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Transverse projective structures of foliations and infinitesimal derivatives of the Godbillon-Vey class

by

Taro ASUKE

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

TRANSVERSE PROJECTIVE STRUCTURES OF FOLIATIONS AND INFINITESIMAL DERIVATIVES OF THE GODBILLON-VEY CLASS

TARO ASUKE

ABSTRACT. We study transverse projective structures of foliations and construct an invariant, which is a homomorphism from a foliated cohomology to the ordinary one. It is shown that the infinitesimal derivatives of the Godbillon-Vey class and the Bott class are determined by the invariant. As a corollary, a rigidity theorem for the Godbillon-Vey class and the Bott class is shown.

INTRODUCTION

It is known that the Godbillon-Vey class and some secondary characteristic classes for foliations admit continuous deformations, namely, they can vary continuously under deformation of foliations. If smooth families of foliations are given, the derivatives of these classes with respect to families can be considered. The derivatives are defined indeed not only with respect to actual deformations but with respect to infinitesimal deformations [10] (see also [4], [3]). We call such derivatives infinitesimal derivatives for short. It is known that infinitesimal derivatives of the Godbillon-Vey class and the Bott class are represented in terms of projective Schwarzians and their Ricci curvatures. A formula was found by Maszczyk for codimension-one foliations and by the author for those of codimension greater than one [18], [3]. The formulae suggest that there exists an invariant associated with transverse projective structures of foliations which determines the infinitesimal derivatives. Transverse projective structures are usually studied under

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the assumption that they are invariant under the holonomy (cf. [19], [5]), however, it is insufficient in the study of infinitesimal derivatives. In this paper, we first clarify transverse projective structures not necessarily invariant under the holonomy. It is an adaptation of Cartan's and Thomas' theory [23], [16]. We largely rely on the Robert version of Thomas' theory [21] but we also need a version of the Hlavatý connection [12] in order to construct appropriate connections. After discussing transverse projective structures, we will recall some relevant notions on the infinitesimal derivatives. Then, we will construct an invariant, which is a homomorphism from a foliated cohomology to the ordinary one, and describe infinitesimal derivatives by means of it. As a corollary, we obtain a rigidity theorem for the Godbillon-Vey class and the Bott class. A part of this work is done while the author enjoys his visit to the 'Institute de Mathématiques de Toulouse'. He would like to express his gratitude for their warm hospitality, especially to J. Rebelo.

1. Torsion-free and Thomas-Whitehead connections for foliations

Throughout this paper, we will assume that manifolds and foliations are smooth, namely, of class C^∞. We refer to [4] for generalities of transversely holomorphic foliations.

1.1. **Transversal torsion and Christoffel symbols.**

Let M be a manifold, $\mathcal{F}$ a foliation of M and $T\mathcal{F}$ the tangent bundle of $\mathcal{F}$. If $\mathcal{F}$ is a real foliation, then we set $E(\mathcal{F}) = T\mathcal{F}$. If $\mathcal{F}$ is transversely holomorphic, then we define $E(\mathcal{F})$ as follows. Let $(x, y) = (x^1, \dots, x^p, y^1, \dots, y^q)$ be coordinates on a foliation chart, where $y = (y^1, \dots, y^q)$ are coordinates in the transversal direction so that $y \in \mathbb{C}^q$. We define $E(\mathcal{F})$ to be the complex vector bundle locally spanned by $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}$ and $\frac{\partial}{\partial \bar{y}^1}, \dots, \frac{\partial}{\partial \bar{y}^q}$. By abuse of notations we denote by TM the complexification $TM \otimes \mathbb{C}$ of TM. In the both cases, let $Q(\mathcal{F}) = TM/E(\mathcal{F})$ be the (complex) normal bundle of $\mathcal{F}$ and $\pi\colon TM \to Q(\mathcal{F})$ the projection. If F is a vector bundle over M and if U is an open subset of M, then we denote by $\mathit{\Gamma}_U(F)$ the set of smooth (even in the transversely holomorphic case) sections of F over U. If $U = M$, then $\mathit{\Gamma}_U(F)$ is also denoted by $\mathit{\Gamma}(F)$. In what follows, we mostly deal with transversely holomorphic foliations. The arguments in the real case are easier and almost parallel.

Notation. We will frequently compare coefficients of tensors, connections, etc. in what follows. Once a chart is chosen and coefficients are defined, the symbol '$\widehat{\ }$' is used to express another chart and the coefficients on it. For example, if (U,φ) is a chart and if $a_1,\dots,a_q$ are coefficients of a tensor on (U,φ), then $(\widehat{U},\widehat{\varphi})$ represents a chart such that $U\cap\widehat{U}\neq\varnothing$ and $\widehat{a}_1,\dots,\widehat{a}_q$ represent the coefficients on $(\widehat{U},\widehat{\varphi})$. The coefficients are often considered as entries of matrices, and the multiplication rule of matrices is applied. For example, if $\omega^1,\dots,\omega^q$ are coefficients of a $\mathbb{C}^q$-valued 1-form and if a^i_j, where $1\leq i,j\leq q$, are coefficients of a $\mathfrak{gl}_q(\mathbb{C})$-valued 2-form, then we set $\omega={}^t(\omega^1\ \cdots\ \omega^q)$, $A=(a^i_j)$ and define $A\wedge\omega$ to be a $\mathbb{C}^q$-valued 3-form of which the i-th entry is given by $\sum\limits_j a^i_j\wedge\omega^j$. Finally, the Roman indices will begin from one, while the Greek indices will begin from zero.

Definition 1.1. A connection ∇ on $Q(\mathcal{F})$ is said to be a Bott connection if $\nabla_XY=\pi[X,\widetilde{Y}]$ for $X\in\Gamma(E(\mathcal{F}))$ and $Y\in\Gamma(Q(\mathcal{F}))$, where $\widetilde{Y}$ is any lift of Y to $\Gamma(TM)$. A connection $\mathcal{D}$ on $\bigwedge^q Q(\mathcal{F})$ is said to be a Bott connection if $\mathcal{D}_XY=L_XY$ if $X\in\Gamma(E(\mathcal{F}))$ and $Y\in\Gamma(\bigwedge^q Q(\mathcal{F}))$, where L_X denotes the Lie derivative.

It is well-known that Bott connections always exist. Note that a Bott connection on $Q(\mathcal{F})$ induces a Bott connection on $\bigwedge^q Q(\mathcal{F})$.

Definition 1.2. We denote by $K_{\mathcal{F}}$ the line bundle $\bigwedge^q Q(\mathcal{F})^*$, where q is the codimension of $\mathcal{F}$. If $\mathcal{F}$ is transversely holomorphic, then q is the complex codimension and $K_{\mathcal{F}}$ is called the canonical bundle of $\mathcal{F}$. We denote $\bigwedge^q Q(\mathcal{F})$ by $K_{\mathcal{F}}^{-1}$.

Definition 1.3 ([24])**.** Let ∇^b a Bott connection on $Q(\mathcal{F})$. We define a skew-symmetric $(0,2)$-tensor field T on $Q(\mathcal{F})$ by

$$T(X,Y)=\nabla^b_{\widetilde{X}}Y-\nabla^b_{\widetilde{Y}}X-\pi[\widetilde{X},\widetilde{Y}],$$

where $\widetilde{X},\widetilde{Y}$ are lifts of X,Y to TM. We call T the *transversal torsion* of ∇^b. A Bott connection is said to be transversely torsion-free if $T=0$.

Lemma 1.4. The transversal torsion is a well-defined $(0,2)$-tensor field.

Proof. First we fix $\widetilde{Y}$. Let $\widetilde{X}'$ be also a lift of X and set $\widetilde{U} = \widetilde{X}' - \widetilde{X}$. Then we have $\nabla^b_{\widetilde{X}'} Y = \nabla^b_{\widetilde{X}+\widetilde{U}} Y = \nabla^b_{\widetilde{X}} Y + \pi[\widetilde{U}, \widetilde{Y}]$. On the other hand, we have $\pi[\widetilde{X}', \widetilde{Y}] = \pi[\widetilde{X}, \widetilde{Y}] + \pi[\widetilde{U}, \widetilde{Y}]$. Hence $T(X,Y)$ is independent of the choice of $\widetilde{X}$. Similarly, we can show that $T(X,Y)$ is independent of the choice of $\widetilde{Y}$. If we replace X by fX, where f is a function, then we may assume that $\widetilde{fX} = f\widetilde{X}$. Hence $T(fX,Y) = \nabla^b_{f\widetilde{X}} Y - \nabla^b_{\widetilde{Y}} fX - \pi[f\widetilde{X}, \widetilde{Y}] = f\nabla^b_{\widetilde{X}} Y - (\widetilde{Y}f)X - f\nabla^b_{\widetilde{Y}} X - f\pi[\widetilde{X}, \widetilde{Y}] + (\widetilde{Y}f)X = fT(X,Y)$. Similarly, we can show that $T(X, fY) = fT(X,Y)$. Therefore T is a tensor. $\square$

Remark 1.5. If we set

$$\widetilde{T}(X,Y) = \nabla^b_X \pi(Y) - \nabla^b_Y \pi(X) - \pi[X,Y],$$

then, $\widetilde{T}$ is the *torsion* in the sense of [24]. We have $\widetilde{T} = \pi^* T$.

Let $U \times T \subset \mathbb{R}^q \times \mathbb{C}^q$ be a foliation chart, and let x and y be local coordinates in the leaf and transversal directions, respectively. We set

$$e_i = \pi\left(\frac{\partial}{\partial y^i}\right),$$

and let $\omega = (\omega^i_j)$ be the connection matrix of a Bott connection on $Q(\mathcal{F})$ with respect to $\{e_1, \dots, e_q\}$. Then, each ω^i_j involves only $dy^1, \dots, dy^q$, especially not $d\bar{y}_1, \dots, d\bar{y}_q$ in the transversely holomorphic case.

Definition 1.6. Let ∇^b be a Bott connection on $Q(\mathcal{F})$ and let (x,y) be coordinates on a foliation chart. We set $\Gamma^i_{jk} = dy^i\left(\nabla^b_{\frac{\partial}{\partial y^j}} e_k\right)$ and call $\left\{\Gamma^i_{jk}\right\}_{i,j,k}$ the *Christoffel symbols* of ∇^b by abuse of notations.

Lemma 1.7. Let ∇^b be a Bott connection on $Q(\mathcal{F})$.

1) $\Gamma^i_{jk} = dy^i\left(\nabla^b_{\frac{\partial}{\partial y^j}+X} e_k\right)$ if $X \in \Gamma(E(\mathcal{F}))$.
2) ∇^b is transversely torsion-free if and only if $\Gamma^i_{jk} = \Gamma^i_{kj}$ holds for any i,j,k.

Proof. We first show 1). We have $\left[X, \frac{\partial}{\partial y^k}\right] \in \Gamma(E(\mathcal{F}))$ because $X \in \Gamma(E(\mathcal{F}))$. Hence we have $\nabla^b_X e_k = \pi\left[X, \frac{\partial}{\partial y^k}\right] = 0$. Therefore, $\nabla^b_{\frac{\partial}{\partial y^j}+X} e_k = \nabla^b_{\frac{\partial}{\partial y^j}} e_k + \nabla^b_X e_k = \nabla^b_{\frac{\partial}{\partial y^j}} e_k$. Next we show 2). By Lemma 1.4, ∇^b is transversely torsion-free if and only if $T(e_j, e_k) = 0$ for any j, k. We have

$$T(e_j, e_k) = \nabla^b_{\frac{\partial}{\partial y^j}} e_k - \nabla^b_{\frac{\partial}{\partial y^k}} e_j - \pi\left[\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right] = \sum_i (\Gamma^i_{jk} - \Gamma^i_{kj}) e_i.$$

The right hand side is identically equal to 0 if and only if $\Gamma^i_{jk} = \Gamma^i_{kj}$ holds for any i, j, k. □

We set $T^i_{jk} = \frac{\Gamma^i_{jk} - \Gamma^i_{kj}}{2}$. We have then $T = \sum\limits_{i,j,k} e_i T^i_{jk} dy^j \wedge dy^k$.

Let $U \times T$ and $\widehat{U} \times \widehat{T}$ be foliation charts with coordinates (x, y) and $(\widehat{x}, \widehat{y})$. We have then $\widehat{y} = \gamma(y)$ for some (biholomorphic) diffeomorphism γ defined on an open subset of T. We refer γ as the *transversal component* of the transition function.

Lemma 1.8 (cf. [17, Proposition 7.9])**.** Transversely torsion-free Bott connections exist.

Proof. Let ∇^b be a Bott connection. If we denote by $\left\{\Gamma^i_{jk}\right\}$ the Christoffel symbols of ∇^b with respect to $\{e_1, \ldots, e_q\}$, then

$$\Gamma^i_{jk} = \sum_l \frac{\partial y^i}{\partial \widehat{y}^l} \frac{\partial^2 \widehat{y}^l}{\partial y^j \partial y^k} + \sum_{l,m,n} \frac{\partial y^i}{\partial \widehat{y}^l} \widehat{\Gamma}^l{}_{mn} \frac{\partial \widehat{y}^m}{\partial y^j} \frac{\partial \widehat{y}^n}{\partial y^k}.$$

Hence, if we set $\Gamma'^i_{jk} = \Gamma^i_{kj}$, then the family $\{\Gamma'^i_{jk}\}$ satisfies the same relation as above. Therefore, if we set $\omega^i_{jk} = \frac{\Gamma^i_{jk} + \Gamma^i_{kj}}{2}$, then $\{\omega^i_{jk}\}$ determines a connection. It is easy to see that thus defined connection is a Bott connection and transversely torsion-free. □

We refer to [24] for more on the differential geometry of foliations.

1.2. **Bundle of transversal volume elements and Thomas-Whitehead connections.**

We will introduce a version of the Thomas-Whitehead connections in the sense of [21] for foliations. First of all, we recall relevant Lie algebras.

Definition 1.9. We denote by $\mathfrak{pgl}_{q+1}(\mathbb{C}) = \mathfrak{gl}_{q+1}(\mathbb{C})/\mathbb{C}$ the Lie algebra of $\mathrm{PGL}_{q+1}(\mathbb{C}) = \mathrm{GL}_{q+1}(\mathbb{C})/\mathbb{C}^*$ and set $\mathfrak{m} = \mathbb{C}^q$.

Let $\mathfrak{m} \oplus \mathfrak{gl}_q(\mathbb{C}) \oplus \mathfrak{m}^*$ be the Lie algebra of which the Lie bracket is given by $[(x^i, x^i_j, x_j), (y^i, y^i_j, y_j)] = (z^i, z^i_j, z_j)$, where

$$\begin{cases} z^i = \sum_k x^i_k y^k - \sum_k y^i_k x^k, \\ z^i_j = \sum_k (x^i_k y^k_j - y^i_k x^k_j) + x^i y_j - y^i x_j - \sum_k (x_k y^k - y_k x^k) I_q, \\ z_j = \sum_k x_k y^k_j - \sum_k y_k x^k_j \end{cases} \tag{1.10}$$

for $(x^i, x^i_j, x_j), (y^i, y^i_j, y_j) \in \mathfrak{m} \oplus \mathfrak{gl}_q(\mathbb{C}) \oplus \mathfrak{m}^*$. Then, $\mathfrak{pgl}_{q+1}(\mathbb{C})$ is isomorphic to $\mathfrak{m} \oplus \mathfrak{gl}_q(\mathbb{C}) \oplus \mathfrak{m}^*$. Indeed, let $X \in \mathfrak{pgl}_{q+1}(\mathbb{C})$ and $A = \begin{pmatrix} a & b_i \\ c^i & D^i_j \end{pmatrix} \in \mathfrak{gl}_{q+1}(\mathbb{C})$ a representative of X. If we associate X with $(c^i, D^i_j - a, b_i)$, then it gives an isomorphism. If $\{\omega\}$ is a family of $\mathfrak{m}$-valued r-forms such that $\widehat{\omega} = D\gamma\,\omega$, where γ is the transversal component of a transition function, then $\{\omega\}$ is naturally a $Q(\mathcal{F})$-valued r-form and vice versa. Similarly, a family $\{\mu\}$ of $\mathfrak{m}^*$-valued s-forms such that $\widehat{\mu} D\gamma = \mu$ corresponds to a $Q(\mathcal{F})^*$-valued s-form.

Definition 1.11. The $\mathbb{C}^*$-principal bundle ($\mathbb{R}_{>0}$-principal bundle in the real case, where $\mathbb{R}_{>0} = \{t \in \mathbb{R} \,|\, t > 0\}$) associated with $K_{\mathcal{F}}^{-1}$ is called the bundle of *transversal volume elements* and denoted by $\mathcal{E}_{\mathcal{F}}$. We denote by $\widetilde{p}$ the projection from $\mathcal{E}_{\mathcal{F}}$ to M.

The bundle $\mathcal{E}_{\mathcal{F}}$ admits natural local coordinates as follows. Let $U \times T \subset \mathbb{R}^p \times \mathbb{C}^q$ be a foliation chart and (x, y) be coordinates on $U \times T$. If $U \times T$ is small enough, then we have a local trivialization $\mathcal{E}_{\mathcal{F}}|_{U\times T} \cong U \times T \times \mathbb{C}^*$ (or $\mathcal{E}_{\mathcal{F}}|_{U\times T} \cong U \times T \times \mathbb{R}_{>0}$). We locally identify $U \times T \times \mathbb{C}^*$ with $U \times T \times \mathbb{C}$ by the correspondence $(x, y, u) \mapsto (x, y, \log u)$, where we take refinements and choose branches of the logarithm (in the real case, we can identify $U \times T \times \mathbb{R}_{>0}$ with $U \times T \times \mathbb{R}$). By changing the order, we may use $(x, y^0, y) \in U \times \mathbb{C} \times T$ as local coordinates for $\mathcal{E}_{\mathcal{F}}$. Then, transition functions are given as follows. Let (V, φ) and $(\widehat{V}, \widehat{\varphi})$ be foliation charts such that $V \cap \widehat{V} \neq \varnothing$. If we identify V with $\varphi(V)$ and if we assume that $V = U \times T$, then the transition function from $\varphi(V \cap \widehat{V})$ to $\widehat{\varphi}(V \cap \widehat{V})$ is of the form (ψ, γ). Then a point $(x, y^0, y) \in \varphi(V) \times \mathbb{C}$ (the order of the coordinates is changed) is identified with $\widetilde{\varphi}(x, y^0, y) = (\psi(x, y), y^0 + \log\det D\gamma(y), \gamma(y))$, where log represents the fixed branch. Therefore $\mathcal{E}_{\mathcal{F}}$ is naturally equipped with a foliation which we denote by $\widetilde{\mathcal{F}}$. Indeed, $E(\widetilde{\mathcal{F}})$ is the subbundle of $T\mathcal{E}_{\mathcal{F}}$ locally spanned by

1) in the transversely holomorphic case, $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \bar{y}^\mu}$, where $1 \leq i \leq p$ and $0 \leq \mu \leq q$,
2) in the real case, $\frac{\partial}{\partial x^i}$, where $1 \leq i \leq p$.

Let $\mathcal{D}$ be a Bott connection on $K_{\mathcal{F}}^{-1}$. We denote the associated connection on $\mathcal{E}_{\mathcal{F}}$ also by $\mathcal{D}$. Then, $E(\widetilde{\mathcal{F}})$ is the horizontal lift of $E(\mathcal{F})$ with respect to $\mathcal{D}$. If $X \in TM$, we denote by X^h the horizontal lift of X to $T\mathcal{E}_{\mathcal{F}}$. We set $Q(\widetilde{\mathcal{F}}) = T\mathcal{E}_{\mathcal{F}}/E(\widetilde{\mathcal{F}})$ and denote by $\widetilde{\pi}$ the projection from $T\mathcal{E}_{\mathcal{F}}$ to $Q(\widetilde{\mathcal{F}})$. We set $\epsilon_\mu = \widetilde{\pi}\left(\frac{\partial}{\partial y^\mu}\right)$. Then $\{\epsilon_0, \dots, \epsilon_q\}$ is a local trivialization of $Q(\widetilde{\mathcal{F}})$. If we set $J = \det D\gamma$, $\frac{\partial \log J}{\partial y} = \begin{pmatrix} \frac{\partial \log J}{\partial y^1} & \cdots & \frac{\partial \log J}{\partial y^q} \end{pmatrix}$ and $\widetilde{D}\gamma = \begin{pmatrix} 1 & \frac{\partial \log J}{\partial y} \\ 0 & D\gamma \end{pmatrix}$, then the transition function on $Q(\widetilde{\mathcal{F}})$ is given by $\widetilde{D}\gamma$. The bundle $Q(\widetilde{\mathcal{F}})$ also deserves to be called the horizontal lift of $Q(\mathcal{F})$.

Definition 1.12. We denote by $p\colon Q(\widetilde{\mathcal{F}}) \to Q(\mathcal{F})$ the projection induced from the projection $\widetilde{p}_*\colon T\mathcal{E}_{\mathcal{F}} \to TM$. If $v \in Q(\mathcal{F})$, then let $\widetilde{v}$ be a lift of v to TM and set $v^h = \widetilde{\pi}(\widetilde{v}^h)$. We call v^h the *horizontal lift* of v to $Q(\widetilde{\mathcal{F}})$ with respect $\mathcal{D}$.

The following commutative diagram commutes:

$$\begin{array}{ccc} T\mathcal{E}_{\mathcal{F}} & \xrightarrow{\widetilde{\pi}} & Q(\widetilde{\mathcal{F}}) \\ {\scriptstyle \widetilde{p}_*}\downarrow & & \downarrow{\scriptstyle p} \\ TM & \xrightarrow{\pi} & Q(\mathcal{F}). \end{array}$$

Lemma 1.13. The horizontal lift of $v \in Q(\mathcal{F})$ is independent of the choice of $\widetilde{v}$.

Proof. Let $\widetilde{v}'$ be also a lift of v to TM. Since $\widetilde{v}' - \widetilde{v} \in E(\mathcal{F})$, we have $(\widetilde{v}')^h - \widetilde{v}^h \in E(\widetilde{\mathcal{F}})$ so that $\widetilde{\pi}((\widetilde{v}')^h) = \widetilde{\pi}(\widetilde{v}^h)$. □

Let $f_1 dy^1 + \cdots + f_q dy^q$ be the connection form of $\mathcal{D}$ with respect to $e_1 \wedge \cdots \wedge e_q$. Then we have

$$e_i^h = \epsilon_i - f_i \epsilon_0. \tag{1.14}$$

We formally set $e_0^h = \epsilon_0$. As $Q(\widetilde{\mathcal{F}})$ is the horizontal lift of $Q(\mathcal{F})$, it is natural to choose $\left\{e_0^h, e_1^h, \dots, e_q^h\right\}$ as a local trivialization of $Q(\widetilde{\mathcal{F}})$. We have

$$(\epsilon_0, \dots, \epsilon_q) = \left(e_0^h, \dots, e_q^h\right) \begin{pmatrix} 1 & f \\ 0 & I_q \end{pmatrix},$$

where $f = \begin{pmatrix} f_1 & \cdots & f_q \end{pmatrix}$. Note that $(\widehat{e}_\mu)^h = \widehat{e}_\mu^h$ and that $\left(\widehat{e}_0^h, \dots, \widehat{e}_q^h\right) \begin{pmatrix} 1 & 0 \\ 0 & D\gamma \end{pmatrix} = \left(e_0^h, \dots, e_q^h\right)$. This trivialization will be useful for a characterization of TW-connections (Theorem 1.20).

Definition 1.15. Let ξ be the vector field on $\mathcal{E}_\mathcal{F}$ locally given by $\frac{\partial}{\partial y^0}$. We call ξ the *canonical fundamental vector field* after [21].

Note that $\widetilde{\mathcal{F}}$ is invariant under the $\mathbb{C}$-action generated by ξ. Note also that $\widetilde{\pi}(\xi) = e_0^h = \epsilon_0$.

We introduce a foliated version of Thomas-Whitehead connections as follows.

Definition 1.16. A linear connection ∇ on $Q(\widetilde{\mathcal{F}})$ is called a *transverse Thomas-Whitehead projective connection* (transverse TW-connection, or even simply TW-connection for short) if ∇ satisfies the following conditions, namely,

1) $\nabla_X \epsilon_0 = -\frac{1}{q+1}\widetilde{\pi}(X)$ for any $X \in \Gamma(T\mathcal{E}_\mathcal{F})$.
2) $\nabla_\xi Y = -\frac{1}{q+1}Y$ for any $Y \in \Gamma(Q(\widetilde{\mathcal{F}}))$.
3) ∇ is invariant under the right $\mathbb{C}$-action generated by ξ.

Let $\mathcal{D}$ be a Bott connection on $K_\mathcal{F}^{-1}$ and ∇^b a Bott connection for $\mathcal{F}$. A transverse TW-connection ∇ is called a *transverse TW-connection for* $(\nabla^b, \mathcal{D})$ if ∇ satisfies the following additional condition, namely,

4) If $X \in \Gamma(TM)$ and $Y \in \Gamma(Q(\mathcal{F}))$, then $p(\nabla_{X^h} Y^h) = \nabla^b_X Y$, where $p\colon Q(\widetilde{\mathcal{F}}) \to Q(\mathcal{F})$ is the projection.

If the following stronger condition is satisfied instead of 4), then ∇ is said to be *standard*:

4') $\nabla_{X^h} Y^h = (\nabla^b_X Y)^h$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(Q(\mathcal{F}))$.

If $\mathcal{D}$ is induced from ∇^b, then we omit mentioning $\mathcal{D}$.

Remark 1.17. 1) The condition 1) in Definition 1.16 implies that $\nabla_{X^h}\epsilon_0 = 0$ for $X \in E(\mathcal{F})$.

2) We will later introduce the torsion and curvature of a TW-connection in Definition 1.28, which are different from those as a linear connection.
3) If $\nabla_X = 0$ for $X \in \Gamma(E(\widetilde{\mathcal{F}}))$, then we can ask ∇ to be transversely torsion-free as a linear connection on $Q(\widetilde{\mathcal{F}})$ as in [23]. Then, the conditions 1) and 2) in Definition 1.16 are equivalent. See also 3) of Remark 1.22.

Let V be a foliation chart for $\mathcal{F}$. Let $\left\{\Gamma^i_{jk}\right\}$ be the Christoffel symbols of ∇^b on V with respect to $\{e_1, \dots, e_q\}$, and let $\theta = \sum_i f_i dy^i$ be the connection form of $\mathcal{D}$ with respect to $e_1 \wedge \cdots \wedge e_q$. We consider $f = (f_1 \ \cdots \ f_q)$ as a locally defined $\mathfrak{m}^*$-valued 1-form. The following Lemma is immediate from the definition.

Lemma 1.18. Let ∇ be a TW-connection and ω the connection form of ∇ with respect to $\{\epsilon_0, \dots, \epsilon_q\}$. Then we have

$$\omega = -\frac{1}{q+1}\begin{pmatrix} dy^0 & 0 \\ dy & dy^0 I_q \end{pmatrix} + \begin{pmatrix} 0 & \nu \\ 0 & \mu \end{pmatrix}$$

for some μ and ν, where $\iota_\xi \mu = L_\xi \mu = 0$ and $\iota_\xi \nu = L_\xi \nu = 0$. We refer (μ, ν) as the *components* of ∇ with respect to $\{\epsilon_0, \dots, \epsilon_q\}$.

More precise description of μ and ν when ∇ is a TW-connection for $(\nabla^b, \mathcal{D})$ will be later given in Theorem 1.20.

Under the identification $\mathfrak{pgl}_{q+1}(\mathbb{C}) = \mathfrak{m} \oplus \mathfrak{gl}_q(\mathbb{C}) \oplus \mathfrak{m}^*$, the connection form ω as in Lemma 1.18 corresponds to $\left(-\frac{1}{q+1}dy, \mu, \nu\right)$, where $dy = {}^t\left(dy^1 \ \cdots \ dy^q\right)$. Indeed the connection form of a Cartan connection is obtained from (μ, ν) and the Maurer-Cartan form of $H^2(q)$ which will appear in §3. We refer to [7] for details.

Definition 1.19. We set, for $1 \leq i, j, k \leq q$,

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{\delta^i_j}{q+1} f_k - \frac{\delta^i_k}{q+1} f_j,$$

$$L(q)_j = df_j - \frac{1}{q+1} \sum_k f_j f_k dy^k - \sum_{i,k} f_i \Pi^i_{jk} dy^k,$$

where δ^i_j denotes the Dirac delta. We set $\Pi^i_j = \sum_k \Pi^i_{jk} dy^k$, $\Pi = (\Pi^i_j)$ and $L(q) = (L(q)_j)$.

Note that $L(q) = df - \frac{1}{q+1} f\theta - f\Pi$ and that if we know $\mathcal{D}$ then we can recover Γ from Π. By modifying the constructions of [23] and [12], we can always find a TW-connection.

Theorem 1.20. Let ∇^b be a Bott connection for $\mathcal{F}$ and $\mathcal{D}$ a Bott connection on $K_{\mathcal{F}}^{-1}$. If ∇ is a transverse TW-connection for $(\nabla^b, \mathcal{D})$, then the components of ∇ with respect to $\{\epsilon_0, \dots, \epsilon_q\}$ is of the form $(\Pi, L(q) + \alpha)$, where α is the pull-back by $\widetilde{p}$ of a $Q(\mathcal{F})^*$-valued 1-form on M. A transverse TW-connection ∇ for $(\nabla^b, \mathcal{D})$ is standard if and only if $\alpha = 0$.

Proof. First, we show that $(\Pi, L(q) + \alpha)$ as above gives rise to a connection on $Q(\widetilde{\mathcal{F}})$. Let $U = V \times \mathbb{C}$ and $\widehat{U} = \widehat{V} \times \mathbb{C}$ be foliation charts for $\widetilde{\mathcal{F}}$, and let $(\Pi, L(q)+\alpha)$ and $(\widehat{\Pi}, \widehat{L}(q)+\widehat{\alpha})$ be as above. Let γ be the transversal component of the transition function from V to $\widehat{V}$. We set $\omega = -\dfrac{1}{q+1}\begin{pmatrix} dy^0 & 0 \\ dy & dy^0 I_q \end{pmatrix} + \begin{pmatrix} 0 & L(q)+\alpha \\ 0 & \Pi \end{pmatrix}$ and define $\widehat{\omega}$ in the same way. Then, ω and $\widehat{\omega}$ are local connection forms of a connection on $Q(\widetilde{\mathcal{F}})$ if and only if they satisfy the equality $\omega = (\widetilde{D}\gamma)^{-1} d\widetilde{D}\gamma + (\widetilde{D}\gamma)^{-1}\widehat{\omega}\widetilde{D}\gamma$. We have

$$(\widetilde{D}\gamma)^{-1} d\widetilde{D}\gamma = \begin{pmatrix} 0 & d\frac{\partial \log J}{\partial y} - \frac{\partial \log J}{\partial y}(D\gamma)^{-1} dD\gamma \\ 0 & (D\gamma)^{-1} dD\gamma \end{pmatrix},$$

$$\begin{aligned}
(\widetilde{D}\gamma)^{-1}\begin{pmatrix} d\widehat{y}^0 & 0 \\ d\widehat{y} & d\widehat{y}^0 I_q \end{pmatrix}\widetilde{D}\gamma &= \begin{pmatrix} d\widehat{y}^0 & 0 \\ 0 & d\widehat{y}^0 I_q \end{pmatrix} \\
&\quad + \begin{pmatrix} -\frac{\partial \log J}{\partial y}(D\gamma)^{-1} d\widehat{y} & -\frac{\partial \log J}{\partial y}(D\gamma)^{-1} d\widehat{y}\frac{\partial \log J}{\partial y} \\ (D\gamma)^{-1} d\widehat{y} & (D\gamma)^{-1} d\widehat{y}\frac{\partial \log J}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} dy^0 & -d\log J\frac{\partial \log J}{\partial y} \\ dy & d\widehat{y}^0 + dy\frac{\partial \log J}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} dy^0 & 0 \\ dy & dy^0 I_q \end{pmatrix} + \begin{pmatrix} 0 & -d\log J\frac{\partial \log J}{\partial y} \\ 0 & (d\log J) I_q + dy\frac{\partial \log J}{\partial y} \end{pmatrix},
\end{aligned}$$

$$(\widetilde{D}\gamma)^{-1}\begin{pmatrix} 0 & \widehat{L}(q)+\widehat{\alpha} \\ 0 & \widehat{\Pi} \end{pmatrix}\widetilde{D}\gamma = \begin{pmatrix} 0 & (\widehat{L}(q)+\widehat{\alpha})D\gamma - \frac{\partial \log J}{\partial y}(D\gamma)^{-1}\widehat{\Pi}D\gamma \\ 0 & (D\gamma)^{-1}\widehat{\Pi}D\gamma \end{pmatrix}.$$

Since $\Gamma = (\Gamma^i_j)$ and $\widehat{\Gamma}$ are connection forms, we have

$$\Gamma = (D\gamma)^{-1} dD\gamma + (D\gamma)^{-1} \widehat{\Gamma} D\gamma,$$
$$\sum_j f_j dy^j = d\log J + \sum_j \widehat{f}_j d\widehat{y}^j.$$

Therefore,

$$\begin{aligned}
(\Pi^i_j) &= (D\gamma)^{-1} dD\gamma + (D\gamma)^{-1}(\widehat{\Gamma}^i_j) D\gamma \\
&\qquad - \frac{1}{q+1}\left(d\log J + \sum_j \widehat{f}_j d\widehat{y}^j \right) I_q \\
&\qquad - \frac{1}{q+1}(D\gamma)^{-1} d\widehat{y} \left(\frac{\partial \log J}{\partial y} + (\widehat{f}_j) D\gamma \right) \\
&= (D\gamma)^{-1}(\widehat{\Pi}^i_j) D\gamma \\
&\qquad + (D\gamma)^{-1} dD\gamma - \frac{1}{q+1}(d\log J) I_q - \frac{1}{q+1} dy \left(\frac{\partial \log J}{\partial y} \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&L(q) \\
&= d\frac{\partial \log J}{\partial y} + d\widehat{f} D\gamma + \widehat{f} dD\gamma - \frac{1}{q+1}\left(\frac{\partial \log J}{\partial y} + \widehat{f} D\gamma \right)(d\log J + \widehat{\theta}) \\
&\quad - \left(\frac{\partial \log J}{\partial y} + \widehat{f} D\gamma \right)(D\gamma)^{-1}(\widehat{\Pi}^i_j) D\gamma \\
&\quad - \left(\frac{\partial \log J}{\partial y} + \widehat{f} D\gamma \right)\left((D\gamma)^{-1} dD\gamma - \frac{1}{q+1}(d\log J) I_q - \frac{1}{q+1} dy \left(\frac{\partial \log J}{\partial y} \right) \right) \\
&= \widehat{L}(q) D\gamma + d\frac{\partial \log J}{\partial y} - \frac{\partial \log J}{\partial y}(D\gamma)^{-1}(\widehat{\Pi}^i_j) D\gamma \\
&\quad - \frac{\partial \log J}{\partial y}\left((D\gamma)^{-1} dD\gamma - \frac{1}{q+1} dy \left(\frac{\partial \log J}{\partial y} \right) \right).
\end{aligned}$$

Thus a connection is defined. It is easy to see that the conditions other than 4) in Definition 1.16 are satisfied. To see that the condition 4) is also satisfied and the uniqueness of the standard TW-connection, let $\{e^h_0, e^h_1, \dots, e^h_q\}$ be the local trivialization of $Q(\widetilde{\mathcal{F}})$ given by (1.14). Let ∇ be a TW-connection for $(\nabla^b, \mathcal{D})$, ω the connection form of ∇ with respect to $\{\epsilon_0, \dots, \epsilon_q\}$ and (μ, ν) the components. If we set $F =$

$\begin{pmatrix} 1 & -f \\ 0 & I_q \end{pmatrix}$, where $f = \begin{pmatrix} f_1 & \cdots & f_q \end{pmatrix}$, then we have

$$\begin{aligned} & F^{-1}dF + F^{-1}\omega F \\ &= \begin{pmatrix} -\frac{1}{q+1}dy^0 - \frac{1}{q+1}fdy & -df + \frac{1}{q+1}fdyf + \nu + f\mu \\ -\frac{1}{q+1}dy & -\frac{1}{q+1}(-dyf + dy^0 I_q) + \mu \end{pmatrix}, \end{aligned}$$

which is the connection form of ∇ with respect to $\{e_0^h, e_1^h, \dots, e_q^h\}$. Therefore, if we denote by $d\eta^0 = dy^0 + \theta, d\eta^1 = dy^1, \dots, d\eta^q = dy^q$ the dual to $\epsilon_0, \dots, \epsilon_q$, then

$$\begin{aligned} & F^{-1}dF + F^{-1}\omega F \\ &= \begin{pmatrix} -\frac{1}{q+1}d\eta^0 & -L(q) + \nu + f(\mu - \Pi) \\ -\frac{1}{q+1}d\eta & -\frac{1}{q+1}d\eta_0 I_q + \mu + \frac{1}{q+1}(fd\eta I_q + d\eta f) \end{pmatrix}. \end{aligned}$$

The connection ∇ is a transverse TW-connection for $(\nabla^b, \mathcal{D})$ if and only if

$$F^{-1}dF + F^{-1}\omega F = \begin{pmatrix} -\frac{1}{q+1}d\eta^0 & \widetilde{\alpha} \\ -\frac{1}{q+1}d\eta & -\frac{1}{q+1}d\eta^0 I_q + \Gamma \end{pmatrix},$$

where $\widetilde{\alpha}$ is an $\mathfrak{m}^*$-valued 1-form which does not involve dy^0 (and $d\bar{y}^0$). This holds if and only if $\mu = \Pi$. Moreover, ∇ is standard if and only if in addition $\widetilde{\alpha} = 0$, namely, $\nu = L(q)$. □

Definition 1.21. A TW-connection ∇ is said to be *invariant under the holonomy* if ω is the connection form of ∇ with respect to $\{\epsilon_0, \dots, \epsilon_q\}$, then $\iota_X\omega = \mathcal{L}_X\omega = 0$ for any $X \in E(\widetilde{\mathcal{F}})$, where ι and $\mathcal{L}$ denotes the inner product and the Lie derivative, respectively.

Remark 1.22. 1) If $\mathcal{D}$ is induced from ∇^b, then we have $\sum\limits_i \Pi^i_{ik} = 0$. In particular, $\Pi = 0$ if $q = 1$.

2) Let T be the transverse torsion of ∇^b. If $X, Y \in TM$, then let Z be a lift of $T(\pi(X), \pi(Y))$ to TM and set $T_\theta(X, Y) = \theta(Z)$. Then, $L(q) \wedge dy = R(\mathcal{D}) + T_\theta$, where $R(\mathcal{D})$ is the curvature of $\mathcal{D}$. If in particular ∇^b is transversely torsion-free, then $L(q) \wedge dy = R(\mathcal{D})$.
3) A standard transversal TW-connection is not always torsion-free as a linear connection on $Q(\widetilde{\mathcal{F}})$ even if $\nabla_X = 0$ for $X \in \mathit{\Gamma}(E(\widetilde{\mathcal{F}}))$. Indeed, e^h_μ are not necessarily associated with transversal coordinates. Suppose that ∇^b is transversely torsion-free and that $\theta = \sum\limits_i f_i dy^i = dg$ locally holds for some function g such that $X(g) = 0$ for any

$X \in \Gamma(E(\mathcal{F}))$. Such a g is a function only in $y^1, \ldots, y^q$, in particular, transversely holomorphic in the transversely holomorphic case. If we set $\eta^0 = y^0 + g(y^1, \ldots, y^q), \eta^1 = y^1, \ldots, \eta^q = y^q$, then, $\frac{\partial}{\partial \eta^\mu} = e^h_\mu$ and the standard TW-connection is torsion-free.

Definition 1.23. Let ∇, ∇' be transverse TW-connections. We say that ∇ and ∇' are *projectively equivalent* if there exists a section β of $Q(\widetilde{\mathcal{F}})^* \otimes T^*\mathcal{E}_{\mathcal{F}}$ with the following properties. We regard β as a section of $(T^*\mathcal{E}_{\mathcal{F}})^{\otimes 2}$ by the pull-back if necessary.

i) $\beta(\epsilon_0, X) = \beta(X, \epsilon_0)$ holds for all $X \in T\mathcal{E}^*_{\mathcal{F}}$.
ii) $\beta(\epsilon_0, \epsilon_0) = 0$ and $\beta(\epsilon_0, \cdot\,) = 0$ on $E(\widetilde{\mathcal{F}})$.
iii) β is invariant under the $\mathbb{C}$-action generated by ξ.
iv) $\nabla' - \nabla = (\iota_{\epsilon_0}\beta) \otimes \mathrm{id} + \mathrm{id} \otimes (\iota_{\epsilon_0}\beta) - \beta \otimes \epsilon_0$, where $\mathrm{id} = \sum_\mu \epsilon_\mu dy^\mu$ denotes the identity map on $Q(\widetilde{\mathcal{F}})$ and $(\iota_{\epsilon_0}\beta)(X) = \beta(\epsilon_0, X)$.

If moreover β is foliated in the sense that $\iota_X\beta = L_X\beta = 0$ if $X \in E(\widetilde{\mathcal{F}})$, then we say that ∇ and ∇' are *$\mathcal{F}$-equivalent*.

Let ∇ and ∇' be equivalent transverse TW-connections. We will calculate an explicit form of $\nabla' - \nabla$ for later use. If β is an equivalence, then we can represent

$$\beta = \sum_i b_i dy^i \otimes dy^0 + \sum_j b_j dy^0 \otimes dy^j + \sum_i dy^i \otimes \beta_i,$$

where $\iota_{\epsilon_0}\beta_i = 0$ and $L_{\epsilon_0}\beta_i = L_{\epsilon_0} b_i = 0$ for any i and j. We have

$$(\iota_{\epsilon_0}\beta) \otimes \mathrm{id} = \begin{pmatrix} 0 & b_j dy^0 \\ 0 & b_j dy^i \end{pmatrix},$$

$$\mathrm{id} \otimes (\iota_{\epsilon_0}\beta) = \begin{pmatrix} \sum_j b_j dy^j & & 0 \\ & \ddots & \\ 0 & & \sum_j b_j dy^j \end{pmatrix},$$

$$\beta \otimes \epsilon_0 = \begin{pmatrix} \sum_j b_j dy^j & b_1 dy^0 + \beta_1 & \cdots & b_q dy^0 + \beta_q \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, if we set $B = (\beta_1 \ \cdots \ \beta_q)$ and $b = (b_1 \ \cdots \ b_q)$, then

$$\nabla' - \nabla = \begin{pmatrix} 0 & -B \\ 0 & b\,dy I_q + dy\,b \end{pmatrix}. \tag{1.24}$$

We also clarify the condition that the coefficients of β should satisfy. Let $\beta = \sum_i \widehat{b}_i d\widehat{y}^i \otimes d\widehat{y}^0 + \sum_j \widehat{b}_j d\widehat{y}^0 \otimes d\widehat{y}^j + \sum_i d\widehat{y}^i \otimes \widehat{\beta}_i$ be the representation of β on another foliation chart. Then $b_i = \sum_k \widehat{b}_k (D\gamma)^k_i$ and $\beta_i = \sum_{k,l} \widehat{b}_k (D\gamma)^k_i \frac{\partial \log J}{\partial y^l} dy^l + \sum_{j,k} \widehat{b}_j \frac{\partial \log J}{\partial y^i} (D\gamma)^j_k dy^k + \sum_k \widehat{\beta}_k D\gamma^k_i$.

The following definition is classical [15].

Definition 1.25. Let ∇^b and $\nabla^{b\prime}$ be Bott connections for $\mathcal{F}$. We say that ∇^b and $\nabla^{b\prime}$ are *projectively equivalent* if ${\nabla^{b\prime}}_X Y - \nabla^b_X Y = \lambda(X)Y + \lambda(Y)X$ holds for some section λ of $Q^*(\mathcal{F})$. We call λ an *equivalence* between ∇ and ∇'. If λ is invariant under the holonomy, then we say that ∇ and ∇' are $\mathcal{F}$-equivalent.

Let ρ be the canonical form on the frame bundle $P(\mathcal{F})$ of $Q(\mathcal{F})$. We denote by ω and ω' the connection forms on $P(\mathcal{F})$ of the connections associated with ∇^b and $\nabla^{b\prime}$. Then, ρ and λ can be naturally regarded as an $\mathfrak{m}$-valued 1-form and an $\mathfrak{m}^*$-valued function on $P(\mathcal{F})$, respectively. If we represent $\rho = {}^t(\rho^1 \ \cdots \ \rho^q)$ and $\lambda = (\lambda_1 \ \cdots \ \lambda_q)$, then $\omega' - \omega = [\rho, \lambda]$, where the right hand side is defined by (1.10).

1.3. **Normal and Hlavatý connections.**

There are classical constructions on manifolds of dimension $q > 1$. If a torsion-free connection is given on a manifold, then there is a *normal*[†1] TW-connection. In this case, a transversely torsion-free Bott connection is a torsion-free connection in the usual sense. The definition of normal TW-connections is almost the same as in Theorem 1.20 but $\mathcal{D}$ is assumed to be induced from ∇^b, and $L(q)_{jk}$ is replaced by

$$L'(q)_{jk} = \frac{q+1}{q-1}\left(\sum_l \frac{\partial \Pi^l_{jk}}{\partial y^l} - \sum_{l,m} \Pi^l_{mj}\Pi^m_{lk}\right). \tag{1.26}$$

The most difference is that normal ones are Ricci flat but standard ones are in general not. Note that the Ricci curvature does not necessarily

[†1] We call the connection normal because it corresponds to the normal projective connection [15]. See 'Fundamental Theorem for TW-connections' in [21] and also [23].

make a sense for a transverse TW-connection if it is not holonomy invariant. Normal connections are useful when transversely projective foliations (Definition 2.1) are considered.

On the other hand, Hlavatý constructed a connection similar to the TW-connection in [12]. The original one is slightly different from projective connections in the sense of [23] but it can be modified as follows. We work on manifolds so that $(y^1,\ldots,y^q)$ denote local coordinates. First let

$$\Phi^i_{jk} = \Gamma^i_{jk} + \frac{1}{q^2-1}\sum_a \left(\delta^i_j(\Gamma^a_{ka} - q\Gamma^a_{ak}) + \delta^i_k(\Gamma^a_{aj} - q\Gamma^a_{ja})\right).$$

Note that if ∇ is torsion-free, then $\Phi^i_{jk} = \Pi^i_{jk}$. Let ω be a volume form and locally represent ω as $\omega = \mu\, dy^1 \wedge \cdots \wedge dy^q$. If we set

$$\gamma_j = \frac{1}{q+1}\frac{\partial \log \mu}{\partial y^j},$$

then

$$\gamma_j = \widehat{\gamma}_j + \frac{1}{q+1}\frac{\partial \log J}{\partial y^j}.$$

Note that $(q+1)\gamma_j dy^j$ is the connection form with respect to $\frac{\partial}{\partial y^1}\wedge\cdots\wedge\frac{\partial}{\partial y^q}$ of the connection on $K_{\mathcal{F}}^{-1}$ flat with respect to $\frac{1}{\mu}\frac{\partial}{\partial y^1}\wedge\cdots\wedge\frac{\partial}{\partial y^q}$. Indeed, Hlavatý made use of volume forms but connection forms suffice. If $\sum_j f_j dy^j$ is the connection form of a connection on $\bigwedge^q TM$ with respect to $\frac{\partial}{\partial y^1}\wedge\cdots\wedge\frac{\partial}{\partial y^q}$, then

$$\frac{1}{q+1}f_j = \frac{1}{q+1}\widehat{f}_j + \frac{1}{q+1}\frac{\partial \log J}{\partial y^j}.$$

Conversely, Definition 1.19 also works if we define f_j from a volume form. We will proceed as if γ_j is defined from a volume form, but we can modify the argument even if γ_j is defined by a connection. Let m, l be a function, and set

$$m_j = \frac{1}{q+1}\frac{\partial m}{\partial y^j}, \quad l_j = \frac{1}{q+1}\frac{\partial l}{\partial y^j},$$

and

$$\begin{aligned}\pi_{jk} &= \frac{\partial\gamma_j}{\partial y^k} - \sum_a \gamma_a \Phi^a_{jk} + \gamma_j m_k + l_j\gamma_k - \gamma_j\gamma_k \\ &= \frac{1}{q+1}\left(\frac{\partial^2 \log\mu}{\partial y^j \partial y^k} - \sum_a \frac{\partial \log\mu}{\partial y^a}\Phi^a_{jk} + \frac{\partial\log\mu}{\partial y^j} m_k + l_j \frac{\partial\log\mu}{\partial y^k}\right. \\ &\qquad\qquad \left. - \frac{1}{q+1}\frac{\partial\log J}{\partial y^j}\frac{\partial \log J}{\partial y^k}\right).\end{aligned}$$

Let P_{jk} be a tensor in the sense that $P_{jk} = \sum_{a,b} \widehat{P}_{ab}\frac{\partial\widehat{y}^a}{\partial y^j}\frac{\partial\widehat{y}^b}{\partial y^k}$. We set

$$\begin{aligned}\omega^0_{jk} &= P_{jk} + (q+1)\pi_{jk} \\ &= P_{jk} + \frac{\partial^2\log\mu}{\partial y^j\partial y^k} - \sum_a \frac{\partial\log\mu}{\partial y^a}\Phi^a_{jk} + \frac{\partial\log\mu}{\partial y^j} m_k + l_j\frac{\partial\log\mu}{\partial y^k} \\ &\qquad - \frac{1}{q+1}\frac{\partial\log\mu}{\partial y^j}\frac{\partial\log\mu}{\partial y^k}.\end{aligned}$$

We can show, in a similar way to prove Theorem 1.20, the following.

Theorem 1.27 (cf. [12])**.** We can define a TW-connection by locally setting

$$\omega = -\frac{1}{q+1}\begin{pmatrix} dy^0 & 0 \\ dy & dy^0 I_q\end{pmatrix} + \begin{pmatrix} \sum_k m_k dy^k & l_j dy^0 + \sum_k P_{jk} dy^k + (q+1)\sum_k \pi_{jk} dy^k \\ 0 & \sum_k \Phi^i_{jk} dy^k \end{pmatrix}.$$

Compared with the original one, $\omega^0_{0\gamma}, \omega^0_{j0}$ and π are modified. As a consequence, ω^0_{jk} are also modified. The connection given in Theorem 1.20 can be regarded as a variant of the Hlavatý connections.

1.4. **Curvature and torsion of TW-connections.**

Let ∇ be a TW-connection, ω the connection form of ∇ with respect to $\{\epsilon_0, \dots, \epsilon_q\}$ and (μ, ν) the components. Then, the curvature form of ∇ as a linear connection on $Q(\widetilde{\mathcal{F}})$ is given by

$$R = (R^i_j) = \begin{pmatrix} -\frac{1}{q+1}\nu\wedge dy & d\nu + \nu\wedge\mu \\ -\frac{1}{q+1}\mu\wedge dy & d\mu + \mu\wedge\mu - \frac{1}{q+1}dy\wedge\nu. \end{pmatrix}$$

Under the identification $\mathfrak{pgl}_{q+1}(\mathbb{C}) = \mathfrak{m} \oplus \mathfrak{gl}_q(\mathbb{C}) \oplus \mathfrak{m}^*$, R corresponds to $(\Omega^i, \Omega^i_j, \Omega_j)$, where

$$\begin{aligned}
\Omega^i &= -\frac{1}{q+1}\sum_k \mu^i_k \wedge dy^k,\\
\Omega^i_j &= (d\mu + \mu \wedge \mu)^i_j - \frac{1}{q+1}dy^i \wedge \nu_j + \frac{1}{q+1}\sum \nu_k \wedge dy^k,\\
\Omega_j &= d\nu_j + \sum_k \nu_k \wedge \mu^k_j.
\end{aligned}$$

Definition 1.28. We call (Ω^i_j, Ω_j) the *curvature* of ∇ and Ω^i the *torsion* of ∇. We say that ∇ is *torsion-free* if $\Omega^i = 0$.

Let ∇ be a TW-connection for $(\nabla^b, \mathcal{D})$. Then, the components of ∇ with respect to $\{\epsilon_0, \dots, \epsilon_q\}$ is given by $(\Pi, L(q) + \alpha)$. Therefore we have

$$\begin{aligned}
\Omega^i &= -\frac{1}{q+1}\sum_k \Pi^i_k \wedge dy^k\\
&= \sum_{j,k} \frac{1}{q+1}T^i_{jk}dy^j \wedge dy^k,\\
\Omega^i_j &= (d\Pi + \Pi \wedge \Pi)^i_j - \frac{1}{q+1}dy^i \wedge (L(q)_j + \alpha_j)\\
&\qquad + \frac{1}{q+1}\sum_k (L(q)_k + \alpha_k) \wedge dy^k,\\
\Omega_j &= d(L(q) + \alpha)_j + \sum_k (L(q)_k + \alpha_k) \wedge \Pi^k_j\\
&= -\sum_i f_i\Omega^i_j + d\alpha_j + \sum_i \alpha_i \wedge \Gamma^i_j - \frac{1}{q+1}\sum_{k,l,m} f_j f_l T^l_{km} dy^k \wedge dy^m,
\end{aligned}$$

where $T^i_{jk} = \frac{\Gamma^i_{jk} - \Gamma^i_{kj}}{2}$ are the coefficients of the transversal torsion T of ∇^b. Note that $\sum\limits_\mu R^\mu_\mu = 0$ and

$$R^0_0 = -\frac{1}{q+1}\left(d\theta + \sum_{i,j,k} f_i T^i_{jk} dy^j \wedge dy^k + \sum_j \alpha_j \wedge dy^j\right).$$

These equalities can be shown by lengthy calculations which we omit.

2. TRANSVERSE PROJECTIVE STRUCTURES

Let $G^2(q)$ the set of 2-frames at $0 \in \mathbb{C}^q$. It is well-known [16] that $G^2(q) = \{(s^i_j; s^i_{jk}) \,|\, (s^i_j) \in \mathrm{GL}_q(\mathbb{C}),\ s^i_{jk} = s^i_{kj}\}$ and $G^2(q)$ has the group structure such that

$$(s'^i_j; s'^i_{jk})(s^i_j; s^i_{jk}) = \left(s'^i_p s^p_j; \sum_p s'^i_p s^p_{jk} + \sum_{q,r} s'^i_{qr} s^q_j s^r_k \right).$$

We set

$$H^2(q) = \{(s^i_j; s^i_{jk}) \in G^2(q) \,|\, \exists\, s_i \text{ s.t. } s^i_{jk} = -(s^i_j s_k + s^i_k s_j)\}$$

and

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{PGL}_q(\mathbb{C}) \right\}.$$

The group H is naturally isomorphic to $H^2(q)$ via linear fractional transformations.

Definition 2.1. Let $P^2(\mathcal{F})$ be the transversal 2-jet bundle [19]. A *transverse projective structure* of $\mathcal{F}$ is a subbundle of $P^2(\mathcal{F})$ of which structure group is $H^2(q)$ [19], [5]. If $\mathcal{F}$ admits a transverse projective structure invariant under the holonomy, then $\mathcal{F}$ is said to be *transversely projective*. If in addition the transverse projective structure is given by that of $\mathbb{C}P^q$, then the transverse projective structure is said to be *flat*.

Transverse projective structures are often assumed to be flat and the term 'flat' is omitted. However, we consider the both in this paper.

Theorem 2.2. Let ∇^b and $\nabla^{b\prime}$ be Bott connections for $\mathcal{F}$, and let $\mathcal{D}$ and $\mathcal{D}'$ be Bott connections on $K_{\mathcal{F}}^{-1}$.

1) Transverse TW-connections ∇ for $(\nabla^b, \mathcal{D})$ and ∇' for $(\nabla^{b\prime}, \mathcal{D}')$ are projectively equivalent in the sense of Definition 1.23 if and only if ∇^b and $\nabla^{b\prime}$ are projectively equivalent in the sense of Definition 1.25. Therefore, the projective equivalence classes of Bott connections on $Q(\mathcal{F})$ are in one-to-one correspondence to the projective equivalence classes of transverse TW-connections on $Q(\widetilde{\mathcal{F}})$.
2) Any transverse TW-connection for $(\nabla^b, \mathcal{D})$ is equivalent to the standard one for $(\nabla^b, \mathcal{D})$ and also to TW-connections for ∇^b.
3) Transverse TW-connections ∇ for ∇^b and ∇' for $\nabla^{b\prime}$ are projectively equivalent if and only if ∇^b and $\nabla^{b\prime}$ are projectively equivalent.

4) A torsion-free TW-connection determines a transverse projective structure. Therefore, transverse projective structures are in one-to-one correspondence to the projective equivalence class of transversely torsion-free Bott connections.
5) Any TW-connection is projectively equivalent to a TW-connection for some $(\nabla^b, \mathcal{D})$.

Proof. We will denote by $\theta = \sum_i f_i dy^i$ and $\theta' = \sum_i f'_i dy^i$ the connection forms of $\mathcal{D}$ and $\mathcal{D}'$ with respect to $e_1 \wedge \cdots \wedge e_q$. First, we show 1). If an equivalence between ∇ and ∇' is given by β, then we can define a 1-form λ on M by setting $\lambda = \sum_j b_j dy^j$. By (1.24), $\lambda + \dfrac{1}{q+1}(\theta' - \theta)$ gives a projective equivalence between ∇^b and $\nabla^{b\prime}$. Conversely let λ be a projective equivalence between ∇^b and $\nabla^{b\prime}$. We set $b_i = \lambda_i - \dfrac{1}{q+1}(f'_i - f_i)$, $\beta_i = -(L(q)' - \alpha')_i + (L(q) - \alpha)_i$ and $\beta = \sum_i b_i dy^i \otimes dy^0 + \sum_j b_j dy^0 \otimes dy^j + \sum_i dy^i \otimes \beta_i$. Then β is well-defined and an equivalence between ∇ and ∇'. The part 2) follows from 1) at once. The part 3) follows from 1) and 2). The part 4) is well-known. Indeed, if (μ, ν) be the components of ∇ with respect to $\{\epsilon_0, \ldots, \epsilon_q\}$, then $\left(-\frac{1}{q+1} dy, \Pi, L(q) + \alpha\right)$ viewed as a family of $\mathfrak{m} \oplus \mathfrak{gl}_q(\mathbb{C}) \oplus \mathfrak{m}^*$-valued forms, namely viewed as the connection form of a Cartan connection, determines a projective structure and vice versa (cf. [16], [7]). In order to show 5), first assume that ∇ is torsion-free. Then we have a subbundle of $P^2(\mathcal{F})$ with structure group $H^2(q)$. Therefore, we have a section, say σ, to $P^2(\mathcal{F})/H^2(q)$. Since $H^2(q)/\mathrm{GL}_q(\mathbb{C})$ is contractible, we can find a lift of σ to a section to $P^2(\mathcal{F})/\mathrm{GL}_q(\mathbb{C})$. The last section gives a transversely torsion-free Bott connection which we denote by ∇^b. Let now (μ, ν) be the components of ∇ with respect to $\{\epsilon_0, \ldots, \epsilon_q\}$. As ∇^b is given by a lift of σ, we can find a connection $\mathcal{D}$ of $K_{\mathcal{F}}^{-1}$ of which the connection form is given by $\theta = \sum_i f_i dy^i$ and that $\mu^i_{jk} = \Gamma^i_{jk} - \frac{\delta^i_j}{q+1} f_k - \frac{\delta^i_k}{q+1} f_j$, where Γ^i_{jk} are the transverse Christoffel symbols of ∇^b. Let ∇' be the standard TW-connection for $(\nabla^b, \mathcal{D})$. If we define β as in the proof of 1), β is an equivalence between ∇ and ∇'. Suppose now that ∇ is not necessarily torsion-free. In this case, we set ${\mu'}^i_{jk} = \frac{\mu^i_{jk}+\mu^i_{kj}}{2}$. If we regard (μ', ν) as the components with respect to $\{\epsilon_0, \dots, \epsilon_q\}$, then a TW-connection without torsion is defined. If we denote this connection by ∇', then we can find Bott connections ∇^b and $\mathcal{D}$ such that the standard TW-connection for $(\nabla^b, \mathcal{D})$ is projectively equivalent to ∇'. Let Γ^i_{jk} be the transverse Christoffel symbols of ∇^b and set ${\Gamma'}^i_{jk} = \Gamma^i_{jk} + \frac{\mu^i_{jk}-\mu^i_{kj}}{2}$. Then $\{{\Gamma'}^i_{jk}\}$ defines a Bott connection which we denote by ∇'^b, and ∇ is projectively equivalent to the standard TW-connection for $(\nabla'^b, \mathcal{D})$ by the part 2). □

Theorem 2.3. 1) A foliation $\mathcal{F}$ is transversely projective if there is a transverse TW-connection on $Q(\widetilde{\mathcal{F}})$ which is invariant under holonomy.

2) Suppose conversely that $\mathcal{F}$ is transversely projective and that $q > 1$. Then, $\mathcal{F}$ admits a *normal* transverse TW-connection for a transversely torsion-free Bott connection ∇^b, namely, $\mathcal{F}$ admits a transverse TW-connection of which the components with respect to $\{\epsilon_0, \dots, \epsilon_q\}$ is given by $(\Pi, L'(q))$, where Π is given by the formula in Definition 1.19 with $f_j = \sum\limits_i \Gamma^i_{ij}$ and $L'(q)$ is given by the formula (1.26) in §§ 1.3. Moreover, the standard transverse TW-connection for ∇^b is projectively equivalent to the normal transverse TW-connection.

3) Suppose that $\mathcal{F}$ admits a transverse TW-connection ∇ on $Q(\widetilde{\mathcal{F}})$ which is invariant under the holonomy. If ∇ is torsion-free and the curvature of ∇ is equal to zero, then the transverse projective structure determined by ∇ is flat.

4) Suppose that $\mathcal{F}$ is transversely projective and let ∇ be a holonomy invariant TW-connection for $(\nabla^b, \mathcal{D})$, where ∇^b is torsion-free and $\mathcal{D}$ is induced by ∇^b. If ∇ is Ricci-flat as a linear connection on $Q(\widetilde{\mathcal{F}})$, then ∇ is transversely normal connection.

Proof. If a transverse TW-connection invariant under the holonomy exits, then we can modify it to be transversely torsion-free as in the proof of 5) of Theorem 2.2. Once we have a transverse TW-connection invariant under the holonomy, the projective structure found by Theorem 2.2 is also invariant under the holonomy. This shows 1). In order to show 2), suppose that $\mathcal{F}$ is transversely projective. We can then find a projective structure on $T = \coprod\limits_\lambda T_\lambda$ invariant under the holonomy. By locally pulling-back the normal projective connection on

T, we obtain a normal transverse TW-connection. The part 3) can be shown in a similar way to the proof of Theorem 15 in [16]. Finally suppose again that $\mathcal{F}$ is transversely projective and ∇ is holonomy invariant. Suppose in addition that ∇^b is torsion-free and $\mathcal{D}$ is induced by ∇^b. Then, the Ricci curvature makes a sense, and we have $\sum_i R^i_{jik} = -L'(q)_{jk} + (L(q) + \alpha)_{jk}$. □

Remark 2.4. The connection ∇^b in 1) is not necessarily invariant under the holonomy. Note also that ∇ in 4) is Ricci-flat if the transverse projective structure is indeed flat.

If we consider only transverse projective structures invariant under the holonomy, it is natural to restrict ourselves to $\mathcal{F}$-equivalences in Definition 1.23 and Definition 1.25.

3. Bott, Godbillon-Vey classes and their infinitesimal derivatives

We will introduce the Godbillon-Vey and the Bott classes and infinitesimal derivatives of them. For the sake of simplicity, we assume for a while that $K_{\mathcal{F}} = \bigwedge^q Q(\mathcal{F})^*$ is trivial. We refer to [2] for detailed accounts. Let Ω be a trivialization of $K_{\mathcal{F}}$ and regard Ω as a q-form on M. By virtue of Frobenius' theorem, there is a 1-form, say θ, such that $d\Omega + \theta \wedge \Omega = 0$. This θ is essentially the connection form of a Bott connection on $K_{\mathcal{F}}^{-1}$ with respect to the dual of Ω, and the Bott vanishing theorem [6] shows that the differential form $\theta \wedge (d\theta)^q$ is closed. It can be shown that the differential form represents a cohomology class independent of the choice of Ω and θ.

Definition 3.1. 1) In the real case, the cohomology class of $H^{2q+1}(M;\mathbb{R})$ represented by $\left(\frac{-1}{2q}\right)^{2q+1} \theta \wedge (d\theta)^q$ is called the Godbillon-Vey class and denoted by $\mathrm{GV}_q(\mathcal{F})$.

2) In the transversely holomorphic case, the cohomology class of $H^{2q+1}(M;\mathbb{C})$ represented by $\left(\frac{-1}{2q\sqrt{-1}}\right)^{2q+1} \theta \wedge (d\theta)^q$ is called the Bott class and denoted by $\mathrm{Bott}_q(\mathcal{F})$. If $K_{\mathcal{F}}$ is not necessarily trivial, then the Bott class is defined as an element of $H^{2q+1}(M;\mathbb{C}/\mathbb{Z})$.

If $K_{\mathcal{F}}$ is trivial, then the Bott class in $H^{2q+1}(M;\mathbb{C}/\mathbb{Z})$ coincides with the natural image of $\mathrm{Bott}_q(\mathcal{F})$.

Let $\{\mathcal{F}_s\}$ be a one-parameter smooth family of foliations of codimension q such that $\mathcal{F}_0 = \mathcal{F}$. If transversely holomorphic foliations are considered, we assume that transversal holomorphic structures also vary smoothly (see [3] for a precise definition). We still assume that $K_{\mathcal{F}_s}$ is trivial for each s for simplicity. If s is small enough, then we may assume that there is a smooth family of trivializations $\{\Omega_s\}$ and 1-forms $\{\theta_s\}$ such that $d\Omega_s + \theta_s \wedge \Omega_s = 0$. If we set $\theta = \theta_0$, $\dot{\theta} = \frac{\partial}{\partial s}\theta_s\big|_{s=0}$, then we have $\frac{\partial}{\partial s}(\theta_s \wedge (d\theta_s)^q)\big|_{s=0} = \dot{\theta} \wedge (d\theta)^q + q\theta \wedge d\dot{\theta} \wedge (d\theta)^{q-1}$. Since we have $d(\theta \wedge \dot{\theta}) = d\theta \wedge d\dot{\theta} - \theta \wedge d\dot{\theta}$, the right hand side is cohomologous to $(q+1)\dot{\theta} \wedge (d\theta)^q$. The differential form $(q+1)\dot{\theta} \wedge (d\theta)^q$ makes a sense even if $K_{\mathcal{F}}$ is non-trivial, and represents the derivative of the Godbillon-Vey class or the Bott class with respect to $\{\mathcal{F}_s\}$.

The derivatives of the Godbillon-Vey class and Bott class with respect to infinitesimal deformations of foliations are defined as a generalization of the above construction. In order to explain it, we need some definitions.

Definition 3.2. Let $\Omega^r(U) = \mathit{\Gamma}_U(\bigwedge^r T^*M)$ be the set of $\mathbb{C}$-valued differential forms of class C^∞ on an open subset U of M. If E is a vector bundle over M, we denote by $\Omega^r(U;E) = \mathit{\Gamma}_U(\bigwedge^r T^*M \otimes E)$ the set of E-valued r-forms on U. We denote by $I^*_k(U;E)$ the ideal of $\Omega^*(U;E)$ locally generated by $dy^{i_1} \wedge \cdots \wedge dy^{i_k} \otimes s$, where $i_1 < \cdots < i_k$ and $s \in \mathit{\Gamma}_U(E)$. If E is a trivial line bundle, then we denote $I^r_k(U;E)$ by $I^r_k(U)$. We set $C^r_{\mathcal{F}}(U;Q(\mathcal{F})) = \Omega^r(U;Q(\mathcal{F}))/I^r_1(U;Q(\mathcal{F}))$.

Note that naturally $C^r_{\mathcal{F}}(U;Q(\mathcal{F})) \cong \mathit{\Gamma}_U(\bigwedge^r E(\mathcal{F})^* \otimes Q(\mathcal{F}))$.

Definition 3.3. Let ∇^b be a Bott connection on $Q(\mathcal{F})$. Let $\{e_1, \ldots, e_q\}$ be a local trivialization of $Q(\mathcal{F})$ and τ the connection form of ∇^b with respect to $\{e_1, \ldots, e_q\}$. If $c \in C^r_{\mathcal{F}}(U;Q(\mathcal{F}))$, then we locally represent $c = \sum_i c^i \otimes e_i$ and set

$$d_{\mathcal{F}}\, c = \sum_i \left(dc^i + \sum_j \tau^i_j \wedge c^j \right) \otimes e_i \mod I^{r+1}_1(U;Q(\mathcal{F})).$$

We denote by $H^*_{\mathcal{F}}(M;Q(\mathcal{F}))$ the (co)homology of $(C^*_{\mathcal{F}}(M;Q(\mathcal{F})), d_{\mathcal{F}})$.

Note that if we choose $\{dy^1, \ldots, dy^q\}$ as a local trivialization, then $\tau = \Gamma$.

Lemma 3.4. $(C^*_{\mathcal{F}}(M; Q(\mathcal{F})), d_{\mathcal{F}})$ is indeed a cochain complex, namely, we have $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$.

Proof. Let $c = \sum_i c^i \otimes e_i \in C^r_{\mathcal{F}}(U; Q(\mathcal{F}))$. We have

$$\begin{aligned}
&d_{\mathcal{F}}(d_{\mathcal{F}} c)\\
&= d_{\mathcal{F}}\left(\sum_i \left(dc^i + \sum_j \tau^i_j \wedge c^j\right) \otimes e_i\right)\\
&= \sum_i \left(\sum_j d\Theta^i_j \wedge c^j - \sum_j \tau^i_j \wedge dc^j + \sum_j \tau^i_j \wedge dc^j + \sum_{j,k} \tau^i_j \wedge \tau^j_k \wedge c^k\right) \otimes e_i\\
&= \sum_{i,j} (d\tau + \tau \wedge \tau)^i_j \wedge c^j \otimes e_i\\
&= 0
\end{aligned}$$

because $d\tau + \tau \wedge \tau \in I^2_1(U; Q(\mathcal{F}) \otimes Q(\mathcal{F})^*)$. □

Let $\Theta_{\mathcal{F}}$ be the sheaf of germs of vector fields which preserves $\mathcal{F}$. If $\mathcal{F}$ is transversely holomorphic, then we assume that vector fields preserve the transverse holomorphic structure. A vector field X is such a one if $[X, Y] \in \Gamma(E(\mathcal{F}))$ for any $Y \in \Gamma(E(\mathcal{F}))$. The following is known.

Theorem 3.5 (Heitsch [9], Duchamp-Kalka [8])**.** The complex $(C^*_{\mathcal{F}}(M; Q(\mathcal{F})), d_{\mathcal{F}})$ is a resolution of $\Theta_{\mathcal{F}}$ so that $H^*_{\mathcal{F}}(M; Q(\mathcal{F})) \cong H^*(M; \Theta_{\mathcal{F}})$.

Definition 3.6 (Heitsch [9], Duchamp-Kalka [8])**.** The elements of $H^1(M; \Theta_{\mathcal{F}})$ are called *infinitesimal deformations* of $\mathcal{F}$.

Derivatives of the Godbillon-Vey class and the Bott class with respect to infinitesimal deformations of $\mathcal{F}$ are defined as follows [9], [10], [3], [4]. Let $\alpha \in H^1(M; \Theta_{\mathcal{F}})$. By Theorem 3.5, $-\alpha$ is represented by a $Q(\mathcal{F})$-valued 1-form σ such that $d\sigma + \tau \wedge \sigma + \dot{\tau} \wedge \omega = 0$ for some $\mathfrak{gl}_q(\mathbb{C})$-valued ($\mathfrak{gl}_q(\mathbb{R})$-valued) 1-form $\dot{\tau}$. We set $\dot{\theta} = \operatorname{tr} \dot{\tau}$ and $\theta = \operatorname{tr} \tau$. It can be shown that $\dot{\theta} \wedge (d\theta)^q$ is closed and represents a class independent of the choices.

Definition 3.7. 1) In the real case, the element of $H^{2q+1}(M; \mathbb{R})$ represented by $\left(\frac{-1}{2q}\right)^{2q+1} (q+1)\dot{\theta} \wedge (d\theta)^q$ is called the *infinitesimal derivative* of the Godbillon-Vey class with respect to α.

2) In the transversely holomorphic case, the element of $H^{2q+1}(M;\mathbb{C})$ represented by $\left(\frac{-1}{2q\sqrt{-1}}\right)^{2q+1}(q+1)\dot{\theta}\wedge(d\theta)^q$ is called the *infinitesimal derivative* of the Bott class with respect to α.
3) The Godbillon-Vey class or the Bott class is said to be *infinitesimally rigid* if their infinitesimal derivatives vanish with respect to any infinitesimal deformation.

It is known that a smooth family $\{\mathcal{F}_s\}$ of foliations induces a class in $H^1(M;\Theta_{\mathcal{F}})$, and that if α is induced from $\{\mathcal{F}_s\}$, then the infinitesimal derivative of the Godbillon-Vey class (or the Bott class) with respect to α coincides with the derivative of the Godbillon-Vey class (or the Bott class) with respect to s at $s=0$.

4. Transverse projective structures and infinitesimal derivatives

Definition 4.1. Let γ be a (biholomorphic) diffeomorphism from an open subset, say U, of $\mathbb{C}^q$ to an open subset of $\mathbb{C}^q$. We set $\widehat{y}=\gamma(y)$ for $y\in U$, and

$$\begin{aligned}
\Sigma(\gamma)^i_{jk} &= \sum_l \frac{\partial y^i}{\partial \widehat{y}^l}\frac{\partial^2\widehat{y}^l}{\partial y^j\partial y^k} - \frac{\delta^i_j}{q+1}\frac{\partial\log J}{\partial y^k} - \frac{\delta^i_k}{q+1}\frac{\partial\log J}{\partial y^j},\\
\Lambda(\gamma)_{jk} &= \frac{-1}{q+1}\frac{\partial^2\log J}{\partial y^j\partial y^k} - \frac{1}{(q+1)^2}\frac{\partial\log J}{\partial y^j}\frac{\partial\log J}{\partial y^k}\\
&\qquad + \sum_{l,m}\frac{1}{q+1}\frac{\partial\log J}{\partial y^l}\frac{\partial y^l}{\partial\widehat{y}^m}\frac{\partial^2\widehat{y}^m}{\partial y^j\partial y^k},
\end{aligned}$$

where $J=\det D\gamma$. We set $\Sigma(\gamma)^i_j=\sum\limits_k \Sigma(\gamma)^i_{jk}dy^k$ and $\Sigma(\gamma)=(\Sigma(\gamma)^i_j)$. Similarly, we set $\Lambda(\gamma)_j=\sum\limits_k \Lambda(\gamma)_{jk}dy^k$ and $\Lambda(\gamma)=(\Lambda(\gamma)_j)$. We call $\Sigma(\gamma)$ the (projective) Schwarzian of γ, and $\Lambda(\gamma)$ the curvature of the Schwarzian, respectively.

Remark 4.2. 1) We have $\Sigma(\gamma)^i_{jk}=\Sigma(\gamma)^i_{kj}$ and $\Lambda(\gamma)_{jk}=\Lambda(\gamma)_{kj}$.
2) We have $\sum\limits_i \Sigma(\gamma)^i_{ik}=0$. In particular, if $q=1$, then we have $\Sigma(\gamma)^1_{11}=0$. On the other hand, $\Lambda(\gamma)_{11}=\dfrac{-1}{2}\left(\dfrac{\gamma'''}{\gamma'}-\dfrac{3}{2}\left(\dfrac{\gamma''}{\gamma'}\right)^2\right)$ which is the classical Schwarzian.

3) If $q > 1$, then $\Lambda(\gamma)$ is equal to be the Ricci curvature of $\Sigma(\gamma)$ multiplied by $\dfrac{-1}{q-1}$ in the sense that we have

$$\Lambda(\gamma)_{jk} = \frac{-1}{q-1}\left(\sum_l \frac{\partial\Sigma(\gamma)^l_{jk}}{\partial y^l} - \sum_{m,n}\Sigma(\gamma)^m_{nj}\Sigma(\gamma)^n_{mk}\right).$$

It is known that

Definition 4.3. Let ∇ be a TW-connection and $\mathcal{B}$ be a Bott connection on $K_{\mathcal{F}}^{-1}$. Let ω and $\zeta = \sum_j g_j dy^j$ be connection forms of ∇ and $\mathcal{B}$ with respect to $\{\epsilon_0,\dots,\epsilon_q\}$ and $e_1\wedge\cdots\wedge e_q$, respectively. Let (μ,ν) be the components of ∇ with respect to $\{\epsilon_0,\dots,\epsilon_q\}$ and set $N = N(\nabla,\mathcal{B}) = \nu + g\mu$, where $g = (g_1 \ \cdots \ g_q)$.

If ∇ is a TW-connection for $(\nabla^b,\mathcal{D})$, then $(\mu,\nu) = (\Pi, L(q)+\alpha)$ and $N(\nabla,\mathcal{D}) = L(q)+\alpha+f\Pi$, where $\theta = \sum_j f_j dy^j$ is the connection form of $\mathcal{D}$.

Proposition 4.4. Let ∇^b, $\mathcal{D}$ and $\mathcal{B}$ be Bott connections and ∇ the standard TW-connection for $(\nabla^b,\mathcal{D})$. Then $(d\theta)^q = (N(\nabla,\mathcal{B})\wedge dy)^q$.

Proof. By Theorem 1.20, we have $N\wedge dy = (L(q)-g\Pi)\wedge dy = d\theta - g\Pi\wedge dy$. Since $-g\Pi\wedge dy \in I_2^2(U)$, we have $(N\wedge dy)^q = (d\theta)^q$. □

A study on relationship between projective connections and Chern forms can be found in [14].

Lemma 4.5. We have $N = (\widehat{N} + (q+1)\Lambda(\widehat{\gamma}) - \widehat{f}\Sigma(\widehat{\gamma}))D\gamma$, where $\widehat{\gamma} = \gamma^{-1}$. If ∇' is also a TW-connection, then $(N(\nabla',\mathcal{B})-N(\nabla,\mathcal{B}))\otimes dy$ is globally well-defined.

Proof. We make use of the same notation as in the proof of Theorem 1.20. Let (μ,ν) and $(\widehat{\mu},\widehat{\nu})$ be the components of ∇ with respect to $\{\epsilon_0,\dots,\epsilon_q\}$ and $\{\widehat{\epsilon}_0,\dots,\widehat{\epsilon}_q\}$, respectively. As in the proof of Theorem 1.20, we have

$$\begin{aligned}\nu &= d\frac{\partial\log J}{\partial y} - \frac{\partial\log J}{\partial y}(D\gamma)^{-1}dD\gamma + \frac{1}{q+1}d\log J\frac{\partial\log J}{\partial y}\\ &\quad + \widehat{\lambda}D\gamma - \frac{\partial\log J}{\partial y}(D\gamma)^{-1}\widehat{\mu}D\gamma,\\ \mu &= (D\gamma)^{-1}dD\gamma - \frac{1}{q+1}(d\log J)I_q - \frac{1}{q+1}dy\frac{\partial\log J}{\partial y} + (D\gamma)^{-1}\widehat{\mu}D\gamma.\end{aligned}$$

Since $g = \widehat{g}D\gamma + \frac{\partial \log J}{\partial y}$, we have

$$\begin{aligned}
N &= \nu + g\mu \\
&= \widehat{N}D\gamma + d\frac{\partial \log J}{\partial y} - \frac{\partial \log J}{\partial y}(D\gamma)^{-1}dD\gamma + \frac{1}{q+1}d\log J\frac{\partial \log J}{\partial y} \\
&\quad - \frac{\partial \log J}{\partial y}(D\gamma)^{-1}\widehat{\mu}D\gamma \\
&\quad + \widehat{g}D\gamma\left((D\gamma)^{-1}dD\gamma - \frac{1}{q+1}(d\log J)I_q - \frac{1}{q+1}dy\frac{\partial \log J}{\partial y}\right) \\
&\quad + \frac{\partial \log J}{\partial y}\left((D\gamma)^{-1}dD\gamma - \frac{1}{q+1}(d\log J)I_q - \frac{1}{q+1}dy\frac{\partial \log J}{\partial y} + (D\gamma)^{-1}\widehat{\mu}D\gamma\right) \\
&= \widehat{N}D\gamma + d\frac{\partial \log J}{\partial y} - \frac{1}{q+1}\frac{\partial \log J}{\partial y}(d\log J) \\
&\quad + \widehat{g}D\gamma\left((D\gamma)^{-1}dD\gamma - \frac{1}{q+1}(d\log J)I_q - \frac{1}{q+1}dy\frac{\partial \log J}{\partial y}\right).
\end{aligned}$$

On the other hand, if we set $\widehat{\gamma} = \gamma^{-1}$ and $\widehat{J} = \det D\widehat{\gamma}$, then $\frac{\partial \log \widehat{J}}{\partial \widehat{y}}D\gamma + \frac{\partial \log J}{\partial y} = 0$. Therefore, we have

$$\begin{aligned}
&d\frac{\partial \log J}{\partial y} - \frac{1}{q+1}\frac{\partial \log J}{\partial y}(d\log J) \\
&= -d\frac{\partial \log J}{\partial \widehat{y}}(D\widehat{\gamma})^{-1} + \frac{\partial \log \widehat{J}}{\partial y}(D\widehat{\gamma})^{-1}dD\widehat{\gamma}(D\widehat{\gamma})^{-1} \\
&\quad - \frac{1}{q+1}\frac{\partial \log \widehat{J}}{\partial \widehat{y}}(D\widehat{\gamma})^{-1}(d\log \widehat{J}) \\
&= (q+1)\Lambda(\widehat{\gamma})D\widehat{\gamma}^{-1},
\end{aligned}$$

and

$$\begin{aligned}
&(D\gamma)^{-1}dD\gamma - \frac{1}{q+1}(d\log J)I_q - \frac{1}{q+1}dy\frac{\partial \log J}{\partial y} \\
&= -D\widehat{\gamma}(D\widehat{\gamma})^{-1}dD\widehat{\gamma}(D\widehat{\gamma})^{-1} + \frac{1}{q+1}(d\log \widehat{J})I_q + \frac{1}{q+1}D\widehat{\gamma}d\widehat{y}\frac{\partial \log \widehat{J}}{\partial \widehat{y}}(D\widehat{\gamma})^{-1} \\
&= -D\widehat{\gamma}\Sigma(\widehat{\gamma})(D\widehat{\gamma})^{-1}.
\end{aligned}$$

Consequently, $N = (\widehat{N} + (q+1)\Lambda(\widehat{\gamma}) - \widehat{g}\Sigma(\widehat{\gamma}))D\gamma$. Therefore, $N(\nabla', \mathcal{B}) - N(\nabla, \mathcal{B}) = (\widehat{N}(\nabla', \mathcal{B}) - \widehat{N}(\nabla, \mathcal{B}))D\gamma$ holds and $(N(\nabla', \mathcal{B}) - N(\nabla, \mathcal{B})) \otimes dy$ is globally well-defined. $\square$

Definition 4.6. Let ∇^b be a transversely torsion-free connection on $Q(\mathcal{F})$, $\mathcal{D}$ the induced connection on $K_{\mathcal{F}}^{-1}$ and ∇ be the standard TW-connection for $(\nabla^b, \mathcal{D})$. We define $\mathcal{L}_{\mathcal{P}}\colon H^r_{\mathcal{F}}(M; Q(\mathcal{F})) \to H^{2q+r}(M)$ as follows. Let $N = N(\nabla, \mathcal{D})$ and $N \wedge dy = \sum_i N_i \wedge dy^i$. If $[c] \in H^r_{\mathcal{F}}(M; Q(\mathcal{F}))$, then we locally represent $c = c^i \otimes e_i$ and set $N \wedge c = \sum_i N_i \wedge c^i$. We set then

$$\mathcal{L}_{\mathcal{P}}([c]) = [d((N \wedge c) \wedge (N \wedge dy)^{q-1})] \in H^{2q+r}(M),$$

where $\mathcal{P}$ denotes the projective equivalence class represented by ∇.

In what follows, we always assume that $\mathcal{D}$ and $\mathcal{B}$ are induced by ∇^b, and denote the connection form by θ.

Proposition 4.7. $\mathcal{L}_{\mathcal{P}}$ is well-defined and depends on the projective equivalence class of ∇.

Proof. First, note that $(dc + \Gamma \wedge c)|_{E(\mathcal{F})} = 0$ and that $N_i = df_i - \sum_j \frac{1}{q+1} f_i f_j dy^j$. Therefore, $N \wedge dy = d\theta$ and $d((N \wedge c) \wedge (N \wedge dy)^{q-1}) = d(N \wedge c) \wedge (d\theta)^{q-1}$. We have

$$\begin{aligned}
& d(N \wedge c) \wedge (d\theta)^{q-1} \\
&= d((\widehat{N} + (q+1)\Lambda(\widehat{\gamma}) - \widehat{f}\Sigma(\widehat{\gamma}))D\gamma \wedge (D\gamma)^{-1}\widehat{c}) \wedge (d\widehat{\theta})^{q-1} \\
&= d(\widehat{N} \wedge \widehat{c}) \wedge (d\widehat{\theta})^{q-1} \\
&\qquad + d((q+1)\Lambda(\widehat{\gamma}) - \widehat{f}\Sigma(\widehat{\gamma})) \wedge \widehat{c} \wedge (d\widehat{\theta})^{q-1} \\
&\qquad - ((q+1)\Lambda(\widehat{\gamma}) - \widehat{f}\Sigma(\widehat{\gamma})) \wedge (d\widehat{c}) \wedge (d\widehat{\theta})^{q-1} \\
&= d(\widehat{N} \wedge \widehat{c}) \wedge (d\widehat{\theta})^{q-1} \\
&\qquad + ((q+1)d\Lambda(\widehat{\gamma}) - \widehat{f}d\Sigma(\widehat{\gamma})) \wedge \widehat{c} \wedge (d\widehat{\theta})^{q-1} \\
&\qquad - (d\widehat{f}) \wedge \Sigma(\widehat{\gamma}) \wedge \widehat{c} \wedge (d\widehat{\theta})^{q-1} \\
&\qquad - ((q+1)\Lambda(\widehat{\gamma}) - \widehat{f}\Sigma(\widehat{\gamma})) \wedge (-\Gamma \wedge \widehat{c}) \wedge (d\widehat{\theta})^{q-1} \\
&= d(\widehat{N} \wedge \widehat{c}) \wedge (d\widehat{\theta})^{q-1} - d\widehat{f} \wedge \Sigma(\widehat{\gamma}) \wedge \widehat{c} \wedge (d\widehat{\theta})^{q-1}.
\end{aligned}$$

by 1) of Lemma 4.5. On the other hand, we have

$$\begin{aligned}
&d\widehat{f} \wedge \Sigma(\widehat{\gamma}) \wedge \widehat{c} \wedge (d\widehat{\theta})^{q-1} \\
&= \sum_{i,j,k} d\widehat{f}_i \wedge \Sigma(\widehat{\gamma})^i_{jk} d\widehat{y}^k \wedge \widehat{c}^j \wedge \left(\sum_l d\widehat{f}_l \wedge d\widehat{y}^l\right)^{q-1} \\
&= \frac{1}{q} \sum_{i,j} \Sigma(\widehat{\gamma})^i_{ji} \widehat{c}^j \wedge \left(\sum_l d\widehat{f}_l \wedge d\widehat{y}^l\right)^{q} \\
&= 0.
\end{aligned}$$

Moreover, $d(d(N \wedge c) \wedge (d\theta)^{q-1}) = 0$ so that a closed form on M is defined. Next, suppose that $c = d_{\mathcal{F}} h$ for some $h \in C^{r-1}_{\mathcal{F}}(M; Q(\mathcal{F}))$, that is, $c^i = dh^i + \sum_j \Gamma^i_j \wedge h^j + k^i$ for some $(r-1)$-forms h^i such that $\widehat{h}^i = \sum_j D\gamma^i_j h^j$ and $k \in I^{r-1}_1(M; Q(\mathcal{F}))$. Then,

$$\begin{aligned}
&d(N \wedge c) \wedge (d\theta)^{q-1} \\
&= \sum_i dN_i \wedge \left(dh^i + \sum_j \Gamma^i_j \wedge h^j\right) \wedge (d\theta)^{q-1} - N \wedge dc \wedge (d\theta)^{q-1} \\
&= dN \wedge dh \wedge (d\theta)^{q-1} - df \wedge d(\Gamma \wedge h + k) \wedge (d\theta)^{q-1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
df \wedge \Gamma \wedge h \wedge (d\theta)^{q-1} &= \sum_{i,j,k} df_i \wedge \Gamma^i_{jk} dy^k \wedge h^j \wedge (d\theta)^{q-1} \\
&= \sum_{i,j} \frac{1}{q} \Gamma^i_{ji} h^j \wedge (d\theta)^q \\
&= \sum_j \frac{1}{q} f_j h^j \wedge (d\theta)^q
\end{aligned}$$

because $\mathcal{D}$ is induced from ∇^b, and also have

$$\begin{aligned}
&dN \wedge h \wedge (d\theta)^{q-1} \\
&= -\sum_{j,k} \frac{1}{q+1} (df_j \wedge f_k dy^k \wedge h_j + f_j df_k \wedge dy^k \wedge h^j) \wedge (d\theta)^{q-1} \\
&= -\sum_j \frac{1}{q} f_j h^j \wedge (d\theta)^q.
\end{aligned}$$

Therefore,

$$d(N \wedge c) \wedge (d\theta)^{q-1} = d(-2dN \wedge h \wedge (d\theta)^{q-1} + df \wedge k \wedge (d\theta)^{q-1}).$$

We will show that $dN \wedge h \wedge (d\theta)^{q-1}$ and $df \wedge k \wedge (d\theta)^{q-1}$ are globally well-defined. We have

$$\begin{aligned}
&dN \wedge h \wedge (d\theta)^{q-1} \\
&= (d\widehat{N} + (q+1)d\Lambda(\widehat{\gamma}) - d\widehat{f} \wedge \Sigma(\widehat{\gamma}) - \widehat{f}d\Sigma(\widehat{\gamma}))D\gamma \wedge (D\gamma)^{-1}\widehat{h} \wedge (d\widehat{\theta})^{q-1} \\
&= (d\widehat{N} - d\widehat{f} \wedge \Sigma(\widehat{\gamma})) \wedge \widehat{h} \wedge (d\widehat{\theta})^{q-1} \\
&= d\widehat{N} \wedge \widehat{h} \wedge (d\widehat{\theta})^{q-1}.
\end{aligned}$$

The last equality follows from the following one, namely,

$$\begin{aligned}
d\widehat{f} \wedge \Sigma(\widehat{\gamma}) \wedge \widehat{h} \wedge (d\widehat{\theta})^{q-1} &= \sum_{i,j} d\widehat{f}_i \wedge \Sigma(\widehat{\gamma})^i_{ji} d\widehat{y}^i \wedge \widehat{h}^j \wedge (d\widehat{\theta})^{q-1} \\
&= \frac{1}{q}\sum_{i,j} \Sigma(\widehat{\gamma})^i_{ji} \wedge \widehat{h}^j \wedge (d\widehat{\theta})^q \\
&= 0.
\end{aligned}$$

We also have

$$\begin{aligned}
df \wedge k \wedge (d\theta)^{q-1} &= (d\widehat{f} + d\partial \log J)D\gamma \wedge ((D\gamma)^{-1})\widehat{k} \wedge (d\widehat{\theta})^{q-1} \\
&= d\widehat{f} \wedge \widehat{k} \wedge (d\widehat{\theta})^{q-1}.
\end{aligned}$$

Therefore $dN \wedge h \wedge (d\theta)^{q-1}$ and $df \wedge k \wedge (d\theta)^{q-1}$ are globally well-defined, and $d(N \wedge c) \wedge (d\theta)^{q-1}$ represents the trivial class in $H^{2q+r}(M)$. Now let ∇' be a TW-connection for $(\nabla^{b\prime}, \mathcal{D}')$ which is projectively equivalent to ∇ and $N' = N(\nabla', \mathcal{D}')$. We denote by $\pi_{[0,1]}$ the projection from $M \times [0,1] \to M$ and by $\iota_t \colon M \to M \times \{t\}$ the inclusion, where $t \in [0,1]$. Let $\mathcal{F} \times [0,1]$ be the foliation of $M \times [0,1]$ of which the leaves are given by $\{L \times [0,1] \mid L \text{ is a leaf of } \mathcal{F}\}$. We set $\nabla^b_t = (1-t)\nabla^b + t\nabla^{b\prime}$, $\mathcal{D}_t = (1-t)\mathcal{D} + t\mathcal{D}'$ and let ∇_t be the standard TW-connection for $(\nabla^b_t, \mathcal{D}_t)$ on $Q(\widetilde{\mathcal{F} \times [0,1]})$. Let $[c] \in H^r_{\mathcal{F}}(M; Q(\mathcal{F}))$ and $\widetilde{c} = \pi^*_{[0,1]}[c] \in H^r_{\mathcal{F}\times[0,1]}(M \times [0,1]; Q(\mathcal{F} \times [0,1]))$. If we set $N_t = N(\nabla_t, \mathcal{D}_t)$ and $\rho = d(N_t \wedge \widetilde{c}) \wedge (d\theta_t)^{q-1}$, then ρ is closed and we have $\iota_0^*\rho = d(N \wedge c) \wedge (d\theta)^{q-1}$ and $\iota_1^*\rho = d(N' \wedge c) \wedge (d\theta')^{q-1}$. We represent $d\rho = \alpha + \beta \wedge dt$, where α and β do not involve dt, and set $U = \displaystyle\int_0^1 \beta dt$. It is well-known and easy to show that we have $dU = \iota_0^*\rho - \iota_1^*\rho$. Therefore $\mathcal{L}$ depends only on the projective equivalence class of ∇. $\square$

Remark 4.8. The equality $\nabla_t = (1-t)\nabla + t\nabla'$ fails in general, and concrete forms of the coboundary between $d(N \wedge c) \wedge (d\theta)^{q-1}$ and $d(N' \wedge c) \wedge (d\theta')^{q-1}$ are quite complicated.

Remark 4.9. Secondary characteristic classes for transversely projective foliations have been well-studied. See e.g. [20], [19], [22]. See also [1] for a related study.

Lemma 4.10. We set $N' = (N'_i)$, where $N'_i = df_i - \dfrac{1}{q+1} \sum_j f_i f_j dy^j$. Then $d(N' \wedge \dot\omega) \wedge (d\theta)^{q-1} = -\frac{1}{q}\dot\theta \wedge (d\theta)^q$. In addition, we have

$$d_{\mathcal{D}} N' \wedge \dot\omega = (dN' + N' \wedge \Theta) \wedge \dot\omega \wedge (d\theta)^{q-1} = 0,$$

$$N \wedge \dot\Theta \wedge \omega \wedge (d\theta)^{q-1} = -\frac{1}{q}\dot\theta \wedge (d\theta)^q.$$

Proof. We set $\Omega = \omega^1 \wedge \cdots \wedge \omega^q = dy^1 \wedge \cdots \wedge dy^q$ and $\dot\Omega = \dot\omega^1 \wedge dy^2 \wedge \cdots \wedge dy^q + \cdots + dy^1 \wedge \cdots \wedge dy^{q-1} \wedge \dot\omega^q$. We have

$$\begin{aligned}
&\dot\theta \wedge d\theta^q \\
&= q!(-1)^{\frac{q(q-1)}{2}} \dot\theta \wedge df_1 \wedge \cdots \wedge df_q \wedge \Omega \\
&= -q!(-1)^{\frac{q(q+1)}{2}} (df_1 \wedge \cdots \wedge df_q \wedge \theta \wedge \dot\Omega + df_1 \wedge \cdots \wedge df_q \wedge d\dot\Omega) \\
&= q!(-1)^{\frac{q(q-1)}{2}} df_1 \wedge \cdots \wedge df_q \wedge \Omega \wedge (f_1\dot\omega^1 + \cdots + f_q\dot\omega^q) \\
&\qquad - q!(-1)^{\frac{q(q+1)}{2}} df_1 \wedge \cdots \wedge df_q \wedge d\dot\Omega.
\end{aligned}$$

On the other hand, we have the following equalities, namely,

$$
\begin{aligned}
&d\left(\sum_i\left(df_i-\frac{1}{q+1}\sum_j f_if_jdy^j\right)\wedge\dot\omega^i\right)\wedge(d\theta)^{q-1}\\
&=-\frac{1}{q+1}\sum_{i,j}(df_if_j+f_idf_j)\wedge dy^j\wedge\dot\omega^i\wedge(d\theta)^{q-1}\\
&\quad-\sum_i\left(df_i-\frac{1}{q+1}\sum_j f_if_jdy^j\right)\wedge d\dot\omega^i\wedge(d\theta)^{q-1}\\
&=-\frac{(q-1)!}{(q+1)}(-1)^{\frac{q(q-1)}{2}}(f_1\dot\omega^1+\cdots+f_q\dot\omega^q)\wedge df_1\wedge\cdots\wedge df_q\wedge\Omega\\
&\quad-\frac{1}{(q+1)}(f_1\dot\omega^1+\cdots+f_q\dot\omega^q)\wedge(d\theta)^q\\
&\quad-\sum_i df_i\wedge d\dot\omega^i\wedge(d\theta)^{q-1}\\
&=-(q-1)!(-1)^{\frac{q(q-1)}{2}}(f_1\dot\omega^1+\cdots+f_q\dot\omega^q)\wedge df_1\wedge\cdots\wedge df_q\wedge\Omega\\
&\quad+(-1)^{\frac{q(q+1)}{2}}(q-1)!df_1\wedge\cdots\wedge df_q\wedge d\dot\Omega\\
&=-\frac{1}{q}\dot\theta\wedge(d\theta)^q.
\end{aligned}
$$

Hence the first equality is shown. We have $d(N'\wedge\dot\omega)\wedge(d\theta)^{q-1}=(dN'\wedge\dot\omega+N'\wedge\Theta\wedge\dot\omega+N'\wedge\dot\Theta\wedge\omega)\wedge(d\theta)^{q-1}$. On the other hand, $N'\wedge\dot\Theta\wedge\omega\wedge(d\theta)^{q-1}=\sum_{i,j}N'_i\wedge\dot\Theta^i_j\wedge dy^j\wedge(d\theta)^{q-1}=-\frac{1}{q}\dot\theta\wedge(d\theta)^q$. □

Theorem 4.11. We have $\mathcal{L}_{\mathcal{P}}([\dot\omega])=-\frac{1}{q}\dot\theta\wedge d\theta^q$ in $H^{2q+1}(M;\mathbb{C})$.

Since $d(N'\wedge\dot\omega)\wedge(d\theta)^{q-1}=d(N'\wedge\dot\omega\wedge(d\theta)^{q-1})$, the Čech-de Rham class $\{\delta((N'\otimes dy)\wedge(d\theta)^{q-1})\}$ makes a sense. This class is denoted by $\mathcal{L}$ and studied in [3], where it is shown that $\mathcal{L}$ can be represented only in terms of $\Lambda(\gamma)$.

Proposition 4.12. Let ∇ be the standard TW-connection for ∇^b. If ∇^b is transversely torsion-free and if ∇ is invariant under holonomy, then $\mathcal{L}_{\mathcal{P}}=0$.

Proof. Let $(\Pi, L(q))$ be the components of ∇ with respect to $\{\epsilon_0,\dots,\epsilon_q\}$ and $[c]\in H^*_{\mathcal{F}}(M;Q(\mathcal{F}))$, where $c=\sum_i c^i\otimes e_i$. Then $d(N\wedge c)\wedge(d\theta)^{q-1}=d((L(q)+f\Pi)\wedge c)\wedge(d\theta)^{q-1}$. As ∇ is holonomy invariant, $dL(q)_i\in I^2_2(\mathcal{F})$

and $dL(q)\wedge c\wedge(d\theta)^{q-1}=0$. We also have $(L(q)+f\Pi)\wedge dc\wedge(d\theta)^{q-1}=0$ because c is $d_{\mathcal{F}}$-closed. Finally, we have

$$\begin{aligned} d(f\Pi)\wedge c\wedge(d\theta)^{q-1} &= df\wedge\Pi\wedge c\wedge(d\theta)^{q-1} \\ &= \sum_{i,j,l} df_l\wedge\Pi^l_{ij}dy^j\wedge c^i\wedge(df_l\wedge dy^l)^{q-1} \\ &= -\sum_{i,l}\frac{1}{q}\Pi^l_{il}c^i\wedge(d\theta)^q \end{aligned}$$

Since ∇^b is assumed to be transversely torsion-free, we have $\Pi^l_{il}=\Pi^l_{li}=0$. $\square$

Combining with Theorem 4.11, we obtain the following

Corollary 4.13. 1) Let $\mathcal{F}$ be a (real) foliation. If $\mathcal{F}$ admits a transverse projective structure, then the Godbillon-Vey class of $\mathcal{F}$ is infinitesimally rigid.

2) Let $\mathcal{F}$ be a transversely holomorphic foliation. If $\mathcal{F}$ admits a transverse holomorphic projective structure, then the Bott class of $\mathcal{F}$ is infinitesimally rigid.

Corollary 4.13 is shown in [3] for transverse flat projective structures.

Remark 4.14. 1) The homomorphism $\mathcal{L}_{\mathcal{P}}$ is non-trivial if $\mathcal{F}$ admits deformations with respect to which the Godbillon-Vey class or the Bott class vary continuously.

2) Suppose that $\mathcal{F}$ is transversely projectively flat. Then, $\mathcal{F}$ admits an foliation atlas such that every transition function γ_{ji} in the transversal direction is a projective transformation. If we make use of such a foliation atlas, then we have $\Sigma(\gamma_{ji})=0$ and $\Lambda(\gamma_{ji})=0$. Therefore $N\wedge c$ is globally well-defined by Lemma 4.5, and $(N\wedge c)\wedge(N\wedge dy)^{q-1}$ is a cocycle. We do not know if this cocycle leads to an invariant of transverse flat projective structures. For example, the class represented by this cocycle depends a priori the cocycle c.

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Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
E-mail address: asuke@ms.u-tokyo.ac.jp

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012