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Abstract. In this paper, we study the relation between the Shilov boundaries of the pluricomplex Green function's level sets and the support of currents defined by the pluricomplex Green function. We also study the set which intersects the support of any non-zero closed positive current by applying our main result.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and let $x \in \Omega$. Let $\mathrm{Psh}(\Omega)$ denote the plurisubharmonic functions on Ω. The (Klimek) pluricomplex Green function of Ω with logarithmic pole at x is defined as

$$K_{\Omega,x} = \sup\{u(z); u \in \mathrm{Psh}(\Omega), u < 0, \limsup_{z \to x} u(z) - \log \|z - x\| < \infty\}.$$

We denote by $d = d' + d''$ the usual decomposition of the exterior derivative in terms of its $(1,0)$ and $(0,1)$ parts, and we set $d^c = (2\sqrt{-1}\pi)^{-1}(d' - d'')$. We shall say that a (bounded) pseudoconvex domain $\Omega \subset \mathbb{C}^n$ is hyperconvex if there exists a continuous plurisubharmonic function $v : \Omega \to (-\infty, 0)$ such that the set $\{z \in \Omega; v(z) < c\}$ is a relatively compact subset of Ω for each $c \in (-\infty, 0)$. If Ω is a bounded hyperconvex domain, Demailly [4] showed that $K_{\Omega,x}$ is the unique solution for the following homogeneous Monge-Ampère equation:

$$\begin{cases} u \in \mathrm{Psh}(\Omega) \cap L^{\infty}_{loc}(\bar{\Omega} \setminus \{x\}), \\ (dd^c u)^n \equiv 0 \quad \text{in} \quad \Omega \setminus \{x\}, \\ u(z) - \log \|z - x\| = O(1) \quad \text{for} \quad z \to x, \\ \lim_{z \to p} u(z) = 0 \quad \text{for all} \quad p \in \partial\Omega. \end{cases}$$

In this paper, we study the relation between the Shilov boundaries of the pluricomplex Green function's level sets and the support of the measure $dK_{\Omega,x} \wedge d^c K_{\Omega,x} \wedge (dd^c K_{\Omega,x})^{n-1}$ in $\Omega \setminus \{x\}$. Note that $dK_{\Omega,x} \wedge d^c K_{\Omega,x} \wedge (dd^c K_{\Omega,x})^{n-1}$ is well defined in $\Omega \setminus \{x\}$. Let E be a compact subset of $\mathbb{C}^n$. We define the Shilov boundary of E by the smallest closed subset $\partial_S E$ of E such that, for each function f which is holomorphic on a neighborhood of E the equality $\max_E |f| = \max_{\partial_S E} |f|$ holds. Let φ be an upper semi-continuous function on Ω. We define $S_\varphi(r) = \{z \in \Omega; \varphi(z) = r\}$, $B_\varphi(r) = \{z \in \Omega; \varphi(z) < r\}$, $\overline{B}_\varphi(r) = \{z \in \Omega; \varphi(z) \leq r\}$. Our main theorem is the following:

Theorem 1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $x \in \Omega$. Let $K_{\Omega,x}$ be the pluricomplex Green function with logarithmic pole at x. The support of the measure $dK_{\Omega,x} \wedge d^c K_{\Omega,x} \wedge (dd^c K_{\Omega,x})^{n-1}$ in $\Omega \setminus \{x\}$ is equal to the closure of*

$$\bigcup_{-\infty < r < 0} \partial_S \overline{B}_{K_{\Omega,x}}(r) \quad \text{in } \Omega \setminus \{x\}.$$

Let $\delta : \mathbb{C}^n \to [0, \infty)$ be a continuous function such that $\delta^{-1}(0) = \{0\}$, $\delta(\xi z) = |\xi| h(z)$ for all $\xi \in \mathbb{C}, z \in \mathbb{C}^n$. Let $\Omega = \{z \in \mathbb{C}^n; h(z) < 1\}$. The function δ is called a Minkowski function of Ω and Ω is called a balanced domain. Assume that $\log \delta \in \mathrm{Psh}(\mathbb{C}^n)$. Then $\log \delta$ is equal to the pluricomplex Green function on Ω with logarithmic pole at $\{0\}$. In the case of a balanced domain, we can prove a more strong result (see the last section). We also show that the support of any non-zero closed positive current of bidegree $(1, 1)$ intersects the closure of $\bigcup_{0 \le r < 1} \partial_S \overline{B}_\delta(r)$ (see Corollary 2).

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2. Monge-Ampère operator and Integration

The Monge-Ampère operator is well defined on locally bounded plurisubharmonic functions (see [1], [2]). In [2], the following convergence of currents is proved.

Theorem 2 ([2])**.** *Let Ω be an open set in $\mathbb{C}^n$ and let $u_0, \dots, u_n \in \mathrm{Psh}(\Omega) \cap L^\infty(\Omega)$. Let $\{u_0^j\}_j, \dots, \{u_n^j\}_j$ are sequences of uniformly bounded plurisubharmonic functions such that u_i^j converges monotonically almost everywhere, either increasing or decreasing, to u_i $(i = 0, \dots, n)$ when j goes to ∞. Then*

$$(1) \quad dd^c u_1^j \wedge \cdots \wedge dd^c u_n^j \to dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

$$(2) \quad u_0^j dd^c u_1^j \wedge \cdots \wedge dd^c u_n^j \to u_0 dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

$$(3) \quad du_0^j \wedge d^c u_1^j \wedge dd^c u_2^j \wedge \cdots \wedge dd^c u_n^j \to du_0 \wedge d^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_n$$

in the weak topology on the space of currents.

Let $\varphi \in \mathrm{Psh}(\Omega)$. For $r \in \mathbb{R}$, we define $\varphi_{\ge r} = \max\{\varphi, r\}$. We show the following equation.

Proposition 1. *Let Ω be an open set in $\mathbb{C}^n$ and let $u \in \mathrm{Psh}(\Omega) \cap L^\infty(\Omega)$. Take $a, b \in \mathbb{R}$, $a < b$ such that $a < u < b$ in Ω. Let h be a smooth function which has a compact support in Ω. Then we have*

$$\int_a^b dr \int_\Omega h (dd^c u_{\ge r})^n = \int_\Omega h du \wedge d^c u \wedge (dd^c u)^{n-1} + \int_\Omega (u - a) h (dd^c u)^n.$$

Note that $\int_\Omega h(dd^c u_{\ge r})^n$ is a continuous function of $r \in \mathbb{R}$ by Theorem 2.

Proof. We first show that the equation is true when u is smooth. Let $E \subset [a, b]$ be the critical values of u. Assume that $r \notin E$. Then $S_u(r)$ is a smooth oriented real hypersurface and

$$\int_\Omega h(dd^c u_{\ge r})^n = \int_{S_u(r)} h d^c u \wedge (dd^c u)^{n-1} + \int_{\Omega \setminus B_u(r)} h (dd^c u)^n$$

(see Proposition 4.4 of [5]). By Sard's theorem, E is measure zero set. Hence

$$\int_a^b dr \int_\Omega h (dd^c u_{\ge r})^n = \int_{[a,b] \setminus E} dr \left(\int_{S_u(r)} h d^c u \wedge (dd^c u)^{n-1} + \int_{\Omega \setminus B_u(r)} h (dd^c u)^n \right)$$

Put $\sigma = hd^c u \wedge (dd^c u)^{n-1}$. By Stokes' theorem, Sard's theorem and Fubini's theorem, We have that

$$\begin{aligned}\int_{[a,b]\setminus E} dr \int_{S_u(r)} \sigma &= \int_{[a,b]\setminus E} dr \int_{B_u(r)} d\sigma = \int_a^b dr \int_{B_u(r)} d\sigma \\ &= \int_\Omega (b-u) d\sigma = \int_\Omega du \wedge \sigma = \int_\Omega h du \wedge d^c u \wedge (dd^c u)^{n-1}\end{aligned}$$

and

$$\begin{aligned}\int_{[a,b]\setminus E} dr \int_{\Omega\setminus B_u(r)} h(dd^c u)^n &= \int_a^b dr \int_{\Omega\setminus B_u(r)} h(dd^c u)^n \\ &= \int_\Omega (u-a) h(dd^c u)^n.\end{aligned}$$

Proposition 1 is thus proved when u is smooth. If u is not smooth, one can take a decreasing sequence of smooth plurisubharmonic functions $\{u_i\}$ on a neighborhood of the support of h such that u_i converges to u (see Chapter 2 of [7]). By Theorem 2,

$$\int_\Omega h(dd^c u_{\geq r})^n = \lim_{i\to\infty} \int_\Omega h\{dd^c(u_i)_{\geq r}\}^n.$$

Note that $|\int_\Omega h\{dd^c(u_i)_{\geq r}\}^n|$ is bounded above by $(b-a)^n(\sup|h|)C(\operatorname{supp} h, \Omega)$. Here $C(\operatorname{supp} h, \Omega)$ is the relative Monge-Ampère capacity of the support of h (see [2] for the definition of the relative Monge-Ampère capacity). By Lebesgue's dominated convergence theorem, Proposition 1 for a smooth plurisubharmonic function and Theorem 2, it follows that

$$\begin{aligned}\int_a^b dr \int_\Omega h(dd^c u_{\geq r})^n &= \lim_{i\to\infty} \int_a^b dr \int_\Omega h\{dd^c(u_i)_{\geq r}\}^n \\ &= \lim_{i\to\infty} \left(\int_\Omega h du_i \wedge d^c u_i \wedge (dd^c u_i)^{n-1} + \int_\Omega (u_i - a) h (dd^c u_i)^n \right) \\ &= \int_\Omega h du \wedge d^c u \wedge (dd^c u)^{n-1} + \int_\Omega (u-a) h (dd^c u)^n.\end{aligned}$$

This completes the proof of Proposition 1. □

An immediate consequence of Proposition 1 is the following:

Theorem 3. *Let Ω be an open set in $\mathbb{C}^n$ and let $u \in \mathrm{Psh}(\Omega) \cap L^\infty(\Omega)$. Then $(dd^c u_{\geq r})^n \equiv 0$ for all $r \in \mathbb{R}$ if and only if $(dd^c u)^n \equiv 0$ and $du \wedge d^c u \wedge (dd^c u)^{n-1} \equiv 0$.*

3. Pluricomplex Green Functions and Relative Extremal Functions

Let $\Omega \subset \mathbb{C}^n$ be an open subset and let E be a subset of Ω. We consider the extremal function associated to E in Ω. We define

$$u^*_{E,\Omega}(z) = \limsup_{\zeta\to z} u_{E,\Omega}(\zeta)$$

where

$$u_{E,\Omega}(z) = \sup\{v(z); v \in \mathrm{Psh}(\Omega), v|_E \leq -1, v < 0\} \qquad (z \in \Omega).$$

Then $u^*_{E,\Omega}$ is plurisubharmonic function on Ω and $-1 \leq u^*_{E,\Omega} \leq 0$. The following theorem is proved in [3].

Theorem 4 (Theorem 7.1 of [3])**.** *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain and let $E \subset \Omega$ be a compact subset. Let $A(E)$ denote the subalgebra of the Banach algebra of continuous functions on E which is the closure of the holomorphic functions on Ω. We define E_0 as the closure of*

$$\{z \in E; u^*_{E,\Omega} = -1\}.$$

Then the Shilov boundary of E_0 for $A(E)$ is equal to the support of the measure $(dd^c u^*_{E,\Omega})^n$.

Remark 1. One can check that Theorem 4 is also true when $\Omega \subset \mathbb{C}^n$ is a bounded hyperconvex domain by the same argument as that used on the proof of Theorem 7.1 of [3].

We have the following lemma.

Lemma 1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and let $x \in \Omega$. For the simplicity of notation, we just write $\overline{B}_{K_{\Omega,x}}(r)$ as $\overline{B}(r)$. Then*

$$\frac{1}{|r|}(K_{\Omega,x})_{\geq r} = u^*_{\overline{B}(r),\Omega} \quad \text{for any } r < 0$$

and the support of the measure $\{dd^c(K_{\Omega,x})_{\geq r}\}^n$ is equal to $\partial_S \overline{B}(r)$.

Proof. Note that $\overline{B}(r)$ is compact since $K_{\Omega,x}$ is continuous (See [4]). Let v be a plurisubharmonic function on Ω such that $v < 0$ in Ω and $v \leq -1$ in $\overline{B}(r)$. We show that $|r|^{-1}(K_{\Omega,x})_{\geq r} \geq v$. Let $\psi : \Omega \to (-\infty, 0)$ be a continuous plurisubharmonic function such that the set $\{z \in \Omega; \psi(z) < c\}$ is a relatively compact subset of Ω for any $c \in (-\infty, 0)$ and $\psi \leq -1$ on $\overline{B}(r)$. By replacing v by $\max\{v, \psi\}$, we may assume that $\lim_{z \to p} v(z) = 0$ for all $p \in \partial\Omega$. Since $U := \{z \in \Omega; |r|^{-1}(K_{\Omega,x})_{\geq r} < v(z)\}$ is contained in $\Omega \setminus \overline{B}(r)$, it follows that $\int_U \{dd^c|r|^{-1}(K_{\Omega,x})_{\geq r}\}^n = 0$. Then $|r|^{-1}(K_{\Omega,x})_{\geq r} \geq v$ in Ω by the domination principle (see [2]). Since $|r|^{-1}(K_{\Omega,x})_{\geq r} < 0$ in Ω and $|r|^{-1}(K_{\Omega,x})_{\geq r} \leq -1$ in $\overline{B}(r)$, we have that $|r|^{-1}(K_{\Omega,x})_{\geq r} = u^*_{\overline{B}(r),\Omega}$. Let $A(\overline{B}(r))$ be the Banach subalgebra of continuous functions on $\overline{B}(r)$ which is the closure of the holomorphic functions on Ω. Since holomorphically convex hull of $\overline{B}(r)$ is equal to $\overline{B}(r)$ itself, $A(\overline{B}(r))$ is equal to the Banach subalgebra of continuous functions on $\overline{B}(r)$ which is the closure of functions which are holomorphic on a neighborhood of $\overline{B}(r)$ (see Theorem 4.3.2 of [7]). Therefore the support of the measure $\{dd^c(K_{\Omega,x})_{\geq r}\}^n$ is equal to $\partial_S \overline{B}(r)$. □

We have a corollary of Lemma 1

Corollary 1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and let $x \in \Omega$. The relative Monge-Ampère capacity $C(\overline{B}_{K_{\Omega,x}}(r), \Omega)$ is $|r|^{-n}$.*

Proof. By Proposition 6.5 of [2], we have that

$$C(\overline{B}_{K_{\Omega,x}}(r), \Omega) = \int_\Omega (dd^c u^*_{\overline{B}(r),\Omega})^n.$$

Then, by Stokes' theorem and Lemma 1, it follows that

$$\begin{aligned} C(\overline{B}_{K_{\Omega,x}}(r), \Omega) &= \frac{1}{|r|^n} \int_\Omega \{dd^c(K_{\Omega,x})_{\geq r}\}^n \\ &= \frac{1}{|r|^n} \int_\Omega (dd^c K_{\Omega,x})^n = \frac{1}{|r|^n}. \end{aligned}$$

The last equality holds by Theorem 4.3 of [4]. □

Now we prove Theorem 1.

Proof of Theorem 1. Let E be the closure of $\bigcup_{-\infty\le r<0}\partial_S\overline{B}_{K_{\Omega,x}}(r)$ in $\Omega\setminus\{x\}$. Let $z\in\Omega\setminus\{x\}$. Then $z\notin E$ if and only if $\{dd^c(K_{\Omega,x})_{\ge r}\}^n\equiv 0$ for any $r<0$ in a neighborhood of z by Lemma 1. We have that $\{dd^c(K_{\Omega,x})_{\ge r}\}^n\equiv 0$ for any $r<0$ in a neighborhood of z if and only if $dK_{\Omega,x}\wedge d^cK_{\Omega,x}\wedge(dd^cK_{\Omega,x})^{n-1}\equiv 0$ in a neighborhood of z by Theorem 3. This completes the proof. □

4. The case of a Balanced domain

Proposition 2. *Let $\delta:\mathbb{C}^n\to[0,\infty)$ be a continuous function such that $\delta^{-1}(0)=\{0\}$, $\delta(\xi z)=|\xi|\delta(z)$ for all $\xi\in\mathbb{C}, z\in\mathbb{C}^n$ and $\log\delta\in\mathrm{Psh}(\mathbb{C}^n)$. Let $\Omega=\{z\in\mathbb{C}^n;\delta(z)<1\}$. Then the support of the current $(dd^c\log\delta)^{n-1}$ in Ω is equal to the closure of*

$$\bigcup_{0\le r<1}\partial_S\overline{B}_\delta(r).$$

Proof. Note that $\log\delta$ is equal to the pluricomplex Green function on Ω with logarithmic pole at $\{0\}$ (see Lemma 6.1.3 of [8]) and $(dd^c\log\delta)^{n-1}$ is well defined on Ω. By Theorem 1, it is enough to show that the support of the current $(dd^c\log\delta)^{n-1}$ is equal to the support of the current $d\log\delta\wedge d^c\log\delta\wedge(dd^c\log\delta)^{n-1}$ in $\Omega\setminus\{0\}$. Let $z^0=(z_1^0,\ldots,z_n^0)\in\Omega\setminus\{0\}$. Without loss of generality, we may assume that $z_1^0\neq 0$. Let $\phi:\mathbb{C}^n\to\mathbb{C}^n$ be a holomorphic map such that $\phi((\gamma,w_2,\ldots,w_n))=((1+\gamma)z_1^0,(1+\gamma)(w_2+z_2^0),\ldots,(1+\gamma)(w_n+z_n^0))$ for $(\gamma,w_2,\ldots,w_n)\in\mathbb{C}^n$. Then $\phi(0)=z^0$ and the restriction of ϕ to a small neighborhood of $\{0\}\in\mathbb{C}^n$ is a biholomorphic map. We have that

$$\log\delta\circ\phi((\gamma,w'))=\log|1+\gamma|+\log\delta\circ\phi((0,w'))$$

where $w'=(w_2,\ldots,w_n)\in\mathbb{C}^{n-1}$. We put $h(w')=\log\delta\circ\phi((0,w'))$. Let $\{\rho_\varepsilon\}$ be a family of smoothing kernels. It follows that

$$(\log\delta\circ\phi)*\rho_\varepsilon((\gamma,w'))=\log|1+\gamma|+h*\rho_\varepsilon(w')$$

since $\log|1+\gamma|$ is a harmonic function in a neighborhood of $\{0\}\in\mathbb{C}^n$. Then

$$\begin{aligned} d\{(\log\delta\circ\phi)*\rho_\varepsilon\}\wedge d^c\{(\log\delta\circ\phi)*\rho_\varepsilon\} &= \frac{\sqrt{-1}}{\pi}d'\{(\log\delta\circ\phi)*\rho_\varepsilon\}\wedge d''\{(\log\delta\circ\phi)*\rho_\varepsilon\} \\ &= \frac{\sqrt{-1}}{4\pi}\frac{d\gamma\wedge d\bar\gamma}{|1+\gamma|^2}+\eta \end{aligned}$$

where η is a smooth $(1,1)$-form which does not contain a term of $d\gamma\wedge d\bar\gamma$. Since $dd^c\{(\log\delta\circ\phi)*\rho_\varepsilon\}=dd^ch*\rho_\varepsilon$ depends only on w', it follows that

$$\begin{aligned} &d\{(\log\delta\circ\phi)*\rho_\varepsilon\}\wedge d^c\{(\log\delta\circ\phi)*\rho_\varepsilon\}\wedge[dd^c\{(\log\delta\circ\phi)*\rho_\varepsilon\}]^{n-1} \\ &=\frac{\sqrt{-1}}{4\pi}\frac{d\gamma\wedge d\bar\gamma}{|1+\gamma|^2}\wedge[dd^c\{(\log\delta\circ\phi)*\rho_\varepsilon\}]^{n-1}. \end{aligned}$$

We have that

$$\begin{aligned} &\lim_{\varepsilon\to 0}d\{(\log\delta\circ\phi)*\rho_\varepsilon\}\wedge d^c\{(\log\delta\circ\phi)*\rho_\varepsilon\}\wedge[dd^c\{(\log\delta\circ\phi)*\rho_\varepsilon\}]^{n-1} \\ &=d(\log\delta\circ\phi)\wedge d^c(\log\delta\circ\phi)\wedge\{dd^c(\log\delta\circ\phi)\}^{n-1} \end{aligned}$$

and

$$\lim_{\varepsilon\to 0}[dd^c\{(\log\delta\circ\phi)*\rho_\varepsilon\}]^{n-1} = \{dd^c(\log\delta\circ\phi)\}^{n-1}$$

in the weak topology on the space of currents by Theorem 2 and Theorem 2.1 of [2]. Hence we have that

$$d(\log\delta\circ\phi)\wedge d^c(\log\delta\circ\phi)\wedge\{dd^c(\log\delta\circ\phi)\}^{n-1} = \frac{\sqrt{-1}}{4\pi}\frac{d\gamma\wedge d\bar{\gamma}}{|1+w|^2}\wedge\{dd^c(\log\delta\circ\phi)\}^{n-1}.$$

Therefore the support of the current $d\log\delta\wedge d^c\log\delta\wedge(dd^c(\log\delta)^{n-1}$ is equal to the support of the current $(dd^c\log\delta)^{n-1}$ in $\Omega\setminus\{0\}$. □

For a balanced domain Ω, there exists a S^1 action on the boundary $\partial\Omega = \{z\in\mathbb{C};\delta(z)=1\}$ such that $(e^{\sqrt{-1}\theta}, z)\mapsto e^{\sqrt{-1}\theta}z$ for $\theta\in(0,2\pi]$ and $z\in\partial\Omega$. The Shilov boundary $\partial_S\Omega$ is invariant by this action and the quotient space $\partial\Omega/S^1$ with quotient topology is a compact Hausdorff space. Let $i:\partial\Omega\to\partial\Omega/S^1$ be the canonical map and let $\tau:\Omega\setminus\{0\}\to\partial\Omega/S^1$ be a continuous function such that $\tau(z) = i((\delta(z))^{-1}z)$. Then $\tau:\Omega\setminus\{0\}\to\partial\Omega/S^1$ is a fiber space whose fiber is isomorphic to $D^* = \{z\in\mathbb{C}; 0 < |z| < 1\}$.

Theorem 5. *There exists a unique measure μ in $\partial\Omega/S^1$ whose support is $\partial_S\Omega/S^1$ such that*

$$(dd^c\log\delta)^{n-1} = 1_{\Omega\setminus\{0\}}\int_{\partial\Omega/S^1}[\tau^{-1}(t)]d\mu(t).$$

Here $\int_{\partial\Omega/S^1}[\tau^{-1}(t)]d\mu(t)$ is a current on $\Omega\setminus\{0\}$ such that

$$\langle\int_{\partial\Omega/S^1}[\tau^{-1}(t)]d\mu(t), u\rangle = \int_{\partial\Omega/S^1}\left(\int_{\tau^{-1}(t)} u\right)d\mu(t)$$

for all smooth $(1,1)$-form u which has a compact support in $\Omega\setminus\{0\}$ and

$$1_{\Omega\setminus\{0\}}\int_{\partial\Omega/S^1}[\tau^{-1}(t)]d\mu(t)$$

is a trivial extension of the current $\int_{\partial\Omega/S^1}[\tau^{-1}(t)]d\mu(t)$ on $\Omega\setminus\{0\}$ to the current of Ω.

Proof. To prove the uniqueness of μ, it is enough to prove the uniqueness of μ locally. Hence we may assume that $\tau:\Omega\to\partial\Omega/S^1$ is a trivial D^*-fiber bundle and it is easy to prove the uniqueness of μ in this case. Now let us prove the existence of μ. We use the same notation as in the proof of Proposition 2. In a neighborhood of $z^0\in\Omega\setminus\{0\}$, we have that $\{dd^c(\log\delta\circ\phi)\}^{n-1} = \lim_{\varepsilon\to 0}\{dd^c(\log\delta\circ\phi)*\rho_\varepsilon\}^{n-1}$ depend only on w' (see the proof of Proposition 2). Therefore, in a neighborhood U of $\tau(z^0)\in\partial\Omega$, there exists a measure μ_U such that $(dd^c\log\delta)^{n-1}|_{\tau^{-1}(U)} = \int_U[\tau^{-1}(t)]d\mu_U(t)$. By gluing measures $\{\mu_U\}$, we obtain the measure μ in $\partial\Omega$ and such that $(dd^c\log\delta)^{n-1}|_{\Omega\setminus\{0\}} = \int_{\partial\Omega/S^1}[\tau^{-1}(t)]d\mu(t)$. Note that $1_{\Omega\setminus\{0\}}\int_{\partial\Omega/S^1}[\tau^{-1}t]d\mu(t)$ is a closed current on Ω by El Mir's extension theorem(see [5]) and the support of the closed current $(dd^c\log\delta)^{n-1} - 1_{\Omega\setminus\{0\}}\int_{\partial\Omega/S^1}[\tau^{-1}t]d\mu(t)$ is contained in $\{0\}\in\Omega$. Then the current is zero by the First theorem of support (see [6]). By Proposition 2, the support of μ is equal to $\partial_S\Omega/S^1$ and this completes the proof.

□

We need the following lemma to apply our results to an intersection theory of the supports of currents.

Lemma 2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and let $\varphi \in \mathrm{Psh}(\Omega)$ such that $\lim_{z\to p}\varphi(z) = 0$ for all $p \in \partial\Omega$ and $\overline{B}_\varphi(r)$ is relatively compact for all $r < 0$. Let T be a closed positive current of bidegree (k,k) $(1 \le k < n)$ on Ω. Assume that*

$$\int_\Omega (dd^c\varphi)^{n-k} \wedge T = 0.$$

Then $T = 0$.

Note that the measure $(dd^c\varphi)^{n-k} \wedge T$ is well defined by Proposition 2.1 of [5].

Proof. Let R be a positive number such that Ω is relatively compact in $B_{\|z\|}(R) = \{z \in \mathbb{C}^n; \|z\| = |z_1|^2 + \cdots + |z_n|^2 < R\}$. For $r < 0$, take a small positive number ε such that $\varepsilon(\|z\|^2 - R^2) > \varphi$ on $B_\varphi(r)$. By Stokes' theorem, we have that

$$\begin{aligned} 0 = \int_\Omega (dd^c\varphi)^{n-k} \wedge T &= \int_\Omega (dd^c \max\{\varphi, \varepsilon(\|z\|^2 - R^2)\})^{n-k} \wedge T \\ \ge \int_{B_\varphi(r)} (dd^c \max\{\varphi, \varepsilon(\|z\|^2 - R^2)\})^{n-k} \wedge T &= \int_{B_\varphi(r)} \{dd^c\varepsilon(\|z\|^2 - R^2)\}^{n-k} \wedge T. \end{aligned}$$

Hence $(dd^c\|z\|^2)^{n-k} \wedge T = 0$ on $B_\varphi(r)$ for any $r < 0$. This implies that $T = 0$ on Ω. □

By Proposition 2 and Lemma 2, the following corollary holds.

Corollary 2. *Let $\delta : \mathbb{C}^n \to [0,\infty)$ be a continuous function such that $\delta^{-1}(0) = \{0\}$, $\delta(\xi z) = |\xi|\delta(z)$ for all $\xi \in \mathbb{C}, z \in \mathbb{C}^n$ and $\log\delta \in \mathrm{Psh}(\mathbb{C}^n)$. Let $\Omega = \{z \in \mathbb{C}^n; \delta(z) < 1\}$. Then the support of a non-zero closed positive current of bidegree $(1,1)$ on Ω intersects the closure of*

$$\bigcup_{0\le r<1} \partial_S \overline{B}_\delta(r).$$

In particular, any plurisubharmonic function on Ω which is pluriharmonic on a neighborhood of the closure of $\bigcup_{0\le r<1} \partial_S \overline{B}_\delta(r)$ is pluriharmonic on Ω.

Example 1. Let $\delta = \log\max\{|z_1|, \dots, |z_n|\}$ and let $P(1) = \{z \in \mathbb{C}^n; |z_1| < 1, \dots, |z_n| < 1\}$. Then the support of the current $(dd^c\log\delta)^{n-1}$ in $P(1)$ is $X = \{z \in P(1); |z_1| = \cdots = |z_n|\}$. The support of any non-zero closed positive current of bidegree $(1,1)$ intersects X and X is the minimal set which satisfies this property, that is:

Let T be a closed positive current of bidegree $(n-1, n-1)$. We denote the support of T by $\mathrm{supp}\, T$. Assume that $\mathrm{supp}\, T \subset X$ and the support of any non-zero closed positive current of bidegree $(1,1)$ intersects $\mathrm{supp}\, T$. Then $\mathrm{supp}\, T = X$.

Proof. Let $\tau : P(1)\backslash\{0\} \to \partial P(1)/S^1$ be the continuous map as in Theorem 5. It is easy to check that $X \setminus \{0\}$ is a CR submanifold whose CR dimension is *one*. By the Second theorem of support (see [6]), there exists a unique measure ν on $\partial_S P(1)/S^1 = \{z \in \mathbb{C}^n; |z_1| = \cdots = |z_n| = 1\}/S^1$ such that $T|_{P(1)\backslash\{0\}} = \int_{\partial_S P(1)/S^1} [\tau^{-1}(t)] d\nu(t)$. Assume that $\mathrm{supp}\, T|_{P(1)\backslash\{0\}} \ne X \setminus \{0\}$. Without loss of generality, we may assume that the support of ν does not contain $i((1, \cdots, 1)) \in \partial_S P(1)/S^1$. Here $i : \partial P(1) \to \partial P(1)/S^1$ is the canonical map. Then there exists a small $\varepsilon > 0$ such that $\mathrm{supp}\, T$ does not intersect $Y = \{(z_1, e^{\sqrt{-1}\theta_2} z_1, \dots, e^{\sqrt{-1}\theta_n} z_1) \in P(1); |z_1| < 1, |\theta_2| \le \varepsilon, \dots, |\theta_n| \le \varepsilon\}$. Let $Q(z_1, \dots, z_n) = z_1 + z_2 - |1 + e^{\sqrt{-1}\varepsilon}|$. Then $Q(0) \neq 0$ and

$$Q(w, \dots, w) = 2w - |1 + e^{\sqrt{-1}\varepsilon}| = 0$$

has a solution in $|w| < 1$. Hence the hypersurface defined by $Q = 0$ in $\mathbb{C}^n$ intersects $P(1) \setminus \{0\}$ and intersects $\operatorname{supp} T|_{P(1)\setminus\{0\}}$. However,

$$Q(w, e^{\sqrt{-1}\theta_2} w, \dots, e^{\sqrt{-1}\theta_n} w) = (1 + e^{\sqrt{-1}\theta_2})w - |1 + e^{\sqrt{-1}\varepsilon}| = 0$$

does not have a solution in $|w| < 1$ if $|\theta_2| > \varepsilon$. Hence $\operatorname{supp} T$ does not intersect the hypersurface defined by $Q = 0$. This is a contradiction. Hence $\operatorname{supp} T|_{P(1)\setminus\{0\}} = X \setminus \{0\}$. Since $\operatorname{supp} T$ is closed, we have that $\operatorname{supp} T = X$ $\square$

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