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Curves in quadric and cubic surfaces whose complements are Kobayashi hyperbolically imbedded

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CURVES IN QUADRIC AND CUBIC SURFACES WHOSE COMPLEMENTS ARE KOBAYASHI HYPERBOLICALLY IMBEDDED

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ABSTRACT. We construct smooth irreducible curves of the lowest possible degrees in quadric and cubic surfaces whose complements are Kobayashi hyperbolically imbedded into those surfaces. Moreover we characterize line bundles on quadric and cubic surfaces such that the complete linear systems of the line bundles have a smooth irreducible curve whose complement is Kobayashi hyperbolically imbedded.

1. Introduction and main result

Kobayashi [9], [10] introduced the Kobayashi hyperbolicity and proposed the following famous conjecture.

Conjecture 1 (Kobayashi conjecture). *Let X be a general hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$.*

(i) *If $d \geq 2n-1$, X is Kobayashi hyperbolic for $n \geq 3$.*
(ii) *If $d \geq 2n+1$, $\mathbb{P}^n(\mathbb{C}) \setminus X$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$.*

A lot of researchers have studied about Kobayashi conjecture. For example, Păun [16] showed that a (very) general smooth surface of degree $d \geq 18$ in $\mathbb{P}^3(\mathbb{C})$ is Kobayashi hyperbolic and Rousseau [17] showed that $\mathbb{P}^2(\mathbb{C}) \setminus X$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^2(\mathbb{C})$ if X is a very general curve of degree $d \geq 14$ improving the results of [19], [15], [3] and [6]. It seems that there exist some difficulties to prove Kobayashi conjecture with optimal degrees even in the case of low dimensions. We note that Fujimoto [7] proved that the statement of Kobayashi conjecture (ii) is true if X is a union of d-hyperplanes with simple normal crossing.

On the other hand, concrete examples of Kobayashi hyperbolic hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ are known. Masuda and Noguchi [11] constructed algebraic families of hyperbolic hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ for large degrees. Duval [5] constructed smooth Kobayashi hyperbolic hypersurfaces of degree six in $\mathbb{P}^3(\mathbb{C})$. Shiffman and Zaidenberg [18] constructed smooth Kobayashi hyperbolic hypersurfaces of degree d in $\mathbb{P}^3(\mathbb{C})$ for $d \geq 8$. A smooth Kobayashi hyperbolic hypersurface of degree five in $\mathbb{P}^3(\mathbb{C})$ is not known yet. Zaidenberg [21] showed that, for each degree $d \geq 5$, there exists a smooth irreducible curve of degree d in $\mathbb{P}^2(\mathbb{C})$ such that the complement of the curve is Kobayashi hyperbolically imbedded into $\mathbb{P}^2(\mathbb{C})$. In this paper, we construct smooth irreducible curves of the lowest possible degrees in quadric and cubic surfaces whose complements are Kobayashi hyperbolically imbedded into those surfaces.

Theorem 1. *Let $Q = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $L = \mathcal{O}_{\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})}(m, n)$ be a line bundle on Q. There exists a smooth irreducible curve X in the linear system $|L|$ such that $Q \setminus X$*

is Kobayashi hyperbolically imbedded into Q if and only if m and n are larger than or equal to four.

Remark 1. We cannot remove the irreducibility of X. For example, $(\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\})\times(\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\})$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$.

Theorem 2. *Let S be a general cubic surface in $\mathbb{P}^3(\mathbb{C})$ and let L be a holomorphic line bundle on S. There exists a smooth curve X in $|L|$ such that $S\setminus X$ is Kobayashi hyperbolically imbedded into S if and only if the intersection number $L.\, l$ is larger than or equal to three for any line l in S.*

Remark 2. It is known that there exist 27-lines in S, where a line is a smooth rational curve of degree one in $\mathbb{P}^3(\mathbb{C})$ (cf. Proposition IV. 12 of [1]).

Remark 3. Let M be a compact complex manifold and let L be a line bundle on M. Let $D\in|L|$ be a smooth divisor on M whose complement is Kobayashi hyperbolically imbedded into M. Then there exists a family of divisors in $|L|$ whose complements are Kobayashi hyperbolically imbedded into M. More precisely, there exists a small neighborhood U of $D\in|L|$ in the sense of classical topology such that the complement of any element of U is Kobayashi hyperbolically imbedded into M (see [21]).

We note that any smooth quadric surface in $\mathbb{P}^3(\mathbb{C})$ is isomorphic to $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ imbedded by $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})}(1,1)$. By Theorem 1 and Theorem 2, there exists a hypersurface Y in $\mathbb{P}^3(\mathbb{C})$ of degree d such that $Y|_Q$ (resp. $Y|_S$) is a smooth curve and $Q\setminus Y|_Q$ (resp. $S\setminus Y|_S$) is Kobayashi hyperbolically imbedded into Q (resp. S) if $d\geq 4$ (resp. $d\geq 3$). However, there exists no such hypersurface of degree d if $d\leq 3$ (resp. $d\leq 2$). Furthermore, the above $Y|_Q$ for $d=4$ (resp. $Y|_S$ for $d=3$) is a smooth irreducible curve of the lowest possible degree whose complement is Kobayashi hyperbolically imbedded into Q (resp. S). This follows from the nefness of $L-\mathcal{O}_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})}(4,4)$ (resp. $L-\mathcal{O}_S(3)$) for any line bundle L on Q (resp. S) which satisfies the condition in Theorem 1 (resp. Theorem 2). The nefness of $L-\mathcal{O}_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})}(4,4)$ is clear, and see the paragraph following *Fourth Step* of Section 3 for the nefness of $L-\mathcal{O}_S(3)$.

The plan of this paper is as follows. Since the proof of Theorem 1 is similar to and simpler than that of Theorem 2, we first prove Theorem 2. In Section 2, we prove the necessity of Theorem 2 and prove that there exists a non-irreducible curve of degree nine in S such that the complement of the curve is Kobayashi hyperbolically imbedded into S. In Section 3, we deform the above non-irreducible curve to a smooth irreducible curve preserving the Kobayashi hyperbolic imbedding as [21] and we complete the proof of Theorem 2. In Section 4, we prove Theorem 1.

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2. Kobayashi hyperbolic imbedding of the complement of lines in S

Let $S\subset\mathbb{P}^3(\mathbb{C})$ be a general smooth cubic surface and set $\mathcal{O}_S(1)=\mathcal{O}_{\mathbb{P}^3(\mathbb{C})}(1)|_S$. Then S is isomorphic to the blow-up of $\mathbb{P}^2(\mathbb{C})$ at six points (cf. Theorem IV.13 of [1]). Let $\pi: S\to\mathbb{P}^2(\mathbb{C})$ be the blow-up at six points $p_1,\dots,p_6\in\mathbb{P}^2(\mathbb{C})$. Let E_i $(1\leq i\leq 6)$ be the exceptional divisor over p_i, let $L_{ij}\subset S$ $(1\leq i<j\leq 6)$ be the strict transform of

the hyperplane in $\mathbb{P}^2(\mathbb{C})$ through p_i and p_j under π, and let C_i $(1 \le i \le 6)$ be the strict transform of the conic in $\mathbb{P}^2(\mathbb{C})$ through five points p_j $(j \ne i)$ under π (such conic is uniquely exist since S is a smooth cubic surface). It is known that E_i $(1 \le i \le 6)$, L_{ij} $(1 \le i < j \le 6)$, and C_i $(1 \le i \le 6)$ are all the lines on S (cf. Proposition IV.12 of [1]).

The necessity of Theorem 2 is easily shown as follows.

Proof of the necessity of Theorem 2. Assume that there exists a line l on S such that $L.\, l \le 2$. Let $X \in |L|$ be a smooth irreducible curve in S. If $X = l$, X intersects another line l' at most one point. Hence, there exists a non-constant holomorphic map from $\mathbb{C}$ to $l' \setminus X|_{l'} \subset S \setminus X$ and $S \setminus X$ is not Kobayashi hyperbolically imbedded into S. If $X \ne l$, $X \cap l$ contains at most two points. Hence there exists a non-constant holomorphic map from $\mathbb{C}$ to $l \setminus X|_l$. This implies that $S \setminus X$ is not Kobayashi hyperbolically imbedded into S. $\square$

To prove the sufficiency of Theorem 2, we show the following proposition.

Proposition 1. *Let D be a divisor on S such that*

$$D = E_4 + E_5 + E_6 + L_{45} + L_{46} + L_{56} + C_4 + C_5 + C_6.$$

Then $S \setminus D$ is Kobayashi hyperbolically imbedded into S.

Before proving Proposition 1, we recall some theorems.

Theorem 3 (Logarithmic Bloch-Ochiai's theorem [12], [13])**.** *Let A be a semi-abelian variety and let $f : \mathbb{C} \to A$ be a holomorphic map. Then the Zariski closure of the image $f(\mathbb{C})$ is a translation of a semi-abelian subvariety.*

Theorem 4 (Theorem 5.2 of [2])**.** *Let A be a semi-abelian variety and let B be an effective reduced divisor on A. Let $f : \mathbb{C} \to A$ be a non-constant holomorphic map whose image $f(\mathbb{C})$ is Zariski dense in A. Assume that the dimension of the stabilizer*

$$\mathrm{Stab}(B) = \{x \in A : x + B = B\}$$

is zero. Then there exists at least one irreducible component B' of B such that the intersection $f(\mathbb{C}) \cap B'$ is Zariski dense in B'. In particular, if $\dim A \ge 2$, $f(\mathbb{C}) \cap B$ is infinite.

We use Theorem 3 and Theorem 4 only for the case when A is an algebraic torus.

Proof of Proposition 1. Let D_i $(1 \le i \le 9)$ be irreducible components of D. Hence D_i is equal to E_k or L_{kl} or C_l. Assume that $S \setminus D$ is not Kobayashi hyperbolically imbedded into S. Then there exists a partition of indices $I \cup J = \{1, 2, \ldots, 9\}$ and a non-constant holomorphic map from $\mathbb{C}$ to $\bigcap_{i \in I} D_i \setminus \bigcup_{j \in J} D_j$ (cf. Theorem (1.8.3) of [14]). Here I may be the empty set and we define $\bigcap_{i \in I} D_i = S$ in that case. Since each D_i is isomorphic to $\mathbb{P}^1(\mathbb{C})$ and each D_i intersects $\bigcup_{j \ne i} D_j$ at four points, there exists no non-constant holomorphic map from $\mathbb{C}$ to $D_i \setminus \bigcup_{j \ne i} D_j$ by the small Picard theorem. Therefore I must be empty and there exists a non-constant holomorphic map f from $\mathbb{C}$ to $S \setminus \bigcup_{i=1}^9 D_i = S \setminus D$. Put $g = \pi \circ f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$. It follows that the image of g is contained in $T := \mathbb{P}^2(\mathbb{C}) \setminus \pi(L_{45} \cup L_{46} \cup L_{56})$ and T is isomorphic to the algebraic torus $(\mathbb{C}^*)^2$. Since π is isomorphic over $T \setminus \{p_1, p_2, p_3\}$ and $f(\mathbb{C}) \cap \bigcup_{i=4}^6 C_i = \emptyset$, we have $g(\mathbb{C}) \cap \bigcup_{i=4}^6 \pi(C_i) \subset \{p_1, p_2, p_3\}$. It is easy to see that the dimension of

$$\mathrm{Stab}\left(\sum_{i=4}^{6} \pi(C_i)|_T\right)$$

is zero. Since $g(\mathbb{C}) \cap \bigcup_{i=4}^6 \pi(C_i)$ is a finite set, $g(\mathbb{C})$ is not Zariski dense in T by Theorem 4. We take a homogeneous coordinate $[x : y : z]$ of $\mathbb{P}^2(\mathbb{C})$ such that

$$\pi(L_{45} \cup L_{46} \cup L_{56}) = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : xyz = 0\}.$$

By Theorem 3, $g(\mathbb{C})$ is contained in a translation of a subtorus in T. Hence we may assume that there exist non-negative integers m, n $(m \leq n)$ and a non-zero complex number β such that $g(\mathbb{C})$ is contained in

$$G = \{[x : y : z] \in T \subset \mathbb{P}^2(\mathbb{C}) : x^m y^n - \beta z^{m+n} = 0\} \simeq \mathbb{C}^*.$$

Let q be a point in $G \cap \pi(C_i)$ for $4 \leq i \leq 6$. We show that $q \in \{p_1, p_2, p_3\}$. Otherwise, $g(\mathbb{C})$ does not contain q since $g(\mathbb{C}) \cap \pi\left(\bigcup_{i=4}^6 C_i\right) \subset \{p_1, p_2, p_3\}$. Hence $g(\mathbb{C})$ is in the complement of one points in $G \simeq \mathbb{C}^*$ and g is a constant map by the small Picard theorem. This is a contradiction. Therefore $q \in \{p_1, p_2, p_3\}$.

If G is tangent to $\pi(C_i)$ $(i = 4, 5, 6)$ at p_j $(j = 1, 2, 3)$, the strict transform of G under π intersects C_i over p_j. Then $g(\mathbb{C})$ does not contain p_j. By the same argument as above, g is a constant map. This is a contradiction. Hence G intersects $\pi(C_i)$ $(i = 4, 5, 6)$ transversally.

We denote by $\overline{G}$ the Zariski closure of G in $\mathbb{P}^2(\mathbb{C})$, that is, $\overline{G} = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : x^m y^n - \beta z^{m+n} = 0\}$. Assume that $m = 0$, i.e. $\overline{G}$ is a hyperplane in $\mathbb{P}^2(\mathbb{C})$. Two of $\{p_4, p_5, p_6\}$ are not contained in $\overline{G}$, say $p_4, p_5 \notin \overline{G}$. It follows that $\overline{G} \cap \pi(C_6) = G \cap \pi(C_6)$. By Bézout's theorem, $\pi(C_6)$ intersects $\overline{G}$ at two points of $\{p_1, p_2, p_3\}$. Since $p_6 \in \overline{G}$, three points of $\{p_1, \ldots, p_6\}$ are in the hyperplane $\overline{G}$ and such case does not occur since S is a general cubic surface. This is a contradiction.

We have $m > 0$. Then $\overline{G}$ contains two points of $\{p_4, p_5, p_6\}$, say p_4 and p_5. We may assume that $p_4 = [1 : 0 : 0]$ in the homogeneous coordinate $[x : y : z]$ in $\mathbb{P}^2(\mathbb{C})$. Let $U = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : x \neq 0\}$ and let $u = \frac{y}{x}, v = \frac{z}{x}$ be a local coordinate on U. On U, $\pi(C_5)$ is defined by

$$au + bv + P(u, v) = 0,$$

where $a, b \in \mathbb{C}$ and $P(u, v)$ is a homogeneous polynomial of u, v of degree two. Since S is a general cubic surface, $\pi(C_5)$ intersects $\pi(L_{45})$ and $\pi(L_{46})$ transversally. Therefore a and b are non-zero. Since $\overline{G}$ is defined by $u^n = \beta v^{m+n}$ on U, the intersection multiplicity of $\overline{G}$ and $\pi(C_5)$ at p_4 is equal to n. By Bézout's theorem, $\overline{G}$ intersects $\pi(C_5)$ at $2m+n$ different points in T (recall that G intersects $\pi(C_5)$ transversally in T and $\pi(C_5)$ does not contain p_5). Since $\overline{G} \cap \pi(C_5) \cap T = G \cap \pi(C_5) \subset \{p_1, p_2, p_3\}$, it follows that $2m + n \leq 3$. Since $0 < m \leq n$, we have $m = n = 1$. Then $\overline{G}$ is a conic which contains $\{p_1, p_2, p_3, p_4, p_5\}$, which means $\overline{G}$ coincides with $\pi(C_6)$ by the uniqueness of such conic. This is a contradiction and the proof of Proposition 1 is completed. $\square$

3. Deformation of curves

We show the sufficiency of Theorem 2. From *First Step* to *Fourth Step*, we deform the nine lines in Proposition 1 to three smooth elliptic curves keeping the Kobayashi hyperbolic imbedding.

First Step. We first deform E_4 and L_{45} to an irreducible conic in S keeping other lines. Since L_{ij} is linearly equivalent to $\pi^*\mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(1) - E_i - E_j$ and C_i is linearly equivalent to $\pi^*\mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(2) - \sum_{j\neq i} E_j$, $E_i + L_{ij} + C_j$ $(4 \leq i < j \leq 6)$ is linearly equivalent to $\mathcal{O}_S(1) = \pi^*\mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(3) - \sum_{i=1}^6 E_i$ (cf. Chap. IV of [1]). Hence there exists a hyperplane H_{ij} in $\mathbb{P}^3(\mathbb{C})$ such that $H_{ij}|_S = E_i + L_{ij} + C_j$. Let H_t $(t \in \mathbb{C}, |t| < 1)$ be a deformation family of hyperplanes in $\mathbb{P}^2(\mathbb{C})$ such that $C_5 \subset H_t$ for all t and $H_0 = H_{45}$. Since there exist only finite lines in S, $H_t|_S$ is a union of C_5 and an irreducible conic for small $t \neq 0$. We show that there exists small $\delta > 0$ such that $S \setminus (H_t \cup H_{46} \cup H_{56})|_S$ is Kobayashi hyperbolically imbedded into S for $|t| < \delta$. Otherwise, there exists a sequence $\{t_\nu\}_{\nu\in\mathbb{N}}$ such that $t_\nu \to 0, t_\nu \neq 0$ and $S \setminus (H_{t_\nu} \cup H_{46} \cup H_{56})|_S$ is not Kobayashi hyperbolically imbedded into S. Then there exist holomorphic maps f_ν from $\Delta(\nu) = \{z \in \mathbb{C} : |z| < \nu\}$ to $S \setminus (H_{t_\nu} \cup H_{46} \cup H_{56})|_S$ such that f_ν converges uniformly to a non-constant holomorphic map $f : \mathbb{C} \to S$ on compact subsets (cf. Section 2 of [21]). Let $\mathcal{D}$ be the hypersurface in $S \times \Delta(1)$ such that $\mathcal{D}|_{S\times\{t\}} = (H_t \cup H_{46} \cup H_{56})|_S$. Then we may consider f_ν (resp. f) as a holomorphic map from $\mathbb{C}$ to $S \times \{t_\nu\} \setminus \mathcal{D}|_{S\times\{t_\nu\}}$ (resp. $S \times \{0\} \setminus \mathcal{D}|_{S\times\{0\}}$). Let $\mathcal{C}$ be an irreducible component of $\mathcal{D}$. By Hurwitz' theorem, $f(\mathbb{C}) \subset \mathcal{C}|_{S\times\{0\}}$ or $f(\mathbb{C}) \cap \mathcal{C}|_{S\times\{0\}} = \emptyset$. Because there exists no non-constant holomorphic map from $\mathbb{C}$ to $S \setminus D$, $f(\mathbb{C})$ is contained in an irreducible component of $H_{45}|_S$ or $H_{46}|_S$ or $H_{56}|_S$. Since any irreducible component of D intersects other components of D at four points, $f(\mathbb{C})$ must be contained in E_4 or L_{45}. Without loss of generality, we may assume that $f(\mathbb{C}) \subset E_4$. Since E_4 intersects the divisor $D - (E_4 + L_{45})$ at three points, it follows that f is constant by the small Picard theorem. This is a contradiction and we have that $S \setminus (H_{t_0} \cup H_{46} \cup H_{56})|_S$ is Kobayashi hyperbolically imbedded into S for small t_0.

Second Step. Next we deform $H_{t_0}|_S$ to an irreducible curve with a node in S. Since two components of $H_{t_0}|_S$ intersect at two points, there exist two nodes in $H_{t_0}|_S$. Let p be one of those nodes. Then H_{t_0} is the tangent plane of S at p. Let $\epsilon : \mathbb{P}^3(\mathbb{C}) \times S \to S$ be the second projection and let $\widetilde{H}$ be the divisor on $\mathbb{P}^3(\mathbb{C}) \times S$ such that $\widetilde{H}_q = \widetilde{H}|_{\mathbb{P}^3(\mathbb{C})\times\{q\}}$ $(q \in S)$ is the tangent plane of S at q in $\mathbb{P}^3(\mathbb{C})$. Then $\widetilde{H}|_{S\times S}$ is a divisor on $S \times S$.

Lemma 1. *Let U be a sufficiently small neighborhood of p in S. Then the divisor $\widetilde{H}|_{S\times S}$ consists of two smooth transversally intersecting irreducible components in $U \times U$.*

Proof. Let (x, y) be a local coordinate on U such that $x(p) = y(p) = 0$ and $H_{t_0}|_S$ is equal to the divisor defined by xy in U. Let $\pi_i : S \times S \to S$ be the i-th projection. We will still write $x = \pi_1^*x, y = \pi_1^*y$ by abuse of notation. Let $u = \pi_2^*x$, $v = \pi_2^*y$. There exists a holomorphic function $f(x, y, u, v)$ on $U \times U$ which defines the divisor $\widetilde{H}|_{U\times U}$. Then we write

$$\begin{aligned} f(x,y,u,v) &= f_{1,0}(u,v)(x-u) + f_{0,1}(u,v)(y-v) + f_{2,0}(u,v)(x-u)^2 \\ &\quad + f_{1,1}(u,v)(x-u)(y-v) + f_{0,2}(u,v)(y-v)^2 + O((|x-u|+|y-v|)^3). \end{aligned}$$

Let $q = (a, b)$ be any point in U. The divisor $\widetilde{H}|_{S\times\{q\}}$ is defined by $f(x, y, a, b)$ on U. Because of the singularity of $\widetilde{H}|_{S\times\{q\}}$ at q, we have $f_{1,0}(a, b) = f_{0,1}(a, b) = 0$. Since

(a, b) is any point in U, we have

$$f(x,y,u,v) = f_{2,0}(u,v)(x-u)^2 + f_{1,1}(u,v)(x-u)(y-v) + f_{0,2}(u,v)(y-v)^2 \\ + O((|x-u|+|y-v|)^3).$$

Since $H_{t_0}|_S$ is defined by xy on U, we may assume $f(x,y,0,0) = xy$. In particular, we have that $f_{2,0}(0,0) = f_{0,2}(0,0) = 0$ and $f_{1,1}(0,0) = 1$. Hence $f_{1,1}(u,v)^2 - 4f_{2,0}(u,v)f_{0,2}(u,v) \neq 0$ on the small open set U, and we take the branch of the function $\sqrt{f_{1,1}(u,v)^2 - 4f_{2,0}(u,v)f_{0,2}(u,v)}$ as

$$-\frac{\pi}{2} < \arg\left(\sqrt{f_{1,1}(u,v)^2 - 4f_{2,0}(u,v)f_{0,2}(u,v)}\right) < \frac{\pi}{2}.$$

It follows that

$$\begin{aligned} f(x,y,u,v) = {} & \left(x' + \frac{2f_{0,2}(u,v)}{f_{1,1}(u,v) + \sqrt{f_{1,1}(u,v)^2 - 4f_{2,0}(u,v)f_{0,2}(u,v)}}y'\right) \\ & \times \left(f_{2,0}(u,v)x' + \frac{1}{2}\left(f_{1,1}(u,v) + \sqrt{f_{1,1}(u,v)^2 - 4f_{2,0}(u,v)f_{0,2}(u,v)}\right)y'\right) \\ & + O((|x'|+|y'|)^3), \end{aligned}$$

where $x' = x-u$, $y' = y-v$. We take

$$\begin{aligned} x'' &= x' + \frac{2f_{0,2}(u,v)}{f_{1,1}(u,v) + \sqrt{f_{1,1}(u,v)^2 - 4f_{2,0}(u,v)f_{0,2}(u,v)}}y', \\ y'' &= f_{2,0}(u,v)x' + \frac{1}{2}\left(f_{1,1}(u,v) + \sqrt{f_{1,1}(u,v)^2 - 4f_{2,0}(u,v)f_{0,2}(u,v)}\right)y'. \end{aligned}$$

We have that (x'', y'', u, v) is a local coordinate on $U \times U$ and $f = x''y'' + O((|x''|+|y''|)^3)$. It is easy to see that $f = \tilde{x}\tilde{y}$ for suitable $\tilde{x} = x'' + O((|x''|+|y''|)^2)$ and $\tilde{y} = y'' + O((|x''|+|y''|)^2)$ by a similar argument to Example 5.6.3 in Chapter 1 of [8]. Since $(\tilde{x}, \tilde{y}, u, v)$ is a local coordinate on $U \times U$, this completes the proof of the lemma. □

We show that $S \setminus (\widetilde{H}_q \cup H_{46} \cup H_{56})|_S$ is Kobayashi hyperbolically imbedded into S if q is sufficiently close to p. Otherwise, there exists a non-constant holomorphic map $f : \mathbb{C} \to S$ such that $f(\mathbb{C}) \subset \widetilde{H}_p|_S \setminus (H_{46} \cup H_{56})|_S$ by the same argument as *First Step.* We note that $\widetilde{H}_p = H_{t_0}$. Because of Hurwitz' theorem and Lemma 1, we have that $f(\mathbb{C})$ does not contain the point p (cf. Lemma-Definition 3.2 of [21]). Each component of $\widetilde{H}_p|_S$ intersects $H_{46}|_S \cup H_{56}|_S$ at more than or equal to two points. Hence $f(\mathbb{C})$ is contained in the complement of three points in a rational curve and f is a constant map by the small Picard theorem. This is a contradiction and $S \setminus (\widetilde{H}_q \cup H_{46} \cup H_{56})|_S$ is Kobayashi hyperbolically imbedded into S if q is sufficiently close to p.

Third Step. Let q_0 be a point of S such that q_0 is sufficiently close to p and $\widetilde{H}_{q_0}|_S$ is irreducible. We deform the nodal rational curve $\widetilde{H}_{q_0}|_S$ to a smooth elliptic curve in S. Let E' be a smooth irreducible elliptic curve in S which is sufficiently close to $\widetilde{H}_{q_0}|_S$ in $|\mathcal{O}_S(1)|$. Since $\widetilde{H}_{q_0}|_S$ is a small deformation of $H_{45}|_S$, $\widetilde{H}_{q_0}|_S$ intersects $H_{46}|_S \cup H_{56}|_S$ at six points and $S \setminus (E' \cup H_{46}|_S \cup H_{56}|_S)$ is Kobayashi hyperbolically imbedded into S by the same argument as *First Step.*

Fourth Step. Deforming $H_{46}|_S$ and $H_{56}|_S$ as $H_{45}|_S$, there exist smooth irreducible elliptic curves E'', E''' in $|\mathcal{O}_S(1)|$ such that $S \setminus (E' \cup E'' \cup E''')$ is Kobayashi hyperbolically imbedded into S.

Let L be a holomorphic line bundle on S such that $L.\ l \geq 3$ for any line l on S. Let $L' = L - \mathcal{O}_S(3)$. Then $L'.\ l \geq 0$ for any line l on S and this implies that L' is spanned by its global sections (cf. [4]). We take a general curve D' in $|L'|$. We have that $S \setminus (E' \cup E'' \cup E''' \cup D')$ is Kobayashi hyperbolically imbedded into S.

It follows that D is linearly equivalent to $\pi^*\mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(9) - 3\sum_{i=1}^{6} E_i$ and this is linearly equivalent to $\mathcal{O}_S(3)$ (cf. Chap. IV of [1]). Hence $E' + E'' + E''' + D'$ is linearly equivalent to L.

In the rest steps, we deform $E' \cup E'' \cup E''' \cup D'$ to a smooth irreducible curve in S keeping the Kobayashi hyperbolic imbedding.

Fifth Step. Let G be a smooth irreducible curve in S which is a small deformation of $E' \cup E''$. Since both elliptic curves E' and E'' intersect $E''' \cup D'$ at more than or equal to three points respectively, $S \setminus (G \cup E''' \cup D')$ is Kobayashi hyperbolically imbedded into S by the same argument as *First Step.* Let G' be a smooth irreducible curve in S which is a small deformation of $E''' \cup D'$. Here we put $G' = E'''$ if $D' = 0$. Note that D' is an irreducible smooth curve which is not a line if $D' \neq 0$. We have that E''' intersects G at six points and D' also intersects G at more than or equal to four points if $D' \neq 0$. Hence $S \setminus (G \cup G')$ is Kobayashi hyperbolically imbedded into S by the same argument as *First Step.*

Sixth Step. We deform $G \cup G'$ to an irreducible curve with a node in S. It follows that G intersects G' transversally at any point $s \in G \cap G'$. Let $\mu : \widetilde{S} \to S$ be the blow-up at s and let Z be the exceptional divisor of μ. The strict transform $\widetilde{G}$ of $G + G'$ under μ is linearly equivalent to $\mu^*L - 2Z$, and $\mu^*L - 2Z$ is a very ample line bundle on $\widetilde{S}$. We can take a smooth irreducible divisor $\widetilde{G}'$ in $|\widetilde{G}|$ which is sufficiently close to $\widetilde{G}$ and intersects Z transversally at two points. The image $\mu(\widetilde{G}')$ is an irreducible curve in S with a node at s and $S \setminus \mu(\widetilde{G}')$ is Kobayashi hyperbolically imbedded into S by the same argument as *Second Step* (note that the genera of G and G' are larger than or equal to one. Hence there exists no non-constant holomorphic map from $\mathbb{C}$ to $G \setminus s$ and $G' \setminus s$).

Seventh Step. Finally we deform $\mu(\widetilde{G}')$ to a smooth irreducible curve in S. Since the genus of the normalization of $\mu(\widetilde{G}')$ is larger than or equal to two, there exists no non-constant holomorphic map from $\mathbb{C}$ to $\mu(\widetilde{G}')$. Hence the complement of a small deformation of $\mu(\widetilde{G}')$ in S is Kobayashi hyperbolically imbedded into S by the same argument as *First Step.* This completes the proof of Theorem 2.

4. The case of quadric surfaces

Proof of the sufficiency of Theorem 1. Let $Q = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $[X_0 : X_1]$ and $[Y_0 : Y_1]$ be the homogeneous coordinates on the first and second factors of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $H_1, \ldots, H_4$ be divisors on Q defined by $X_0 = 0$, $X_1 = 0$, $Y_0 = 0$ and $Y_1 = 0$ respectively. Let D be a general divisor in $|\mathcal{O}_Q(m-2, n-2)|$ $(m, n \geq 4)$. Then $\sum_{i=1}^{4} H_i + D$ is linearly equivalent to $L = \mathcal{O}_Q(m, n)$. It follows that

$$Q \setminus \left(\bigcup_{i=1}^{4} H_i \cup D \right)$$

is Kobayashi hyperbolically imbedded into Q (cf. Theorem 1 of [20]). Let C be a smooth irreducible curve in Q which is sufficiently close to $H_1 + H_3$ in $|\mathcal{O}_Q(1, 1)|$. We have that H_i $(i = 1, 3)$ intersects $H_2 + H_4 + D$ at more than or equal to three points. Then $Q \setminus (C \cup H_2 \cup H_4 \cup D)$ is Kobayashi hyperbolically imbedded into Q by the same argument as *First Step* of Section 3. Let C' be a smooth irreducible curve in Q which is sufficiently close to H_2+H_4 in $|\mathcal{O}_Q(1, 1)|$. As the case of C, it follows that $Q\setminus(C\cup C'\cup D)$ is Kobayashi hyperbolically imbedded into Q. Let D' be a smooth irreducible curve in Q which is sufficiently close to $C + C'$ in $|\mathcal{O}_Q(2, 2)|$. Then both C and C' intersect D at more than or equal to four points respectively and $Q \setminus (D \cup D')$ is Kobayashi hyperbolically imbedded into Q by the same argument as the *First Step* of Section 3. Let $p \in D \cap D'$. We have that D intersects D' transversally at p. Let $\mu : \widetilde{Q} \to Q$ be the blowing up at p and let Z be the exceptional divisor of μ. Let $\widetilde{D}$ be the strict transform of $D + D'$ under μ. We have that $\widetilde{D}$ is an element of $|\mu^*\mathcal{O}_Q(m, n) - 2Z|$ and the line bundle $\mu^*\mathcal{O}_Q(m, n) - 2Z$ is very ample. Note that genera of D and D' are at least one. By the same arguments as *Sixth Step* and *Seventh Step* of Section 3, there exists a smooth irreducible curve in Q whose complement is Kobayashi hyperbolically imbedded into Q. $\square$

Proof of the necessity of Theorem 1. Let $L = \mathcal{O}_Q(m, n)$. Without loss of generality, we may assume that $m \le n$. Assume that $m \le 3$. Let $X \in |L|$ be a smooth irreducible curve in Q. Let $\pi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be the second projection.

If $m \le 2$, a general fiber F of π intersects X at less than or equal to two points. Then there exists a non-constant holomorphic map from $\mathbb{C}$ to $F \setminus X$, and $Q \setminus X$ is not Kobayashi hyperbolically imbedded into Q.

If $m = 3$, $\pi|_X : X \to \mathbb{P}^1(\mathbb{C})$ is ramified because of the Riemann–Hurwitz formula. Let $q \in \mathbb{P}^1(\mathbb{C})$ be a branch point of $\pi|_X$. The fiber $\pi^{-1}(q)$ intersects X at less than or equal to two points. There exists a non-constant holomorphic map from $\mathbb{C}$ to $\pi^{-1}(q) \setminus X$. This implies that $Q \setminus X$ is not Kobayashi hyperbolically imbedded into Q. $\square$

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