\oplus$ in Figure 5.) Then by the very definition of $Q\square Q'$, every incoming arrow α to v comes from "vertical" directions; α is of the form $(i, j') \to (i, i')$ where j' is adjacent to i' in the underlying graph $\underline{Q'}$. This means that the k- and s-variables are related as

$$k_{(i,i')} = 2s_{(i,i')} - \sum_{j':i'\sim j'\in \underline{Q'}} s_{(i,j')} \tag{6.9}$$

Next we take up the mutation sequence $\boldsymbol{m}_-$ (marked with $\ominus$ in Figure 5). Since the mutation $\boldsymbol{m}_-$ is applied only after $\boldsymbol{m}_+$ is over, it is convenient to consider $\mu_{\boldsymbol{m}_+}(Q\square Q') = (Q\square Q')^{op}$ as the initial quiver. Then the connections around the mutating vertex is exactly the same as before: all incoming arrows again come from "vertical" directions. Thus, (6.9) is true for $v = (i, i') \in \boldsymbol{m}_-$ as well. Thus we have the relation

$$\boldsymbol{k} = (I_Q \otimes C_{Q'})\boldsymbol{s}. \tag{6.10}$$

Since Q' is of ADE type, $C_{Q'}$ is a positive definite symmetric matrix. Thus the linear relation (6.10) is invertible:

$$\boldsymbol{s} = (I_Q \otimes C_{Q'}^{-1})\boldsymbol{k}. \tag{6.11}$$

In particular, the mutation loop γ is nondegenerate.

We have seen that all incoming arrows to the mutating vertices run "vertically." This means that all outgoing arrows run "horizontally." Therefore, $k^\vee$-variables, which are introduced in (3.8), are related with s-variables as

$$k^\vee_{(i,i')} = 2s_{(i,i')} - \sum_{j:i\sim j\in \underline{Q}} s_{(j,i')}, \tag{6.12}$$

or equivalently,

$$\boldsymbol{k}^\vee = (C_Q \otimes I_{Q'})\boldsymbol{s}. \tag{6.13}$$

The weight of the whole mutation sequence is then

$$W(\gamma) = \prod_v \frac{q^{\frac{1}{2}k_v k_v^\vee}}{(q)_{k_v}} = \frac{q^{\frac{1}{2}\sum_v k_v k_v^\vee}}{(q)_{\boldsymbol{k}}}. \tag{6.14}$$

Note that the sum in the numerator is written as

$$\begin{aligned}\sum_v k_v k_v^\vee &= \boldsymbol{k}^T\boldsymbol{k}^\vee = \boldsymbol{k}^T\,(C_Q \otimes I_{Q'})\boldsymbol{s} \\ &= \boldsymbol{k}^T\,(C_Q \otimes I_{Q'})(I_Q \otimes C_{Q'}^{-1})\boldsymbol{k} = \boldsymbol{k}^T\,(C_Q \otimes C_{Q'}^{-1})\boldsymbol{k} \\ &= (C_Q \otimes C_{Q'}^{-1})[\boldsymbol{k}].\end{aligned} \tag{6.15}$$

Combining (6.14), (6.15), and summing over the k-variables, we obtain the desired formula (6.4).

□

Remark 6.3. With the same initial quiver $Q\square Q'$, we can construct another mutation loop $\gamma' = (\boldsymbol{m}_-\boldsymbol{m}_+, \mathrm{id})$ by exchanging $\boldsymbol{m}_+$ and $\boldsymbol{m}_-$. Analysis similar to that in the proof of Theorem 6.1 show that Q and Q' exchange their roles; the partition q-series is now given by

$$Z(\gamma') = \sum_{\boldsymbol{k}\geq 0} \frac{q^{\frac{1}{2}\left(C_Q^{-1}\otimes C_{Q'}\right)[\boldsymbol{k}]}}{(q)_{\boldsymbol{k}}}. \tag{6.16}$$

References

[1] S. Cecotti, A. Neitzke, and C. Vafa. R-twisting and 4d/2d correspondences. *arXiv preprint arXiv:1006.3435*, 2010.

[2] B. Feigin and A. Stoyanovsky. Quasi-particles models for the representations of Lie algebras and geometry of flag manifold. *arXiv preprint hep-th/9308079*, 1993.

[3] S. Fomin and A. Zelevinsky. Cluster algebras I: Foundations. *Journal of the American Mathematical Society*, 15(2):497–529, 2002.

[4] S. Fomin and A. Zelevinsky. Cluster algebras IV: Coefficients. *Compositio Mathematica*, 143(01):112–164, 2007.

[5] G. Georgiev. Combinatorial constructions of modules for infinite-dimensional Lie algebras, I. Principal subspace. *Journal of Pure and Applied Algebra*, 112(3):247–286, 1996.

[6] G. Georgiev. Combinatorial constructions of modules for infinite-dimensional Lie algebras, II. Parafermionic space. *arXiv preprint q-alg/9504024*, 1995.

[7] G. Hatayama, A. N. Kirillov, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Character formulae of $\widehat{sl}_n$-modules and inhomogeneous paths. *Nuclear Physics B*, 536(3):575 – 616, 1998.

[8] V. G. Kač and D. H. Peterson. Infinite-dimensional Lie algebras, theta functions and modular forms. *Advances in Mathematics*, 53:125 – 264, 1984.

[9] R. Kedem, T. Klassen, B. McCoy, and E. Melzer. Fermionic quasi-particle representations for characters of $(G^{(1)})_1 \times (G^{(1)})_1/(G^{(1)})_2$. *Physics Letters B*, 304(3):263 – 270, 1993.

[10] B. Keller. The periodicity conjecture for pairs of Dynkin diagrams. *Ann. of Math.(2)*, 177(1):111–170, 2013.

[11] A. Kuniba, T. Nakanishi, and J. Suzuki. Characters in conformal field theories from thermodynamic Bethe ansatz. *Modern Physics Letters A*, 8(18):1649–1659, 1993.

[12] J. Lepowsky and M. Primc. *Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$*, volume 46 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1985.

[13] A. Stoyanovsky and B. Feigin. Functional models for representations of current algebras and semi-infinite Schubert cells. *Functional Analysis and its applications*, 28(1):55–72, 1994.

[14] Y. Terashima and M. Yamazaki. 3d $N = 2$ theories from cluster algebras. *arXiv preprint arXiv:1301.5902*, 2013.

[15] M. Terhoeven. Lift of dilogarithm to partition identities. *arXiv preprint hep-th/9211120*, (1992).

[16] D. Zagier. The dilogarithm function. In *Frontiers in number theory, physics, and geometry II*, pages 3–65. Springer, 2007.

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