UTMS 2013–9 July 11, 2013

A second-order time-discretization scheme for a system of nonlinear Schrödinger equations

by

Takiko SASAKI

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

A SECOND-ORDER TIME-DISCRETIZATION SCHEME FOR A SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS

Takiko Sasaki[1]

Graduate School of Mathematical Sciences, The University of Tokyo
3-8-1 Komaba Meguro-ku Tokyo 153-8914, Japan

Keywords. time-discretization, finite difference method, nonlinear Schrödinger equation, error analysis

AMS subject classifications. 65L12, 65G99, 35Q55

Abstract

We study a linearly semidiscrete-in-time finite difference method for the system of nonlinear Schrödinger equations that is a model of the interaction of a non-relativistic particles with different masses. The main aim is to show that the scheme is second-order convergent.

1 Introduction and main results

We consider the following system of nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u + \alpha\Delta u = \lambda\bar{u}v, \ t \geq 0, \ x \in \mathbb{R}^d, \\ i\partial_t v + \beta\Delta v = \mu u^2, \ t \geq 0, \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where u and v are complex-valued functions, Δ is the Laplacian in $\mathbb{R}^d$, α and β are positive constants, and λ and μ are complex constants. This system is a model of the interaction of a non-relativistic particles with different masses.

The mathematical study for (1) is well developed. Throghout this paper, we suppose that

$$s > d/2, \ s : \text{integer}.$$

For some $s_1 \geq s$, there exists a constant $T^* = T^*(u_0, v_0) \in (0, \infty]$ such that the system (1) admits a unique maximal solution

$$(u, v) \in C^{s_1}([0, T^*); H^{s_1}(\mathbb{R}^d)),$$

for any initial data $(u_0, v_0) \in H^{s_1}(\mathbb{R}^d)$; see, e.g., Cazenave [3]. Moreover, the aymptotic profiles of solutions of (1) are studied, for example, in [5] and [6].

In this paper, we are concerned with a time discretization method for (1). As is well-known, we need to consider implicit schemes to obtain stable numerical solutions for Schrödinger equations. Especially, the Crank-Nicolson scheme is useful and widely applied, since it is stable and second order convergent. However, if applying the Crank-Nicolson scheme to a nonlinear Schrödinger equation, we deduce a nonlinear elliptic equation at each time step as the the resulting equation in order to maintain the second order convergence (cf. [1], [4]). On a consequence, we meet another difficulty for solving nonlinear elliptic equations. This can be quite time-consuming when the size of a fully discretized problem

[1]tsasaki@ms.u-tokyo.ac.jp

is very large. In this connection, Besse's relaxation scheme ([2]) is a method worthy of note. He considers a nonlinear Schrödinger equation and studies a linear scheme by considering both the main time step t_n and the dual one $t_{n+1/2}$. Here, by a linear scheme, we mean a time discretization method whose resulting equations consist of linear elliptic equations. His relaxation scheme is shown to be convergent but the proof of the second order convergence is still open at present.

In this paper, we propose a linear scheme for (1) that is motivated by the relaxation scheme. The main contribution of this paper is to show that it is actually second order convergent. As stated above, we restrict our attention within a time discretization scheme and not discuss about space discretizations. However, the resulting equations of our scheme is linear so that the standard space discretization methods, for example, the finite difference, finite element, spectral methods are readily applicable. As a matter of fact, fully discrete schemes for (1) by those methods under various boundary conditions will be studied in forthcoming papers. Moreover, numerical examples will be reported there.

Now let us state the time discretization scheme for (1) to be considered. let h be a time step size. We propose the following scheme for (1).

$$\begin{cases} i\dfrac{u^{n+\frac{3}{2}}-u^{n+\frac{1}{2}}}{h}+\alpha\Delta\dfrac{u^{n+\frac{3}{2}}+u^{n+\frac{1}{2}}}{2}=\lambda\left(\overline{\dfrac{u^{n+\frac{3}{2}}+u^{n+\frac{1}{2}}}{2}}\right)v^{n+1}, \\ i\dfrac{v^{n+1}-v^{n}}{h}+\beta\Delta\dfrac{v^{n+1}+v^{n}}{2}=\mu(u^{n+\frac{1}{2}})^2 \end{cases} \tag{2}$$

for $n=0,1,2,\ldots$ Namely the first and the second equations of (1) are discretized at times $t_{n+1}=(n+1)h$ and $t_{n+\frac{1}{2}}=(n+\frac{1}{2})h$, respectively.

The scheme (2) consists of two linear equations for solving $u^{n+3/2}$ and v^{n+1} at each time step. More specifically, the first equation of (2) is equivalently written as

$$K_{n+1}\begin{pmatrix} u^{n+\frac{3}{2}} \\ \bar{u}^{n+\frac{3}{2}} \end{pmatrix}=\begin{pmatrix} \left(1+i\frac{\alpha h}{2}\Delta\right)u^{n+\frac{1}{2}}-i\frac{\lambda h}{2}\bar{u}^{n+\frac{1}{2}}v^{n+1} \\ \left(1-i\frac{\alpha h}{2}\Delta\right)\bar{u}^{n+\frac{1}{2}}+i\frac{\lambda h}{2}u^{n+\frac{1}{2}}\bar{v}^{n+1} \end{pmatrix}$$

where

$$K_{n+1}=\begin{pmatrix} 1-i\frac{\alpha h}{2}\Delta & i\frac{\lambda h}{2}v^{n+1} \\ -i\frac{\lambda h}{2}\bar{v}^{n+1} & 1+i\frac{\alpha h}{2}\Delta \end{pmatrix}.$$

Since the operator K_{n+1} is defined in terms of the solution v^{n+1} at the previous time step, it is not certain that K_{n+1} is invertible at this stage. However, as we will state in Propositon 1 and Theorem 2 below, the scheme (2) has a unique solution in $t_{n+3/2}<T^*$ for a suitably chosen h so that K_{n+1} is actually invertible.

Below, we use the usual Lebesgue spaces $L^2=L^2(\mathbb{R}^d)$, $L^\infty=L^\infty(\mathbb{R}^d)$ and Sobolev spaces $H^k=H^k(\mathbb{R}^d)$ for an integer k together with their standard norms. We write as

$$\|\cdot\|_{L^2}=\|\cdot\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\cdot\|_{H^s}=\|\cdot\|_{H^s(\mathbb{R}^d)}.$$

First, we state the following local stability result which plays an important role.

Proposition 1 *Let a, $b \in H^s$, and put $R \geq \|a\|_{H^s} + \|b\|_{H^s}$. Then there exist a constant $T_R > 0$, which depends only on R, λ, μ, s and d, a unique solution $(u^{n+\frac{1}{2}}, v^n)_n$ of (2) with initial condition $(u^{\frac{1}{2}}, v^0) = (a, b)$ satisfies*

$$\|u^{n+\frac{1}{2}}\|_{H^s} + \|v^n\|_{H^s} \leq 2R \tag{3}$$

for all $n \in \mathbb{N}$ with $nh \leq T_R$.

It should be kept in mind that, since $h \in (0, T_{R/2}]$, the set $\{n \geq 1 \mid nh \leq T_R\}$ is not an empty set. This proposition will be proved in section 2, after having prepared a few prelimimary results.

We are now in a position to state the main results of this paper.

Theorem 2 *Let $s \in \mathbb{N}$ and $s > d/2$. Let u_0, $v_0 \in H^{s+6}$, and let $T^* = T^*(u_0, v_0)$ be the maximal existence time of the solution (u, v) of (1) as mentioned before. Then (u, v) further satisfies*

$$(u, v) \in \bigcap_{k=0}^{3} C^k\left([0, T^*); \left(H^{s+6-2k}\right)^2\right). \tag{4}$$

Let $T \in (0, T^*)$ be arbitary, and set $M^* = \max_{0 \leq k \leq 3}\{M_k\}$, where

$$M_k = \max_{t \in [0,T]} \left\{\|\partial_t^k u(t)\|_{H^{s+6-2k}} + \|\partial_t^k v(t)\|_{H^{s+6-2k}}\right\} \quad (k = 0, 1, 2, 3). \tag{5}$$

Moreover, let $(u^{n+\frac{1}{2}}, v^n)_n$ be the solution of (2) with initial condition

$$u^{\frac{1}{2}} = u_0 + \frac{ih}{2}(\alpha\Delta u_0 - \lambda\overline{u_0}v_0), \;\; v^0 = v_0. \tag{6}$$

Then there exist positive constants h_0 and K_0, which depend only on $\alpha, \beta, \lambda, \mu$, T and M_0, such that the problem (2) is solvable and the solution $(u^{n+\frac{1}{2}}, v^n)_n$ satisfies

$$\|u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}\|_{H^s} + \|v(t_n) - v^n\|_{H^s} \leq K_0 h^2 \tag{7}$$

for all $h \in (0, h_0)$ and $n \in \mathbb{N}$ satisfying $(n+1)h \leq T$.

2 Proof of Proposition 1

First, we collect preliminary results, which we use in the proof of Proposition 1 and Theorem 2. We introduce operators, for positive constants α and β,

$$A_\alpha = \left(I + i\frac{\alpha h}{2}\Delta\right)\left(I - i\frac{\alpha h}{2}\Delta\right)^{-1}, \quad B_\beta = \left(I - i\frac{\beta h}{2}\Delta\right)^{-1}$$

which are primary defined on L^2, where I denotes the identity operator in L^2. An application of the Fourier transformation, we can deduce the following lemma.

Lemma 1 *1. A_α is a unitary operator on H^s and we can write*

$$A_\alpha = \left(I - i\frac{\alpha h}{2}\Delta\right)^{-1}\left(I + i\frac{\alpha h}{2}\Delta\right).$$

2. B_β is a bounded operator on H^s.

The following estimates are readily obtainable cosequeces of the Taylor's theorem.

Lemma 2 *1. Let $f(t) \in C^3([0,T];H^s)$, $h > 0$, $t+h \in [0,T]$, and $t-h \in [0,T]$. Then we have*

$$\|h^{-1}(f(t+h) - f(t-h)) - 2\partial_t f(t)\|_{H^s} \le \frac{1}{3}\|f\|_{C^3([0,T];H^s)}h^2.$$

2. Let $f(t) \in C^2([0,T];H^s)$, $h \ge 0$, $t+h \in [0,T]$, and $t-h \in [0,T]$. Then we have

$$\|f(t+h) + f(t-h) - 2f(t)\|_{H^s} \le \|f\|_{C^2([0,T];H^s)}h^2.$$

We will make use of the well-known Sobolev inequality.

Lemma 3 *There exists a positive constant C which depends only on d and s such that*

$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}$$

for all u, $v \in H^s$.

Now we can state the following proof.

Proof of Proposition 1. It is based on the contraction mapping principle. Let $a, b \in H^s$ be arbitrary and set $R \ge \|a\|_{H^s} + \|b\|_{H^s}$.

First, Equation (2) with initial condition $(u^{\frac{1}{2}}, v^0) = (a, b)$ can be written in the following form with Duhamel's principle:

$$\begin{cases} u^{n+\frac{3}{2}} = A_\alpha^{n+1}a - i\lambda h \displaystyle\sum_{j=0}^{n} A_\alpha^{n-j} B_\alpha \left(\overline{\frac{u^{j+\frac{3}{2}} + u^{j+\frac{1}{2}}}{2}} \right) v^{j+1}, \\ v^{n+1} = A_\beta^{n+1}b - i\mu h \displaystyle\sum_{j=0}^{n} A_\beta^{n-j} B_\beta \left(u^{j+\frac{1}{2}}\right)^2 \end{cases} \tag{8}$$

for $n = 0, 1, 2, \ldots$

For the time being, we fix $N \in \mathbb{N}$ and set $\hat{N} = \{1, 2, \ldots, N\}$. Then, we consider a Banach space

$$\mathcal{X}_N = \{(w^{n+\frac{1}{2}}, \hat{w}^n)_{n\in\hat{N}} \mid w^{n+\frac{1}{2}}, \hat{w}^n \in H^s \ \ \forall n \in \hat{N}\}$$

with the norm

$$\|(w^{n+\frac{1}{2}}, \hat{w}^n)_n\|_{\mathcal{X}_N} = \sup_{n\in\hat{N}} \left(\|w^{n+\frac{1}{2}}\|_{H^s} + \|\hat{w}^n\|_{H^s} \right),$$

for $(w^{n+\frac{1}{2}}, \hat{w}^n)_n \in \mathcal{X}_N$. We introduce $\mathcal{T} : \mathcal{X}_N \to \mathcal{X}_N$ by setting

$$(\tilde{u}^{n+\frac{1}{2}}, \tilde{v}^n)_n = \mathcal{T}(u^{n+\frac{1}{2}}, v^n)_n, \tag{9}$$

where

$$\begin{cases} \tilde{u}^{n+\frac{3}{2}} = A_\alpha^{n+1}a - i\lambda h \displaystyle\sum_{j=0}^{n} A_\alpha^{n-j} B_\alpha \left(\overline{\frac{u^{j+\frac{3}{2}} + u^{j+\frac{1}{2}}}{2}} \right) v^{j+1}, \\ \tilde{v}^{n+1} = A_\beta^{n+1}b - i\mu h \displaystyle\sum_{j=0}^{n} A_\beta^{n-j} B_\beta \left(u^{j+\frac{1}{2}}\right)^2 \end{cases} \tag{10}$$

for $n = 0, 1, ..., N-1$. Here, we set $u^{\frac{1}{2}} = a$.

Below, we will show that $\mathcal{T}$ is a contraction operator from a closed ball $\mathcal{B}_{2R}$ into itself, with a suitably chosen h, where

$$\mathcal{B}_{2R} = \left\{ (w^{n+\frac{1}{2}}, \hat{w}^n)_n \in \mathcal{X}_N \mid \|(w^{n+\frac{1}{2}}, \hat{w}^n)_n\|_{\mathcal{X}_N} \le 2R \right\}.$$

First, let $(u^{n+\frac{1}{2}}, v^n)_n \in \mathcal{B}_{2R}$, and set $(\tilde{u}^{n+\frac{1}{2}}, \tilde{v}^n)_n = \mathcal{T}(u^{n+\frac{1}{2}}, v^n)_n$. By using Lemma 1 and Lemma 3, we have

$$\begin{aligned}
\|\tilde{u}^{n+\frac{3}{2}}\|_{H^s} &\le \|a\|_{H^s} + Ch \sum_{j=0}^{n} \left(\|u^{j+\frac{3}{2}}\|_{H^s} + \|u^{j+\frac{1}{2}}\|_{H^s} \right) \|v^{j+1}\|_{H^s} \\
&\le \|a\|_{H^s} + CNhR^2, \\
\|\tilde{v}^{n+1}\|_{H^s} &\le \|b\|_{H^s} + Ch \sum_{j=0}^{n} \|u^{j+\frac{1}{2}}\|_{H^s}^2 \le \|b\|_{H^s} + CNhR^2.
\end{aligned}$$

for $n = 0, 1, \ldots, N-1$. Hence there exists a positive constant C_1, which depends only on d, s, λ and μ, such that

$$\|\tilde{u}^{n+\frac{3}{2}}\|_{H^s} + \|\tilde{v}^{n+1}\|_{H^s} \le R + C_1 NhR^2$$

for all $n = 0, 1, \ldots, N-1$. Therefore, if

$$C_1 NhR \le 1 \tag{11}$$

then we have $\|(\tilde{u}^{n+\frac{1}{2}}, \tilde{v}^n)_n\|_{\mathcal{X}_N} \le 2R$, which implies that $\mathcal{T}(\mathcal{B}_{2R}) \subset \mathcal{B}_{2R}$.

Next, let $(u_1^{n+\frac{1}{2}}, v_1^n)_n$, $(u_2^{n+\frac{1}{2}}, v_2^n)_n \in \mathcal{B}_{2R}$, and let

$$(\tilde{u}_1^{n+\frac{1}{2}}, \tilde{v}_1^n)_n = \mathcal{T}(u_1^{n+\frac{1}{2}}, v_1^n)_n, \quad (\tilde{u}_2^{n+\frac{1}{2}}, \tilde{v}_2^n)_n = \mathcal{T}(u_2^{n+\frac{1}{2}}, v_2^n)_n.$$

Then, we have

$$\begin{aligned}
&\|\tilde{u}_1^{n+\frac{3}{2}} - \tilde{u}_2^{n+\frac{3}{2}}\|_{H^s} \\
&\le Ch \sum_{j=0}^{n} \Big\{ \left(\|u_1^{j+\frac{3}{2}} - u_2^{j+\frac{3}{2}}\|_{H^s} + \|u_1^{j+\frac{1}{2}} - u_2^{j+\frac{1}{2}}\|_{H^s} \right) \|v^{j+1}\|_{H^s} \\
&\qquad\qquad + \left(\|u_2^{j+\frac{3}{2}}\|_{H^s} + \|u_2^{j+\frac{1}{2}}\|_{H^s} \right) \|v_1^{j+1} - v_2^{j+1}\|_{H^s} \Big\} \\
&\le CNhR \|(u_1^{k+\frac{1}{2}}, v_1^k)_k - (u_2^{k+\frac{1}{2}}, v_2^k)_k\|_{\mathcal{X}_N},
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{v}_1^{n+1} - \tilde{v}_2^{n+1}\|_{H^s} &\le Ch \sum_{j=0}^{n} \left(\|u_1^{j+\frac{1}{2}}\|_{H^s} + \|u_2^{j+\frac{1}{2}}\|_{H^s} \right) \|u_1^{j+\frac{1}{2}} - u_2^{j+\frac{1}{2}}\|_{H^s} \\
&\le CNhR \|(u_1^{k+\frac{1}{2}}, v_1^k)_k - (u_2^{k+\frac{1}{2}}, v_2^k)_k\|_{\mathcal{X}_N}
\end{aligned}$$

for $n = 0, 1, \ldots, N-1$. Hence there exists a positive constant C_2, which depends only on d, s, λ and μ, such that

$$\begin{aligned}
&\|\tilde{u}_1^{n+\frac{3}{2}} - \tilde{u}_2^{n+\frac{3}{2}}\|_{H^s} + \|\tilde{v}_1^{n+1} - \tilde{v}_2^{n+1}\|_{H^s} \\
&\le C_2 NhR \|(u_1^{k+\frac{1}{2}}, v_1^k)_k - (u_2^{k+\frac{1}{2}}, v_2^k)_k\|_{\mathcal{X}_N}
\end{aligned}$$

for all $n = 0, 1, ..., N-1$. Therefore, if

$$C_2 NhR \leq \frac{1}{2},$$

then we have

$$\|\mathcal{T}(u_1^{k+\frac{1}{2}}, v_1^k)_k - \mathcal{T}(u_2^{k+\frac{1}{2}}, v_2^k)_k\|_{\mathcal{X}_N} \leq \frac{1}{2}\|(u_1^{k+\frac{1}{2}}, v_1^k)_k - (u_2^{k+\frac{1}{2}}, v_2^k)_k\|_{\mathcal{X}_N},$$

which implies that $\mathcal{T} : \mathcal{B}_{2R} \to \mathcal{B}_{2R}$ is a contraction mapping. At this stage, we define as

$$T_R = \min\left\{\frac{1}{C_1 R}, \frac{1}{2C_2 R}\right\}.$$

Moreover, from now on, and chose N as $N = \max\{n \mid n \geq 1, nh \leq T_R\}$. Then, the mapping $\mathcal{T}$ turns out to be a cotraction mapping of $B_{2R} \to B_{2R}$. As the result, $\mathcal{T}$ has a unique fixed point $(u^{n+\frac{1}{2}}, v^n)_{n\in\hat{N}}$ which obviously satisfies (10) and (3) for $1 \leq n \leq N$. This completes the proof of Proposition 1. $\square$

3 Proof of Theorem 2

This section is devoted to the proof of Theorem 2.

Let u_0, $v_0 \in H^{s+6}$ and let $(u^{n+\frac{1}{2}}, v^n)_n$ be the solution of (2) with initial condition (6). Then there exists a positive constant C^* which depends s, d and α, such that

$$\|u^0\|_{H^s} + \|v^0\|_{H^s} \leq C^* M^*(1 + M^*)$$

Put $M' := \max\{M^*, C^*(M^* + 1)M^*\}$. From Propsition 1, there exists a constant $T_{M'} > 0$, which depends only on R, λ, μ, s and d, a unique solution $(u^{n+\frac{1}{2}}, v^n)_n$ of (2) with initial condition (6) satisfies

$$\|u^{n+\frac{1}{2}}\|_{H^s} + \|v^n\|_{H^s} \leq 2M'$$

for all $n \in \mathbb{N}$ with $nh \leq T_{M'}$. We define

$$\nu_h = \sup\{n \in \mathbb{N} \mid \|u^{n+\frac{1}{2}}\|_{H^s} + \|v^n\|_{H^s} \leq 3M'\}.$$

We divide the proof into two steps.

Step 1. First, we show that there exist positive constants h_1 and K_0, which depend only on T and M_0, such that the estimate (7) holds for all $h \in (0, h_1)$ and $n \in \mathbb{N}$ satisfying

$$(n+1)h \leq T, \quad n \leq \nu_h. \tag{12}$$

We define as

$$N_h = \min\left\{\left[\frac{T}{h}\right] - 1, \nu_h\right\},$$

where $[T/h]$ denotes the integer part of T/h.

For $n = 0, 1, 2, ...,$ we set

$$\theta^{n+\frac{1}{2}} = u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}, \quad \rho^n = v(t_n) - v^n.$$

Then we have

$$\theta^{n+\frac{3}{2}} - \theta^{n+\frac{1}{2}} - i\frac{\alpha h}{2}\Delta(\theta^{n+\frac{3}{2}} + \theta^{n+\frac{1}{2}}) = ih\Phi^{n+1},$$

or equivalently,

$$\theta^{n+\frac{3}{2}} = A_\alpha \theta^{n+\frac{1}{2}} + ihB_\alpha \Phi^{n+1},$$

where $\Phi^{n+1} = \phi_1^{n+1} + \phi_2^{n+1} + \phi_3^{n+1}$,

$$\phi_1^{n+1} = i\left\{\partial_t u(t_{n+1}) - \frac{u(t_{n+\frac{3}{2}}) - u(t_{n+\frac{1}{2}})}{h}\right\},$$

$$\phi_2^{n+1} = \alpha\Delta\left\{u(t_{n+1}) - \frac{u(t_{n+\frac{3}{2}}) + u(t_{n+\frac{1}{2}})}{2}\right\},$$

$$\phi_3^{n+1} = -\lambda\left\{\overline{u(t_{n+1})}v(t_{n+1}) - \left(\overline{\frac{u^{n+\frac{3}{2}} + u^{n+\frac{1}{2}}}{2}}\right)v^{n+1}\right\}.$$

It follows from Lemma 1 that

$$\|\theta^{n+\frac{3}{2}}\|_{H^s} \leq \|\theta^{n+\frac{1}{2}}\|_{H^s} + h\|\Phi^{n+1}\|_{H^s}.$$

Next, we estimate $\|\Phi^{n+1}\|_{H^s}$. Fisrt, from Lemma 2, we have

$$\|\phi_1^{n+1}\|_{H^s} \leq CM_3h^2, \quad \|\phi_2^{n+1}\|_{H^s} \leq CM_2h^2$$

for $n = 0, 1, ..., N_h - 1$, where M_2 and M_3 are constants defined by (5).

Moreover, since

$$\begin{aligned}
&\overline{u(t_{n+1})}v(t_{n+1}) - \left(\overline{\frac{u^{n+\frac{3}{2}} + u^{n+\frac{1}{2}}}{2}}\right)v^{n+1} \\
&= \left\{\overline{u(t_{n+1})} - \left(\overline{\frac{u(t_{n+\frac{3}{2}}) + u(t_{n+\frac{1}{2}})}{2}}\right)\right\}v(t_{n+1}) \\
&+ \left\{\left(\overline{\frac{u(t_{n+\frac{3}{2}}) + u(t_{n+\frac{1}{2}})}{2}}\right) - \left(\overline{\frac{u^{n+\frac{3}{2}} + u^{n+\frac{1}{2}}}{2}}\right)\right\}v(t_{n+1}) \\
&+ \left(\overline{\frac{u^{n+\frac{3}{2}} + u^{n+\frac{1}{2}}}{2}}\right)\{v(t_{n+1}) - v^{n+1}\},
\end{aligned}$$

it follows from Lemma 2 that

$$\begin{aligned}
\|\phi_3^{n+1}\|_{H^s} &\leq CM_2h^2\|v(t_{n+1})\|_{H^s} \\
&\quad + C(\|u(t_{n+\frac{3}{2}}) - u^{n+\frac{3}{2}}\|_{H^s} + \|u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}\|_{H^s})\|v(t_{n+1})\|_{H^s} \\
&\quad + C(\|u^{n+\frac{3}{2}}\|_{H^s} + \|u^{n+\frac{1}{2}}\|_{H^s})\|v(t_{n+1}) - v^{n+1}\|_{H^s} \\
&\leq CM'(M_2h^2 + \|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s})
\end{aligned}$$

for $n = 0, 1, ..., N_h - 1$. Thus, we obtain

$$\begin{aligned}
\|\Phi^{n+1}\|_{H^s} &\leq CM'h^2 \\
&\quad + CM'\left(\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s}\right),
\end{aligned}$$

and cosequently

$$\begin{aligned}\|\theta^{n+\frac{3}{2}}\|_{H^s} &\le \|\theta^{n+\frac{1}{2}}\|_{H^s} + h\|\Phi^{n+1}\|_{H^s} \\ &\le \|\theta^{n+\frac{1}{2}}\|_{H^s} + CM'h^3 \\ &\quad + CM'h\left(\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s}\right). \qquad (13)\end{aligned}$$

for $n = 0, 1, \dots, N_h - 1$.

Similarly, we have

$$\rho^{n+1} - \rho^n - i\frac{\beta h}{2}\Delta(\rho^{n+1} + \rho^n) = ih\Psi^{n+\frac{1}{2}}, \qquad (14)$$

or equivalently,

$$\rho^{n+1} = A_\beta \rho^n + ihB_\beta \Psi^{n+\frac{1}{2}},$$

where $\Psi^{n+\frac{1}{2}} = \psi_1^{n+\frac{1}{2}} + \psi_2^{n+\frac{1}{2}} + \psi_3^{n+\frac{1}{2}}$,

$$\begin{aligned}\psi_1^{n+\frac{1}{2}} &= i\left\{\partial_t v(t_{n+\frac{1}{2}}) - \frac{v(t_{n+1}) - v(t_n)}{h}\right\}, \\ \psi_2^{n+\frac{1}{2}} &= \beta\Delta\left\{v(t_{n+\frac{1}{2}}) - \frac{v(t_{n+1}) + v(t_n)}{2}\right\}, \\ \psi_3^{n+\frac{1}{2}} &= -\mu\left\{\left(u(t_{n+\frac{1}{2}})\right)^2 - \left(u^{n+\frac{1}{2}}\right)^2\right\}.\end{aligned}$$

Again, from Lemma 2, we have

$$\|\psi_1^{n+\frac{1}{2}}\|_{H^s} \le CM_3h^2, \quad \|\psi_2^{n+\frac{1}{2}}\|_{H^s} \le CM_2h^2$$

for $n = 0, 1, ..., N_h - 1$. Moreover, we have

$$\begin{aligned}\|\psi_3^{n+\frac{1}{2}}\|_{H^s} &\le C(\|u(t_{n+\frac{1}{2}})\|_{H^s} + \|u^{n+\frac{1}{2}}\|_{H^s})\|u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}\|_{H^s} \\ &\le CM'\|\theta^{n+\frac{1}{2}}\|_{H^s}\end{aligned}$$

for $n = 0, 1, ..., N_h - 1$. Thus, we obtain

$$\|\rho^{n+1}\|_{H^s} \le \|\rho^n\|_{H^s} + h\|\Psi^{n+\frac{1}{2}}\|_{H^s} \le \|\rho^n\|_{H^s} + CM^*h^3 + CM'h\|\theta^{n+\frac{1}{2}}\|_{H^s} \qquad (15)$$

for $n = 0, 1, ..., N_h - 1$.

Summing up estimates (13) and (15), we deduce

$$\begin{aligned}&\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s} \\ &\le \|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s} + C_3M'h^3 \\ &\qquad + C_4M'h\left(\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s} + \|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s}\right)\end{aligned}$$

where C_3 and C_4 denote positive constants depending only on $d, s, \alpha, \beta, \lambda$ and μ. Therefore

$$\begin{aligned}&(1 - C_4M'h)\left(\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s}\right) \\ &\le (1 + C_4M'h)\left(\|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s}\right) + C_3M'h^3\end{aligned}$$

for $n = 0, 1, ..., N_h - 1$.

At this stage, we define a positive constant h_1 by

$$h_1 = \frac{1}{2C_4 M'} \tag{16}$$

and we assume that $h \in (0, h_1]$. Then, we have

$$\begin{aligned}
&\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s} \\
&\le (1 + 4C_4 M' h) \left(\|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s} \right) + 2C_3 M' h^3 \\
&\le e^{4C_4 M' h} \left(\|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s} \right) + 2C_3 M' h^3
\end{aligned}$$

for $n = 0, 1, ..., N_h - 1$. Thus, we have

$$\begin{aligned}
&\|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s} \\
&\le e^{4C_2 M' nh} \left(\|\theta^{\frac{1}{2}}\|_{H^s} + \|\rho^0\|_{H^s} \right) + 2C_1 M' h^3 \sum_{j=0}^{n-1} e^{4C_2 M' jh} \\
&\le e^{4C_2 M' T} \|\theta^{\frac{1}{2}}\|_{H^s} + 2C_1 M' T e^{4C_2 M' T} h^2
\end{aligned} \tag{17}$$

$n \in \mathbb{N}$ satisfying (12).

In view of the regularity property (4), we have

$$\partial_t u(0) = i(\alpha \Delta u_0 - \lambda \bar{u}_0 v_0).$$

Hence, using the Taylor theorem, we can calculated as

$$\begin{aligned}
\theta^{\frac{1}{2}} = u(t_{\frac{1}{2}}) - u^{\frac{1}{2}} &= \left\{ u(0) + \frac{h}{2} \partial_t u(0) + \int_0^{\frac{h}{2}} \left(\frac{h}{2} - \tau \right) \partial_\tau^2 u(\tau)\, d\tau \right\} \\
&\quad - \left\{ u_0 + i\frac{h}{2} \left(\alpha \Delta u_0 - \lambda \overline{u_0} v_0 \right) \right\} \\
&= \int_0^{\frac{h}{2}} \left(\frac{h}{2} - \tau \right) \partial_\tau^2 u(\tau)\, d\tau.
\end{aligned}$$

This gives

$$\|\theta^{\frac{1}{2}}\|_{H^s} \le \int_0^{\frac{h}{2}} \left(\frac{h}{2} - \tau \right) \|\partial_\tau^2 u(\tau)\|_{H^s}\, d\tau \le \frac{h^2}{8} \max_{t \in [0,T]} \|\partial_t^2 u(t)\|_{H^s} \le \frac{M'}{8} h^2.$$

Therefore, taking

$$K_0 = \frac{M'}{8} e^{4C_2 M' T} + 2C_1 M' T e^{4C_2 M' T},$$

we have shown that the desired estimate (7) holds for all $h \in (0, h_1]$ and $n \in \mathbb{N}$ satisfying (12).

Step 2. We prove that there exists a positive constant h_0 such that

$$\left[\frac{T}{h} \right] - 1 \le \nu_h$$

holds for all $h \in (0, h_0)$. We argue it by contradiction.

Assume that $\nu_h < \left[\frac{T}{h}\right] - 1$. Then, we have $N_h = \nu_h$ and in view of Step 1,

$$\|u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}\|_{H^s} + \|v(t_n) - v^n\|_{H^s} \leq K_0 h^2$$

for all $n = 1, ..., \nu_h$ and $h \in (0, h_1)$. Moreover, since $(\nu_h + 1)h \leq T$, it follows from the definition of M' that

$$\max_{n=1,\dots,\nu_h} \left(\|u(t_{n+\frac{1}{2}})\|_{H^s} + \|v(t_n)\|_{H^s} \right) \leq M'.$$

Combinig those inequalities, we get

$$\|u^{n+\frac{1}{2}}\|_{H^s} + \|v^n\|_{H^s} \leq M' + K_0 h^2$$

for all $n = 1, ..., \nu_h$.

We define a positive constant h_0 by

$$h_0 = \min \left\{ h_1, \sqrt{\frac{M'}{2K_0}}, \frac{1}{2} T_{\frac{3}{2}M'} \right\},$$

where $T_{\frac{3}{2}M'}$ is the constant introduced in Proposition 1 with $R = \frac{3}{2}M'$.

From now on, soppose that $h \in (0, h_0]$. Then,

$$\|u^{\nu_h + \frac{1}{2}}\|_{H^s} + \|v^{\nu_h}\|_{H^s} \leq M' + K_0 h^2 \leq \frac{3}{2} M'.$$

We apply Proposition 1 with $a = u^{\nu_h + \frac{1}{2}}$, $b = v^{\nu_h}$ and $R = \frac{3}{2}M'$ and obtain

$$\|u^{\nu_h + \frac{3}{2}}\|_{H^s} + \|v^{\nu_h + 1}\|_{H^s} \leq 3M'.$$

This contradicts the definition of ν_h. Therefore, $\left[\frac{T}{h}\right] - 1 \leq \nu_h$ holds for all $h \in (0, h_0]$. That is, we have $N_h = \left[\frac{T}{h}\right] - 1$ for all $h \in (0, h_0]$. Hence, by the result of Step 1, we see that the desired estimate (7) holds for all $h \in (0, h_0]$ and $n \in \mathbb{N}$ satisfying $(n+1)h \leq T$. This completes the proof of Theorem 2. □

Acknowledgement. I thank Professors M. Ohta and N. Saito who encouraged me through valuable disccussions.

References.

[**1**] G. T. Akrivis, V. A. Dougali and O. A. Karakashian : *On a fully discrete Galerkin methods of second-order temporal accuracy for the nonlinear Schrödinger equation*, Numer. Math. **59**(1991), 31-53

[**2**] C. Besse : *A relaxation scheme for the nonliear Schrödinger equation*, SIAM J. Numer. Anal. **42**(2004), no.3, 934-95

[**3**] T. Cazenave : *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics vol. **10**, American Mathematical Society (2003).

[**4**] D. Furihata and T. Matsuo : *Discrete variational discrete method. A structure-preserving numerical method for partial differential equations*, Chapman & Hall / CRC Numerical Analysis and Scientific Computing. CRC press, Boca Raton, FL(2011)

[**5**] N. Hayashi, C. Li and T. Ozawa : *Small data scattering for a system of nonlinear Schrödinger equations*, Differ. Equ. Appl. **3**(2011), no. 3, 415-426

[**6**] N. Hayashi, C. Li and P. I. Naumkin : *Modified wave operator for a system of nonlinear Schrödinger equations in 2d*, Comm. Partial Differential Equations **37**(2012), no.6, 947-968

UTMS

2012–16 Nao Hamamuki: *A discrete isoperimetric inequality on lattices.*

2012–17 Nao Hamamuki: *Asymptotically self-similar solutions to curvature flow equations with prescribed contact angle and their applications to groove profiles due to evaporation-condensation.*

2012–18 Guanyu Zhou: *Analysis of the fictitious domain method with L^2-penalty for elliptic and parabolic problems.*

2013–1 Yasuko Hasegawa: *The critical values of exterior square L-functions on $GL(2)$.*

2013–2 Toshio Oshima: *An elementary approach to the Gauss hypergeometric function.*

2013–3 Kazuki Okamura: *Large deviations for simple random walk on percolations with long-range correlations .*

2013–4 Kiyoomi Kataoka and Nobuko Takeuchi: *A system of fifth-order PDE's describing surfaces containing 2 families of circular arcs and the reduction to a system of fifth-order ODE's .*

2013–5 Takushi Amemiya: *Orders of meromorphic mappings into Hopf and Inoue surfaces.*

2013–6 Yasuhito Miyamoto: *Structure of the positive radial solutions for the supercritical Neumann problem $\varepsilon^2\Delta u - u + u^p = 0$ in a ball.*

2013–7 Atsushi Ito and Yusaku Tiba: *Curves in quadric and cubic surfaces whose complements are Kobayashi hyperbolically imbedded.*

2013–8 Norbert Pozar: *Homogenization of the Hele-Shaw problem in periodic spatiotemporal media.*

2013–9 Takiko SASAKI: *A second-order time-discretization scheme for a system of nonlinear Schrödinger equations.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012