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Finite volume methods for a Keller-Segel system: discrete energy, error estimates and numerical blow-up analysis

by

Guanyu ZHOU and Norikazu SAITO

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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Guanyu Zhou · Norikazu Saito

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Abstract We are concerned with the finite volume approximation for a nonlinear parabolic-elliptic system, which describes the aggregation of slime molds resulting from their chemotactic features, called a simplified Keller-Segel system. First, we present a linear finite volume scheme that satisfies both positivity and mass conservations, which are important features of the original system. The discrete free energy for the finite volume approximation is derived. Then, under some regularity assumptions of solution and admissible mesh, we establish error estimates in L^p norm with a suitable $p > 2$ for the two dimensional case. In the last part of this paper, we restrict our attention to the radially symmetric solution of chemotaxis system, and we derive some analysis of a-priori estimates and the discrete moment to study the blow-up phenomenon of numerical solution. Several numerical experiments are presented to validate our theoretical results.

Keywords finite volume method · error analysis · nonlinear parabolic system · blow-up

Mathematics Subject Classification (2000) 65M15 · 65M08 · 35K55 · 92C17

G. Zhou
Graduate School of Mathematical Sciences, Univ. of Tokyo, 3-8-1 Komaba Meguro-ku Tokyo 153-8914, Japan
Tel.: +81-03-54657001, Fax: +81-03-54657001
E-mail: koolewind@gmail.com

N. Saito
Graduate School of Mathematical Sciences, Univ. of Tokyo, 3-8-1 Komaba Meguro-ku Tokyo 153-8914, Japan
E-mail: norikazu@ms.u-tokyo.ac.jp

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1 Introduction

We consider the finite volume methods to a simplified Keller-Segel model (cf. [20]) for the functions $u = u(x,t)$ and $v = v(x,t)$ of $(x,t) \in \overline{\Omega} \times [0,T]$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times [0,T], \\ -\Delta v + v = u & \text{in } \Omega \times [0,T], \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma \times [0,T], \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with the boundary $\Gamma = \partial\Omega$, ν is the outer unit normal vector to Γ, $\partial/\partial\nu$ represents differentiation along ν on Γ, and $u_0 \geq 0$, $u_0 \not\equiv 0$. Although we will deal with the one space-dimensional problem (5.1) in Section 5, we mainly study the two space-dimensional problem (1.1).

The system (1.1) describes the aggregation of slime molds resulting from their chemotactic features. Therein, u is defined to be the density of the cellular slime molds and v the concentration of the chemical substance.

The mathematical study for (1.1) is well developed. The unique solvability locally in time when Γ and u_0 are sufficiently smooth has been showed by Biler [3] and Yagi [36]. The solution (u,v) to (1.1) has the properties of conservation of positivity

$$u(x,t) > 0, \quad (x,t) \in \overline{\Omega} \times (0,T], \tag{1.2}$$

and the conservation of total mass

$$\int_\Omega u(x,t)\ dx = \int_\Omega u_0(x)\ dx, \quad t \in [0,T], \tag{1.3}$$

which imply the conservation of L^1 norm,

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}, \quad t \in [0,T]. \tag{1.4}$$

Besides of the fundamental existence results, relevant properties of the solutions, including blow-up, chemotactic collapse and aggregation, have been well concerned by many researchers, and one can refer to survey articles and monographs Horstmann [17,18] and Suzuki [31,32] for those mathematical results. For example, it is showed in [22,23] that the value of $\|u_0\|_{L^1(\Omega)}$ plays a crucial role in the blow-up and globally existence of solutions. For example, if $\|u_0\|_{L^1(\Omega)} < 4\pi$, the solution exists globally in time; whereas if $\|u_0\|_{L^1(\Omega)} > 8\pi$ and $\displaystyle\int_\Omega u_0|x - x_0|^2\ dx$ is sufficiently small with some $x_0 \in \Omega$, then the solution blows up in finite time.

Another important feature of (1.1) is the existence of the free energy (cf. [24]) which is expressed as

$$\frac{d}{dt} W(u(\cdot,t), v(\cdot,t)) \leq 0, \quad t \in [0,T], \tag{1.5}$$

where

$$
\begin{aligned}
W(u(\cdot,t),v(\cdot,t)) &= \int_\Omega (u\log u - u)\ dx - \frac{1}{2}\int_\Omega uv\ dx \\
&= \int_\Omega (u\log u - u)\ dx - \frac{1}{2}\int_\Omega (|\nabla v|^2 + |v|^2)\ dx.
\end{aligned} \tag{1.6}
$$

We have triple purpose in this paper that are briefly summarized as follows. As mentioned before, the L^1 conservation (1.4) and the free energy (1.5) are essential requirements; it is desirable that solutions of numerical schemes preserve these properties. After introducing the admissible mesh, we first consider a linear finite volume scheme for (1.1) satisfying the discrete analogues of (1.2), (1.3) and (1.5) (cf. Theorems 2.1, 2.2 and 2.3). Our second aim is to establish the error estimates of the finite volume scheme (cf. Theorem 3.1), with some additional assumptions on the solution and admissible mesh. An important and interesting aspect of the Keller-Segel system (1.1) is the possibility of blow-up of solutions in finite time. Our last motivation is to give analysis of the blow-up phenomenon for numerical solutions.

Below, we explain our work in detail. Several numerical schemes with conservation laws (the discrete version of (1.2) and (1.3)) have been proposed for (1.1). An upwind finite difference scheme proposed by Saito and Suzuki [30] is proved to satisfy the conservation of positivity with a time-step size control. Filbert [15] introduced a nonlinear finite volume scheme. The existence of positive solution is proved under some condition of time-step size. Saito [27] then proposed a conservation upwind finite element scheme with adjusting a time step increment τ_n at every discrete time-step $t_n = \tau_1 + \cdots + \tau_n$ to guarantee the positivity of the solution. A conservative upwind approximation of Baba and Tabata [2] is applied to the finite element method to obtain the conservation of mass. Bessemoulin-Chatard and Jüngel [5] considered a nonlinear finite volume scheme for the Keller-Segel model with additional cross-diffusion, which is obtained by replacing the second equation of (1.1) with

$$
-\Delta v - \delta\Delta u + v - u = 0, \quad \delta > 0, \quad x \in \Omega, \quad t \in [0,T].
$$

The existence of solution to the nonlinear scheme with the conservation laws are proved.

For the discrete free energy, a time-discrete version has been proved in [30]. However, any space-discretization is not undertaken. An entropy stability (similar to the free energy) of discrete solution of finite volume method is derived in [5] for the system with additional cross-diffusion.

For the convergence of numerical scheme, Filbert [15] proved the convergence of the nonlinear finite volume scheme for the Keller-Segel model (1.1). However, the error estimate with explicit convergence rate is still open at present. A similar convergence result is obtained in [5] for Keller-Segel model with additional cross-diffusion. Saito [27] succeeded in deriving explicit error estimates of the form

$$
\sup_{0\le n\le l} \|u(t_n) - u_h^n\|_p,\ \sup_{0\le n\le l} \|v(t_n) - \hat{v}_h^n\|_{1,\infty} \le C(h^{1-2/p} + \tau^\sigma), \tag{1.7}
$$

where u_h^n and v_h^n denote the finite element approximation of $u(x,t_n)$ and $v_(x,t_n)$, respectively, τ the size of the time discretization, $\sigma \in (0,1]$, and $p \in (2,\mu)$ with the shape constant $\mu > 2$ of Ω (cf. Paragraph 2.1).

With a slight change of the nonlinear scheme in [15], we propose a linear finite volume scheme that preserves the conservation law *without* time-step control (cf. Theorems 2.1 and 2.2). The existence and uniqueness of a solution of the linear finite volume scheme are obvious, and since the linear scheme reduces the complexity of computation, it is well applicable to numerical experiments. Then, we introduce a discrete version of the energy function to the linear finite volume scheme, and show the time-decreasing property of it (cf. Theorem 2.3). In the error analysis, we consider the case where the admissible Voronoi mesh has a dual triangulation, since the convergence is proved in [15] for general admissible meshes. By introducing a mass-lumping operator based on the circumcentric domain, we can employ the technique of error analysis in [27] for finite element method to our case, and develop the error analysis. Let (u_h^n, v_h^n) be the solution of finite volume scheme (2.4) at a discrete time step t_n. Under some assumptions on the regularity of solution (u,v) to (1.1) and some a priori estimates on discrete solution u_h^n, we have (cf. Theorem 3.1)

$$\sup_{0\le t_n\le T} \|u(t_n) - u_h^n\|_{L^p(\Omega)} \le C(h^{1-2/p} + \tau^\sigma). \tag{1.8}$$

Moreover, we have the error estimate for $v(t_n) - v_h^n$ that is described as

$$\sup_{0\le n\le l} \|v(t_n) - \hat{v}_h^n\|_{1,\infty} \le C_2 h^{1-2/p} + C_3 \sup_{0\le n\le l} \|u(t_n) - u_h^n\|_p, \tag{1.9}$$

where $\hat{v}_h^n$ is the projection of v_h^n to a linear piecewise continuous function which subordinate to the dual triangulation of the Voronoi admissible mesh. The precise definition of $\hat{v}_h^n$ is presented below. As mentioned above, our error analysis extending from the method of finite element approximation proposed by [27] is different from the argument of error analysis for finite volume method in [14]. Some L^p error estimates results in [7] for finite volume approximation to elliptic problems with Dirichlet boundary condition on a mesh of barycentric type has been extended to the case of Neumann boundary value problem on Voronoi mesh (cf. Lemma 4.2), which can be applied to our problem. The convergence rate $h^{1-2/p}$ in (1.8) is due to the approximation of the nonlinear term $\nabla \cdot (u\nabla v)$ in (1.1). We can summarize the strategy of establishing the error estimates as follows (Sections 3 and 4):

(i) A dual triangulation of Voronoi mesh and a mass-lumping operator M_h is introduced (Section 3.1 and 3.2);
(ii) Operators A_h and L_h are denoted as the finite volume and finite element approximation of $-\Delta+1$, respectively, and we present some error estimates on A_h^{-1} and L_h^{-1} (Paragraph 4.1.3);
(iii) We show that $-A_h$ satisfies the resolvent estimates in a modified L^p norm, which is an analogue to the results in Crouzeix and Thomée (cf. [13]) (Paragraph 4.1.4);

(iv) Some estimates about time discretization error $\partial\tau_n u - u$ and the error of approximating to the nonlinear term $b - B_h$ are derived (Section 4.2 and 4.3);

(vi) With the aid of M_h, A_h, the projection operator R_h, P_h, and the estimates of (iv), we reformulate the variational equation of error $\hat{w}_h^n = \hat{u}_h^n - R_h u(t_n)$ into a single equation. Then we make the use of Duhamel's principle, fractional powers of operators and the smoothing properties of semigroup to derive the error estimates (Section 4.4).

Our final topic is the blow-up of solution of Keller-Segel model (1.1). We see from previous works [15,21,27] that numerical solutions seem to reproduce the "blow-up" phenomenon (the mass concentration at some elements) under the large initial condition. On the other hand, the solution of conservative numerical schemes cannot blow up in finite time (the solution goes to infinity at some points in finite time), since any norm (in space) are equivalent in a finite dimensional space. Therefore, it is natural to ask whether numerical solutions can reproduce the blow-up phenomena. There are plenty mathematical theory for the blow-up study of (1.1) (cf. [11,8,22,23]). In [22], to show the chemotactic collapse is not possible in the one dimension, it is proved

$$0 \le v(x,t) \le \frac{1}{2L}(1+4L^2)\theta, \tag{1.10}$$

$$|v_x(x,t)| \le \theta, \tag{1.11}$$

where $\Omega = (-L, L)$ and $\theta = \int_\Omega u_0 dx$. Moreover, when considering only radially symmetric solutions in the ball $\Omega = B(0, L)$ with the radius L the and center at the origin and $u_0(x) = u_0(r)$ with $r = |x|$, the global existence of a solution for $\theta = \int_{B(0,L)} u_0(x)\, dx < 8\pi$ is proved by showing (cf. [22])

$$0 \le v(x,t) \le 2l + \frac{1}{2}\theta, \tag{1.12}$$

$$|v_x(x,t)| \le L(l+4k), \tag{1.13}$$

where $l, k > 0$ depend on θ. To establish the blow-up results, setting the moment

$$M_2(t) = \int_\Omega u(x,t)|x|^2\, dx = 2\pi \int_0^L u(r,t) r^3\, dr, \tag{1.14}$$

it is proved that

$$\frac{d}{dt}M_2(t) \le 4\theta - \frac{1}{2\pi}\theta^2 + \frac{1}{\pi L^2}\theta M_2(t) + \frac{1}{2e\pi}\theta^{\frac{3}{2}} M_2(t)^{\frac{1}{2}}. \tag{1.15}$$

From (1.15), if $\theta > 8\pi$ and $M_2(0)$ is sufficiently small, then we have

$$\frac{d}{dt}M_2(t) < 0, \quad t > 0,$$

which implies $M_2(t)$ goes to 0 at some time $t = t_{max}$. In view of $u > 0$ and $\int_\Omega u(x,t) = \theta$, the function u actually blows up in finite time t_{max}.

In this article, the conservative finite volume scheme is applied to the one dimensional and two dimensional system. We derive the discrete versions of (1.10), (1.11), (1.12) and (1.13) (cf. Theorems 5.1 and 5.2). Moreover, we define the discrete moment M_2^n for the n-th time step, and obtain a discrete analogue of (1.15) (cf. Theorem 5.3) for a nonconservative scheme, and draw a remark to explain the difference of moment equation between conservative and nonconservative scheme.

The numerically "blow-up" study is important. It is not only for understanding the properties of numerical solution, but it also shows some hints to approximating the blow-up time of exact solution. For nonlinear parabolic and wave equations, there are well-developed theories of approximating the blow-up time (cf. [9,10,29]); however, those techniques are not applicable to conservative numerical schemes. Our final motivation is to develop the method to approximate the blow-up time and this paper is the first step towards this end.

The paper is organized as follows. In Section 2, we briefly introduce the admissible mesh and present the linear finite volume scheme. The conservation laws and discrete free energy are proved. In Section 3, we consider the Voronoi admissible mesh with dual triangulation and define the associated mass-lumping operator M_h. Some properties of M_h are given, and the main theorem of error estimates is presented. The proof of main theorem of error estimates is presented in Section 4, including the discrete Laplace operators, resolvent estimates of $-A_h$ in a modified L^p norm, and the estimates of $\partial\tau_n u - u$ and $b - B_h$. The finite volume scheme is applied to the 1-dimensional system and 2-dimensional symmetric system in Section 5, and some analysis of numerical solution concerned with blow-up theory are presented. Several numerical experiments are performed in Section 6 to verify our theoretical results.

Notation

Throughout this paper, we follow the notation of [1]. Namely we use standard Lebesgue and Sobolev spaces. We set $W^{m,p} = W^{m,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $L^p = L^p(\Omega)$, $\|\cdot\|_{m,p} = \|\cdot\|_{W^{m,p}}$, $\|\cdot\|_p = \|\cdot\|_{L^p}$ for $m \in \mathbb{N}$ and $p \in [1,\infty]$. For $p \in [1,\infty)$,

$$\mathcal{W}^p = \left\{ v \in W^{1,p} \mid \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma \right\}.$$

The inner-product of L^2 is denoted by $(\cdot,\cdot)$. For Banach space X, we denote X^* as its dual space, and $\langle\cdot,\cdot\rangle$ is denoted as their dual product.

General positive constant depending on Ω are denoted as C, C' and so forth. In particular, C does not depend on discretization parameters h and τ described below. We use $C_{\alpha,\beta}$ or $C(\alpha,\beta)$ to specify the C depending on other parameters, say α, β, if necessary.

The d-dimensional Lebesgue measure of $\mathcal{O} \subset \mathbb{R}^d$ is denoted by $m(\mathcal{O}) = m_d(\mathcal{O})$.

2 Finite volume scheme: conservation laws and discrete free energy

2.1 Weak formulation

Throughout the paper (except for Subsection 5.1), we suppose that Ω is a convex polygonal domain in $\mathbb{R}^2$. Then, we have the following elliptic regularity in the L^p sense (cf. [16]): there exists $\mu \in (2,\infty)$ such that, for any $p \in (1,\mu)$, the linear elliptic equation

$$-\Delta v + v = u \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma \tag{2.1}$$

admits a unique solution $v \in \mathcal{W}^p$ satisfying

$$\|v\|_{2,p} \le C(p,\Omega)\|u\|_p. \tag{2.2}$$

We introduce an operator $G : L^p \to \mathcal{W}^p$ by

$$Gu = v,$$

where v denotes the solution of (2.1) for $u \in L^p$. We then introduce the trilinear form b on $L^2 \times H^1 \times H^1$:

$$b(w,u,\chi) = -\int_\Omega u\nabla(Gw)\nabla\chi \quad (w \in L^2, u, \chi \in H^1).$$

The weak form of (1.1) reads as: find $u \in C^1([0,T];H^1)$ such that, for all $t \in (0,T)$,

$$\begin{cases} (\partial_t u(t),\chi) + (\nabla u(t),\nabla\chi) + b(u(t),u(t),\chi) = 0, \quad \forall\chi \in H^1, \\ u(0) = u_0 \in H^1. \end{cases} \tag{2.3}$$

2.2 Admissible mesh

We follow the standard notation of the finite volume method described in [14]. Let $\mathcal{T}$ be an admissible mesh (cf. [14, Definition 9.1]) such that

$$\overline{\Omega} = \bigcup_{K\in\mathcal{T}} \overline{K}.$$

An element $K \in \mathcal{T}$ is called a control volume. We introduce the neighborhood of $K \in \mathcal{T}$ as

$$\mathcal{N}_K := \{L \in \mathcal{T} \mid \overline{L} \cap \overline{K} \neq \emptyset\},$$

We write $K|L$ or $\sigma_{K,L}$ to express the common edge $\overline{L} \cap \overline{K}$ of control volumes K and L, and set

$$\mathcal{E}_{int} = \{K|L \mid \forall K \in \mathcal{T},\ \forall L \in \mathcal{N}_K\}.$$

Moreover, letting $\sigma_{K,\Gamma} = \overline{K} \cap \Gamma$ be the edge of control volume K on Γ, we set

$$\mathcal{E}_{ext} = \{\sigma_{K,\Gamma} \mid \forall K \in \mathcal{T},\ \overline{K} \cap \Gamma \neq \emptyset\}$$

and

$$\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{ext}.$$

For every control volume K, $x_k \in K$ (or denoted as P_K) is the control point, where $P_K P_L$ is the perpendicular to $K|L$ for all $K \in \mathcal{T}$, $L \in \mathcal{N}_K$ (see Figure 2.1). Set

$$d_{K,L} = \mathrm{dist}(x_k, x_L), \quad \tau_{K,L} = \frac{m(K|L)}{d_{K,L}} = \tau_{L,K}, \quad K, L \in \mathcal{T},$$
$$d_{K,\sigma} = \mathrm{dist}(x_k, \sigma_{K,\Gamma}), \quad \tau_{K,\sigma} = \frac{m(\sigma_{K,\Gamma})}{d_{K,\sigma}}, \quad \sigma_{K,\Gamma} \in \mathcal{E}_{ext}.$$

Following [14], we assume there exists $\xi > 0$ such that

$$d_{x_K,\sigma_{K,L}} \geq \xi d_{K,L}, \quad \forall K \in \mathcal{T}, \forall L \in \mathcal{N}_K.$$

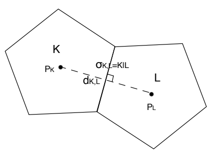

Fig. 2.1 Control volume K, L of admissible mesh $\mathcal{T}$.

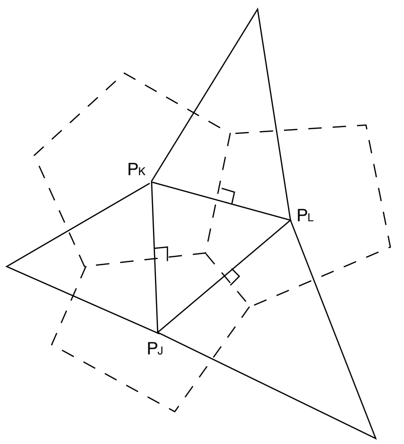

Fig. 2.2 Dual triangulation $\hat{\mathcal{T}}$ to Voronoi mesh $\mathcal{T}$.

2.3 A linear finite volume scheme

Setting $X_h = \mathrm{span}\{\phi_K \mid K \in \mathcal{T}\}$, where ϕ_K the characteristic function of K, *i.e.*,

$$\phi_K = \begin{cases} 1 & \text{on } K, \\ 0 & \text{otherwise.} \end{cases}$$

In general, for $u_h \in X_h$ and $K \in \mathcal{T}$, we write as $u_K = u_h(P_K)$. Given the initial condition

$$u_h^0 \in X_h, \quad u_h^0 \geq 0, \quad \int_\Omega u_h^0 = \sum_{K\in\mathcal{T}} m(K) u_K^0 \equiv M > 0,$$

we state the finite volume scheme for (1.1): find $(u_h^n, v_h^n) \in X_h \times X_h$ for $n \geq 1$, integer, such that

$$\begin{cases} \displaystyle\sum_{L\in\mathcal{N}_K} \tau_{K,L}(v_K^{n-1} - v_L^{n-1}) + m(K)v_K^{n-1} = m(K)u_K^{n-1}, \quad \forall K \in \mathcal{T}, \\ \displaystyle m(K)\partial_{\tau_n} u_K^n + \sum_{L\in\mathcal{N}_K} \tau_{K,L}(u_K^n - u_L^n) \\ \displaystyle\quad + \sum_{L\in\mathcal{N}_K} \tau_{K,L}\left[(Dv_{K,L}^{n-1})_+ u_K^n - (Dv_{K,L}^{n-1})_- u_L^n\right] = 0, \quad \forall K \in \mathcal{T}, \end{cases} \tag{2.4}$$

where $w_+ = \max(w, 0)$, $w_- = \max(-w, 0)$, $w = w_+ - w_-$,

$$Dv_{K,L} = v_L - v_K \text{ for } v_h \in X_h, \quad Dv_{K,\sigma} = 0 \text{ for } \sigma \in \mathcal{E}_{ext}.$$

Here, $\tau_n > 0$ is the time-step increment, $t_n = \tau_1 + \cdot + \tau_n$, and $\partial_{\tau_n} u_K^n$ is the backward Euler difference quotient approximating to $\partial_t u(t_n)$, which is defined by

$$\partial_{\tau_n} u_K^n = \frac{u_K^n - u_K^{n-1}}{\tau_n}.$$

The scheme is a modification of the nonlinear scheme presented in [15] which is obtained by following the basic idea of the finite volume method.

2.4 Conservation laws

Theorem 2.1 (Conservation of total mass) $\{u_h^n\}_{n\geq 0} \subset X_h$ *is the solution of (2.4), then we have*

$$(v_h^n, 1) = (u_h^n, 1) = (u_h^0, 1), \quad \forall n \geq 0. \tag{2.5}$$

Proof Taking a summation with $K \in \mathcal{T}$ of (2.4) leads to (2.5).

Theorem 2.2 (Well-posedness and conservation of positivity) $u_h^0 \geq 0$, $u_h^0 \not\equiv 0$. *(2.4) admits a unique solution* $\{(u_h^n, v_h^n)\}_{n\geq 0} \subset X_h \times X_h$, *and* $u_h^n > 0$ *for* $n \geq 1$, $v_h^n > 0$ *for* $n \geq 0$.

Proof Following the same argument of [5, Section 3], (2.4) is written into

$$\mathbf{A}^n \mathbf{u}^{(n)} = \mathbf{u}^{(n-1)},$$

where

$$\mathbf{u}^{(n)} = (u_K^n)_{K\in\mathcal{T}}, \quad \mathbf{A}^n = (a_{K,L}^n)_{K,L\in\mathcal{T}},$$

$$a_{K,K}^n = \frac{m(K)}{\tau_n} + \sum_{L \in \mathcal{N}_K} \tau_{K,L}(1 + (Dv_{K,L}^{n-1})_+) > 0,$$

$$a_{K,L}^n = -\tau_{K,L}(1 + (Dv_{K,L}^{n-1})_-) \le 0, \quad L \in \mathcal{N}_K.$$

Since $\tau_{K,L} = \tau_{L,K}$ and $(Dv_{K,L})_+ = (Dv_{L,K})_-$, we have

$$a_{K,K}^n + \sum_{L \in \mathcal{N}_K} a_{L,K}^n = \frac{m(K)}{\tau_n} > 0.$$

Consequently, $\mathbf{A}^n$ is irreducibly diagonally dominant with respect to the *columns* so that we can apply [35, Corollary 3.20] to obtain $(\mathbf{A}^n)^{-1} > 0$, which yields,

$$\mathbf{u}^{(n)} = (\mathbf{A}^n)^{-1}\mathbf{u}^{(n-1)} > 0, \quad n \ge 1,$$

under the assumption that $\mathbf{u}^{(0)} \ge 0$ and not identically 0 ($u_h^0 \ge 0$, $u_h^0 \ne 0$). Finally, $v_h^n > 0$ for $n \ge 0$ is proved in the same way.

Remark 2.1 In previous papers, [27,28,30], we proved that corresponding coefficient matrices are irreducibly diagonally dominant with respect to the *row* under a certain restriction on a time-step size. Consequently, we need to employ a time-step size control to ensure the positivity of discrete solutions. However, we can remove such restrictions by considering that corresponding coefficient matrices are irreducibly diagonally dominant with respect to the *column*.

Corollary 2.1 $\{u_h^n\}_{n\ge 0} \subset X_h$ *is the solution of (2.4) as in Theorem 2.2, then we have*

$$\|v_h^n\|_1 = \|u_h^n\|_1 = \|u_h^0\|_1, \quad \forall n \ge 0. \tag{2.6}$$

2.5 Discrete free energy

The free energy for system (1.1) is given by (1.5) and (1.6). For $\{(u_h^n, v_h^n)\}_{n\ge 0}$ the solution to (2.4), we define the discrete free energy function: for all $u_h, v_h \in X_h$,

$$W_h(u_h, v_h) \equiv \sum_{K \in \mathcal{T}} m(K)(u_K \log u_K - u_K) - \frac{1}{2}\sum_{K \in \mathcal{T}} m(K) u_K v_K, \tag{2.7}$$

and set $W_h^n = W_h(u_h^n, v_h^n)$.

Theorem 2.3 (Discrete free energy) *Let* $\{(u_h^n, v_h^n)\}_{n\ge 0}$ *be the solution to (2.4), then we have*

$$\frac{1}{\tau_n}(W_h^n - W_h^{n-1}) \le 0, \quad n \ge 1. \tag{2.8}$$

To state the proof, we need the following lemma, which is a version of [15, Lemma 3.1] and whose proof is exactly the same.

Lemma 2.1 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing C^1-function and $\{(u_h^n, v_h^n)\}_{n\geq 1}$ the solution to finite volume scheme (2.4), then we have*

$$\begin{aligned}\sum_{K\in\mathcal{T}} (u_K^n - u_K^{n-1})\phi(u_K^n) \leq &-\frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K} \tau_{K,L}\left[Du_{K,L}^n\sqrt{\phi'(\tilde{u}_{K,L}^n)}\right]^2 \\ &+ \frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K} \tau_{K,L}\tilde{u}_{K,L}^n\phi'(\tilde{u}_{K,L}^n)Dv_{K,L}^{n-1}Du_{K,L}^n,\end{aligned} \tag{2.9}$$

where $\tilde{u}_{K,L}^n = s_{K,L}^n u_K^n + (1 - s_{K,L}^n)u_L^n$, $s_{K,L}^n \in (0,1)$, $(D\chi_{K,L} = \chi_L - \chi_K)$.

Proof (Proof of Theorem 2.3)

$$\begin{aligned}W_h^n - W_h^{n-1} = &\underbrace{\sum_{K\in\mathcal{T}} m(K)(u_K^n \log u_K^n - u_K^{n-1}\log u_K^{n-1})}_{=I_1} \\ &+ \underbrace{\sum_{K\in\mathcal{T}} m(K)(u_K^n - u_K^{n-1})}_{=0, \quad \because (2.5)} - \underbrace{\frac{1}{2}\sum_{K\in\mathcal{T}} m(K)(u_K^n v_K^n - u_K^{n-1}v_K^{n-1})}_{=I_2}.\end{aligned}$$

Since for all $a, b > 0$, $b(\log a - \log b) \leq a - b$, we see that

$$\begin{aligned}I_1 &= \sum_{K\in\mathcal{T}} m(K)(u_K^n - u_K^{n-1})\log u_K^n + \sum_{K\in\mathcal{T}} m(K)u_K^{n-1}(\log u_K^n - \log u_K^{n-1}), \\ &\leq \sum_{K\in\mathcal{T}} m(K)(u_K^n - u_K^{n-1})\log u_K^n + \underbrace{\sum_{K\in\mathcal{T}} m(K)(u_K^n - u_K^{n-1})}_{=0, \quad \because (2.5)}.\end{aligned}$$

Applying Lemma 2.1 with $\phi(x) = \log(x)(\ \log'(x) = x^{-1})$, we have

$$I_1 \leq -\frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K} \tau_{K,L}\frac{|Du_{K,L}^n|^2}{\tilde{u}_{K,L}^n} + \frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K} \tau_{K,L}Dv_{K,L}^{n-1}Du_{K,L}^n.$$

$$\begin{aligned}I_2 = &\underbrace{\sum_{K\in\mathcal{T}} m(K)(u_K^n - u_K^{n-1})v_K^{n-1}}_{=I_{21}} \\ &+ \underbrace{\frac{1}{2}\sum_{K\in\mathcal{T}} m(K)(u_K^n v_K^n - 2u_K^n v_K^{n-1} + u_K^{n-1}v_K^{n-1})}_{=I_{22}}.\end{aligned}$$

In view of (2.4), we have

$$\begin{aligned}
I_{21} &= -\frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}Dv^{n-1}_{K,L}Du^n_{K,L} \\
&\quad+\frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}Dv^{n-1}_{K,L}\left((Dv^{n-1}_{K,L})_+u^n_K-(Dv^{n-1}_{K,L})_-u^n_L\right) \\
&= -\frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}Dv^{n-1}_{K,L}Du^n_{K,L}+\frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}\tilde{u}^n_{K,L}|Dv^{n-1}_{K,L}|^2 \\
&\quad+\underbrace{\frac{\tau_n}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}\left((Dv^{n-1}_{K,L})^2_+(1-s^n_{K,L})u^n_K+(Dv^{n-1}_{K,L})^2_-s^n_{K,L}u^n_L\right)}_{=I_{213}\geq 0}.
\end{aligned}$$

In view of (2.4), we have

$$\begin{aligned}
I_{22} &= \frac{1}{2}\left(\sum_{K\in\mathcal{T}}m(K)|v^n_K|^2+\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}|Dv^n_{K,L}|^2\right) \\
&\quad-\left(\sum_{K\in\mathcal{T}}m(K)v^n_Kv^{n-1}_K+\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}Dv^n_{K,L}Dv^{n-1}_{K,L}\right) \\
&\quad+\frac{1}{2}\left(\sum_{K\in\mathcal{T}}m(K)|v^{n-1}_K|^2+\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}|Dv^{n-1}_{K,L}|^2\right) \\
&= \frac{1}{2}\left(\sum_{K\in\mathcal{T}}m(K)|v^n_K-v^{n-1}_K|^2+\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}|Dv^n_{K,L}-Dv^{n-1}_{K,L}|^2\right).
\end{aligned}$$

Hence $I_{22}\geq 0$. Combining those estimates, we have

$$\begin{aligned}
&\frac{1}{\tau_n}(W^n_h-W^{n-1}_h)=\frac{1}{\tau_n}I_1-I_2=\frac{1}{\tau_n}I_1-I_{21}-I_{22} \\
&\leq -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}\frac{|Du^n_{K,L}|^2}{\tilde{u}^n_{K,L}}+\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}Dv^{n-1}_{K,L}Du^n_{K,L} \\
&\qquad-\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}\tilde{u}^n_{K,L}|Dv^{n-1}_{K,L}|^2-(I_{213}+I_{22})/\tau_n \\
&= -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K,L}\left|\frac{Du^n_{K,L}}{\sqrt{\tilde{u}^n_{K,L}}}-Dv^{n-1}_{K,L}\sqrt{\tilde{u}^n_{K,L}}\right|^2\leq 0.
\end{aligned}\tag{2.10}$$

Thus, we complete the proof.

3 Error estimate: dual triangulation and mass lumping operator

As mentioned in Introduction, Filbert [15] proved the convergence of the nonlinear finite volume scheme defined in general admissible mesh without any explicit convergence rate. The aim of our error analysis is to derive an error estimate with explicit convergence rate. To this end, we assume that admissible mesh is obtained from a non-obtuse triangulation of the finite element method.

3.1 Voronoi mesh and its dual triangulation

We assume the admissible mesh $\mathcal{T}$ of Voronoi type has a dual triangulation, denoted as $\hat{\mathcal{T}}$. We set

$$\hat{\mathcal{P}} = \{P_K \mid K \in \mathcal{T}\},$$

$$\hat{\mathcal{E}} = \{P_K P_L \mid \forall K \in \mathcal{T},\ L \in \mathcal{N}_K\},$$

where $\hat{\mathcal{P}}$ and $\hat{\mathcal{E}}$ are the sets of vertices and edges of triangles, respectively (see Figure 2.2). $\hat{T}$ is the triangle of $\hat{\mathcal{T}}$.

Assumptions

- $\overline{\Omega} = \cup_{K\in\mathcal{T}} \overline{K} = \cup_{\hat{T}\in\hat{\mathcal{T}}} \overline{\hat{T}}$.
- Every two elements of $\mathcal{T}$ (or $\hat{\mathcal{T}}$) meet only in entire common faces or in vertices.
- There exist two positive constants ξ_1, ξ_2, such that

$$\xi_1 \le \tau_{K,L} \le \xi_2, \quad \forall K \in \mathcal{T},\ L \in \mathcal{N}_K. \tag{3.1}$$

- There exists a positive constants γ_1, such that

$$h_{\hat{T}} \le \gamma_1 \rho_{\hat{T}}, \quad \forall \hat{T} \in \hat{\mathcal{T}}. \tag{3.2}$$

  where $h_{\hat{T}} = \mathrm{diam}(\hat{T})$, $\rho_{\hat{T}} = \max\{\mathrm{diam}(\hat{S}) \mid \hat{S}$ is ball included in $\hat{T}\}$.
- $\hat{h} = \max_{\hat{T}\in\hat{\mathcal{T}}} h_{\hat{T}}$. There exist positive constants $\gamma_2, \hat{\gamma}_2$ such that

$$\gamma_2 h \le h_K, \quad \forall K \in \mathcal{T}, \quad \hat{\gamma}_2 \hat{h} \le h_{\hat{T}}, \quad \forall \hat{T} \in \hat{\mathcal{T}}.$$

Remark 3.1 Those assumptions are fulfilled if admissible mesh is defined as the dual mesh of an non-obtuse triangulation of the finite element method (cf. [14, 19]).

Remark 3.2 The above assumptions imply that for any $K \in \mathcal{T}$, $L \in \mathcal{N}_K$ and $\hat{T} \in \hat{\mathcal{T}}$,

$$h \simeq \hat{h}, \quad m(K), m(\hat{T}) \simeq h^2, \quad d_{K,L}, m(K|L) \simeq h, \tag{3.3}$$

where, for $a, b > 0$, $a \simeq b$ means the existence of positive constants C_1, C_2 satisfying

$$C_1 a \le b \le C_2 a.$$

3.2 A mass-lumping operator

We associate a function $\hat{\phi}_K \in C(\overline{\Omega})$ with P_K (the control point of K or vertex of some triangle in $\hat{\mathcal{T}}$), such that $\hat{\phi}_K$ is linear on each $\hat{T} \in \hat{\mathcal{T}}$, and $\hat{\phi}_{K_i}(P_{K_j}) = \delta_{ij}$ for any $K_i, K_j \in \hat{\mathcal{P}}$, where δ_{ij} is the Kronecker's delta.

$$\hat{X}_h = \mathrm{span}\{\hat{\phi}_K \mid P_K \in \hat{\mathcal{P}}\}.$$

The mass-lumping operator M_h is defined by

$$M_h : \hat{X}_h \to X_h; \quad \hat{u}_h \mapsto u_h = M_h \hat{u}_h,$$

satisfying

$$\hat{u}_h(P_K) = u_h(P_K), \quad \forall P_K \in \hat{\mathcal{P}}.$$

It is easy to verify that M_h is a bijection, $\hat{u}_h = M_h^{-1} u_h$, and we have the following estimates:

$$C\|\hat{u}_h\|_p \le \|M_h\hat{u}_h\|_p \le C'\|\hat{u}_h\|_p \quad (p \in [1,\infty], u_h \in \hat{X}_h). \tag{3.4}$$

$$\|M_h\hat{u}_h - \hat{u}_h\|_p \le Ch\|\nabla\hat{u}_h\|_p \quad (p \in [1,\infty], u_h \in \hat{X}_h). \tag{3.5}$$

We put

$$(\hat{u}_h, \hat{v}_h)_h = (u_h, v_h). \tag{3.6}$$

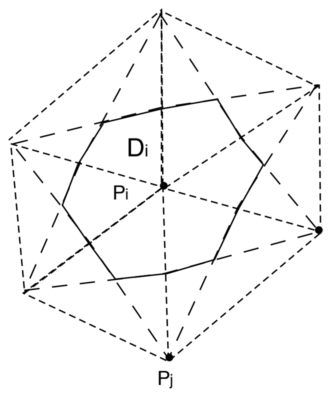

Fig. 3.1 Barycentric domain D_i.

Remark 3.3 All arguments in [2,27,7,13] are associated to a mass-lumping operator $\bar{M}_h$ based on the barycentric coordinate, which is different to our case; to be more specific, we reorder the control points (or the vertices of triangles)

$\hat{\mathcal{P}} = \{P_i\}_{i=1}^N$, the barycentric domain D_i (see Figure 3.1) corresponding to P_i is defined as

$$D_i = \cup_{\hat{T}\in\mathcal{J}_i}\{x \in \hat{T} \mid \varphi_j^{\hat{T}}(x) \le \varphi_i^{\hat{T}}(x)\ (P_j \in \mathcal{V}(\hat{T}),\ P_j \neq P_i)\},$$

where $\mathcal{J}_i = \{\hat{T} \in \hat{\mathcal{T}} \mid P_i \in \hat{T}\}$, $\mathcal{V}(\hat{T}) = \{P_j \in \hat{\mathcal{P}} \mid P_j \in \hat{T}\}$, and $\{\varphi_i^{\hat{T}}\}_{i=1}^{d+1}$ are the barycentric coordinates of $\hat{T}$ with respect to P_i. And we denote the mesh $\bar{\mathcal{T}} = \{D_i\}_{i=1}^N$. Let $\bar{\phi}_i$ be the characteristic function of D_i,

$$\bar{X}_h = \mathrm{span}\{\bar{\phi}_i \mid i = 1, \ldots, N\},$$

then we define

$$\bar{M}_h : \hat{X}_h \to \bar{X}_h; \quad \hat{u}_h \mapsto \bar{u}_h = \bar{M}_h\hat{u}_h, \quad \bar{M}_h\hat{u}_h = \sum_{i=1}^N \hat{u}_h(P_i)\bar{\phi}_i. \tag{3.7}$$

The operator $\bar{M}_h$, which is called lumping operator, has the properties as (3.4) and (3.5); moreover, it satisfies(cf. [34], Lemma 15.1), for all $p \in (1,\infty)$, $p^{-1} + q^{-1} = 1$,

$$|(\bar{M}_h\hat{u}_h, \bar{M}_h\hat{v}_h) - (\hat{u}_h, \hat{v}_h)| \le Ch^2\|\hat{u}_h\|_{1,p}\|\hat{v}_h\|_{1,q}, \quad \forall \hat{u}_h, \hat{v}_h \in \hat{X}_h, \tag{3.8}$$

which follows from the relationship between the barycentric partition and the quadrature formula:

$$\int_{\hat{T}} \hat{u}_h = \int_{\hat{T}} \bar{M}_h\hat{u}_h, \quad \forall \hat{T} \in \hat{\mathcal{T}}. \tag{3.9}$$

For M_h, we have the estimates,

$$|(M_h\hat{u}_h, M_h\hat{v}_h) - (\hat{u}_h, \hat{v}_h)| \le Ch(\|\hat{u}_h\|_p\|\nabla\hat{v}_h\|_q + \|\hat{v}_h\|_p\|\nabla\hat{u}_h\|_q). \tag{3.10}$$

The effect of this difference between M_h and $\bar{M}_h$ to the error analysis for finite-volume approximation will be explained in Sect. 4.1.3(see Remark 4.1).

3.2.1 $\|\cdot\|_{1,p,\mathcal{T}}$ *norm for* X_h

Although the function in X_h is not differentiable, we can define a norm $\|\cdot\|_{1,p,\mathcal{T}}$ (cf. [4,14]) for X_h, which is similar to $W^{1,p}$-norm $\|\cdot\|_{1,p}$ for $\hat{X}_h$. With the assumptions on mesh $\mathcal{T}$ in Sect. 3.1, for $p \in [1,\infty)$, we define, $\forall u_h \in X_h$

$$\|u_h\|_{1,p,\mathcal{T}}^p = \sum_{K|L\in\mathcal{E}_{int}} \tau_{K,L} d_{K,L}^{2-p}|u_K - u_L|^p. \tag{3.11}$$

$$\|u_h\|_{1,\infty,\mathcal{T}} = \max_{K|L\in\mathcal{E}_{int}} \frac{|u_K - u_L|}{d_{K,L}}. \tag{3.12}$$

Lemma 3.1 *Under the assumptions on mesh $\mathcal{T}$ and its dual mesh $\hat{\mathcal{T}}$ in Sect. 3.1, we have, for $p \in [1,\infty]$,*

$$\|u_h\|_{1,p,\mathcal{T}} \backsimeq \|\nabla\hat{u}_h\|_{0,p}, \quad \forall u_h \in X_h, \hat{u}_h = M_h^{-1}u_h. \tag{3.13}$$

Proof With (3.3), we have

$$\|u_h\|_{1,p,\mathcal{T}}^p \backsimeq \sum_{K\in\mathcal{T}} m(K)\left|\frac{u_K-u_L}{d_{K,L}}\right|^p \backsimeq \|\nabla\hat{u}_h\|_{0,p}^p.$$

3.2.2 Variational forms of finite volume scheme (2.4)

For any $u_h,\chi_h\in X_h$,

$$\begin{aligned} A_h(u_h,\chi_h) &:= \sum_{K\in\mathcal{T}}\chi_K\sum_{L\in\mathcal{N}_K}\tau_{K,L}(u_K-u_L)\\ &= \sum_{K|L\in\mathcal{E}_{int}}\tau_{K,L}Du_{K,L}D\chi_{K,L}. \end{aligned} \tag{3.14}$$

For any $u_h\in X_h$, there exists $v_h\in X_h$ such that

$$(v_h,\chi_h)+A_h(v_h,\chi_h)=(u_h,\chi_h),\quad \forall\chi_h\in X_h. \tag{3.15}$$

Then we define the operator G_h which is a discrete version of G, with

$$G_h: X_h\to X_h;\quad u_h\mapsto v_h=G_hu_h,$$

And we have

$$v_h^n=G_hu_h^n,\quad n\geq 0, \tag{3.16}$$

for the finite-volume scheme (2.4). For any $w_h,u_h,\chi_h\in X_h$, we set

$$\begin{aligned} &B_h(w_h,u_h,\chi_h)\\ \equiv&\sum_{K\in\mathcal{T}}\chi_K\sum_{L\in\mathcal{N}_K}\tau_{K,L}\left[(D(G_hw_h)_{K,L})_+u_K-(D(G_hw_h)_{K,L})_-u_L\right]. \end{aligned} \tag{3.17}$$

Now, we can write the scheme (2.4) into a variational form,

$$\begin{cases} \text{find } u_h^n\in X_h, n\geq 1, \text{ such that,}\\ (\partial_{\tau_n}u_h^n,\chi_h)+A_h(u_h^n,\chi_h)+B_h(u_h^{n-1},u_h^n,\chi_h)=0,\quad \forall\chi_h\in X_h. \end{cases} \tag{3.18}$$

Variational form with mass-lumpling operator M_h

To derive error estimates, we need to applying the mass-lumpling operator to scheme (2.4) or (3.18) to rewrite its into a variational form of finite element type. In view of that, for any $u_h,\chi_h\in X_h$, $\hat{u}_h=M_h^{-1}u_h\in\hat{X}_h$, $\hat{\chi}_h=M_h^{-1}\chi_h\in\hat{X}_h$,

$$A_h(u_h,\chi_h)=(\nabla\hat{u}_h,\nabla\hat{\chi}_h), \tag{3.19}$$

(3.18) is equivalent to:

$$\begin{cases} \text{find } \hat{u}_h^n\in \hat{X}_h, n\geq 1, \text{ such that,}\\ (\partial_{\tau_n}\hat{u}_h^n,\hat{\chi}_h)_h+(\nabla\hat{u}_h^n,\nabla\hat{\chi}_h)+b_h(\hat{u}_h^{n-1},\hat{u}_h^n,\hat{\chi}_h)=0,\quad \forall\hat{\chi}_h\in \hat{X}_h, \end{cases} \tag{3.20}$$

where $(\cdot,\cdot)_h$ is defined by (3.6), and for all $\hat{w}_h,\hat{u}_h,\hat{\chi}_h\in X_h$,

$$b_h(\hat{w}_h,\hat{u}_h,\hat{\chi}_h):=B_h(w_h,u_h,\chi_h),\quad w_h=M_h\hat{w}_h,\ u_h=M_h\hat{u}_h,\ \chi_h=M_h\hat{\chi}_h.$$

We then state our main result.

3.3 Results on error estimates

Since Ω is not smooth, L^p solution is considered, and we make the following regularity assumption on the solution u of (1.1) (cf. [33,31,27]):

$$u \in C([0,T]; \mathcal{W}^p), \quad u' \in C([0,T]; W^{1,p}) \cap C^\sigma([0,T]; L^p), \tag{3.21}$$

for some $p \geq 2$, $\sigma \in (0,1]$, and put

$$\alpha_{1,p} = \sup_{t\in[0,T]} \|u(t)\|_{2,p}, \quad \alpha_{2,p} = \sup_{t\in[0,T]} \|u'(t)\|_{1,p},$$

$$\alpha_{3,p} = \sup_{t,s\in[0,T]} \frac{\|u'(t) - u'(s)\|_p}{|t-s|^\sigma},$$

where $u' = du/dt$. In addition, we make the a priori estimates assumption on discrete solution $\{u_h^n\}_{n=1}^J$, $J = \max\{n \in \mathbb{N} \mid t_n < T\}$, that

$$\|u_h^n\|_p, \|u_h^n\|_{1,2,\mathcal{T}} \leq C(u_h^0), \quad (p > d). \tag{3.22}$$

Remark 3.4 The proof of a priori estimates to $\{u_h^n\}_{n=1}^J$ is quite technical. In [15], it shows the results for a non-linear scheme,

$$\|u_h^n\|_2 + \sum_{j=1}^n \|u_h^n\|_{1,2,\mathcal{T}} \leq C_1(\|u_h^0\|_2 + \|u_h^0 \log(u_h^0)\|_1),$$

$$\|u_h^n\|_3 \leq C_2(\|u_h^0\|_2 + \|u_h^0 \log(u_h^0)\|_1).$$

with the similar argument of [15], one can obtain,

$$\|u_h^n\|_4 \leq C_3(\|u_h^0\|_2 + \|u_h^0 \log(u_h^0)\|_1 + \|u_h^0\|_{1,2,\mathcal{T}}).$$

However, the a priori estimate $\|u_h^n\|_{1,2,\mathcal{T}} \leq C(u_h^0)$ has not been verified yet. Assumption (3.22) is used in Lemma 4.1, Lemma 4.9 and Remark 4.2.

Theorem 3.1 (Error estimates) *Assume (1.1) admits a unique solution u satisfying (3.21) for some $p \in (d, \mu)$ and $\sigma \in (0,1]$. Let $u_h^0 \in X_h$ with*

$$\|u_0 - u_h^0\|_p \leq C_0 h^{1-2/p}. \tag{3.23}$$

$\{(u_h^n, v_h^n)\}_{n=1}^J$ is the solution of (2.4) satisfying (3.22). Then we have the error estimates

$$\sup_{0\leq n\leq l} \|u(t_n) - u_h^n\|_p \leq C_1(h^{1-2/p} + \tau^\sigma), \tag{3.24}$$

$$\sup_{0\leq n\leq l} \|v(t_n) - \hat{v}_h^n\|_{1,\infty} \leq C_2 h^{1-2/p} + C_3 \sup_{0\leq n\leq l} \|u(t_n) - u_h^n\|_p, \tag{3.25}$$

where $\tau = \max \tau_n$, $t_n = \sum_{i-1}^n \tau_i$. C_1, C_2, C_3 are the constants depending on Ω, p, d, C_0, α_i, $i = 1,2,3$.

4 Proof of error estimates

To start the error analysis, we need some preliminary results.

4.1 Some preliminary results

4.1.1 Sobolev and inverse inequalities

The following inequalities can be found in [1,12].

$$\|v\|_\infty \leq C\|v\|_{1,p} \quad (p \in (d,\infty], v \in W^{1,p}). \tag{4.1}$$

Let $\hat{T} \in \hat{\mathcal{T}}$, we have the inverse inequality

$$\|\hat{v}_h\|_{W^{l,p}(\hat{T})} \leq Ch_{\hat{T}}^{m-l+min\{0,\frac{d}{p}-\frac{d}{q}\}}\|v_h\|_{W^{m,q}(\hat{T})}, \quad (\hat{v}_h \in \hat{X}_h), \tag{4.2}$$

where $p, q \in [1,\infty]$, l, m are integers, $m < l$.

$$\max_{x,y\in\hat{T}} |\hat{v}_h(x) - \hat{v}_h(y)| \leq Ch_{\hat{T}}^{1-d/p}\|\nabla\hat{v}_h\|_{L^p(\hat{T})} \quad (p \in [1,\infty], v_h \in \hat{X}_h). \tag{4.3}$$

4.1.2 Interpolation and H^1-projection operators

We use the Lagrange interpolation operator Π_h and the H^1-projection operator R_h (cf. [12,34,6]).

$$\Pi_h : C(\overline{\Omega}) \to \hat{X}_h, \quad \Pi_h v(P_i) = v(P_i), \quad \forall P_i \in \hat{\mathcal{P}}. \tag{4.4}$$

For any $\hat{T} \in \hat{\mathcal{T}}$, $v \in W^{2,p}(\hat{T})$, $p \in (d/2,\infty]$, we have

$$\|\Pi_h v - v\|_{L^p(\hat{T})} + h\|\nabla(\Pi_h v - v)\|_{L^p(\hat{T})} \leq Ch_{\hat{T}}^2\|v\|_{W^{2,p}(\hat{T})}, \tag{4.5}$$

and for $p \in (d,\infty]$,

$$\|\Pi_h v - v\|_{L^\infty(\hat{T})} + h\|\nabla(\Pi_h v - v)\|_{L^\infty(\hat{T})} \leq Ch_{\hat{T}}^{2-d/p}\|v\|_{W^{2,p}(\hat{T})}. \tag{4.6}$$

H^1-projection operator R_h is defined by

$$R_h : H^1 \to \hat{X}_h, \quad v \mapsto R_h v,$$

satisfying

$$(\nabla R_h v - \nabla v, \nabla\hat{\chi}_h) + (R_h v - v, \hat{\chi}_h) = 0, \quad \forall\hat{\chi}_h \in \hat{X}_h. \tag{4.7}$$

Under the assumption that for any $v \in W^{r,p}$, $p \geq 2$

$$\inf_{\hat{\chi}_h\in\hat{X}_h} (\|v - \hat{\chi}_h\|_p + h\|\nabla(v - \hat{\chi}_h)\|_p) \leq Ch^r\|v\|_{r,p}, \quad r = 1, 2,$$

we have

$$\|R_h v - v\|_p + h\|\nabla(R_h v - v)\|_p \leq Ch^r\|v\|_{r,p}, \quad r = 1, 2. \tag{4.8}$$

Also, we have

$$\|R_h v\|_{1,p} \leq C\|v\|_{1,p} \quad (p \in (1,\infty], v \in W^{1,p}). \tag{4.9}$$

4.1.3 Discrete Laplace operators

We define two discrete Laplace operators: L_h and G_h^{-1}.(G_h is defined by (3.15).)

$$L_h : \hat{X}_h \to \hat{X}_h, \quad \hat{w}_h \mapsto L_h \hat{w}_h = \hat{u}_h,$$

$$(\nabla \hat{w}_h, \nabla \hat{\chi}_h) + (\hat{w}_h, \hat{\chi}_h) = (\hat{u}_h, \hat{\chi}_h), \quad \forall \hat{\chi}_h \in \hat{X}_h. \tag{4.10}$$

$$G_h^{-1} : X_h \to X_h, \quad v_h \mapsto G_h^{-1} v_h = u_h,$$

$$A_h(v_h, \chi_h) + (v_h, \chi_h) = (u_h, \chi_h), \quad \forall \chi_h \in X_h. \tag{4.11}$$

For any $v_h, u_h \in X_h$ satisfying $u_h = G_h^{-1} v_h$, or equivalently (4.11), setting $\hat{v}_h = M_h^{-1} v_h$, $\hat{u}_h = M_h^{-1} u_h$, we define the operator

$$A_h : \hat{X}_h \to \hat{X}_h, \quad \hat{v}_h \mapsto A_h \hat{v}_h = \hat{u}_h. \tag{4.12}$$

In view of (3.19) and (3.6), we have

$$(\nabla \hat{v}_h, \nabla \hat{\chi}_h) + (\hat{v}_h, \hat{\chi}_h)_h = (\hat{u}_h, \hat{\chi}_h)_h, \quad \forall \hat{\chi}_h \in \hat{X}_h. \tag{4.13}$$

Lemma 4.1 *For $u_h \in X_h$, let $\hat{u}_h = M_h^{-1} u_h$, $\hat{w}_h = L_h^{-1} \hat{u}_h$, $v_h = G_h u_h$, $\hat{v}_h = M_h^{-1} v_h$, we have*

$$\|\nabla(\hat{v}_h - \hat{w}_h)\|_2 + \|\hat{v}_h - \hat{w}_h\|_2 \le Ch(\|\nabla \hat{u}_h\|_2 + \|\hat{u}_h\|_2), \tag{4.14}$$

Proof Subtracting (4.10) from (4.13), it yields, for any $\hat{\chi}_h \in \hat{X}_h$,

$$(\nabla(\hat{v}_h - \hat{w}_h), \nabla \hat{\chi}_h) + (\hat{v}_h - \hat{w}_h, \hat{\chi}_h) = (\hat{u}_h + \hat{w}_h, \hat{\chi}_h) - (\hat{u}_h + \hat{w}_h, \hat{\chi}_h)_h. \tag{4.15}$$

Substituting $\hat{\chi}_h = \hat{v}_h - \hat{w}_h$ into (4.15), (4.14) follows form (3.4) and (3.5).

Applying (4.2) to (4.14), we obtain

$$\|\nabla(\hat{v}_h - \hat{w}_h)\|_p = \|\nabla(M_h^{-1} G_h u_h - L_h^{-1} M_h^{-1} u_h)\|_p \le Ch^{1+d/p-d/2}. \tag{4.16}$$

The estimate (4.16) is not sharp, applying which we will obtain the error estimate of order $h^{d/p} + h^{1-d/p}$ instead of $h^{1-d/p}$ in (3.24). We extend the method of [7] to obtain a sharper error estimate for $M_h^{-1} G_h - L_h^{-1} M_h^{-1}$.

Lemma 4.2 (An analogue of Theorem 2.1 of [7]) *Let $u_h \in X_h$, set $v_h = G_h u_h$, $\hat{v}_h = M_h^{-1} v_h$. Let $V = G u_h$, be the solution of*

$$-\Delta V + V = u_h \text{ in } \Omega, \quad \partial_\nu V = 0 \text{ on } \Gamma.$$

Then, we have

$$\|\hat{v}_h - V\|_{1,p} \le Ch \|V\|_{2,p}, \quad 2 \le p < \mu. \tag{4.17}$$

Proof $\hat{v}_h = M_h^{-1} G_h u_h \in \hat{X}_h$ is the solution of

$$a^*(\hat{v}_h, \chi_h) = (u_h, \chi_h), \quad \chi_h \in X_h, \tag{4.18}$$

where

$$\begin{aligned} a^*(\hat{w}_h, \chi_h) &:= \sum_{P_i \in \hat{\mathcal{P}}} \chi_h(P_i)(-\int_{\partial K_{P_i}} \nabla \hat{w}_h \cdot \nu ds) + (w_h, \chi_h), \\ &= A_h(w_h, \chi_h) + (w_h, \chi_h), \quad \forall \hat{w}_h \in \hat{X}_h,\ w_h = M_h \hat{w}_h,\ \chi_h \in X_h. \end{aligned}$$

Let $p \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} &|a(v - \hat{v}_h, \Pi_h v) - a^*(v - \hat{v}_h, M_h \Pi_h v)| \\ \le& Ch(\|\nabla(v - \hat{v}_h)\|_p + \|v\|_{2,p})\|\nabla v\|_{1,q}, \end{aligned} \tag{4.19}$$

where

$$a(u, v) = (\nabla u, \nabla v) + (u, v), \quad \forall u, v \in H^1.$$

The proof of (4.19) is the same to (2.1) in [7](see Lemma 2.1 of [7]).

In view of Theorem 2.1 of [7], (4.19) implies (4.17).

Remark 4.1 In [7], the authors consider lumping operator $\bar{M}_h$(defined by (3.7)) instead of M_h. However, (4.19) and (4.17) hold for both M_h and $\bar{M}_h$, since the proof of (4.19), (4.17) only use the properties (3.5), (3.6) and (3.10), which are verified for both M_h and $\bar{M}_h$. Different to M_h, $\bar{M}_h$ satisfies (3.9), (3.8), which gives a higher-order error estimate

$$\|\bar{M}_h^{-1} - V\|_{0,p} \le Ch^2 \|V\|_{3,p}.$$

This higher-order estimates is not necessary to our case.

Following form Lemma 4.2 and the L^p error estimates for $G - L_h$, we have

Lemma 4.3 *For $u_h \in X_h$, $\hat{u}_h = M_h^{-1} u_h$,*

$$\|\nabla(M_h^{-1} G_h - L_h^{-1} M_h^{-1}) u_h\|_p \le Ch(\|\nabla \hat{u}_h\|_p + \|\hat{u}_h\|_p), \tag{4.20}$$

Proof Set $v_h = G_h u_h$, $\hat{w}_h = L_h^{-1} M_h^{-1} u_h$, $V = G u_h$. We have by Lemma 4.2,

$$\|\hat{v}_h - V\|_{1,p} \le Ch\|V\|_{2,p} \le Ch\|u_h\|_p, \quad (\hat{v}_h = M_h^{-1} v_h).$$

For finite element method, we have

$$\|(L_h^{-1} - G) M_h^{-1} u_h\|_{1,p} \le Ch\|G(M_h^{-1} u_h)\|_{2,p} \le Ch\|M_h^{-1} u_h\|_p \le Ch\|u_h\|_p.$$

It is obvious that

$$\|G(u_h - M_h^{-1} u_h)\|_{2,p} \le C\|u_h - M_h^{-1} u_h\|_p \le Ch\|\nabla M_h^{-1} u_h\|_p.$$

Combining those inequalities, we prove the lemma.

4.1.4 L^p estimates of A_h

We state some results on L^p estimates for A_h(defined by (4.12)). Setting a new norm $\|\cdot\|_{h,p}$ for $\hat{X}_h$, with

$$\|\hat{v}_h\|_{h,p} := \|v_h\|_p, \quad v_h = M_h \hat{v}_h, \quad \forall \hat{v}_h \in X_h, \tag{4.21}$$

we denote $\mathcal{X}_{h,p}$ as the Banach space $\hat{X}_h$ equipped with norm $\|\cdot\|_{h,p}$. Furthermore, we extend A_h to space $\mathcal{X}_{h,p}$, with

$$\|A_h\|_{h,p} = \sup_{\hat{v}_h \in X_h} \frac{\|A_h \hat{v}_h\|_{h,p}}{\|\hat{v}_h\|_{h,p}}.$$

We have the following properties.

Lemma 4.4 *Let $p \in (1, \infty)$, then*

(i) A_h is sectorial in $\mathcal{X}_{h,p}$, and its fractional powers A_h^α, $\alpha \in (0,1)$, are defined.
(ii) A_h and A_h^α, $\alpha \in (0,1)$, are positive and self-adjoint in $\mathcal{X}_{h,2}$.
(iii) For any $\theta \in (0,1)$ and $\{\tau_j\}_{j=1}^n$, $\tau_j > 0$, we have

$$\|r(\tau_n A_h) \cdots r(\tau_1 A_h) A_h^\theta\|_{h,p} \le C_\theta (\tau_n + \cdots + \tau_1)^{-\theta}, \tag{4.22}$$

where $r(\tau_j A_h) = (I + \tau_j A_h)^{-1}$.

Proof It is not difficult to verify that, the result (i)(cf. [13]), (ii)(cf. [26]), and (iii)(cf. [25]), which are all derived originally for mass lumping operator $\bar{M}_h$ can be extended to M_h.

Lemma 4.5 (Lemma 4.4 of [27]) *Under the assumptions on mesh $\hat{\mathcal{T}}$ in Sect. 3.1, we have*

$$\|\hat{v}_h\|_{1,p} \le C \|A_h^\theta \hat{v}_h\|_{h,p} \quad (\hat{v}_h \in \hat{X}_h) \tag{4.23}$$

for $p \in (\mu/(\mu-1), \mu)$ and $\theta \in (1/2, 1]$.

Now, we can state the error estimates. In the following, we separate the error

$$u_h^n - u(t_n) = u_h^n - M_h R_h u(t_n) + M_h R_h u(t_n) - u(t_n). \tag{4.24}$$

Following from (3.5), (4.9) and (4.8) for $r = 2$ we have

$$\|M_h R_h u(t_n) - u(t_n)\|_p \le C(h + h^2)\alpha_{1,p}. \tag{4.25}$$

Setting

$$w_h^n = u_h^n - M_h R_h u(t_n),$$

the following argument is aim to estimate $\|w_h^n\|_p$.

4.2 Estimates on $\partial_{\tau_n} u_h^n - u'(t_n)$

Lemma 4.6 *For any $\chi_h \in X_h$, $\hat{\chi}_h = M_h^{-1}\chi_h$, we have, $p \in (2,\infty)$, $1/p + 1/q = 1$,*

$$|(\partial_{\tau_n} M_h R_h u(t_n), \chi_h) - (u'(t_n), \hat{\chi}_h)| \le C(\alpha_{2,p} + \alpha_{3,p})(h + \tau_n^\sigma)\|\hat{\chi}_h\|_{1,q}. \tag{4.26}$$

Proof The argument is quite standard. We make the separation,

$$\begin{aligned}
&(\partial_{\tau_n} M_h R_h u(t_n), \chi_h) - (u'(t_n), \hat{\chi}_h)\\
=&((M_h - I)\partial_{\tau_n} R_h u(t_n), \chi_h) + ((R_h - I)\partial_{\tau_n} u(t_n), \chi_h)\\
&+ (\partial_{\tau_n} u(t_n) - u'(t_n), \chi_h) + (u'(t_n), \chi_h - \hat{\chi}_h) = \sum_{i=1}^4 I_i,
\end{aligned}$$

By (3.5) and (4.9) we have

$$\begin{aligned}
I_1 &\le Ch\|\nabla R_h \partial_{\tau_n} u(t_n)\|_p \|\chi_h\|_q \le Ch\|\partial_{\tau_n} u(t_n)\|_{1,p}\|\chi_h\|_q\\
&\le Ch\left\|\tau_n^{-1}\int_{t_{n-1}}^{t_n} u'(s)ds\right\|_{1,p}\|\chi_h\|_q \le C\alpha_{2,p} h\|\chi_h\|_q.
\end{aligned} \tag{4.27}$$

$$I_4 \le Ch\alpha_{2,p}\|\nabla\hat{\chi}_h\|_q. \tag{4.28}$$

$$\begin{aligned}
I_3 &\le C\left\|\frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} u'(t_n) - u'(s)ds\right\|_p \|\chi_h\|_q\\
&\le C\tau_n^{1/q-1}\left(\int_{t_{n-1}}^{t_n}\|u'(t_n) - u'(s)\|_p^p ds\right)^{1/p}\|\chi_h\|_q\\
&\le C\alpha_{3,p}\tau_n^{1/q-1}\left(\int_{t_{n-1}}^{t_n}|t-s|^{\sigma p}\right)^{1/p} ds\|\chi_h\|_q \le C\alpha_{3,p}\tau_n^\sigma\|\chi_h\|_q.
\end{aligned} \tag{4.29}$$

The estimate of I_2 follows from (4.8) for $r=1$,

$$\begin{aligned}
I_2 &\le C\left\|(R_h - I)\partial_{\tau_n} u(t_n)\right\|_p \|\chi_h\|_q\\
&\le Ch\|\partial_{\tau_n} u(t_n)\|_{1,p}\|\chi_h\|_q \le Ch\alpha_{2,p}\|\chi_h\|_q.
\end{aligned} \tag{4.30}$$

The proof is completed.

4.3 Estimates on $b - B_h$

We shall estimate $b(u(t_n), u(t_n), \hat{\chi}_h) - B_h(u_h^{n-1}, u_h^n, \chi_h)$.

Lemma 4.7 *Let $\{u_n\}_{n=0}^J$ be the solution of (2.4), and u be the solution of (1.1). For any $\chi_h \in X_h$, $\hat{\chi}_h = M_h^{-1}\chi_h$, we have*

$$\begin{aligned}
&|B_h(u_h^{n-1}, u_h^n, \chi_h) - b(u(t_n), u(t_n), \hat{\chi}_h)|\\
\le& C(\tau_n + h^{1-d/p} + \|\hat{w}_h^n\|_p + \|\hat{w}_h^{n-1}\|_p)\|\hat{\chi}_h\|_{1,q},
\end{aligned} \tag{4.31}$$

where $p \in (2,\infty)$, $1/p + 1/q = 1$, $w_h^n = u_h^n - M_h R_h u(t_n)$, and $\hat{w}_h^n = M_h^{-1} w_h^n$.

To prove Lemma 4.7, we first state some lemmas on b, $\tilde{b}_h$ and B_h, where the definition of $\tilde{b}_h$ is given below.

4.3.1 Lemmas on b, $\tilde{b}_h$ and B_h

Lemma 4.8 *For $b(u,v,w) = \int_\Omega \nabla(Gu)v\nabla w$, we have $\forall u \in L^p, v \in L^\infty, w \in L^q$,*

$$|b(u,v,w)| \le C\|u\|_p\|v\|_p\|\nabla w\|_q, \quad p \in (d,\mu), \frac{1}{p}+\frac{1}{q}=1. \tag{4.32}$$

Lemma 4.9 *For any $u_h, v_h, \chi_h \in X_h$, set $\hat{u}_h = M_h^{-1}u_h$, $\hat{v}_h = M_h^{-1}v_h$, $\hat{\chi}_h = M_h^{-1}\chi_h$. $B(u_h, v_h, \chi_h)$ is defined by (3.17), then we have*

$$|B(u_h,v_h,\chi_h)| \le C\|\hat{u}_h\|_{1,p}\|\hat{v}_h\|_p\|\nabla\hat{\chi}\|_q, \quad p \in (d,\mu), \frac{1}{p}+\frac{1}{q}=1. \tag{4.33}$$

Proof From (3.17), we have

$$\begin{aligned}
|B(u_h,v_h,\chi_h)| \le{}& C \max_{K|L\in\mathcal{E}_{int}} \left\{\frac{|D(G_hu_h)_{K,L}|}{d_{K,L}}\right\} \\
&\cdot \left(\sum_{K|L\in\mathcal{E}_{int}} \tau_{K,L}d_{K,L}^{2-q}|\chi_K-\chi_L|^q\right)^{1/q} \left(\sum_{K|L\in\mathcal{E}_{int}} \tau_{K,L}d_{K,L}^{2}|v_K|^p\right)^{1/p} \\
\le{}& C\|G_hu_h\|_{1,\infty,\mathcal{T}}\|v_h\|_p\|\chi_h\|_{1,q,\mathcal{T}}.
\end{aligned}$$

Applying Lemma 3.1, we have

$$\|\chi_h\|_{1,q,\mathcal{T}} \le C\|\nabla\hat{\chi}_h\|_q, \quad \|G_hu_h\|_{1,\infty,\mathcal{T}} \le C\|\nabla M_h^{-1}G_hu_h\|_\infty.$$

To estimate $\|\nabla M_h^{-1}G_hu_h\|_\infty$, we see that,

$$\|\nabla M_h^{-1}G_hu_h\|_\infty \le \|\nabla(M_h^{-1}G_h - L_h^{-1}M_h^{-1})u_h\|_\infty + \|\nabla L_h^{-1}M_h^{-1}u_h\|_\infty.$$

By (4.2) and Lemma 4.3 we have

$$\begin{aligned}
&\|\nabla(M_h^{-1}G_h - L_h^{-1}M_h^{-1})u_h\|_\infty \\
\le{}& h^{-d/p}\|\nabla(M_h^{-1}G_h - L_h^{-1}M_h^{-1})u_h\|_p \le Ch^{1-d/p}\|\hat{u}\|_{1,p}.
\end{aligned}$$

Applying Lemma 4.5 in [27], we obtain the estimate

$$\|\nabla L_h^{-1}M_h^{-1}u_h\|_\infty \le C\|\hat{u}_h\|_p.$$

The Lemma is proved.

Remark 4.2 In the proof of Lemma 4.9, we obtain that

$$|B(u_h, v_h, \chi_h)| \leq C(\|\hat{u}\|_p + \underbrace{h^{1-d/p}\|\nabla \hat{u}_h\|_p}_{\leq C\|\nabla \hat{u}_h\|_2,\ \because\ (4.2),\ d=2})\|\hat{v}_h\|_p\|\nabla\hat{\chi}\|_q, \tag{4.34}$$

In [15], for (u_h^n, v_h^n) the solution of finite-volume scheme, the a-prior estimates

$$\|u_h^n\|_2 + \sum_{i=1}^{l} \tau_n \|u_h^i\|_{1,2,\mathcal{T}} \leq C, \quad \|u_h^n\|_3 \leq C,$$

are proved, where C is some constant depending on initial data u_h^0. However, the a priori estimate of $\|u_h^n\|_{1,2,\mathcal{T}}$ has not been proved yet. In view of (4.34), to obtain

$$|B(u_h^n, v_h, \chi_h)| \leq C\|\hat{v}_h\|_p\|\nabla\hat{\chi}\|_q,$$

we need assumption on the a-prior estimates that

$$\|u_h^n\|_p, \|u_h^n\|_{1,2,\mathcal{T}} \leq C. \tag{4.35}$$

Next, we introduce $\tilde{b}_h$. Let

$$\tilde{\beta}_{K,L}^{\pm}(u_h) := \int_{K|L} [\nabla(L_h^{-1}M_h^{-1}u_h)\cdot\nu_{K,L}]_{\pm} ds,$$

where $\mu_{K,L}$ is the unit outer normal vector to the edge $K|L$ of element K, and thanks to the property of Voronoi mesh, that is $x_K x_L \perp K|L$, we have

$$\tilde{\beta}_{K,L}^{\pm}(u_h) = m(K|L)[(L_h^{-1}M_h^{-1}u_h)(x_L) - (L_h^{-1}M_h^{-1}u_h)(x_K)].$$

Then we set, for any $u_h \in X_h$, $\hat{v}_h, \hat{\chi}_h \in \hat{X}_h$, setting $v_h = M_h\hat{v}_h$, $\chi_h = M_h\hat{\chi}_h$,

$$\tilde{b}_h(u_h, \hat{v}_h, \hat{\chi}_h) := \sum_{K\in\mathcal{T}} \chi_K \sum_{L\in\mathcal{N}_K} [\tilde{\beta}_{K,L}^{+}(u_h)v_K - \tilde{\beta}_{K,L}^{-}(u_h)v_L].$$

We have the following lemma.

Lemma 4.10 *For $p \in (d, \mu)$, $1/p + 1/q = 1$,*

$$|\tilde{b}_h(u_h, \hat{v}_h, \hat{\chi}_h) - b(u_h, \hat{v}_h, \hat{\chi}_h)| \leq Ch^{1-d/p}\|u_h\|_p\|\hat{v}_h\|_{1,p}\|\nabla\hat{\chi}_h\|_q, \tag{4.36}$$

for any $u_h \in X_h$, $\hat{v}_h, \hat{\chi}_h \in \hat{X}_h$.

Proof We set the finite-element approximation operator L_h the same to [27], and our $\tilde{b}_h$ is equivalent to b_h of [27]. It is a direct result from Lemma 5.4 in [27].

4.3.2 Estimates on $b - B_h$(proof of Lemma 4.7)

Proof (Proof of Lemma 4.7) With $w_h^n = u_h^n - M_h R_h u(t_n)$, we write as

$$\begin{aligned}
&B_h(u_h^{n-1}, u_h^n, \chi_h) - b(u(t_n), u(t_n), \hat{\chi}_h) \\
&= b(u(t_{n-1}) - u(t_n), u(t_n), \hat{\chi}_h) + b(R_h u(t_{n-1}) - u(t_{n-1}), u(t_n), \hat{\chi}_h) \\
&\quad + b(M_h R_h u(t_{n-1}) - R_h u(t_{n-1}), u(t_n), \hat{\chi}_h) \\
&\quad + b(M_h R_h u(t_{n-1}), R_h u(t_n) - u(t_n), \hat{\chi}_h) \\
&\quad + b(-w_h^{n-1}, R_h u(t_n), \hat{\chi}_h) \\
&\quad + \big(B_h(u_h^{n-1}, u_h^n, \chi_h) - b(u_h^{n-1}, R_h u(t_n), \hat{\chi}_h)\big) \\
&\equiv \sum_{i=1}^{6} I_i.
\end{aligned}$$

Using Lemma 4.8, (4.8), (3.5) and (3.4), we have

$$\begin{aligned}
|I_1| &\le C\|u(t_n)\|_\infty \|u(t_{n-1}) - u(t_n)\|_p \|\nabla\hat{\chi}_h\|_q \\
&= C\alpha_{1,p} \left\| \int_{t_{n-1}}^{t_n} u'(s)ds \right\|_p \|\nabla\hat{\chi}_h\|_q \\
&\le C\alpha_{1,p}\tau_n \|u'\|_{C([0,T],L^p)} \|\nabla\hat{\chi}_h\|_q \\
&\le C\alpha_{2,p}\alpha_{1,p}\tau_n \|\nabla\hat{\chi}_h\|_q; \\
|I_2| &\le C\|R_h u(t_{n-1}) - u(t_{n-1})\|_p \|u(t_n)\|_p \|\nabla\hat{\chi}_h\|_q \\
&\le C\alpha_{1,p}^2 h^2 \|\nabla\hat{\chi}_h\|_q; \\
|I_3| &\le C\|M_h R_h u(t_{n-1}) - R_h u(t_{n-1})\|_p \|u(t_n)\|_p \|\nabla\hat{\chi}_h\|_q \\
&\le C\alpha_{1,p}^2 h \|\nabla\hat{\chi}_h\|_q; \\
|I_4| &\le C\|M_h R_h u(t_{n-1})\|_p \|R_h u(t_n) - u(t_n)\|_p \|\nabla\hat{\chi}_h\|_q \\
&\le C\alpha_{1,p}^2 h^2 \|\nabla\hat{\chi}_h\|_q; \\
|I_5| &\le C\|w_h^{n-1}\|_p \|R_h u(t_n)\|_p \|\nabla\hat{\chi}_h\|_q \\
&\le C\alpha_{1,p} \|w_h^{n-1}\|_p \|\nabla\hat{\chi}_h\|_q.
\end{aligned}$$

To estimate I_6, we divide it as

$$I_6 = I_{61} + I_{63} + I_{63},$$

where

$$\begin{aligned}
I_{61} &= B_h(u_h^{n-1}, u_h^n, \chi_h) - B_h(u_h^{n-1}, M_h R_h u(t_n), \chi_h), \\
I_{62} &= B_h(u_h^{n-1}, M_h R_h u(t_n), \chi_h) - \tilde{b}_h(u_h^{n-1}, R_h u(t_n), \hat{\chi}_h), \\
I_{63} &= \tilde{b}_h(u_h^{n-1}, R_h u(t_n), \hat{\chi}_h) - b(u_h^{n-1}, R_h u(t_n), \hat{\chi}_h).
\end{aligned}$$

Under the assumption (4.35), applying Lemma 4.9 (or (4.34)) to I_{61}, it yields

$$|I_{61}| = -B_h(u_h^{n-1}, w_h^n, \chi_h) \le C\|w_h^n\|_p \|\nabla\hat{\chi}_h\|_q.$$

Setting

$$\beta^{\pm}_{K,L}(u_h^{n-1}) := \int_{K|L} [\nabla(M_h^{-1} G_h u_h^{n-1}) \cdot \nu_{K,L}]_{\pm} ds,$$

we calculate as

$$\begin{aligned}
|I_{62}| &= \sum_{K\in\mathcal{T}} \chi_K \sum_{L\in\mathcal{N}_K} \Big\{[\beta^{+}_{K,L}(u_h^{n-1}) - \tilde{\beta}^{+}_{K,L}(u_h^{n-1})] R_h u(t_n)(x_K) \\
&\quad - [\beta^{-}_{K,L}(u_h^{n-1}) - \tilde{\beta}^{-}_{K,L}(u_h^{n-1})] R_h u(t_n)(x_L)\Big\} \\
&= \sum_{K|L\in\mathcal{E}_{int}} (\chi_K - \chi_L) \Big\{[\beta^{+}_{K,L} - \tilde{\beta}^{+}_{K,L}] R_h u(t_n)(x_K) - [\beta^{-}_{K,L} - \tilde{\beta}^{-}_{K,L}] R_h u(t_n)(x_L)\Big\} \\
&\le \max_{K|L\in\mathcal{E}_{int}} \frac{|\beta_{K,L} - \tilde{\beta}_{K,L}|}{d_{K,L}} \|M_h R_h u(t_n)\|_p \|\chi_h\|_{1,q,\mathcal{T}}.
\end{aligned}$$

To estimate I_{62}, we see by (4.3)

$$\begin{aligned}
&\frac{|\beta_{K,L} - \tilde{\beta}_{K,L}|}{d_{K,L}} \\
&= \frac{1}{d_{K,L}} \int_{K|L} \nabla(L_h^{-1} M_h^{-1} - M_h^{-1} G_h) u_h^{n-1} \cdot \nu_{K,L} \\
&= \frac{m(K|L)}{d_{k,L}} [(L_h^{-1} M_h^{-1} - M_h^{-1} G_h) u_h^{n-1}(x_K) - (L_h^{-1} M_h^{-1} - M_h^{-1} G_h) u_h^{n-1}(x_L)] \\
&\le C h^{1-d/p} \|\nabla(L_h^{-1} M_h^{-1} - M_h^{-1} G_h) u_h^{n-1}\|_p.
\end{aligned}$$

Applying Lemma 4.3, we have

$$\|(L_h^{-1} M_h^{-1} - M_h^{-1} G_h) u_h^{n-1}\|_p \le C h \|\hat{u}_h^{n-1}\|_{1,p} \le C h^{d/p} \|\hat{u}_h^{n-1}\|_{1,2},$$

which gives

$$\frac{|\beta_{K,L} - \tilde{\beta}_{K,L}|}{d_{K,L}} \le C h \|\hat{u}_h^{n-1}\|_{1,2}.$$

Furthermore, under the assumption (4.35), we obtain,

$$|I_{62}| \le C \alpha_{1,p} h \|\hat{\chi}_h\|_{1,q}.$$

Following from Lemma 4.10, we have

$$I_{63} \le C h^{1-d/p} \|\hat{\chi}_h\|_{1,q}.$$

Combining the estimates of I_{6j}, $j = 1, 2, 3$ and I_m, $m = 1, \ldots, 6$, we obtain (4.31).

4.4 Error estimates(proof of Theorem 3.1)

Proof (Proof of Theorem 3.1) In views of (4.24) and (4.25), we are aim to estimate

$$w_h^n = u_h^n - M_h R_h u(t_n), \text{ or equivalently } \hat{w}_h^n = M_h^{-1} w_h^n.$$

Recalling the equation (2.3) (the weak form of system (1.1)), and the equation (3.20) (the equivalent form to (2.4)), we have, for any $\hat{\chi}_h \in \hat{X}_h$,

$$\begin{aligned}
&(\partial_{\tau_n} \hat{w}_h^n, \hat{\chi}_h)_h + (\nabla \hat{w}_h^n, \nabla \hat{\chi}_h) + (\hat{w}_h^n, \hat{\chi}_h)_h \\
=&(\partial_{\tau_n} w_h^n, \chi_h) + (\nabla \hat{w}_h^n, \nabla \hat{\chi}_h) + (w_h^n, \chi_h) \\
=&\underbrace{(\partial_{\tau_n} M_h R_h u(t_n), \chi_h) - (u'(t_n), \hat{\chi}_h)}_{=J_1} + \underbrace{(w_h^n, \chi_h) - (R_h u(t_n) - u(t_n), \hat{\chi}_h)}_{=J_2} \\
&+ \underbrace{(u'(t_n), \hat{\chi}_h) - (\partial_{\tau_n} u_h^n, \chi_h) + (\nabla u(t_n) - \nabla \hat{u}_h^n, \nabla \hat{\chi}_h)}_{=B_h - b = J_3} \\
&+ \underbrace{(\nabla R_h u(t_n) - \nabla u(t_n), \nabla \hat{\chi}_h) + (R_h u(t_n) - u(t_n), \hat{\chi}_h)}_{=0}
\end{aligned} \tag{4.37}$$

From Lemma 4.6, we have

$$|J_1| \le C(\alpha_{2,p} + \alpha_{3,p})(h + \tau_n^\sigma)\|\hat{\chi}_h\|_{1,q}.$$

$$|J_2| \le C\|\hat{w}_h^n\|_p \|\hat{\chi}_h\|_q + ch^2 \alpha_{1,p} \|\hat{\chi}_h\|_q.$$

Follows from Lemma 4.7,

$$|J_3| \le C(\tau_n + h^{1-d/p} + \|\hat{w}_h^n\|_p + \|\hat{w}_h^{n-1}\|_p)\|\hat{\chi}_h\|_{1,q}.$$

Combining these estimates of J_i, $i = 1, 2, 3$, we rewrite (4.37) into

$$(\partial_{\tau_n} \hat{w}_h^n, \hat{\chi}_h)_h + (\nabla \hat{w}_h^n, \nabla \hat{\chi}_h) + (\hat{w}_h^n, \hat{\chi}_h)_h = \langle F^n, \hat{\chi}_h \rangle, \tag{4.38}$$

for some $F^n \in \hat{X}_h^*$ such that

$$\langle F^n, \hat{\chi}_h \rangle \le C(\tau_n^\sigma + h^{1-d/p} + \|\hat{w}_h^n\|_p + \|\hat{w}_h^{n-1}\|_p)\|\hat{\chi}_h\|_{1,q}. \tag{4.39}$$

Furthermore, we formulate (4.38) into a single equation, with the help of (4.13),

$$\partial_{\tau_n} \hat{w}_h^n + A_h \hat{w}_h^n = F^n. \tag{4.40}$$

Applying Lemma 4.4, we obtain

$$\hat{w}_h^n = E_{n,1} \hat{w}_h^0 + \sum_{j=1}^{n} \tau_j E_{n,j} F^j, \tag{4.41}$$

where

$$E_{n,j} = r(\tau_n A_h) \cdots r(\tau_j A_h), \quad r(\tau_j A_h) = (I + \tau_j A_h)^{-1}.$$

Now, applying norm (4.21) and (4.22) in Lemma 4.4, we have

$$
\begin{aligned}
\|\hat{w}_h^n\|_{h,p} = \|w_h^n\|_p &\le \|E_{n,1}\hat{w}_h^0\|_{h,p} + \|\sum_{j=1}^n \tau_j E_{n,j} F^j\|_{h,p} \\
&\le \xi_1^{-\theta}\|A_h^{-\theta}\hat{w}_h^0\|_{h,p} + \sum_{j=1}^n \tau_j \xi_1^{-\theta}\|A_h^{-\theta}F^j\|_{h,p},
\end{aligned}
$$

where $\xi_j = (\tau_n + \cdots + \tau_j) = (t_n - t_j)$. Thanks to the following property of $A_h^{-\theta}$, which is a consequence of Lemma 4.5,

$$
\|A_h^{-\theta}\hat{v}_h\|_{1,q} \le C\|\hat{v}_h\|_{h,q}, \quad \forall \hat{v}_h \in \hat{X}_h,\ \theta \in (1/2, 1],\ q \in (\mu/(\mu-1), \mu),\ (\mu > d),
$$

we obtain

$$
\begin{aligned}
\|A_h^{-\theta}F^j\|_{h,p} &\le \sup_{\hat{\chi}_h \in \mathcal{X}_{h,q}} \frac{\langle A_h^{-\theta}F^j, \hat{\chi}_h\rangle}{\|\hat{\chi}_h\|_{h,p}} = \sup_{\hat{\chi}_h \in \mathcal{X}_{h,q}} \frac{\langle F^j, A_h^{-\theta}\hat{\chi}_h\rangle}{\|\hat{\chi}_h\|_{h,p}} \\
&\le C(\tau_j^\sigma + h^{1-d/p} + \|\hat{w}_h^j\|_p + \|\hat{w}_h^{j-1}\|_p) \sup_{\hat{\chi}_h \in \mathcal{X}_{h,q}} \frac{\|A_h^{-\theta}\hat{\chi}_h\|_{1,q}}{\|\hat{\chi}_h\|_q} \\
&\le C(\tau_j^\sigma + h^{1-d/p} + \|\hat{w}_h^j\|_{h,p} + \|\hat{w}_h^{j-1}\|_{h,p}).
\end{aligned}
$$

Obviously, $\|A_h^{-\theta}\hat{w}_h^0\|_{h,p} \le C\|\hat{w}_h^0\|_{h,p}$, and (3.23) gives

$$
\|\hat{w}_h^0\|_{h,p} \le C\|R_h u_0 - \hat{u}_h^0\|_p \le Ch^{1-d/p}.
$$

Noticing that $\sum_{j=1}^n \tau_j \xi_n^{-\theta} \le T^{1-\theta}/(1-\theta)$, $(\xi_j = t_n - t_j)$, we obtain

$$
\|\hat{w}_h^n\|_{h,p} \le C(\tau_n^\sigma + h^{1-d/p}) + C\sum_{j=1}^n \tau_j \xi_n^{-\theta}(\|\hat{w}_h^{j-1}\|_{h,p} + \|\hat{w}_h^j\|_{h,p}).
$$

We then introduce a discrete inequality of Volterra type (Lemma 4.7 [27]). For $\{Z_n\}_{n=1}^l$ satisfying

$$
0 < Z_n \le c_1 + c_2 \sum_{j=1}^n \tau_j (t_n - t_j)^{-\theta}(z_{j-1} + z_j),
$$

then, we have

$$
z_n \le c_1 c_3 \exp(c_4 c_2^{1/(1-\theta)} t_n),
$$

where c_3, c_4 only depending on θ. Putting $z_n = \|\hat{w}_h^n\|_{h,p} = \|w_h^n\|_p$, we obtain the error estimate (3.24). And the error estimate (3.25) of $v_h^n - v(t_n)$ follows directly from Lemma 4.2 and 4.3.

5 Analysis on numerically "blow-up"

5.1 One-dimensional system

We consider the one-dimensional Keller-Segel system of (1.1) and its finite volume approximation. For all $t \in (0,T)$, $(u(x,t), v(x,t))$ satisfies

$$\begin{cases} u_t - u_{xx} + (uv_x)_x = 0, \quad -v_{xx} + v = u & \text{in } \Omega = (-L, L), \\ u_x = v_x = 0 & \text{on } x = 0, L, \\ u(x,0) = u_0(x) & \text{on } \Omega. \end{cases} \tag{5.1}$$

We consider the finite volume scheme for (5.1) with mesh $\mathcal{T}$:

$$-L = x_{\frac{1}{2}} < x_{1+\frac{1}{2}} < \cdots < x_{N-1+\frac{1}{2}} < x_{N+\frac{1}{2}} = L,$$

where $0 < N \in \mathbb{N}$ is the number of control volumes, $h = \frac{2L}{N}$ is the mesh size. $x_{i+\frac{1}{2}} = ih - L$. $(x_{i+\frac{1}{2}}, x_{i+1+\frac{1}{2}})$ is the control volume with control point $x_{i+1} = (i + \frac{1}{2})h$, $i = 0, 1, \ldots, N-1$. We set $u_i^0 = u_0(x_i)$, $i = 1, \ldots, N$. Let u_i^n, v_i^n be the approximation of $u(t_n, x_i)$, $v(t_n, x_i)$, respectively. The finite volume scheme is to find $u^n = (u_i^n)_{i=1}^N$, $v^n = (v_i^n)_{i=1}^N$ for $n = 1, 2, \ldots, J$, such that

$$-\frac{v_{i+1}^{n-1} - 2v_i^{n-1} + v_{i-1}^{n-1}}{h^2} + v_i^{n-1} = u_i^{n-1}, \tag{5.2a}$$

$$\begin{aligned} \partial_\tau u_i^n - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} & \\ + \frac{1}{h}\Bigg\{ \left(\frac{[v_{i+1}^{n-1} - v_i^{n-1}]_+}{h} u_i^n - \frac{[v_{i+1}^{n-1} - v_i^{n-1}]_-}{h} u_{i+1}^n \right) & \\ + \left(\frac{[v_{i-1}^{n-1} - v_i^{n-1}]_+}{h} u_i^n - \frac{[v_{i-1}^{n-1} - v_i^{n-1}]_-}{h} u_{i-1}^n \right) \Bigg\} &= 0 \end{aligned} \tag{5.2b}$$

$$v_0^{n-1} = v_1^{n-1}, \; v_N^{n-1} = v_{N+1}^{n-1}, \quad u_0^n = u_1^n, \; u_N^n = u_{N+1}^n, \tag{5.2c}$$

where $\tau > 0$ is the time-step increment, $\partial_\tau u_i^n = (u_i^n - u_i^{n-1})/\tau$. $\{u_i^0\}_{i=1}^N \geq 0$, and not identically zero. It is easy to verify that scheme (5.2) satisfies the conservation of mass and positivity,

$$\sum_{i=1}^N u_i^n h = \sum_{i=1}^N u_i^0 h =: \theta, \quad u_i^n > 0, \quad \forall n \geq 1, \tag{5.3}$$

$$\sum_{i=1}^N v_i^n h = \theta, \quad v_i^n > 0, \quad n \geq 0. \tag{5.4}$$

As it mentioned in Sect. 1, the global existence of u follows from the boundedness of $\|v\|_{1,\infty}$. For 1-dimensional problem, (1.10) and (1.11) are proved. For the discrete system, We have the following theorem of v^n, which is an analogue to (1.10) and (1.11).

Theorem 5.1 *We have, for any $n \geq 0$ and $i = 1, \dots, N-1$,*

$$v_{i+1}^n \leq \frac{1}{2L}(1+4L^2)\theta, \tag{5.5}$$

$$\frac{|v_{i+1}^n - v_i^n|}{h} \leq \theta. \tag{5.6}$$

Proof From (5.2a), we see that

$$\frac{v_{i+1}^n - v_i^n}{h} = \sum_{j=1}^{n}(v_i^n - u_i^{n+1})h.$$

In view of conservation laws (5.3) and (5.4), we have $|\sum_{j=1}^n (v_i^n - u_i^{n+1})h| \leq \theta$, which implies (5.6). To derive (5.5), we see tha

$$\underbrace{2Lv_{i+1}^n}_{=\sum_{j=1}^N v_{i+1}^n h} - \sum_{j=1}^{N} v_j^n h = \sum_{j=1}^{N} h \underbrace{\sum_{k=j}^{i}(v_{j+1} - v_j)}_{=v_{i+1}-v_j} = \underbrace{\sum_{j=1}^{N} h \sum_{k=j}^{i} \frac{v_{k+1} - v_k}{h} h}_{\leq \sum_{j=1}^N h \sum_{k=j}^i \theta h \leq 4L^2\theta}.$$

Hence, $2Lv_{i+1}^n - \theta \leq 4L^2\theta$, which gives (5.5)

5.2 Two-dimensional system

The global existence and blow-up of solution for 2-dimensional Keller-Segel system (1.1) is briefly discussed in Sect. 1. For the global existence of u, it is sufficient to prove (1.12) and (1.13) under the condition $\theta < 8\pi$; for the blow-up of solution, It is crucial to estimate the moment $M_2(t) = \int_\Omega u(t)|x|^2 dx$. Setting $\Omega = B(0, L) \in \mathbb{R}^d$, $u_0(x) = u_0(r)$, the Keller-Segel system with radially symmetric solution reads as

$$\begin{aligned} &u_t = r^{1-d}(r^{d-1}(u_r - uv_r))_r, && r \in (0, L), \quad t \in (0, T), \\ &0 = r^{1-d}(r^{d-1}v_r)_r - v + u, && r \in (0, L), \quad t \in (0, T), \\ &u_r = v_r = 0, && r = 0, L, \quad t \in (0, T), \\ &u(r, 0) = u_0(r), && r \in (0, L). \end{aligned} \tag{5.7}$$

Setting $N \in \mathbb{N}_+$, we define mesh $\mathcal{T}$:

$$0 = r_{\frac{1}{2}} < r_{1+\frac{1}{2}} < \cdots < r_{N-1+\frac{1}{2}} < r_{N+\frac{1}{2}} = L,$$

where $r_{i+\frac{1}{2}} = ih$. $(r_{i+\frac{1}{2}}, r_{i+1+\frac{1}{2}})$ is the control volume, with control point $r_{i+1} = (i+\frac{1}{2})h$, $i = 0, \dots, N-1$. Let $d = 2$, and we set the initial data $u_i^0 = u_0(r_i)$, $u^0 = (u_i^0)_{i=1}^N \geq 0$, and not identically zero. Let u_i^n and v_i^n be the approximation of $u(t_n, r_i)$ and $v(t_n, r_i)$, respectively. The finite volume scheme reads as: finding $u^n = (u_i^n)_{i=1}^N$, $v^n = (v_i^n)_{i=1}^N$, for $n = 0, 1, \ldots, J$, such that,

$$-\left[r_{i+\frac{1}{2}}\frac{v_{i+1}^n - v_i^n}{h} - r_{i-\frac{1}{2}}\frac{v_i^n - v_{i-1}^n}{h}\right] = hr_i(u_i^n - v_i^n), \tag{5.8a}$$

$$\begin{aligned}\partial_\tau u_i^{n+1} r_i h &- \left[r_{i+\frac{1}{2}}\frac{u_{i+1}^{n+1} - u_i^{n+1}}{h} - r_{i-\frac{1}{2}}\frac{u_i^{n+1} - u_{i-1}^{n+1}}{h}\right] \\ &+ \left\{r_{i+\frac{1}{2}}\left(\frac{[v_{i+1}^n - v_i^n]_+}{h}u_i^{n+1} - \frac{[v_{i+1}^n - v_i^n]_-}{h}u_{i+1}^{n+1}\right)\right. \\ &+ \left. r_{i-\frac{1}{2}}\left(\frac{[v_{i-1}^n - v_i^n]_+}{h}u_i^{n+1} - \frac{[v_{i-1}^n - v_i^n]_-}{h}u_{i-1}^{n+1}\right)\right\} = 0.\end{aligned} \tag{5.8b}$$

$$v_{N+1}^n = v_N^n, \quad u_{N+1}^{n+1} = u_N^{n+1}, \tag{5.8c}$$

In view of $\int_\Omega u_0 dx = 2\pi \int_0^L u_0(r) r dr$, we set the mass of initial data

$$\theta = 2\pi \sum_{i=1}^N r_i u_i^0 h.$$

The conservation of mass and positivity of u^n, v^n are easy to verify:

$$u_i^n > 0, \quad \sum_{i=1}^N u_i^n r_i h = \frac{\theta}{2\pi}, \quad n \geq 1, \tag{5.9}$$

$$v_i^n > 0, \quad \sum_{i=1}^N v_i^n r_i h = \frac{\theta}{2\pi}, \quad n \geq 0. \tag{5.10}$$

We have the following theorem of v^n, which is an analogue to (1.12) and (1.13).

Theorem 5.2 *Suppose $\theta < 8\pi$, we have, $n \geq 0$, $i = 1, \ldots, N-1$,*

$$v_{i+1}^n \leq \frac{\theta(i+1)}{4\pi i} + 2l, \tag{5.11}$$

$$\frac{|v_{i+1}^n - v_i^n|}{h} \leq L(l + 4k), \tag{5.12}$$

where constant $k > 0$ is sufficiently large such that, for $U_i^n = \sum_{j=1}^i r_j u_j^n h$, $V_i^n = \sum_{j=1}^i r_j v_j^n h$,

$$U_i^0, V_i^0 \leq W_i := \frac{4kr_{i+\frac{1}{2}}^2}{1 + kr_{i+\frac{1}{2}}^2}, \quad i = 1, \ldots, N, \tag{5.13}$$

and constant $l \geq 4k$.

Remark 5.1 We see that $V_i^n, U_i^n \geq 0$, $U_N^n = V_N^n = \frac{\theta}{2\pi}$, and $W_i < W_{i+1}$. We also have $W_i < 4$, and $W_i \to 4$ as $k \to \infty$. Hence, in order to find some k satisfying (5.13), we have to assume $\theta < 8\pi$.

Proof Setting $U_0^n = V_0^n = 0$, $U_{N+1}^n = U_N^n$, $V_{N+1}^n = V_N^n$, and in view of

$$U_N^n = V_N^n = \frac{\theta}{2\pi}, \quad u_i^n = \frac{U_i^n - U_{i-1}^n}{r_i h}, \quad -r_{i+\frac{1}{2}} \frac{v_{i+1}^n - v_i^n}{h} = U_i - V_i, \tag{5.14}$$

we have, for $n = 0, 1, \ldots, J$,

$$-\frac{r_{i+\frac{1}{2}}}{h}\left(\frac{V_{i+1}^n - V_i^n}{hr_{i+1}} - \frac{V_i^n - V_{i-1}^n}{hr_i}\right) + V_i^n = U_i^n, \tag{5.15a}$$

$$\partial_\tau U_i^{n+1} - \frac{r_{i+\frac{1}{2}}}{h}\left(\frac{U_{i+1}^{n+1} - U_i^{n+1}}{hr_{i+1}} - \frac{U_i^{n+1} - U_{i-1}^{n+1}}{hr_i}\right) + \left([U_i^n - V_i^n]_- \frac{U_i^{n+1} - U_{i-1}^{n+1}}{r_i h} - [U_i^n - V_i^n]_+ \frac{U_{i+1}^{n+1} - U_i^{n+1}}{r_{i+1} h}\right) = 0. \tag{5.15b}$$

In view of (5.13), setting $W_0 = 0$, $W_{N+1} = W_N$, we obtain, for all $i = 1, \ldots, N$,

$$\frac{r_{i+\frac{1}{2}}}{h}\left(\frac{W_{i+1} - W_i}{hr_{i+1}} - \frac{W_i - W_{i-1}}{hr_i}\right) + (W_i - V_i^n)\frac{W_{i+1} - W_i}{r_{i+1}h} \le 0. \tag{5.16}$$

Setting $Y_i^n := U_i^n - W_i$, and adding (5.15) to (5.16), it yields, for $i = 1, \ldots, N$,

$$\begin{aligned}
&\partial_\tau Y_i^{n+1} - \frac{r_{i+\frac{1}{2}}}{h}\left(\frac{Y_{i+1}^{n+1} - Y_i^{n+1}}{hr_{i+1}} - \frac{Y_i^{n+1} - Y_{i-1}^{n+1}}{hr_i}\right) \\
&\quad + [U_i^n - V_i^n]_- \frac{Y_i^{n+1} - Y_{i-1}^{n+1}}{hr_i} - [U_i^n - V_i^n]_+ \frac{Y_{i+1}^{n+1} - Y_i^{n+1}}{hr_{i+1}} \\
&\quad + [U_i^n - V_i^n]_- \underbrace{\left(-\frac{W_{i+1} - W_i}{hr_{i+1}} + \frac{W_i - W_{i-1}}{hr_i}\right)}_{\ge 0} + Y_i^n \underbrace{\frac{W_{i+1} - W_i}{hr_{i+1}}}_{\ge 0} \le 0,
\end{aligned}$$

which implies,

$$Y_i^{n+1} \le 0 \text{ if } Y_i^n \le 0 \quad \forall i = 1, \ldots, N. \tag{5.17}$$

From (5.16) and (5.17), we obtain

$$U_i^n \le W_i, \quad i = 1, \ldots, N, \quad n = 0, 1, \ldots, J. \tag{5.18}$$

Setting $Z_i = l r_{i+\frac{1}{2}}^2$, $i = 1, \ldots, N$, $Z_0 = 0$, $Z_{N+1} = Z_N = L^2 l$, where $l > 4k$. $Z_N = lL^2 > \frac{\theta}{2\pi}$. We see that $(Z_i)_{i=1}^N \ge (W_i)_{i=1}^N$, and

$$\frac{r_{i+\frac{1}{2}}}{h}\left(\frac{Z_{i+1} - Z_i}{hr_{i+1}} - \frac{Z_i - Z_{i-1}}{hr_i}\right) - Z_i + U_i^n = -Z_i + U_i^n \le -Z_i + W_i \le 0,$$

which shows, in view of (5.15a),

$$V_i^n \le Z_i, \quad i = 1, \ldots, N, \quad n = 0, 1, \ldots, J. \tag{5.19}$$

From (5.14), (5.18) and (5.19), we have

$$|v_{i+1}^n - v_i^n|/h = |U_i^n - V_i^n|/r_{i+\frac{1}{2}} \le (W_i^n + Z_i^n)/r_{i+\frac{1}{2}},$$

which implies (5.12). In view of (5.15a) and $r_{i+1/2}^2 - r_{i-1/2}^2 = 2r_i h$, we get

$$\begin{aligned}
&r_{i+\frac{1}{2}}^2 \frac{V_{i+1}^n - V_i^n}{r_{i+1}h} = r_{i+\frac{1}{2}}^2 \frac{V_i^n - V_{i-1}^n}{r_i h} + hr_{i+\frac{1}{2}}(V_i^n - U_i^n) \\
=&r_{i-\frac{1}{2}}^2 \frac{V_i^n - V_{i-1}^n}{r_i h} + 2(V_i^n - V_{i-1}^n) + hr_{i+\frac{1}{2}}(V_i^n - U_i^n) \\
=&\cdots = \sum_{j=1}^{i} hr_{j+\frac{1}{2}}(V_j^n - \underbrace{U_j^n}_{\ge 0}) + 2V_i^n - \underbrace{2V_0^n}_{=0} \\
\le&\sum_{j=1}^{i} hr_{j+\frac{1}{2}} V_j^n + 2V_i^n \le \sum_{j=1}^{i} hr_{j+\frac{1}{2}} \frac{\theta}{2\pi} + 2lr_{i+\frac{1}{2}}^2, \quad (\because\ (5.19)) \\
\le&\frac{1}{2} r_{i+\frac{1}{2}} r_{i+\frac{3}{2}} \frac{\theta}{2\pi} + 2lr_{i+\frac{1}{2}}^2,
\end{aligned}$$

which gives

$$v_{i+1}^n = \frac{V_{i+1}^n - V_i^n}{r_{i+1}h} \le \frac{r_{i+\frac{3}{2}}}{r_{i+\frac{1}{2}}} \frac{\theta}{4\pi} + 2l = \frac{i+1}{i} \frac{\theta}{4\pi} + 2l.$$

Thus, we proved (5.11).

In view of $M_2(t) = \int_\Omega u(t,x)|x|^2 dx = 2\pi \int_0^L u(t,r) r^3 dr$, we define a discrete moment

$$M_2^n = \sum_{i=1}^{N} r_i^3 u_i^n h.$$

To explain the blow-up of solution to (1.1), it proves the decreasing of $M_2(t)$ under the condition of large mass θ and small moment $M_2(0)$(see (1.15)). Our motivation lies to show a discrete analogue of inequality (1.15), such as, for $n = 1, \ldots, J$,

$$\frac{M_2^n - M_2^{n-1}}{\tau} \le \frac{4\theta}{2\pi} - \left(\frac{\theta}{2\pi}\right)^2 + C_1\theta M_2^{n-1} + C_2\theta^{\frac{3}{2}}\sqrt{M_2^{n-1}} + C_3 h\theta^2, \quad (5.20)$$

where C_1, C_2, C_3 are independent of h, θ, and M_2^{n-1}. However, (5.20) is impossible for conservation scheme. If θ is sufficiently large and M_2^0 is small enough, then M_2^n is decreasing and goes to zero or negative value, which is contradict to the conservation laws. In the following, we consider a numerical scheme without conservation laws but satisfying (5.20), and we draw a remark to compare the moment equations between conservative scheme and nonconservative scheme. The nonconservative scheme is to replace (5.8b) by

$$
\begin{aligned}
&\partial_\tau u_i^{n+1} r_i h - \left(r_{i+\frac{1}{2}} \frac{u_{i+1}^{n+1} - u_i^{n+1}}{h} - r_{i-\frac{1}{2}} \frac{u_i^{n+1} - u_{i-1}^{n+1}}{h} \right) \\
&\quad + \left(r_{i+\frac{1}{2}} \frac{v_{i+1}^n - v_i^n}{h} u_i^n - r_{i-\frac{1}{2}} \frac{v_i^n - v_{i-1}^n}{h} u_{i-1}^n \right) = 0,
\end{aligned}
\tag{5.21}
$$

which satisfies the conservation of total mass, but not conservation of positivity, which means u_i^n, v_i^n may lose positivity after some time step n.

Theorem 5.3 *Let $(u^n, v^n)_{n=1}^J$ be the positive solution of nonconservative scheme (5.21) and (5.8a), then we have the moment inequality (5.20).*

Proof Follows from (5.21), we have

$$
\frac{M_2^n - M_2^{n-1}}{\tau} = \underbrace{-2 \sum_{i=1}^{N-1} r_{i+\frac{1}{2}}^2 (u_{i+1}^{n+1} - u_i^{n+1})}_{=I_A} + \underbrace{2 \sum_{i=1}^{N-1} r_{i+\frac{1}{2}}^2 h u_i^n \frac{v_{i+1}^n - v_i^n}{h}}_{=I_B}.
$$

It is not difficult to derive that

$$
I_A = 4 \sum_{i=1}^{N} r_i h u_i^n - 2L^2 u_N^n \le \frac{4\theta}{2\pi}.
$$

In view of (5.8a), we get

$$
I_B = 2 \sum_{i=1}^{N-1} r_{i+\frac{1}{2}} h(-U_i^n + V_i^n) u_i^n = I_{B1} + I_{B2},
$$

where V_i^n, U_i^n are defined in Theorem 5.2.

$$
\begin{aligned}
I_{B1} &= -2 \sum_{i=1}^{N-1} r_{i+\frac{1}{2}} U_i^n h u_i^n \le -2 \sum_{i=1}^{N-1} r_i h U_i^n u_i^n \\
&= -\Big(\sum_{i=1}^{N} r_i h u_i^n\Big)^2 - \frac{1}{2} \sum_{i=1}^{N} r_i^2 h^2 (u_i^n)^2 + r_N h u_N^n U_N^n \\
&\le -\left(\frac{\theta}{2\pi}\right)^2 + r_N h u_N^n \frac{\theta}{2\pi}.
\end{aligned}
$$

To estimate $I_{B2} = 2 \sum_{i=1}^{N-1} r_{i+\frac{1}{2}} V_i^n h u_i^n$, we introduce Φ_i^n and Ψ_i,

$$
\Phi_i^n = V_i^n - \frac{\theta}{2\pi} \left(\frac{r_{i+\frac{1}{2}}}{L} \right)^2, \quad i = 1, \dots, N, \quad \Phi_0^n = 0, \quad \Phi_{N+1}^n = \Phi_N^n = 0,
$$

$$
\Psi_i = -\frac{10}{9} \frac{\theta}{4\pi} r_{i+\frac{1}{2}}^2 \log \frac{r_{i+\frac{1}{2}}}{L}, \quad i = 1, \dots, N, \quad \Psi_0 = 0, \quad \Psi_{N+1} = \Psi_N = 0.
$$

One can verify that

$$\frac{1}{h}\left(\frac{\Phi^n_{i+1}-\Phi^n_i}{h}-\frac{\Phi^n_i-\Phi^n_{i-1}}{h}\right)-\frac{1}{2}\left(\frac{\Phi^n_{i+1}-\Phi^n_i}{r_{i+1}h}-\frac{\Phi^n_i-\Phi^n_{i-1}}{r_ih}\right)$$
$$-\Phi^n_i=-U^n_i+\frac{\theta}{2\pi}\left(\frac{r_{i+\frac{1}{2}}}{L}\right)^2>-\frac{\theta}{2\pi},$$

$$\frac{1}{h}\left(\frac{\Psi_{i+1}-\Psi_i}{h}-\frac{\Psi_i-\Psi_{i-1}}{h}\right)-\frac{1}{2}\left(\frac{\Psi_{i+1}-\Psi_i}{r_{i+1}h}-\frac{\Psi_i-\Psi_{i-1}}{r_ih}\right)$$
$$-\Psi_i<-\frac{\theta}{2\pi}-W_i\le-\frac{\theta}{2\pi},$$

which implies $\Phi^n_i\le\Psi_i$ and

$$V^n_i\le\Psi+\frac{\theta}{2\pi}\left(\frac{r_{i+\frac{1}{2}}}{L}\right)^2.$$

We have, with some calculation

$$\begin{aligned}I_{B2}&\le\frac{\theta}{L^2\pi}M^n_2+\frac{3}{2L^2\pi}\theta\sum_{i=1}^N r_i^2h^2u^n_i+\frac{\theta^2h^2}{L^2\pi}-\frac{\theta L}{\pi}hu^n_N+\frac{5L}{9\pi}\theta\sum_{i=1}^N r_i^2hu^n_i+\frac{10Lh}{9\pi}\theta^2\\&\le\frac{\theta M^n_2}{L^2\pi}+\frac{3h\theta^2}{2L^2\pi}+\frac{10Lh\theta^2}{9\pi}+\frac{\theta^2h^2}{L^2\pi}-\frac{\theta L}{\pi}hu^n_N+\frac{5L}{9\pi}\theta\underbrace{(\sum_{i=1}^N r_ihu^n_i)^{\frac{1}{2}}}_{=\theta^{1/2}}\underbrace{(\sum_{i=1}^N r_i^3hu^n_i)^{\frac{1}{2}}}_{=(M^n_2)^{1/2}}.\end{aligned}$$

Combining the estimates of I_A, I_{B1} and I_{B2}, we obtain (5.20).

Remark 5.2 For the conservative scheme (5.8), assuming $v^n_{i+1}\le v^n_i$ (the blow-up point is at origin, and u,v are decreasing by r. The numerical solution is showed in Figure 6.4), we have,

$$\frac{M^n_2-M^{n-1}_2}{\tau}=I_A+\underbrace{2\sum_{i=1}^{N-1}r_{i+\frac{1}{2}}h(V^n_i-U^n_i)u^{n+1}_{i+1}}_{=I_{B'}}.$$

The difference between I_B and $I_{B'}$ is slight but crucial. The numerical experiment of conservative scheme also reproduce the decreasing of moment M^n_2(see Figure 6.6).

6 Numerical experiments

6.1 Numerical example: error, Lyapunov's property and blow-up phenomenon

Let $\Omega = (-0.5, 0.5)^2$, $0 < N \in \mathbb{N}$, and $h = 1/(N-1)$. We take $\{(-0.5 + ih, -0.5 + jh) \mid i, j = 0, \dots, N-1\}$ as the set of control point for the admissible mesh of rectangles. Setting

$$u_0 = 100 e^{-\frac{x^2+y^2}{0.04}} + 60 e^{-\frac{(x-0.2)^2+y^2}{0.05}} + 30 e^{-\frac{x^2+(y-0.2)^2}{0.05}},$$

where $\|u_0\|_1 \approx 26.26 > 8\pi$, so that the corresponding solution u may blow-up. The mass concentrating is well captured by numerical simulation(see Figure 6.1, $N = 61$). The conservation laws are verified, and we show the discrete discrete Lyapunov's property(see Figure 6.3). Taking $N = 81$ and $t = 0.0625$, $h = 0.0125$ $\tau_n = \tau = 0.2h$, we obtain the numerical solution $u_h^J (=: \tilde{U})$, $J = t/\tau = 10$. Since the exact solution u is not obvious, we think of $\tilde{U}$ as the solution closed to u. Then, for $N = 11, 21, \dots, 61, 71$, we exam the error $\frac{\|u_h^J - \tilde{U}\|_p}{\|\tilde{U}\|_p}$, for $p = 2, 3, 4, 5$(see Figure 6.2).

6.2 Numerical examples for radially symmetric solution

Both conservative scheme (5.8) and nonconservative scheme (5.21) are applied to Keller-Segel system (5.7). $\Omega = B(0, L)$, $L = 1$. Assume

$$u_0(r) = 10 e^{-\frac{r^2}{0.5}} + 20 e^{-\frac{(r-0.3)^2}{0.5}},$$

with $\theta \approx 9.31 > 4$. The solution u is expected to blow-up at the origin. Setting $N = 100$, $h = 1/N$, $\tau = \frac{1}{50} h$, we show the mass concentration of numerical solutions(see Figure 6.4,6.5) and the decreasing of discrete moment(see Figure 6.6,6.7). For the nonconservative scheme, we stop the computation when positivity of solution broken.

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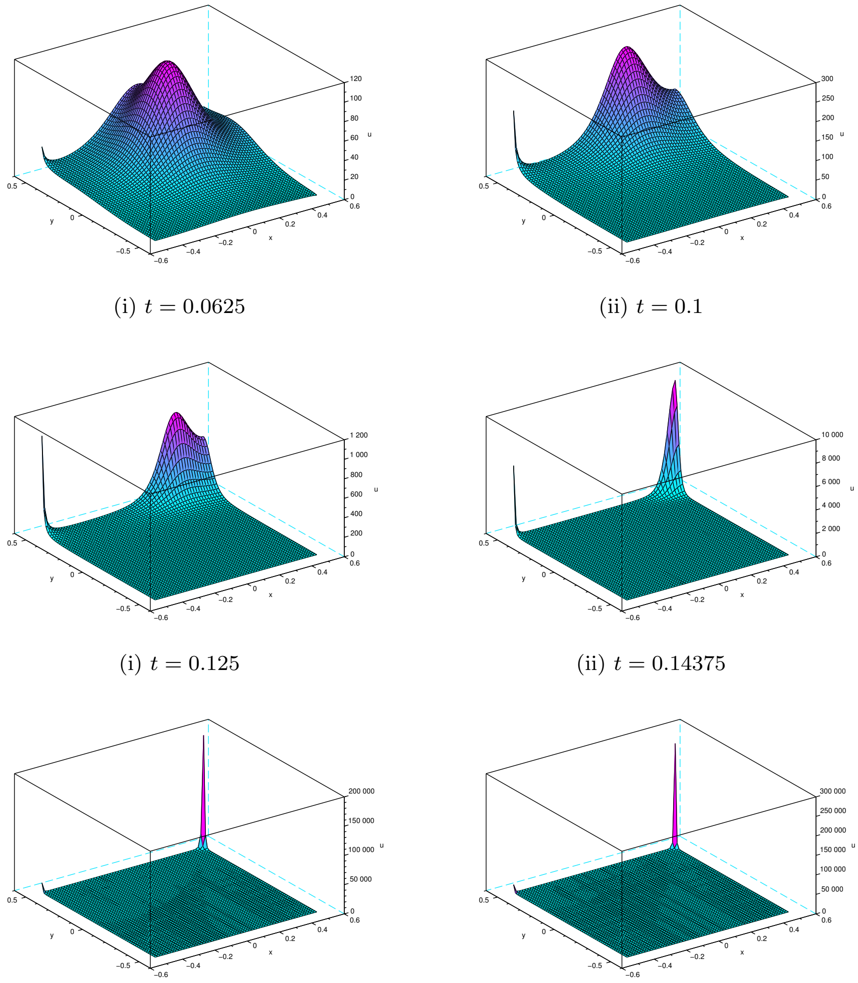

(i) $t = 0.0625$ (ii) $t = 0.1$

(i) $t = 0.125$ (ii) $t = 0.14375$

(i) $t = 0.1625$ (ii) $t = 0.18125$

Fig. 6.1 u_h^n: $N = 61$, $\|u_h^0\|_1 \approx 26.26$.

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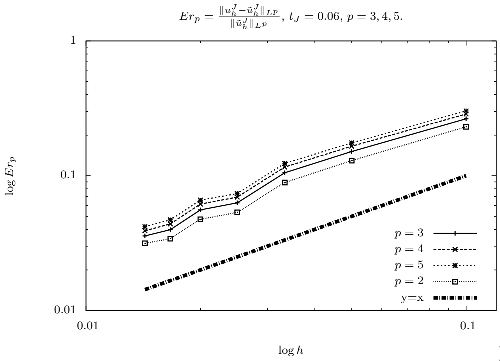

Fig. 6.2 L^p error at $t = 0.0625$.

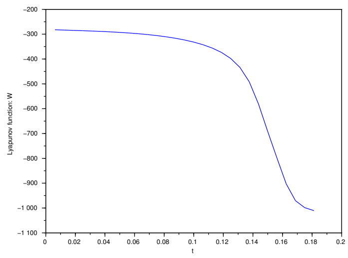

Fig. 6.3 $W_h^n : N = 61$, $t_J = 0.18125$.

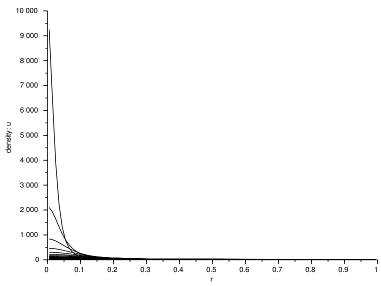

Fig. 6.4 u^n of (5.8). $t_J = 0.223$.

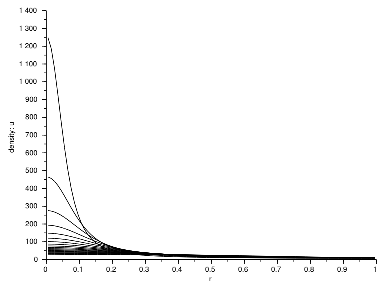

Fig. 6.5 u^n of (5.21). $t_J = 0.1868$.

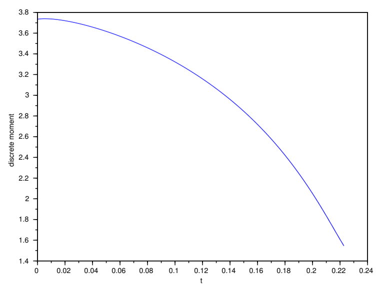

Fig. 6.6 M_2^n of (5.8). $t_J = 0.223$.

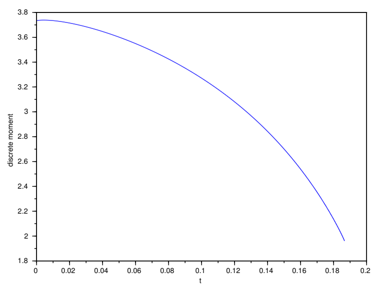

Fig. 6.7 M_2^n of (5.21). $t_J = 0.1868$.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
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