UTMS 2013–5 March 25, 2013

Orders of meromorphic mappings into Hopf and Inoue surfaces

by

Takushi Amemiya

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

Orders of Meromorphic Mappings into Hopf and Inoue Surfaces

Takushi Amemiya

Abstract

In a late paper of J. Noguchi and J. Winkelmann [7] (J. Math. Soc. Jpn., Vol. **64** No.4 (2012), 1169-1180) they showed the condition of being Kähler or non-Kähler of the image space to make a difference in the value distribution theory of meromorphic mappings into compact complex manifolds. In this paper, we will investigate orders of meromorphic mappings to a Hopf surface which is more general than dealt with by Noguchi-Winkelmann, and an Inoue surface. They are non-Kähler surfaces and belong to VII_0-class. For a general Hopf surface S, we prove that there exists a differentiably non-degenerate holomorphic mapping $f : \mathbb{C}^2 \to S$ with order at most one. For any Inoue surface S', we prove that every non-constant meromorphic mapping $f : \mathbb{C}^n \to S'$ is holomorphic, differentiably degenerate and its order satisfies $\rho_f \geq 2$.

1 Main Results

In Nevanlinna theory, there are many studies on the value distribution of meromorphic mappings whose image spaces are Kähler, especially complex projective algebraic manifolds. On the other hand, however, little are known for the non-Kähler case. J. Noguchi and J. Winkelmann gave the first phenomena where Kähler or non-Kähler condition of the image space make difference in value distribution theory by focusing on orders of meromorphic mappings [7]. The purpose of this paper is to investigate orders of meromorphic mappings into Hopf surfaces and Inoue surfaces, both of which are non-Kähler surfaces. The two main theorems are as follows.

Main Theorem 1.1. *Let $S_{a,b}$ be a Hopf surface defined by the action,*

$$n : (x, y) \in \mathbb{C}^2 \setminus \{(0,0)\} \mapsto (a^n x, b^n y) \in \mathbb{C}^2 \setminus \{(0,0)\}, \quad n \in \mathbb{Z},$$

where a, b are complex numbers with $|a|, |b| > 1$. Then there exists a differentiably non-degenerate holomorphic mapping $f : \mathbb{C}^2 \to S_{a,b}$ with order at most one.

We are going to prove this theorem by branched covering argument and applying some estimates introduced by J. Noguchi-J. Winkelmann [7] to prove the case of $a = b$.

N.B. In general, whether there exists a differentiably non-degenerate meromorphic mapping with order less than two or not are big difference. Because if there is such a map, every global contravariant holomorphic tensor on the manifold must vanish ([7]).

Main Theorem 1.2. *Let S be an Inoue surface. Let n be an arbitrary natural number. Then every non-constant meromorphic mapping $f : \mathbb{C}^n \to S$ is holomorphic, differentially degenerate and its order satisfies $\rho_f \geq 2$.*

Acknowledgement. I would like to express my deep gratitude to my advisor Professor Junjiro Noguchi for his great advice, helpful comments and warm encouragements. I would also thank Dr. Yusaku Tiba for giving me a number of invaluable comments.

2 Preliminaries

2.1 Notation

We fix the following notaion.

- Let X be a compact complex manifold.
- Let $f : \mathbb{C}^n \to X$ be a meromorphic mapping. We denote by $I(f)$ the *indeterminancy locus of f*.
- If the differential df is generically of maximal rank, f is said to be *differentiably non-degenerate*.
- For $z = (z_j) \in \mathbb{C}^n$, we set

  (2.1) $$\alpha = dd^c \|z\|^2,$$

  (2.2) $$\zeta = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1},$$

  where $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$ and $\|z\|^2 = \sum_{j=1}^n |z_j|^2$.
- $B(r) = \{z \in \mathbb{C}^n : \|z\| < r\}, \qquad S(r) = \{z \in \mathbb{C}^n : \|z\| = r\} \quad (r > 0).$

Definition 2.3. Let $f : \mathbb{C}^n \to X$ be a meromorphic mapping and let ω be a Hermitian metric form on X. We define a function

$$T_f(r, \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} f^*\omega \wedge \alpha^{n-1}$$

which is called the characteristic function of f with respect to ω.

Definition 2.4. In above setting we define the order of f as follows,

(2.5) $$\rho_f = \overline{\lim_{r \to \infty}} \frac{\log T_f(r, \omega)}{\log r}.$$

Since X is compact, ρ_f is independent of the choice of a metric form ω on X.

2.2 Relations between orders and one-dimensional image spaces

Possible values of orders are affected by image spaces. To get a better comprehension of our results, we recall the following facts of one dimensional case.

Fact 2.6. *Let X be a closed Riemann surface of genus g.*

(i) *Let $g \geq 2$ and let f be a holomorphic mapping from $\mathbb{C}$ into X. Then f is constant.*

(ii) *Let $g = 1$ and let f be a non-constant holomorphic mapping from $\mathbb{C}$ into X. Then the order satisfies $\rho_f \geq 2$.*

(iii) *Let $g = 0$ and let $s \geq 0$ be a given real number. Then there exists a non-constant holomorphic mapping $f : \mathbb{C} \to X$ with order s. ([5],Theorem 7.5.9, p.241.)*

Here it is noted that every meromorphic mapping from $\mathbb{C}$ into a compact complex manifold is holomorphic since $\operatorname{codim} I(f) \geq 2$ *($I(f) = \emptyset$ in this case).*

2.3 Difference between Kähler and non-Kähler surfaces

J. Noguchi and J. Winkelmann proved the following theorems, giving the first phenomena where Kähler or non-Kähler conditions of image spaces make difference in value distribution.

Theorem 2.7 (J. Noguchi-J. Winkelmann [7])**.** *Let X be a compact Kähler surface. Assume that there is a differentiably non-degenerate meromorphic mapping $f : \mathbb{C}^2 \to X$. If $\rho_f < 2$, then X is rational.*

The Kähler condition is necessary by the following:

Theorem 2.8 (J. Noguchi-J. Winkelmann [7])**.** *Let a be a complex number with $|a| > 1$. Let $S_{a,a}$ be a Hopf surface defined as the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by a $\mathbb{Z}$-action $n : (x, y) \mapsto (a^n x, a^n y)$. Then there exists a differentiably non-degenerate holomorphic mapping $f : \mathbb{C}^2 \to S_{a,a}$ with order at most one.*

3 General Hopf surfaces : Proof of Main Theorem 1.1.

Our Main Theorem 1.1. asserts that Theorem 2.8 still holds for more general Hopf surfaces. We are going to prove Main Theorem 1.1. in two steps.

Proof. We may assume $1 < |b| \leq |a|$. In the first step, we prove the holomorphic mapping $f : \mathbb{C}^2 \to S_{a,b}$ induced by

$$\tilde{f} : \mathbb{C}^2 \to \mathbb{C}^2 \setminus \{(0,0)\} \quad (z, w) \mapsto (z, 1 + zw)$$

is diffetentiably non-degenerate and its order satisfies $\rho_f \leq 1$, assuming that

$$1 < |b| \leq |a| \leq |b|^{\frac{3}{2}}. \tag{3.1}$$

In the second step, we prove that the same result holds for all $a, b \in \mathbb{C}$ with $|a|, |b| > 1$ by using a branched covering argument.

3.1 The first step with assumption (3.1)

Let α be as in (2.1). Setting $\gamma = \frac{\log|a|}{\log|b|} - 1$ and $\delta = 1 - \frac{\log|b|}{\log|a|}$, we have $0 \leq \delta \leq \gamma \leq \frac{1}{2}$ by (3.1). We define a continuous positive Hermitian form on $\mathbb{C}^2 \setminus \{(0,0)\}$ which is invariant under the above $\mathbb{Z}$-action as follows,

$$\tilde{\omega} = \frac{i}{2\pi} \cdot \frac{dx \wedge d\bar{x} + (|y|^{2\gamma} + |x|^{2\delta}) dy \wedge d\bar{y}}{|x|^2 + (|y|^{2\gamma} + |x|^{2\delta})|y|^2}.$$

It induces a continuous positive Hermitian form on the quotient space $S_{a,b}$ which is denoted by ω.

Since $S_{a,b}$ is compact, the order ρ_f is independent of choices of smooth Hermitian metrics. In addition to this, a continuous Hermitian metric is bounded from below by a positive constant multiple of a smooth Hermitian metric by the compactness. Therefore it suffices to show

$$\overline{\lim_{r\to\infty}} \frac{1}{\log r} \log \int_1^r \frac{dt}{t^3} \int_{B(t)} f^*\omega \wedge \alpha \leq 1.$$

Note that

$$f^*\omega \wedge \alpha = \frac{1 + (|z|^2 + |w|^2)(|1 + zw|^{2\gamma} + |z|^{2\delta})}{|z|^2 + |1 + zw|^2(|1 + zw|^{2\gamma} + |z|^{2\delta})} \alpha^2.$$

We define

$$I'_r = \int_{S(r)} \frac{r^2 + \frac{1}{|1+zw|^{2\gamma} + |z|^{2\delta}}}{\frac{|z|^2}{|1+zw|^{2\gamma} + |z|^{2\delta}} + |1 + zw|^2} dV, \qquad r = \|(z, w)\|,$$

$$I_r = \int_{S(r)} \frac{r^2}{\frac{|z|^2}{|1+zw|^{2\gamma} + |z|^{2\delta}} + |1 + zw|^2} dV, \qquad r = \|(z, w)\|.$$

Here dV is the euclidean volume element on $S(r)$. Then we have

$$I'_r \leq 2 I_r. \tag{3.2}$$

Indeed,

- When $|z| \geq r^{-\frac{1}{\delta}}$, we have $|1 + zw|^{2\gamma} + |z|^{2\delta} \geq |z|^{2\delta} \geq r^{-2}$.
- When $|z| \leq r^{-\frac{1}{\delta}}$, we have $|zw| \leq r^{1-\frac{1}{\delta}} \leq r^{-1}$. This implies

  $$|1 + zw|^{2\gamma} + |z|^{2\delta} \geq |1 + zw|^{2\gamma} \geq (1 - r^{1-\frac{1}{\delta}})^{2\gamma} \geq (1 - r^{1-\frac{1}{\delta}}) \geq r^{-2}$$

  for all large r.

In both cases, $\frac{1}{|1+zw|^{2\gamma}+|z|^{2\delta}} \leq r^2$ for all large r. Hence it is sufficient to show

$$I_r = O(r^{2+\varepsilon}), \quad \forall \varepsilon > 0. \tag{3.3}$$

In fact, from this and (3.2), we obtain

$$\int_{B(r)} \frac{r^2 + \frac{1}{|1+zw|^{2\gamma}+|z|^{2\delta}}}{\frac{|z|^2}{|1+zw|^{2\gamma}+|z|^{2\delta}} + |1+zw|^2} \alpha^2 = O\left(\int^r I'_r dr\right) = O(r^{3+\varepsilon}), \quad \forall \varepsilon > 0,$$

which implies

$$T_f(r) = \int_1^r \frac{dt}{t^3} \int_{B(t)} \frac{r^2 + \frac{1}{|1+zw|^{2\gamma}+|z|^{2\delta}}}{\frac{|z|^2}{|1+zw|^{2\gamma}+|z|^{2\delta}} + |1+zw|^2} \alpha^2 = O(r^{1+\varepsilon}), \quad \forall \varepsilon > 0.$$

Therefore we have

$$\rho_f = \varlimsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \leq 1.$$

To show (3.3), we set

$$\eta = \frac{r^2}{\frac{|z|^2}{|1+zw|^{2\gamma}+|z|^{2\delta}} + |1+zw|^2}.$$

To estimate $I_r = \int_{S(r)} \eta dV$, we divide $S(r)$ into eleven regions, A, B, C, D_{-2}, D_{-1}, D_0, D_1, E, F, G, H which are defined later, and estimate the volume and the integrand on each region. We introduce useful some geometric and arithmetic estimates used in [7].

Geometric estimates.

For $(z,w) \in \mathbb{C}^2$ with $z \neq 0$ and $w \neq 0$, set $\theta \in [0, 2\pi)$ by $e^{i\theta}|zw| = zw$. For $K > 0$, $-\infty < \lambda < 1$ and $\mu \geq 0$, we set

$$\Omega_{K,\lambda,\mu} = \{(z,w) \in S(r) \,|\, z = 0 \,\text{or}\, (0 < |z| \leq Kr^\lambda, \ |\sin\theta| \leq r^{-\mu})\}.$$

We define a mapping $\Phi : \mathbb{C}^2 \setminus \{z = 0 \,\text{or}\, w = 0\} \to \mathbb{C} \times \mathbb{R}^2$ as follows,

$$\Phi : (z,w) \mapsto (z, r\arg(zw), r)$$

where $r = ||(z,w)|| = \sqrt{|z|^2 + |w|^2}$. To show the Jacobian of Φ is identically -1 we set $z = x + \sqrt{-1}y$, $w = u + \sqrt{-1}v$ and write Φ with real coordinates as follows,

$$\Phi : (x,y,u,v) \mapsto (x, y, r(\arg z + \arg w), r) \in \mathbb{R}^4$$

with $r = \sqrt{x^2 + y^2 + u^2 + v^2}$. The Jacobian of Φ is

$$\begin{aligned}
|J_\Phi| &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & \frac{u}{r}(\arg z + \arg w) + r\frac{\partial}{\partial u}\arg w & \frac{v}{r}(\arg z + \arg w) + r\frac{\partial}{\partial v}\arg w \\ * & * & \frac{u}{r} & \frac{v}{r} \end{vmatrix} \\
&= \begin{vmatrix} r\frac{\partial}{\partial u}\arg w & \frac{u}{r} \\ r\frac{\partial}{\partial v}\arg w & \frac{v}{r} \end{vmatrix} \\
&\equiv -1
\end{aligned}$$

Furthermore the gradient grad(r) is of length one and normal on the level set $S(r)$.

Hence the euclidean volume of $\Omega_{K,\lambda,\mu}$ is the same as the euclidean volume of

$$\{z \in \mathbb{C} : |z| \le K r^\lambda\} \times \{\theta r : \theta \in [0, 2\pi), |\sin\theta| \le r^{-\mu}\} \times \{r\}$$

becuase

$$\begin{aligned}
\mathrm{vol}(\Omega_{K,\lambda,\mu}) &= \mathrm{vol}(\Omega_{K,\lambda,\mu} \setminus \{z = 0\}) \\
&= \mathrm{vol}(\Phi(\Omega_{K,\lambda,\mu} \setminus \{z = 0\})) \\
&= \mathrm{vol}(\{z \in \mathbb{C} : 0 < |z| \le K r^\lambda\} \times \{\theta r : \theta \in [0, 2\pi), |\sin\theta| \le r^{-\mu}\} \times \{r\}) \\
&= \mathrm{vol}(\{z \in \mathbb{C} : |z| \le K r^\lambda\} \times \{\theta r : \theta \in [0, 2\pi), |\sin\theta| \le r^{-\mu}\} \times \{r\})
\end{aligned}$$

Using $\sin(\theta) \ge \frac{2}{\pi}\theta$ ($\theta \in [0, \frac{\pi}{2})$), it follows that for $r \ge 1$ the volume of $\Omega_{K,\lambda,\mu}$ is bounded from above by

$$\pi \left(K r^\lambda\right)^2 \cdot 2r^{-\mu}\pi r = 2K^2\pi^2 r^{2\lambda+1-\mu}.$$

In particular,

$$\mathrm{vol}(\Omega_{K,\lambda,\mu}) = O(r^{2\lambda+1-\mu}). \tag{3.4}$$

Arithmetic estimates.

Besides the Landau O-symbols we also use the notation $\gtrsim$: If f, g are functions of a real parameter r, then $f(r) \gtrsim g(r)$ indicates that

$$\liminf_{r\to+\infty} \frac{f(r)}{g(r)} \ge 1.$$

Similarly $f \sim g$ indicates

$$\lim_{r\to+\infty} \frac{f(r)}{g(r)} = 1.$$

In the sequel, we will work with domains $\Omega \subset S(r)$ (i.e. for each $r > 0$ some subset $\Omega = \Omega_r \subset S(r)$ is chosen). In this context, given functions f, g on $\mathbb{C}^2$ we say $f(z,w) \gtrsim g(z,w)$ holds on Ω if for every sequence $(z_n, w_n) \in \Omega_r$ $(r = ||(z_n, w_n)||)$ with

$$\lim_{n\to\infty} ||(z_n, w_n)|| = +\infty$$

and we have

$$\liminf_{n\to\infty} \frac{f(z_n, w_n)}{g(z_n, w_n)} \ge 1.$$

We show some estimates for $\eta = \frac{r^2}{\frac{|z|^2}{|1+zw|^{2\gamma}+|z|^{2\delta}}+|1+zw|^2}$. Fix $-\infty < \lambda < 1$.

(i) Suppose $(z,w) \in S(r)$ and $|z| \le \frac{1}{2r}$. Since $|w| \le r$, we have $|zw| \le \frac{1}{2}$, implying $|1+zw| \ge \frac{1}{2}$. Therefore $\eta \le \frac{r^2}{|1+zw|^2} \le 4r^2$.

(ii) Suppose $|z| \le r^\lambda$. We have $|w| \sim r$.

(iii) Suppose $|z| \ge \frac{3}{2r}$ and $|z| \le r^\lambda$. We have $|1+zw|^2 \gtrsim \frac{1}{9}|zw|^2$. Since $|w| \sim r$, we have $|zw| \gtrsim \frac{3}{2}$ (equivalently, $1 \lesssim \frac{2}{3}|zw|$), implying $|1+zw| \ge |zw| - 1 \gtrsim \frac{1}{3}|zw|$. Therefore $\eta \le \frac{cr^2}{|zw|^2}$. (Here c is a positive constant greater than nine)

(iv) For all z and w, $\frac{|z|^2}{|1+zw|^{2\gamma}+|z|^{2\delta}} + |1+zw|^2 \ge |\mathrm{Im}(1+zw)|^2 = (|zw|\sin\theta)^2$.

Estimates on each regions.

We are going to prove the following claim

$$I_r = O(r^{2+\varepsilon}), \quad \forall \varepsilon > 0$$

by dividing $S(r)$ into eleven regions A, B, C, D_{-2}, D_{-1}, D_0, D_1, E, F, G, H, each of which is investigated separately.

- $A = \{(z,w) \in S(r) |\ |z| \le \frac{1}{2r}\}$, i.e., $A = \Omega_{\frac{1}{2},-1,0}$. By (3.4), we have $\mathrm{vol}(A) = O(r^{-1})$. Due to (i), restriction of integrand η to A is $\eta|_A = O(r^2)$. Thus

$$\int_A \eta\, dV \le \mathrm{vol}(A) \cdot \sup_{(z,w)\in A} \eta(z,w) \le O(r).$$

Hence the contribution of A to the integral $I_r = \int_{S(r)} \eta\, dV$ is bounded by $O(r)$.

- $B = \{(z,w) \in S(r) |\ \frac{1}{2r} \le |z| \le \frac{3}{2r}$ and $|\sin\theta| < \frac{1}{r}\}$. Thus $B \subset \Omega_{3/2,-1,1}$. Due to (3.4), we have $\mathrm{vol}(B) = O(r^{-2})$. Since $|zw| \le \frac{3}{2}$, the function $|1+zw|^{2\gamma}$ is bounded on B. Therefore we obtain

$$\eta|_B \le \frac{r^2}{|z|^2}(|1+zw|^{2\gamma} + |z|^{2\delta}) = O(r^4).$$

At the last estimate we used the inequality $|z| \ge \frac{1}{2r}$. Hence we have

$$\int_B \eta\, dV \le \mathrm{vol}(B) \cdot \sup_{(z,w)\in B} \eta(z,w) = O(r^2),$$

which implies the contribution of B to the integral I_r is bounded by $O(r^2)$.

- $C = \{(z,w) \in S(r) |\ \frac{1}{2r} \le |z| \le \frac{3}{2r}$ and $|\sin\theta| > \frac{1}{r}\}$. Then its image by Φ is

$$\Phi(C) = \Big\{z \in \mathbb{C} \Big|\ \frac{1}{2r} \le |z| \le \frac{3}{2r}\Big\} \times \Big\{\theta r \Big|\ \theta \in [0,2\pi), |\sin\theta| > \frac{1}{r}\Big\} \times \Big\{r\Big\}.$$

For $z \in \mathbb{C}$ with $\frac{1}{2r} \leq |z| \leq \frac{3}{2r}$, we define

$$J_r(z) := \int_{0<\theta<2\pi,\, |\sin\theta| > \frac{1}{r}} \eta(\Phi^{-1}(z, r\theta, r))\, r d\theta.$$

Since $|w| \sim r$, we obtain $\frac{1}{2} \lesssim |zw| \lesssim \frac{3}{2}$. Using arithmetic estimate (iv), we get

$$\eta \leq \frac{r^2}{|1+zw|^2} \leq \frac{r^2}{\left|\sin^2\theta\right| |zw|^2} \leq \frac{c \cdot r^2}{\left|\sin^2\theta\right|}.$$

Here c is a constant greater than four. Hence we obtain

$$J_r(z) \leq \int_{0<\theta<2\pi,\, |\sin\theta| > \frac{1}{r}} \frac{c \cdot r^2}{\left|\sin^2\theta\right|} r d\theta = 4 \int_{\arcsin\frac{1}{r}}^{\frac{\pi}{2}} \frac{c \cdot r^3}{\left|\sin^2\theta\right|} d\theta = 4c \cdot r^4 \sqrt{1 - \frac{1}{r^2}} \leq 4c \cdot r^4.$$

Therefore it follows that

$$\int_C \eta\, dV = \int_{\frac{1}{2r} \leq |z| \leq \frac{3}{2r}} J_r \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \leq c' r^2 \tag{3.5}$$

where c' is a positive constant. Thus the contribution of C to the integral I_r is bounded by $O(r^2)$.

- For $n \in \{-2, -1, 0, 1\}$, set $D_n = \{(z, w) \in S(r) |\, |z| \geq \frac{3}{2r},\, |z| \leq r^{1-\varepsilon}$ and $r^{\frac{n}{2}} \leq |z| \leq r^{\frac{n+1}{2}}\}$. For each n, the integrand η is bounded by $O(r^{-n})$ on D_n due to (iii), and $\mathrm{vol}(D_n) = O(r^{2+n})$ because $D_n \subset \Omega_{1, \frac{n+1}{2}, 0}$. Thus the contribution of D_n to the integral I_r is bounded by $O(r^2)$.

- $E = \{(z, w) \in S(r) |\, |z| \geq r^{1-\varepsilon}, |w| \geq r^{\frac{1}{2}}\}$. Since $|zw| \geq r^{\frac{3}{2}-\varepsilon}$, we have

$$\eta|_E \leq \frac{r^2}{|1+zw|^2} \leq \frac{r^2}{(|zw|-1)^2} \leq \frac{r^2}{(r^{\frac{3}{2}-\varepsilon}-1)^2} = O(r^{2\varepsilon-1}).$$

  Because $\mathrm{vol}(E)$ is bounded by the total volume of $S(r)$, $\mathrm{vol}(E) = O(r^3)$. Thus the contribution of E to I_r is bounded by $O(r^{2+2\varepsilon})$.

- $F = \{(z, w) \in S(r) |\, 1 \leq |w| \leq r^{\frac{1}{2}}\}$. Since $|z| = \sqrt{r^2 - |w|^2} \geq \sqrt{r^2 - r} > 1$, we have

$$\eta|_F \leq \frac{r^2}{|1+zw|^2} \leq \frac{r^2}{(\sqrt{r^2-r}-1)^2} = O(1).$$

  Because the volume of F agrees with the volume of $\{(z, w) \in S(r) |\, 1 \leq |z| \leq r^{\frac{1}{2}}\} \subset \Omega_{1, \frac{1}{2}, 0}$, we obtain

$$\mathrm{vol}(F) \leq \mathrm{vol}(\Omega_{1, \frac{1}{2}, 0}) = O(r^2).$$

  Thus the contribution of F to I_r is bounded by $O(r^2)$.

- $G = \{(z,w) \in S(r) | \, r^{-1} \leq |w| \leq 1\}$. Since $|z| \leq r$, we have $|zw| \leq r$. This implies $|1+zw|^{2\gamma} \leq (r^2+2r+1)^{\gamma}$. Hence we obtain

$$\eta|_G \leq \frac{r^2}{|z|^2}(|1+zw|^{2\gamma}+|z|^{2\delta}) \leq O(r^{2\gamma}) \leq O(r).$$

  Here we used $|z| \sim r$ and $0 \leq \delta \leq \gamma \leq \frac{1}{2}$. Because $\mathrm{vol}(G) \leq \mathrm{vol}(\Omega_{1,0,0}) = O(r)$, the contribution of G to I_r is bounded by $O(r^2)$.

- $H = \{(z,w) \in S(r) | \, 0 \leq |w| \leq r^{-1}\}$. Since $|w| \leq r^{-1}$, we have $|z| \sim r$ and $|zw| \leq 1$. Hence we obtain

$$\eta|_H \leq \frac{r^2}{|z|^2}(|1+zw|^{2\gamma}+|z|^{2\delta}) \leq O(r^{2\delta}) \leq O(r).$$

  Because $\mathrm{vol}(H) \leq O(r^{-1})$, the contribution of H to the integral I_r is bounded by $O(1)$.

Eleven regions A, B, C, D_{-2}, D_{-1}, D_0, D_1, E, F, G, H cover the sphere $S(r)$. On each such region Ω we have verified

$$\int_{\Omega} \eta \, dV = O(r^{2+\varepsilon}), \quad \varepsilon > 0.$$

Therefore those establish our claim

$$I_r = O(r^{2+\varepsilon}), \quad \varepsilon > 0.$$

As a consequence, the holomorphic mapping $f : \mathbb{C}^2 \to S_{a,b}$ induced by $\tilde{f} : (z,w) \mapsto (z, 1+zw)$ is of order at most one.

3.2 The second step: To remove assumption (3.1)

We show by branched covering argument that for every $a, b \in \mathbb{C}$ with $1 < |b| \leq |a|$, there exists a differentiably non-degenerate meromorphic mapping from $\mathbb{C}^2$ into Hopf surface $S_{a,b}$ with order at most one.

Take $a, b \in \mathbb{C}$ with $1 < |b| \leq |a|$. Then there exist $p, q \in \mathbb{N}$ such that $|b|^q \leq |a|^p \leq |b|^{\frac{3}{2}q}$. Let $\Pi_{a,b}$ be the universal covering of $S_{a,b}$, and Π_{a^p,b^q} be the one of S_{a^p,b^q}. We define a holomorphic mapping $\tilde{\Psi}$ as follows,

$$\tilde{\Psi} : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{C}^2 \setminus \{(0,0)\}, \quad (x,y) \mapsto (x^q, y^p).$$

Then $\tilde{\Psi}$ induces a branched covering Ψ,

$$\begin{array}{ccc} \mathbb{C}^2 \setminus \{(0,0)\} & \xrightarrow{\tilde{\Psi}} & \mathbb{C}^2 \setminus \{(0,0)\} \\ \Pi_{a^p,b^q} \big\downarrow & & \Pi_{a,b} \big\downarrow \\ S_{a^p,b^q} & \xrightarrow[\Psi]{} & S_{a,b} \end{array} .$$

Note that a^p and b^q satisfy (3.1). By the first step, there exists a differentiably non-degenerate holomorphic mapping $g : \mathbb{C}^2 \to S_{a^p, b^q}$ with order at most one. Then $\Psi \circ g$ is also a differentiably non-degenrate holomorphic mapping from $\mathbb{C}^2$ into $S_{a,b}$ with order at most one since $d\Psi$ is generically rank 2. $\square$

4 Inoue Surfaces : Proof of the Main Theorem 1.2.

M. Inoue constructed in [2], three type of surfaces S_M, $S^{(+)}_{N,p,q,r;t}$ and $S^{(-)}_{N,p,q,r}$, which are called Inoue surfaces. It is known that a VII_0 surface with second betti number zero is either an Inoue surface or a Hopf surface, and that an Inoue surface contains no closed curve. In this section we recall the definition of S_M, $S^{(+)}_{N,p,q,r;t}$, $S^{(-)}_{N,p,q,r}$ and prove the Main Theorem 1.2. as $S = S_M$, $S^{(+)}_{N,p,q,r;t}$, $S^{(-)}_{N,p,q,r}$ respectively.

The case of $S = S_M$: Let $\mathbb{H} = \{x \in \mathbb{C} \mid \mathrm{Im} x > 0\}$ be the upper half plane. Let $M = (m_{ij}) \in SL(3, \mathbb{Z})$ be a unimodular matrix with one real eigenvalue $\lambda_1 > 1$ and two complex conjugate eigenvalues $\lambda_2 \neq \bar{\lambda}_2$. Note that $\lambda_1 |\lambda_2|^2 = 1$ and that real number λ_1 is necessarily irrational. Let (a_1, a_2, a_3) be a real eigenvector with eigenvalue λ_1 and let (b_1, b_2, b_3) be an eigen vector with eigen value λ_2. Since (a_1, a_2, a_3), (b_1, b_2, b_3), $(\bar{b_1}, \bar{b_2}, \bar{b_3})$ are $\mathbb{C}$-linearly independent, it follows that (a_1, b_1), (a_2, b_2), and (a_3, b_3) are $\mathbb{R}$-linearly independent. Let G_M be the group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0(x, y) =& (\lambda_1 x, \lambda_2 y), \\ g_j(x, y) =& (x + a_j, y + b_j), 1 \leq j \leq 3. \end{aligned}$$

Then G_M acts on $\mathbb{H} \times \mathbb{C}$ properly discontinuously without fixed points. Hence

$$S_M = (\mathbb{H} \times \mathbb{C}) / G_M$$

is a complex surface. Furthermore by the definition of the action, S_M becomes a compact complex surface, which is diffeomorphic to a 3-torus bundle over a circle. Relations between the generators g_0, g_1, g_2, g_3 of G_M are as follows:

$$\begin{aligned} g_i g_j &= g_j g_i & \text{for } i, j = 1, 2, 3, \\ g_0 g_i {g_0}^{-1} &= {g_1}^{m_{i1}} {g_2}^{m_{i2}} {g_3}^{m_{i3}} & \text{for } i = 1, 2, 3. \end{aligned}$$

It follows that

$$H_1(S_M, \mathbb{Z}) \cong \pi_1(S_M)/[\pi_1(S_M), \pi_1(S_M)] \cong G_M/[G_M, G_M] = \mathbb{Z} \oplus \mathbb{Z}_{e_1} \oplus \mathbb{Z}_{e_2} \oplus \mathbb{Z}_{e_3},$$

where $e_1, e_2, e_3 \neq 0$ are the elementary divisors of $M - I$. Hence $b_1(S_M) = 1$. Thus we deduce $b_2(S_M) = 0$, since Euler characteristic of S_M is zero.

Proof. We first prove that meromorphic mapping $f : \mathbb{C}^n \to S$ is holomorphic. Let $p : \mathbb{H}\times\mathbb{C} \to S$ be the universal covering mapping. Since $\operatorname{codim} I(f) \geq 2$, $\mathbb{C}^n \setminus I(f)$ is simply connected. Then we get a holomorphic lift

$$\widetilde{f_{\mathbb{C}^n\setminus I(f)}} : \mathbb{C}^n \setminus I(f) \to \mathbb{H}\times\mathbb{C}$$

of

$$f|_{\mathbb{C}^n\setminus I(f)} : \mathbb{C}^n \setminus I(f) \to S.$$

Since $\operatorname{codim} I(f) \geq 2$, the holomorphic mapping $\widetilde{f_{\mathbb{C}^n\setminus I(f)}} : \mathbb{C}^n \setminus I(f) \to \mathbb{H}\times\mathbb{C}$ extends to a holomorphic mapping $\tilde{f} : \mathbb{C}^n \to \mathbb{H}\times\mathbb{C}$. Because $f = p \circ \tilde{f}$, we deduce that f is holomorphic.

We now calculate the order of f. Since S_M is compact, the order is independent of the choice of Hermitian metric forms on S_M. We define a Hermitian metric form on $\mathbb{H}\times\mathbb{C}$ which is invariant under the action of G_M as follows

$$\tilde{\omega} = \frac{\sqrt{-1}}{2\pi}\Big(\frac{1}{(\mathrm{Im}x)^2}\, dx\wedge d\bar{x} + (\mathrm{Im}x)\, dy\wedge d\bar{y}\Big).$$

Note that $\lambda_1 |\lambda_2|^2 = 1$. Let $\tilde{f} = (f_1, f_2) : \mathbb{C}^n \to \mathbb{H}\times\mathbb{C}$ be a holomorphic lift of f. Then f_1 is constant. Set $\mathrm{Im} f_1 = c$. Since

$$\tilde{f}^*\tilde{\omega} = \frac{\sqrt{-1}}{2\pi}(\frac{1}{c^2} df_1\wedge d\bar{f}_1 + c\, df_2\wedge d\bar{f}_2) = \frac{\sqrt{-1}}{2\pi}(c\, df_2\wedge d\bar{f}_2) = \frac{\sqrt{-1}}{2\pi}(c\,\partial f_2\wedge\bar{\partial}\bar{f}_2),$$

we obtain

$$\tilde{f}^*\tilde{\omega}\wedge\alpha^{n-1} = c\, dd^c\,|f_2|^2\wedge\alpha^{n-1}.$$

Therefore we have

$$T_f(r,\omega) = \int_1^r \frac{dt}{t^{2n-1}}\int_{B(t)} f^*\omega\wedge\alpha^{n-1} = \int_1^r \frac{dt}{t^{2n-1}}\int_{B(t)} c\, dd^c|f_2|^2\wedge\alpha^{n-1}.$$

From Jensen's formula we obtain

$$\int_1^r \frac{dt}{t^{2n-1}}\int_{B(t)} c\, dd^c|f_2|^2\wedge\alpha^{n-1} = \frac{c}{2}\int_{S(r)} |f_2|^2\,\zeta - \frac{c}{2}\int_{S(1)} |f_2|^2\zeta.$$

Let $f_2(z) = \sum_{k\geq 0} P_k(z_1, ..., z_n)$ be the expansion with homogeneous polynomials P_k of degree k. Since f_2 is not constant, there exists $k_0 \geq 1$ such that $P_{k_0} \neq 0$. Hence we obtain

$$\frac{c}{2}\int_{S(r)} |f_2|^2\zeta = \frac{c}{2}\sum_{k\geq 0} r^{2k}\int_{S(1)} |P_k|^2\,\zeta \geq \frac{c\cdot r^{2k_0}}{2}\int_{S(1)} |P_{k_0}|^2\,\zeta \geq \frac{c\cdot r^2}{2}\int_{S(1)} |P_{k_0}|^2.$$

Therefore we deduce the order of f satisfies $\rho_f \geq 2$, since $c \neq 0$ and $\int_{S(1)} |P_{k_0}|^2 \neq 0$. $\square$

The case of $S = S^{(+)}_{N,p,q,r;t}$: Here we study Inoue surface $S^{(+)}_{N,p,q,r;t}$. Let $N = (n_{ij}) \in SL(2,\mathbb{Z})$ be a matrix with two real eigenvalues $\lambda > 1$ and $\frac{1}{\lambda}$. Let (a_1, a_2) and (b_1, b_2) be two real eigen vectors of N corresponding to λ and $\frac{1}{\lambda}$ respectively (λ is necessarily irrational).

Fix integers p, q, r with $r \neq 0$ and a complex number t. Set real numbers (c_1, c_2) as the solution of the following linear equation

$$(c_1, c_2) = (c_1, c_2) \cdot^t N + (e_1, e_2) + \frac{b_1a_2 - b_2a_1}{r}(p, q),$$

where

$$e_i = \frac{1}{2}n_{i1}(n_{i1} - 1)a_1b_1 + \frac{1}{2}n_{i2}(n_{i2} - 1)a_2b_2 + n_{i1}n_{i2}b_1a_2, \quad i = 1, 2.$$

Let $G^{(+)}_{N,p,q,r,t}$ be the group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0(x, y) &= (\lambda x, y + t), \\ g_j(x, y) &= (x + a_j, y + b_jx + c_j), \quad j = 1, 2, \\ g_3(x, y) &= \left(x, y + \frac{b_1a_2 - b_2a_1}{r}\right). \end{aligned}$$

They satisfy the following relations:

$$\begin{aligned} g_3g_i &= g_ig_3 \quad \text{for } i = 0, 1, 2, \\ g_1g_2 &= g_2g_1g_3^r, \\ g_0g_1g_0{}^{-1} &= g_1^{n_{11}}g_2^{n_{12}}g_3^p, \\ g_0g_2g_0{}^{-1} &= g_1^{n_{21}}g_2^{n_{22}}g_3^q. \end{aligned} \tag{4.1}$$

Then $S^+_{N,p,q,r;t} = (\mathbb{H} \times \mathbb{C})/G^{(+)}_{N,p,q,r,t}$ is an Inoue surface. Since the action is properly discontinuously with no fixed points, $S^{(+)}_{N,p,q,r;t}$ becomes a complex surface. Moreover it is a compact complex surface. It is known that $S^{(+)}_{N,p,q,r;t}$ is diffeomorphic to a fiber bundle over a circle whose fiber is a circle bundle over a two torus ([2]). It is known that $b_1(S^+_{N,p,q,r;t}) = 1$ and $b_2(S^+_{N,p,q,r;t}) = 0$.

Proof. Let $p : \mathbb{H} \times \mathbb{C} \to S^{(+)}_{N,p,q,r;t}$ be the universal covering. As in the case of S_M, every meromorphic mapping $f : \mathbb{C}^n \to S^{(+)}_{N,p,q,r;t}$ is holomorphic. We construct an Hermitian metric on $\mathbb{H} \times \mathbb{C}$ which is invariant under the the action of $G^{(+)}_{N,p,q,r,t}$ and which makes it easier to calculate the order of f. Take an arbitrary Hermitian metric form ω on $S^{(+)}_{N,p,q,r,t}$. Let $\tilde{\omega}$ be the pull-back $p^*\omega$. Then $\tilde{\omega}$ is invariant under the action. Write $\tilde{\omega}$ in coordinates,

$$\tilde{\omega} = \frac{\sqrt{-1}}{2\pi}(h_{11}dx \wedge d\bar{x} + h_{12}dx \wedge d\bar{y} + h_{21}dy \wedge d\bar{x} + h_{22}dy \wedge d\bar{y}).$$

Then $h_{22} \neq 0$ since $\tilde{\omega}$ is a positive Hermitian metric form. Therefore we can define a Hermitian metric form $\tilde{\sigma} = \frac{\tilde{\omega}}{h_{22}}$. Note that the coefficient of $dy \wedge d\bar{y}$ of $\tilde{\sigma}$ is one. Since $g_i^*\tilde{\omega} = \tilde{\omega}$, we obtain

$h_{22}(g_i(x,y)) = h_{22}(x,y)$ for $i = 0,1,2,3$. This implies

$$g_i^*\tilde{\sigma} = g_i^*(\frac{\tilde{\omega}}{h_{22}}) = \frac{\tilde{\omega}}{h_{22}} = \tilde{\sigma}.$$

Let $\tilde{f} = (f_1, f_2) : \mathbb{C}^n \to \mathbb{H} \times \mathbb{C}$ be a holomorphic lift of $f : \mathbb{C}^n \to S^{(+)}_{N,p,q,r;t}$. We calculate the order of $\tilde{f}$ with respect to $\tilde{\sigma}$. Since f_1 is constant, we have

$$\begin{aligned}\tilde{f}^*\tilde{\sigma} =& \frac{\sqrt{-1}}{2\pi}(\frac{h_{11}}{h_{22}}(\tilde{f})df_1 \wedge d\bar{f}_1 + \frac{h_{12}}{h_{22}}(\tilde{f})df_1 \wedge d\bar{f}_2 + \frac{h_{21}}{h_{22}}(\tilde{f})df_2 \wedge d\bar{f}_1 + \frac{h_{22}}{h_{22}}(\tilde{f})df_2 \wedge d\bar{f}_2) \\ =& \frac{\sqrt{-1}}{2\pi}(df_2 \wedge d\bar{f}_2).\end{aligned}$$

Hence we obtain

$$T_{\tilde{f}}(r;\tilde{\sigma}) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} \tilde{f}^*\tilde{\sigma} \wedge \alpha^{n-1} = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} dd^c|f_2|^2 \wedge \alpha^{n-1}. \tag{4.2}$$

Note that f_2 is not constant. As in the case of S_M or in the case of complex torus, we deduce from (4.2) that the order of f satisfies $\rho_f \geq 2$. □

The case of $S = S^{(-)}_{N,p,q,r}$: We define an Inoue surface $S^{(-)}_{N,p,q,r}$ as follows. Let $N = (n_{ij}) \in GL(2,\mathbb{Z})$ be a matrix with $\det N = -1$ and with two real eigenvalues λ and $-\frac{1}{\lambda}$. Let (a_1, a_2) and (b_1, b_2) be two real eigenvectors for N with eigenvalues λ and $-\frac{1}{\lambda}$ respectively. Fix integers p, q, r, with $r \neq 0$. Define two real numbers (c_1, c_2) as the solution of the following linear equation

$$-(c_1, c_2) = (c_1, c_2)\ {}^tN + (e_1, e_2) + \frac{b_1a_2 - b_2a_1}{r}(p, q),$$

where e_i are the same as for the surface $S^{(+)}_{N,p,q,r;t}$. Let $G^{(-)}_{N,p,q,r}$ be a group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0(x,y) &= (\lambda x, -y), \\ g_j(x,y) &= (x + a_j, y + b_jx + c_j), \quad j = 1,2, \\ g_3(x,y) &= \left(x, y + \frac{b_1a_2 - b_2a_1}{r}\right). \end{aligned}$$

Then $S^{(-)}_{N,p,q,r} = (\mathbb{H} \times \mathbb{C})/G^{(-)}_{N,p,q,r}$ is an Inoue surface.

Proof. As we have seen in other Inoue surfaces, the meromorphic mapping f is holomorphic. As in the case of $S^{(+)}_{N,p,q,r;t}$, we can construct a Hermitian metric form $\tilde{\sigma}$ on $\mathbb{H} \times \mathbb{C}$ which is invariant under the action of $G^{(-)}_{(N,p,q,r)}$ and is written in coordinates as follows,

$$\tilde{\sigma} = \frac{\sqrt{-1}}{2\pi}(h_{11}dx \wedge d\bar{x} + h_{12}dx \wedge d\bar{y} + h_{21}dy \wedge d\bar{x} + dy \wedge d\bar{y}).$$

Note that the coefficient of $dy \wedge d\bar{y}$ is one. This implies that the order of a non-constant holomorphic mapping $f : \mathbb{C}^n \to S^{(-)}_{N,p,q,r}$ satisfies $\rho_f \geq 2$. □

5 Inoue surfaces : Restriction of the universal covering to a leaf

We now prove that the restriction of the universal covering mapping to a leaf $\{x_0\} \times \mathbb{C}$ ($\forall x_0 \in \mathbb{H}$) is of order two.

Proposition 5.1. *Let S be an Inoue surface and let $p : \mathbb{H} \times \mathbb{C} \to S$ be the universal covering mapping. Fix an arbitrary $x_0 \in \mathbb{H}$. Let $\tilde{f}$ be a holomorphic mapping $w \in \mathbb{C} \mapsto (x_0, w) \in \mathbb{H} \times \mathbb{C}$. Then $p \circ \tilde{f}$ has order two.*

Proof. The case of $S = S_M$. Take the following Hermitian metric form on $\mathbb{H} \times \mathbb{C}$

$$\tilde{\omega} = \frac{\sqrt{-1}}{2\pi} \left(\frac{1}{(\mathrm{Im} x)^2} \, dx \wedge d\bar{x} + (\mathrm{Im} x) \, dy \wedge d\bar{y} \right).$$

Let ω be the induced Hermitian metric form on S by $\tilde{\omega}$. We calculate the characteristic function of $p \circ \tilde{f}$ with respect to ω. Since $\tilde{f}^* \tilde{\omega} = (\mathrm{Im} x_0)\alpha$,

$$T_{p \circ \tilde{f}}(r, \omega) = T_{\tilde{f}}(r, \tilde{\omega}) = \int_1^r \frac{dt}{t} \int_{B(t)} \tilde{f}^* \tilde{\omega} = \frac{1}{2}(\mathrm{Im} x_0) r^2 - \frac{1}{2}(\mathrm{Im} x_0).$$

Hence we obtain $\rho_{p \circ \tilde{f}} = 2$.

The case of $S = S^{(+)}_{N,p,q,r;t}, S^{(-)}_{N,p,q,r}$. Take the following Hermitian metric form on $\mathbb{H} \times \mathbb{C}$

$$\tilde{\sigma} = \frac{\sqrt{-1}}{2\pi} (h_{11} dx \wedge d\bar{x} + h_{12} dx \wedge d\bar{y} + h_{21} dy \wedge d\bar{x} + dy \wedge d\bar{y}).$$

Let σ be the induced Hermitian metric form on S. We calculate the characteristic function of $p \circ \tilde{f}$ with respect to σ. Since $\tilde{f}^* \tilde{\sigma} = \alpha$, we have

$$T_{p \circ \tilde{f}}(r, \sigma) = T_{\tilde{f}}(r, \tilde{\sigma}) = \int_1^r \frac{dt}{t} \int_{B(t)} \tilde{f}^* \tilde{\sigma} = \frac{1}{2} r^2 - \frac{1}{2}.$$

Hence we deduce $\rho_{p \circ \tilde{f}} = 2$. □

Remark 5.2. By similar calculations, we get the order of the holomorphic mapping from $\mathbb{C}^n$ to an Inoue surface S induced by $(z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n \mapsto (x_0, w^d) \in \mathbb{H} \times \mathbb{C}$ is $2d$.

Remark 5.3. Let S be an Inoue surface. Let $p : \mathbb{H} \times \mathbb{C} \to S$ be the universal covering mapping. Fix an arbitrary $x_0 \in \mathbb{H}$. Then its image $p(\{x_0\} \times \mathbb{C}) \subset S$ is Zariski dense, but not dense with respect to the differential topology, for there is no closed curves on an Inoue surface (see [8]). The differential structure of an Inoue surface S is as follows:

If $S = S_M$, S is diffeomorphic to a real 3-torus bundle over a circle parametrized by the imaginary part $\mathrm{Im}\, x$ of $x \in \mathbb{H}$.

If $S = S^{(+)}_{N,p,q,r;t}$, S is diffeomorphic to a fiber bundle over a circle parametrized by $\mathrm{Im}\, x$, whose fiber is a real three dimensional compact manifold. According to [2], this three dimensional compact manifold is a circle bundle over a real 2-torus.

If $S = S^{(-)}_{N,p,q,r}$, S is diffeomorphic to a fiber bundle over a circle parametrized by $\operatorname{Im} x$, whose fiber is a real three dimensional compact manifold.

6 Problems

Finally we pose some interesting questions related to characteristic functions of meromorphic mappings from $\mathbb{C}^2$ into Hopf surfaces.

Problem 6.1. Let $S_{a,b}$ be a Hopf surface defined in Main Theorem 1.1. We define a non-negative number $\rho(S_{a,b})$ as follows,

$$\rho(S_{a,b}) = \inf\{\rho_f | \, f : \mathbb{C}^2 \to S_{a,b} \, \text{differentiably non-degenerate meromorphic mapping}\}.$$

Which number is $\rho(S_{a,b})$? Since there exists a holomorphic mapping from $\mathbb{C}^2$ into $S_{a,b}$ with order at most one, we have at least $\rho(S_{a,b}) \le 1$.

Problem 6.2. Let $S_{a,a}$ be a Hopf surface defined in theorem 2.8. Let $f : \mathbb{C}^2 \to S_{a,a}$ be a holomorphic mapping, and let $\tilde{f} = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2 \backslash \{(0,0)\}$ be its lift. Let $\tilde{\omega} = \frac{\sqrt{-1}}{2\pi} \frac{dx \wedge d\bar{x} + dy \wedge d\bar{y}}{|x|^2 + |y|^2}$ be a Hermitian metric form on $\mathbb{C}^2 \backslash \{(0,0)\}$ and let ω be the induced Hermitian metric form on $S_{a,a}$. Let ω_0 be Fubini-Study metric form on $\mathbb{P}^1(\mathbb{C})$ and let $\pi : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1(\mathbb{C})$, $(x, y) \mapsto [x : y]$ be the Hopf mapping. Set $F = \pi \circ \tilde{f}$. Then we found the following decomposition of the characteristic function of f with respect to ω,

$$T_f(r, \omega) = T_F(r, \omega_0) + \int_1^r \frac{dt}{t^3} \int_{B(t)} d \log(|f_1|^2 + |f_2|^2) \wedge d^c \log(|f_1|^2 + |f_2|^2) \wedge \alpha.$$

Let $R_f(r)$ denote the second term of the above formula. It is interesting to compare the growths of $T_F(r, \omega_0)$ and $R_f(r)$ as $r \to \infty$ or the growths of $T_F(r, \omega_0)$ and $T_f(r, \omega)$ as $r \to \infty$.

References

[1] W. Barth, C. Peters and A. Van de Ven, Compact Complex Surfaces, Ergebnisse Math. und ihrer Grenzgebiete **4**, Springer-Verlag, 1984.

[2] M. Inoue, On surfaces of class VII_0, Invent. Math. **24** (1974), 269-310.

[3] J. Li, S.-T. Yau, F. Zheng, On projectively flat Hermitian manifolds, Comm. Anal. Geom. **2** (1994), no.1, 103-109.

[4] J. Noguchi and T. Ochiai, Geometric Function Theory in Several Complex Variables, Math. Monog. Vol. **80**, Amer. Math. Soc., Providence R.I., 1990 (translated from Japanese version published from Iwanami, Tokyo 1984).

[5] J. Noguchi, Introduction to Complex Analysis (in Japanese), Shokabo.Co.,1993: Translated to English, Math. Monog. **168**, Amer. Math. Soc., Providence R.I., 1997

[6] J. Noguchi, Nevanlinna Theory in Several Complex Variables and Diophantine Approximation (in Japanese), Kyoritsu Publ. Co., 2003.

[7] J. Noguchi and J. Winkelmann, Order of meromorphic maps and rationality of the image space, J. Math. Soc. Jpn., Vol. **64** No.4 (2012), 1169-1180.

[8] A. Teleman, Projectively flat surfaces and Bogomolov's theorem on class VII_0 surfaces, Internat. J. Math. **5** (1994), no.2, 253-264.

Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Meguroku, Tokyo 153-8914
e-mail: amemiya@ms.u-tokyo.ac.jp
t.amemiya@11.alumni.u-tokyo.ac.jp

UTMS

2012–12 Hajime Fujita, Mikio Furuta and Takahiko Yoshida: *Geodesic flows on spheres and the local Riemann-Roch numbers.*

2012–13 H. Kawakami, A. Nakamura, and H. Sakai: *Degeneration scheme of 4-dimensional Painlevé-type equations.*

2012–14 Yusaku Tiba: *The second main theorem for entire curves into Hilbert modular surfaces.*

2012–15 O. Yu. Imanuvilov and M. Yamamoto: *Inverse boundary value problem for Schrödinger equation in cylindrical domain by partial boundary data.*

2012–16 Nao Hamamuki: *A discrete isoperimetric inequality on lattices.*

2012–17 Nao Hamamuki: *Asymptotically self-similar solutions to curvature flow equations with prescribed contact angle and their applications to groove profiles due to evaporation-condensation.*

2012–18 Guanyu Zhou: *Analysis of the fictitious domain method with L^2-penalty for elliptic and parabolic problems.*

2013–1 Yasuko Hasegawa: *The critical values of exterior square L-functions on $GL(2)$.*

2013–2 Toshio Oshima: *An elementary approach to the Gauss hypergeometric function.*

2013–3 Kazuki Okamura: *Large deviations for simple random walk on percolations with long-range correlations .*

2013–4 Kiyoomi Kataoka and Nobuko Takeuchi: *A system of fifth-order PDE's describing surfaces containing 2 families of circular arcs and the reduction to a system of fifth-order ODE's .*

2013–5 Takushi Amemiya: *Orders of meromorphic mappings into Hopf and Inoue surfaces.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012