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Analysis of the fictitious domain method with H^1-penalty for parabolic problem

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ANALYSIS OF THE FICTITIOUS DOMAIN METHOD WITH H^1-PENALTY FOR PARABOLIC PROBLEM

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ABSTRACT. We consider the fictitious domain method with H^1-penalty for a parabolic problem. First, the sharp error estimate for the H^1-penalty problem approximating the original problem is derived. Then, we present some regularity analysis and prior estimate to the H^1-penalty problem. The finite element approximation is investigated for discrete problems of two types, "continuous-time" and "single-step backward". To perform the numerical computation, we provide an approximation scheme for the "single-step backward" discrete problem as well as its error estimate. The theoretical result is verified using our numerical experiment.

1. INTRODUCTION

The purpose of this paper is to establish a mathematical study of the fictitious domain method for parabolic problems. The fictitious domain method is well known to be based on a reformulation of the original problem in a larger spatial domain, called the fictitious domain, with a simple shape. Then, the fictitious domain can be discretized by a uniform mesh, independent of the original boundary. The advantage of this approach is that we can avoid the time-consuming construction of a boundary-fitted mesh. Furthermore, this approach will be useful to solve time-dependent moving-boundary problems. Several means exist to introduce a reformulation in the fictitious domain ([3]). We restrict ourselves however to the H^1-penalty method described below because of its wide applicability. In a previous report ([25]), we developed a mathematical theory for the H^1-penalty fictitious domain method for elliptic problems that can be applied easily to parabolic and moving-boundary problems while maintaining the sharpness of error estimates. Herein, we study the H^1-penalty fictitious domain method and its fully discrete finite element approximations for parabolic problems in a fixed spatial domain. Analysis for the H^1-penalty method is also given for moving-boundary problems.

To be more specific, presuming that Ω is a bounded domain in $\mathbb{R}^2$ with the smooth boundary Γ and $T > 0$ is a fixed constant, we consider a parabolic problem

$$\begin{cases} u_t - \Delta u = f \text{ in } & Q_T, \\ u(x,t) = 0 \text{ on } & \Sigma_T, \\ u(\cdot, 0) = u_0 \text{ on } & \Omega, \end{cases} \tag{1.1}$$

where

$$Q_T = \Omega \times [0,T],\ \Sigma_T = \Gamma \times (0,T]. \tag{1.2}$$

To state the weak form of (1.1), we introduce

$$H_0^{1,0}(Q_T) = L^2(0,T; H_0^1(\Omega)),\ H^{-1,0}(Q_T) = L^2(0,T; H^{-1}(\Omega)), \tag{1.3}$$

where $H_0^1(\Omega)$ is the standard Sobolev space of order one in $L^2(\Omega)$ with boundary value zero in Γ and where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. Moreover, $\langle \cdot, \cdot \rangle_{Q_T}$ denotes the duality pairing between $H_0^{1,0}(Q_T)$ and $H^{-1,0}(Q_T)$, and $(\cdot, \cdot)_{Q_T}$ is the scalar product of $L^2(Q_T)$. The weak form of (1.1), which we will call the original problem, reads as

$$\begin{cases} \text{Find } u \in H_0^{1,0}(Q_T) \text{ with } u_t \in H^{-1,0}(Q_T)\ s.t. \\ \langle u_t, v \rangle_{Q_T} + (\nabla u, \nabla v)_{Q_T} = \langle f, v \rangle_{Q_T},\ \forall v \in H_0^{1,0}(Q_T) \\ u(\cdot, 0) = u_0. \end{cases} \tag{1.4}$$

Hereinafter, we assume that

$$f \in L^2(Q_T),\ u_0 \in L^2(\Omega). \tag{1.5}$$

We have $u \in C([0,T]; L^2(\Omega))$ for $u \in H_0^{-1,0}(Q_T)$ and $u_t \in H^{-1,0}(Q_T)$ ([10]). Therefore, the condition $u(\cdot, 0) = u_0 \in L^2(\Omega)$ is meaningful.

The fictitious domain method is used to find a larger domain (the *fictitious domain*) of a simple shape, which we designate as D satisfying $D \supset \Omega$. $D_T = D \times [0,T]$, $Q_T' = D_T \backslash \overline{Q_T}$. Then we introduce the H^1-penalty problem (Q_ϵ) to approximate the original problem (1.4), where ϵ is the penalty parameter. Setting $0 < \epsilon \ll 1$ and $M_\epsilon = \chi_{Q_T} + \frac{1}{\epsilon}\chi_{Q_T'}$, where χ_{Q_T} is the indicator function of Q_T, (Q_ϵ) reads as

$$\begin{cases} \text{Find } u_\epsilon \in H_0^{1,0}(D_T) \text{ with } M_\epsilon u_{\epsilon t} \in H^{-1,0}(D_T)\ s.t. \\ \langle M_\epsilon u_{\epsilon t}, v \rangle_{D_T} + (\nabla u_\epsilon, \nabla v)_{Q_T} + \dfrac{1}{\epsilon}(\nabla u_\epsilon, \nabla v)_{Q_T'} \\ \quad = (\tilde{f}, v)_{D_T},\ \forall v \in H_0^{1,0}(Q_T), \\ u_\epsilon(\cdot, 0) = \tilde{u}_0, \end{cases} \tag{1.6}$$

where $\tilde{f}$ and $\tilde{u}_0$ respectively denote the zero extension of f onto D_T and u_0 onto D (In fact, $\tilde{f}$ and $\tilde{u}_0$ can be any extension function satisfying $\|\tilde{u}_0\|_{0,\Omega_1} \leq$

$C\|u_0\|_{0,\Omega}\epsilon$ and $\|\tilde{f}\|_{0,Q'_T} \le C\|f\|_{0,Q_T}$, see Remark 1). The first main result is the error estimate of the form (Theorem 2.1)

$$\|u_\epsilon|_{Q_T} - u\|_{H^{1,0}(Q_T)} + \|u_{\epsilon t}|_{Q_T} - u_t\|_{H^{-1,0}(Q_T)} \le C(\|f\|_{0,Q_T} + \|u_0\|_{0,\Omega})\epsilon, \tag{1.7}$$

which maintains the sharpness of the estimate for the elliptic problem (see [14, 22, 25]).

For the finite element approximation of H^1-penalty problem, no analysis is given in an earlier paper [15]. The H^1-penalty problem, which is equivalent to an interface problem, can also be viewed as a parabolic problem with a discontinuous coefficient. However, our coefficients of both contain the penalty parameter ϵ ($\epsilon \ll 1$). For this reason, although the finite element approximation for parabolic interface problem has been examined in many articles (for example, [1]), as well as the problem with discontinuous coefficient (see [2] for the elliptic case), we will examine explicit error estimates on our own.

Finite element analysis for the H^1-penalty elliptic problem has been well developed [14, 23, 25]. In one of those papers [25], the error estimate of both H^1 and L^2 norms was obtained. Therefore, here, we will apply those finite element results of elliptic problem in [25] to our H^1-penalty parabolic problem.

To implement the finite element method to solve the H^1-penalty problem, we take some uniform triangulation $\mathcal{T}_h$ to domain D, where h is the maximum diameter of the triangles of $\mathcal{T}_h$. $V_h(D)$ is the subspace of all piecewise linear continuous functions subordinate to $\mathcal{T}_h$. We set $\Omega_1 = D\backslash\overline{\Omega}$, $\Gamma = \partial\Omega$. Then, we introduce "continuous-time" discrete problems ($CQ_{\epsilon,h}$), which reads as

$$\begin{cases} \text{Find } u_{\epsilon,h} \in C^1([0,T], V_h(D)),\ \ s.t. \\ (u_{\epsilon,ht}, v_h)_\Omega + \dfrac{1}{\epsilon}(u_{\epsilon,ht}, v_h)_{\Omega_1} + (\nabla u_{\epsilon,h}, \nabla v_h)_\Omega \\ \quad + \dfrac{1}{\epsilon}(\nabla u_{\epsilon,h}, \nabla v_h)_{\Omega_1} = (\tilde{f}, v_h)_D,\ \forall v_h \in V_h,\ t \in (0,T], \\ u_{\epsilon,h}(0) = \tilde{u}_{0h}, \end{cases} \tag{1.8}$$

where $\tilde{u}_{0h} \in V_h$ is an approximation to $\tilde{u}_0$. Several analytical methods exist for finite element approximation for a parabolic problem (e.g., [8, 9, 13, 18, 20]). Then we mainly apply the analysis method from [13].

After giving some regularity results and *apriori* estimates for the H^1-penalty problem, we start to derive the finite element error analysis, and obtain the following results. For $f \equiv 0,\ u_0 \in H_0^1(\Omega) \cap H^2(\Omega),\ t \in (0,T]$, we have (see Theorem 4.8)

$$\begin{aligned} &\|e(t)\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|e(t)\|_{0,\Omega_1} \\ \le &C\left((\sqrt{\epsilon} + \sqrt{h})^2\|u_0\|_{2,\Omega} + \|\tilde{u}_0 - \tilde{u}_{0h}\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|\tilde{u}_0 - \tilde{u}_{0h}\|_{0,\Omega_1}\right). \end{aligned}$$

If $f \in L^2(Q_T),\ u_0 \equiv 0,\ t \in (0,T]$, then (see Theorem 4.9)

$$\begin{aligned}&\|e(t)\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|e(t)\|_{0,\Omega_1}\\ \leq& C(\sqrt{\epsilon}+\sqrt{h})^2 \ln\frac{1}{\sqrt{\epsilon}+\sqrt{h}} \max_{[0,t]} \|f(s)\|_{0,\Omega}.\end{aligned}$$

For the H^1-norm error with $f \in C^1([0,T],L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$, we show that (see Theorem 4.10)

$$\begin{aligned}&\|e(t)\|_{1,\Omega}^2 + \frac{1}{\epsilon}\|e(t)\|_{1,\Omega_1}^2 \leq \frac{C}{t}(\sqrt{\epsilon}+\sqrt{h})(\|u_0\|_{2,\Omega} + \|f\|_{C^1([0,T],L^2(\Omega))})\\ &+ C(\sqrt{\epsilon}+\sqrt{h}) + C(\|\tilde{u}_0 - \tilde{u}_{0h}\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|\tilde{u}_0 - \tilde{u}_{0h}\|_{0,\Omega_1}).\end{aligned}$$

Then, setting time-step $k = \frac{T}{N}$, N is a positive integer, we consider the "single-step backward" discrete problem $(SQ_{\epsilon,h})$, which reads as

$$\begin{cases}\text{Find } U^{n+1} \in V_h,\ n = 0,1,\ldots,N-1,\ s.t.\\ (\partial_t U^{n+1}, v_h)_\Omega + \frac{1}{\epsilon}(\partial_t U^{n+1}, v_h)_{\Omega_1} + (\nabla U^{n+1}, \nabla v_h)_\Omega\\ \quad + \frac{1}{\epsilon}(\nabla U^{n+1}, \nabla v_h)_{\Omega_1} = (\bar{f}^{n+1}, v_h)_D,\ \forall v_h \in V_h\\ U^0 = \tilde{u}_{0h},\end{cases} \tag{1.9}$$

where

$$\partial_t U^{n+1} = \frac{U^{n+1}-U^n}{k} \text{ and } \bar{f}^{n+1} = \frac{1}{k}\int_{t_n}^{t_{n+1}} \tilde{f}(s)ds.$$

Letting $\zeta^n \equiv u_{\epsilon,h}(t_n) - U^n$, then for $f \equiv 0$, we prove that (see Theorem 4.14)

$$\|\zeta^n\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|\zeta^n\|_{0,\Omega_1} \leq C\frac{k}{t_n}(\|\tilde{u}_{0h}\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|\tilde{u}_{0h}\|_{0,\Omega_1}).$$

For $\tilde{u}_{0h} \equiv 0,\ f \in C([0,T];L^2(\Omega))$, we have obtained (see Theorem 4.15)

$$\|u_{\epsilon,h}(t_n) - U^n\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|u_{\epsilon,h}(t_n) - U^n\|_{0,\Omega_1} \leq Ck\ln\frac{1}{k}\max_{s\in[0,T]}\|f(s)\|_0.$$

For the H^1-norm error with $f \in C([0,T];L^2(\Omega))$, we have (see Remark 8)

$$\|\zeta^n\|_{1,\Omega}^2 + \frac{1}{\epsilon}\|\zeta^n\|_{1,\Omega_1}^2 \leq C(\|\zeta^n\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|\zeta^n\|_{0,\Omega_1})$$

Because (1.9) contains an integral on curved domain Ω and Ω_1, which comes out to be a problem when doing computations, we propose an approximation scheme here. Setting $\hat{\Omega}$ as a polygon approximating to Ω, and $\hat{\Omega}_1 = D\backslash\overline{\hat{\Omega}}$, we consider problem $(S\hat{Q}_{\epsilon,h})$:

$$\begin{cases} \text{Find } \hat{U}^{n+1} \in V_h,\ n = 0, 1, \ldots, N-1,\ s.t. \\ (\partial_t \hat{U}^{n+1}, v_h)_{\hat{\Omega}} + \dfrac{1}{\epsilon}(\partial_t \hat{U}^{n+1}, v_h)_{\hat{\Omega}_1} + (\nabla \hat{U}^{n+1}, \nabla v_h)_{\hat{\Omega}} \\ \quad + \dfrac{1}{\epsilon}(\nabla \hat{U}^{n+1}, \nabla v_h)_{\hat{\Omega}_1} = (\bar{f}^{n+1}, v_h)_D,\ \forall v_h \in V_h \\ \hat{U}^0 = \tilde{u}_{0h}, \end{cases} \tag{1.10}$$

With some assumption of $\hat{\Omega}$, we obtain the error estimate of (see Theorem 5.3)

$$\|U^n - \hat{U}^n\|_{0,\Omega\cup\hat{\Omega}} + \frac{1}{\sqrt{\epsilon}}\|U^n - \hat{U}^n\|_{0,\Omega_1\cap\hat{\Omega}_1} \le C(\sqrt{\epsilon} + \sqrt{h})^2,$$

$$\|U^n - \hat{U}^n\|_{1,\Omega\cup\hat{\Omega}} + \frac{1}{\sqrt{\epsilon}}\|U^n - \hat{U}^n\|_{1,\Omega_1\cap\hat{\Omega}_1} \le C(\sqrt{\epsilon} + \sqrt{h}).$$

To this point, we have stated only the time-independent domain Ω. At this stage, we consider a time-dependent (bounded) domain $\Omega_t \subset \mathbb{R}^2$ with the smooth boundary $\Gamma_t,\ 0 \le t \le T$. We propose a redefinition

$$Q_T = \cup_{t\in[0,T]}\Omega_t,\ \Sigma_T = \cup_{t\in[0,T]}\Gamma_t, \tag{1.11}$$

instead of (1.2). As usual, we respectively call (1.11) and (1.2) cylindrical and non-cylindrical domains. We assume that $Q_T \subset \mathbb{R}^2_x \times \mathbb{R}_t$ is sufficiently smooth, and we consider the original problem (1.4) and its H^1-penalty problem (1.6) again. Then, we will prove that the error estimate (1.7) holds true(cf. Theorem 7.1). The key point is to modify the standard extension theorem (Lemma 2.2) for a cylindrical domain to this non-cylindrical domain. However, we must confront some obstacles to accomplish the analysis of the finite element approximation for a non-cylindrical domain. Therefore, we will postpone the detailed analysis for discussion in a future paper.

The remainder of this paper is organized as follows. We study the error estimate for H^1-penalty approximation in Section 2. Furthermore, in Section 3, we present some regularity analysis for H^1-penalty problems. We will show that the H^1-penalty problem is, in a sense, equivalent to a kind of interface problem. Section 4 is devoted to finite element approximation for the H^1-penalty problem. We consider a scheme approximating the discrete problem in Section 5. We give some numerical experiments to verify our theoretical results in Section 6. Finally, we conclude this paper by stating some results for moving-boundary problem.

2. ERROR ESTIMATE FOR H^1-PENALTY APPROXIMATION

For problem (1.4) and (1.6), we have the following theorem.

Theorem 2.1. *There exist unique solution u and u_ϵ for (1.4) and (1.6), respectively. Then we have*

$$\|u_\epsilon|_{Q_T} - u\|_{H^{1,0}(Q_T)} + \|u_{\epsilon t}|_{Q_T} - u_t\|_{H^{-1,0}(Q_T)} \le C(\|f\|_{0,Q_T} + \|u_0\|_{0,\Omega})\epsilon, \tag{2.1}$$

$$\|u_\epsilon\|_{H^{1,0}(Q'_T)} + \|u_{\epsilon t}\|_{H^{-1,0}(Q'_T)} \le C(\|f\|_{0,Q_T} + \|u_0\|_{0,\Omega})\epsilon. \tag{2.2}$$

Before stating the proof, we state an extension theorem, which is a direct result following from Lemma 12.2 of Chapter 1 in [10].

Lemma 2.2. (Extension Theorem) $Q_T = \Omega \times [0,T]$, *as defined before. For $u \in H^{1,0}(Q_T)$ with $u_t \in H^{-1,0}(Q_T)$ and $u(t) \in L^2(\Omega), \forall t \in [0,T]$, there exists an extension operator $\mathbf{P}$ such that*

$$(i)\ \mathbf{P} \in \mathcal{L}\left(H^{1,0}(Q_T); H^{1,0}(\mathbb{R}^2_x \times [0,T])\right);$$

$$(ii)\ \frac{d}{dt}(\mathbf{P}u) = \mathbf{P}(u_t),\ \mathbf{P} \in \mathcal{L}\left(H^{-1,0}(Q_T); H^{-1,0}(\mathbb{R}^2_x \times [0,T])\right);$$

$$\langle \mathbf{P}u_t, v\rangle_{\mathbb{R}^2\times[0,T]} \le C\|u_t\|_{H^{-1,0}(Q_T)}\|v\|_{H^{1,0}(\mathbb{R}^2_x\times[0,T])}, \forall v \in H^{1,0}(\mathbb{R}^2 \times [0,T]);$$

$$(iii)\ Pu(t) \equiv (\mathbf{P}u)(t),\ \forall t \in [0,T],\ P \in \mathcal{L}(L^2(\Omega); L^2(\mathbb{R}^2_x)).$$

proof of theorem 2.1. The unique existence of the solutions for (1.4) and (1.6) might be readily apparent in view of the standard theory of parabolic equations ([10]).

Substituting $v = u_\epsilon$ into (1.6), we obtain the expressions shown below:

$$\begin{aligned}
&\frac{1}{2}(\|u_\epsilon(T)\|^2_{0,\Omega} + \frac{1}{\epsilon}\|u_\epsilon(T)\|^2_{0,\Omega_1} - \|\tilde{u}_\epsilon(0)\|^2_{0,\Omega} - \frac{1}{\epsilon}\|\tilde{u}_\epsilon(0)\|^2_{0,\Omega_1})\\
&+\|\nabla u_\epsilon\|^2_{0,Q_T} + \frac{1}{\epsilon}\|\nabla u_\epsilon\|^2_{0,Q'_T}\\
&=(\tilde{f}, u_\epsilon)_{0,D_T} \le \frac{1}{2C}(\|\tilde{f}\|^2_{0,Q_T} + \epsilon\|\tilde{f}\|^2_{0,Q'_T}) + \frac{C}{2}(\|u_\epsilon\|^2_{0,Q_T} + \frac{1}{\epsilon}\|u_\epsilon\|^2_{0,Q'_T}).
\end{aligned}$$

Following from *Friedrichs'* inequality, we have the a priori estimate as

$$\begin{aligned}
&\|u_\epsilon(T)\|^2_{0,\Omega} + \frac{1}{\epsilon}\|u_\epsilon(T)\|^2_{0,\Omega_1} + \|u_\epsilon\|^2_{1,Q_T} + \frac{1}{\epsilon}\|u_\epsilon\|^2_{1,Q'_T}\\
\le&C(\|\tilde{u}_\epsilon(0)\|^2_{0,\Omega} + \frac{1}{\epsilon}\|\tilde{u}_\epsilon(0)\|^2_{0,\Omega_1} + \|\tilde{f}\|^2_{0,Q_T} + \epsilon\|\tilde{f}\|^2_{0,Q'_T}).
\end{aligned}$$

Next, considering the $H^{-1,0}$ norm for $u_{\epsilon,t}$. Setting $v \in H^{1,0}_0(D_T)$, satisfying $v \equiv 0$ in Q_T(*resp.* Q'_T) in (1.6), then

$$\left\langle \frac{d}{dt}u_\epsilon, v\right\rangle_{Q_T} = -(\nabla u_\epsilon, \nabla v)_{Q_T} + (\tilde{f}, v)_{Q_T},\ \forall v \in H^{1,0}_0(Q_T),$$

$$(resp.\ \left\langle \frac{d}{dt}u_\epsilon, v\right\rangle_{Q'_T} = -(\nabla u_\epsilon, \nabla v)_{Q'_T} + \epsilon(\tilde{f}, v)_{Q'_T},\ \forall v \in H^{1,0}_0(Q'_T))$$

which gives

$$\left\|\frac{d}{dt}u_\epsilon\right\|_{H^{-1,0}(Q_T)} \leq C(\|u_\epsilon\|_{H^{1,0}(Q_T)} + \|\tilde{f}\|_{0,Q_T}), \tag{2.3}$$

$$(resp.\ \left\|\frac{d}{dt}u_\epsilon\right\|_{H^{-1,0}(Q'_T)} \leq C(\|u_\epsilon\|_{H^{1,0}(Q'_T)} + \epsilon\|\tilde{f}\|_{0,Q'_T}).) \tag{2.4}$$

Using the extension theorem (Lemma 2.2) and setting $v = \mathbf{P}(u_\epsilon|_{Q'_T}) \in H_0^{1,0}(D_T)$ (the extension of u_ϵ from domain Q'_T onto D_T) into (1.6), then the following is obtained:

$$\frac{1}{\epsilon}(\|u_\epsilon(T)\|^2_{0,\Omega_1} - \|u_\epsilon(0)\|^2_{0,\Omega_1}) + (u_\epsilon(T), v(T))_\Omega - (u_\epsilon(0), v(0))_\Omega - \langle u_\epsilon, v_t\rangle_{D_T}$$
$$+(\nabla u_\epsilon, \nabla v)_{Q_T} + \frac{1}{\epsilon}\|u_\epsilon\|^2_{0,Q'_T} = (\tilde{f}, v)_{D_T}.$$

Following from $(i)(ii)(iii)$ of Lemma 2.2,

$$\langle u_\epsilon, v_t\rangle_{D_T} \leq C\|u_\epsilon\|_{1,D_T}\|u_{\epsilon t}\|_{H^{-1,0}(Q'_T)},$$
$$\|v(T)\|_{0,\Omega} \leq C\|u_\epsilon(T)\|_{0,\Omega_1},$$
$$\|v\|_{H^{1,0}(Q_T)} \leq C\|u_\epsilon\|_{H^{1,0}(Q'_T)}.$$

Therefore, combining *Friedrichs'* inequality, it is possible to obtain the following:

$$\frac{1}{\epsilon}\|u_\epsilon(T)\|^2_{\Omega_1} + \frac{1}{\epsilon}\|u_\epsilon\|^2_{H^{1,0}(Q'_T)}$$
$$\leq C(\frac{1}{\epsilon}\|u_\epsilon(0)\|^2_{0,\Omega_1} + \|u_\epsilon(T)\|_{0,\Omega}\|u_\epsilon(T)\|_{0,\Omega_1} + \|u_\epsilon(0)\|_\Omega\|u_\epsilon(0)\|_{0,\Omega_1}$$
$$+\|\tilde{f}\|_{0,D_T}\|u_\epsilon\|_{H^{1,0}(Q'_T)} + \|u_\epsilon\|_{1,D_T}\|u_{\epsilon t}\|_{H^{-1,0}(Q'_T)} + \|\nabla u_\epsilon\|_{0,Q_T}\|u_\epsilon\|_{H^{1,0}(Q'_T)}).$$

With (2.4), it is apparent that, if $\|u_\epsilon(0)\|_{0,\Omega_1} = \|\tilde{u}_0\|_{0,\Omega_1} \leq C\|u_0\|_{0,\Omega}\epsilon$ and $\|\tilde{f}\|_{0,Q'_T} \leq C\|f\|_{0,Q_T}$, (which are satisfied because we let $\tilde{u}_0$ and $\tilde{f}$ be the zero extension of u_0 and f respectively), then

$$\|u_\epsilon(T)\|_{\Omega_1} + \|u_\epsilon\|_{H^{1,0}(Q'_T)} \leq C(\|u_0\|_{0,\Omega} + \|f\|_{0,Q_T})\epsilon,$$

which is the (2.2).

Setting $w \equiv \mathbf{P}(u_\epsilon|_{Q'_T})$, $\xi = u_\epsilon|_{Q_T} - u - w|_{Q_T}$, it is apparent that $\xi \in H_0^{1,0}(Q_T)$ and $\xi_t \in H^{-1,0}(Q_T)$. Then, we consider $\frac{d}{dt}\xi + A\xi = \frac{d}{dt}(u_\epsilon|_{Q_T} - u) + A(u_\epsilon|_{Q_T} - u) - \frac{d}{dt}w|_{Q_T} - Aw|_{Q_T}$, where A is the operator of $H^{1,0}(Q_T) \to H^{1,0}(Q_T)$, $g \mapsto Ag$ defined as $\langle Ag, v\rangle = (\nabla g, \nabla v)$, $\forall v \in H_0^{1,0}(Q_T)$.

First, following from (1.6) and (1.4), it is readily apparent that

$$\frac{d}{dt}(u_\epsilon|_{Q_T} - u) + A(u_\epsilon|_{Q_T} - u) = 0 \in H^{-1,0}(Q_T).$$

For $Aw|_{Q_T}$ and $\frac{d}{dt}w|_{Q_T}$, we have

$$\|Aw|_{Q_T}\|_{H^{1,0}(Q_T)} = \sup_{v\in H_0^{1,0}(Q_T)} \frac{\langle Aw|_{Q_T}, v\rangle}{\|v\|_{H^{1,0}(Q_T)}} \le C\|w\|_{H^{1,0}(Q_T)},$$

Subsequently, by extension theorem(Lemma 2.2),

$$\|w\|_{H^{1,0}(Q_T)} + \left\|\frac{d}{dt}w|_{Q_T}\right\|_{H^{-1,0}(Q_T)} \le C\left(\|u_\epsilon\|_{H^{1,0}(Q'_T)} + \left\|\frac{d}{dt}u_\epsilon\right\|_{H^{-1,0}(Q'_T)}\right), \tag{2.5}$$

Therefore, $\frac{d}{dt}\xi + A\xi = -\frac{d}{dt}w|_{Q_T} - Aw|_{Q_T} \equiv F \in H^{-1,0}(Q_T)$, and from (2.5), (2.4) and (2.2), which yield

$$\|F\|_{H^{-1,0}(Q_T)} \le C(\|u_\epsilon\|_{H^{1,0}(Q'_T)} + \epsilon\|\tilde{f}\|_{0,Q'_T}) \le C(\|u_0\|_{0,\Omega} + \|f\|_{0,Q_T})\epsilon.$$

On the other hand, ξ satisfies the initial condition $\xi(0) = w(0)|_\Omega$ and $\|w(0)\|_{0,\Omega} \le C\|u_\epsilon(0)\|_{0,\Omega_1} = \|\tilde{u}_0\|_{0,\Omega_1}$. With the assumption that $\|\tilde{u}_0\|_{0,\Omega_1} \le C\|u_0\|_{0,\Omega}\epsilon$, then applying Theorem 4.1 of Chapter 3 in [10],

$$\begin{aligned} \|\xi\|_{H^{1,0}(Q_T)} + \left\|\frac{d}{dt}\xi\right\|_{H^{-1,0}(Q_T)} &\le C(\|F\|_{H^{-1,0}(Q_T)} + \|w(0)\|_{0,\Omega}) \\ &\le C(\|u_0\|_{0,\Omega} + \|f\|_{0,Q_T})\epsilon. \end{aligned} \tag{2.6}$$

Because

$$\begin{aligned} &\|u_\epsilon - u\|_{H^{1,0}(Q_T)} + \|u_{\epsilon t} - u_t\|_{H^{-1,0}(Q_T)} \\ &\le \|\xi\|_{H^{1,0}(Q_T)} + \left\|\frac{d}{dt}\xi\right\|_{H^{-1,0}(Q_T)} + \|w\|_{H^{1,0}(Q_T)} + \left\|\frac{d}{dt}w\right\|_{H^{-1,0}(Q_T)}, \end{aligned}$$

(2.1) follows from (2.6) and (2.5), and the proof is completed. □

Remark 1. Actually, $\tilde{f}$ and $\tilde{u}_0$ was the zero extension at the beginning. However, in the proof, it is shown that $\|\tilde{u}_0\|_{0,\Omega_1} \le C\|u_0\|_{0,\Omega}\epsilon$ and $\|\tilde{f}\|_{0,Q'_T} \le C\|f\|_{0,Q_T}$ are sufficient to deduce the sharp error estimate.

Remark 2. For the original parabolic parabolic problem with Neumann or the mixed boundary condition, one can apply a similar analytical method in [25], which is for elliptic problems, to obtain the sharp error estimate.

3. Regularity theorem for H^1-penalty problem

In this section, we derive some regularity analysis for H^1-penalty problem (1.6). First, we recall the regularity theorem for the parabolic problem (1.4).

Theorem 3.1. (regularity theorem for problem (1.4), Theorem 5.3 of Chapter 4 in [11]) *For* $f \in H^{2k,k}(Q_T)$, ($2k \ge 0$, *integer*), $u_0 \in H^{2k+1}(\Omega_0)$ *satisfying* *the* $(\mathcal{R}.\mathcal{L}.)$ *condition of*

$$\begin{bmatrix} \exists w \in H^{2k+2,k+1}(Q_T) \cap H_0^{1,0}(Q_T) \text{ with} \\ w(x,0) = u_0(x), \\ D_t^r[Aw + D_t w]|_{t=0} = D_t^r f(x,0),\ 0 \le r < k - \dfrac{1}{2}. \end{bmatrix}$$

Then, a unique solution $u \in H^{2k+k,k+1}(Q_T) \cap H_0^{1,0}(Q_T)$ *exists for problem (1.4).*

Applying *Green's* formula, it is apparent that the H^1-penalty problem (1.6) is equivalent to the interface problem (P_ϵ):

$$\begin{cases} \dfrac{d}{dt}u_\epsilon + \Delta u_\epsilon = f \text{ in } Q_T, \quad \dfrac{d}{dt}u_\epsilon + \Delta u_\epsilon = \epsilon \tilde{f} \text{ in } Q'_T, \\ u_\epsilon|_{Q_T} = u_\epsilon|_{Q'_T} \text{ on } \Sigma_T, \quad u_\epsilon = 0 \text{ on } \partial D \times (0,T), \\ \dfrac{\partial u_\epsilon|_{Q_T}}{\partial n} = \dfrac{1}{\epsilon}\dfrac{\partial u_\epsilon|_{Q'_T}}{\partial n} \text{ on } \Sigma_T, \\ u_\epsilon = u_0 \text{ in } \Omega, \quad u_\epsilon = 0 \text{ in } \Omega_1. \end{cases}$$

To study the regularity of the H^1-penalty problem, we state some spaces and utilities as

$$A_\epsilon(u,v) : H_0^1(D) \times H_0^1(D) \to \mathbb{R},\ A_\epsilon(u,v) \equiv (\nabla u, \nabla v)_\Omega + \frac{1}{\epsilon}(\nabla u, \nabla v)_{\Omega,1}.$$

$$A_\epsilon : H_0^1(D) \to H^{-1}(D),\ \langle A_\epsilon u, v\rangle \equiv A_\epsilon(u,v).$$

$$H^k(\Omega, \Omega_1) \equiv \{u \mid u \in L^2(D),\ u|_\Omega \in H^k(\Omega),\ u|_{\Omega_1} \in H^k(\Omega_1)\},$$

with norm $\|u\|_{k,\Omega,\Omega_1} \equiv \|u\|_{k,\Omega} + \|u\|_{k,\Omega_1}$,for arbitrary $k \ge 0$, where k is an integer.

For arbitrary $k \ge 1$, where k is an integer, we define

$$D_k(A_\epsilon) \equiv \left\{ u \mid u \in H_0^1(D),\ u|_\Omega \in H^k(\Omega),\ u|_{\Omega_1} \in H^k(\Omega_1),\ \left.\frac{\partial u}{\partial n}\right|_{\Omega,\Gamma} = \frac{1}{\epsilon}\left.\frac{\partial u}{\partial n}\right|_{\Omega_1,\Gamma} \right\},$$

with norm $\|u\|_{D_k(A_\epsilon)} \equiv \|u\|_{k,\Omega} + \frac{1}{\epsilon}\|u\|_{k,\Omega_1}$;

Remark 3. In an earlier report [25] (see Theorem 3.1 and Lemma 7), we showed that A_ϵ is an isomorphism of $D_k(A_\epsilon) \to H^{k-2}(\Omega,\Omega_1),\ k \ge 2,\ k$ is an integer.

We also define the space

$$H_\epsilon^k(\Omega,\Omega_1) \equiv \{v \mid v \in H^k(\Omega,\Omega_1)\},\ k \ge 0,\ k \text{ is integer},$$

with norm $\|v\|_{H_\epsilon^k(\Omega,\Omega_1)} \equiv \|v\|_{k,\Omega} + \frac{1}{\epsilon}\|v\|_{k,\Omega_1}$.

For arbitrary $k \geq 1$, where k is an integer, we define

$$\begin{aligned}&D_k(\mathcal{A}_\epsilon) \equiv \{u \mid \ u|_\Omega \in H^k(\Omega),\ u|_{\Omega_1} \in H^k(\Omega_1),\\ &\gamma_0(\Omega,\Gamma)u = \frac{1}{\epsilon}\gamma_0(\Omega_1,\Gamma)u,\ \ \frac{\partial u}{\partial n}\bigg|_{\Omega,\Gamma} = \frac{\partial u}{\partial n}\bigg|_{\Omega_1,\Gamma},\ \gamma_0(D,\partial D)u = 0\},\end{aligned}$$

with norm $\|u\|_{D_k(\mathcal{A}_\epsilon)} \equiv \|u\|_{k,\Omega} + \|u\|_{k,\Omega_1}$. We also define operator $\mathcal{A}_\epsilon$: $D_1(\mathcal{A}_\epsilon) \mapsto H^{-1}(D)$ by $\langle \mathcal{A}_\epsilon\phi,\psi\rangle = (\nabla\phi,\nabla\psi)_\Omega + (\nabla\phi,\nabla\psi)_{\Omega_1},\ \forall\psi \in H_0^1(D)$.

Remark 4. $A_\epsilon = \mathcal{A}_\epsilon M_\epsilon$, and M_ϵ is an isomorphism from $D_k(\mathcal{A}_\epsilon)$ onto $D_k(A_\epsilon),\ k \geq 2$, such that $\forall v \in D_k(A_\epsilon),\ \|v\|_{D_k(A_\epsilon)} = \|M_\epsilon v\|_{D_k(\mathcal{A}_\epsilon)}$.

We define operator $\mathcal{A}_\epsilon : D_1(\mathcal{A}_\epsilon) \mapsto H^{-1}(D)$ by $\langle \mathcal{A}_\epsilon\phi,\psi\rangle = (\nabla\phi,\nabla\psi)_\Omega + (\nabla\phi,\nabla\psi)_{\Omega_1},\ \forall\psi \in H_0^1(D)$, then we have the following lemma.

Lemma 3.2. *$\forall w \in D_{k+2}(\mathcal{A}_\epsilon), k \geq 0, k$ is an integer, there exists $\xi_0 \in \mathbb{R}$ and $c > 0$, such that $\forall p = \xi + i\eta,\ \xi > \xi_0$, we have*

$$\begin{aligned}&\|(\mathcal{A}_\epsilon + p)w\|_{k,\Omega,\Omega_1} + (1 + |p|^{\frac{k}{2}})\|(\mathcal{A}_\epsilon + p)w\|_{0,\Omega,\Omega_1}\\ \geq &c(\|w\|_{D_{k+2}(\mathcal{A}_\epsilon)} + |p|^{1+\frac{k}{2}}\|w\|_{H_\epsilon^0(\Omega,\Omega_1)}.)\end{aligned}$$

Proof. This lemma is similar to Theorem 5.1 of Chapter 4 in [11]. For arbitrary $k \geq 1$, where k is an integer, we define

$$\begin{aligned}D_k(\Lambda_{\epsilon,\theta}) \equiv &\{w \mid w \in H^1(\mathbb{R}_t; H_0^1(D)),\ w|_{\Omega\times\mathbb{R}_t} \in H^k(\Omega\times\mathbb{R}_t),\\ &w|_{\Omega_1\times\mathbb{R}_t} \in H^k(\Omega_1\times\mathbb{R}_t)\},\end{aligned}$$

with norm $\|w\|_{D_k(\Lambda_{\epsilon,\theta})} \equiv \|w\|_{k,\Omega\times\mathbb{R}_t} + \frac{1}{\epsilon}\|w\|_{k,\Omega_1\times\mathbb{R}_t}$. In addition, we define the operator

$$\Lambda_{\epsilon,\theta} : H^1(\mathbb{R}_t; H_0^1(D)) \to H^{-1}(\mathbb{R}_t; H^{-1}(D)),\ w \mapsto \Lambda_{\epsilon,\theta}w,$$

$$\langle \Lambda_{\epsilon,\theta}w, v\rangle \equiv \int_{\Omega\times\mathbb{R}_t} e^{i\theta}\nabla_{(x,t)}w\cdot\nabla_{(x,t)}v dxdt + \frac{1}{\epsilon}\int_{\Omega_1\times\mathbb{R}_t} e^{i\theta}\nabla_{(x,t)}w\cdot\nabla_{(x,t)}v dxdt.$$

It might be readily apparent that there exists C independent of $\theta \in [-\frac{\pi}{2},\frac{\pi}{2}]$ such that, $\forall w \in D_k(\Lambda_{\epsilon,\theta})$ with *supp* $w \in D\times(-1,1)$, one obtains

$$\|w\|_{D_k(\Lambda_{\epsilon,\theta})} \leq C\|\Lambda_{\epsilon,\theta}w\|_{H^{k-2}(\Omega\times\mathbb{R}_t,\Omega_1\times\mathbb{R}_t)},\ k \geq 2,$$

which follows from the isomorphism of A_ϵ (see Remark 3). Then, with an analogue of proof of Theorem 4.1 and Theorem 5.1 of Chapter 4 in [11], we can obtain that $\forall v \in D_k(A_\epsilon),\ k \geq 0,\ k$ is an integer, there exists $\xi_0 \in \mathbb{R}$ and $c > 0$, such that $\forall p = \xi + i\eta,\ \xi > \xi_0$,

$$\begin{aligned}&\|(A_\epsilon + pM_\epsilon)v\|_{k,\Omega,\Omega_1} + (1 + |p|^{\frac{k}{2}})\|(A_\epsilon + pM_\epsilon)v\|_{0,\Omega,\Omega_1}\\ \geq &c(\|v\|_{D_{k+2}(A_\epsilon)} + |p|^{1+\frac{k}{2}}\|M_\epsilon v\|_{0,\Omega,\Omega_1}.)\end{aligned},$$

which shows our result (see Remark 4). ∎

Next, we apply Theorem 5.2 of Chapter 4 in [11] to obtain the following theorem

Theorem 3.3. *$k \geq 0$, k is an integer, $\beta = \frac{k}{2}$, and*

$$\begin{aligned}
&f \in L^2(0,T;H^k(\Omega,\Omega_1)) \cap H^\beta(0,T;H^0(\Omega,\Omega_1)),\\
&f^{(j)}(0) = 0, \text{ if } j < \beta - \frac{1}{2}, \beta \neq \text{ integer } + \frac{1}{2},\\
&\frac{1}{\sqrt{t}} f^{(\mu)}(t) \in L^2(0,T;H^0(\Omega,\Omega_1)) \text{ if } \beta = \mu + \frac{1}{2}, \ \mu \text{ is an integer,}
\end{aligned}$$

then, we have

$$\begin{cases}
\exists^1 w \in L^2(0,T;D_{k+2}(\mathcal{A}_\epsilon)) \cap H^{\beta+1}(0,T;H^0(\Omega,\Omega_1) \text{ satisfying}\\
\mathcal{A}_\epsilon w + w_t = f,\\
w(0) = 0.
\end{cases}$$

Proof. Setting $H = H^k(\Omega,\Omega_1)$ and $\mathcal{H} = H^0(\Omega,\Omega_1)$ in the proof of Theorem 5.2 of Chapter 4 in [11], then because $\mathcal{A}_\epsilon$ satisfies Lemma 3.2, we can obtain the result immediately. □

Combining Remark 4, finally one is able to obtain

Theorem 3.4. *$k \geq 0$, $2k$ is an integer, for*

$$\begin{aligned}
&f_\epsilon \in L^2(0,T;H^{2k}(\Omega,\Omega_1)) \cap H^k(0,T;H^0(\Omega,\Omega_1)),\\
&u_{\epsilon 0} \in H_\epsilon^{2k+1}(\Omega,\Omega_1),
\end{aligned}$$

which satisfy the $(\mathcal{R}.\mathcal{L}.^*)$ condition

$$\left[\begin{aligned}
&\exists w \in L^2(0,T;D_{2k+2}(A_\epsilon)) \cap H^{k+1}(0,T;H_\epsilon^0(\Omega,\Omega_1)) \text{ with}\\
&w(x,0) = u_{\epsilon 0},\\
&D_t^r[A_\epsilon w + D_t w]|_{t=0} = D_t^r f_\epsilon(x,0), \ 0 \leq r < k - \frac{1}{2}.
\end{aligned}\right.$$

Then, there exists a unique solution

$$u_\epsilon \in L^2(0,T;D_{2k+2}(A_\epsilon)) \cap H^{k+1}(0,T;H_\epsilon^0(\Omega,\Omega_1))$$

for problem

$$\begin{cases}
M_\epsilon u_{\epsilon t} + A_\epsilon u_\epsilon = f_\epsilon,\\
u_\epsilon(0) = u_{\epsilon 0}.
\end{cases}$$

4. FINITE ELEMENT APPROXIMATION AND ITS ERROR ESTIMATION

We consider the finite element methods with $P1$ function approximation for the H^1-penalty problem (1.6). First, we define the *Ritz projection* operator

$$R_{\epsilon,h}: H_0^1(D) \to V_h(D),\ v \mapsto R_{\epsilon,h}v,\ A_\epsilon(v - R_{\epsilon,h}v, \phi) = 0, \forall \phi \in V_h(D).$$

From the result for elliptic problem in [25] (see Theorem 4.4 and notice that $R_{\epsilon,h}u_\epsilon = u_{\epsilon,h}$), one obtains the following.

$$\|u - R_{\epsilon,h}u\|_{1,\Omega} + \frac{1}{\sqrt{\epsilon}}\|u - R_{\epsilon,h}u\|_{1,\Omega_1} \le C(\sqrt{h} + \sqrt{\epsilon})(\|u\|_{2,\Omega} + \|u\|_{2,\Omega_1}) \tag{4.1}$$

Remembering that the right-hand-side function $\tilde{f}$ need not to be the zero extension of f (see Remark 1), then neither does the elliptic problem with the Dirichlet boundary condition. Therefore, we consider the adjoint boundary value problem (4.10) in [25] with $\tilde{f}$, such that $\|\tilde{f}\|_{0,D} \le C\|f\|_{0,\Omega}$. Then, taking $\tilde{f} = M_\epsilon(u_\epsilon - u_{\epsilon,h})$ in the proof of Theorem 4.5 in [25], one finds that

$$\begin{aligned}
&\|u - R_{\epsilon,h}u\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|u - R_{\epsilon,h}u\|_{0,\Omega_1} \\
\le& C(\sqrt{h} + \sqrt{\epsilon})(\|u - R_{\epsilon,h}u\|_{1,\Omega} + \frac{1}{\sqrt{\epsilon}}\|u - R_{\epsilon,h}u\|_{1,\Omega_1}) \\
\le& C(\sqrt{h} + \sqrt{\epsilon})^2(\|u\|_{2,\Omega} + \|u\|_{2,\Omega_1}),
\end{aligned} \tag{4.2}$$

for $u \in H_0^1(d)$, with $u|_\Omega \in H^2(\Omega),\ u|_{\Omega_1} \in H^2(\Omega_1),\ \|u\|_{2,\Omega} \le C,\ \|u\|_{2,\Omega_1} \le C\epsilon$.

4.1. **Some prior estimates.** We derive some prior estimates for our H^1-penalty problem. $u_{\epsilon,h}$ is the solution of (1.8). In the following, we designate $\|v\|^2_{k,\sqrt{\epsilon},\Omega,\Omega_1} \equiv \|v\|^2_{k,\Omega} + \frac{1}{\epsilon}\|v\|^2_{k,\Omega_1}$ and $\|v\|_{k,\sqrt{\epsilon},\Omega,\Omega_1} \equiv \|v\|_{k,\Omega} + \frac{1}{\sqrt{\epsilon}}\|v\|_{k,\Omega_1}$.

Lemma 4.1. *If $u_0 \in L^2(\Omega),\ \tilde{f} \in L^2(D_T)$, then*

$$\|u_\epsilon(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \int_0^t \|u_\epsilon(s)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} ds \le C(\|u_0\|^2_{0,\Omega} + \|f\|^2_{0,Q_T} + \epsilon\|\tilde{f}\|^2_{0,Q'_T}). \tag{4.3}$$

Proof. Substituting $v = u_\epsilon$ into (1.6), we obtain the result immediately. □

Lemma 4.2. *If $\tilde{u}_{0h} \in V_h(D),\ f \equiv 0$, then*

$$\|u_{\epsilon,ht}(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \int_0^t \|u_{\epsilon,ht}\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} ds \le C\|u_{\epsilon,ht}(0)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}. \tag{4.4}$$

For $\tilde{f} \in L^2(D_T)$, the following is true:

$$\begin{aligned}
&\int_0^t \|u_{\epsilon,hs}(s)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} ds + \|u_\epsilon(t)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} \\
\le& C(\|\tilde{u}_{0h}\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} + \|f\|^2_{0,Q_T} + \epsilon\|\tilde{f}\|^2_{0,Q'_T}).
\end{aligned} \tag{4.5}$$

Proof. To prove (4.4), we take the derivative with respect to t of (1.8). To prove (4.5), we substitute $v_h = u_{\epsilon,ht}$ into (1.8). We have

$$A_\epsilon(u_{\epsilon,h}, u_{\epsilon,ht}) = \frac{1}{2}\frac{d}{dt}A_\epsilon(u_{\epsilon,h}, u_{\epsilon,h}),$$

which gives

$$\begin{aligned}&\|u_{\epsilon,ht}(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \frac{1}{2}\frac{d}{dt}A_\epsilon(u_{\epsilon,h}(t), u_{\epsilon,h}(t)) = (\tilde{f}(t), u_{\epsilon,ht}(t))_{0,D}\\ \leq&\frac{1}{2}(\|f\|^2_{0,\Omega} + \|u_{\epsilon,ht}(t)\|^2_{0,\Omega}) + \frac{1}{2}(\epsilon\|\tilde{f}\|^2_{0,\Omega_1} + \frac{1}{\epsilon}\|u_{\epsilon,ht}(t)\|^2_{0,\Omega_1}).\end{aligned}$$

Then, taking the integration from 0 to t of the above inequality, and with *Poincáre's* inequality for $u_\epsilon(t)$, we obtain (4.5). □

Remark 5. From the proof of (4.5), it is apparent that replacing $u_{\epsilon,ht},\ u_{\epsilon,h}$ in (4.5) can be done by $u_{\epsilon t},\ u_\epsilon$ respectively for $u_0 \in H^1_0(\Omega)$.

Lemma 4.3. *If $\tilde{u}_{0h} \in V_h(D),\ f \equiv 0$, then we have*

$$t\|u_{\epsilon,h}(t)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} + \int_0^t s\|u_{\epsilon,hs}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}ds \leq C\|\tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}. \tag{4.6}$$

Lemma 4.4. *If $\tilde{u}_{0h} \in V_h(D),\ f \equiv 0$, then we have*

$$t^2\|u_{\epsilon,ht}(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \int_0^t s^2\|u_{\epsilon,hs}\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1}ds \leq C\|\tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}. \tag{4.7}$$

Lemma 4.5. *If $u_0 \in H^1_0(\Omega) \cap H^2(\Omega),\ f \equiv 0$, then*

$$t\|u_{\epsilon t}(t)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} + \int_0^t s\|u_{\epsilon ss}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}ds \leq C\|u_0\|^2_{2,\Omega}. \tag{4.8}$$

The proofs of Lemma 4.3, 4.4, and 4.5 are, respectively, the analogues to the proofs of Lemmas 2.3, 2.4, and 2.5 in [13]. (To show Lemma 4.5, we require regularity results in the previous section for $\tilde{u}_0 \in H^2_\epsilon(\Omega, \Omega_1)$.)

4.2. **"continuous-time" discrete problem.** Setting $e(t) = u_\epsilon(t) - u_{\epsilon,h}(t)$, for $t \in (0, T]$ yields the following estimates.

Theorem 4.6. *If $f \in L^2(Q_T),\ u_0 \in H^1_0(\Omega)$, then*

$$\begin{aligned}&\|e(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \int_0^t \|e(s)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1}ds\\ \leq&C(\sqrt{\epsilon} + \sqrt{h})^2(\|u_0\|^2_{1,\Omega} + \|f\|^2_{0,Q_T}) + \|\tilde{u}_0 - \tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}.\end{aligned} \tag{4.9}$$

Proof. Subtracting (1.6) from (1.8), then

$$(e(t), \phi)_{0,\Omega} + \frac{1}{\epsilon}(e(t), \phi)_{0,\Omega_1} + A_\epsilon(e(t), \phi) = 0,\ \forall\phi \in V_h(D).$$

Substituting $\phi = R_{\epsilon,h}e(t) = R_{\epsilon,h}u_\epsilon(t) - u_{\epsilon,h}$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + |e(t)|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} \\
=&\left(\frac{d}{dt}e(t), u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t)\right)_{0,\Omega} + \frac{1}{\epsilon}\left(\frac{d}{dt}e(t), u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t)\right)_{0,\Omega_1} \\
&+ (\nabla e(t), \nabla(u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t)))_{0,\Omega} + \frac{1}{\epsilon}(\nabla e(t), \nabla(u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t)))_{0,\Omega_1},
\end{aligned}$$

where $|v|_{1,\Omega} \equiv \|\nabla v\|_{0,\Omega}$, is the semi-norm of $H^1(\Omega)$. Applying *Poincáre's* inequality yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + C_1\|e(t)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} \\
\le& C\|e_t(t)\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}\|u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t)\|_{0,\sqrt{\epsilon},\Omega,\Omega_1} \\
&+ \frac{C_1}{2}\|e(t)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} + \frac{1}{2C_1}\|u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1}.
\end{aligned}$$

Taking integration with respect to t of the inequality above, (4.9) follows from (4.1), (4.2), the regularity result, and Lemma 4.2. □

Remark 6. If we assume additionally that $u \in H^2(\Omega)$, setting $\tilde{u}_{0h} = R_{\epsilon,h}\tilde{u}_0$, then

$$\|\tilde{u}_0 - \tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} \le C(\sqrt{\epsilon} + \sqrt{h})^4\|u_0\|^2_{2,\Omega}.$$

Remark 7. If $u \in C^1(\Omega)$, setting $\tilde{u}_{0h} \in V_h(D)$, with

$$\tilde{u}_{0h}(\nu) = \bar{I}_h u_0 \equiv \begin{cases} u_0(\nu) \text{ on } \nu \in K,\ K \subset \Omega,\ K \cap \Gamma = \emptyset, \\ 0 \text{ on others,} \end{cases}$$

where K is the closed triangle of $\mathcal{T}_h$, and ν is the vertex of K. Then, we can obtain that

$$\|\tilde{u}_0 - \tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} \le C(\sqrt{\epsilon} + \sqrt{h})^2\|u_0\|^2_{1,\Omega}.$$

Theorem 4.7. *Assuming* $f \equiv 0,\ u_0 \in H^1_0(\Omega)$, *then*

$$\int_0^t \|e(s)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} ds \le C\|\tilde{u}_0 - \tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + C(\sqrt{\epsilon} + \sqrt{h})^4\|u_0\|^2_{1,\Omega}. \tag{4.10}$$

Proof. We consider adjoint problems of finding $w : [0,T) \to H^1_0(D)$ and $w_h : [0,T) \to V_h$ such that

$$\begin{cases} (\phi, w_t)_\Omega + \dfrac{1}{\epsilon}(\phi, w_t)_{\Omega_1} - A_\epsilon(\phi, w) = (\phi, M_\epsilon e(t))_D, \\ \forall\phi \in H^1_0(D),\ 0 \le t \le T,\quad w(T) = 0, \end{cases} \tag{4.11}$$

$$\begin{cases} (\phi_h, w_{ht})_\Omega + \dfrac{1}{\epsilon}(\phi_h, w_{ht})_{\Omega_1} - A_\epsilon(\phi_h, w_h) = (\phi_h, M_\epsilon e(t))_D, \\ \forall\phi_h \in W_h(D),\ 0 \le t \le T,\quad w_h(T) = 0, \end{cases} \tag{4.12}$$

Following from (1.6) and (1.8) with the assumption that $f \equiv 0$, we can infer the following.

$$\begin{aligned}&\frac{d}{dt}\left((e(t), w_h(t))_\Omega + \frac{1}{\epsilon}(e(t), w_h(t))_{\Omega_1}\right)\\ =& - A_\epsilon(u_\epsilon(t), w_h(t)) + A_\epsilon(u_{\epsilon,h}(t), w_h(t)) + (u_\epsilon(t), w_{ht}(t))_\Omega + \frac{1}{\epsilon}(u_\epsilon(t), w_{ht}(t))_{\Omega_1}\\ &- A_\epsilon(u_{\epsilon,h}(t), w_h(t)) - (u_{\epsilon,h}(t), M_\epsilon e(t))_D.\end{aligned}$$

Because

$$A_\epsilon(u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t), w_h(t)) = 0,$$

following from (1.6), (1.8), (4.11), and (4.12), and which yields

$$\begin{aligned}&\frac{d}{dt}\left((e(t), w_h(t))_\Omega + \frac{1}{\epsilon}(e(t), w_h(t))_{\Omega_1}\right)\\ =&\|e(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} - (u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t), w_t(t) - w_{ht}(t))_\Omega\\ &- \frac{1}{\epsilon}(u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t), w_t(t) - w_{ht}(t))_{\Omega_1} + A_\epsilon(u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t), w(t) - w_h(t)).\end{aligned}$$

Taking integration from 0 to T of the equation presented above, and recalling that $w(T) = w_h(T) = 0$, then

$$\begin{aligned}I =& \int_0^T \|e(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} dt = -(e(0), w_h(0))_\Omega - \frac{1}{\epsilon}(e(0), w_h(0))_{\Omega_1}\\ &+ \int_0^T (u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t), w_t(t) - w_{ht}(t))_\Omega + \frac{1}{\epsilon}(u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t), w_t(t) - w_{ht}(t))_{\Omega_1} dt\\ &- \int_0^T A_\epsilon(u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t), w(t) - w_h(t))dt.\end{aligned}$$

It is readily apparent that

$$\begin{aligned}I \leq& \|\tilde{u}_0 - \tilde{u}_{0h}\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}\|w_h(0)\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}\\ &+ \frac{C}{\eta}\int_0^T (\|u_\epsilon - R_{\epsilon,h}u_\epsilon\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + (\sqrt{\epsilon} + \sqrt{h})^2\|u_\epsilon - R_{\epsilon,h}u_\epsilon\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1})dt\\ &+ \eta\int_0^T (\|w_t - w_{ht}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \frac{1}{(\sqrt{\epsilon} + \sqrt{h})^2}\|w_t - w_{ht}\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1})dt\end{aligned}$$

Applying Lemma 4.1, Lemma 4.2, and Theorem 4.6, the right-hand-side of the previous inequality is bounded by

$$C\|\tilde{u}_0 - \tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + C(\frac{1}{2} + \eta)\int_0^T \|e(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} dt + \frac{C}{\eta}(\sqrt{\epsilon} + \sqrt{h})^4\|u_0\|^2_{1,\Omega}.$$

Here, we can choose $\eta > 0$ such that $C(\frac{1}{2}+\eta) < 1$. Furthermore, it is apparent that the T can be replaced by any $t_1 \in (0, T]$. Therefore, we complete the proof. □

Theorem 4.8. *If* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega),\ f \equiv 0$, *then we have*

$$\|e(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} \leq C(\sqrt{\epsilon}+\sqrt{h})^4\|u_0\|^2_{2,\Omega} + C\|\tilde{u}_0 - \tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}. \tag{4.13}$$

Proof. Setting $\xi(t) = R_{\epsilon,h}e(t)$, with

$$A_\epsilon(R_{\epsilon,h}u_\epsilon(t),\xi) = A_\epsilon(u_\epsilon(t),\xi),$$

yields the following:

$$\begin{aligned}&(\xi_t(t),\xi(t))_\Omega + \frac{1}{\epsilon}(\xi_t(t),\xi(t))_{\Omega_1} + A_\epsilon(\xi(t),\xi(t))\\ =&-\left(u_{\epsilon t} - \frac{d}{dt}R_{\epsilon,h}u_\epsilon(t),\xi\right)_\Omega - \frac{1}{\epsilon}\left(u_{\epsilon t} - \frac{d}{dt}R_{\epsilon,h}u_\epsilon(t),\xi\right)_{\Omega_1},\end{aligned} \tag{4.14}$$

from which, with multiplication by t, we obtain the following:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(t(\|\xi(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}) + tA_\epsilon(\xi(t),\xi(t))\\ =&\frac{1}{2}t\|\xi(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} - t(u_{\epsilon t} - \frac{d}{dt}R_{\epsilon,h}u_\epsilon(t),\xi)_\Omega - \frac{1}{\epsilon}t(u_{\epsilon t} - \frac{d}{dt}R_{\epsilon,h}u_\epsilon(t),\xi)_{\Omega_1}.\end{aligned}$$

With integration from 0 to t of the above equation, we obtain the following:

$$t\|\xi(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} \leq \int_0^t s\|\xi(s)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}ds + \int_0^t s^2(\|(I-R_{\epsilon,h})u_{\epsilon t}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}ds,$$

where $R_{\epsilon,h}\frac{d}{dt} = \frac{d}{dt}R_{\epsilon,h}$. Noting that $e(t) = u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t) + \xi(t)$, then

$$\begin{aligned}&t\|e(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}\\ \leq&Ct(\sqrt{\epsilon}+\sqrt{h})^4(\|u_\epsilon(t)\|^2_{2,\Omega} + \|u_\epsilon(t)\|^2_{2,\Omega_1}) + Ct\int_0^t \|e(s)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}ds.\\ &+Ct(\sqrt{\epsilon}+\sqrt{h})^4\int_0^t s(\|u_{\epsilon s}(s)\|^2_{2,\Omega} + \|u_{\epsilon s}(s)\|^2_{2,\Omega_1} + \|u_\epsilon(s)\|^2_{2,\Omega} + \|u_\epsilon(s)\|^2_{2,\Omega_1})ds.\end{aligned}$$

For $f \equiv 0$,

$$\begin{aligned}&\|u_{\epsilon s}(s)\|_{2,\Omega} + \|u_{\epsilon s}(s)\|_{2,\Omega_1} \leq C(\|A_\epsilon u_{\epsilon t}(t)\|_{0,\Omega} + \|A_\epsilon u_{\epsilon t}(t)\|_{0,\Omega_1})\\ \leq&C(\|u_{\epsilon tt}(t)\|_{0,\Omega} + \|u_{\epsilon tt}(t)\|_{0,\Omega_1}).\end{aligned}$$

Then, (4.13) follows from regularity result, Lemma 4.5, and Theorem 4.7. □

The method of those proofs fundamentally derives from [13]. With the similar analogue of [13], one can obtain the following theorems, which are the analogues of Lemma 3.10 and 3.7 in [13].

Theorem 4.9. *If* $u_0 \equiv 0$ *and* $f \in C([0,T];L^2(\Omega))$, *then*

$$\|e(t)\|_{0,\sqrt{\epsilon},\Omega,\Omega_1} \leq C(\sqrt{\epsilon}+\sqrt{h})^2 \ln\frac{1}{\sqrt{\epsilon}+\sqrt{h}}\max_{[0,t]}\|f(t)\|_{0,\Omega}. \tag{4.15}$$

Theorem 4.10. *If $f \equiv 0$ and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, then*

$$\begin{aligned} \|e(t)\|_{1,\Omega}^2 + \frac{1}{\epsilon}\|e(t)\|_{1,\Omega_1}^2 \leq \frac{C}{t}(\sqrt{\epsilon}+\sqrt{h})(\|u_0\|_{2,\Omega} + \|f\|_{C^1([0,T],L^2(\Omega))}) \\ + C(\sqrt{\epsilon}+\sqrt{h}) + C(\|\tilde{u}_0 - \tilde{u}_{0h}\|_{0,\Omega} + \frac{1}{\sqrt{\epsilon}}\|\tilde{u}_0 - \tilde{u}_{0h}\|_{0,\Omega_1}). \end{aligned} \tag{4.16}$$

Proof. Because we have $A_\epsilon(e(t), e(t)) = A(e(t), u_\epsilon - R_{\epsilon,h}u_\epsilon) + (M_\epsilon e_t(t), u_\epsilon(t) - R_{\epsilon,h}u_\epsilon(t))_D - (M_\epsilon e_t(t), e(t))_D$. Furthermore, following the same method of Lemma 3.6 of [13], it is possible to show that $t(M_\epsilon e_t(t), g) \leq C(\|u_0\|_{2,\Omega} + \|f\|_{C^1([0,T],L^2(\Omega))})h\|g\|_{1,\sqrt{\epsilon},\Omega,\Omega_1}$. Consequently, using Theorem 4.8, the result can be obtained. □

4.3. **"single-step backward" discrete problem.** Setting $\zeta^n = u_{\epsilon,h}^n - U^n$, for $n = 0, 1, \ldots, N$, (here, $\{U^n\}$ is the solution of (1.9)), before we derive some error estimates we define,for $w_h \in V_h(D)$,

$$\|w_h\|_{-1,\epsilon,h,\Omega,\Omega_1} = \sup_{v_h \in V_h(D)} \frac{(w_h, v_h)_\Omega + \frac{1}{\epsilon}(w_h, v_h)_{\Omega_1}}{\|v_h\|_{1,\Omega} + \frac{1}{\sqrt{\epsilon}}\|v_h\|_{1,\Omega_1}}.$$

Following from (1.8), one obtains the following.

$$\|u_{\epsilon,ht}(t)\|_{-1,\epsilon,h,\Omega,\Omega_1} \leq C(\|u_{\epsilon,h}(t)\|_{1,\Omega} + \frac{1}{\sqrt{\epsilon}}\|u_{\epsilon,h}(t)\|_{1,\Omega_1} + \|f(t)\|_{0,\Omega} + \epsilon\|\tilde{f}(t)\|_{0,\Omega_1}).$$

Theorem 4.11. *If $f \equiv 0$, and $\tilde{u}_{0h} = U^0$, then*

$$\|\zeta^n\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}^2 + \sum_{m=1}^{m=n+1} k\|\zeta^m\|_{1,\sqrt{\epsilon},\Omega,\Omega_1}^2 \leq C\frac{k^2}{3}\|u_{\epsilon,ht}(0)\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}^2. \tag{4.17}$$

(Here, $k = \frac{T}{N}$.)

Proof. Setting $r^{n+1} = \partial_t u_{\epsilon,h}^{n+1} - \frac{d}{dt}u_{\epsilon,h}(t_{n+1})$ yields

$$\begin{aligned} &(\partial_t \zeta^{n+1}, \phi)_\Omega + \frac{1}{\epsilon}(\partial_t \zeta^{n+1}, \phi)_{\Omega_1} + A_\epsilon(\zeta^{n+1}, \phi) \\ =&(r^{n+1}, \phi)_\Omega + \frac{1}{\epsilon}(r^{n+1}, \phi)_{\Omega_1} + (\tilde{f}(t_{n+1}) - \bar{f}^{n+1}, \phi)_D, \ \forall \phi \in V_h(D). \end{aligned}$$

Substituting $\phi = \zeta^{n+1}$, we get, with assumption that $f \equiv 0$

$$\begin{aligned} &\|\zeta^{n+1}\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}^2 + 2kA_\epsilon(\zeta^{n+1}, \zeta^{n+1}) + \|\zeta^{n+1} - \zeta^n\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}^2 \\ \leq&\|\zeta^n\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}^2 + 2k(r^{n+1}, \zeta^{n+1})_\Omega + 2k\frac{1}{\epsilon}(r^{n+1}, \zeta^{n+1})_{\Omega_1} \\ \leq&\|\zeta^n\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}^2 + Ck\|r^{n+1}\|_{-1,\epsilon,h,\Omega,\Omega_1}^2 + k\|\zeta^{n+1}\|_{1,\sqrt{\epsilon},\Omega,\Omega_1}^2. \end{aligned}$$

To estimate $\|r^{n+1}\|_{-1,\epsilon,h.\Omega,\Omega_1}$, because

$$r^{n+1} = \frac{1}{k}\int_{t_n}^{t_{n+1}} (s - t_n)u_{\epsilon,hss}(s)ds,$$

it is possible to derive that

$$\|r^{n+1}\|^2_{-1,\epsilon,h,\Omega,\Omega_1} \leq \frac{k}{3}\int_{t_n}^{t_{n+1}} \|u_{\epsilon,hss}(s)\|^2_{-1,\epsilon,h,\Omega,\Omega_1} ds,$$

and

$$\|u_{\epsilon,hss}(s)\|_{-1,\epsilon,h,\Omega,\Omega_1} \leq C\|u_{\epsilon,hs}(s)\|_{1,,\sqrt{\epsilon},\Omega,\Omega_1}.$$

Therefore,

$$\begin{aligned}
&\|\zeta^{n+1}\|^2_{0,,\sqrt{\epsilon},\Omega,\Omega_1} + \sum_{m=1}^{m=n+1} k\|\zeta^m\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1}\\
\leq& Ck\sum_{m=1}^{m=n+1} \|r^{n+1}\|^2_{-1,\epsilon,h,\Omega,\Omega_1} \leq Ck^2\int_0^{t_{n+1}} \|u_{\epsilon,hss}(s)\|^2_{-1,\epsilon,h,\Omega,\Omega_1} ds.
\end{aligned}$$

The use of Lemma 4.2 produces the result. □

We apply the method in [13], and define an adjoint problem for $(SQ_{\epsilon,h})$. Let $\{F^j\}_{j=1}^n \subset V_h(D)$, and $\{W\}_{m=1}^n \subset V_h(D)$ be the solution of

$$\begin{cases} (\phi, \partial W^m)_\Omega + \dfrac{1}{\epsilon}(\phi, \partial W^m)_{\Omega,1} - A_\epsilon(\phi, W^{m-1}) = (\phi, F^m), \\ \forall\phi \in V_h(D),\ m = n,\dots,1, \quad W^n = 0. \end{cases} \tag{4.18}$$

Consequently, we obtain the following lemma.

Lemma 4.12.

$$\|W^m\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} \leq Ck\sum_{j=m+1}^{n} \|F^j\|^2_{0,\frac{1}{\sqrt{\epsilon}},\Omega,\Omega_1}. \tag{4.19}$$

Proof. The proof is similar to that of Lemma 4.2 in [13]. Setting $\phi = \partial W^m$ in (4.18), we can calculate the following.

$$\begin{aligned}
&\|\partial W^m\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} - A_\epsilon(\partial W^m, W^{m-1}) = (\partial W^m, F^m)_D\\
&\leq \frac{1}{2}\|\partial W^m\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \frac{1}{2}\|F^m\|^2_{0,\frac{1}{\sqrt{\epsilon}},\Omega,\Omega_1}.
\end{aligned}$$

It is noteworthy that

$$-A_\epsilon(\partial W^m, W^{m-1}) \geq \frac{1}{2}\partial_t(A_\epsilon(W,W)^m).$$

Therefore, we have proved the result. □

Using this lemma, we can derive the following theorems.

Theorem 4.13. *If $f \equiv 0$, then*

$$\|\zeta^{n+1}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} \leq C\frac{k}{t_{n+1}}\|\tilde{u}_{0h}\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}.$$

Proof. Setting $F^m = M_\epsilon \zeta^m$ in (4.18), we can obtain our result using the same method of Lemma 4.3 in [13]. $\square$

The following theorems are analogues of Lemma 4.4 and Lemma 4.6 in [13], respectively, the proof of which is similar to those in [13] and which is not shown here.

Theorem 4.14. *If $f \equiv 0$, then*

$$\|\zeta^{n+1}\|_{0,\sqrt{\epsilon},\Omega,\Omega_1} \le C\frac{k}{t_{n+1}}\|\tilde{u}_{0h}\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}. \tag{4.20}$$

Theorem 4.15. *If $\tilde{u}_{0h} = U^0 \equiv 0,\ f \in C([0,T];L^2(\Omega))$, then*

$$\|\zeta^n\|_{0,\sqrt{\epsilon},\Omega,\Omega_1} \le Ck\ln\frac{1}{k}\max_{s\in[0,t_n]}\|f(s)\|_{L^2(\Omega)}. \tag{4.21}$$

Remark 8. For H^1-norm error $\|\zeta^n\|_{1,\sqrt{\epsilon},\Omega,\Omega_1}$, from

$$A_\epsilon(\zeta^n,\zeta^n) = \left(M_\epsilon(\partial_t U^n - \frac{d}{dt}u_{\epsilon,h}(t_n)),\zeta^n\right)_D + (\tilde{f}(t_n) - \bar{f}^n,\zeta^n)_D,$$

$$= \left(\partial_t(U^n - u_{\epsilon,h}(t^n)) + \partial u_{\epsilon,h}(t^n) - \frac{d}{dt}u_{\epsilon,h}(t_n), M_\epsilon\zeta^n\right)_D + (\tilde{f}(t_n) - \bar{f}^n,\zeta^n)_D,$$

we have $\|\zeta^n\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} \le C\|\zeta^n\|_{0,\sqrt{\epsilon},\Omega,\Omega_1} + Ck^2$, if $\|\tilde{f}(t_n) - \bar{f}^n\|_{0,\frac{1}{\sqrt{\epsilon}},\Omega,\Omega_1} \le Ck$. (The boundedness of $\|\partial_t U^n\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}$ can be shown in the same way as the proof of Lemma 4.12).

5. Approximation to a discrete problem in numerical computation

From examination of the approximation scheme proposed in Section 1, it is possible to derive the error between (1.9) and (1.10). First, it is assumed that $\hat{\Omega}$ satisfies

$$\|U\|_{1,K\cap(\Omega\setminus\hat{\Omega})} \le C\sqrt{h}\|U\|_{1,K\cap\Omega(\text{ also } K\cap\Omega_1)},\ \forall U \in V_h(D), \tag{5.1}$$

$$\|U\|_{1,K\cap(\Omega_1\setminus\hat{\Omega}_1)} \le C\sqrt{h}\|U\|_{1,K\cap\Omega(\text{ also } K\cap\Omega_1)},\ \forall U \in V_h(D), \tag{5.2}$$

for K is the open triangle of $\mathcal{T}_h$ and $K \cap \Gamma \ne \emptyset$. For example, if $\Omega \in C^2$, and $\forall K \cap \Gamma \ne \emptyset$ such that Γ cuts K into two parts with points of intersection on two sides of K respectively, then one can connect those points of intersection to form a polygon $\hat{\Omega}$. It is not difficult to see that this $\hat{\Omega}$ satisfies the assumption described above by Lemma 5.1 in [25].

To derive the error of $U^n - \hat{U}^n$, as a first step, the following lemmas are presented, which are the prior estimates for (1.9) similar to Lemma 4.1 and Lemma 4.2.

Lemma 5.1. *$\{U^n\}_{n=1}^N$ is the solution of (1.9). Consequently,*

$$\|U^n\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + Ck\sum_{j=1}^n \|U^n\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} \le \|U^0\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + k\sum_{j=1}^n \|\bar{f}^j\|^2_{0,\frac{1}{\sqrt{\epsilon}},\Omega,\Omega_1}.$$

Proof. Substituting $v_h = U^{n+1}$ into (1.9), the result is obtained immediately. □

Lemma 5.2. *$\{U^n\}_{n=1}^N$ is the solution of (1.9). Then*

$$k\sum_{j=1}^n \|\partial_t U^j\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \|\nabla U^n\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} \le k\sum_{j=1}^n \|\bar{f}^j\|^2_{0,\frac{1}{\sqrt{\epsilon}},\Omega,\Omega_1} + \|\nabla U^0\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1}.$$

Proof. Substituting $v_h = \partial_t U^{n+1}$ in (1.9) yields the result immediately. □

Remark 9. It is possible to obtain Lemma 5.1 and Lemma 5.2 for (1.10) with $U,\ \Omega, \Omega_1$ replaced by $\hat{U},\ \hat{\Omega},\ \hat{\Omega}_1$, respectively.

Theorem 5.3. *$\{U^n\}_{n=1}^N$ and $\{\hat{U}^n\}_{n=1}^N$ respectively denote the solutions of (1.9) and (1.10). If*

$$k\sum_{j=1}^n \|\bar{f}^j\|_{1,\frac{1}{\sqrt{\epsilon}},\Omega,\Omega_1} + \|\nabla U^0\|_{0,\sqrt{\epsilon},\Omega,\Omega_1} \le C,$$

then denoting $Z^n \equiv U^n - \hat{U}^n$,

$$\begin{aligned} &\|Z^n\|^2_{0,\sqrt{\epsilon},\Omega\cup\hat{\Omega},\Omega_1\cap\hat{\Omega}_1} + k\sum_{j=1}^n (\|Z^n\|^2_{1,\sqrt{\epsilon},\Omega\cup\hat{\Omega},\Omega_1\cap\hat{\Omega}_1} \\ \le& Ch + \|Z^0\|^2_{0,\sqrt{\epsilon},\Omega\cup\hat{\Omega},\Omega_1\cap\hat{\Omega}_1}. \end{aligned} \tag{5.3}$$

Proof. Subtracting (1.9) from (1.10) yields

$$\begin{aligned} &(\partial_t(Z^{n+1}), v_h)_{\Omega\cap\hat{\Omega}} + \frac{1}{\epsilon}(\partial_t(Z^{n+1}), v_h)_{\Omega_1\cap\hat{\Omega}_1} + (\partial_t U^{n+1} - \frac{1}{\epsilon}\partial_t\hat{U}^{n+1}, v_h)_{\Omega\backslash\hat{\Omega}} \\ &+ (\frac{1}{\epsilon}\partial_t U^{n+1} - \partial_t\hat{U}^{n+1}, v_h)_{\Omega_1\backslash\hat{\Omega}_1} + (\nabla(Z^{n+1}), \nabla v_h)_{\Omega\cap\hat{\Omega}} + \frac{1}{\epsilon}(\nabla(Z^{n+1}), \nabla v_h)_{\Omega_1\cap\hat{\Omega}_1} \\ &+ (\nabla U^{n+1} - \frac{1}{\epsilon}\nabla\hat{U}^{n+1}, \nabla v_h)_{\Omega\backslash\hat{\Omega}} + (\frac{1}{\epsilon}\nabla U^{n+1} - \nabla\hat{U}^{n+1}, \nabla v_h)_{\Omega_1\backslash\hat{\Omega}_1} = 0. \end{aligned}$$

Substituting $v_h = U^{n+1} - \hat{U}^{n+1}$ into the equation above yields

$$
\begin{aligned}
&\|Z^{n+1}\|^2_{0,\sqrt{\epsilon},\Omega\cup\hat{\Omega},\Omega_1\cap\hat{\Omega}_1} + k\sum_{j=0}^{n}(\|\nabla(Z^{j+1})\|^2_{0,\sqrt{\epsilon},\Omega\cup\hat{\Omega},\Omega_1\cap\hat{\Omega}_1} \\
&+ k(\frac{1}{\epsilon}-1)\sum_{j=0}^{n}((\partial_t U^{j+1}, Z^{j+1})_{\Omega_1\setminus\hat{\Omega}_1} + (\nabla U^{j+1}, \nabla Z^{j+1})_{\Omega_1\setminus\hat{\Omega}_1}) \\
&- k(\frac{1}{\epsilon}-1)\sum_{j=0}^{n}((\partial_t \hat{U}^{j+1}, Z^{j+1})_{\Omega\setminus\hat{\Omega}} + (\nabla \hat{U}^{j+1}, \nabla Z^{j+1})_{\Omega\setminus\hat{\Omega}}) \\
=&\|U^0 - \hat{U}^0\|^2_{0,\Omega\cup\hat{\Omega}} + \frac{1}{\epsilon}\|U^0 - \hat{U}^0\|^2_{0,\Omega_1\cap\hat{\Omega}_1}
\end{aligned}
$$

Following from (5.1) and (5.2),

$$
\begin{aligned}
&(\partial_t U^{j+1}, Z^{j+1})_{\Omega_1\setminus\hat{\Omega}_1} + (\nabla U^{j+1}, \nabla Z^{j+1})_{\Omega_1\setminus\hat{\Omega}_1} \\
\leq&\frac{C}{2}(h\|\partial_t U^{j+1}\|^2_{0,\Omega_1} + h\|U^{j+1}\|^2_{0,\Omega_1} + h\|\hat{U}^{j+1}\|^2_{0,\hat{\Omega}_1}) \\
&+ \frac{C}{2}(h\|U^{j+1}\|^2_{1,\Omega_1} + h\|U^{j+1}\|^2_{1,\Omega_1} + h\|\hat{U}^{j+1}\|^2_{1,\hat{\Omega}_1}).
\end{aligned}
$$

Then for $(\partial_t \hat{U}^{j+1}, Z^{j+1})_{\Omega\setminus\hat{\Omega}} + (\nabla \hat{U}^{j+1}, \nabla Z^{j+1})_{\Omega\setminus\hat{\Omega}}$, one can obtain a similar estimate. Then, following from Lemma 5.1 and Lemma 5.2, we prove the result. □

Remark 10. It is readily apparent that the error of $U^n - \hat{U}^n$ is independent of k.

Remark 11. Following from a similar mode of Remark 8, we can derive the H^1-norm error of $U^n - \hat{U}^n$:

$$
\|Z^{n+1}\|_{1,\sqrt{\epsilon},\Omega\cup\hat{\Omega},\Omega_1\cap\hat{\Omega}_1} \leq C\|Z^{n+1}\|^2_{0,\sqrt{\epsilon},\Omega\cup\hat{\Omega},\Omega_1\cap\hat{\Omega}_1}.
$$

6. Numerical experiment

In this section, we present a numerical example. We consider the parabolic problem in a cylinder domain $Q_T = \Omega \times [0,1]$, where $\Omega = \{(x,y,t) \mid x^2 + y^2 < 8\}$. $\Sigma_T = \partial\Omega \times (0,1]$. The original problem (Q) reads as

$$
\begin{cases}
u_t - \Delta u = f \equiv 3 - \dfrac{x^2+y^2}{4} + t, \text{ for } t \in (0,T], \text{ in } \Omega, \\
u = 0 \text{ on } \Sigma_T \\
u(x,y,0) = 2 - \dfrac{x^2+y^2}{4} \text{ in } \Omega.
\end{cases}
\tag{6.1}
$$

The exact solution is

$$
u = (t+1)(2 - \frac{x^2+y^2}{4}).
$$

Setting $D = \{(x,y) \mid -3 < x, y < 3\}$, we have H^1-penalty problem (Q_ϵ) in $D_T = D \times [0,1]$. Introducing the Cartesian mesh to D, we solve the approximation "single-step backward" discrete problem $(S\hat{Q}_{\epsilon,h})$ with $\hat{U}^0 = \tilde{u}_{0h}$ defined in Remark 7 and replace $\bar{f}^n$ in (1.10) by $\bar{I}_h f(t_n)$ (see Remark 7 for the definition of $\bar{I}_h$). We define

$$F \in L^2(Q_T),\ F(t) = \bar{I}_h f(t_n) \text{ if } t \in (t_{n-1}, t_n],\ n = 1, 2, \ldots, N.$$

Then, following from Lemma 4.1,

$$\|\tilde{f} - F\|^2_{0,Q_T} + \epsilon \|\tilde{f} - F\|^2_{0,Q'_T} \le C(k + \sqrt{\epsilon} + \sqrt{h})^2.$$

Consequently, the solution u_F of (1.6) with $\tilde{f}$ replaced by F satisfies

$$\|u_F(t) - u_\epsilon(t)\|^2_{0,\sqrt{\epsilon},\Omega,\Omega_1} + \int_0^t \|u_F(s) - u_\epsilon(s)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} ds \le C(k + \sqrt{\epsilon} + \sqrt{h})^2$$

Therefore, it is readily apparent that $\|u_F(t) - u_\epsilon(t)\|^2_{1,\sqrt{\epsilon},\Omega,\Omega_1} \le C\|u_F(t) - u_\epsilon(t)\|_{0,\sqrt{\epsilon},\Omega,\Omega_1}$. Moreover, it is apparent that $\hat{U}^0$ satisfies the assumption of Theorem 5.3. Consequently, recalling the results of the previous sections, we have $\|\tilde{u}(t) - u_\epsilon(t)\|_{1,D} \le C\epsilon$, where $\tilde{u}(t)$ is the zero extension of $u(t)$ from Ω to D. And,

$$\|u_\epsilon(t) - u_{\epsilon,h}(t)\|_{0,D} + \|u_\epsilon(t) - u_{\epsilon,h}(t)\|^2_{1,D} \le C(\sqrt{\epsilon} + \sqrt{h})^2 \ln \frac{1}{\sqrt{\epsilon} + \sqrt{h}},$$

$$\|u_{\epsilon,h}(t_n) - U^n\|_{0,D} \le Ck \ln \frac{1}{k},\ \|u_{\epsilon,h}(t_n) - U^n\|_{1,D} \le Ck \ln \frac{1}{k},$$

$$\|\hat{U}^n - U^n\|_{0,D} \le Ch,\ \|\hat{U}^n - U^n\|_{1,D} \le C\sqrt{h}.$$

We present our numerical results, which are the error $U^n - \tilde{u}(t_n)$ in L^2 and H^1 norms at $t_n = 1$ in the following graphs. The error is independent of k (see Figure 1 and 2). Next, we show the error with respect to h and ϵ. setting $\epsilon = h$ when doing numerical experiments, we find the L^2-norm error is bounded by h (see Figure 3), and the H^1-norm error is bounded by $\sqrt{h}$ (see Figure 4), which is the same as the elliptic finite element error estimate (see [25]). Therefore, we believe our finite element error estimate is also sharp.

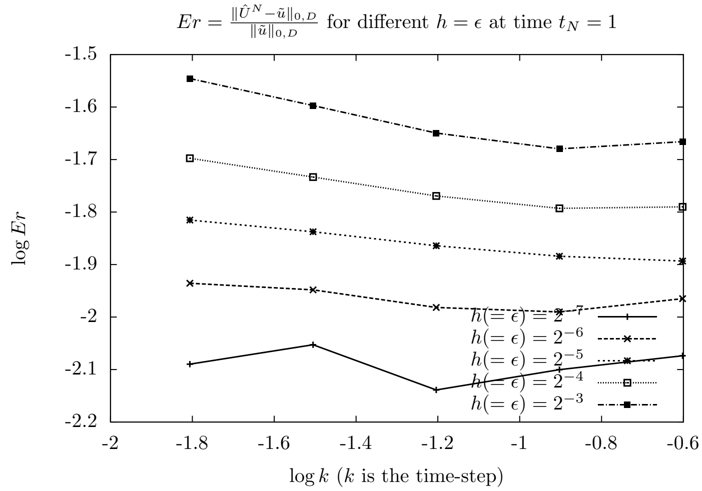

FIGURE 1. $\frac{\|\hat{U}^N - \tilde{u}(t_N)\|_{0,D}}{\|\tilde{u}(t_N)\|_{0,D}}$ with different $h(=\epsilon)$ and k

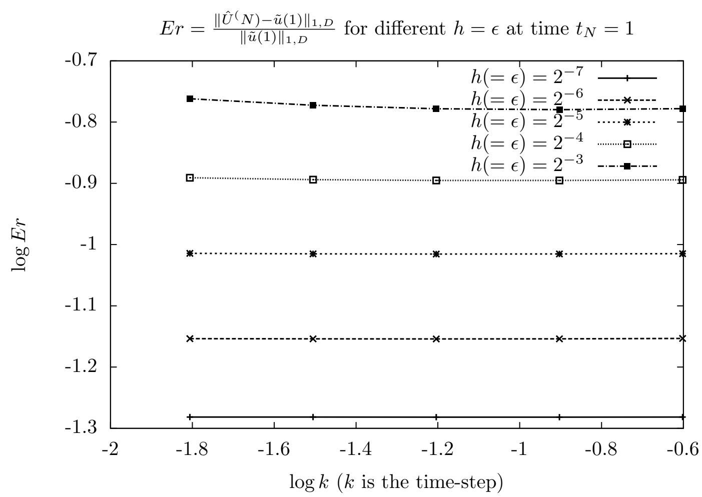

FIGURE 2. $\frac{\|\hat{U}^N - \tilde{u}(t_N)\|_{1,D}}{\|\tilde{u}(t_N)\|_{1,D}}$ with different $h(=\epsilon)$ and k.

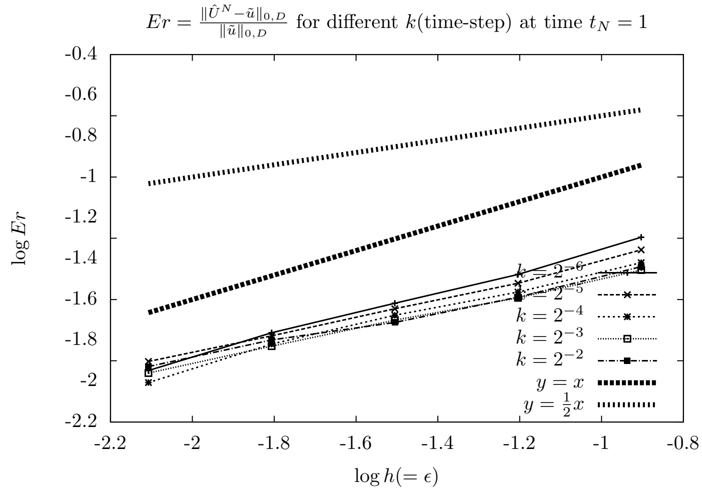

FIGURE 3. $\frac{\|\hat{U}^N - \tilde{u}(t_N)\|_{0,D}}{\|\tilde{u}(t_N)\|_{0,D}}$ with different k and $h(=\epsilon)$.

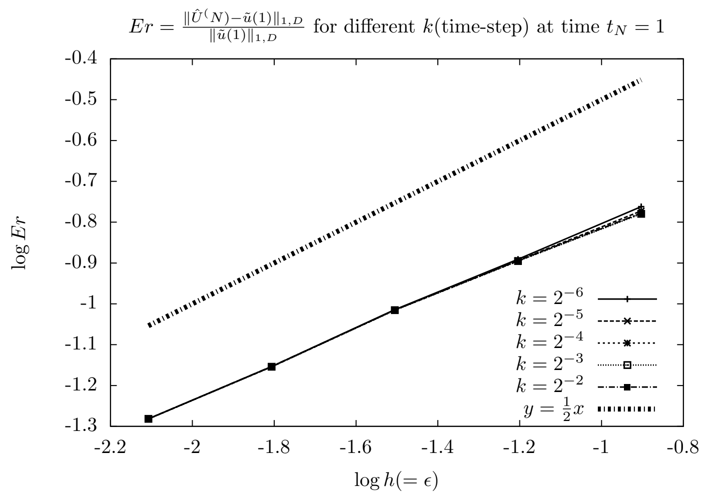

FIGURE 4. $\frac{\|\hat{U}^N - \tilde{u}(t_N)\|_{1,D}}{\|\tilde{u}(t_N)\|_{1,D}}$ with different k and $h(=\epsilon)$.

7. Non-cylindrical domain

We consider the original problem (1.4) in non-cylindrical domain (1.11). Presuming that there exists $D \supset \Omega_t$, $\forall t \in [0,T]$, denoting $\Omega_{t,1} = D\backslash\overline{\Omega}_t$, we can write the H^1-penalty problem (1.6). We have, as in Theorem 2.1,

Theorem 7.1. *For $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega_0)$, satisfying $\|\tilde{f}\|_{0,Q'_T} \leq C\|f\|_{0,Q_T}$, $\|\tilde{u}_0\|_{0,\Omega_{0,1}} \leq C\epsilon\|u_0\|_{0,\Omega_0}$, there exist unique solution u and u_ϵ for (1.4) and (1.6), respectively. Then we have*

$$\|u_\epsilon|_{Q_T} - u\|_{H^{1,0}(Q_T)} + \|u_{\epsilon t}|_{Q_T} - u_t\|_{H^{-1,0}(Q_T)} \leq C(\|f\|_{0,Q_T} + \|u_0\|_{0,\Omega_0})\epsilon, \tag{7.1}$$

$$\|u_\epsilon\|_{H^{1,0}(Q'_T)} + \|u_{\epsilon t}\|_{H^{-1,0}(Q'_T)} \leq C(\|f\|_{0,Q_T} + \|u_0\|_{0,\Omega_0})\epsilon. \tag{7.2}$$

The proof of Theorem 7.1 is the same as that for Theorem 2.1 because we have the extension theorem in a non-cylindrical domain, as the following.

Lemma 7.2. *Ω, Q_T are defined above. For every integer l, m, n, there exists an extension operator $\mathbf{P}$ such that, for all $u \in L^2(0,T;H^l(\Omega))$,*

$$(i)\ \mathbf{P} \in \mathcal{L}\left(L^2(0,T;H^l(\Omega_t)); L^2(0,T;H^l(\mathbb{R}^n))\right),$$

and if $l < 0$, then

$$\langle \mathbf{P}u, v\rangle_{\mathbb{R}^n\times[0,T]} \leq C\|u\|_{L^2(0,T;H^l(\Omega_t))}\|v\|_{L^2(0,T;H^{-l}(\mathbb{R}^n))}, \forall v \in L^2(0,T;H^{-l}(\mathbb{R}^n)).$$

Moreover, if $u_t \in L^2(0,T;H^m(\Omega_t))$, $u(t) \in H^n(\Omega_t)$, $t \in [0,T]$, $\mathbf{P}$ satisfies

$$(ii)\ \frac{d}{dt}(\mathbf{P}u) = \mathbf{P}(u_t),\ \mathbf{P} \in \mathcal{L}\left(L^2(0,T;H^m(\Omega_t)); L^2(0,T;H^m(\mathbb{R}^n))\right),$$

$$(iii)\ Pu(t) \equiv (\mathbf{P}u)(t),\ P \in \mathcal{L}(H^n(\Omega_t); H^n(\mathbb{R}^n)).$$

We can replace $\mathbb{R}^n$ by some domain $D \supset \Omega_t$, $\forall t \in [0,T]$. Additionally, we can extend u to $L^2(0,T;H_0^l(D))$ maintaining the properties (i)(ii)(iii).

Proof. Because Q_T is sufficiently smooth, there exists O_j, $\Phi_j(x,t)$ and β_j, $j = 0,1,\ldots,\nu$ satisfying the following properties:

(1)O_j is a bounded set in $\mathbb{R}^n \times [t_1,t_2]$, such that $\cup_{j=0}^{\nu} O_j \supset Q_T$, $\cup_{j=1}^{\nu} O_j \supset \Sigma_T$;

(2)$\Phi_j(x,t) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ transforms O_j to some cylinder domain $L_+ \times [t_1,t_2]$, where $L_+ = \{y_n > 0\} \cap L \equiv \{y = (y', y_n) \mid |y| < 1\}$, and $\Phi_j(x,t)$ has the invert $\Phi_j^{-1}(y,t) \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}_t)$;

(3)$\beta_j \in C^\infty(Q_T)$ with $supp\beta_j \subset O_j \cap Q_T$, $\sum_{j=0}^{\nu} \beta_j = 1$.

We define $\Phi_j^* : u(y,t) \mapsto \Phi_j^* u \equiv u(\Phi_j(x,t))$ and $\Phi_j^{-1} : u(x,t) \mapsto \Phi_j^{*-1}u \equiv u(\Phi_j^{*-1}(y,t))$, then

$$\mathbf{P}u \equiv \sum_{j=0}^{\nu} \Phi_j^{*-1}(P(\widetilde{\Phi_j^*(\beta_j u)})),$$

where $\widetilde{\Phi_j^*(\beta_j u)} \in L^2 0, T; H^l(\mathbb{R}^n_+)$ is the zero extension of $\Phi_j^*(\beta_j u) \in L^2 0, T; H^l(L_+)$ because $supp(\beta_j u) \in O_j \cap Q_T$, and that P is the extension operator from $H^l(L_+)$ to $H^l(\mathbb{R}^n)$ for $\widetilde{\Phi_j^*(\beta_j u)}(t), \forall t \in [0,T]$. ($P$ is defined in the Lemma 12.2 of Chapter 1 in [10].) Consequently, $\mathbf{P}$ satisfies $(i)(iii)$, and for $\sum_{j=0}^{\nu} \beta_j = 1$, one can confirm that $\mathbf{P}$ satisfies (ii).

The cases of extending to $H^1(D)$ and $H^l_0(D)$ are trivial now. □

It must be mentioned that some difficulties exist for finite element analysis for the case of non-cylindrical domain. It is readily apparent that $R_{\epsilon,h}$ is dependent on t. The analysis becomes more complicated because we must address $\frac{d}{dt}(A_\epsilon(t; u_\epsilon(t), v))$, $\frac{d}{dt}R_{\epsilon,h}(t)$, and so on, where

$$A_\epsilon(t; u_\epsilon(t), v) \equiv (\nabla u_\epsilon(t), \nabla v)_{\Omega_t} + \frac{1}{\epsilon}(\nabla u_\epsilon(t), \nabla v)_{\Omega_{t1}}.$$

Furthermore, for a "single-step backward" discrete problem, the equations of (1.9)) include terms such as $(\nabla U^{n+1}, \nabla v_h)_{\Omega_{t_{n+1}}}$, the integration domains of which are different on every time step t_{n+1}, which makes the finite element error estimate more difficult.

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