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Abstract

In the present paper the authors discuss the efficiency of stochastic mesh methods introduced by Broadie and Glasserman [4]. The authors apply stochastic mesh methods to certain type of Hörmander type diffusion processes and show the following. (1) If one carefully takes partitions, the estimated price of American option converges to the real price with probability one. (2) One can obtain better estimates by re-simulation methods discussed in Belomestny [3], although the order is not so sharp as his result.

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1 Introduction

Stochastic mesh methods were introduced by Broadie and Glasserman [4], and Avramidis and Hyden [1] and Avramidis and Matzinger[2] proved the efficiency of them in some cases (see [5] also). Also, Belomestny [3] showed in Bermuda options that once we have estimated functions for the so-called continuation values, we have a better estimated value if we construct a pre-optimal stopping time by using these estimated functions and estimate the expectation of pay-off functionals based on this stopping time by re-simulation.

In the present paper, we consider the efficiency of stochastic mesh methods and re-simulation in the case that we apply them to Hörmander type diffusion processes.

Let $N, d \geqq 1$. Let $W_0 = \{w \in C([0,\infty); \mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ be the Borel algebra over W_0 and μ be the Wiener measure on $(W_0, \mathcal{F})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1, \dots, d$, be given by $B^i(t, w) = w^i(t)$, $(t, w) \in [0,\infty) \times W_0$. Then $\{(B^1(t), \dots, B^d(t); t \in [0,\infty)\}$ is a d-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $V_0, V_1, \dots, V_d \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N; \mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined

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in $\mathbf{R}^N$ whose derivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t\in[0,\infty)$, $x\in\mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x)=x+\sum_{i=0}^{d}\int_0^t V_i(X(s,x))\circ dB^i(s). \tag{1}$$

Then there is a unique solution to this equation. Moreover we may assume that $X(t,x)$ is continuous in t and smooth in x and $X(t,\cdot):\mathbf{R}^N\to\mathbf{R}^N$, $t\in[0,\infty)$, is a diffeomorphism with probability one.

Let $\mathcal{A}=\{\emptyset\}\cup\bigcup_{k=1}^\infty\{0,1,\dots,d\}^k$ and for $\alpha\in\mathcal{A}$, let $|\alpha|=0$ if $\alpha=\emptyset$, let $|\alpha|=k$ if $\alpha=(\alpha^1,\dots,\alpha^k)\in\{0,1,\dots,d\}^k$, and let $\|\alpha\|=|\alpha|+\mathrm{card}\{1\leqq i\leqq|\alpha|;\ \alpha^i=0\}$. Let $\mathcal{A}^*$ and $\mathcal{A}^{**}$ denote $\mathcal{A}\setminus\{\emptyset\}$ and $\mathcal{A}\setminus\{\emptyset,0\}$, respectively. Also, for each $m\geqq 1$, $\mathcal{A}^{**}_{\leqq m}$, $\{\alpha\in\mathcal{A}^{**};\ \|\alpha\|\leqq m\}$.

We define vector fields $V_{[\alpha]}$, $\alpha\in\mathcal{A}$, inductively by

$$V_{[\emptyset]}=0,\qquad V_{[i]}=V_i,\quad i=0,1,\dots,d,$$

$$V_{[\alpha*i]}=[V_{[\alpha]},V_i],\qquad i=0,1,\dots,d.$$

Here $\alpha*i=(\alpha^1,\dots,\alpha^k,i)$ for $\alpha=(\alpha^1,\dots,\alpha^k)$ and $i=0,1,\dots,d$.

We say that a system $\{V_i;i=0,1,\dots,d\}$ of vector fields satisfies the following condition (UFG).

(UFG) There are an integer ℓ_0 and $\varphi_{\alpha,\beta}\in C_b^\infty(\mathbf{R}^N)$, $\alpha\in\mathcal{A}^{**}$, $\beta\in\mathcal{A}^{**}_{\leqq\ell_0}$, satisfying the following.

$$V_{[\alpha]}=\sum_{\beta\in\mathcal{A}^{**}_{\leqq\ell_0}}\varphi_{\alpha,\beta}V_{[\beta]},\qquad \alpha\in\mathcal{A}^{**}.$$

Let $A(x)=(A^{ij}(x))_{i,j=1,\dots,N}$, $t>0$ $x\in\mathbf{R}^N$ be a $N\times N$ symmetric matrix given by

$$A^{ij}(x)=\sum_{\alpha\in\mathcal{A}^{**}_{\leqq\ell_0}}V^i_{[\alpha]}(x)V^j_{[\alpha]}(x),\qquad i,j=1,\dots,N.$$

Let $h(x)=\det A(x), x\in\mathbf{R}^N$ and $E=\{x\in\mathbf{R}^N;\ h(x)>0\}$. By Kusuoka-Stroock [7], we see that if $x\in E$, the distribution law of $X(t,x)$ under μ has a smooth density function $p(t,x,\cdot):\mathbf{R}^N\to[0,\infty)$ for $t>0$. Moreover, we will show in that $\int_E p(t,x,y)dy=1$, $x\in E$.

Now let $x_0\in E$ and fix it throughout this paper. Let $(\Omega,\mathcal{F},P)$ be a probability space, and $X^\ell:[0,\infty)\times\Omega\to\mathbf{R}^N$, $\ell=1,2,\dots$, be continuous stochastic processes such that the probability laws on $C([0,\infty);\mathbf{R}^N)$ of $X_\ell(\cdot)$ under P and of $X(\cdot,x_0)$ under μ are the same for all $\ell=1,2,\dots$, and that $\sigma\{X_\ell(t);\ t\geqq0\}$, $\ell=1,2,\dots$, are independent.

Let $q^{(L)}_{s,t}:E\times\Omega\to[0,\infty)$, $t>s\geqq0$, $L\geqq1$, be given by

$$q^{(L)}_{s,t}(y,\omega)=\frac{1}{L}\sum_{\ell=1}^{L}p(t-s,X_\ell(s,\omega),y),\qquad y\in E,\ \omega\in\Omega,$$

Let $m(E)$ denote the space of measurable functions on E.

We define a random linear operator $Q^{(L)}_{s,t}$, $t > s \geqq 0$, $L \geqq 1$, defined in $m(E)$ by

$$(Q^{(L)}_{s,t} f)(x) = \frac{1}{L} \sum_{\ell=1}^{L} \frac{p(t-s,x,X_\ell(t)) f(X_\ell(t))}{q^{(L)}_{s,t}(X_\ell(t))}, \qquad x \in E, \ f \in m(E).$$

Now let $T > 0$, and $g : [0,T] \times \mathbf{R}^N \to \mathbf{R}$ be a continuous function with $\sup\{(1+|x|)^{-1}|g(t,x)|;\ x \in \mathbf{R}^N, t \in [0,T]\} < \infty$. For any $n \geqq 1$, and $0 = t_0 < t_1 < \ldots < t_n = T$, we define $c_{t_k,t_{k+1},\ldots,t_n} : E \to \mathbf{R}$, and $\tilde{c}^{(L)}_{t_k,t_{k+1},\ldots,t_n} : E \times \Omega \to \mathbf{R}$, $k = n, n-1, \ldots, 0$, $L \geqq 1$, inductively by $c_{t_n}(x) = \tilde{c}^{(L)}_{t_n}(x) = g(T,x)$, $x \in E$, and

$$c_{t_k,t_{k+1},\ldots,t_n}(x) = \int_E p(t_{k+1}-t_k,x,y)(g(t_{k+1},y) \vee c_{t_{k+1},\ldots,t_n}(y))dy,$$

and

$$\tilde{c}^{(L)}_{t_k,t_{k+1},\ldots,t_n}(x) = Q^{(L)}_{t_k,t_{k+1}}(g(t_{k+1},\cdot) \vee \tilde{c}^{(L)}_{t_{k+1},\ldots,t_n}(\cdot))(x)$$

for $x \in E$ and $k = n-1, \ldots, 0$.

Then we will show the following.

Theorem 1 *Suppose that $n(L) \geqq 1$, $0 = t^{(L)}_0 < t^{(L)}_1 < \ldots < t^{(L)}_{n(L)} = T$. If there is an $\varepsilon > 0$ such that*

$$L^{-(1-\varepsilon)/2} \sum_{k=1}^{n(L)} (t^{(L)}_k - t^{(L)}_{k-1})^{-(N+1)\ell_0/4} \to 0,$$

then

$$E[|\tilde{c}^{(L)}_{t^{(L)}_0,t^{(L)}_1,\ldots,t^{(L)}_{n(L)}}(x_0) - c_{t^{(L)}_0,t^{(L)}_1,\ldots,t^{(L)}_{n(L)}}(x_0)|^2] \to 0, \qquad L \to \infty.$$

Let $n \geqq 1$, and $0 = T_0 < T_1 < \ldots < T_n = T$ and fix them. For each $\omega \in \Omega$, let $\hat{\tau}_{L,\omega} W_0 \to \{T_1, \ldots, T_n\}$ be a stopping time given by

$$\hat{\tau}_{L,\omega} = \min\{T_k;\ k = 1,2,\ldots,n,\ \tilde{c}^{L}_{T_k,T_{k+1},\ldots,T_n}(X(T_k,x_0),\omega) \leqq g(T_k,X(T_k,x_0))\}.$$

Let $\hat{c} : \Omega \to \mathbf{R}$ be given by

$$\hat{c}(\omega) = E^\mu[g(\hat{\tau}_{L,\omega}, X(\hat{\tau}_{L,\omega}, x_0)].$$

Then we have the following.

Theorem 2 *Suppose that $\gamma \in (0,1]$. If*

$$\sum_{k=1}^{n} \mu(|c_{T_k,T_{k+1},\ldots,T_n}(X(T_k,x_0)) - g(T_k,X(T_k,x_0))| < \varepsilon) = O(\varepsilon^\gamma), \ \text{as } \varepsilon \downarrow 0,$$

then for any $\alpha \in (1/2, (1+\gamma)/(2+\gamma))$, there are $\Omega_L \in \mathcal{F}$, $L \geqq 1$, and $C > 0$ such that $P(\Omega_L) \to 1$, $L \to \infty$, and

$$|\hat{c}(\omega) - c_{T_0,T_1,\ldots,T_n}| \leqq CL^{-\alpha} \ \text{for any } \omega \in \Omega_L \text{ and } L \geqq 1.$$

2 The basic property of Hörmander diffusion processes

Let $J : [0,\infty) \times \mathbf{R}^N \times W_0 \to \mathbf{R}^N \otimes \mathbf{R}^N$, $J(t,x) = (J^i_j(t,x))_{i,j=1,\ldots,N}$ be given by

$$J^i_j(t,x) = \frac{\partial}{\partial x^j} X^i(t,x)$$

Then it has been shown in [6] Section 2 that there are $b^\beta_\alpha : [0,\infty) \times \mathbf{R}^N \times W_0 \to \mathbf{R}$, $\alpha, \beta \in \mathcal{A}^{**}_{\leqq \ell_0}$, such that

$$V_{[\alpha]}(x) = \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} b^\beta_\alpha(t,x) J(t,x)^{-1} V_\beta(X(t,x)), \qquad \alpha \in \mathcal{A}^{**}_{\leqq \ell_0},$$

and

$$\sup_{x \in \mathbf{R}^N, t \in [0,T]} E^\mu[|b^\beta_\alpha(t,x)|^p] < \infty \qquad \alpha, \beta \in \mathcal{A}^{**}_{\leqq \ell_0},\ T > 0,\ p \geqq 1.$$

So we see that for any $\xi \in \mathbf{R}^N$,

$$(A(x)\xi, \xi) = \sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} (V_{[\alpha]}(x), \xi)^2$$

$$\leqq \sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} (\sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} b^\beta_\alpha(t,x)^2)(\sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} (J(t,x)^{-1} V_{[\beta]}(X(t,x)), \xi)^2)$$

$$= (\sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} b^\beta_\alpha(t,x)^2)(J(t,x) A(X(t,x))^t J(t,x)\xi, \xi)$$

Therefore we see that

$$h(x) \leqq (\sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} b^\beta_\alpha(t,x)^2)^N det(J(t,x))^2 h(X(t,x)). \tag{2}$$

Then we have the following.

Proposition 3 (1) $\mu(X(t,x) \in E) = 1$ *for any* $x \in E$ *and* $t > 0$. *In particular,* $p(t,x,y) = 0$, $y \in \mathbf{R}^N \setminus E$, $x \in E$.
(2) *For any* $p > 1$ *and* $T > 0$, *there exists a* $C > 0$ *such that*

$$E[h(X(t,x))^{-p}] \leqq C h(x)^{-p}, \qquad x \in E,\ t \in [0,T].$$

(3) *For any* $n, m \geqq 0$, $p \in (1,\infty)$, *and* $T > 0$, *there exists a* $C > 0$ *such that*

$$||h(X(t,x))^{-m}||_{W^{n,p}} \leqq C h(x)^{-(n+m)} \qquad x \in E,\ t \in [0,T].$$

Proof. The assertions (1) and (2) are easy consequence of Equation (2). Note that

$$D(h^{-m}(X(t,x))) = -m h^{-(m+1)}(X(t,x))) D(h(X(t,x))).$$

Thus we easily obatain the assertion (3) by induction. ∎

By Kusuoka-Stroock [7], we have the following.

Proposition 4 *Let $\delta_0 > 0$ be given by*

$$\delta_0 = (3N(\sup_{x\in\mathbf{R}^N} \sum_{k=1}^d |V_k(x)|^2))^{-1}$$

Then we have the following.
(1) *For any $T > 0$,*

$$\sup_{t\in(0,T],\, x\in\mathbf{R}^N} E[\exp(\frac{2\delta_0}{t}|X(t,x) - x|^2)] < \infty.$$

(2) *For any $T > 0$, $n \geqq 1$, and $p \in (1, \infty)$,*

$$\sup_{t\in(0,T],\, x\in\mathbf{R}^N} t^{n/2}||\exp(\frac{\delta_0}{t}|X(t,x) - x|^2)||_{W^{n,p}} < \infty.$$

Proposition 5 *For any $\gamma \in \mathbf{Z}^N_{\geqq 0}$, there are $g_{\gamma,\alpha_1,\ldots,\alpha_k} \in C_b^\infty(\mathbf{R}^N)$, $k = 1, \ldots, |\gamma|$, $\alpha_i \in \mathcal{A}^{**}_{\leqq \ell_0}$, $i = 1, \ldots, k$, such that*

$$h(x)^{|\gamma|}\frac{\partial^{|\gamma|}}{\partial x^\gamma}f(x) = \sum_{k=1}^{|\gamma|} \sum_{\alpha_1,\ldots,\alpha_k\in\mathcal{A}^{**}_{\leqq\ell_0}} g_{\gamma,\alpha_1,\ldots,\alpha_k}(x)(V_{[\alpha_1]}\cdots V_{[\alpha_k]}f)(x), \qquad x \in \mathbf{R}^N$$

for any $f \in C_b^\infty(\mathbf{R}^N)$,

Proof. Let $\tilde{A}(x) = (\tilde{A}_{ij}(x))_{i,j=1,\ldots,N}$ be the cofactor matrix of the matrix $A(x)$ for $x \in \mathbf{R}^N$. Also, let $c_{\alpha,i}(x)$, $x \in \mathbf{R}^N$, $\alpha \in \mathcal{A}^{**}_{\leqq\ell_0}$, $i = 1, \ldots, N$, be given by

$$c_{\alpha,i}(x) = \sum_{j=1}^N \tilde{A}_{ij}(x)V^j_{[\alpha]}(x).$$

Then we see that $h, c_{\alpha,i} \in C_b^\infty(\mathbf{R}^N)$, and

$$\sum_{\alpha\in\mathcal{A}^{**}_{\leqq\ell_0}} c_{\alpha,i}(x)(V_{[\alpha]}f)(x) = h(x)\frac{\partial f}{\partial x^i}(x), \qquad i = 1, \ldots, N.$$

So we have the assertion for the case that $|\gamma| = 1$. Since

$$h(x)^{|\gamma|+1}(x)\frac{\partial}{\partial x^i}\frac{\partial^{|\gamma|}}{\partial x^\gamma}f(x)$$

$$= h(x)\frac{\partial}{\partial x^i}(h^{|\gamma|}\frac{\partial^{|\gamma|}}{\partial x^\gamma}f)(x) - |\gamma|\frac{\partial h}{\partial x^i}(x)h^{|\gamma|}(x)\frac{\partial^{|\gamma|}}{\partial x^\gamma}f(x),$$

we have our assertion by induction. ∎

Now we have the following lemma.

Lemma 6 *For any $t > 0$, $x \in E$ and $\gamma_0, \gamma_1 \in \mathbf{Z}^N_{\geqq 0}$, there are $k_{\gamma_0,\gamma_1}(t,x) \in W^{\infty,\infty-}$ such that*

$$\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y)f(y)dy = E[h(X(t,x))^{-2(|\gamma_0|+|\gamma_1|)\ell_0} f(X(t,x))k_{\gamma_0,\gamma_1}(t,x)], \quad f \in C_0^\infty(\mathbf{R}^N),$$

and

$$\sup_{t\in(0,T],x\in E} t^{(|\gamma_0|+|\gamma_1|)\ell_0/2}||k_{\gamma_0,\gamma_1}(t,x)||_{W^n,p} < \infty, \qquad T > 0,\ n \in \mathbf{N},\ p \in (1,\infty).$$

Here $\partial_x^\gamma = \partial^{|\gamma|}/\partial x^\gamma$ and $\partial_y^\gamma = \partial^{|\gamma|}/\partial y^\gamma$.

Proof. First, by the argument in Shigekawa [9] we see that for $\gamma \in \mathbf{Z}^N_{\geqq 0}$, there are $J_{\gamma,\beta}(t,x) \in W^{\infty,\infty-}$, $t \geqq 0$, $x \in \mathbf{R}^N$, $\beta \in \mathbf{Z}^N_{\geqq 0}$, $|\beta| \leqq |\gamma|$, such that

$$\partial_x^\gamma (f(X(t,x)) = \sum_{\beta\in\mathbf{Z}^N_{\geqq 0},\ |\beta|\leqq|\gamma|} (\partial_x^\beta f)(X(t,x))J_{\gamma,\beta}(t,x),$$

and

$$\sup_{t\in(0,T],x\in\mathbf{R}^N} ||J_{\gamma,\beta}(t,x)||_{W^n,p} < \infty, \qquad T > 0,\ n \in \mathbf{N},\ p \in (1,\infty).$$

Then we have for any $x \in E$ and $f \in C_0^\infty(\mathbf{R}^N)$,

$$\begin{aligned}
&\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y)f(y)dy \\
&= (-1)^{|\gamma_1|}\int_{\mathbf{R}^N} \partial_x^{\gamma_0} p(t,x,y)(\partial_y^{\gamma_1} f)(y)dy \\
&= (-1)^{|\gamma_1|}\partial_x^{\gamma_0} E[(\partial_y^{\gamma_1} f)(X(t,x))] \\
&= (-1)^{|\gamma_1|}\sum_{\beta\in\mathbf{Z}^N_{\geqq 0},\ |\beta|\leqq|\gamma_0|} E[(\partial_x^{\gamma_1+\beta} f)(X(t,x))J_{\gamma_0,\beta}(t,x)] \\
&= (-1)^{|\gamma_1|}\sum_{\beta\in\mathbf{Z}^N_{\geqq 0},\ |\beta|\leqq|\gamma_0|}\sum_{k=0}^{|\gamma_1|+|\beta|}\sum_{\alpha_1,\dots,\alpha_k\in\mathcal{A}^{**}_{\leqq \ell_0}} E[h(X(t,x))^{-(|\gamma_1|+|\beta|)} g_{\gamma_1+\beta,\alpha_1,\dots,\alpha_k}(X(t,x)) \\
&\qquad\times J_{\gamma_0,\beta}(t,x)(V_{[\alpha_1]}\cdots V_{[\alpha_k]}f)(X(t,x))].
\end{aligned}$$

So by the integration parts formula in [6] Lemma 8 and by Proposition 3, we have our assertion. ∎

Proposition 7 *For any $t > 0$, $x \in E$ and $\gamma_0, \gamma_1 \in \mathbf{Z}^N_{\geqq 0}$,*

$$\partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) = 0 \ a.e. y \in \mathbf{R}^N \setminus E.$$

Moreover, for any $\gamma_0, \gamma_1 \in \mathbf{Z}^N_{\geqq 0}$, $p \in (1,\infty)$, $T > 0$, and $m \in \mathbf{Z}$ with $m \leqq 2(|\gamma_0| + |\gamma_1|)$,

$$\sup\{t^{(|\gamma_0|+\gamma_1|)\ell_0/2} h(x)^{2(|\gamma_0|+|\gamma_1|)\ell_0-m}(\int_E h(y)^{pm}\exp(\frac{p\delta_0}{t}|y-x|^2)\frac{|\partial_x^{\gamma_0}\partial_y^{\gamma_1}p(t,x,y)|^p}{p(t,x,y)^{p-1}}dy)^{1/p};$$
$$t \in (0,T],\ x \in E\} < \infty.$$

Proof. Let

$$\varphi_{t,x}(y) = \exp(\frac{\delta_0}{t}|y-x|^2), \qquad x,y \in \mathbf{R}^N,\ t>0.$$

Then we have for any $\varepsilon > 0$, $f \in C_0^\infty(\mathbf{R}^N)$ and $x \in E$

$$\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) f(y)(\varepsilon + h(y))^m \varphi_{t,x}(y)dy$$

$$= E[h(X(t,x))^{-2(|\gamma_0|+|\gamma_1|)\ell_0} f(X(t,x))(\varepsilon + h(X(t,x)))^m \varphi_{t,x}(X(t,x)))k_{\gamma_0,\gamma_1}(t,x)]$$

By Propositions 3 and 4, we see that

$$\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) f(y) h(y)^m \varphi_{t,x}(y)dy$$

$$= E[h(X(t,x))^{m-2(|\gamma_0|+|\gamma_1|)\ell_0} f(X(t,x))\varphi_{t,x}(X(t,x)))k_{\gamma_0,\gamma_1}(t,x)].$$

Let $k'(t,x) = h(X(t,x))^{m-2(|\gamma_0|+|\gamma_1|)\ell_0}\varphi_{t,x}(X(t,x))k_{\gamma_0,\gamma_1}(t,x)$. Then we see that

$$\sup_{t\in(0,T],x\in E} t^{(|\gamma_0|+|\gamma_1|)\ell_0/2} h(x)^{2(|\gamma_0|+|\gamma_1|)\ell_0-m} E[|k'(t,x)|^p]^{1/p} < \infty, \qquad T>0,\ p\in(1,\infty).$$

Note that there is a Borel function $\tilde{k}(t,x) : \mathbf{R}^N \to \mathbf{R}$, $t \in (0,T]$, $x \in E$, such that

$$E[k'(t,x)|\sigma\{X(t,x))\}] = \tilde{k}(t,x)(X(t,x)), \qquad t\in(0,T],\ x\in E.$$

Then we have

$$\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) f(y) h(y)^m \varphi_{t,x}(y)dy$$

$$= E[k'(t,x)f(X(t,x))] = E[\tilde{k}(t,x)(X(t,x))f(X(t,x))] = \int_{\mathbf{R}^N} f(y)\tilde{k}(t,x)(y)p(t,x,y)dy,$$

for any $f \in C_0^\infty(\mathbf{R}^N)$. This implies that $\partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y)h(y)^m\varphi_{t,x}(y) = \tilde{k}(t,x)(y)p(t,x,y)$ *a.e.y*, $t \geqq 0$, $x \in E$. Therefore letting $m = 0$, we have the first assertion. Since

$$\int_E h(y)^{pm}\varphi_{t,x}(y)^p \frac{|\partial_y^\gamma p(t,x,y)|^p}{p(t,x,y)^{p-1}} dy = \int_E |\tilde{k}(t,x,y)|^p p(t,x,y)dy$$

$$= E[|\tilde{k}(t,x)(X(t,x))|^p] \leqq E[|k'(t,x)|^p],$$

we have our assertion. ∎

Proposition 8 *For any $T > 0$, there is a $C > 0$ such that*

$$p(t,x,y) \leqq Ct^{-(N+1)\ell_0/2} h(x)^{-2(N+1)\ell_0} \exp(-\frac{2\delta_0}{t}|y-x|^2), \qquad t\in(0,T],\ x,y\in E$$

and

$$p(t,x,y) \leqq Ct^{-(N+1)\ell_0/2} h(y)^{-2(N+1)\ell_0} \exp(-\frac{2\delta_0}{t}|y-x|^2), \qquad t\in(0,T],\ x,y\in E.$$

In particular, for any $T > 0$ and $m \geqq 1$, there is a $C > 0$ such that

$$p(t,x,y) \leqq Ct^{-(N+1)\ell_0/2} h(x)^{-2(N+1)\ell_0} (1+|x|^2)^m (1+|y|^2)^{-m}, \qquad t\in(0,T],\ x,y\in E$$

Proof. Let C_0

$$= \sup\{t^{\ell_0/2}h(x)^2(\int_E \exp(\frac{2(N+1)\delta_0}{t}|y-x|^2)\frac{|\partial_{y^i}p(t,x,y)|^{N+1}}{p(t,x,y)^N}dy)^{1/(N+1)}; t \in (0,T],\ x \in E, \varepsilon > 0\}.$$

Let

$$\rho_\varepsilon(t,x,y) = (p(t,x,y) + \varepsilon \exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^{1/(N+1)}.$$

Then we see that

$$(\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)|\frac{\partial}{\partial y^i}\rho_\varepsilon(t,x,y)|^{N+1}dy)^{1/(N+1)}$$

$$= (N+1)^{-1}(\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)\frac{|\partial_{y^i}(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))|^{N+1}}{(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^N}dy)^{1/(N+1)}$$

$$\leqq (\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)\frac{|\partial_{y^i}p(t,x,y)|^{N+1}}{(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^N}dy)^{1/(N+1)}$$

$$+(\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)\frac{|\partial_{y^i}(\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))|^{N+1}}{(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^N}dy)^{1/(N+1)}$$

$$\leqq C_0 t^{-\ell_0/2}h(x)^{-2} + (\varepsilon\int_{\mathbf{R}^N}(2|y^i - x^i|)^{N+1}(1+\frac{1}{t})^{N+1}\exp(-|y-x|^2))dy)^{1/(N+1)}.$$

Also, we have

$$(\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)\rho_\varepsilon(t,x,y)^{N+1}dy)^{1/(N+1)}$$

$$= (\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))dy)^{1/(N+1)}$$

$$= (E[\exp(\frac{2\delta_0}{t}|X(t,x)-x|^2)] + \pi^N\varepsilon)^{1/(N+1)},$$

and

$$(\int_{\mathbf{R}^N}(|\partial_{y_i}(\exp(\frac{2\delta_0}{(N+1)t}|y-x|^2))|\rho_\varepsilon(t,x,y))^{N+1}dy)^{1/(N+1)}$$

$$= (\int_{\mathbf{R}^N}(\frac{4\delta_0|y_i - x_i|}{t})^{N+1}\exp(\frac{2\delta_0}{(N+1)t}|y-x|^2)(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))dy)^{1/(N+1)}$$

$$\leqq \frac{4\delta_0}{t}E[|X(t,x)-x|^{N+1}\exp(\frac{2\delta_0}{t}|X(t,x)-x|^2)]^{1/(N+1)} + \varepsilon\frac{4\delta_0}{t}(\int_{\mathbf{R}^N}|y_i|^{N+1}\exp(-|y|^2))dy)^{1/(N+1)}.$$

Then by Sobolev's inequality, we see that there is a constant $C > 0$ such that

$$\sup_{y\in\mathbf{R}^N}(\exp(\frac{2\delta_0}{t}|y-x|^2)((p(t,x,y)+\varepsilon\exp(-|y|^2)))^{1/(N+1)}$$

$$\leqq C(C_0 t^{-\ell_0/2}h(x)^{-2\ell_0} + Ct^{-1/2} + C\varepsilon(1+\frac{1}{t})).$$

So letting $\varepsilon \downarrow 0$, we have our first assertion.

Let

$$\tilde{\rho}_\varepsilon(t,x,y) = (p(t,x,y)h(y)^{2(N+1)\ell_0} + \varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^{1/(N+1)}.$$

Then similarly we can show that

$$\int_{\mathbf{R}^N}(\exp(\frac{2\delta_0}{(N+1)t}|y-x|^2)\tilde{\rho}_\varepsilon(t,x,y))^{N+1}$$

$$+\sum_{i=1}^N|\partial_{y_i}(\exp(\frac{2\delta_0}{(N+1)t}|y-x|^2)\tilde{\rho}_\varepsilon(t,x,y))|^{N+1})dy \leqq Ct^{-\ell_0/2}, \qquad t\in(0,T],\ x\in E.$$

So we have our second assertion.

Finally note that

$$|\log(1+|x|^2)-\log(1+|y|^2)| \leqq |\int_{|y|}^{|x|}\frac{2t}{1+t^2}dt| \leqq |x-y| \leqq \frac{1}{\varepsilon}+\varepsilon|x-y|^2, \qquad x,y\in\mathbf{R}^N,\ \varepsilon>0.$$

So we have the final assertion. ∎

Proposition 9 *Let $\delta\in(0,1/N)$, $\alpha,\beta\in\mathbf{Z}^N_{\geqq 0}$ and $T>0$. Then there are $C>0$ and $q>0$ such that*

$$|\partial_x^\alpha\partial_y^\beta p(t,x,y)| \leqq Ct^{-(|\alpha|+|\beta|+1)\ell_0/2}h(x)^{-2(|\alpha|+|\beta|+1)\ell_0}p(t,x,y)^{1-\delta}, \qquad x,y\in E,\ t\in(0,T],$$

and

$$|\partial_x^\alpha\partial_y^\beta p(t,x,y)| \leqq Ct^{-(|\alpha|+|\beta|+1)\ell_0/2}h(y)^{-2(|\alpha|+|\beta|+1)\ell_0}p(t,x,y)^{1-\delta}, \qquad x,y\in E,\ t\in(0,T].$$

Proof. Let $p=1/\delta>N$, and let

$$\rho_\varepsilon(t,x,y) = \frac{\partial_x^\alpha\partial_y^\beta p(t,x,y)}{(p(t,x,y)+\varepsilon)^{1-\delta}}$$

for $\varepsilon>0$. Then we see by Proposition 7 that there is a $C_1>0$ such that

$$(\int_{\mathbf{R}^N}|\rho_\varepsilon(t,x,y)|^p dy)^{1/p} = (\int_{\mathbf{R}^N}\frac{|\partial_x^\alpha\partial_y^\beta p(t,x,y)|^p}{(p(t,x,y)+\varepsilon)^{p-1}}dy)^{1/p}$$

$$\leqq C_1t^{-(|\alpha|+|\beta|)\ell_0/2}h(y)^{-2(|\alpha|+|\beta|)\ell_0}, \qquad \varepsilon>0,\ t\in(0,T],\ x\in E.$$

Also, we have

$$(\int_{\mathbf{R}^N}|\partial_{y^i}\rho_\varepsilon(t,x,y)|^p dy)^{1/p}$$

$$\leqq (\int_{\mathbf{R}^N}\frac{|\partial_x^\alpha\partial_y^\beta\partial_{y^i}p(t,x,y)|^p}{(p(t,x,y)+\varepsilon)^{p-1}}dy)^{1/p}$$

$$+(1-\delta)(\int_{\mathbf{R}^N}\frac{|\partial_x^\alpha\partial_y^\beta p(t,x,y)|^{2p}}{(p(t,x,y)+\varepsilon)^{2p-1}}dy)^{1/(2p)}(\int_{\mathbf{R}^N}\frac{|\partial_{y_i}p(t,x,y)|^{2p}}{(p(t,x,y)+\varepsilon)^{2p-1}}dy)^{1/(2p)}.$$

So we see by Proposition 7 that there is a $C_2 > 0$ such that

$$(\int_{\mathbf{R}^N} |\partial_{y_i}\rho_\varepsilon(t,x,y)|^p dy)^{1/p}$$

$$\leqq C_2 t^{-(|\alpha|+|\beta|+1)\ell_0/2} h(y)^{-2(|\alpha|+|\beta|+1)\ell_0}, \qquad \varepsilon > 0,\ t \in (0,T],\ x \in E.$$

So by Sobolev's inequality, we see that there is a $C_3 > 0$ such that

$$\sup_{y\in\mathbf{R}^N} |\rho_\varepsilon(t,x,y)|$$

$$\leqq C_3 t^{-(|\alpha|+|\beta|+1)\ell_0/2} h(x)^{-2(|\alpha|+|\beta|+1)}, \qquad \varepsilon > 0,\ t \in (0,T],\ x \in E.$$

Letting $\varepsilon \downarrow 0$, we have the first assertion.

Let

$$\tilde\rho_\varepsilon(t,x,y) = \frac{\partial_x^\alpha \partial_y^\beta p(t,x,y)}{(p(t,x,y)+\varepsilon)^{1-\delta}} h(y)^{2(|\alpha|+|\beta|+1)}$$

for $\varepsilon > 0$. Then a similar argument implies that there is a $C_4 > 0$ such that

$$\sup_{y\in\mathbf{R}^N} |\tilde\rho_\varepsilon(t,x,y)|$$

$$\leqq C_4 t^{-(|\alpha|+|\beta|+1)\ell_0/2}, \qquad \varepsilon > 0,\ t \in (0,T],\ x \in E.$$

So we have the second assertion. ∎

Proposition 10 *Let $m \geqq 0$, $\alpha, \beta \in \mathbf{Z}^N_{\geqq 0}$, $p \in [1,\infty)$, $\delta \in (0,1)$ and $T > 0$. Then there is a $C > 0$ such that*

$$\int_{\mathbf{R}^N} |\partial_t^m \partial_x^\alpha \partial_y^\beta p(t-s,x,y)|^p p(s,x_0,x) dx$$
$$\leqq C(t-s)^{p(|\alpha|+|\beta|+2m+2)\ell_0/2} p(t,x_0,y)^{1-\delta}$$

for any $t \in (0,T],\ s \in [0,t),\ y \in \mathbf{R}^N$.

Proof. First note that

$$\partial_t p(t,x,y) = L_x p(t,x,y), \qquad \text{where } L = \frac{1}{2}\sum_{k=1}^d V_k^2 + V_0.$$

So it is sufficient to prove the case $m = 0$.

Let $r = 1/(1-\delta)$. Since $p > 1-\delta$, we see by Propositions 8and 9, that there is a $C > 0$ and $b > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta p(t-s,x,y)|^p \leqq C(t-s)^{p(|\alpha|+|\beta|+2)\ell_0/2} h(x)^{-b} p(t-s,x,y)^{1-\delta},$$

for any $t \in (0,T],\ s \in [0,t),\ x \in E,\ y \in \mathbf{R}^N$. So we see that

$$\int_{\mathbf{R}^N} |\partial_x^\alpha \partial_y^\beta p(t-s,x,y)|^p p(s,x_0,x) dx$$

$$\leqq C(t-s)^{p(|\alpha|+|\beta|+2)\ell_0/2}\int_{\mathbf{R}^N} h(z)^{-b}p(t-s,z,y)^{1/r}p(s,x_0,z)dz$$

$$\leqq C(t-s)^{p(|\alpha|+|\beta|+2)\ell_0/2}(\int_{\mathbf{R}^N}(h(z)^{-b/\delta}p(s,x_0,z)dz)^{\delta}(\int_{\mathbf{R}^N} p(t-s,z,y)p(s,x_0,z)dz)^{1-\delta}.$$

Since

$$\int_{\mathbf{R}^N} p(t-s,z,y)p(s,x_0,z)dz = p(t,x_0,y),$$

we have our assertion. ∎

Proposition 11 *Let $a \in (0,1]$, and $b \in (0,a)$. Then we have*

$$\int_{\mathbf{R}^N} p(s,x_0,x)^a p(t-s,x,y)^b\phi(x)dx \leqq p(t,x_0,y)^b(\int_E dxp(s,x_0,x)^{(a-b)/(1-b)}\phi(x)^{1/(1-b)})^{1-b}$$

for any $t > s \geqq 0$, and non-negative measurable function $\phi : E \to [0,\infty)$.

Proof. Let $\delta = (a-b)/(1-b)$, $p = 1/b$, and $q = 1/(1-b)$. Then we see that $1-\delta = (1-a)/(1-b)$ and $a-\delta = b(1-a)/(1-b)$, and so we have

$$\int_{\mathbf{R}^N} p(s,x_0,x)^a p(t-s,x,y)^b\phi(x)dx = \int_{\mathbf{R}^N} p(s,x_0,x)^{\delta}p(s,x_0,x)^{(1-\delta)/p}p(t-s,x,y)^{1/p}\phi(x)dx$$

$$\leqq (\int_E p(s,x_0,x)^{\delta}p(s,x_0,x)^{1-\delta}p(t-s,x,y)dx)^{1/p}(\int_E p(s,x_0,x)^{\delta}\phi(x)^q dx)^{1/q}$$

$$= p(s,x_0,y)^b(\int_E p(s,x_0,x)^{(a-b)/(1-b)}\phi(x)^{1/(1-b)}dx)^{1-b}.$$

This proves our assertion. ∎

Proposition 12 *Let $p \geqq 1$, $m \geqq 1$. $\alpha,\beta \in \mathbf{Z}^N_{\geqq 0}$, $T > 0$, $a \in (0,1/p]$ and $b \in (a-1/N,a)$. tThen here are $C > 0$ such that*

$$\int_{\mathbf{R}^N} |\partial_x^{\alpha}(p(s,x_0,x)^a)|^p|\partial_x^{\beta}p(t-s,x,y)|^p dx$$

$$\leqq Cs^{-p(|\alpha|+1)\ell_0/2}(t-s)^{-p(|\beta|+2)\ell_0/2}p(t,x_0,y)^{pb}(1+|y|^2)^{-m}$$

for any $y \in E$ and $s,t \in (0,T]$ with $s < t$.

Proof. Let $\delta = (a-b)/2 < 1/N$. Note that $\partial_x^{\alpha}(p(s,x_0,x)^a)$ is a linear conbination of $a(a-1)\cdots(a-m+1)p(s,x_0,x)^{a-m}\partial_x^{\alpha_1}p(s,x_0,x)\cdots\partial_x^{\alpha_m}p(s,x_0,x)$, $m = 1,\ldots,|\alpha|$, $\alpha_k \in \mathbf{Z}_{\geqq 0}$, $|\alpha_k| \geqq 1$, $k = 1,\ldots,m$, $\alpha_1+\cdots+\alpha_m = \alpha$.

Then by Propositions 9, we see that there is a $C_1 > 0$ such that

$$|\partial_x^{\alpha}(p(s,x_0,x)^a)||\partial_x^{\beta}p(t-s,x,y)|$$

$$\leqq C_1 s^{-(|\alpha|+1)\ell_0/2}(t-s)^{-(|\beta|+1)\ell_0/2}h(x)^{-2(|\beta|+1)\ell_0}p(s,x_0,x)^{a-\delta}p(t-s,x,y)^{1-\delta}$$

for any $a \in (0,1/p]$, $b \in (a-1/N,a)$, $x,y \in E$ and $s,t \in [0,T]$ with $s < t$. By Propositions 8, we see that there is a $C_2 > 0$ such that

$$p(t-s,x,y)^{1-\delta-b} \leqq C_2(t-s)^{-\ell_0/2}h(x)^{-2(N+1)\ell_0}(1+|x|^2)^m(1+|y|^2)^{-(1-\delta-b)m}$$

for any $x,y\in E$ and $s,t\in[0,T]$ with $s<t$. So we have

$$|\partial_x^\alpha(p(s,x_0,x)^a)||\partial_x^\beta p(t-s,x,y)|$$
$$\leqq C_1C_2 s^{-(|\alpha|+1)\ell_0/2}(t-s)^{-(|\beta|+2)\ell_0/2}h(x)^{-2(|\beta|+N+2)\ell_0}p(s,x_0,x)^{a-\delta}p(t-s,x,y)^b$$
$$\times(1+|x|^2)^m(1+|y|^2)^{-(1-(a+b)/2)m}$$

Note that $pb<p(a-\delta)<1$, and so we have

$$\int_E (h(x)^{-2(|\beta|+N+2)\ell_0}p(s,x_0,x)^{a-\delta}p(t-s,x,y)^b(1+|x|^2)^m)^p dx$$
$$=\int_E p(s,x_0,x)^{p(a-\delta-b)/(1-pb)}p(s,x_0,y)^{pb(1-p(a-\delta))/(1-pb)}p(t-s,x,y)^{pb}$$
$$\times(h(x)^{-2p(|\beta|+N+2)\ell_0}(1+|x|^2)^{mp}\,dx$$
$$\leqq(\int_E p(s,x_0,x)^{p(a-\delta-b)/(1-pb)}p(s,x_0,y)^{(1-p(a-\delta))/(1-pb)}p(t-s,x,y)\,dx)^{pb}$$
$$\times(\int_E p(s,x_0,x)^{p(a-\delta-b)/(1-pb)}h(x)^{-p(|\beta|+N+2)/(1-pb)}(1+|x|^2)^{mp/(1-pb)}\,dx)^{1-pb}$$
$$=p(t,x_0,y)^{pb}(\int_E(1+|x|^2)^{-N}p(s,x_0,x)^{p(a-\delta-b)/(1-pb)}h(x)^{-2p(|\beta|+N+2)/(1-pb)}$$
$$\times(1+|x|^2)^{mp/(1-pb)+N}\,dx)^{1-pb}$$
$$\leqq p(t,x_0,y)^{pb}(\int_E(1+|x|^2)^{-N}p(s,x_0,x)h(x)^{-p(|\beta|+N+2)/\ell_0(p(a-\delta-b))}$$
$$\times(1+|x|^2)^{(mp+N(1-pb))/(p(a-\delta-b))}\,dx)^{p(a-\delta-b)}(\int_E(1+|x|^2)^{-N}\,dx)^{(1-p(a-\delta-b))/(1-pb)}$$
$$=p(t,x_0,y)^{pb}(\int_E(1+|x|^2)^{-N}\,dx)^{(1-p\delta)/(1-pb)}$$
$$\times E^\mu[h(X(s,x_0))^{-(|\beta|+N+2)\ell_0/\delta}(1+|X(s,x_0)|^2)^{(mp+N(1-pb))/(p\delta)}]^{p\delta}.$$

So by Proposition 3, we have our assertion. ∎

3 Stochastic mesh and random norms

Let $\mathcal{F}_t^{(L)}$, $t\geqq 0$, $L=0,1,\ldots,\infty$ be sub σ-algebra of $\mathcal{F}$ given by

$$\mathcal{F}_t^{(L)}=\sigma\{X_\ell(s);\ s\in[0,t],\ \ell=1,2,\ldots.L\},$$

and

$$\mathcal{F}_t^{(\infty)}=\sigma\{X_\ell(s);\ s\in[0,t],\ \ell=1,2,\ldots\}.$$

Let ν_t, $t\geqq 0$, be the probability law of $X(t,x_0)$ under μ. Then we see that ν_0 is the probability measure concentrated in x_0, and $\nu_t(dx)=p(t,x_0,x)dx$, $t>0$.

Then for any $t>s\geqq 0$, we can define a linear contraction map $P_{s,t}:L^1(E;d\nu_t)\to L^1(E;d\nu_s)$ by

$$(P_{s,t}f)(x)=\int_E p(t-s,x,y)f(y)dy,\qquad x\in E,\ f\in L^1(E;d\nu_t).$$

Proposition 13 *Let $t > s \geqq 0$, $\alpha \in \mathbf{Z}^N_{\geqq 0}$ and bounded measurable function $f : E \to \mathbf{R}$. Then we have*

$$E[\partial_x^\alpha (Q_{s,t}^{(L)} f)(x)|\mathcal{F}_s^{(\infty)}] = \partial_x^\alpha (P_{s,t} f)(x), \qquad \nu_s - a.e.x.$$

and

$$E[|\partial_x^\alpha (Q_{s,t}^{(L)} f)(x) - \partial_x^\alpha (P_{s,t} f)(x)|^2|\mathcal{F}_s^{(\infty)}] \leqq \frac{1}{L}\int_E \frac{(\partial_x^\alpha p(t-s,x,y))^2|f(y)|^2}{q_{s,t}^{(L)}(y)}\, dy.$$

Proof. Note that

$$E[\partial_x^\alpha (Q_{s,t}^{(L)} f)(x)|\mathcal{F}_s^{(\infty)}] = \frac{1}{L}\sum_{\ell=1}^L \int_E \frac{\partial_x^\alpha p(t-s,x,y)f(y)}{q_{s,t}^{(L)}(y)} p(t-s, X_\ell(s), y)\, dy$$

$$= \int_E \partial_x^\alpha p(t-s,x,y) f(y)\, dy = \partial_x^\alpha (P_{s,t} f)(x).$$

This implies the first assertion.

Let

$$m_\ell = \frac{1}{L}\int_E \frac{\partial_x^\alpha p(t-s,x,y)f(y)}{q_{s,t}^{(L)}(y)} p(t-s, X_\ell(s), y)\, dy$$

and

$$d_\ell = \frac{1}{L}\frac{\partial_x^\alpha p(t-s,x,X_\ell(t))f(X_\ell(t))}{q_{s,t}^{(L)}(X_\ell(t))} - m_\ell$$

for $\ell = 1, \ldots, L$. Then we see that

$$E[d_\ell|\mathcal{F}_s^{(\infty)} \vee \mathcal{F}_t^{(\ell-1)}] = 0, \qquad \ell = 1, \ldots, L.$$

Here we let $\mathcal{F}_t^{(0)} = \{\emptyset, \Omega\}$. Moreover, we have

$$\sum_{\ell=1}^L d_\ell = \partial_x^\alpha (Q_{s,t}^{(L)} f)(x) - \partial_x^\alpha (P_{s,t} f)(x)$$

So we see that

$$E[|\partial_x^\alpha (Q_{s,t}^{(L)} f)(x) - \partial_x^\alpha (P_{s,t} f)(x)|^2|\mathcal{F}_s^{(\infty)}] \leqq E[(\sum_{\ell=1}^L |d_\ell|^2)|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \sum_{\ell=1}^L E[(\frac{1}{L}\frac{\partial_x^\alpha p(t-s,x,X_\ell(t))f(X_\ell(t))}{q_{s,t}^{(L)}(X_\ell(t))})^2|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{1}{L^2}\sum_{\ell=1}^L \int_E \frac{(\partial_x^\alpha p(t-s,x,y))^2|f(y)|^2}{q_{s,t}^{(L)}(y)^2} p(t-s, X_\ell(s), y)\, dy$$

$$= \frac{1}{L}\int_E \frac{(\partial_x^\alpha p(t-s,x,y))^2|f(y)|^2}{q_{s,t}^{(L)}(y)}\, dy.$$

So we have the second assertion. ∎

Now let $M_t^{(L)} : m(E) \times \Omega \to \mathbf{R}$, and $N_t^{(L)} : m(E) \times \Omega \to [0,\infty)$, $t \geqq 0$, $L \geqq 1$, be a random functional given by

$$M_t^{(L)}(f) = M_t^{(L)}(f;\omega) = \frac{1}{L}\sum_{\ell=1}^{L} f(X_\ell(t)), \qquad f \in m(E),$$

and

$$N_t^{(L)}(f) = N_t^{(L)}(f;\omega) = M_t^{(L)}(|f|) = \frac{1}{L}\sum_{\ell=1}^{L} |f(X_\ell(t))|, \qquad f \in m(E).$$

Then we see that $M_t^{(L)}$ is a linear function and $N_t^{(L)}$ is a semi-norm in $m(E)$.

Proposition 14 *Let $t > s \geqq 0$ and $L \geqq 1$* (1) *For any $f \in m(E)$,*

$$M_s^{(L)}(Q_{s,t}^{(L)} f) = M_t(f).$$

(2) *For any $f \in m(E)$*

$$N_s^{(L)}(Q_{s,t}^{(L)} f) \leqq N_t(f).$$

Proof. Suppose that $f \in m(E)$. Then we have

$$M_s^{(L)}(Q_{s,t}^{(L)} f) = \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{L}\sum_{k=1}^{L}\frac{p(t-s,X_\ell(s),X_k(t))f(X_k(t))}{q_{s,t}^{(L)}(X_k(t))}$$

$$= \frac{1}{L}\sum_{k=1}^{L}(\frac{1}{L}\sum_{\ell=1}^{L}\frac{p(t-s,X_\ell(s),X_k(t))f(X_k(t))}{q_{s,t}^{(L)}(X_k(t))}) = M_t(f).$$

So we have the assertion (1).

The second assertion is an easy consequence of the assertion (1). ∎

Proposition 15 (1) *Let $T > 0$ and $m \geqq 1$. Then there is a $C > 0$ such that*

$$\frac{1}{L}\sum_{\ell=1}^{L} E[((Q_{s,t}^{(L)} f)(X_\ell(s)) - (P_{s,t}f)(X_\ell(s))^2|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{C}{L}(t-s)^{-(N+1)\ell_0/2}\max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m)\int_E f(y)^2(1+|y|^2)^{-m}\,dy \qquad a.s.$$

for any $L \geqq 1$ and $s,t \in [0,T]$ with $s < t$.

In particular,

$$E[N_s^{(L)}(Q_{s,t}^{(L)} f - P_{s,t}f)^2]$$

$$\leqq \frac{C}{L}(t-s)^{-(N+1)\ell_0/2}E[\max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m]\int_E f(y)^2(1+|y|^2)^{-m}\,dy$$

for any $L \geqq 1$ and $s,t \in [0,T]$ with $s < t$.
(2) *For any $\varepsilon > 0$ and $T > 0$,*

$$\overline{\lim_{L\to\infty}}\, L^{-\varepsilon}\sup_{s\in[0,T]} E[\max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m] = 0$$

Proof. By Proposition 13, we see that

$$\frac{1}{L}\sum_{\ell=1}^{L} E[((Q_{s,t}^{(L)}f)(X_\ell(s)) - (P_{s,t}f)(X_\ell(s))^2|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{1}{L^2}\sum_{\ell=1}^{L}\int_E \frac{p(t-s,X_\ell(s),y)^2 f(y)^2}{q_{s,t}^{(L)}(y)}dy$$

$$\leqq \frac{1}{L^2}\sum_{\ell=1}^{L}\int_E (\max_{\ell'=1,\dots,L} p(t-s,X_{\ell'}(s),y))\frac{p(t-s,X_\ell(s),y)f(y)^2}{q_{s,t}^{(L)}(y)}dy$$

$$= \frac{1}{L}\int_E (\max_{\ell=1,\dots,L} p(t-s,X_\ell(s),y))f(y)^2 dy.$$

Then by Proposition 8 we have the assertion (1).

Let $\varepsilon > 0$. Let us take $p > 1/\varepsilon$. Then we have

$$E[\max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m]$$

$$\leqq E[(\sum_{\ell=1}^{L}(h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m)^p)^{1/p}]$$

$$\leqq E[\sum_{\ell=1}^{L}(h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m))^p]^{1/p}$$

$$= L^{1/p}E^\mu[h(X(s,x_0))^{-2p(N+1)}(1+|X(s,x_0)|^2)^{mp}]^{1/p}$$

$$\leqq L^{1/p}E^\mu[h(X(s,x_0))^{-4p(N+1)}]^{1/(2p)}E^\mu[(1+|X(s,x_0)|^2)^{2pm}]^{1/(2p)}$$

So we have the assertion (2) by Proposition 3.

4 Application 1

Let $r \geqq 0$, and let $\mathcal{B}_r$ be the set of Borel measrurable functions $f : \mathbf{R}^N \to \mathbf{R}$ such that $\sup_{x\in\mathbf{R}^N}(1+|x|^2)^{-r/2}|f(x)| < \infty$.

Then we see that $Q_{s,t}^{(L)}$ and $P_{s,t}$, $t > s \geqq 0$, can be regarded linear operators on $\mathcal{B}_r$.

Now let $\phi_{s,t} : \mathbf{R}^n \times \mathbf{R}$, $s,t \in [0,\infty)$, $s < t$, be measurable functions. We assume that there is a $\lambda \geqq 0$, such that

$$|\phi_{s,t}(x,y) - \phi_{s,t}(x,z)| \leqq \exp(\lambda(t-s))|y-z|, \qquad x \in \mathbf{R}^N,\ y,z \in \mathbf{R},\ t > s \geqq 0.$$

Also, we assume that $\phi_{s,t}(\cdot,0) \in \mathcal{B}_r$, $t > s \geqq 0$.

Let us define a nonlinear operator $\Phi_{s,t} : \mathcal{B}_r \to \mathcal{B}_r$, $s,t \in [0,\infty)$, $s < t$, by

$$(\Phi_{s,t}f)(x) = \phi_{s,t}(x,f(x)), \qquad x \in E,\ f \in \mathcal{B}_r.$$

Then we have

$$N_s^{(L)}(\Phi_{s,t}f - \Phi_{s,t}g) \leqq \exp(\lambda(t-s))N_s^{(L)}(f-g)$$

for any $f,g \in \mathcal{B}_r$.

Let us define operators $\tilde{Q}_{s,t}^{(L)}$ and $\tilde{P}_{s,t}$ on $\mathcal{B}_r$ by $\tilde{Q}_{s,t}^{(L)} = \Phi_{s,t} \circ Q_{s,t}^{(L)}$ and $\tilde{P}_{s,t} = \Phi_{s,t} \circ P_{s,t}$.

Then we have the following easily from Propositions 14 and 15.

Proposition 16 (1)

$$N_s^{(L)}(\tilde{Q}_{s,t}^{(L)} f - \tilde{Q}_{s,t}^{(L)} g) \leqq \exp(\lambda(t-s)) N_t^{(L)}(f-g)$$

for any $f, g \in \mathcal{B}_r$.
(2) *Let* $T > 0$ *and* $m \geqq 1$. *Then there is a* $C > 0$ *such that*

$$E[N_s^{(L)}(\tilde{Q}_{s,t}^{(L)} f - \tilde{P}_{s,t} f)^2]$$

$$\leqq \frac{C}{L} a(L) \exp(2\lambda(t-s))(t-s)^{-(N+1)\ell_0/2} \int_E f(y)^2 (1+|y|^2)^{-(r+N)}\, dy$$

for any $L \geqq 1$ *and* $s, t \in [0,T]$ *with* $s < t$. *Here*

$$a(L) = \sup_{s \in [0,T]} E[\max_{\ell=1\ldots,L} h(X_\ell(s))^{-2(N+1)} (1 + |X_\ell(s)|^2)^m]$$

Note that by Proposition 15(2), we see that for any $\delta > 0$,

$$L^{-\delta} a(L) \to 0, \qquad L \to \infty.$$

So we have the following.

Theorem 17 *For* $T > 0$, *there is a* $C > 0$ *satisfying the following. For any* $n \geqq 1$, *and* $0 = t_0 < t_1 < \cdots < t_n \leqq T$,

$$E[|(\tilde{Q}_{t_0,t_1}^{(L)} \cdots \tilde{Q}_{t_{n-1},t_n}^{(L)} f)(x_0) - (\tilde{P}_{t_0,t_1} \cdots \tilde{P}_{t_{n-1},t_n} f)(x_0)|^2]^{1/2}$$

$$\leqq \frac{C}{L^{1/2}} a(L)^{1/2} \exp(\lambda t_n) \sum_{k=1}^n (t_k - t_{k-1})^{-(N+1)\ell_0/4}$$

$$(\int_E (\tilde{P}_{t_k,t_{k+1}} \cdots \tilde{P}_{t_{n-1},t_n} f)(y)^2 (1+|y|^2)^{-(r+N)}\, dy)^{1/2}$$

Proof. Note that

$$|(\tilde{Q}_{t_0,t_1}^{(L)} \cdots \tilde{Q}_{t_{n-1},t_n}^{(L)} f)(x_0) - (\tilde{P}_{t_0,t_1} \cdots \tilde{P}_{t_{n-1},t_n} f)(x_0)|$$

$$= N_0^{(L)}((\tilde{Q}_{t_0,t_1}^{(L)} \cdots \tilde{Q}_{t_{n-1},t_n}^{(L)} f) - (\tilde{P}_{t_0,t_1} \cdots \tilde{P}_{t_{n-1},t_n} f))$$

$$\leqq \sum_{k=1}^n N_0^{(L)}((\tilde{Q}_{t_0,t_1}^{(L)} \cdots \tilde{Q}_{t_{k-1},t_k}^{(L)} \tilde{P}_{t_k,t_{k+1}} \cdots \tilde{P}_{t_{n-1},t_n} f) - (\tilde{Q}_{t_0,t_1}^{(L)} \cdots \tilde{Q}_{t_{k-2},t_{k-1}}^{(L)} \tilde{P}_{t_{k-1},t_k} \cdots \tilde{P}_{t_{n-1},t_n} f))$$

$$\leqq \sum_{k=1}^n \exp(\lambda t_{k-1}) N_{t_{k-1}}^{(L)}(\tilde{Q}_{t_{k-1},t_k}^{(L)} \tilde{P}_{t_k,t_{k+1}} \cdots \tilde{P}_{t_{n-1},t_n} f) - (\tilde{P}_{t_{k-1},t_k} \cdots \tilde{P}_{t_{n-1},t_n} f)).$$

Also, we have by Propostion 16

$$E[N_{t_{k-1}}^{(L)}(\tilde{Q}_{t_{k-1},t_k}^{(L)} \tilde{P}_{t_k,t_{k+1}} \cdots \tilde{P}_{t_{n-1},t_n} f) - (\tilde{P}_{t_{k-1},t_k} \cdots \tilde{P}_{t_{n-1},t_n} f))^2]^{1/2}$$

$$\leqq \frac{C^{1/2}}{L^{1/2}} a(L)^{1/2} \exp(\lambda(t_k - t_{k-1}))(t_k - t_{k-1})^{-(N+1)\ell_0/4}$$

$$\times \left(\int_E (\tilde{P}_{t_k,t_{k+1}} \cdots \tilde{P}_{t_{n-1},t_n} f)(y)^2 (1+|y|^2)^{-(r+N)}\, dy\right)^{1/2}.$$

These imply our theorem. ∎

Now we apply the above theorem to American option. Let $g : [0,T] \times \mathbf{R}^n \to \mathbf{R}$ be a continuous function such that there are $r \geqq 1$ and $C_1 > 0$ such that $|g(t,x)| \leqq C_1(1+|x|^2)^{r/2}$, $t \in [0,T]$, $x \in \mathbf{R}^n$. Let $\phi_{s,t}(x,y) = g(s,x) \vee y$, for $x \in \mathbf{R}^n$, $y \in \mathbf{R}$, and $s,t \in [0,T]$ with $s < t$. Then we have $\phi_{s,t}(x,y) - \phi_{s,t}(x,z)| \leqq |y-z|$. It is easy to see that there is a $a \geqq 0$ such that

$$E[\sup_{t\in[0,T]} (1+|X(t,x)|^2)^{r/2}] \leqq \exp(aT)(1+|x|^2)^{r/2}, \qquad x \in \mathbf{R}^n.$$

So we see that

$$\sup_{x\in\mathbf{R}^n} (1+|x|^2)^{-r/2} |\tilde{P}_{s,t} f(x)| \leqq \exp C_1 \vee \exp(a(t-s)) \sup_{x\in\mathbf{R}^n} (1+|x|^2)^{-r/2} |f(x)|, \qquad f \in \mathcal{B}_r.$$

Then we see that

$$\left(\int_E (\tilde{P}_{t_k,t_{k+1}} \cdots \tilde{P}_{t_{n-1},t_n} g(t_n,\cdot))(y)^2 (1+|y|^2)^{-(r+N)}\, dy\right)^{1/2} \leqq C_1 \exp(a(t_n - t_k)) \int_E (1+|y|^2)^{-N}\, dy)^{1/2}.$$

So we have by Theorem 17, we see that there is a $C_2 > 0$ such that

$$E[|(\tilde{Q}^{(L)}_{t_0,t_1} \cdots \tilde{Q}^{(L)}_{t_{n-1},t_n} g(t_n,\cdot)(x_0) - (\tilde{P}_{t_0,t_1} \cdots \tilde{P}_{t_{n-1},t_n} g(t_n,\cdot))(x_0)|^2]^{1/2}$$

$$\leqq \frac{C_2}{L^{1/2}} a(L)^{1/2} \sum_{k=1}^n (t_k - t_{k-1})^{-(N+1)\ell_0/4}$$

for any $n \geqq 1$, and $0 = t_0 < t_1 < \cdots < t_n \leqq T$. So if we take $n_L \geqq 1$ and $0 = t_0^{(L)} < t_1^{(L)} < \cdots < t_n^{(L)} = T$ for each $L \geqq 1$, and there is a $\delta_0, \delta_1 > 0$, with $\delta_0 < \delta_1 < 1/2$ such that

$$\lim_{L\to\infty} L^{-\delta_0} \sum_{k=1}^{n_L} (t_k^{(L)} - t_{k-1}^{(L)})^{-(N+1)\ell_0/4} = 0,$$

then we see that

$$L^{-(1-\delta_1)/2} |(\tilde{Q}^{(L)}_{t_0^{(L)},t_1^{(L)}} \cdots \tilde{Q}^{(L)}_{t^{(L)}_{n_L-1},t^{(L)}_{n_L}} g(T,\cdot)(x_0) - (\tilde{P}_{t_0^{(L)},t_1^{(L)}} \cdots \tilde{P}_{t^{(L)}_{n_L-1},t^{(L)}_{n_L}} gT,\cdot))(x_0)| \to 0$$

in probability.

5 Preparations for estimates of functions

Proposition 18 *Let Z_k, $k = 1,2\ldots$ be independent integrable random variables.*
(1) *For any $p \geqq 1$, there is a $C > 0$ only depend on p such that*

$$E[|\sum_{k=1}^n (Z_k - E[Z_k])|^{2p}] \leqq C(E[(\sum_{k=1}^n Z_k^2)^p] + (\sum_{k=1}^n |E[Z_k]|)^{2p}), \; n \geqq 1.$$

(2) *For any* $p \geqq 1$, *there is a* $C > 0$ *only depend on* p *such that*

$$E[|\sum_{k=1}^{n} Z_k|^{2p}] \leqq C(E[(\sum_{k=1}^{n} Z_k^2)^p] + (\sum_{k=1}^{n} |E[Z_k]|)^{2p}),\ n \geqq 1.$$

(3) *For any* $m \in \mathbf{N}$, *there is a* $C > 0$ *only depend on* m *such that*

$$E[|\sum_{k=1}^{n} Z_k^2|^{2^m}] \leqq C \sum_{r=1}^{m+1} (\sum_{k=1}^{n} E[Z_k^{2^r}])^{2^{m+1-r}}\ n \geqq 1.$$

Proof. (1) If $\sum_{k=1}^n E[|Z_k|^{2p}] = \infty$, the right hand side is infinity, and so the inequality is valid. So we assume that $\sum_{k=1}^n E[|Z_k|^{2p}] < \infty$. Then by Burkholder's inequality we have

$$E[|\sum_{k=1}^{n} (Z_k - E[Z_k])|^{2p}] \leqq C_{2p} E[(\sum_{k=1}^{n} (Z_k - E[Z_k])^2)^p].$$

Since we have

$$E[(\sum_{k=1}^{n} (Z_k - E[Z_k])^2)^p] \leqq 2^p E[(\sum_{k=1}^{n} (Z_k^2 + E[Z_k]^2))^p]$$

$$\leqq 2^{2p} E[(\sum_{k=1}^{n} Z_k^2)^p] + 2^{2p} (\sum_{k=1}^{n} E[Z_k]^2)^p \leqq 2^{2p} E[(\sum_{k=1}^{n} Z_k^2)^p] + 2^{2p} (\sum_{k=1}^{n} |E[Z_k]|)^{2p},$$

we have our assertion.
(2) Note that

$$E[|\sum_{k=1}^{n} Z_k|^{2p}] = E[|\sum_{k=1}^{n} ((Z_k - E[Z_k]) + E[Z_k])|^{2p}]$$

$$\leqq 2^{2p} (E[|\sum_{k=1}^{n} (Z_k - E[Z_k]|^{2p}] + |\sum_{k=1}^{n} E[Z_k]|^{2p}).$$

So we have our assertion by the assertion (1).

We can show the assertion (3) easily by induction and the assertion (2). ∎

Proposition 19 *For any* $m \geqq 1$, $j \geqq 0$, $\alpha \in \mathbf{Z}_{\geqq 0}^N$, $\delta \in (0,1)$, *and* $T > 0$, *there is a* $C > 0$ *such that*

$$E[\sup_{s \in [0, t-\varepsilon]} |(\frac{1}{L} \sum_{\ell=1}^{L} \partial_t^j \partial_y^\alpha p(t-s, X_\ell(s), y)) - \partial_t^j \partial_y^\alpha p(t, x_0, y)|^{2^{m+1}}]$$

$$\leqq C \varepsilon^{-2^m (j+|\alpha|+3)\ell_0} L^{-2^m} L p(t, x_0, y)^{1-\delta} (L^{-1} + p(t, x_0, y)^{1-\delta})^{2^m},$$

for any $y \in \mathbf{R}^N$, $L \geqq 1$, $t \in (0, T]$, $\varepsilon \in (0, t)$.

Proof. Let us note that

$$\frac{\partial}{\partial t} \partial_t^j \partial_y^\alpha p(t, x, y) = L_x \partial_t^j \partial_y^\alpha p(t, x, y), \qquad t > 0,\ x \in E,\ y \in \mathbf{R}^N,$$

where

$$L_x = \frac{1}{2}\sum_{k=1}^d V_k^2 + V_0.$$

So we see that $\partial_t^j\partial_y^\alpha p(t-s, X_\ell(s), y)$, $s \in [0,t)$, $h>0$, is a martingale, and

$$\langle \partial_t^j\partial_y^\alpha p(t-\cdot, X_\ell(\cdot), y)\rangle_s$$

$$= \sum_{k=1}^d \int_0^s |\partial_t^j\partial_y^\alpha V_{k,x} p(t-r, X_\ell(r), y)|^2 dr.$$

So we have by Burkholder's inequality and Proposition 18 (3),

$$E[\sup_{s\in[0,t-\varepsilon]} |\sum_{\ell=1}^L (\partial_t^j\partial_y^\alpha p(t-s, X_\ell(s), y) - \partial_t^j\partial_y^\alpha p(t, x_0, y))|^{2^{m+1}}]$$

$$\leqq C_{2^{m+1}} E[(\sum_{\ell=1}^L\sum_{k=1}^d \int_0^{t-\varepsilon} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, X_\ell(s), y)|^2 ds)^{2^m}]$$

$$\leqq C_{2^{m+1}} d^{2^m} \sum_{k=1}^d E[(\sum_{\ell=1}^L \int_0^{t-\varepsilon} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, X_\ell(s), y)|^2 ds)^{2^m}]$$

$$\leqq C\sum_{k=1}^d\sum_{r=0}^m (\sum_{\ell=1}^L E[(\int_0^{t-\varepsilon} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, X_\ell(s), y)|^2 ds)^{2^r}])^{2^{m-r}}$$

$$\leqq C\sum_{k=1}^d\sum_{r=0}^m t^{2^m-2^{m-r}} (\sum_{\ell=1}^L E[(\int_0^{t-\varepsilon} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, X_\ell(s), y)|^{2^{r+1}} ds])^{2^{m-r}}$$

$$= C\sum_{k=1}^d\sum_{r=0}^m t^{2^m-2^{m-r}} L^{2^{m-r}} (\int_0^{t-\varepsilon} (\int_{\mathbf{R}^N} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, z, y)|^{2^{r+1}} p(s, x_0, z) dz) ds)^{2^{m-r}}.$$

Then by Proposition 10, we have

$$E[\sup_{s\in[0,t-\varepsilon]} |\sum_{\ell=1}^L (\partial_t^j\partial_y^\alpha p(t-s, X_\ell(s), y) - \partial_t^j\partial_y^\alpha p(t, x_0, y))|^{2^{m+1}}]$$

$$\leqq C' t^{2^m} \varepsilon^{-2^m(j+|\alpha|+3)\ell_0} \sum_{r=0}^m L^{2^{m-r}} p(t, x_0, y)^{2^{m-r}(1-\delta)}.$$

$$\leqq C' t^{2^m} \varepsilon^{-2^m(j+|\alpha|+3)\ell_0} L^{2^m} L p(t, x_0, y)^{1-\delta} (L^{-1} + p(t, x_0, y)^{1-\delta})^{2^m}.$$

This implies our assertion. ∎

Proposition 20 *For any $\delta \in (0, 1/2)$, $T > 0$ and $p \in [2, \infty)$, there is a $C > 0$ such that*

$$E[(\sup_{y\in\mathbf{R}^N} \sup_{t\in[\varepsilon,T], s\in[0,t-\varepsilon]} (\frac{|q_{s,t}^{(L)}(y) - p(t, x_0, y)|}{(L^{-1/(1-\delta)} + p(t, x_0, y))^{(1-\delta)/2}})^p]^{1/p} \leqq C\varepsilon^{-5\ell_0} L^{-1/2+1/p}, \quad L \geqq 1,\ \varepsilon \in (0,1).$$

Proof. Let us take an $m \geqq 1$ such that $p+N<2^m$. Note that

$$L^{-1}+p(t,x_0,y)^{1-\delta} \leqq 2(L^{-1/(1-\delta)}+p(t,x_0,y))^{1-\delta}.$$

Let

$$\rho_L(s,t,y)=\frac{q^{(L)}_{s,t}(y)-p(t,x_0,y)}{(L^{-1/(1-\delta)}+p(t,x_0,y))^{(1-\delta)/2}}, \qquad 0 \leqq s<t \leqq T,\ y \in \mathbf{R}^N.$$

We see by Proposition 9, we see that for any $a>0$, $j \geqq 0$, and $\alpha \in \mathbf{Z}^N_{\geqq 0}$, there is a $C>0$ such that

$$(L^{-1/(1-\delta)}+p(t,x_0,y))^{-a+2j+|\alpha|}|\partial_t^j\partial_y^\alpha((L^{-1/(1-\delta)}+p(t,x_0,y))^{-a})|$$
$$\leqq Ct^{-(2j+|\alpha|)\ell_0}p(t,x_0,y)^{-\delta}, \qquad y \in \mathbf{R}^N,\ t \in (0,T].$$

So we see that by Proposition 18, for any $a>0$, $j=0,1$, and $\alpha \in \mathbf{Z}^N_{\geqq 0}$ with $|\alpha| \leqq 1$, there is a $C>0$ such that

$$E[\sup_{s\in[0,t-\varepsilon]}|\partial_t^j\partial_y^\alpha\rho_L(s,t,y)|^{2^{m+1}}]$$
$$\leqq C\varepsilon^{-2^{m+1}4\ell_0}L^{-2^m}Lp(t,x_0,y)^{1-2\delta}, \qquad y \in \mathbf{R}^N,\ L \geqq 1,\ \varepsilon \in (0,1),\ t \in (\varepsilon,T].$$

Therefore we see that

$$E[\int_{\mathbf{R}^N}dy \sup_{s\in[0,t-\varepsilon]}|\partial_t^j\partial_y^\alpha\rho_L(s,t,y)|^{2^{m+1}}]$$
$$\leqq C\varepsilon^{-2^{m+3}\ell_0}L^{-2^m}L\int_{\mathbf{R}^N}p(t,x_0,y)^{1-2\delta}dy, \qquad L \geqq 1,\ \varepsilon \in (0,1),\ t \in (\varepsilon,T].$$

Note by Proposition 8 that there is a $C>0$ such that

$$\int_{\mathbf{R}^N}p(t,x_0,y)^{1-2\delta}dy \leqq Ct^{(N+1)\ell_0\delta}, \qquad t \in (0,T].$$

Also, note that

$$\partial_y^\alpha\rho_L(s,t,y)=\partial_y^\alpha\rho_L(s,T,y)-\int_t^T\partial_r\partial_y^\alpha\rho_L(s,r,y)dr,$$

and so we see that

$$\sup_{t\in[\varepsilon,T],s\in[0,t-\varepsilon]}\int_{\mathbf{R}^N}|\partial_y^\alpha\rho_L(s,t,y)|^{2^m}dy$$
$$\leqq 2^{m+1}\int_{\mathbf{R}^N}dy\sup_{s\in[0,T-\varepsilon]}|\partial_y^\alpha\rho_L(s,T,y)|^{2^m}$$
$$+2^{m+1}(T+1)^{2^m}\int_t^T dr\int_{\mathbf{R}^N}dy\sup_{s\in[0,r-\varepsilon)}|\partial_y^\alpha\rho_L(s,r,y)|^{2^m}.$$

Then by Sobolev's inequality, we see that there is a $C>0$ such that

$$E[\sup_{y\in\mathbf{R}^N}\sup_{t\in[\varepsilon,T],s\in[0,t-\varepsilon]}|\rho_L(s,t,y)|^{2^{m+1}}]^{1/2^{m+1}} \leqq C\varepsilon^{-(4+(N+1)/2^{m+1})\ell_0}L^{-1/2+1/2^{m+1}},$$

$L \geqq 1$, $\varepsilon \in (0,1)$. This implies our assertion. ∎

Let

$$Z_L(s,t;\delta) = \sup_{y\in\mathbf{R}^N} \frac{|q_{s,t}^{(L)}(y) - p(t,x_0,y)|}{(L^{-1/(1-\delta)} + p(t,x_0,y))^{(1-\delta)/2}}, \qquad t>0,\ s\in[0,t)$$

and

$$\tilde{Z}_L(\varepsilon,\delta,T) = \sup_{t\in[\varepsilon,T], s\in[0,t-\varepsilon]} Z_L(s,t;\delta)$$

for $T>0$, $\varepsilon\in(0,T]$, and $\delta\in(0,1)$. Note that $Z_L(s,t;\delta)$ is $\mathcal{F}_s^{(\infty)}$-measurable.

Then we have the following.

Proposition 21 (1) *Let $T>0$, $\varepsilon\in(0,T]$, and $\delta\in(0,1)$. Then for any $p>1$, there is a $C>0$ such that*

$$E[(L^{(1-\delta^2)/2}\tilde{Z}_L(\varepsilon,\delta,T))^p]^{1/p} \leqq C\varepsilon^{-5\ell_0}L^{-p\delta^2/2+1/p}, \qquad L\geqq 1.$$

(2)*Let $\delta\in(0,1)$, $t>0$, and $s\in(0,t)$. If $L^{(1-\delta^2)/2}Z_L(s,t;\delta)\leqq 1/4$, and $p(t,x_0,y)\geqq L^{-(1-\delta)}$, then*

$$\frac{1}{2}\leqq \frac{q_{s,t}^{(L)}(y)}{p(t,x_0,y)} \leqq 2, \qquad t\in(\varepsilon,T],\ s\in[0,t-\varepsilon]..$$

Proof. The assertion (1) is an immediate consequence of Proposition 20. Note that

$$|q_{s,t}^{(L)}(y) - p(t,x_0,y)| \leqq Z_L(s,t;\delta)(L^{-1/(1-\delta)} + p(t,x_0,y))^{(1-\delta)/2}$$

. for any $y\in\mathbf{R}^N$, $t\in[\varepsilon,T]$ and $s\in[0,t-\varepsilon]$.

If $p(t,x_0,y)\geqq L^{-(1-\delta)}$, we have

$$|\frac{q_{s,t}^{(L)}(y)}{p(t,x_0,y)} - 1| \leqq Z_L(s,t;\delta)(L^{-1/(1-\delta)}p(t,x_0,y)^{-1}+1)^{(1-\delta)/2}p(t,x_0,y)^{-(1+\delta)/2}$$

$$\leqq Z_L(s,t;\delta)(L^{-1/(1-\delta)}L^{1-\delta}+1)^{(1-\delta)/2}L^{(1-\delta^2)/2} \leqq 2L^{(1-\delta^2)/2}Z_L(s,t;\delta).$$

This implies our second assertion. ∎

Proposition 22 *Let $T>0$, and $\delta\in(0,1)$. Let $B_L(s,t)\in\mathcal{F}$, $L\geqq 1$, be given by*

$$B_L(s,t) = \{\omega\in\Omega:\ L^{(1-\delta^2)/2}Z_L(s,t;\delta)\leqq 1/4\}, \qquad t > ands\in(0,t),$$

and $\varphi_{t,L}: E\to\{0,1\}$, $t\in(0,T]$, $L\geqq 1$, be given by

$$\varphi_{t,L} = 1_{\{y\in E; p(t,x_0,y)>L^{-(1-\delta)}\}}, \qquad t>0.$$

(1) *Let $a\in(1/(2N),1/2)$, $b\in(a-1/(2N),a)$, and $m\geqq 1$. Then there is a $C>0$ such that*

$$1_{B_L(s,t)}E[(\sup_{x\in E} p(s,x_0,x)^a|(Q_{s,t}^{(L)}(\varphi_{t,L}f))(x) - (P_{s,t}(\varphi_{t,L}f))(x)|)^2|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{C}{L}s^{-(N+2)\ell_0}(t-s)^{-(N+2)\ell_0}\int_E p(t,x_0,y)^{-1+2b}(1+|y|^2)^{-m}\varphi_{t,L}(y)f(y)^2dy \qquad a.s.$$

for $t \in (0,T]$, $s \in (0,t)$, $L \geqq 1$, *and any bounded measurable function* f *defined in* E.
(2) *Let* $a \in (0, 1/2)$, *and* $m \geqq 1$. *Then there is a* $C > 0$ *such that*

$$1_{B_L(s,t)} E[(\sup_{x\in E} p(s,x_0,x)^{1/2-\delta/4} |Q^{(L)}_{s,t}(\varphi_{t,L} f))(x) - (P_{s,t}(\varphi_{t,L} f))(x)|)^2 | \mathcal{F}^{(\infty)}_s]$$

$$\leqq \frac{C}{L^{1-\delta}} s^{-(N+2)\ell_0} (t-s)^{-(N+2)\ell_0} \int_E (1+|y|^2)^{-m} \varphi_{t,L}(y) f(y)^2 dy \qquad a.s.$$

for $t \in (0,T]$, $s \in (0,t)$, $L \geqq 1$, *and any bounded measurable function* f *defined in* E.
(3) *Let* $a \in (0,1/2)$ *and* $b \in (a - 1/(2N), a)$. *Then there is a* $C > 0$ *such that*

$$1_{B_L(s,t)} E[(\sup_{x\in E} p(s,x_0,x)^a |Q^{(L)}_{s,t}(\varphi_{t,L} p(t,x_0,\cdot)^{-b}))(x) - (P_{s,t}(\varphi_{t,L} p(t,x_0,\cdot)^{-b}))(x)|)^2 | \mathcal{F}^{(\infty)}_s]$$

$$\leqq \frac{C}{L^{\delta}} s^{-(N+2)\ell_0} (t-s)^{-(N+2)\ell_0} \qquad a.s.$$

for $t \in (0,T]$, $s \in (0,t)$, $L \geqq 1$.

Proof. Note that for $\alpha \in \mathbf{Z}^N_{\geqq 0}$

$$1_{B_L(s,t)} E[|\partial^\alpha_x (p(s,x_0,x)^a (Q^{(L)}_{s,t}(\varphi_{t,L} f))(x) - (P_{s,t}(\varphi_{t,L} f))(x)))|^2 \mathcal{F}^{(\infty)}_s]$$

$$\leqq \frac{1}{L} 1_{B_L(s,t)} \int_E \frac{|\partial^\alpha_x (p(s,x_0,x)^a p(t-s,x,y))|^2}{q^{(L)}_{s,t}(y)} \varphi_{t,L(y)} f(y)^2 \, dy$$

$$\leqq \frac{2}{L} 1_{B_L(s,t)} \int_E |\partial^\alpha_x (p(s,x_0,x)^a p(t-s,x,y))|^2 p(t,x_0,y)^{-1} \varphi_{t,L(y)} f(y)^2 \, dy$$

So we have by Proposition 12 there is a $C > 0$ such that

$$1_{B_L(s,t)} E[\int_{\mathbf{R}^N} dx \, |\partial^\alpha_x (p(s,x_0,x)^a (Q^{(L)}_{s,t}(\varphi_{t,L} f))(x) - (P_{s,t}(\varphi_{t,L} f))(x)))|^2 \mathcal{F}^{(\infty)}_s]$$

$$\leqq \frac{C}{L} s^{-(N+2)\ell_0} (t-s)^{-(N+2)\ell_0} \int_E p(t,x_0,y)^{-1+2b} (1+|y|^2)^{-m} \varphi_{t,L}(y) f(y)^2 dy.$$

This and Sobolev's inequality imply the assertion (1).

In the assertion (1), if $a = 1 - \delta/4$ and $b > 1/2 - \delta/2$, then we have

$$p(t,x_0,y)^{-1+2b} \varphi_{t,L}(y) \leqq L^{-\delta}.$$

This implies the assertion (2).

In the assertion (1), if $m = N+1$ and $f = p(t,x_0,\cdot)^{-b}$ then we have

$$\int_E p(t,x_0,y)^{-1+2b} (1+|y|^2)^{-m} \varphi_{t,L}(y) f(y)^2 dy \leqq L^{1-\delta} \int_{\mathbf{R}^N} (1+|y|^2)^{-(N+1)} dy$$

This implies the assertion (3). ∎

Similarly by using Proposition 12, we have the following.

Proposition 23 *Let* $a \in (1/(2N), 1/2)$ *and* $b \in (a - 1/(2N), a)$. *Then there is a* $C > 0$ *such that*

$$\sup_{x\in E} p(s,x_0,x)^a |(P_{s,t} f)(x)|$$

$$\leqq C s^{-(N+2)\ell_0/2} (t-s)^{-(N+3)\ell_0/2} \sup_{y\in E} p(t,x_0,y)^b |f(y)|$$

for $t \in (0,T]$, $s \in (0,t)$, *and any bounded measurable function* f *defined in* E.

6 Application to Bermuda type problem

Let us think of the situation in Section 4. Then we have the following.

Theorem 24 *Let $0 = T_0 < T_1 < \ldots < T_n < T$, $\delta \in (0, 1/2)$, and $f \in \mathcal{B}_r$, for some $r \geqq 0$. Then there are $C > 0$, $\Omega^L \in \mathcal{F}$, $L \geqq 1$, and measurable functions $d^{(L)}_{m,i} : E \times \Omega \to [0, \infty)$, $m = 1, \ldots, n-1$, $i = 1, 2$, $L \geqq 1$, such that*

$$\lim_{L\to\infty} L^p(1 - P(\Omega^L)) = 0, \qquad p \in (1, \infty),$$

$$\begin{aligned} &1_{\Omega^L}|(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{n-1},T_n} f)(x) - (\tilde{P}_{T_m,T_{m+1}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x)| \\ &\leqq d^{(L)}_{m,1}(x) + d^{(L)}_{m,2}(x), \qquad x \in E,\ m = 1, \ldots, n-1,\ L \geqq 1 \end{aligned}$$

and

$$E[\int_E d^{(L)}_{m,1}(x) p(T_m, x_0, x) dx] \leqq C L^{-(1-\delta)^2}$$

$$E[\int_E d^{(L)}_{m,2}(x)^2 p(T_m, x_0, x) dx] \leqq C L^{-(1-\delta)}$$

for any $L \geqq 1$, $m = 1, \ldots, n-1$.

Proof. Note that for $f, g \in \mathcal{B}_{r'}$

$$\begin{aligned} &|(\tilde{Q}^{(L)}_{s,t} f)(x) - (\tilde{Q}^{(L)}_{s,t} g)(x)| \\ &= |\phi_{s,t}(x, (Q^{(L)}_{s,t} f)(x)) - \phi_{s,t}(x, (Q^{(L)}_{s,t} g)(x))| \\ &\leqq \exp(\lambda(t-s))(Q^{(L)}_{s,t}(|f-g|))(x) \end{aligned}$$

So we see that

$$\begin{aligned} &|(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{k-1},T_k} f)(x) - (\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{k-1},T_k} g)(x)| \\ &\leqq \exp(\lambda(T_k - T_m))(Q_{T_m,T_{m+1}} \cdots Q^{(L)}_{T_{k-1}.T_k}(|f-g|))(x) \end{aligned}$$

Similary we have

$$|(\tilde{Q}^{(L)}_{s,t} f)(x) - (\tilde{P}_{s,t} g)(x)| \leqq \exp(\lambda(t-s))|(Q^{(L)}_{s,t} f)(x) - (P_{s,t} g)(x)|$$

Let us take a_k, $k = 0, 1, \ldots, n$ such taht $1/2 > a_0 > a_1 > \ldots > a_n > 1/2 - \delta$. Also, let

$$c_m(x) = (\tilde{P}_{T_m,T_{m+1}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x).$$

Note that

$$\begin{aligned} &|(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{n-1},T_n} f)(x) - (\tilde{P}_{T_m,T_{m+1}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x)| \\ &\leqq \sum_{k=1}^{n-m} |(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{m+k-1},T_{m+k}} \tilde{P}_{T_{m+k},T_{m+k+1}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x) \\ &\quad -(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{m+k-2},T_{m+k-1}} \tilde{P}_{T_{m+k-1},T_{m+k}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x)| \end{aligned}$$

$$\leqq \exp(\lambda T)\sum_{k=1}^{n-m}(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}(|Q^{(L)}_{T_{m+k-1},T_{m+k}}c_{m+k}-P_{T_{m+k-1},T_{m+k}}c_{m+k}|))(x)$$

Let

$$R_k = 1_{B_L(T_{k-1},T_k)}\sup_{x\in E} p(T_{k-1},x_0,x)^{a_{k-1}}(|Q^{(L)}_{T_{k-1},T_k}(\varphi_{T_k,L}c_k)-P_{T_{k-1},T_k}(\varphi_{T_k,L}c_k)|)(x),$$

$$Z_k$$

$$= 1_{B_L(T_{k-1},T_k)}\sup_{x\in E} p(T_{k-1},x_0,x)^{a_{k-1}}(|Q^{(L)}_{T_{k-1},T_k}(\varphi_{T_k,L}p(T_k,x_0,\cdot)^{-a_k})-P_{T_{k-1},T_k}(\varphi_{T_k,L}p(T_k,x_0,\cdot)^{-a_k})|)(x),$$

and

$$D_k = \sup_{x\in E} p(T_{k-1},x_0,x)^{a_{k-1}}(P_{T_{k-1},T_k}(\varphi_{T_k,L}p(T_k,x_0,\cdot)^{-a_k}))(x) < \infty, \qquad k=1,\dots,n.$$

Then R_k and Z_k are $\mathcal{F}^{(\infty)}_{T_k}$-measurable for $k=1,\dots,n$, and by Proposition 22 we see that there is a $C>0$ such that

$$E[R_k^2|\mathcal{F}^{(\infty)}_{T_{k-1}}] \leqq CL^{-1}, \qquad E[Z_k^2|\mathcal{F}^{(\infty)}_{T_{k-1}}] \leqq CL^{-(1-\delta)}$$

for any $L \geqq 1$, and $k=1,\dots,n$. So inductively we have

$$E[R_k^2(\prod_{i=\ell+1}^{k}(Z_i+D_i)^2)|\mathcal{F}^{(\infty)}_{T_\ell}] \leqq 2^{k-\ell}C^{k+1-\ell}L^{-1}\prod_{i=\ell+1}^{k}(D_i^2+CL^{-(1-\delta)})$$

for any $L\geqq 1$, and $1\leqq \ell\leqq k\leqq n$. Let $\Omega^L=\bigcap_{k=1}^n B_L(T_{k-1},T_k)$. Then we have

$$1_{\Omega^L}Q^{(L)}_{T_{k-1},T_k}(p(T_k,x_0,\cdot)^{-a_k})(x)$$

$$= 1_{\Omega^L}Q^{(L)}_{T_{k-1},T_k}(\varphi_{T_k,L}p(T_k,x_0,\cdot)^{-a_k})(x) + Q^{(L)}_{T_{k-1},T_k}((1-\varphi_{T_k,L})p(T_k,x_0,\cdot)^{-a_k})(x)$$
$$\leqq 1_{\Omega^L}(Z_k+D_k)p(T_{k-1},x_0,x)^{-a_{k-1}} + 1_{\Omega^L}Q^{(L)}_{T_{k-1},T_k}((1-\varphi_{T_k,L})p(T_k,x_0,\cdot)^{-a_k})(x)$$

Therefore we have

$$1_{\Omega^L}(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}(|Q^{(L)}_{T_{m+k-1},T_{m+k}}c_{m+k}-P_{T_{m+k-1},T_{m+k}}c_{m+k}|)(x)$$
$$\leqq 1_{\Omega^L}R_{m+k}(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}p(T_{m+k-1},x_0,\cdot)^{-a_{m+k-1}})(x)$$
$$+1_{\Omega^L}(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}(Q^{(L)}_{T_{m+k-1},T_{m+k}}((1-\varphi_{T_{m+k},L})|c_{m+k}|)$$
$$+P_{T_{m+k-1},T_{m+k}}((1-\varphi_{T_{m+k},L})|c_{m+k}|)))(x)$$
$$\leqq \tilde{d}_{m,2}(x)+\tilde{d}_{m.1}(x),$$

where

$$\tilde{d}^{(L)}_{m,2}(x) = R_{m+k}(\prod_{i=1}^{k}(Z_{m+i}+D_{m+i}))p(T_m,x_0,x)^{-a_m})$$

and

$$\tilde{d}^{(L)}_{m,1}(x)$$

$$= \sum_{\ell=1}^{k} R_{m+k}(\prod_{i=m+\ell+1}^{m+k}(Z_i+D_i))(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+\ell-2},T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}}))(x))$$

$$+(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}(Q^{(L)}_{T_{m+k-1},T_{m+k}}((1-\varphi_{T_{m+k},L})|c_{m+k}|)$$

$$+P_{T_{m+k-1},T_{m+k}}((1-\varphi_{T_{m+k},L})|c_{m+k}|)))(x)$$

Note that

$$E[R_{m+k}(\prod_{i=m+\ell+1}^{m+k}(Z_i+D_i))(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+\ell-2},T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}})(x))]$$

$$= E[E[R_{m+k}(\prod_{i=m+\ell+1}^{m+k}(Z_i+D_i))|\mathcal{F}^{(\infty)}_{T_m+\ell}]$$

$$\times(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+\ell-2},T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}})(x))]$$

$$\leqq (2^k C^{k+1}L^{-1}\prod_{i=m+\ell+1}^{m+k}(D_i^2+CL^{-(1-\delta)}))^{1/2}$$

$$\times E[Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+\ell-2},T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}})(x))]$$

$$= (2^k C^{k+1}L^{-1}\prod_{i=m+\ell+1}^{m+k}(D_i^2+CL^{-(1-\delta)}))^{1/2}P_{T_m,T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}})(x).$$

Note that for $a \geqq 0$

$$\int_E P_{T_m,T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a})(x)p(T_m,x_0,x)dx$$

$$= \int_E 1_{\{p(T_{m+\ell-1},x_0,x)\leqq L^{-(1-\delta)}\}}p(T_{m+\ell-1},x_0,x)^{1+a}dx$$

$$\leqq L^{-(1-\delta)^2}\int_E p(T_{m+\ell-1},x_0,x)^{\delta+a}dx.$$

Then we have our assertion. ∎

7 re-simulation

We think of application to pricing Bermuda derivatives.

Let $r \geqq 1$ and let $g : [0,T]\times\mathbf{R}^N \to \mathbf{R}$ be a continuous function such that

$$\sup_{x\in\mathbf{R}^N,\, t\in[0,T]}(1+|x|^2)^{-r/2}|g(t,x)| < \infty.$$

Let $\phi_{s,t}(x,y) = g(s,x)\vee y$, $0 \leqq s < t \leqq T$, $x\in\mathbf{R}^N$ and $y\in\mathbf{R}$. Let $0 = T_0 < T_1 < \ldots < T_n < T$, and let $c_m : E \to \mathbf{R}$, $m = 0,1,\ldots,n$, be given by

$$c_m(x) = (\tilde{P}_{T_m,T_{m+1}}\cdots\tilde{P}_{T_{n-1},T_n}g(T_n,\cdot))(x),\ m \leqq n-1,\ \text{and}\ c_n(x) = g(T_n.x).$$

Now let $\tilde{c}_m : E \to \mathbf{R}$, $m = 1, \dots, n-1$, be given and let $\tilde{c}_n = g(T_n, \cdot)$. We regard $\tilde{c}_m$ as estimators of c_m, $m = 1, \dots, n$.

Let us think of the SDE in Introduction. Let $\tau : W_0 \to \{T_1, \dots, T_n\}$ and $\tilde{\tau} : W_0 \to \{T_1, \dots, T_n\}$ be stopping times given by

$$\tau = \min\{T_k;\ c_k(X(T_k, x_0)) \leqq g(T_k, X(T_k, x_0),\ k = 1, \dots, n\}$$

and

$$\tilde{\tau} = \min\{T_k;\ \tilde{c}_k(X(T_k, x_0)) \leqq g(T_k, X(T_k, x_0),\ k = 1, \dots, n\}$$

Let $\bar{c}_m$, $m = 0, \dots, n$, be given by inductively, $\bar{c}_n = g(T_n, \cdot)$, and

$$\bar{c}_{m-1} = P_{T_{m-1},T_m}(g(T_m, \cdot)1_{\{\tilde{c}_m \leqq g(T_m, \cdot)\}} + \bar{c}_m 1_{\{\tilde{c}_m > g(T_m, \cdot)\}}), \quad m = n, n-1, \dots, 1.$$

Then we have the following.

Proposition 25 (1) *For* $m = 0, 1, \dots, n-1$,

$$E^\mu[g(\tau, X(\tau, x_0)|\mathcal{B}_{T_m}]1_{\{\tau \geqq T_{m+1}\}} = c_m(X(T_m, x_0))1_{\{\tau \geqq T_{m+1}\}} \ a.s.$$

and

$$E^\mu[g(\tilde{\tau}, X(\tilde{\tau}, x_0)|\mathcal{B}_{T_m}]1_{\{\tilde{\tau} \geqq T_{m+1}\}} = \bar{c}_m(X(T_m, x_0))1_{\{\tilde{\tau} \geqq T_{m+1}\}} \ a.s.$$

Here $\mathcal{B}_t = \sigma\{B^i(s);\ s \leqq t,\ i = 1, \dots, d\}$.
(2) *For* $m = 0, 1, \dots, n-1$, *and* $x \in E$,

$$0 \leqq c_m(x) - \bar{c}_m(x) \leqq P_{T_m,T_{m+1}}(|c_{m+1} - \tilde{c}_{m+1}|)(x) + P_{T_m,T_{m+1}}(1_{\{\tilde{c}_{m+1} > g_{m+1}\}}(c_{m+1} - \bar{c}_{m+1}))(x).$$

In particular,

$$0 \leqq c_m(x) - \bar{c}_m(x) \leqq \sum_{k=m+1}^{n} P_{T_m,T_k}(|c_k - \tilde{c}_k|)(x), \qquad m = 0, 1, \dots, n.$$

Proof. Since we have

$$\begin{aligned} & E^\mu[g(\tilde{\tau}, X(\tilde{\tau}, x_0)|\mathcal{B}_{T_{m-1}}]1_{\{\tilde{\tau} \geqq T_m\}} \\ &= E^\mu[E^\mu[g(\tilde{\tau}, X(\tilde{\tau}, x_0)1_{\{\tilde{\tau} \geqq T_{m+1}\}}|\mathcal{B}_{T_m}] + g(T_m, X(T_m, x_0))1_{\{\tilde{\tau} = T_m\}}|\mathcal{B}_{T_{m-1}}], \end{aligned}$$

we can easily obtain the assertion (1) by induction.

Note that

$$\begin{aligned} & c_m - \bar{c}_m \\ &= P_{T_m,T_{m+1}}(1_{\{\tilde{c}_{m+1} \leqq g(T_{m+1}, \cdot)\}}((g(T_{m+1}, \cdot) \vee c_{m+1}) - g(T_{m+1}, \cdot))) \\ &\quad + P_{T_m,T_{m+1}}(1_{\{\tilde{c}_{m+1} > g(T_{m+1}, \cdot)\}}((g(T_{m+1}, \cdot) \vee c_{m+1}) - \bar{c}_{m+1})) \\ &= P_{T_m,T_{m+1}}(1_{\{\tilde{c}_{m+1} \leqq g(T_{m+1}, \cdot)\}}((g(T_{m+1}, \cdot) \vee c_{m+1}) - (g(T_{m+1}, \cdot) \vee \tilde{c}_{m+1}))) \\ & + P_{T_m,T_{m+1}}(1_{\{\tilde{c}_{m+1} > g(T_{m+1}, \cdot)\}}((g(T_{m+1}, \cdot) - c_{m+1}) \vee 0) - ((g(T_{m+1}, \cdot) - \tilde{c}_{m+1}) \vee 0) + c_{m+1} - \bar{c}_{m+1})) \\ &\leqq P_{T_m,T_{m+1}}(|c_{m+1} - \tilde{c}_{m+1}|) + P_{T_m,T_{m+1}}(1_{\{\tilde{c}_{m+1} > g(T_{m+1}, \cdot)\}}(c_{m+1}) - \bar{c}_{m+1})) \end{aligned}$$

This implies the first inequlity of the assertion (2). The second inequality follows from this by induction. ∎

Proposition 26 *

$$c_0(x_0) - \bar{c}_0(x_0)$$
$$\leqq \sum_{k=1}^{n} \int_E (|\tilde{c}_k - \bar{c}_k| + |c_k - \bar{c}_k|)(x) 1_{\{|\tilde{c}_k - c_k| + |c_k - \bar{c}_k| \geqq \varepsilon\}}(x) + \varepsilon 1_{\{|g(T_k,\cdot) - c_k| < \varepsilon\}} p(T_k, x_0, x) dx$$

for any $\varepsilon > 0$.

Proof. Note that

$$c_0(x_0) - \bar{c}_0(x_0) = E^\mu[g(\tau, X(\tau, x_0)) - g(\tilde{\tau}, X(\tilde{\tau}, x_0))]$$
$$= E^\mu[g(\tau, X(\tau, x_0)) - g(\tilde{\tau}, X(\tilde{\tau}, x_0)), \tau > \tilde{\tau}] + E^\mu[g(\tau, X(\tau, x_0)) - g(\tilde{\tau}, X(\tilde{\tau}, x_0)), \tau < \tilde{\tau}]$$
$$= E^\mu[E^\mu[g(\tau, X(\tau, x_0))|\mathcal{B}_{\tilde{\tau}}] - g(\tilde{\tau}, X(\tilde{\tau}, x_0), \tau > \tilde{\tau}]$$
$$+ E^\mu[g(\tau, X(\tau, x_0) - E^\mu[g(\tilde{\tau}, X(\tilde{\tau}, x_0))|\mathcal{B}_\tau], \tau < \tilde{\tau}]$$
$$= \sum_{k=1}^{n-1} (E^\mu[c_k(X(T_k, x_0)) - g(T_k, X(T_k, x_0)), \tau > T_k, \tilde{\tau} = T_k]$$
$$+ E^\mu[g(T_k, X(k, x_0)) - \bar{c}_k(X(T_k, x_0)), \tau = T_k, T_k < \tilde{\tau}])$$
$$\leqq \sum_{k=1}^{n-1} (E^\mu[(c_k(X(T_k, x_0)) - g(T_k, X(T_k, x_0)) 1_{\{\tilde{c}_k \leqq g(T_k,\cdot) < c_k\}}(X(T_k, x_0))]$$
$$+ E^\mu[((g(T_k, X(k, x_0)) - \bar{c}_k(X(T_k, x_0)))) \vee 0) 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}}(X(T_k, x_0))]).$$

For any $\varepsilon > 0$, we see that

$$(c_k - g(T_k, \cdot)) 1_{\{\tilde{c}_k \leqq g(T_k,\cdot) < c_k\}}$$
$$\leqq \varepsilon 1_{\{g(T_k,\cdot) < c_k \leqq g(T_k,\cdot) + \varepsilon\}} + (c_k - g(T_k, \cdot)) 1_{\{\tilde{c}_k \leqq g(T_k,\cdot) < c_k\}} 1_{\{g_k + \varepsilon < c_k\}}$$
$$\leqq \varepsilon 1_{\{g(T_k,\cdot) < (T_k,\cdot) + \varepsilon\}} + (c_k - \tilde{c}_k) 1_{\{c_k - \tilde{c}_k > \varepsilon\}} 1_{\{\tilde{c}_k \leqq g(T_k,\cdot) < c_k\}},$$

and

$$((g(T_k, \cdot) - \bar{c}_k) \vee 0) 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}}$$
$$\leqq ((g(T_k, \cdot) - \bar{c}_k) \vee 0) 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}} 1_{\{|\tilde{c}_k - c_k| + |c_k - \bar{c}_k| \geqq \varepsilon\}} + ((g(T_k, \cdot) - \bar{c}_k) \vee 0) 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}} 1_{\{|\tilde{c}_k - c_k| + |c_k - \bar{c}_k| < \varepsilon\}}$$
$$\leqq (|\tilde{c}_k - c_k| + |c_k - \bar{c}_k|) 1_{\{|\tilde{c}_k - c_k| + |c_k - \bar{c}_k| \geqq \varepsilon\}} 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}} + \varepsilon 1_{\{c_k \leqq g(T_k,\cdot) < c_k + \varepsilon\}}$$

So we have our assertion. ∎

Now we have the following.

Lemma 27 *Let $d_{m,i} : E \to [0, \infty)$. $m = 1, \ldots, n$, $i = 1, 2$, be measurable functions. Assume that $|\tilde{c}_m - c_m| \leqq d_{m,1} + d_{m,2}$, $m = 1, \ldots, n$. Then we have the following.*

$$c_0(x_0) - \bar{c}_0(x_0)$$
$$\leqq n \sum_{k=1}^{n} \int_E d_{k,1}(x) p(T_k, x_0, x) dx + n (\sum_{k=1}^{n} (\int_E d_{k,2}(x)^2 p(T_k, x_0, x) dx)^{1/2})$$

$$\times(\varepsilon^{-1/2}(\sum_{k=1}^{n}\int_E d_{k,1}(x)p(T_k,x_0,x)dx)+\varepsilon^{-1}(\sum_{k=1}^{n}(\int_E d_{k,2}(x)^2p(T_k,x_0,x)dx)^{1/2}))$$

$$+2\varepsilon\sum_{k=1}^{n}\int_E 1_{\{|g(T_k,\cdot)-c_k|<2\varepsilon\}}p(T_k,x_0,x)dx$$

for any $\varepsilon>0$.

Proof. Let

$$\tilde{d}_{m,i}(x)=\sum_{k=m}^{n}(P_{T_m,T_k}d_{k,i})(x), m=1,\ldots,n.$$

Then by Proposition 25, we have

$$|\tilde{c}_m(x)-c_m(x)|+|\bar{c}_m(x)-c_m(x)|\leqq\tilde{d}_{m,1}(x)+\tilde{d}_{m,2}(x).$$

Note that

$$\int_E(\tilde{d}_{m,1}(x)+\tilde{d}_{m,2}(x))1_{\{\tilde{d}_{m,1}(x)+\tilde{d}_{m,2}(x)\geqq 2\varepsilon\}}(x)p(T_m,x_0,x)dx$$

$$\leqq\int_E\tilde{d}_{m,1}(x)p(T_m,x_0,x)dx+(\int_E\tilde{d}_{m,2}(x)^2p(T_m,x_0,x)dx)^{1/2}((\int_E 1_{\{\tilde{d}_{m,1}(x)\geqq\varepsilon\}}(x)p(T_m,x_0,x)dx)^{1/2}$$

$$+(\int_E 1_{\{\tilde{d}_{m,2}(x)\geqq\varepsilon\}}(x)p(T_m,x_0,x)dx)^{1/2})$$

$$\leqq\int_E\tilde{d}_{m,1}(x)p(T_m,x_0,x)dx+(\int_E\tilde{d}_{m,2}(x)^2p(T_m,x_0,x)dx)^{1/2}(\varepsilon^{-1/2}(\int_E\tilde{d}_{m,1}(x)p(T_m,x_0,x)dx)^{1/2}$$

$$+\varepsilon^{-1}(\int_E\tilde{d}_{m,2}(x)^2p(T_m,x_0,x)dx)^{1/2})$$

Also, note that

$$\int_E\tilde{d}_{m,1}(x)p(T_m,x_0,x)dx\leqq\sum_{k=m}^{n}\int_E d_{k,1}(x)p(T_k,x_0,x)dx,$$

and

$$(\int_E\tilde{d}_{m,2}(x)^2p(T_m,x_0,x)dx)^{1/2}\leqq\sum_{k=m}^{n}(\int_E d_{k,2}(x)^2p(T_k,x_0,x)dx)^{1/2}.$$

This and Proposition 26 imply our assertion. ∎

Now we apply this Lemma and the results in the previous section to a Bermuda derivative.

Let $\phi_{s,t}(x,y)=g(s,x)\vee y$, $0\leqq s<t\leqq T$, $x\in\mathbf{R}^N$ and $y\in\mathbf{R}$. Let $\tilde{c}_m:E\to\mathbf{R}$, $m=1,\ldots,n-1$, be given by

$$\tilde{c}_m(x)=(\tilde{Q}^{(L)}_{T_m,T_{m+1}}\cdots\tilde{Q}^{(L)}_{T_{n-1},T_n}g(T_n,\cdot))(x).$$

Then by Theorem 24, we see that for any $\delta\in(0,1/2)$, there are $\Omega'_L\in\mathcal{F}$, $L\geqq 1$, $C>0$ and measurable functions $d^{(L)}_{m,i}:E\times\Omega\to[0,\infty)$, $m=1,\ldots,n-1$, $i=1,2$, $L\geqq 1$, such that

$$\lim_{L\to\infty}P(\Omega'_L)=1,$$

$$|\tilde{c}_m(x) - c_m(x)| \leqq d_{m,1}(x) + d_{m,2}(x), \qquad x \in E,\ \omega \in \Omega'_L,\ m = 1, \ldots, n-1,\ L \geqq 1$$

and

$$E[\int_E d_{m,1}(x) p(T_m, x_0, x) dx] \leqq C L^{-(1-\delta)^2} \qquad m = 1, \ldots, n-1,\ L \geqq 1,$$

and

$$E[\int_E d_{m,2}(x)^2 p(T_m, x_0, x) dx] \leqq C L^{-(1-\delta)^2} \qquad m = 1, \ldots, n-1,\ L \geqq 1.$$

Let

$$\Omega''_L = \{\omega \in \Omega;\ \int_E d_{m,1}(x) p(T_m, x_0, x) dx \geqq L^{-(1-\delta)^3} \text{ or } \int_E d_{m,2}(x)^2 p(T_m, x_0, x) dx \geqq L^{-(1-\delta)^3}\}.$$

Then we see that

$$P(\Omega \setminus \Omega''_L) \leqq 2 C L^{-(1-\delta)^2 \delta}, \qquad L \geqq 1.$$

Let $\Omega_L = \Omega'_L \cap \Omega''_L$, $L \geqq 1$. Then we see that $P(\Omega_L) \to 1$, $l \to \infty$. So if we use these $\tilde{c}_m(x)$, $m = 1, \ldots, n-1$, as estimators and use the re-simulation method, we have

$$\begin{aligned} &c_0(x_0) - \bar{c}_0(x_0) \\ \leqq\ & n^2 L^{-(1-\delta)^3} + n^3 L^{-(1-\delta)^3/2} (\varepsilon^{-1/2} L^{-(1-\delta)^3/2} + \varepsilon^{-1} L^{-(1-\delta)^3/2}) \\ &+ \varepsilon \sum_{k=1}^{n} \int_E 1_{\{|g(T_k, \cdot) - c_k| < \varepsilon\}} p(T_k, x_0, x) dx \end{aligned}$$

for any $\varepsilon > 0$, $\omega \in \Omega_L$, and $L \geqq 1$. Suppose that

$$\sum_{k=1}^{n-1} \int_E 1_{\{|g(T_k, \cdot) - c_k| < \varepsilon\}} p(T_k, x_0, x) dx = O(\varepsilon^{\gamma}), \quad \varepsilon \downarrow 0,$$

for some $\gamma \in (0, 1]$. Then letting $\varepsilon = L^{-(1-\delta)^3/(2+\gamma)}$, we see that $c_0(x_0) - \bar{c}_0(x_0) = O(L^{-(1-\delta)^3(1+\gamma)/(2+\gamma)})$ as $L \to \infty$.

Since δ is arbitrary, this proves Theorem 2.

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