UTMS 2014–1 January 17, 2014

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by

Norikazu SAITO and Guanyu ZHOU

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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Norikazu Saito and Guanyu Zhou*
Graduate School of Mathematical Sciences, The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

January 13, 2014

Abstract

The L^2-penalty fictitious domain method is based on a reformulation of the original problem in a larger simple-shaped domain by introducing a discontinuous reaction term with a penalty parameter $\epsilon > 0$. We first derive regularity results and some a priori estimates and then prove several error estimates. We also give several error estimates for discretization problems by the finite element and finite volume methods.

Key words: fictitious domain method ; finite element method; finite volume method

2010 Mathematics Subject Classification: 65N85, 65N30, 65N08

1 Introduction

The fictitious domain method is a powerful technique for solving partial differential equations. It based on a reformulation of the original problem in a larger spatial domain, called the *fictitious domain*, with a simple shape. One of the advantages of this approach is that we can avoid the time-consuming construction of a boundary-fitted mesh. Thus, the fictitious domain is discretized by a simple-shaped mesh, independent of the original boundary. Consequently, we can directly apply a large class of numerical methods, for example, the finite element, finite volume, finite difference methods as well. Furthermore, this approach will be useful to solve time-dependent moving-boundary problems. Actually, the fictitious domain reformulation combined with the finite volume and finite difference discretizations are successfully applied in numerical simulations for real-world problems, for example, a blood flow and fluid-structure interactions in thoracic aorta ([14]) and a simulation of spilled oil on coastal ecosystems ([13]). The aim of our work is to establish a mathematical study of the penalty fictitious

*correspondance: zhoug@ms.u-tokyo.ac.jp

domain method which can be applied to these time-dependent moving-boundary problems. As a primary step towards this final end, herein we examine the error analysis for elliptic problems.

In a previous paper, Zhou and Saito [18], we studied a class of the fictitious domain methods with a penalty for elliptic problems with various boundary conditions. Therein, we introduce a fictitious domain reformulation by considering a discontinuous diffusion coefficient, which we call the *H^1-penalty fictitious domain method* or, simply, the *H^1-penalty method*. As is reported in [18], this reformulation and its finite element discretization enjoy finite mathematical properties. However, it is rather difficult to apply the finite volume and finite difference methods to the H^1-penalty method since the treatment of a discontinuous diffusion coefficient is not straightforward. Moreover, solutions of the H^1-penalty problem are not smooth across the original boundary that may cause some difficulties in actual computations.

In the present paper, we study another type of the fictitious domain method by introducing a *discontinuous reaction term*, which we call the *L^2-penalty fictitious domain method* or, simply, the *L^2-penalty method*. As examined in the present paper, this method can be directly discretized not only by the finite element but also finite volume and finite difference methods. Moreover, the penalty solution has the H^2 regularity in the whole fictitious domain.

Now let us summarize the contents of this paper. In Section 2, we study the L^2-penalty method by examining the H^2 regularity and some estimates for solutions of the L^2-penalty problem. Then, we derive error estimates of H^1 and L^2 norms. In summary, we have (cf. Theorem 2.1) the error estimates

$$\|u - u_\epsilon\|_{H^1(\Omega)} \le C\epsilon^{\frac{1}{4}}\|f\|_{L^2(\Omega)}, \qquad \|u - u_\epsilon\|_{L^2(\Omega)} \le C\epsilon^{\frac{1}{2}}\|f\|_{L^2(\Omega)},$$

where u and u_ϵ denote the solutions of the original elliptic problem (2.1) defined in a bounded domain $\Omega \subset \mathbb{R}^2$ and its L^2-penalty problem (2.18) for a given $f \in L^2(\Omega)$, ϵ is the penalty parameter with $\epsilon \to 0$. Moreover, the Dirichlet boundary condition posed on the original boundary $\Gamma = \partial\Omega$ is approximated in the sense that

$$\|u_\epsilon\|_{H^{\frac{1}{2}}(\Gamma)} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{L^2(\Omega_1)} \le C\epsilon^{\frac{1}{4}}\|f\|_{L^2(\Omega)},$$

where D denotes the fictitious domain such that $\overline{\Omega} \subset D$ and $\Omega_1 = D\backslash\overline{\Omega}$ (see Fig. 1).

Thanks to our regularity results and error estimates, the finite element analysis becomes easy to treat. In Section 3, we derive the error estimates of the finite element approximation of the L^2-penalty problem. We have (cf. Theorem 3.2)

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{L^2(D)} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{L^2(\Omega_1)} \le C\|f\|_{L^2(\Omega)}(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}}),$$
$$\|u_\epsilon - u_{\epsilon h}\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})^2,$$

where $u_{\epsilon h}$ denotes the solution of the finite element approximation (3.1) for the L^2-penalty problem (2.18) with the mesh parameter h.

Consequently, we obtain (cf. Theorem 3.3)

$$\|u-u_{\epsilon h}\|_{H^1(\Omega)} \le C(\epsilon^{\frac{1}{4}}+h^{\frac{1}{2}})\|f\|_{L^2(\Omega)},\ \|u-u_{\epsilon h}\|_{L^2(\Omega)} \le C(\epsilon^{\frac{1}{2}}+h)\|f\|_{L^2(\Omega)},$$
$$\|u_{\epsilon h}\|_{H^{\frac{1}{2}}(\Gamma)}+\frac{1}{\sqrt{\epsilon}}\|u_{\epsilon h}\|_{L^2(\Omega_1)} \le C(h^{\frac{1}{2}}+\epsilon^{\frac{1}{4}})\|f\|_{L^2(\Omega)}.$$

From these results, we see that the optimal choice of ϵ is $\epsilon = h^2$, when h fixed.

According to the fictitious domain method, we solve the discrete L^2-penalty problem (3.1) instead of the original problem of (2.1). Since the domain Ω has smooth boundary, we provide an approximation scheme for the computation of the inner-product $(u_{\epsilon h}, v_h)_{\Omega_1}$. We find a polygon $\hat{\Omega}$ approximating to Ω, with $\max_{x\in\partial\Omega} \operatorname{dist}\ (x,\partial\hat{\Omega}) = O(h^2)$. For example, the $\hat{\Omega}$ is constructed by connecting the intersection points between $\partial\Omega$ and the mesh for every triangle of the mesh. Then, instead of (3.1), we solve its approximation problem (3.6), and we have the error estimate (cf. Theorem 3.4)

$$\|u-\hat{u}_{\epsilon,h}\|_{H^1(\Omega)} \le C(h^{\frac{1}{2}}+\epsilon^{\frac{1}{4}}+\epsilon^{-\frac{1}{2}}h^{\frac{3}{2}})\|f\|_{L^2(\Omega)},$$
$$\|u-\hat{u}_{\epsilon,h}\|_{L^2(\Omega)} \le C(h+\epsilon^{\frac{1}{2}}+\epsilon^{-\frac{1}{2}}h^2+\epsilon^{-\frac{1}{4}}h^{\frac{3}{2}})\|f\|_{L^2(\Omega)},$$

which show the approximation scheme shares the same error order with the error of finite element method for $\epsilon = h^2$; however, $\epsilon \ll h^2$ would enlarge errors and we will verify this phenomenon with the aid of numerical experiments.

As mentioned before, one of the advantages of the L^2-penalty method is that it can be directly applied to the finite volume and finite difference methods (cf. [13], [14]). Therefore, a mathematical study of these problems are of interest. However, it seems that little is known in this direction. Thus, our next aim is to study the finite volume discretization of the L^2-penalty problem. To this end, in Section 4, we first introduce and study a special finite element approximation with the mass-lumping approximation, where the L^2 inner product is replaced by a simple quadrature formula using the Voronoi polygon (cf. [8, §6.2]). Actually, we prove (cf. Theorem 4.1)

$$\|u_\epsilon-u_{\epsilon h}^{\mathrm{ML}}\|_{H^1(D)}+\frac{1}{\sqrt{\epsilon}}\|u_\epsilon-u_{\epsilon h}^{\mathrm{ML}}\|_{L^2(\Omega_1)}$$
$$\le C(h+h\epsilon^{-1/4}+h\epsilon^{-1/2}+h^2\epsilon^{-3/4})\|f\|_{L^2(\Omega)},$$

where $u_{\epsilon h}^{\mathrm{ML}}$ denotes the solution of the finite element approximation with the mass-lumping.

The final section, Section 5, is devoted to the finite volume method. We first verify that the finite element approximation with mass-lumping is equivalent to the finite volume approximation and then derive the following error estimate:

$$|Q_h u_\epsilon - \hat{u}_h|_{1,D,h} \le C(h+h\epsilon^{-1/4}+h\epsilon^{-1/2}+h^2\epsilon^{-3/4})\|f\|_{0,\Omega},$$

where $\hat{u}_h$ denotes the finite volume approximation, $Q_h u_\epsilon$ a suitable projection of u_ϵ into the finite volume trial function space, and $|\cdot|_{1,D,h}$ the discrete H_0^1 norm defined as (5.5). We note that $\hat{u}_h$ and $Q_h u_\epsilon$ are piecewise constant functions. This is the first result concerning error analysis for the fictitious domain method applied to the finite volume method.

The convergence of L^2-penalty for elliptic and parabolic problems has been proved in [10]; however, no error estimate has been found, neither the finite element analysis. A similar penalty problem for the Navier-Stokes equations is considered without any numerical results in [1]. Our error estimates in the H^1 norm maintain the sharpness of those for Navier-Stokes problems in [1]. It should be kept in mind that our method of analysis presented here can also be applied to Stokes and Navier-Stokes problems with little difficulty. Furthermore, the results presented in this paper are applied to analysis of L^2 and H^1-penalty fictitious domain methods for parabolic problems in cylindrical and non-cylindrical domains in [19].

Notation

Throughout this paper, we follow the notation of [9]. Namely we use standard Lebesgue and Sobolev spaces $L^2(\omega)$, $H^m(\omega)$ $(m > 0)$ and $H_0^1(\omega)$, where ω denotes a domain in $\mathbb{R}^2$. We write as

$$\begin{aligned}
(u, v)_\omega &= (u, v)_{L^2(\omega)} = \int_\omega u(x)v(x)\ dx; \\
\|u\|_{0,\omega} &= \|u\|_{L^2(\omega)} = \left(\int_\omega |u(x)|^2\ dx \right)^{1/2}; \\
|u|_{m,\omega} &= \left(\sum_{|\alpha|=m} \|\partial^\alpha u\|_{0,\omega}^2 \right)^{1/2}; \\
\|u\|_{m,\omega} &= \left(\|u\|_{m-1,\omega}^2 + |u|_{m,\omega}^2 \right)^{1/2},
\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2)$ denotes a multi-index with $|\alpha| = \alpha_1 + \alpha_2$ and set $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2}$.

We also use standard Lebesgue and Sobolev spaces $L^2(\gamma)$ and $H^s(\gamma)$ $(s > 0)$ defined on a part γ of the boundary $\partial\omega$. The unit outer normal vector to the boundary under consideration is always denoted by n. Finally, we use the same letter C to express a generic constant independent of the penalty parameter ϵ and the discretization parameter h.

2 Fictitious domain method with L^2-penalty

Throughout this paper, we assume that Ω is a bounded domain in $\mathbb{R}^2$ with the C^2 boundary $\Gamma = \partial\Omega$. As a model problem, we consider the Poisson equation with the homogeneous Dirichlet boundary condition,

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma, \tag{2.1}$$

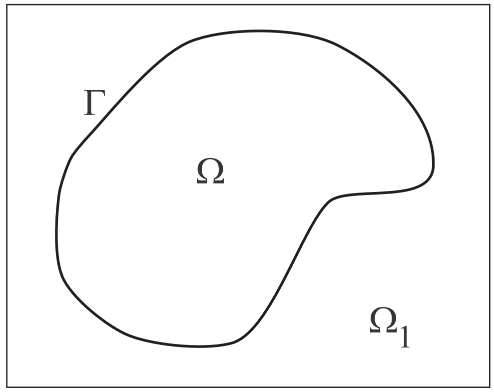

Figure 1: The original domain Ω and the fictitious domain D.

where f is a given function of $L^2(\Omega)$. The weak form reads as

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \qquad \forall v \in H_0^1(\Omega). \end{cases} \tag{2.2}$$

We take a convex polygonal domain $D \subset \mathbb{R}^2$, which we call the fictitious domain, such that $\overline{\Omega} \subset D$ and set $\Omega_1 = D \backslash \overline{\Omega}$. See, for example, Fig. 1. Then, the fictitious domain formulation with the L^2 penalization for (2.2) is given as

$$\begin{cases} \text{Find } u_\epsilon \in H_0^1(D) \text{ such that} \\ (\nabla u_\epsilon, \nabla v)_D + \dfrac{1}{\epsilon}(u_\epsilon, v)_{\Omega_1} = (\tilde{f}, v)_D \qquad \forall v \in H_0^1(D), \end{cases} \tag{2.3}$$

where

$$0 < \epsilon \leq 1 \tag{2.4}$$

is the penalty parameter and $\tilde{f} \in L^2(D)$ is any extension of f into D such that

$$\tilde{f} = f \text{ a.e. in } \Omega, \quad \|\tilde{f}\|_{0,D} \leq C\|f\|_{0,\Omega}$$

with a positive constant C depending only on D and Ω.

According to the Lax and Milgram's theory, there exists a unique solution u_ϵ of (2.3) for any $\epsilon \in (0, 1]$. Substituting $v = u_\epsilon$ in (2.3) and then using Schwarz, Poincaré and Young's inequalities, we have

$$\begin{aligned} &\|\nabla u_\epsilon\|_{0,\Omega}^2 + \|\nabla u_\epsilon\|_{0,\Omega_1}^2 + \frac{1}{\epsilon}\|u_\epsilon\|_{0,\Omega_1}^2 \\ &\qquad\qquad \leq \frac{C^2}{2}\|f\|_{0,\Omega}^2 + \frac{1}{2}\|\nabla u_\epsilon\|_{0,\Omega}^2 + \frac{1}{2}\epsilon\|\tilde{f}\|_{0,\Omega_1}^2 + \frac{1}{2\epsilon}\|u_\epsilon\|_{0,\Omega_1}^2. \end{aligned}$$

This gives

$$\|u_\epsilon\|_{1,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{0,\Omega_1} \leq C\|f\|_{0,\Omega}. \tag{2.5}$$

In particular, we have $\|u_\epsilon\|_{0,\Omega_1} \leq C\sqrt{\epsilon}$.

Furthermore, the function u_ϵ solves the variational problem

$$(\nabla u_\epsilon, \nabla v)_D = \left(\tilde{f} - \frac{1}{\epsilon}\chi u_\epsilon, v\right)_D \qquad \forall v \in H_0^1(D),$$

where $\chi \in L^\infty(D)$ denotes the characteristic function of Ω_1 defined as

$$\chi(x) = \begin{cases} 0 & (x \in \Omega) \\ 1 & (x \in \Omega_1). \end{cases} \tag{2.6}$$

Hence, we can apply regularity results of elliptic problems in convex domains (cf. [5, Theorem 3.2.1.2] for example) to obtain

$$u_\epsilon \in H^2(D) \tag{2.7}$$

and

$$\|u_\epsilon\|_{2,D} \leq C \left\| \tilde{f} - \frac{1}{\epsilon}\chi u_\epsilon \right\|_{0,D} \leq C \left(1 + \frac{1}{\sqrt{\epsilon}}\right) \|f\|_{0,\Omega}. \tag{2.8}$$

This estimate is meaningless for a sufficiently small ϵ; However, we can deduce better a priori bounds for $\|u_\epsilon\|_{2,\Omega}$ and, by using this, we can derive some error estimate for u_ϵ. Actually, the following theorem is the main result of this section.

Theorem 2.1. *Let $u_\epsilon \in H_0^1(D)$ be the solution of* (2.3). *Then, we have $u_\epsilon \in H^2(D)$ and*

$$\begin{aligned} \|u_\epsilon\|_{2,\Omega} &\leq C\|f\|_{0,\Omega}, && (2.9)\\ \|u_\epsilon\|_{2,\Omega_1} &\leq C\epsilon^{-\frac{1}{4}}\|f\|_{0,\Omega}, && (2.10)\\ \|u_\epsilon\|_{1,\Omega_1} &\leq C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}, && (2.11)\\ \|u_\epsilon\|_{0,\Omega_1} &\leq C\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega}. && (2.12) \end{aligned}$$

Furthermore,

$$\begin{aligned} \|u - u_\epsilon\|_{1,\Omega} &\leq \epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}, && (2.13)\\ \|u - u_\epsilon\|_{0,\Omega} &\leq \epsilon^{\frac{1}{2}}\|f\|_{0,\Omega}, && (2.14)\\ \|u_\epsilon\|_{\frac{1}{2},\Gamma} &\leq C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}, && (2.15) \end{aligned}$$

where $u \in H_0^1(\Omega)$ denotes the solution of (2.2).

Remark 2.2. In [10, Theorem I-4], it has already proved

$$\|u_\epsilon - u\|_{1,\Omega} \to 0, \quad \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{0,\Omega_1} \to 0 \quad \text{as} \quad \epsilon \to 0 \tag{2.16}$$

for $\tilde{f}$ being the zero extension of f.

In the proof of Theorem 2.1, we use the following regularity result for a linear elliptic equation. Although it seems not to be new, we give its proof for readers' convenience.

Lemma 2.3. *For $\phi \in L^2(\Omega_1)$ and $g \in H^{1/2}(\Gamma)$, let $w \in H^2(\Omega_1)$ be a solution of*

$$-\Delta w + \frac{1}{\epsilon} w = \phi \text{ in } \Omega_1, \quad \frac{\partial w}{\partial n} = g \text{ on } \Gamma, \quad w = 0 \text{ on } \partial D.$$

Then, we have

$$\begin{aligned} \|w\|_{0,\Omega_1} &\leq C(\epsilon \|\phi\|_{0,\Omega_1} + \epsilon^{\frac{3}{4}} \|g\|_{\frac{1}{2},\Gamma}), \\ \|w\|_{2,\Omega_1} &\leq C(\|\phi\|_{0,\Omega_1} + \epsilon^{-\frac{1}{4}} \|g\|_{\frac{1}{2},\Gamma}). \end{aligned}$$

In order to prove this, we need the following auxiliary lemma. The proof will be given in Appendix A.

Lemma 2.4. *For $g \in H^{\frac{1}{2}}(\Gamma)$ and $\eta > 0$, there exists $v = v_\eta \in H^2(\Omega_1)$ such that,*

$$\frac{\partial v}{\partial n} = g \text{ on } \Gamma, \qquad v = 0 \text{ on } \partial D$$

with estimates

$$\|v\|_{0,\Omega} \leq C\eta^3 \|g\|_{\frac{1}{2},\Gamma}, \quad |v|_{1,\Omega} \leq C\eta \|g\|_{\frac{1}{2},\Gamma}, \quad |v|_{2,\Omega} \leq C\eta^{-1} \|g\|_{\frac{1}{2},\Gamma}.$$

Proof of Lemma 2.3. By Lemma 2.4 with $\eta = \epsilon^{\frac{1}{4}}$, there exists $\psi \in H^2(\Omega)$ such that $\partial\psi/\partial n = g$ on Γ, $\psi = 0$ on ∂D, $\|\psi\|_{0,\Omega_1} \leq C\epsilon^{\frac{3}{4}} \|g\|_{\frac{1}{2},\Gamma}$ and $\|\psi\|_{2,\Omega_1} \leq C\epsilon^{-\frac{1}{4}} \|g\|_{\frac{1}{2},\Gamma}$. Setting $u = w - \psi$, we have

$$-\Delta u + \frac{1}{\epsilon} u = \phi + \Delta\psi + \frac{1}{\epsilon}\psi \text{ in } \Omega_1, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma, \quad u = 0 \text{ on } \partial D.$$

Multiplying the both sides by u and integrating over Ω_1, we have

$$\|\nabla u\|_{0,\Omega_1}^2 + \frac{1}{\epsilon}\|u\|_{0,\Omega_1}^2 \leq \|\phi\|_{0,\Omega_1}\|u\|_{0,\Omega_1} + \left(\|\psi\|_{2,\Omega_1} + \frac{1}{\epsilon}\|\psi\|_{0,\Omega_1} \right) \|u\|_{0,\Omega_1}.$$

Hence,

$$\begin{aligned} \|u\|_{0,\Omega_1} &\leq \epsilon\|\phi\|_{0,\Omega_1} + \epsilon\|\psi\|_{2,\Omega_1} + \|\psi\|_{0,\Omega_1} \\ &\leq \epsilon\|\phi\|_{0,\Omega_1} + \epsilon \cdot C\epsilon^{-\frac{1}{4}}\|g\|_{\frac{1}{2},\Gamma} + C\epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma}. \end{aligned}$$

This implies

$$\|w\|_{0,\Omega_1} \leq \|\psi\|_{0,\Omega_1} + \epsilon\|\phi\|_{0,\Omega_1} + C\epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma} \leq \epsilon\|\phi\|_{0,\Omega_1} + C\epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma}.$$

On the other hand,

$$\begin{aligned} \|w\|_{2,\Omega_1} &\leq C\left\| \phi + \Delta\psi + \frac{1}{\epsilon}\psi \right\|_{0,\Omega_1} + C\|g\|_{\frac{1}{2},\Gamma} \\ &\leq C\|\phi\|_{0,\Omega_1} + C\|\psi\|_{2,\Omega_1} + C\frac{1}{\epsilon}\|\psi\|_{0,\Omega_1} + C\|g\|_{\frac{1}{2},\Gamma} \\ &\leq C\|\phi\|_{0,\Omega_1} + C\epsilon^{-\frac{1}{4}}\|g\|_{\frac{1}{2},\Gamma} + C\|g\|_{\frac{1}{2},\Gamma}, \end{aligned}$$

which implies the desired estimate. □

Now we can state the following proof.

Proof of Theorem 2.1. First, we prove inequalities (2.10)–(2.15) by using (2.9).

Applying Green's formula, we observe that (2.3) is equivalent to the following problem:

$$-\Delta u_\epsilon = f \text{ in } \Omega, \quad u_\epsilon|_\Omega = u_\epsilon|_{\Omega_1} \text{ on } \Gamma; \tag{2.17}$$

$$-\Delta u_\epsilon + \frac{1}{\epsilon}u_\epsilon = \tilde{f} \text{ in } \Omega_1, \quad \left.\frac{\partial u_\epsilon}{\partial n}\right|_\Omega = \left.\frac{\partial u_\epsilon}{\partial n}\right|_{\Omega_1} \text{ on } \Gamma, \quad u_\epsilon = 0 \text{ on } \partial D. \tag{2.18}$$

In view of the trace theorem, we have

$$\left\|\frac{\partial u_\epsilon}{\partial n}\right\|_{\frac{1}{2},\Gamma} \le C\|u_\epsilon\|_{2,\Omega} \le C\|f\|_{0,\Omega}.$$

Hence, we apply Lemma 2.3 to the problem (2.18) in order to obtain

$$\|u_\epsilon\|_{0,\Omega_1} \le C(\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega} + \epsilon\|\tilde{f}\|_{0,\Omega_1}), \tag{2.19}$$

$$\|u_\epsilon\|_{2,\Omega_1} \le C(\epsilon^{-\frac{1}{4}}\|f\|_{0,\Omega} + \|\tilde{f}\|_{0,\Omega_1}) \tag{2.20}$$

which imply (2.10) and (2.12), respectively.

We recall that in general we have (cf. [4, Theorem 7.27])

$$|v|_{1,\Omega_1} \le C(\eta|v|_{2,\Omega_1} + \eta^{-1}\|v\|_{0,\Omega})$$

for any $\eta > 0$ and $v \in H^2(\Omega)$. Setting $\eta = \epsilon^{\frac{1}{2}}$, we deduce (2.11).

Estimates (2.13) and (2.15) are readily obtainable consequences of (2.11) and trace theorems. Thus,

$$\begin{aligned}\|u_\epsilon - u\|_{1,\Omega} &\le C\|u_\epsilon - u\|_{\frac{1}{2},\Gamma} = C\|u_\epsilon\|_{\frac{1}{2},\Gamma} \\ &\le C\|u_\epsilon\|_{1,\Omega_1} \le C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}.\end{aligned}$$

We proceed to derive (2.14). To this end, we introduce the adjoint problems for (2.2) and (2.3) which are given as

$$\begin{cases}\text{Find } u_F \in H_0^1(\Omega) \text{ such that} \\ (\nabla u_F, \nabla v)_\Omega = (F, v)_\Omega \quad \forall v \in H_0^1(\Omega)\end{cases} \tag{2.21}$$

and

$$\begin{cases}\text{Find } u_{F\epsilon} \in H_0^1(D) \text{ such that} \\ (\nabla u_{F\epsilon}, \nabla v)_D + \dfrac{1}{\epsilon}(u_{F\epsilon}, v)_{\Omega_1} = (\tilde{F}, v)_D \quad \forall v \in H_0^1(D),\end{cases} \tag{2.22}$$

for any $F \in L^2(\Omega)$, and the extension of F, $\tilde{F} \in L^2(D)$, satisfying $\|\tilde{F}\|_{0,\Omega_1} \le C\|F\|_{0,\Omega}$.

Apparently, we can obtain the a priori estimates and H^1 norm penalization error estimate, like (2.20), (2.20) and (2.13), for the adjoint problems (2.21) and

(2.22). Thus we have

$$\|u_{F\epsilon}\|_{2,\Omega} \le C(\epsilon^{-\frac{1}{4}}\|F\|_{0,\Omega} + \|\tilde{F}\|_{0,\Omega_1}), \tag{2.23}$$

$$\|u_{F\epsilon}\|_{0,\Omega} \le C(\epsilon^{\frac{3}{4}}\|F\|_{0,\Omega} + \epsilon\|\tilde{F}\|_{0,\Omega_1}), \tag{2.24}$$

$$\|u_{F\epsilon}|_\Omega - u_F\|_{1,\Omega} \le C\epsilon^{\frac{1}{4}}\|F\|_{0,\Omega}. \tag{2.25}$$

Denoting by $\tilde{u}$ and $\tilde{u}_F$ the zero extension of u and u_F, respectively, one can show that

$$\begin{aligned}(\nabla u_\epsilon, \nabla \tilde{u}_F)_D = (\tilde{u}_F, \tilde{f})_D = (u_F, f)_\Omega = (\nabla u_F, \nabla u)_\Omega& \\ = (F, u)_\Omega = (\tilde{F}, \tilde{u})_D = (\nabla u_{F\epsilon}, \nabla \tilde{u})_D&,\end{aligned}$$

and hence

$$(\nabla(u_{F\epsilon} - \tilde{u}_F), \nabla(u_\epsilon - \tilde{u}))_D = (\tilde{F}, u_\epsilon - \tilde{u})_D - \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon)_{\Omega_1}.$$

At this stage, we let $\tilde{F} = u_\epsilon - \tilde{u}$. Then,

$$\|u_\epsilon - \tilde{u}\|_{0,\Omega}^2 + \|u_\epsilon\|_{0,\Omega_1}^2 = (\nabla(u_{F\epsilon} - \tilde{u}_F), \nabla(u_\epsilon - \tilde{u}))_D + \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon)_{\Omega_1}.$$

Combining those estimates, we get

$$\|u_\epsilon|_\Omega - u\|_{0,\Omega} \le C\epsilon^{\frac{1}{2}}\|f\|_{0,\Omega}. \tag{2.26}$$

Thus, we have proved (2.14).

Now, we go back to the beginning of the proof; It remains to show (2.9). To this end, let us consider the interface problem composed of (2.17) and (2.18) and apply the standard method of tangential difference quotients due to Nirenberg; See, for example, [5, Theorem 2.2.2.3], [12, Appendix], or [18, Theorem 3.1].

We take a set $\{U_j\}_{j=1}^N$ of open subsets in $\mathbb{R}^2$ enjoying the following properties. With U_j and $1 \le j \le N$, we associate a C^2 diffeomorphism $\Phi_j : U_j \to \mathbb{R}^2$ that satisfies

$$\overline{\Omega} \subset \bigcup_{j=1}^N \Phi_j(U_j) \subset D,$$

$$U_{j0} = \Psi_j(\Phi_j(U_j) \cap \Omega) = \mathbb{R}^2_+ \cap U_j, \quad U_{j1} = \Psi_j(\Phi_j(U) \cap \Omega_1) = \mathbb{R}^2_- \cap U_j,$$

where $\mathbb{R}^2_\pm = \mathbb{R}^2 \cap \{\pm x_2 > 0\}$ and $\Psi_j = \Phi_j^{-1}$. Further, we take $\{\theta_j\}_{j=1}^N \subset C_0^\infty(\overline{\Omega})$ such that supp $\theta_j \subset \Phi_j(U_j)$ and

$$\sum_{j=1}^N \theta_j = 1 \text{ on } \overline{\Omega} \quad \text{and} \quad \delta = \min_{1 \le j \le N} \text{dist } (\text{supp } \theta_j, \partial\Phi_j(U_j)) > 0.$$

We note that $(\theta_j u_\epsilon) \circ \Phi_j \in H_0^1(U_j)$ for $j = 1, 2, \dots, N$. We drop the subscript j and write $U = U_j$, $U_1 = U_{j1}$, $U_0 = U_{j0}$, $\Phi = \Phi_j$, $\Psi = \Psi_j$, and $\theta = \theta_j$ for short.

Set $u_1 = \theta u_\epsilon$ and $u_2 = (\theta u_\epsilon) \circ \Phi$.

First, if $U_1 = \emptyset$, then $u_1 \in H^2(\Omega)$ and $\|u_1\|_{2,\Omega} \le C\|\tilde{f}\|_{0,D}$ are not new. In what follows, we consider the case $U_0 \neq \emptyset$ and $U_1 \neq \emptyset$. Set $D_i = \partial/\partial x_i$, $(i = 1, 2)$. We observe that $u_2 \in H_0^1(U)$ satisfies

$$\sum_{i,k=1}^{2} \int_U a_{ik} D_i u_2 D_k v dx + \frac{1}{\epsilon} \sum_{i,k=1}^{2} \int_{U_1} D_i u_2 D_k v |D\Phi| dx = (f_2, v) \quad \forall v \in H_0^1(U), \tag{2.27}$$

where $f_2 = (\theta \tilde{f} + \nabla u_\epsilon \nabla \theta + \nabla \cdot (u_\epsilon \nabla \theta)) \circ \Phi |D\Phi|$ and

$$a_{ik} = (\sum_{l=1}^{2} D_l \psi_i D_l \psi_k) \circ \Phi |D\Phi| \quad (i, k = 1, 2), \quad \Psi = (\psi_1, \psi_2).$$

Let $\tilde{u}_2$ be the zero extension of u_2 onto $\mathbb{R}^2$ and let $|h| \le \delta/4$. Substituting $v = \frac{\tau_h - 1}{h} \frac{\tau_{-h} - 1}{h} \tilde{u}_2 \in H_0^1(U)$ into (2.27), where τ_h is the translation operator with $\tau_h \phi(x) = \phi(x_1 + h, x_2)$, $\phi(x) \in L^2(\mathbb{R}^2)$, we have after some calculation

$$\begin{aligned} \sum_{i=1}^{2} \left\| D_i \left(\frac{\tau_h - 1}{h} \tilde{u}_2 \right) \right\|_{0,U}^2 + \frac{1}{\epsilon} \sum_{i=1}^{2} \left\| \frac{\tau_h - 1}{h} \tilde{u}_2 \right\|_{0,U_1}^2 \\ \le C \sum_{i=1}^{2} \left\| D_i \left(\frac{\tau_h - 1}{h} \tilde{u}_2 \right) \right\|_{0,U} + C \frac{1}{\epsilon} \|\tilde{u}_2\|_{0,U_1}^2 + C \|f_2\|_{0,U}^2, \end{aligned}$$

applying (2.16) or (2.5), we have $\sum_{i=1}^{2} \left\| D_i \left(\frac{\tau_h - 1}{h} \tilde{u}_2 \right) \right\|_{U_0} \le C\|f\|_{0,\Omega}$. On letting $h \downarrow 0$, we conclude $D_i D_1 u_2 \in L^2(U_0)$ and $\|D_i D_1 u_2\|_{0,U_0} \le C\|\tilde{f}\|_{0,\Omega}$ for $i = 1, 2$.

Finally, we see that

$$D_2^2 u_2 = \frac{1}{a_{22}} (f_2 - \sum_{k+l \le 3} D_l (a_{kl} D_k u_2) - D_2 a_{22} D_2 u_2) \quad \text{in } U_0.$$

This implies that $D_2^2 u_2 \in L^2(U_0)$ and $\|u_2\|_{2,U_0} \le C\|\tilde{f}\|_{0,\Omega}$.

Summing up, we conclude that $u_\epsilon|_\Omega \in H^2(\Omega)$ and $\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$. This completes the proof of Theorem 2.1. $\square$

Remark 2.5. As stated before, the solution $u_\epsilon \in H_0^1(D)$ of (2.3) has a regularity property $u_\epsilon \in H^2(D)$. Thus, the solution of the penalization problem (2.3) is regular. It seems that this property is an advantage of the L^2-penalty method in contrast to the H^1-penalty method whose solution u_ϵ has only $u_\epsilon|_\Omega \in H^2(\Omega)$ and $u_\epsilon|_{\Omega_1} \in H^2(\Omega_1)$. However, as is reported in Theorem 2.1, we are able to guarantee only $\|u_\epsilon\|_{2,\Omega_1} \le \epsilon^{-\frac{1}{4}} \|f\|_{0,\Omega}$. Thus, $\|u_\epsilon\|_{2,\Omega_1}$ may be very large for a sufficiently small $\epsilon > 0$. This drawback has not been pointed out in the previous studies on the L^2-penalty method including, for example, [1] and [10].

3 Finite element approximation

We introduce a shape-regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$ to the convex polygonal domain D, where h is the maximum diameter of the triangles of $\mathcal{T}_h$.

That is, there exists a positive constant ν_1 such that

$$\frac{h_T}{\rho_T} \leq \nu_1 \qquad (\forall T \in \forall \mathcal{T}_h \in \{\mathcal{T}_h\}_h),$$

where h_T and ρ_T, respectively, denote the diameters of circumscribe and inscribe circles of T. Let $V_h(D) \subset H_0^1(D)$ be the set of all continuous piecewise-affine functions subordinate to $\mathcal{T}_h$. A finite element approximation for (2.3) reads as

$$\begin{cases} \text{Find } u_{\epsilon h} \in V_h(D) \text{ such that} \\ (\nabla u_{\epsilon h}, \nabla v_h)_D + \dfrac{1}{\epsilon}(u_{\epsilon h}, v_h)_{\Omega_1} = (\tilde{f}, v_h)_D \quad \forall v_h \in V_h(D), \end{cases} \tag{3.1}$$

Thus, applying the fictitious domain method, we compute (3.1) instead of (2.2). According to Theorem 2.1, the error satisfies

$$\|u - u_{\epsilon h}\|_{1,\Omega} \leq \|u - u_\epsilon\|_{1,\Omega} + \|u_\epsilon - u_{\epsilon h}\|_{1,D} \leq C\epsilon^{\frac{1}{4}} + C\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D},$$
$$\|u - u_{\epsilon h}\|_{0,\Omega} \leq \|u - u_\epsilon\|_{0,\Omega_1} + \|u_\epsilon - u_{\epsilon h}\|_{0,\Omega} \leq C\epsilon^{\frac{1}{2}} + \|u_\epsilon - u_{\epsilon h}\|_{0,\Omega}.$$

Hence, it suffices to examine $u_\epsilon - u_{\epsilon h}$. First, we give the following lemma.

Lemma 3.1. *Let u_ϵ and $u_{\epsilon h}$ be the solutions of (2.3) and (3.1), respectively. Then, we have*

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega_1} \leq C \inf_{v_h \in V_h(D)} \left(\|\nabla(u_\epsilon - v_h)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - v_h\|_{0,\Omega_1} \right). \tag{3.2}$$

Proof. It is a consequence of the Galerkin orthogonality

$$(\nabla(u_\epsilon - u_{\epsilon h}), \nabla v_h)_D + \frac{1}{\epsilon}(u_\epsilon - u_{\epsilon h}, v_h) = 0 \qquad \forall v_h \in V_h(D).$$

□

Theorem 3.2. *Suppose that u_ϵ and $u_{\epsilon h}$ are the solutions of (2.3) and (3.1), respectively. Then, we have*

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega_1} \leq C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})\|f\|_{0,\Omega}, \tag{3.3}$$

$$\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega} \leq C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})^2\|f\|_{0,\Omega}. \tag{3.4}$$

Proof. We introduce some notations first. A generic (closed) triangle of $\mathcal{T}_h$ is denoted by K, and the set of all vertices of K is denoted by $\Lambda(K) = (\nu_1^K, \nu_2^K, \nu_3^K)$. Set $T_\Gamma = \{K \mid K \cap \Gamma \neq \emptyset\}$ and $T' = \{K \subset \Omega | K \cap T_\Gamma = \emptyset\}$. The standard P1 Lagrange interpolation of $v \in H^2(D)$ is denoted by $I_h v$. We define $v_h \in V_h(D)$ by setting,

$$v_h(\nu) = \begin{cases} 0 & \text{for } \nu \in \Lambda(K), K \subset T_\Gamma \cup \overline{\Omega_1}, \\ u_\epsilon(\nu) & \text{for all other vertices } \nu. \end{cases}$$

substitute v_h into (3.2) and using the a priori estimates in Theorem 2.1, we have

$$\|u_\epsilon - v_h\|_{0,\Omega_1} = \|u_\epsilon\|_{0,\Omega_1} \leq C\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega}$$

and

$$\begin{aligned}
&\|\nabla(u_\epsilon - v_h)\|_{0,\Omega}^2 \\
&\quad\leq\ C(\|\nabla(u_\epsilon - I_h u)\|_{0,T'}^2 + \|\nabla u_\epsilon\|_{0,\Omega\backslash T'}^2 + \|\nabla v_h\|_{0,\Omega\backslash T'}^2) \\
&\quad\leq\ C(\|\nabla(u_\epsilon - u)\|_{0,T'}^2 + \|\nabla(u - I_h u)\|_{0,T'}^2 + \|\nabla u_\epsilon\|_{0,\Omega\backslash T'}^2 + \|\nabla v_h\|_{0,\Omega\backslash T'}^2) \\
&\quad\leq\ C\left(h^2\|u\|_{2,\Omega}^2 + h\|u_\epsilon\|_{2,\Omega}^2 + h\|u\|_{2,\Omega}^2\right) \\
&\quad\leq\ Ch\|f\|_{0,\Omega}^2,
\end{aligned}$$

where $u \in H^2(\Omega)$ is the solution of (2.2). Therefore,

$$\begin{aligned}
\|\nabla(u_\epsilon - v_h)\|_{0,D}^2 &= \|\nabla(u_\epsilon - v_h)\|_{0,\Omega}^2 + \|\nabla(u_\epsilon - v_h)\|_{0,\Omega_1}^2 \\
&= \|\nabla(u_\epsilon - v_h)\|_{0,\Omega}^2 + \|\nabla u_\epsilon\|_{0,\Omega_1}^2 \\
&\leq Ch\|f\|_{0,\Omega}^2 + C\epsilon^{\frac{1}{2}}\|f\|_{0,\Omega}^2,
\end{aligned}$$

which implies (3.3). See the proof of [18, Theorem 4.4] for the detailed proof of this estimate; Especially, the estimate $\|\nabla u_\epsilon\|_{0,\Omega\backslash T'} \leq Ch^{\frac{1}{2}}\|u_\epsilon\|_{2,\Omega}$ follows from [18, Lemma 4.2] or a similar lemma in [17], and for the proof of $\|\nabla v_h\|_{0,\Omega\backslash T'} \leq Ch^{\frac{1}{2}}\|u\|_{2,\Omega}$, one can refer to [17] or the proof of [18, Theorem 4.4], with aware of $u = 0$ on Γ, which gives (3.3).

Then, setting $\tilde{F} = 1_\Omega(u_\epsilon - u_{\epsilon h})$ and $v = u_\epsilon - u_{\epsilon h}$ in the adjoint problem (2.22), where $1_\Omega = 1$ in Ω, and $1_\Omega = 0$ in otherwise, applying (3.3) and the prior estimates in Theorem 2.1, we have for any $v_h \in V_h(D)$

$$\begin{aligned}
\|F\|_{0,\Omega}^2 &= \|u_\epsilon - u_{\epsilon h}\|_{0,\Omega}^2 = (\nabla u_{F\epsilon}, \nabla(u_\epsilon - u_{\epsilon h}))_D + \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon - u_{\epsilon h})_{\Omega_1} \\
&= (\nabla u_{F\epsilon} - v_h, \nabla(u_\epsilon - u_{\epsilon h}))_D + \frac{1}{\epsilon}(u_{F\epsilon} - v_h, u_\epsilon - u_{\epsilon h})_{\Omega_1} \\
&\leq C(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|F\|_{0,\Omega}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|f\|_{0,\Omega} \\
&\qquad + C\frac{1}{\epsilon}\epsilon^{\frac{1}{2}}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|F\|_{0,\Omega}\epsilon^{\frac{1}{2}}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|f\|_{0,\Omega},
\end{aligned}$$

which implies (3.4), and the proof is completed. □

Combining Theorems 2.1 and 3.2, we obtain the following estimates.

Theorem 3.3. *Suppose that u and $u_{\epsilon h}$ are the solutions of (2.2) and (3.1), respectively. Then, we have*

$$\|\nabla(u - u_{\epsilon h})\|_{0,\Omega} \leq C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})\|f\|_{0,\Omega}, \quad \|u - u_{\epsilon h}\|_{0,\Omega} \leq C(h + \epsilon^{\frac{1}{2}})\|f\|_{0,\Omega},$$
$$\|u_{\epsilon h}\|_{\frac{1}{2},\Gamma} + \frac{1}{\sqrt{\epsilon}}\|u_{\epsilon h}\|_{0,\Omega_1} \leq C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})\|f\|_{\Omega}.$$

Due to the smooth boundary of Ω, the inner-product $(u_{\epsilon,h}, v_h)_{\Omega_1}$ cannot be computed exactly. Therefore we need an approximation scheme for computation of the problem (3.1).

As we mentioned in Introduction, we find a polygonal domain $\hat{\Omega}$ for Ω such that the vertices of $\partial\hat{\Omega}$ are situated on $\partial\Omega$ and assume that there are $h_1 > 0$ and $c_0 > 0$ such that

$$\text{dist}\,(\Omega, \hat{\Omega}) \le c_0 h^2 \qquad (h \in (0, h_1)). \tag{3.5}$$

We set $\hat{\Omega}_1 = D \backslash \overline{\hat{\Omega}}$.

Then, we consider

$$\begin{cases} \text{Find } \hat{u}_{\epsilon h} \in V_h(D) \text{ such that} \\ (\nabla \hat{u}_{\epsilon h}, \nabla v_h)_D + \dfrac{1}{\epsilon}(\hat{u}_{\epsilon h}, v_h)_{\hat{\Omega}_1} = (\tilde{f}, v_h)_D \quad \forall v_h \in V_h(D). \end{cases} \tag{3.6}$$

We have the error estimate of the approximation

Theorem 3.4. *Let u and $\hat{u}_{\epsilon,h}$ be the solutions of* (2.2) *and* (3.6)*, respectively. Then, we have*

$$\|u - \hat{u}_{\epsilon,h}\|_{1,\Omega} \le C\|\hat{u}_{\epsilon,h}\|_{\frac{1}{2},\Gamma} \le C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}} + \epsilon^{-\frac{1}{2}} h^{\frac{3}{2}})\|f\|_{0,\Omega},$$
$$\|u - \hat{u}_{\epsilon,h}\|_{0,\Omega} \le C(h + \epsilon^{\frac{1}{2}} + \epsilon^{-\frac{1}{2}} h^2 + \epsilon^{-\frac{1}{4}} h^{\frac{3}{2}})\|f\|_{0,\Omega}.$$

Remark 3.5. For $\epsilon = h^2$, we have $\|u - \hat{u}_{\epsilon,h}\|_{1,\Omega} \le Ch^{\frac{1}{2}} = C\epsilon^{\frac{1}{4}}$ and $\|u - \hat{u}_{\epsilon,h}\|_{0,\Omega} \le Ch = C\epsilon^{\frac{1}{2}}$.

Proof of Theorem 3.4. In view of Theorem 3.3, it suffices to prove

$$\|\hat{u}_{\epsilon,h} - u_{\epsilon,h}\|_{1,\Omega} \le C\epsilon^{-\frac{1}{2}} h^{\frac{3}{2}} \|f\|_{0,\Omega}, \tag{3.7}$$
$$\|\hat{u}_{\epsilon,h} - u_{\epsilon,h}\|_{0,\Omega} \le C(\epsilon^{-\frac{1}{2}} h^2 + \epsilon^{-\frac{1}{4}} h^{\frac{3}{2}})\|f\|_{0,\Omega}. \tag{3.8}$$

Subtracting (3.1) from (3.6), we have

$$(\nabla(u_{\epsilon,h} - \hat{u}_{\epsilon,h}), v_h)_D + \frac{1}{\epsilon}(u_{\epsilon,h} - \hat{u}_{\epsilon,h}, v_h)_{\Omega_1 \cap \hat{\Omega}_1} + \frac{1}{\epsilon}(u_{\epsilon,h}, v_h)_{\Omega_1 \backslash \hat{\Omega}_1} - \frac{1}{\epsilon}(\hat{u}_{\epsilon,h}, v_h)_{\hat{\Omega}_1 \backslash \Omega_1} = 0. \tag{3.9}$$

for any $v_h \in V_h(D)$. We also have

$$\|\hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1} \le C\sqrt{\epsilon}\|f\|_{0,\Omega}, \quad \|u_{\epsilon,h}\|_{0,\Omega_1} \le C\sqrt{\epsilon}\|f\|_{0,\Omega}$$

which be obtained by substituting $v = \hat{u}_{\epsilon,h}$ and $v = u_{\epsilon,h}$, respectively, into (3.6) into (3.1).

Since we assume that (3.5) hold true, we have

$$\|\hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1 \backslash \Omega_1} \le Ch^{\frac{1}{2}} \|\hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1 \cap T_\Gamma},$$
$$\|v_h\|_{0,\hat{\Omega}_1 \backslash \Omega_1} \le Ch^{\frac{1}{2}} \|v_h\|_{0,\hat{\Omega}_1 \cap T_\Gamma} \le Ch\|v_h\|_{1,D},$$
$$\|u_{\epsilon,h}\|_{0,\Omega_1 \backslash \hat{\Omega}_1} \le Ch^{\frac{1}{2}} \|\hat{u}_{\epsilon,h}\|_{0,\Omega_1 \cap T_\Gamma},$$
$$\|v_h\|_{0,\Omega_1 \backslash \hat{\Omega}_1} \le Ch^{\frac{1}{2}} \|v_h\|_{0,\Omega_1 \cap T_\Gamma} \le Ch\|v_h\|_{1,D},$$

where $T_\Gamma = \{K \in \mathcal{T} \mid K \cap \Gamma \neq \emptyset\}$, and these estimates can be found in [16]. Substituting $v_h = u_{\epsilon,h} - \hat{u}_{\epsilon,h}$ into (3.9), and applying these estimates and Poincaré's inequality, we obtain that

$$
\begin{aligned}
& \|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|_{1,D}^2 + \frac{1}{\epsilon}\|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|_{0,\Omega_1\cap\hat{\Omega}_1}^2 \\
\leq \; & (\nabla(u_{\epsilon,h} - \hat{u}_{\epsilon,h}), \nabla(u_{\epsilon,h} - \hat{u}_{\epsilon,h}))_D + \frac{1}{\epsilon}(u_{\epsilon,h} - \hat{u}_{\epsilon,h}, u_{\epsilon,h} - \hat{u}_{\epsilon,h})_{0,\Omega_1\cap\hat{\Omega}_1} \\
\leq \; & \frac{1}{\epsilon}\|\hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1\setminus\Omega_1}\|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1\setminus\Omega_1} + \frac{1}{\epsilon}\|u_{\epsilon,h}\|_{0,\Omega_1\setminus\hat{\Omega}_1}\|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|_{0,\Omega_1\setminus\hat{\Omega}_1} \\
\leq \; & C\frac{1}{\epsilon}h^{\frac{1}{2}}\epsilon^{\frac{1}{2}}h\|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|_{1,D},
\end{aligned}
$$

which gives (3.7). Setting $\tilde{f} = u_{\epsilon,h} - \hat{u}_{\epsilon,h}$ in (3.1) and (3.6), applying (3.7) we finally get (3.8). □

At this stage, we give numerical experiments to show that the L^2-error is bounded by $(\sqrt{\epsilon} + h)$ and the H^1-norm error is bounded by $(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})$, which is according to our analysis on L^2-penalization and finite element error estimates. We consider the problem

$$-\Delta u = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma,$$

where $\Omega = \{(x,y) \mid x^2 + y^2 < 1\}$ and the exact solution is $u = -\frac{1}{4}(x^2 + y^2 - 1)$. To implement the fictitious domain method, we set the domain $D = \{-1.2 < x, y < 1.2\}$. We solve the problem (3.6). First, fixing $h = 0.01$, we show the errors for different ϵ, see Figure 2; then, setting $\epsilon = 10^{-6}$, we observe the errors dependents on different h, see Figure 3. The logarithm is of base 10 for all the figures.

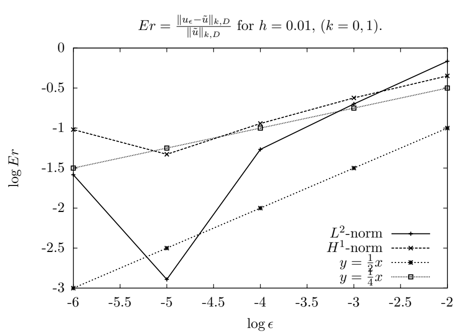

Figure 2: $\frac{\|\hat{u}_{\epsilon,h} - \tilde{u}\|_{k,D}}{\|\tilde{u}\|_{k,D}}$ for $h = 0.01$, $k = 0, 1$

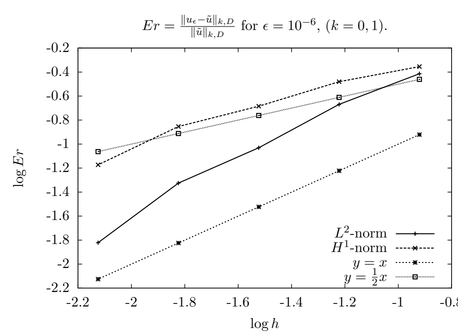

Figure 3: $\frac{\|\hat{u}_{\epsilon,h} - \tilde{u}\|_{k,D}}{\|\tilde{u}\|_{k,D}}$ for $\epsilon = 1e - 6$, $k = 0, 1$.

4 Finite element approximation with mass-lumping

We continue to consider a family of shape-regular triangulations $\{\mathcal{T}_h\}_h$ of the polygonal domain D. Further, we assume that it is of weakly acute type. Thus, each $T \in \mathcal{T}_h$ is a non-obtuse triangle. In this section, we consider a special type of the finite element approximation to (2.3). In the subsequent section, we will show that the scheme considered here coincides with a finite volume scheme.

Let $\mathcal{P}_h$ be the set of all nodes of and $\mathcal{P}_h^0$ be the set of all interior nodes of $\mathcal{T}_h$. Moreover, Let D_P^* be the Voronoi polygon (Dirichlet domain, Wigner-Seitz cell, Thiessen polygon; See for more detail [8, §6.2]) corresponding to $P \in \mathcal{P}_h$. The domain $D_P = D_P^* \cap \Omega$ is called the circum-centric region corresponding to $P \in \mathcal{P}_h$. We introduce, for each $P \in \mathcal{P}_h$,

$$\phi_P \in C(\overline{\Omega}), \quad \phi_P|_T \text{ is an affine function on } T \text{ for all } T \in \mathcal{T}_h,$$
$$\phi_P(Q) = \begin{cases} 1 & (Q \in \mathcal{P}_h,\ Q = P) \\ 0 & (Q \in \mathcal{P}_h,\ Q \neq P), \end{cases}$$

and

$$\hat{\phi}_P = \begin{cases} 1 & \text{in } D_P \\ 0 & \text{otherwise.} \end{cases}$$

Then set

$$X_h = \text{span } \{\phi_P\}_{P \in \mathcal{P}_h}, \quad V_h = \text{span } \{\phi_P\}_{P \in \mathcal{P}_h^0},$$
$$\hat{X}_h = \text{span } \{\hat{\phi}_P\}_{P \in \mathcal{P}_h}, \quad \hat{V}_h = \{v \in \hat{X}_h \mid v \equiv 0 \text{ in } D_P \text{ for } P \in \mathcal{P}_h \backslash \mathcal{P}_h^0\}.$$

(The spaces X_h and V_h are nothing but the standard P1 finite element spaces defined on $\mathcal{T}_h$.)

We introduce the lumping operator $M_h : V_h \to \hat{V}_h$ defined as

$$M_h v = \sum_{P \in \mathcal{P}_h^0} v(P)\hat{\phi}_P \qquad (v \in V_h). \tag{4.1}$$

The inverse operator is obviously given as

$$M_h^{-1}\hat{v} = \sum_{P \in \mathcal{P}_h^0} \hat{v}(P)\phi_P \qquad (\hat{v} \in \hat{V}_h). \tag{4.2}$$

It is well-known that there exists constants C, C' and C'' depending only on D and ν_1 (=the shape-regularity constant) such that

$$C\|v_h\|_{0,D} \le \|M_h v_h\|_{0,D} \le C'\|v_h\|_{0,D} \qquad (v_h \in V_h), \tag{4.3}$$
$$\|v_h - M_h v_h\|_{0,D} \le C'' h \|\nabla v_h\|_{0,D} \qquad (v_h \in V_h). \tag{4.4}$$

See, for example, [6, Lemma 2.1].

Now, we consider the following finite element scheme:

$$\begin{cases} \text{Find } u_{\epsilon h} \in V_h \text{ such that} \\ (\nabla u_{\epsilon h}, \nabla v_h) + \dfrac{1}{\epsilon}(\hat{u}_{\epsilon h}, \hat{v}_h)_{\Omega_1} = (\tilde{f}, \hat{v}_h)_D \quad (\forall v_h \in V_h), \end{cases} \tag{4.5}$$

where $\hat{u}_{\epsilon h} = M_h u_{\epsilon h}$ and $\hat{v}_h = M_h v_h$.

Below, we write as

$$(v_h, w_h)_{\omega,h} = (M_h v_h, M_h w_h)_\omega \qquad (v_h, w_h \in V_h,\ \omega = \Omega_1, D).$$

Theorem 4.1. *Let $u_\epsilon \in H_0^1(D)$ and $u_{\epsilon h} \in V_h$ be solutions of* (2.3) *and* (4.5), *respectively. Then, we have*

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega_1} \leq C(h + h\epsilon^{-1/4} + h\epsilon^{-1/2} + h^2\epsilon^{-3/4})\|f\|_{0,\Omega}. \quad (4.6)$$

In particular,

$$\|u - u_{\epsilon h}\|_{1,\Omega} \leq C(\epsilon^{\frac{1}{4}} + h + h\epsilon^{-1/4} + h\epsilon^{-1/2} + h^2\epsilon^{-3/4})\|f\|_{0,\Omega}$$

where u denotes the solution of (2.2).

Proof. We drop the subscript ϵ for simplicity; $u = u_\epsilon$ and $u_h = u_{\epsilon h}$. Let $w_h \in V_h$ be arbitrary. Then, we have

$$\|\nabla(w_h - u_h)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|w_h - u_h\|_{0,\Omega_1} \leq C \sup_{v_h \in V_h} \frac{(\nabla(w_h - u_h), \nabla v_h)_D + \frac{1}{\epsilon}(w_h - u_h, v_h)_{\Omega_1}}{\|v_h\|_{1,D,\Omega_1,\epsilon}}, \quad (4.7)$$

where

$$\|v_h\|_{1,D,\Omega_1,\epsilon} = \left(\|\nabla v_h\|_{0,D}^2 + \frac{1}{\epsilon}\|v_h\|_{0,\Omega_1}^2\right)^{1/2}. \quad (4.8)$$

We observe

$$\begin{aligned}
&(\nabla(w_h - u_h), \nabla v_h)_D + \frac{1}{\epsilon}(w_h - u_h, v_h)_{\Omega_1} \\
&= (\nabla(w_h - u), \nabla v_h)_D + (\nabla(u - u_h), \nabla v_h)_D + \frac{1}{\epsilon}(w_h - u, v_h)_{\Omega_1} + \frac{1}{\epsilon}(u - u_h, v_h)_{\Omega_1} \\
&= \underbrace{(\nabla(w_h - u), \nabla v_h)_D + \frac{1}{\epsilon}(w_h - u, v_h)_{\Omega_1}}_{=J_1} \\
&\quad \underbrace{+(\nabla u, \nabla v_h)_D + \frac{1}{\epsilon}(u, v_h)_{\Omega_1} - (\nabla u_h, \nabla v_h)_D - \frac{1}{\epsilon}(u_h, v_h)_{\Omega_1,h}}_{=J_2} \\
&\quad \underbrace{-\frac{1}{\epsilon}(u_h, v_h)_{\Omega_1} + \frac{1}{\epsilon}(u_h, v_h)_{\Omega_1,h}}_{=J_3}.
\end{aligned}$$

First,

$$|J_1| \leq \left(\|\nabla(w_h - u)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|w_h - u\|_{0,\Omega_1}\right)\|v_h\|_{1,D,\Omega_1,\epsilon}.$$

Next, since u_h satisfies the equation (4.5), we can deduce from (4.3)

$$\begin{aligned}
|J_2| &= |(\tilde{f}, v_h)_D - (\tilde{f}_h, \hat{v}_h)_D| \\
&\leq C\|\tilde{f}\|_{0,D}\|v_h - \hat{v}_h\|_{0,D} \\
&\leq C\|f\|_{0,\Omega} \cdot Ch\|\nabla v_h\|_{0,D} \\
&\leq Ch\|f\|_{0,\Omega}\|v_h\|_{1,D,\Omega_1,\epsilon}.
\end{aligned}$$

By taking $v_h = u_h$ in (4.5), we can derive

$$\|\nabla u_h\|_{0,D} \le C\|f\|_{0,\Omega}, \qquad \|u_h\|_{0,\Omega_1} \le C\epsilon^{1/2}\|f\|_{0,\Omega}$$

in the same way as the derivation of (2.5).

By using these estimates, together with (4.4), we can perform an estimation as

$$\begin{aligned}
|J_3| &\le \frac{1}{\epsilon}|(u_h, v_h)_{\Omega_1} - (u_h, v_h)_{\Omega_1,h}| \\
&= \frac{1}{\epsilon}|(u_h, v_h)_{\Omega_1} - (\hat{u}_h, \hat{v}_h)_{\Omega_1}| \\
&= \frac{1}{\epsilon}|(u_h - \hat{u}, v_h)_{\Omega_1} - (\hat{u}_h, \hat{v}_h - v_h)_{\Omega_1}| \\
&\le \frac{1}{\epsilon}\Big[|(u_h - \hat{u}, v_h)_{\Omega_1}| + |(\hat{u}_h, \hat{v}_h - v_h)_{\Omega_1}|\Big] \\
&\le \frac{1}{\epsilon}\left(\|u_h - \hat{u}_h\|_{0,\Omega_1}\|v_h\|_{0,\Omega_1} + \|\hat{u}_h\|_{0,\Omega_1}\|v_h - \hat{v}_h\|_{0,\Omega_1}\right) \\
&\le \frac{1}{\epsilon}\left(\|u_h - \hat{u}_h\|_{0,D}\|v_h\|_{0,\Omega_1} + \|\hat{u}_h\|_{0,\Omega_1}\|v_h - \hat{v}_h\|_{0,D}\right) \\
&\le \frac{Ch}{\epsilon}\left(\|\nabla u_h\|_{0,D}\|v_h\|_{0,\Omega_1} + \|\hat{u}_h\|_{0,\Omega_1}\|\nabla v_h\|_{0,D}\right) \\
&\le \frac{Ch}{\sqrt{\epsilon}}\|f\|_{0,\Omega}\cdot\frac{1}{\sqrt{\epsilon}}\|v_h\|_{\Omega_1} + \frac{Ch}{\epsilon}\cdot\epsilon^{-1/2}\|f\|_{0,\Omega}\|\nabla v_h\|_{0,D} \\
&\le Ch\epsilon^{-\frac{1}{2}}\|f\|_{0,\Omega}\|v_h\|_{1,D,\Omega_1,\epsilon}.
\end{aligned}$$

Summing up those estimates and using the triangle inequality, we get

$$\begin{aligned}
&\|\nabla(u - u_h)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u - u_h\|_{0,\Omega_1} \\
&\quad\le \|\nabla(u - w_h)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u - w_h\|_{0,\Omega_1} + \|\nabla(w_h - u_h)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|w_h - u_h\|_{0,\Omega_1} \\
&\quad\le C(h + h\epsilon^{-1/2})\|f\|_{0,\Omega} + C\left(\|\nabla(w_h - u)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|w_h - u\|_{0,\Omega_1}\right).
\end{aligned}$$

Now choosing $w_h = I_h u$ (= the standard P1 Lagrange interpolation of u) and then using (2.9) and (2.10), we obtain

$$\begin{aligned}
\|\nabla(u - I_h u)\|_{0,D}^2 &= \|\nabla(u - I_h u)\|_{0,\Omega}^2 + \|\nabla(u - I_h u)\|_{0,\Omega_1}^2 \\
&\le Ch^2|u|_{2,\Omega}^2 + Ch^2|u|_{2,\Omega_1}^2 \\
&\le Ch^2\|f\|_{0,\Omega}^2 + Ch^2\epsilon^{-\frac{1}{2}}\|f\|_{0,\Omega}^2, \qquad (4.9) \\
\frac{1}{\epsilon}\|u - I_h u\|_{0,\Omega_1}^2 &\le \frac{1}{\epsilon}C(h^4 + h^4\epsilon^{-\frac{1}{2}})\|f\|_{0,\Omega}^2. \qquad (4.10)
\end{aligned}$$

In conclusion, we have

$$\begin{aligned}
&\|\nabla(u - u_h)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u - u_h\|_{0,\Omega_1} \\
&\qquad\qquad \le C(h + h\epsilon^{-1/4} + h\epsilon^{-1/2} + h^2\epsilon^{-3/4})\|f\|_{0,\Omega},
\end{aligned}$$

which completes the proof. □

Remark 4.2. We suppose $\epsilon = h^\alpha$ with $\alpha > 0$. Then, for $h \ll 1$,

$$\|u - u_{\epsilon h}\|_{1,\Omega} \le \begin{cases} Ch^{\alpha/4}\|f\|_{0,\Omega} & (0 < \alpha \le 4/3) \\ Ch^{1-\alpha/2}\|f\|_{0,\Omega} & (4/3 < \alpha < 2). \end{cases} \tag{4.11}$$

The optimal choice of α is $\alpha = 4/3$; then, $\|u - u_{\epsilon h}\|_{1,\Omega} \le Ch^{1/3}\|f\|_{0,\Omega}$. When $\alpha \ge 2$, the error estimate is meaningless.

5 Finite volume approximation

This section is devoted to analysis of the finite volume approximation. We set

$$\hat{\mathcal{T}}_h = \{D_P\}_{P \in \mathcal{P}_h}, \qquad \hat{\mathcal{T}}_h^0 = \{D_P\}_{P \in \mathcal{P}_h^0}.$$

Following the standard notion of the finite volume method (cf. [3]), we write $K \in \hat{\mathcal{T}}_h$ instead of $D_P \in \hat{\mathcal{T}}_h$ to express a general *control volume*. Further, for $K \in \hat{\mathcal{T}}_h$, we write $x_K \in \mathcal{P}_h$ instead of $P \in \mathcal{P}_h$ and call x_K the *corresponding point* to K.

Then, $\hat{\mathcal{T}}_h$ is an actually admissible mesh of D. That is, the following conditions (A1)–(A4) are satisfied:

(A1) Any $K \in \mathcal{T}$ is a convex polyhedral domain and $\overline{D} = \overline{\bigcup_{K \in \hat{\mathcal{T}}_h} K}$;

(A2) For any $K, L \in \hat{\mathcal{T}}_h$ with $K \ne L$, either the $(d-1)$ dimensional Lebesgue measure of $\overline{K} \cap \overline{L}$ is zero or $\overline{K} \cap \overline{L}$ is an entire common side (edge, face) σ. Below we write $\sigma = K|L$ to express the latter case;

(A3) For $K \in \hat{\mathcal{T}}_h$, the corresponding point $x_K \in \mathcal{P}_h$ is in $\overline{K}$. Further, if $\sigma = K|L$, that $x_k \ne x_L$ and the line segment connecting x_K with x_L is orthogonal to σ;

(A4) For $K \in \hat{\mathcal{T}}_h \backslash \hat{\mathcal{T}}_h^0$, we have $x_K \in \partial D$.

Moreover, we use the following notation.

- Each $K \in \hat{\mathcal{T}}_h$ is called a *control volume*. The d dimensional Lebesgue measure of K is denoted by m_K. The set of all sides of $\overline{K}$ is denoted by $\mathcal{E}_K$.

  We set $\mathcal{E} = \bigcup_{K \in \hat{\mathcal{T}}_h} \mathcal{E}_K$, $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E} \mid \sigma \subset D\}$, and $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E} \mid \sigma \subset \partial D\}$.
  Obviously, $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$. For $\sigma \in \mathcal{E}$, the $(d-1)$ dimensional Lebesgue measure of σ is denoted by m_σ.

- For $K \in \hat{\mathcal{T}}_h^0$, the neighbours of K is defined as $\mathcal{N}(K) = \{L \in \hat{\mathcal{T}}_h \mid \sigma = K|L\}$.

- For $\sigma = K|L \in \mathcal{E}$, the distance between x_K and x_L is denoted by d_σ, while the distance between x_K and σ is denoted by $d_{K,\sigma}$.

- The transmissibility coefficient is defined as $\tau_\sigma = m_\sigma/d_\sigma$ for $K \in \hat{\mathcal{T}}_h^0$, $L \in \mathcal{N}(K)$ with $\sigma = K|L$.

- As the size parameter, we continue to employ h. (In [3], h is denoted by size $(\hat{\mathcal{T}}_h)$.)

- Below, we write as $v_K = v|_K = v(x_K)$ for $v \in \hat{X}_h$ and $K \in \hat{\mathcal{T}}_h$.

We introduce interpolation operators $P_h : L^1(D) \to \hat{V}_h$ and $Q_h : C(\overline{D}) \to \hat{V}_h$ which are defined by

$$P_h v \in \hat{V}_h, \quad (P_h v_h)|_K = \frac{1}{m_K}\int_K v(x)\ dx \quad (v \in L^1(D)),$$
$$Q_h w \in \hat{V}_h, \quad (Q_h w)|_K = w(x_K) \quad (w \in H^2(D)).$$

Now we are able to derive the finite volume scheme to $-\Delta u_\epsilon + (1/\epsilon)\chi u_\epsilon = \tilde{f}$ in D with $u_\epsilon|_{\partial D} = 0$, where $\chi \in L^\infty(D)$ denotes the characteristic function of Ω_1 defined as (2.6). First, we integrate the equation over $K \in \hat{\mathcal{T}}_h^0$ to obtain

$$-\int_K \Delta u_\epsilon\ dx + \frac{1}{\epsilon}\int_K \chi u_\epsilon\ dx = \int_K \tilde{f}\ dx.$$

In the finite volume method, the Laplace operator is approximated as

$$\int_K \Delta u_\epsilon\ dx = \sum_{L\in\mathcal{N}(K),\sigma=K|L} \int_\sigma \nabla u_\epsilon \cdot n_{K,\sigma}\ dS \approx \sum_{L\in\mathcal{N}(K),\sigma=K|L} \tau_\sigma[u_\epsilon(x_L) - u_\epsilon(x_K)],$$

where $n_{K,\sigma}$ is the unit normal vector to σ outgoing from K. Other terms are treated as

$$\begin{aligned} \frac{1}{\epsilon}\int_K \chi u_\epsilon\ dx &\approx \frac{1}{\epsilon}\int_K \chi u_\epsilon(x_K)\ dx \approx \frac{1}{\epsilon} u_\epsilon(x_K)\hat{\chi}_K m_K; \\ \int_K \tilde{f}\ dx &= \hat{f}_K m_k, \end{aligned}$$

where

$$\hat{\chi}_h = P_h \chi, \qquad \hat{f}_h = P_h \tilde{f}.$$

Summing up, we can state the finite volume scheme to (2.3) as follows:

$$\begin{cases} \text{Find } \hat{u}_{\epsilon h} \in \hat{V}_h \text{ such that} \\ -\displaystyle\sum_{L\in\mathcal{N}(K),\sigma=K|L} \tau_\sigma(\hat{u}_{\epsilon,L} - \hat{u}_{\epsilon,K}) + \frac{1}{\epsilon}\hat{\chi}_K \hat{u}_K m_K = \hat{f}_K m_K \quad (\forall K \in \hat{\mathcal{T}}_h^0). \end{cases} \tag{5.1}$$

We then derive another expression to (5.1). Multiplying by $\hat{v}_K$ the both sided of the above identity and summing up all $K \in \hat{\mathcal{T}_h}^0$, we obtain

$$-\sum_{K\in\hat{\mathcal{T}}_h} \hat{v}_K \sum_{L\in\mathcal{N}(K)} \gamma_{K|L}(\hat{u}_{\epsilon,L} - \hat{u}_{\epsilon,K}) + \frac{1}{\epsilon}\sum_{K\in\hat{\mathcal{T}}_h} \hat{\chi}_K \hat{u}_K \hat{v}_K m_K = \sum_{K\in\hat{\mathcal{T}}_h} \hat{f}_K \hat{v}_K m_K.$$

We can calculate as

$$\sum_{K\in\hat{\mathcal{T}}_h} \hat{\chi}_K \hat{u}_K \hat{v}_K m_K = \sum_{K\in\hat{\mathcal{T}}_h} \int_K \chi \hat{u}_h \hat{v}_h \, dx = (\hat{u}_h, \hat{v}_h)_{\Omega_1},$$

$$\sum_{K\in\hat{\mathcal{T}}_h} \hat{f}_K \hat{v}_K m_K = (\tilde{f}, \hat{v}_h)_D.$$

Thus, setting

$$a_h(\hat{v}_h, \hat{w}_h) = -\sum_{K\in\hat{\mathcal{T}}_h} \hat{w}_K \sum_{L\in\mathcal{N}(K)} \gamma_{K|L}(\hat{v}_L - \hat{v}_K) \quad (\hat{v}_h, \hat{w}_h \in \hat{V}_h), \tag{5.2}$$

we obtain an equivalent expression to (5.1) as follows:

$$\begin{cases} \text{Find } \hat{u}_{\epsilon h} \in \hat{V}_h \text{ such that} \\ a_h(\hat{u}_{\epsilon h}, \hat{v}_h) + \dfrac{1}{\epsilon}(\hat{u}_{\epsilon h}, \hat{v}_h)_{\Omega_1} = (\tilde{f}, \hat{v}_h)_D \quad (\forall \hat{v}_h \in \hat{V}_h). \end{cases} \tag{5.3}$$

At this stage, we recall the following well-known result (cf. [8, Corollary 6.9], [7, §3]):

Lemma 5.1. *We have $a_h(\hat{v}_h, \hat{w}_h) = (\nabla u_h, \nabla v_h)_D$ for $\hat{v}_h, \hat{w}_h \in \hat{V}_h$ with $v_h = M_h^{-1}\hat{v}_h \in V_h$ and $w_h = M_h^{-1}\hat{w}_h \in V_h$.*

By taking into account of this fact, we see that problem (5.1) is equivalently written as (4.5). Thus, we have the following Lemma.

Lemma 5.2. *Let $u_\epsilon \in H_0^1(D)$ and $\hat{u}_{\epsilon h} \in \hat{V}_h$ be solutions of* (2.3) *and* (5.1)*, respectively. Set $u_{\epsilon h} = M_h^{-1}\hat{u}_{\epsilon h}$. Then, we have*

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega_1} \le C(h + h\epsilon^{-1/4} + h\epsilon^{-1/2} + h^2\epsilon^{-3/4})\|f\|_{0,\Omega}. \tag{5.4}$$

At this stage, we introduce a discrete H_0^1 norm defined as

$$|\hat{v}_h|_{1,D,h} = a_h(\hat{v}_h, \hat{v}_h)^{\frac{1}{2}} \qquad (\hat{v}_h \in \hat{V}_h). \tag{5.5}$$

An error estimate for the finite volume approximation is given as follows.

Theorem 5.3. *Let u_ϵ and $\hat{u}_{\epsilon h}$ be solutions of* (2.3) *and* (5.1)*, respectively. Set $\hat{e}_h = Q_h u_\epsilon - \hat{u}_h \in \hat{V}_h$. Then, we have*

$$|\hat{e}_h|_{1,D,h} \le C(h + h\epsilon^{-1/4} + h\epsilon^{-1/2} + h^2\epsilon^{-3/4})\|f\|_{0,\Omega}. \tag{5.6}$$

Proof. Set $e_h = M_h^{-1}\hat{e}_h \in V_h$. Then, it is explicitly given as $e_h = I_h u_\epsilon - u_h$ with $u_h = M_h^{-1}\hat{u}_h$. Hence, in view of Lemmas 5.1 and 5.3, we have by using (4.9)

$$\begin{aligned} |\hat{e}_h|_{1,D,h} &= a_h(\hat{e}_h, \hat{e}_h)^{\frac{1}{2}} = (\nabla e_h, \nabla e_h) \\ &\le \|\nabla(I_h u_\epsilon - u_\epsilon)\|_{0,D} + \|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} \\ &\le C(h + h\epsilon^{-1/4} + h\epsilon^{-1/2} + h^2\epsilon^{-3/4})\|f\|_{0,\Omega}, \end{aligned}$$

which completes the proof. □

A Proof of Lemma 2.4

It suffices to consider the case $\Omega = \mathbb{R}^N_+$, since then the general case is proved by the standard argument by using partition of the unity and localization technique (see, for example, [15, §20]).

We suppose that $\hat{h}(\xi')$ is the Fourier transform of a function $h(x_1, \ldots, x_{N-1})$, where $\xi' = (\xi_1, \ldots, \xi_{N-1})$. Similarly, let $\hat{w}(\xi)$ be the Fourier transform of a function $w(x)$ in variables $(x_1, \ldots, x_{N-1})$, where $\xi = (\xi', x_N)$. We apply the extension formula in [11, Theorem 5.2, Chapter 2] with a slightly modification. Thus, we propose

$$\hat{v}(\xi', x_N) = x_N \exp\left(-(1 + |\xi'|)\eta^{-2}, x_N\right) \hat{g}(\xi'). \tag{1.1}$$

Indeed, let $|\alpha| \leq 2$, let us consider $w_\alpha = \partial^\alpha v$ in $\mathbb{R}^N_+$ and set $w_\alpha = 0$ for $x_N < 0$. Let us denote $\alpha = (\alpha_1, \ldots, \alpha_N,)$, and $\alpha = (\alpha', \alpha_N)$. Hence $\hat{w_\alpha}(\xi)$ is a finite sum of expressions like

$$aI(\xi) = a \int_0^\infty e^{(-ix_N\xi_N)} (\xi')^{\alpha'} ((1 + |\xi'|)\eta^{-2})^{\alpha_N - j} x_N^{1-j} \cdot \\ \exp\left(-(1 + |\xi'|)\eta^{-2}, x_N\right) \hat{g}(\xi') dx_N,$$

where a is a constant, $j = 0, 1$. We have:

$$I(\xi) = \frac{(\xi')^{\alpha'} ((1 + |\xi'|)\eta^{-2})^{\alpha_N - j} \hat{g}(\xi')}{((1 + |\xi'|)\eta^{-2} + i\xi_N)^{2-j}},$$

and so

$$\begin{aligned} \|I(\xi)\|^2_{0,\mathbb{R}^N} &= C \int_{\mathbb{R}^{N-1}} (\xi')^{2\alpha'} ((1 + |\xi'|)\eta^{-2})^{2\alpha_N - 3} |\hat{g}(\xi')|^2 d\xi' \\ &\leq \begin{cases} C\eta^{-2} \|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 2, \\ C\eta^{2} \|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 1, \\ C\eta^{6} \|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 0. \end{cases} \end{aligned}$$

This completes the proof. □

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012