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An Asymptotic Expansion for Solutions of Cauchy-Dirichlet Problem for Second Order Parabolic PDEs and its Application to Pricing Barrier Options

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Abstract

This paper develops a rigorous asymptotic expansion method with its numerical scheme for the Cauchy-Dirichlet problem in second order parabolic partial differential equations (PDEs). As an application, we propose a new approximation formula for pricing a barrier option under a certain type of stochastic volatility model including the log-normal SABR model.

Keywords : Asymptotic expansion, The Cauchy-Dirichlet problem, Second order parabolic PDEs, Barrier options, Stochastic volatility model.

1 Introduction

Numerical methods for the Cauchy-Dirichlet problem have been a topic of great interest in stochastic analysis and its applications. For example, in mathematical finance the Cauchy-Dirichlet problem naturally arises in valuation of continuously monitoring barrier options:

$$C_{\mathrm{Barrier}}(T,x) = \mathrm{E}[f(X_T^x)1_{\{\tau>T\}}] = \mathrm{E}[f(X_T^x)1_{\{\min_{t\in[0,T]} X_t > L\}}]. \tag{1.1}$$

Here, $T>0$ is a maturity of the option, and $(X_t^x)_t$ denotes a price process of the underlying asset starting from x (usually given as the solution of a certain stochastic differential equation (SDE)). Also, L stands for a constant lower barrier, that is $L<x$, and τ is the hitting time to L:

$$\tau = \inf\{t\in[0,T] : X_t^x \le L\}. \tag{1.2}$$

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It is well-known that a possible approach in computation of $C_{\text{Barrier}}(T,x)$ is the Euler scheme, which stores the sample paths of the process $(X_t^x)_t$ through an n-time discretization with the step size T/n. In applying this scheme to pricing a continuously monitoring barrier option, one kills the simulated process (say, $(\bar{X}_{t_i}^x)_i$) if $\bar{X}_{t_i}^x$ exits from the domain (L,∞) until the maturity T. The usual Eular scheme is *suboptimal* since it does not control the diffusion paths between two successive dates t_i and t_{i+1}: the diffusion paths could have crossed the barriers and come back to the domain without being detected. It is also known that the error between $C_{\text{Barrier}}(T,x)$ and $\bar{C}_{\text{Barrier}}(T,x)$ (the barrier option price obtained by the Euler scheme) is of order $\sqrt{T/n}$, as opposed to the order T/n for standard plain-vanilla options. (See [7]) Therefore, to improve the order of the error, many schemes for the Monte-Carlo method have been proposed. (See [16] for instance.)

One of the other tractable approaches for calculating $C_{\text{Barrier}}(T,x)$ is to derive an analytical approximation. If we obtain a closed form approximation formula, then it is a powerful tool for evaluation of continuously monitoring barrier options because we do not have to rely on Monte-Carlo simulations anymore. However, from a mathematical viewpoint, deriving an approximation formula by applying stochastic analysis is not an easy task since the Malliavin calculus cannot be directly applied, which is due to the non-existence of the Malliavin derivative $D_t\tau$ (see [4]) and to the fact that the minimum (maximum) process of the Brownian motion has only first-order differentiability in the Malliavin sense. Thus, neither approach in [11] nor in [19] can be applied directly to valuation of continuously monitoring barrier options while they are applicable to pricing discrete barrier options. (See [18] for the detail.)

In this paper, we propose a new general method for approximating the solution to the Cauchy-Dirichlet problem. Roughly speaking, our objective is to pricing barrier options when the dynamics of the underlying asset price is described by the following perturbed SDE:

$$\begin{cases} dX_t^{\varepsilon,x} = b(X_t^{\varepsilon,x},\varepsilon)dt + \sigma(X_t^{\varepsilon,x},\varepsilon)dB_t, \\ X_0^{\varepsilon,x} = x, \end{cases} \tag{1.3}$$

where ε is a small parameter, which will be defined precisely later in the paper. In this case, the barrier option price (1.1) is characterized as a solution of the Cauchy-Dirichlet problem:

$$\begin{cases} \dfrac{\partial}{\partial t}u^\varepsilon(t,x) + \mathscr{L}^\varepsilon u^\varepsilon(t,x) = 0, & (t,x) \in [0,T) \times (L,\infty), \\ u^\varepsilon(T,x) = f(x), & x > L, \\ u^\varepsilon(t,L) = 0, & t \in [0,T], \end{cases} \tag{1.4}$$

where the differential operator $\mathscr{L}^\varepsilon$ is determined by the diffusion coefficients b and σ. Next, we introduce an asymptotic expansion formula:

$$u^\varepsilon(t,x) = u^0(t,x) + \varepsilon v_1^0(t,x) + \cdots + \varepsilon^{n-1} v_{n-1}^0(t,x) + O(\varepsilon^n), \tag{1.5}$$

where O denotes the Landau symbol. The function $u^0(t,x)$ is the solution of (1.4) with $\varepsilon = 0$: if $b(x,0)$ and $\sigma(x,0)$ have some simple forms such as constants (as in the Black-Scholes model), we already know the closed form of $u^0(t,x)$ and hence obtain the price. Then, we are able to get the approximate value for $u^\varepsilon(t,x)$ through evaluation of $v_1^0(t,x),\ldots,v_{n-1}^0(t,x)$. In fact, they are also characterized as the solution of a certain PDE with the Dirichlet condition. By formal asymptotic expansions, (1.5) as well as

$$\mathscr{L}^\varepsilon = \mathscr{L}^0 + \varepsilon\tilde{\mathscr{L}}_1^0 + \cdots + \varepsilon^{n-1}\tilde{\mathscr{L}}_{n-1}^0 + \cdots,$$

we can derive the PDEs corresponding to $v_k^0(t,x)$ of the form:

$$\begin{cases} \dfrac{\partial}{\partial t} v_k^0(t,x) + \mathscr{L}^0 v_k^0(t,x) + g_k^0(t,x) = 0, & (t,x) \in [0,T) \times (L,\infty), \\ v_k^0(T,x) = 0, & x > L, \\ v_k^0(t,L) = 0, & t \in [0,T], \end{cases} \tag{1.6}$$

where $g_k^0(t,x)$ will be given explicitly later in this paper. Moreover, by applying the Feynman-Kac approach, we are able to obtain their stochastic representations. We will justify the above argument in a mathematically rigorous way with necessary assumptions in Section 2.

The theory of the Cauchy-Dirichlet problem for this kind of second order parabolic PDE is well understood in the case of bounded domains (see [5], [6] and [14] for instance). As for an unbounded domain case such as (1.4), [17] provides the existence and uniqueness results for a solution of the PDE and the Feynman-Kac type formula (cited as Theorem 1 below). However, some mathematical difficulty exists for applying the results of [17] to the PDE (1.6). More precisely, the function $g_k^0(t,x)$ may be divergent at $t = T$. (If $g_k^0(t,x)$ is continuous on $[0,T] \times [L,\infty)$, the existence and uniqueness of (1.6) are guaranteed: see [5].) To overcome this difficulty, we generalize the Levi's parametrix method (which is used to construct a classical solution of the PDE) in Theorem 2. Furthermore, we show another representation of $v_k^0(t,x)$ by using the corresponding semi-group in Section 3. We notice that such a form is convenient for evaluation of $v_k^0(t,x)$ in concrete examples.

In Section 4, we apply our method to pricing a barrier option in a stochastic volatility model:

$$\begin{aligned} dS_t^\varepsilon &= (c-q)S_t^\varepsilon dt + \sigma_t^\varepsilon S_t^\varepsilon dB_t^1, \ S_0^\varepsilon = S > 0, \\ d\sigma_t^\varepsilon &= \varepsilon\lambda(\theta - \sigma_t^\varepsilon)dt + \varepsilon\nu\sigma_t^\varepsilon(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2), \ \sigma_0^\varepsilon = \sigma > 0, \end{aligned}$$

where $c, q > 0$, $\varepsilon \in [0,1)$, $\lambda, \theta, \nu > 0$, $\rho \in [-1,1]$ and $B = (B^1, B^2)$ is a two dimensional Brownian motion. Then, we obtain a new approximation formula:

$$\begin{aligned} C_{\text{Barrier}}^{SV,\varepsilon}(T,S) &= \mathrm{E}\left[f(S_T^\varepsilon)1_{\{\min_{0\le t\le T} S_t^\varepsilon > L\}}\right] \\ &\simeq P_T^D \bar{f}(\log S) + \varepsilon \int_0^T P_{T-r}^D \tilde{\mathscr{L}}_1^0 P_r^D \bar{f}(\log S) dr, \end{aligned}$$

where $(P_t^D)_t$ is a semi-group defined in Section 3, f is a payoff function and $\bar{f}(x) = f(e^x)$. Here, $P_T^D \bar{f}(\log S)$ is regarded as the down-and-out barrier option price, $C_{\text{Barrier}}^{BS}(T,S)$ in the Black-Scholes model. Moreover, we confirm practical validity of our method through a numerical example given in Section 4. Notice also that our example does not satisfy the assumptions introduced in Section 2. Thus, we generalize our main result and present weaker (but a little bit complicated) version of the assumptions in Section 5.1. Furthermore, Section 5.2–5.4 list the proofs of our results.

Finally, we remark that in the contrast to the previous works ([2], [3], [8], [9] for example), which start with some specific models (the Black-Scholes model or some type of fast mean-reversion model) and derive approximation formulas for (discretely or continuously monitoring) barrier option prices, we firstly develop a general asymptotic expansion scheme for the Cauchy-Dirichlet problem under multi-dimensional diffusion setting; then, as an application, we provide a new approximation formula under a certain class of stochastic volatility model that can be widely applied in practice (e.g. in currency option markets).

2 Main Results

Let $b : \mathbb{R}^d \times I \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times I \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ be Borel measurable functions ($d, m \in \mathbb{N}$,) where I is an interval on $\mathbb{R}$ including the origin 0 (for instance $I = (-1, 1)$.) We consider the SDE (1.3) for any $x \in \mathbb{R}^d$ and $\varepsilon \in I$; we will introduce the assumptions for existence and uniqueness of a weak solution of (1.3) later.

We are interested in evaluation of the following: for a small ε,

$$\begin{aligned} & u^\varepsilon(t,x) \\ = \ & \mathrm{E}\left[\exp\left(-\int_0^{T-t} c(X_r^{\varepsilon,x},\varepsilon)dr\right) f(X_{T-t}^{\varepsilon,x}) 1_{\{\tau_D(X^{\varepsilon,x}) \geq T-t\}}\right], \quad (t,x) \in [0,T] \times \bar{D} \end{aligned} \tag{2.1}$$

for Borel measurable functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ and $c : \mathbb{R}^d \times I \longrightarrow \mathbb{R}$, a positive real number $T > 0$ and a domain $D \subset \mathbb{R}^d$; $\bar{D} \subset \mathbb{R}^d$ is the closure of D and $\tau_D(w)$, $w \in C([0,T]; \mathbb{R}^d)$, stands for the first exit time from D, that is

$$\tau_D(w) = \inf\{t \in [0,T]; w(t) \notin D\}.$$

As mentioned in Section 1, the right-hand side of (2.1) corresponds to a barrier option price of knock-out type with maturity T in finance. We regard $(X_t^{\varepsilon,x})_t$ as the underlying aseet prices and the expectation $\mathrm{E}[\cdot]$ is taken under a risk-neutral probability measure. The boundary ∂D of the domain means the trigger points of the option and f represents a payoff at maturity. The function c represents a short-term interest rate. Our setting includes the case of $D = \mathbb{R}^d$, which corresponds to a price of an European option:

$$u^\varepsilon(t,x) = \mathrm{E}\left[\exp\left(-\int_0^{T-t} c(X_r^{\varepsilon,x},\varepsilon)dr\right) f(X_{T-t}^{\varepsilon,x})\right].$$

For applications to option pricing, see Section 4 for the details.

Now we introduce our assumptions.

[A] There is a positive constant A_1 such that

$$|\sigma^{ij}(x,\varepsilon)|^2 + |b^i(x,\varepsilon)|^2 \leq A_1(1+|x|^2), \quad x \in \mathbb{R}^d, \ \varepsilon \in I, \ i,j = 1,\ldots,d.$$

Moreover, for each $\varepsilon \in I$ it holds that $\sigma^{ij}(\cdot,\varepsilon), b^i(\cdot,\varepsilon) \in \mathcal{L}$ for $i,j = 1,\ldots,d$, where $\mathcal{L}$ is the set of locally Lipschitz continuous functions defined on $\mathbb{R}^d$.

[B] The function $f(x)$ is continuous on $\bar{D}$ and there are $C_f > 0$ and $m \in \mathbb{N}$ such that $|f(x)| \leq C_f(1+|x|^{2m})$, $x \in \mathbb{R}^d$. Moreover, $f(x) = 0$ on $\mathbb{R}^d \setminus D$.

Note that under [A], the existence and uniqueness of a solution of (1.3) are guaranteed on any filtered probability space equipped with a standard d-dimensional Brownian motion, and Corollary 2.5.12 in [10] and Lemma 3.2.6 in [15] imply

$$\mathrm{E}[\sup_{0\leq r\leq t} |X_r^{\varepsilon,x} - x|^{2l}] \leq C_l t^{l-1}(1+|x|^{2l}), \quad (t,x) \in [0,T] \times \mathbb{R}^d, \ l \in \mathbb{N} \tag{2.2}$$

for some $C_l > 0$ which depends only on l and A_1. Moreover, $(X_r^x)_r$ has the strong Markov property. Although the assumptions [A]–[B] are not always satisfied in our example in Section 4, we can weaken them, and will introduce more general conditions in Section 5.1.

We continue to state our assumptions.

[C] There is a positive constant A_2 such that $c(x,\varepsilon) \geq -A_2$ for $x \in \bar{D}$, $\varepsilon \in I$. Moreover, for each $\varepsilon \in I$, it holds that $c(\cdot,\varepsilon) \in \mathcal{L}$.

[D] The boundary ∂D has the outside strong sphere property, that is, for each $x \in \partial D$ there is a closed ball E such that $E \cap D = \phi$ and $E \cap \bar{D} = \{x\}$.

[E] The matrix $(a^{ij}(x,\varepsilon))_{ij}$ is elliptic in the sense that for each $\varepsilon \in I$ and compact set $K \subset \mathbb{R}^d$ there is a positive number $\mu_{\varepsilon,K}$ such that $\displaystyle\sum_{i,j=1}^{d} a^{ij}(x,\varepsilon)\xi^i\xi^j \geq \mu_{\varepsilon,K}|\xi|^2$ for any $x \in K$ and $\xi \in \mathbb{R}^d$. In the case of $\varepsilon = 0$, we further assume

$$\mu_0|\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x,0)\xi^i\xi^j \leq \mu_0^{-1}|\xi|^2, \quad x \in \bar{D},\ \xi \in \mathbb{R}^d$$

for some $\mu_0 > 0$.

Let us define a second order differential operator $\mathscr{L}^\varepsilon$ by

$$\mathscr{L}^\varepsilon = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x,\varepsilon)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x,\varepsilon)\frac{\partial}{\partial x^i} - c(x,\varepsilon),$$

where $a^{ij} = \displaystyle\sum_{k=1}^{d} \sigma^{ik}\sigma^{jk}$. We consider the following Cauchy-Dirichlet problem for a PDE of parabolic type

$$\begin{cases} \dfrac{\partial}{\partial t}u^\varepsilon(t,x) + \mathscr{L}^\varepsilon u^\varepsilon(t,x) = 0, & (t,x) \in [0,T) \times D, \\ u^\varepsilon(T,x) = f(x), & x \in D, \\ u^\varepsilon(t,x) = 0, & (t,x) \in [0,T] \times \partial D. \end{cases} \tag{2.3}$$

The following is obtained by Theorem 3.1 in [17].

Theorem 1. *Assume* [A]–[E]. *For each* $\varepsilon \in I$, $u^\varepsilon(t,x)$ *is a* (*classical*) *solution of* (2.3) *and*

$$\sup_{(t,y)\in[0,T]\times\bar{D}} |u^\varepsilon(t,x)|/(1+|x|^{2m}) < \infty. \tag{2.4}$$

Moreover, if $w^\varepsilon(t,x)$ *is also a solution of* (2.3) *satisfying the growth condition*

$$\sup_{(t,y)\in[0,T]\times\bar{D}} |w^\varepsilon(t,x)|/(1+|x|^{2m'}) < \infty$$

for some $m' \in \mathbb{N}$, *then* $u^\varepsilon = w^\varepsilon$.

To study an asymptotic expansion of $u^\varepsilon(t,x)$, we assume

[F] For each $i,j = 1,\ldots,d$ the functions $\sigma^{ij}(x,0)$, $b^i(x,0)$ and $c(t,x,0)$ are bounded on $[0,T] \times \bar{D}$, and there exist constants $A_3 > 0$ and $\alpha \in (0,1]$ such that

$$|\sigma^{ij}(x,0) - \sigma^{ij}(y,0)| + |b^i(x,0) - b^i(y,0)| + |c(x,0) - c(y,0)| \leq A_3|x-y|^\alpha, \quad x,y \in \bar{D}.$$

[G] Let $n \in \mathbb{N}$. The functions $a^{ij}(x,\varepsilon)$, $b^i(x,\varepsilon)$ and $c(x,\varepsilon)$ are n-times continuously differentiable in ε. Furthermore, each of derivatives $\partial^k a^{ij}/\partial\varepsilon^k$, $\partial^k b^i/\partial\varepsilon^k$, $\partial^k c/\partial\varepsilon^k$, $k = 1,\dots,n-1$, has a polynomial growth rate in $x \in \mathbb{R}^d$ uniformly in $\varepsilon \in I$.

By [G], we can define $\tilde{\mathscr{L}}_k^0$, $k \in \mathbb{N}$, as

$$\tilde{\mathscr{L}}_k^0 = \frac{1}{k!}\left\{\frac{1}{2}\sum_{i,j=1}^d \frac{\partial^k a^{ij}}{\partial\varepsilon^k}(x,0)\frac{\partial^2}{\partial x^i\partial x^j} + \sum_{i=1}^d \frac{\partial^k b^i}{\partial\varepsilon^k}(x,0)\frac{\partial}{\partial x^i} - \frac{\partial^k c}{\partial\varepsilon^k}(x,0)\right\}. \tag{2.5}$$

Our purpose is to present an asymptotic expansion such that

$$u^\varepsilon(t,x) = u^0(t,x) + \varepsilon v_1^0(t,x) + \cdots + \varepsilon^{n-1}v_{n-1}^0(t,x) + O(\varepsilon^n), \quad \varepsilon \to 0. \tag{2.6}$$

Here, $v_k^0(t,x)$, $k = 1,\dots,n-1$, are given as the solution of

$$\begin{cases} \dfrac{\partial}{\partial t}v_k^0(t,x) + \mathscr{L}^0 v_k^0(t,x) + g_k^0(t,x) = 0, & (t,x) \in [0,T) \times D, \\ v_k^0(T,x) = 0, & x \in D, \\ v_k^0(t,x) = 0, & (t,x) \in [0,T] \times \partial D, \end{cases} \tag{2.7}$$

where $g_k^0(t,x)$ is given inductively by

$$g_k^0(t,x) = \tilde{\mathscr{L}}_k^0 u^0(t,x) + \sum_{l=1}^{k-1}\tilde{\mathscr{L}}_{k-l}^0 v_l^0(t,x). \tag{2.8}$$

To state the existence of such a function $v_k^0(t,x)$, we prepare the set $\mathcal{H}^{m,\alpha,p}$ of $g \in C([0,T)\times \bar{D})$ satisfying the following conditions:

- There is some $M^g \in C([0,T)) \cap L^p([0,T),dt)$ such that

$$|g(t,x)| \le M^g(t)(1+|x|^{2m}), \quad t \in [0,T), \ x,y \in \bar{D}. \tag{2.9}$$

- For any compact set $K \subset D$ there is some $\tilde{M}^{g,K} \in C([0,T)) \cap L^p([0,T),dt)$ such that

$$|g(t,x) - g(t,y)| \le \tilde{M}^{g,K}(t)|x-y|^\alpha, \quad t \in [0,T), \ x,y \in K.$$

Then, we have the next theorem of which proof is given in Section 5.2.

Theorem 2. *Assume [A]–[G]. Let $g \in \mathcal{H}^{m,\alpha,p}$ for some $p > 1/\alpha$. Then, the following PDE*

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) + \mathscr{L}^0 v(t,x) + g(t,x) = 0, & (t,x) \in [0,T) \times D, \\ v(T,x) = 0, & x \in D, \\ v(t,x) = 0, & (t,x) \in [0,T] \times \partial D \end{cases} \tag{2.10}$$

has a classical solution v such that

$$|v(t,x)| \le C(1+|x|^{2m}) \tag{2.11}$$

for some $C > 0$ which depends only on $a(\cdot,0), b(\cdot,0), c(\cdot,0)$, D and M^g. Moreover, if w is another classical solution of (2.10) which satisfies $|w(t,x)| \le C'\exp(\beta|x|^2)$, $(t,x) \in [0,T]\times\bar{D}$, for some $C', \beta > 0$, then $v = w$.

We also put the next assumption:

[H] $u^0 \in \mathcal{G}^{m,\alpha,p}$ for some $p > 1/\alpha$, where

$$\begin{aligned}\mathcal{G}^{m,\alpha,p} &= \Big\{ g \in C^{1,2}([0,T) \times D) \cap C([0,T] \times \bar{D}) \ ; \\ &\qquad \frac{\partial g}{\partial x^i} \in \mathcal{H}^{m,0,2}, \ \frac{\partial^2 g}{\partial x^i \partial x^j} \in \mathcal{H}^{m,\alpha,p}, \ i,j = 1,\ldots,d \Big\}.\end{aligned}$$

It is easy to see that the assumptions [F]–[H] imply $g_1^0 \in \mathcal{H}^{m_1,\alpha,p}$ for some $m_1 \in \mathbb{N}$. Therefore (2.7) with $k = 1$ has a unique classical solution v_1^0 under [A]–[H]. Similarly, if $v_1^0, \ldots, v_k^0$ exist and are subject to $\mathcal{G}^{m_k,\alpha,p}$ for some $m_k \in \mathbb{N}$, then the unique classical solution v_{k+1}^0 of (2.7) exists. We introduce our final assumption.

[I] It holds that $v_k^0 \in \mathcal{G}^{m_n,\alpha,p}$, $k = 1, \ldots, n-1$ for some $m_n \in \mathbb{N}$.

We remark that $v_k^0(t,x)$ has the stochastic representation:

$$v_k^0(t,x) = \mathrm{E}\left[\int_0^{(T-t)\wedge\tau_D(X^{0,x})} \exp\left(-\int_0^r c(X_v^{0,x}, 0)dv\right) g_k^0(t+r, X_r^{0,x})dr\right] \tag{2.12}$$

for $k = 1, \ldots, n-1$ under [I]. The proof is almost the same as Theorem 5.1.9 in [13].

Now we are prepared to state our main result whose proof is given in Section 5.3.

Theorem 3. *Assume [A]–[I]. There are positive constants C_n and $\tilde{m}_n$ which are independent of ε such that*

$$\left| u^\varepsilon(t,x) - \left(u^0(t,x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t,x)\right)\right| \le C_n(1 + |x|^{2\tilde{m}_n})\varepsilon^n, \quad (t,x) \in [0,T] \times \bar{D}.$$

3 Semi-Group Representation

In this section we construct a semi-group corresponding to $(X_t^{0,x})_t$ and D, and give another form of (2.12). We always assume [A]–[I] (or the generalized assumptions in Section 5.1.) We only consider the case where $c(t,x,0)$ is non-negative and independent of t; we simply denote $c(x,0) = c(x)$. Let $C_b^0(\bar{D})$ be the set of bounded continuous functions $f : \bar{D} \longrightarrow \mathbb{R}$ such that $f(x) = 0$ on ∂D. Obviously, $C_b^0(\bar{D})$ equipped with the sup-norm becomes a Banach space. For $t \in [0,T]$ and $f \in C_b^0(\bar{D})$, we define $P_t^D f : \bar{D} \longrightarrow \mathbb{R}$ by

$$P_t^D f(x) = \mathrm{E}\left[\exp\left(-\int_0^t c(X_v^{0,x})dv\right) f(X_t^{0,x}) 1_{\{\tau_D(X^{0,x}) \ge t\}}\right].$$

We notice that $P_t^D f(x)$ is equal to $u^0(T-t,x)$ with the payoff function f.

Theorem 4. *The mapping $P_t^D : C_b^0(\bar{D}) \longrightarrow C_b^0(\bar{D})$ is well-defined and $(P_t^D)_{0\le t\le T}$ is a contraction semi-group.*

Proof. Let $f \in C_b^0(\bar{D})$. The relations $P_0^D f = f$, $P_t^D f|_{\partial D} = 0$ and $\sup_{\bar{D}} |P_t^D f| \le \sup_{\bar{D}} |f|$ are obvious. The continuity of $P_t^D f$ is by Lemma 4.3 in [17]. The semi-group property is verified by a straightforward calculation. ■

Note that $(P_t^D)_t$ also becomes a semi-group on the set $C_p^0(\bar{D})$ of continuous functions f, each of which has a polynomial growth rate and satisfies $f(x) = 0$ on ∂D.

Let $g \in \mathcal{H}^{m,\alpha,p}$. Observe that

$$\begin{aligned}
&\int_0^{(T-t)\wedge\tau_D(X^{0,x})} \exp\left(-\int_0^r c(X_v^{0,x})dv\right) g(t+r, X_r^{0,x})dr \\
&= \int_0^{T-t} \exp\left(-\int_0^r c(X_v^{0,x})dv\right) g(t+r, X_r^{0,x}) 1_{\{\tau_D(X^{0,x})\ge r\}}dr,
\end{aligned}$$

and we obtain

$$\begin{aligned}
&\mathrm{E}\left[\int_0^{(T-t)\wedge\tau_D(X^{0,x})} \exp\left(-\int_0^r c(X_v^{0,x})dv\right) g(t+r, X_r^{0,x})dr\right] \\
&= \int_0^{T-t} \mathrm{E}\left[\exp\left(-\int_0^r c(X_v^{0,x})dv\right) g(t+r, X_r^{0,x}) 1_{\{\tau_D(X^{0,x})\ge r\}}\right] dr \\
&= \int_0^{T-t} P_r^D g(t+r,\cdot)(x)dr.
\end{aligned}$$

Thus, under the assumption [H], we see

$$\begin{aligned}
v_1^0(T-t,x) &= \int_0^t P_r^D \tilde{\mathscr{L}}_1^0 u^0(T-t+r,\cdot)(x)dr \\
&= \int_0^t P_r^D \tilde{\mathscr{L}}_1^0 P_{t-r}^D f(x)dr = \int_0^t P_{t-r}^D \tilde{\mathscr{L}}_1^0 P_r^D f(x)dr. \qquad (3.1)
\end{aligned}$$

Similarly we get the following.

Theorem 5. *For each $k = 1, \dots, n-1$*

$$\begin{aligned}
&v_k^0(T-t,x) \\
&= \sum_{l=1}^k \sum_{(\beta^i)_{i=1}^l \subset \mathbb{N}^l, \sum_i \beta^i = k} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} P_{t-t_1}^D \tilde{\mathscr{L}}_{\beta^1}^0 P_{t_1-t_2}^D \tilde{\mathscr{L}}_{\beta^2}^0 \cdots P_{t_{l-1}-t_l}^D \tilde{\mathscr{L}}_{\beta^l}^0 P_{t_l}^D f(x)dt_l \cdots dt_1.
\end{aligned} \qquad (3.2)$$

Proof. By (3.1), we have the assertion for $k = 1$. If the assertion holds for $1, \dots, k-1$, then

$$\begin{aligned}
v_k^0(T-t,x) &= \int_0^t P_{t_0}^D \{\tilde{\mathscr{L}}_k^0 u^0 + \sum_{l=1}^{k-1} \tilde{\mathscr{L}}_{k-l}^0 v_l^0\}(T-t+t_0,\cdot)(x)dt_0 \\
&= \int_0^t P_{t-t_0}^D \tilde{\mathscr{L}}_k^0 P_{t_0}^D f(x)dt_0
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{k-1}\sum_{m=1}^{l}\sum_{(\beta^i)_{i=1}^m\subset\mathbb{N}^m,\sum_i\beta^i=l}\int_0^t\int_0^{t_0}\int_0^{t_1}\cdots\int_0^{t_{l-1}}\\
&\qquad\qquad P^D_{t-t_0}\tilde{\mathscr{L}}^0_{k-l}P^D_{t_0-t_1}\tilde{\mathscr{L}}^0_{\beta^1}P^D_{t_1-t_2}\tilde{\mathscr{L}}^0_{\beta^2}\cdots P^D_{t_{l-1}-t_l}\tilde{\mathscr{L}}^0_{\beta^l}P^D_{t_l}f(x)dt_l\cdots dt_1dt_0\\
=&\int_0^t P^D_{t-t_0}\tilde{\mathscr{L}}^0_kP^D_{t_0}f(x)dt_0\\
&+\sum_{l=2}^{k}\sum_{m=1}^{l}\sum_{(\beta^i)_{i=1}^m\subset\mathbb{N}^m,\sum_i\beta^i=k}\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{l-1}}\\
&\qquad\qquad P^D_{t-t_1}\tilde{\mathscr{L}}^0_{\beta^1}P^D_{t_1-t_2}\tilde{\mathscr{L}}^0_{\beta^2}P^D_{t_2-t_3}\tilde{\mathscr{L}}^0_{\beta^3}\cdots P^D_{t_{l-1}-t_l}\tilde{\mathscr{L}}^0_{\beta^l}P^D_{t_l}f(x)dt_l\cdots dt_1\\
=&\sum_{l=1}^{k}\sum_{(\beta^i)_{i=1}^l\subset\mathbb{N}^l,\sum_i\beta^i=k}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{l-1}}P^D_{t-t_1}\tilde{\mathscr{L}}^0_{\beta^1}P^D_{t_1-t_2}\tilde{\mathscr{L}}^0_{\beta^2}\cdots P^D_{t_{l-1}-t_l}\tilde{\mathscr{L}}^0_{\beta^l}P^D_{t_l}f(x)dt_l\cdots dt_1.
\end{aligned}
$$

Thus, our assertion is also true for k. Then we complete the proof of Theorem 5 by mathematical induction. ■

In particular, when $d=1$, $D=(l,\infty)$, $b(x,0)\equiv\mu$, $\sigma(x,0)\equiv\sigma$ and $c(x)\equiv 0$ with constants $l,\mu\in\mathbb{R}$ and $\sigma>0$, the process $X_t^{0,x}$ is explicitly represented as $X_t^{0,x}=x+\mu t+\sigma B_t$, and it is well-known that

$$
P(\tau_D(X^{0,x})\geq t|X_t^{0,x})=1-\exp\left(-\frac{2(x-l)(X_t^{0,x}-l)}{\sigma^2 t}\right)\quad\text{on }\{X_t^{0,x}>l\}.
$$

Therefore, for $g\in C_p^0(\bar{D})$ we have

$$
P_t^Dg(x)=\mathrm{E}[P(\tau_D(X^{0,x})\geq t|X_t^x)g(X_t^{0,x})1_{\{X_t^{0,x}>l\}}]=\int_l^\infty g(y)p(t,x,y)dy, \tag{3.3}
$$

where

$$
p(t,x,y)=\frac{1}{\sqrt{2\pi\sigma^2t}}(1-e^{-\frac{2(x-l)(y-l)}{\sigma^2t}})e^{-\frac{(y-x-\mu t)^2}{2\sigma^2t}}. \tag{3.4}
$$

We remark that (3.3) is useful for explicit evaluation of (3.1), which is demonstrated in the next section.

4 Application to Barrier Option Pricing under Stochastic Volatility

Consider the following stochastic volatility model:

$$
\begin{aligned}
dS_t^\varepsilon &= (c-q)S_t^\varepsilon dt+\sigma_t^\varepsilon S_t^\varepsilon dB_t^1,\ S_0^\varepsilon=S,\\
d\sigma_t^\varepsilon &= \varepsilon\lambda(\theta-\sigma_t^\varepsilon)dt+\varepsilon\nu\sigma_t^\varepsilon(\rho dB_t^1+\sqrt{1-\rho^2}dB_t^2),\ \sigma_0^\varepsilon=\sigma,
\end{aligned} \tag{4.1}
$$

where $c,q>0$, $\varepsilon\in[0,1)$, $\lambda,\theta,\nu>0$, $\rho\in[-1,1]$ and $B=(B^1,B^2)$ is a two dimensional Brownian motion. Here c and q represent a domestic interest rate and a foreign interest rate, respectively when we consider the currency options. Clearly, applying Itô's formula, we have its logarithmic process:

$$\begin{aligned} dX_t^\varepsilon &= (c - q - \frac{1}{2}(\sigma_t^\varepsilon)^2)dt + \sigma_t^\varepsilon dB_t^1,\ X_0^\varepsilon = x = \log S, \\ d\sigma_t^\varepsilon &= \varepsilon\lambda(\theta - \sigma_t^\varepsilon)dt + \varepsilon\nu\sigma_t^\varepsilon(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2),\ \sigma_0^\varepsilon = \sigma. \end{aligned} \tag{4.2}$$

Also, its generator is expressed as

$$\mathscr{L}^\varepsilon = \left(c - q - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} + \varepsilon\rho\nu\sigma^2\frac{\partial^2}{\partial x \partial\sigma} + \varepsilon\lambda(\theta - \sigma)\frac{\partial}{\partial\sigma} + \varepsilon^2\frac{1}{2}\nu^2\sigma^2\frac{\partial^2}{\partial\sigma^2}. \tag{4.3}$$

In this case, $\tilde{\mathscr{L}}_1^0$ defined by (2.5) is given as

$$\tilde{\mathscr{L}}_1^0 = \rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma} + \lambda(\theta - \sigma)\frac{\partial}{\partial\sigma}. \tag{4.4}$$

We will apply Theorem 12 to (4.1) with $d = 2$ and $d' = 1$ and give an approximation formula for a barrier option of which value is given under a risk-neutral probability measure as

$$C_{\text{Barrier}}^{SV,\varepsilon}(T - t, e^x) = \mathrm{E}\left[e^{-c(T-t)}f(S_{T-t}^{\varepsilon,e^x})1_{\{\tau_{(L,\infty)}(S^{\varepsilon,e^x}) > T-t\}}\right],$$

where f stands for a payoff function and $L(< S)$ is a barrier price.

$u^\varepsilon(t,x) = C_{\text{Barrier}}^{SV,\varepsilon}(T-t, e^x)$ satisfies the following PDE:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathscr{L}^\varepsilon - c\right)u^\varepsilon(t,x) = 0, & (t,x) \in (0,T] \times D, \\ u^\varepsilon(T,x) = \bar{f}(x), & x \in \bar{D}, \\ u^\varepsilon(t,l) = 0, & t \in [0,T]. \end{cases} \tag{4.5}$$

where $\bar{f}(x) = \max\{e^x - K, 0\}$, $D = (l, \infty)$ and $l = \log L$. We obtain the 0-th order u^0 as

$$u^0(t,x) = P_{T-t}^D\bar{f}(x) = \mathrm{E}[e^{-c(T-t)}\bar{f}(X_{T-t}^{x,0})1_{\{\tau_D(X^{0,x}) > T-t\}}]. \tag{4.6}$$

Set $\alpha = c - q$. Then $P_{T-t}^D\bar{f}(x) = C_{\text{Barrier}}^{BS}(T-t, e^x, \alpha, \sigma, L)$ is the price of the down-and-out barrier call option under the Black-Scholes model:

$$C_{\text{Barrier}}^{BS}(T-t, e^x, \alpha, \sigma, L) = C^{BS}(T-t, e^x, \alpha, \sigma) - \left(\frac{e^x}{L}\right)^{1-\frac{2\alpha}{\sigma^2}} C^{BS}\left(T-t, \frac{L^2}{e^x}, \alpha, \sigma\right). \tag{4.7}$$

Here, we recall that the price of the plain vanilla option under the Black-Scholes model is given as

$$C^{BS}(T-t, e^x, \alpha, \sigma) = e^{-q(T-t)}e^xN(d_1(T-t,x,\alpha)) - e^{-c(T-t)}KN(d_2(T-t,x,\alpha)), \tag{4.8}$$

where

$$d_1(t,x,\alpha) = \frac{x - \log K + \alpha t}{\sigma\sqrt{t}} + \frac{\sigma^2}{2}\sqrt{t},$$

$$
\begin{aligned}
d_2(t,x,\alpha) &= d_1(t,x,\alpha) - \sigma\sqrt{t} \\
N(x) &= \int_{-\infty}^{x} n(y)dy, \\
n(y) &= \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}.
\end{aligned}
$$

We show the following main result in this section.

Theorem 6. *We obtain an approximation formula for the down-and-out barrier call option under the stochastic volatility model (4.1):*

$$
C^{SV,\varepsilon}_{\text{Barrier}}(T, e^x) = C^{BS}_{\text{Barrier}}(T, e^x, \alpha, \sigma, L) + \varepsilon v_1^0(0,x) + O(\varepsilon^2), \tag{4.9}
$$

where

$$
v_1^0(0,x) = e^{-cT} \int_0^T \int_l^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} (1 - e^{-\frac{2(x-l)(y-l)}{\sigma^2 s}}) e^{-\frac{(y-x-(\alpha-\frac{1}{2}\sigma^2)s)^2}{2\sigma^2 s}} \vartheta(s,y) dy ds, \tag{4.10}
$$

$$
\begin{aligned}
& \vartheta(t,x) \\
= \; & e^{\alpha(T-t)} \rho\nu\sigma e^x n(d_1(t,x,\alpha))(-d_2(t,x,\alpha)) \\
+ \; & 2e^{\alpha(T-t)} \rho\nu\alpha \left(\frac{e^x}{L}\right)^{-\frac{2\alpha}{\sigma^2}} Ln(c_1(t,x,\alpha))\sqrt{T-t} \\
- \; & e^{\alpha(T-t)} \rho\nu\sigma \left(\frac{e^x}{L}\right)^{-\frac{2\alpha}{\sigma^2}} Ln(c_1(t,x,\alpha)) c_1(t,x,\alpha) \\
- \; & e^{c(T-t)} \rho\nu \frac{4\alpha}{\sigma} \left(\frac{e^x}{L}\right)^{1-\frac{2\alpha}{\sigma^2}} \\
\times \; & \left\{ C^{BS}\left(T-t, \frac{L^2}{e^x}, \alpha, \sigma\right) \left\{ 1 + (x - \log L)\left(1 - \frac{2\alpha}{\sigma^2}\right) \right\} + (x-\log L) e^{-q(T-t)} \frac{L^2}{e^x} N(c_1(t,x,\alpha)) \right\} \\
+ \; & \lambda(\theta-\sigma) e^{\alpha(T-t)} e^x n(d_1(t,x,\alpha))\sqrt{T-t} \\
- \; & \lambda(\theta-\sigma) \left(\frac{e^x}{L}\right)^{-\frac{2\alpha}{\sigma^2}} e^{\alpha(T-t)} Ln(c_1(t,x,\alpha))\sqrt{T-t} \\
- \; & e^{c(T-t)} \lambda(\theta-\sigma) \frac{4\alpha}{\sigma^3} \left(\log\frac{e^x}{L}\right) \left(\frac{e^x}{L}\right)^{1-\frac{2\alpha}{\sigma^2}} C^{BS}\left(T-t, \frac{L^2}{e^x}, \alpha, \sigma\right),
\end{aligned} \tag{4.11}
$$

and

$$
c_1(t,x,\alpha) = \frac{2l - x - \log K + \alpha t}{\sigma\sqrt{t}} + \frac{\sigma^2}{2}\sqrt{t}.
$$

Proof. By Theorem 12 and the equality (3.1), we see the expansion

$$
C^{SV,\varepsilon}_{\text{Barrier}}(T-t, e^x) = C^{BS}_{\text{Barrier}}(T-t, e^x, \alpha, \sigma, L) + \varepsilon \int_0^{T-t} P_s^D \tilde{\mathscr{L}}_1^0 P^D_{T-t-s} \bar{f}(x) ds + O(\varepsilon^2). \tag{4.12}
$$

The first-order approximation term $v_1^0(t,x) = \int_0^{T-t} P_s^D \tilde{\mathscr{L}}_1^0 P_{T-t-s}^D \bar{f}(x)ds$ is given by

$$\begin{aligned} v_1^0(t,x) &= \int_0^{T-t} E[e^{-cs}\tilde{\mathscr{L}}_1^0 P_{T-t-s}^D \bar{f}(X_{T-t-s})]ds \\ &= \int_0^{T-t} E[e^{-cs}\tilde{\mathscr{L}}_1^0 e^{-c(T-t-s)}\bar{P}_{T-t-s}^D \bar{f}(X_{T-t-s})]ds \\ &= e^{-c(T-t)}\int_0^{T-t} \bar{P}_s^D \tilde{\mathscr{L}}_1^0 \bar{P}_{T-t-s}^D \bar{f}(x)ds, \end{aligned}$$

where $\bar{P}_t^D$ is defined by (3.3) with the density (3.4), that is,

$$\bar{P}_t^D \bar{f}(x) = \int_l^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}}(1 - e^{-\frac{2(x-l)(y-l)}{\sigma^2 s}})e^{-\frac{(y-x-(\alpha-\frac{1}{2}\sigma^2)s)^2}{2\sigma^2 s}}\bar{f}(y)dy.$$

Define $\vartheta(t,x)$ as

$$\begin{aligned} \vartheta(t,x) &= \tilde{\mathscr{L}}_1^0 \bar{P}_{T-t}^D f(e^x) \\ &= e^{c(T-t)}\rho\nu\sigma^2\frac{\partial^2}{\partial x\partial\sigma}C_{\text{Barrier}}^{BS}(T-t,e^x,\alpha,\sigma,L) + e^{c(T-t)}\lambda(\theta-\sigma)\frac{\partial}{\partial\sigma}C_{\text{Barrier}}^{BS}(T-t,e^x,\alpha,\sigma,L). \end{aligned}$$

A straightforward calculation shows that the above fucntion agrees with the right-hand side of (4.11). Then we get the assertion. ■

Remark that through numerical integrations with respect to time s and space y in (4.10), we easily obtain the first order approximation of the down-and-out option prices.

Next, as a special case of (4.1) we consider the following stochastic volatility model with no drifts:

$$\begin{aligned} dS_t^\varepsilon &= \sigma_t^\varepsilon S_t^\varepsilon dB_t^1, \quad S_0^\varepsilon = S > 0, \\ d\sigma_t^\varepsilon &= \varepsilon\nu\sigma_t^\varepsilon(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2), \quad \sigma_0^\varepsilon = \sigma > 0. \end{aligned} \tag{4.13}$$

where $\varepsilon \in [0,1)$, $\rho \in [-1,1]$ and $B = (B^1, B^2)$ is a two dimensional Brownian motion. In this case, we can give a slightly simple approximation formula compared with Theorem 6.

By Itô's formula, the following logarithmic model is obtained.

$$\begin{aligned} dX_t^\varepsilon &= -\frac{1}{2}(\sigma_t^\varepsilon)^2 dt + \sigma_t^\varepsilon dB_t^1, \quad X_0^\varepsilon = x = \log S, \\ d\sigma_t^\varepsilon &= \varepsilon\nu\sigma_t^\varepsilon(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2), \quad \sigma_0^\varepsilon = \sigma, \end{aligned} \tag{4.14}$$

This model is regarded as a SABR model with $\beta = 1$ and known as the log-normal SABR (see [12]). Again, the barrier option price is given by

$$C_{\text{Barrier}}^{SV,\varepsilon}(T,e^x) = \mathrm{E}\left[f(S_T^\varepsilon)1_{\{\min_{0\le u\le T} S_u^\varepsilon > L\}}\right],$$

where f stands for a payoff function and $L(< S)$ is a barrier price.

The differentiation operators $\mathscr{L}^\varepsilon$, $\tilde{\mathscr{L}}_1^0$ and the PDE are same as (4.3)–(4.5) with $c = q = 0$ and $\lambda = 0$. Also, the barrier option price in the Black-Scholes model coincides with (4.7) with no drift, that is,

$$C^{BS}_{\text{Barrier}}(T,S) = C^{BS}(T,S) - \left(\frac{S}{L}\right) C^{BS}\left(T, \frac{L^2}{S}\right),$$

where $C^{BS}(T,S)$ is the driftless Black-Scholes formula of the European call option given by

$$C^{BS}(T,S) = SN(d_1(T,\log S)) - KN(d_2(T,\log S))$$

with

$$\begin{aligned} d_1(t,x) &= d_1(t,x,0) = \frac{x - \log K + \sigma^2 t/2}{\sigma\sqrt{t}}, \\ d_2(t,x) &= d_2(t,x,0) = d_1(t,x) - \sigma\sqrt{t}. \end{aligned}$$

Then, we reach the following expansion formula which only needs 1-dimensional numerical integration.

Theorem 7. $C^{SV,\varepsilon}_{\text{Barrier}}(T,e^x) = C^{BS}_{\text{Barrier}}(T,e^x) + \varepsilon v_1^0(0,x) + O(\varepsilon^2)$, *where*

$$\begin{aligned} v_1^0(0,x) &= -\frac{1}{2}T\nu\rho\sigma\left\{e^x n(d_1(T,x))d_2(T,x) + Ln(c_1(T,x))c_1(T,x)\right\} \\ &\quad + \frac{\nu\rho L(x-l)\log(L/K)}{2\pi\sigma}\int_0^T \frac{(T-s)^{1/2}}{s^{3/2}}\exp\left(-\frac{c_2(T-s,L/K)+c_2(s,L/e^x)}{2}\right)ds, \\ c_1(t,x) &= \frac{\log(L^2/e^x K) + \sigma^2 t/2}{\sigma\sqrt{t}}, \quad c_2(t,y) = \left(\frac{\log y + \sigma^2 t/2}{\sigma\sqrt{t}}\right)^2. \end{aligned}$$

Proof. By Theorem 12 and the equality (3.1), we see that the expansion

$$C^{SV,\varepsilon}_{\text{Barrier}}(T,e^x) = C^{BS,\varepsilon}_{\text{Barrier}}(T,e^x) + \varepsilon v_1^0(0,x) + O(\varepsilon^2)$$

holds with

$$v_1^0(t,x) = \int_0^{T-t} P^D_{T-t-r}\tilde{\mathscr{L}}_1^0 P^D_r \bar{f}(x)dr. \tag{4.15}$$

Then, we have the following proposition for an expression of $v_1^0(0,x)$. The proof is given in Section 5.5.

Proposition 1.

$$v_1^0(0,x) = \frac{T}{2}\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_T\bar{f}(x) - \frac{1}{2}\,\mathrm{E}[(T-\tau_D(X^{0,x}))\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_{T-\tau_D(X^{0,x})}\bar{f}(l)1_{\{\tau_D(X^{0,x})<T\}}].$$

We remark that the expectation in the above equality can be represented as

$$\frac{1}{2}\,\mathrm{E}[(T-\tau_D(X^{0,x}))\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_{T-\tau_D(X^{0,x})}\bar{f}(l)1_{\{\tau_D(X^{0,x})<T\}}]$$

$$= \int_0^T \frac{(T-s)}{2}\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-s}^D \bar{f}(l) h(s, x-l) ds, \tag{4.16}$$

where $h(s, x-l)$ is the density function of the first hitting time to l defined by

$$h(s, x-l) = \frac{-(l-x)}{\sqrt{2\pi\sigma^2 s^3}} \exp\left(-\frac{\{l-x+\sigma^2 s/2\}^2}{2\sigma^2 s}\right). \tag{4.17}$$

Now we evaluate

$$\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_t^D \bar{f}(x) = \nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C^{BS}(t, e^x) - \nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} \left\{ \left(\frac{e^x}{L}\right) C^{BS}\left(t, \frac{L^2}{e^x}\right) \right\}.$$

Note that

$$\frac{\partial}{\partial \sigma} C^{BS}(t, e^x) = e^x n(d_1(t,x))\sqrt{t}, \tag{4.18}$$

and

$$\frac{\partial}{\partial \sigma} \left\{ \left(\frac{e^x}{L}\right) C^{BS}\left(t, \frac{L^2}{e^x}\right) \right\} = Ln(c_1(t,x))\sqrt{t}. \tag{4.19}$$

Then we have

$$\begin{aligned} \nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C^{BS}(t, e^x) &= \nu\rho\sigma^2 e^x n(d_1(t,x))\sqrt{t} \left\{1 - \frac{d_1(t,x)}{\sigma\sqrt{t}}\right\} \\ &= -\nu\rho\sigma e^x n(d_1(t,x)) d_2(t,x) \end{aligned} \tag{4.20}$$

and

$$\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} \left\{ \left(\frac{e^x}{L}\right) C^{BS}\left(t, \frac{L^2}{e^x}\right) \right\} = \nu\rho\sigma Ln(c_1(t,x)) c_1(t,x). \tag{4.21}$$

Combining (4.18), (4.20) and (4.21), we get

$$\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_t^D \bar{f}(x) = \nu\rho\sigma \left\{ e^x n(d_1(t,x))(-d_2(t,x)) - Ln(c_1(t,x)) c_1(t,x) \right\}. \tag{4.22}$$

Substituting (4.22) into (4.16), we have

$$\begin{aligned} \nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_t^D \bar{f}(l) &= \nu\rho\sigma Ln(d_1(t,l))(-d_2(t,l)) - \rho\sigma Ln(c_1(t,l)) c_1(t,l) \\ &= \nu\rho\sigma Ln(d_1(t,l))(-(d_1(t,l) + d_2(t,l))) \\ &= \nu\rho\sigma \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\{l - \log K + \frac{1}{2}\sigma^2 t\}^2}{2\sigma^2 t}\right) \left(\frac{-2(l - \log K)}{\sigma\sqrt{t}}\right). \end{aligned}$$

Thus we obtain

$$-\frac{1}{2}\,\mathrm{E}[(T - \tau_D(X^{0,x}))\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X^{0,x})}^D \bar{f}(l) 1_{\{\tau_D(X^{0,x})<T\}}]$$

$$
\begin{aligned}
&= -\int_0^T \frac{(T-s)}{2}\nu\rho\sigma L \frac{1}{\sqrt{2\pi}} e^{-\frac{\left\{l-\log K+\frac{1}{2}\sigma^2(T-s)\right\}^2}{2\sigma^2(T-s)}} \left(\frac{-2(l-\log K)}{\sigma\sqrt{T-s}}\right) \\
&\quad \times \frac{-(l-x)}{\sqrt{2\pi\sigma^2 s^3}} e^{-\frac{\left\{(l-x)+(\sigma^2/2)s\right\}^2}{2\sigma^2 s}} ds \\
&= \frac{\nu\rho L(x-l)\log(L/K)}{2\pi\sigma} \int_0^T \frac{(T-s)^{1/2}}{s^{3/2}} \exp\left(-\frac{c_2(T-s, L/K)+c_2(s, L/e^x)}{2}\right) ds.
\end{aligned} \tag{4.23}
$$

By Proposition 1, (4.16), (4.22) and (4.23), we reach the assertion. ■

Finally, we show a simple numerical example of European down-and-out barrier call prices as an illustrative purpose. Denote $u^0 = C^{BS}_{\text{Barrier}}(T, S)$ and $v_1^0 = v_1^0(0, \log S)$. Then we see

$$C^{SV,\varepsilon}_{\text{Barrier}}(T, S) \simeq u^0 + \varepsilon v_1^0.$$

We list the numerical examples below, where the numbers in the parentheses show the error rates (%) relative to the benchmark prices of $C^{SV,\varepsilon}_{\text{Barrier}}(T, S)$; they are computed by Monte Carlo simulations with 100,000 time steps and 1,000,000 trials. We check the accuracy of our approximations by changing the model parameters. Case 1–6 show the results for the stochastic volatility model with drifts (4.1), and case 7 shows the result for the lognormal SABR model (4.13).

Apparently, our approximation formula $u^0 + \varepsilon v_1^0$ improves the accuracy for $C^{SV,\varepsilon}_{\text{Barrier}}(T, S)$, and it is observed that εv_1^0 accurately compensates for the difference between $C^{SV,\varepsilon}_{\text{Barrier}}(T, S)$ and $C^{BS}_{\text{Barrier}}(T, S)$, which confirms the validity of our method.

1.

$$
\begin{aligned}
&S = 100,\ \sigma = 0.15,\ c = 0.01,\ q = 0.0, \varepsilon\nu = 0.2,\ \rho = -0.5, \\
&\varepsilon\lambda = 0.00,\ \theta = 0.00,\ L = 95,\ T = 0.5,\ K = 100,\ 102,\ 105.
\end{aligned}
$$

Table 1: Down-and-Out Barrier Option

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.468	3.466 (-0.05%)	3.495 (0.80%)
102	2.822	2.822 (0.00%)	2.866 (1.57%)
105	1.986	1.986 (0.01%)	2.052 (3.36%)

2.

$$
\begin{aligned}
&S = 100,\ \sigma = 0.15,\ c = 0.01,\ q = 0.0, \varepsilon\nu = 0.35,\ \rho = -0.7, \\
&\varepsilon\lambda = 0.00,\ \theta = 0.00,\ L = 95,\ T = 0.5,\ K = 100,\ 102,\ 105.
\end{aligned}
$$

Table 2: Down-and-Out Barrier Option

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.421	3.423 (0.07%)	3.495 (2.18%)
102	2.753	2.757 (0.18%)	2.866 (4.13%)
105	1.885	1.890 (0.23%)	2.052 (8.88%)

3.

$$S = 100,\ \sigma = 0.15,\ c = 0.05,\ q = 0.0, \varepsilon\nu = 0.35,\ \rho = -0.7,$$
$$\varepsilon\lambda = 0.00,\ \theta = 0.00,\ L = 95,\ T = 0.5,\ K = 100,\ 102,\ 105.$$

Table 3: Down-and-Out Barrier Option

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	4.352	4.349 (-0.07%)	4.399 (1.06%)
102	3.585	3.586 (0.02%)	3.665 (2.24%)
105	2.560	2.563 (0.11%)	2.696 (5.31%)

4.

$$S = 100,\ \sigma = 0.15,\ c = 0.05,\ q = 0.1, \varepsilon\nu = 0.2,\ \rho = -0.5,$$
$$\varepsilon\lambda = 0.00,\ \theta = 0.00,\ L = 95,\ T = 0.5,\ K = 100,\ 102,\ 105.$$

Table 4: Down-and-Out Barrier Option

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	2.231	2.224 (-0.31%)	2.268 (1.64%)
102	1.758	1.754 (-0.27%)	1.812 (3.02%)
105	1.172	1.168 (-0.31%)	1.243 (6.05%)

5.

$$S = 100,\ \sigma = 0.15,\ c = 0.01,\ q = 0.0, \varepsilon\nu = 0.2,\ \rho = -0.5,$$
$$\varepsilon\lambda = 0.2,\ \theta = 0.25,\ L = 95,\ T = 0.5,\ K = 100,\ 102,\ 105.$$

Table 5: Down-and-Out Barrier Option

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.523	3.517 (-0.16%)	3.495 (-0.77%)
102	2.891	2.888 (-0.09%)	2.866 (-0.85%)
105	2.066	2.065 (-0.06%)	2.052 (-0.64%)

6.

$$S = 100,\ \sigma = 0.15,\ c = 0.01,\ q = 0.0, \varepsilon\nu = 0.2,\ \rho = -0.5,$$
$$\varepsilon\lambda = 0.5,\ \theta = 0.25,\ L = 95,\ T = 0.5,\ K = 100,\ 102,\ 105.$$

Table 6: Down-and-Out Barrier Option

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.587	3.594 (0.20%)	3.495 (-2.55%)
102	2.976	2.987 (0.39%)	2.866 (-3.68%)
105	2.170	2.183 (0.59%)	2.052 (-5.41%)

7.

$$S = 100,\ \sigma = 0.15,\ c = 0.0,\ q = 0.0, \varepsilon\nu = 0.2,\ \rho = -0.5,$$
$$\varepsilon\lambda = 0.0,\ \theta = 0.0,\ L = 95,\ T = 0.5,\ K = 100,\ 102,\ 105.$$

Table 7: Down-and-Out Barrier Option

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.261	3.258 (-0.09%)	3.290 (0.90%)
102	2.640	2.639 (-0.02%)	2.686 (1.78%)
105	1.841	1.841 (0.01%)	1.911 (3.77%)

5 Appendix

5.1 Generalization of Main Results

There are several cases in practice that our assumptions [A]–[B] are not satisfied. Hence, in this section we weaken the assumptions. Let $d' \in \{1, \ldots, d\}$, and we regard $X_t^{\varepsilon,x,i}$ as logarithm of the underlying asset prices for $i \leq d'$, and as parameter processes such as those for a stochastic volatility and a stochastic interest rate for $i > d'$. For a technical reason introduced later, we assume $I \subset [0, \infty)$ in this section.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ be a filtered space equipped with a standard Brownian motion $(B_t)_t$. Set

$$\begin{aligned} \hat{b}^i(y, \varepsilon) &= \begin{cases} y^i \left\{ b^i(\pi(y), \varepsilon) + \dfrac{1}{2} \displaystyle\sum_{j=1}^{d} (\sigma^{ij}(\pi(y), \varepsilon))^2 \right\}, & i \le d', \\ b^i(\pi(y), \varepsilon), & i > d', \end{cases} \\ \hat{\sigma}^{ij}(y, \varepsilon) &= \begin{cases} y^i \sigma^{ij}(\pi(y), \varepsilon), & i \le d', \\ \sigma^{ij}(\pi(y), \varepsilon), & i > d', \end{cases} \end{aligned}$$

where $\pi(y) = (\log y^1, \ldots, \log y^{d'}, y^{d'+1}, \ldots, y^d) \in \mathbb{R}^d$.

[A'] For each $\varepsilon \in I$ it holds that $\sigma^{ij}(\cdot, \varepsilon), b^i(\cdot, \varepsilon) \in \mathcal{L}$ and that $\hat{\sigma}^{ij}(\cdot, \varepsilon)$, $\hat{b}^i(\cdot, \varepsilon)$ and $c(\pi(\cdot), \varepsilon)$ are also in $\mathcal{L}$, that is, they are extended to be locally Lipschitz functions (with respect to the parabolic distance) defined on $\mathbb{R}^d$. Moreover, there exists a solution $(X_t^{\varepsilon,x})_t$ of SDE (1.3) and for any $m > 0$ there are $m', C > 0$ such that

$$\sup_{0 \le r \le t} \mathrm{E}[|Y_t^{\varepsilon,y}|^{2m}] \le C t^{m-1} (1 + |y|^{2m'}), \quad (t, y) \in [0, T] \times [0, \infty)^{d'} \times \mathbb{R}^{d-d'}, \ \varepsilon \in I, \quad (5.1)$$

where $Y_t^{\varepsilon,y} = \iota(X_t^{\varepsilon,\pi(y)})$ and $\iota(x) = (e^{x^1}, \ldots, e^{x^{d'}}, x^{d'+1}, \ldots, x^d) \in \mathbb{R}^d$.

[B'] The function $f(x)$ is represented by the continuous function $\hat{f} : \mathbb{R}^d \longrightarrow \mathbb{R}$ as $f(x) = \hat{f}(\iota(x))$. There exists $C_{\hat{f}} > 0$ such that $|\hat{f}(y)|^2 \le C_{\hat{f}}(1 + |y|^{2m})$, $y \in \mathbb{R}^d$. Moreover, $f(x) = 0$ on $\mathbb{R}^d \setminus D$.

[C'] In addition to the condition [C], there is a constant $A_4^\varepsilon > 0$ such that $c(t, x, \varepsilon) \le A_4^\varepsilon (1 + |x|^{2m})$.

Note that Ito's formula implies that $(Y_t^{\varepsilon,y})_t$ is a solution of

$$\begin{cases} dY_t^{\varepsilon,y} = \hat{b}(Y_t^{\varepsilon,y}, \varepsilon) dt + \hat{\sigma}(Y_t^{\varepsilon,y}, \varepsilon) dB_t, \\ Y_0^{\varepsilon,y} = y. \end{cases}$$

Although Theorem 3.1 in [17] no longer works under [A']–[B'], we can charaterize $u^\varepsilon(t, x)$ as the solution of (2.3) in the viscosity sense. To see this, set

$$\hat{\mathscr{L}}^\varepsilon = \frac{1}{2} \sum_{i,j=1}^{d} \hat{a}^{ij}(y, \varepsilon) \frac{\partial^2}{\partial y^i \partial y^j} + \sum_{i=1}^{d} \hat{b}^i(y, \varepsilon) \frac{\partial}{\partial y^i} - c(\pi(y), \varepsilon),$$

where $\hat{a}^{ij} = \displaystyle\sum_{k=1}^{d} \hat{\sigma}^{ik} \hat{\sigma}^{jk}$. Moreover, define

$$\hat{D} = \{ y \in \mathbb{R}^d \ ; \ y^i > 0, \ i = 1, \ldots, d' \text{ and } \pi(y) \in D \}$$

and $\hat{u}^\varepsilon(t, y) = u^\varepsilon(t, \pi(y))$ $((t, y) \in [0, T] \times \hat{D})$, 0 $((t, y) \in [0, T] \times \partial \hat{D})$.

Theorem 8. *Assume* $[A']$–$[B']$*. Then,* $u^\varepsilon(t,x)$ *is a viscosity solution of* (2.3)*. Moreover,* $\hat{u}^\varepsilon(t,y)$ *is a viscosity solution of*

$$\begin{cases} -\dfrac{\partial}{\partial t}\hat{u}^\varepsilon(t,y) - \hat{\mathscr{L}}^\varepsilon\hat{u}^\varepsilon(t,y) = 0, & (t,y) \in [0,T) \times \hat{D}, \\ \hat{u}^\varepsilon(T,y) = \hat{f}(y), & x \in \hat{D}, \\ \hat{u}^\varepsilon(t,y) = 0, & (t,y) \in [0,T] \times \partial\hat{D} \end{cases} \tag{5.2}$$

satisfying

$$\sup_{(t,y)\in[0,T]\times\bar{\hat{D}}} |\hat{u}^\varepsilon(t,y)|/(1+|y|^{2m'}) < \infty. \tag{5.3}$$

Proof. The latter assertion is by the similar argument to the proof of Proposition 6. Then, the simple calculation gives the former assertion. ■

Applying Theorem 8.2 in [1] and Theorem 7.7.2 in [15] for (5.2), we have the following theorem.

Theorem 9. *Assume* $[A']$–$[C']$ *and* $[D]$–$[E]$*. If* $\hat{w}^\varepsilon(t,y)$ *is a viscosity solution of* (5.2) *satisfying the growth condition* (5.3)*, then* $\hat{u}^\varepsilon = \hat{w}^\varepsilon$*.*

Let $\hat{\mathcal{H}}^{m,\alpha,p}$ be the same as $\mathcal{H}^{m,\alpha,p}$ replacing (2.9) with

$$|g(t,x)| \le M^g(t)\left\{1+|\iota(x)|^{2m}\right\}, \quad t\in[0,T),\ x,y\in\bar{D}.$$

Moreover we define $\hat{\mathcal{G}}^{m,\alpha,p}$ similarly to $\mathcal{G}^{m,\alpha,p}$, replacing $\mathcal{H}^{m,0,2}$ and $\mathcal{H}^{m,\alpha,p}$ in the definition with $\hat{\mathcal{H}}^{m,0,2}$ and $\hat{\mathcal{H}}^{m,\alpha,p}$, respectively.

[H'] The condition [H] holds replacing $\mathcal{G}^{m,\alpha,p}$ with $\hat{\mathcal{G}}^{m,\alpha,p}$.

[I'] The condition [I] holds replacing $\mathcal{G}^{m,\alpha,p}$ with $\hat{\mathcal{G}}^{m,\alpha,p}$.

The following theorem gives a generalization of Theorem 3. The proof is in Section 5.4.

Theorem 10. *Assume* $[A']$–$[C']$*,* $[D]$–$[G]$ *and* $[H']$–$[I']$*. Then there are positive constants* C_n *and* $\tilde{m}_n$ *which are independent of* ε *such that*

$$\left| u^\varepsilon(t,x) - (u^0(t,x) + \sum_{k=1}^{n-1}\varepsilon^k v_k^0(t,x)) \right| \le C_n(1+|\iota(x)|^{2\tilde{m}_n})\varepsilon^n, \quad (t,x)\in[0,T]\times\bar{D}.$$

Here we give another version of generalized assumptions.

[D'] The domain D is given as $D = U \times \mathbb{R}^{d-d'}$, where U is a domain in $\mathbb{R}^{d'}$ whose boundary ∂U satisfies the outside strong shpere property.

[E'] The condition [E] holds for $\varepsilon \neq 0$. Moreover $\sigma^{ij}(x,0) = b^i(x,0) = 0$ for $i = d'+1,\dots,d$, $j = 1,\dots,d$ and for each compact set $K \subset D$ there is a positive constant $\mu_{0,K}$ such that $\mu_{0,K}|\xi|^2 \le \sum_{i,j=1}^{d'} a^{ij}(x,0) \le \mu_{0,K}^{-1}|\xi|^2$ for $x\in K$ and $\xi \in \mathbb{R}^d$.

[F'] For each $y \in \mathbb{R}^{d-d'}$, the inequality

$$\max_{i,j} \sup_{x \in \bar{U}} \{|\sigma^{ij}((x,y),0)| + |b^i((x,y),0)| + |c((x,y),0)|\} < \infty$$

holds, where $(x,y) = (x^1, \dots, x^{d'}, y^1, \dots, y^{d-d'}) \in \mathbb{R}^d$ and there exist $A_3(y) > 0$ and $\alpha \in (0,1]$ such that

$$\begin{aligned} &|\sigma^{ij}((x,y),0) - \sigma^{ij}((x',y),0)| + |b^i((x,y),0) - b^i((x',y),0)| + |c((x,y),0) - c((x',y),0)| \\ &\le A_3(y)|x-x'|^\alpha, \quad (t,x),(s,x') \in [0,T] \times \bar{U}, y \in \mathbb{R}^{d-d'}. \end{aligned}$$

[H"] The condition [H] holds replacing $\mathcal{G}^{m,\alpha,p}$ with $\overline{\mathcal{G}}^{m,\alpha,p}$, where

$$\begin{aligned} \overline{\mathcal{G}}^{m,\alpha,p} \;=\; & \Big\{ g \in C^{1,2}([0,T) \times D) \cap C([0,T] \times \bar{D}) \; ; \\ & \frac{\partial g}{\partial x^i}(\cdot,\cdot;y) \in \hat{\mathcal{H}}_U^{m,0,2}, \; \frac{\partial^2 g}{\partial x^i \partial x^j}(\cdot,\cdot;y) \in \hat{\mathcal{H}}_U^{m,\alpha,p}, \; i,j = 1, \dots, d, \;\; y \in \mathbb{R}^{d-d'} \Big\} \end{aligned}$$

and $\hat{\mathcal{H}}_U^{m,\alpha,p}$ is the same as $\hat{\mathcal{H}}^{m,\alpha,p}$ replacing $D \subset \mathbb{R}^d$ in the definition with $U \subset \mathbb{R}^{d'}$. Here $h(\cdot,\cdot;y)$ denotes the function $[0,T] \times \bar{U} \ni (t,x) \longmapsto h(t,(x,y)) \in \mathbb{R}$ for $h = \partial g/\partial x^i, \partial^2 g/\partial x^i \partial x^j$.

[I"] The condition [I] holds replacing $\mathcal{G}^{m,\alpha,p}$ with $\overline{\mathcal{G}}^{m,\alpha,p}$.

Theorem 2 implies the next theorem.

Theorem 11. *Assume [A']–[F'] and [G]. Let $g \in \mathcal{H}_U^{m,\alpha,p}$ for some $p > 1/\alpha$. Then for each fixed $y = (x^i)_{i=d'+1}^d$, the following PDE*

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) + \mathscr{L}_y^0 v(t,x) + g(t,x;y) = 0, & (t,x) \in [0,T) \times U, \\ v(T,x) = 0, & x \in U, \\ v(t,x) = 0, & (t,x) \in [0,T] \times \partial U \end{cases}$$

has a classical solution v satisfying (2.11), where

$$\mathscr{L}_y^0 = \frac{1}{2} \sum_{i,j=1}^{d'} a^{ij}((x,y),0) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d'} b^i((x,y),0) \frac{\partial}{\partial x^i} - c((x,y),0), \quad x \in \mathbb{R}^{d'}.$$

Moreover, if w is another classical solution of (2.10) which satisfies $|w(t,x)| \le C' \exp(\beta |x|^2)$, $(t,x) \in [0,T] \times \bar{D}$, for some $C', \beta > 0$, then $v = w$.

Using the above theorem instead of Theorem 2 itself, we can prove the following theorem similarly to Theorem 10.

Theorem 12. *Assume [A']–[F'], [G] and [H'']–[I'']. Then, the same assertion of Theorem 10 holds.*

5.2 Proof of Theorem 2

We consider the following PDE which is equivalent to (2.10) with changing variable t to $T-t$

$$\begin{cases} -\dfrac{\partial}{\partial t}v(t,x) + \mathscr{L}^0 v(t,x) + g(t,x) = 0, & (t,x) \in (0,T] \times D, \\ v(0,x) = 0, & x \in D, \\ v(t,x) = 0, & (t,x) \in [0,T] \times \partial D. \end{cases} \tag{5.4}$$

We define $\tilde{\mathcal{H}}^{m,\alpha,p}$ as the same as $\mathcal{H}^{m,\alpha,p}$ replacing $[0,T)$ in the definition with $(0,T]$.

We divide the proof of Theorem 2 into the following two propositions.

Proposition 2. *For any g, a classical solution of* (5.4) *is unique in the following sense: if v and w are classical solutions of* (5.4) *and $|v(t,x)|+|w(t,x)| \le C\exp(\beta|x|^2)$ for some $C, \beta > 0$, then $v = w$.*

Proposition 2 is obtained by the same argument as the proof of Theorem 2.4.9 in [5].

Proposition 3. *There exists a classical solution v of* (5.4) *for $g \in \tilde{\mathcal{H}}^{m,\alpha,p}$ with $p > 1/\alpha$. Moreover,* (2.11) *holds.*

Proof. By Levi's parametrix method, we can construct the fundamental solution $\Gamma(t,x;\tau,\xi)$ for the operator $L = -\partial/\partial t + \mathscr{L}^0$, that is,

$$W_g(t,x) = \int_0^t \int_D \Gamma(t,x;\tau,\xi) g(\tau,\xi) d\xi d\tau$$

is continuous in (t,x), continuously differentiable in x for $g \in C([0,T] \times \bar{D})$. When g is Hölder continuous in x uniformly in $t \in [0,T]$, then we see that W_g is a solution of (5.4) (See Theorem 1.5.8–1.6.10 in [5]. For more details, please refer to Chapter 1, Section 2–6.) However functions in $\tilde{\mathcal{H}}^{m,\alpha,p}$ may not have the regularity at $t = 0$. So we generalize the argument in Chapter 1 of [5]. We remark that $\Gamma(t,x;\tau,\xi)$ is given by

$$\Gamma(t,x;\tau,\xi) = Z(t,x;\tau,\xi) + \int_\tau^t \int_D Z(t,x;\sigma,\eta)\Phi(\sigma,\eta;\tau,\xi) d\eta d\sigma,$$

where

$$Z(t,x;\tau,\xi) = \frac{\sqrt{\det(a(t,x,0))}}{(4\pi(t-\tau))^{d/2}} \exp\left(-\frac{\sum_{i,j=1}^d a^{ij}(\tau,\xi)(x^i-\xi^i)(x^j-\xi^j)}{4(t-\tau)} \right)$$

and $\Phi(t,x;\tau,\xi)$ is the solution of

$$\Phi(t,x;\tau,\xi) = LZ(t,x;\tau,\xi) + \int_\tau^t \int_D LZ(t,x;\sigma,\eta)\Phi(\sigma,\eta;\tau,\xi) d\eta d\sigma.$$

Fix any $g \in \tilde{\mathcal{H}}^{m,\alpha,p}$. We can divide W_g as $W_g = V_g + U_g$, where

$$V_g(t,x) = \int_0^t \int_D Z(t,x;\tau,\xi) g(\tau,\xi) d\xi d\tau,$$

$$U_g(t,x) = \int_0^t \int_D Z(t,x;\tau,\xi)\hat{g}(\tau,\xi)d\xi d\tau, \quad \hat{g}(t,x) = \int_0^t \int_D \Phi(t,x;\tau,\xi)g(\tau,\xi)d\xi d\tau.$$

We remark that V_g, U_g and $\hat{g}$ are well-defined by virtue of (4.9) and (4.15) in [5] and the property of g. Take $\beta \in (\alpha - 1/p, \alpha)$. By Theorem 1.4.8 in [5], we see that

$$|\Phi(t,x;\tau,\xi) - \Phi(t,y;\tau,\xi)| \leq \frac{C|x-y|^\beta}{(t-\tau)^{(d+2-(\alpha-\beta))/2}} \left\{ \exp\left(-\frac{\lambda|x-\xi|^2}{t-r}\right) + \exp\left(-\frac{\lambda|y-\xi|^2}{t-r}\right) \right\}$$

for some $C, \lambda > 0$. Hence,

$$\begin{aligned} |\hat{g}(t,x) - \hat{g}(t,y)| &\leq C' \int_0^t \frac{M^g(\tau)}{(t-\tau)^{1-(\alpha-\beta)/2}} d\tau |x-y|^\beta \\ \leq\ C' &\left(\int_0^T (M^g(\tau))^p d\tau \right)^{1/p} \left(\int_0^t (t-\tau)^{-(1-(\alpha-\beta)/2)q} d\tau \right)^{1/q} |x-y|^\beta, \quad t \in (0,T],\ x,y \in D \end{aligned}$$

for some $C' > 0$ by virtue of the Hölder inequality, where $q > 1$ is given by $1/p + 1/q = 1$. Since $(1-(\alpha-\beta)/2)q$ is smaller than 1, we see that $\hat{g}(t,x)$ is β-Hölder continuous in x uniformly in $t \in (0,T]$. Then, Theorem 1.3.3–1.3.6 and the equality (4.2) in Chapter 1 of [5] imply that $U_g(t,x) \in C^{1,2}((0,T] \times D)$ and

$$\begin{aligned} LU_g(t,x) &= -\hat{g}(t,x) + \int_0^t \int_D \left\{ \frac{1}{2} \sum_{i,j=1}^d (a^{ij}(x,0) - a^{ij}(\xi,0)) \frac{\partial^2}{\partial x^i \partial x^j} \right. \\ &\qquad \left. + b^i(x,0)\frac{\partial}{\partial x^i} - c(x,0) \right\} Z(t,x;\tau,\xi)\hat{g}(\tau,\xi)d\xi d\tau \\ &= -\hat{g}(t,x) + \int_0^t \int_D LZ(t,x;\tau,\xi)\hat{g}(\tau,\xi)d\xi d\tau. \end{aligned} \tag{5.5}$$

For the volume potential V_g, we follow the proof of Theorem 1.3.4 in [5] to find that for any compact set $K \subset D$

$$\sum_{i,j=1}^d \left| \frac{\partial^2}{\partial x^i \partial x^j} J_g(t,x,\tau) \right| \leq \frac{C_K \tilde{M}^{g,K}(\tau)}{(t-\tau)^\mu}, \quad t \in (0,T],\ x \in K,\ \tau \in (0,t),\ \mu \in (1-\alpha/2, 1)$$

for some $C_K > 0$, where

$$J_g(t,x,\tau) = \int_D Z(t,x;\tau,\xi)g(\tau,\xi)d\xi.$$

Hence, the dominated convergence theorem implies that $V_g(t,x)$ is twice continuously differentiable in x. Similarly, we get $V_g \in C^{1,2}((0,T] \times D)$ and

$$LV_g(t,x) = -g(t,x) + \int_0^t \int_D LZ(t,x;\tau,\xi)g(\tau,\xi)d\xi d\tau. \tag{5.6}$$

Combining (5.5)–(5.6), we obtain

$$LW_g(t,x) = -g(t,x) - \int_0^t \int_D \{\Phi(t,x;\tau,\xi) - LZ(t,x;\tau,\xi)\} g(\tau,\xi)d\xi d\tau$$

$$
\begin{aligned}
& \quad + \int_0^t \int_D \int_0^\tau \int_D LZ(t,x;\tau,\xi) g(\sigma,\eta) d\eta d\sigma d\xi d\tau \\
&= -g(t,x) - \int_0^t \int_D \Big\{ \Phi(t,x;\tau,\xi) - LZ(t,x;\tau,\xi) \\
& \qquad - \int_\tau^t \int_D LZ(t,x;\sigma,\eta)\Phi(\sigma,\eta;\tau,\xi) d\eta d\sigma \Big\} g(\tau,\xi) d\xi d\tau \\
&= -g(t,x),
\end{aligned}
$$

which implies that W_g is a solution of (5.4). Moreover, since $g \in \mathcal{H}^{m,\alpha,p}$, using the inequality (6.12) in p.24 of [5], we get

$$
\begin{aligned}
|v(t,x)| &\leq C'' \int_0^t \frac{M^g(\tau)(1+|\xi|^{2m})}{(t-\tau)^{d/2}} \exp\left(-\frac{\lambda'|x-\xi|^2}{t-\tau} \right) d\xi d\tau \\
&\leq C''' \left(\int_0^T (M^g(\tau))^p d\tau \right)^{1/p} (1+|x|^{2m})
\end{aligned}
$$

for some $C'', C''', \lambda' > 0$. Then, we complete the proof of Proposition 3. ■

5.3 Proof of Theorem 3

First, we generalize the definitions of $\tilde{\mathscr{L}}_k^0$, g_k^0 and v_k^0. We define

$$
\begin{aligned}
\tilde{\mathscr{L}}_k^\varepsilon &= \frac{1}{(k-1)!} \Bigg\{ \frac{1}{2} \sum_{i,j=1}^d \int_0^1 (1-r)^{k-1} \frac{\partial^k a^{ij}}{\partial \varepsilon^k}(x, r\varepsilon) dr \frac{\partial^2}{\partial x^i \partial x^j} \\
& \qquad + \sum_{i=1}^d \int_0^1 (1-r)^{k-1} \frac{\partial^k b^i}{\partial \varepsilon^k}(x, r\varepsilon) dr \frac{\partial}{\partial x^i} - \int_0^1 (1-r)^{k-1} \frac{\partial^k c}{\partial \varepsilon^k}(x, r\varepsilon) dr \Bigg\}, \\
g_n^\varepsilon(t,x) &= \tilde{\mathscr{L}}_n^\varepsilon u^0(t,x) + \sum_{k=1}^{n-1} \tilde{\mathscr{L}}_{n-k}^0 v_k^0(t,x) + \sum_{k=1}^{n-2} \varepsilon^k \left\{ \tilde{\mathscr{L}}_n^\varepsilon v_k^0(t,x) + \sum_{l=k+1}^{n-1} \tilde{\mathscr{L}}_{n+k-l}^0 v_k^0(t,x) \right\} \\
& \quad + \varepsilon^{n-1} \tilde{\mathscr{L}}_n^\varepsilon v_{n-1}^0(t,x).
\end{aligned}
$$

We consider the following Cauchy-Dirichlet problem:

$$
\begin{cases}
-\dfrac{\partial}{\partial t} v(t,x) - \mathscr{L}^\varepsilon v(t,x) - g_n^\varepsilon(t,x) = 0, & (t,x) \in [0,T) \times D, \\
v(T,x) = 0, & x \in D, \\
v(t,x) = 0, & (t,x) \in [0,T] \times \partial D.
\end{cases} \tag{5.7}
$$

For $\varepsilon \neq 0$, we define $v_n^\varepsilon = [u^\varepsilon - \{u^0 + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t,x)\}]/\varepsilon^n$. Obviously, we see

$$
u^\varepsilon(t,x) = u^0(t,x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t,x) + \varepsilon^n v_n^\varepsilon(t,x). \tag{5.8}
$$

Proposition 4. *The function v_n^ε is a solution of* (5.7)*.*

Proof. It is obvious that $v_n^\varepsilon(T,x) = 0$ for $x \in D$ and $v_n^\varepsilon(t,x) = 0$ for $(t,x) \in [0,T] \times \partial D$. Apply Taylor's theorem to (2.3) to ovserve

$$\mathscr{L}^\varepsilon u^\varepsilon(t,x) = \left\{ \mathscr{L}^0 + \sum_{k=1}^{n-1} \varepsilon^k \tilde{\mathscr{L}}_k^0 + \varepsilon^n \tilde{\mathscr{L}}_n^\varepsilon \right\} u^\varepsilon(t,x). \tag{5.9}$$

Since u^0 is the solution of (2.3) with $\varepsilon = 0$, we get

$$\frac{\partial}{\partial t} u^0(t,x) + \mathscr{L}^0 u^0(t,x) = 0. \tag{5.10}$$

Similarly, by Theorem 2, we have

$$\frac{\partial}{\partial t} v_k^0(t,x) + \mathscr{L}^0 v_k^0(t,x) + \tilde{\mathscr{L}}_k^0 u^0(t,x) + \sum_{l=1}^{k-1} \tilde{\mathscr{L}}_{k-l}^0 v_l^0(t,x) = 0. \tag{5.11}$$

Combining (5.8)–(5.11) and Theorem 1, we obtain

$$\begin{aligned}
&\varepsilon^n \left\{ \frac{\partial}{\partial t} v_n^\varepsilon(t,x) + \mathscr{L}^0 v_n^\varepsilon(t,x) + \tilde{\mathscr{L}}_n^\varepsilon u^0(t,x) + \sum_{l=1}^{n-1} \tilde{\mathscr{L}}_{n-l}^0 v_l^0(t,x) \right\} \\
&+ \sum_{k=n+1}^{2n-2} \varepsilon^k \left\{ \tilde{\mathscr{L}}_{k-n}^0 v_n^\varepsilon(t,x) + \tilde{\mathscr{L}}_n^\varepsilon v_{k-n}^0(t,x) + \sum_{l=k-n+1}^{n-1} \tilde{\mathscr{L}}_{k-l}^0 v_l^0(t,x) \right\} \\
&+ \varepsilon^{2n-1} \left\{ \tilde{\mathscr{L}}_{n-1}^0 v_n^\varepsilon(t,x) + \tilde{\mathscr{L}}_n^\varepsilon v_{n-1}^0(t,x) \right\} + \varepsilon^{2n} \tilde{\mathscr{L}}_n^\varepsilon v_n^\varepsilon(t,x) = 0,
\end{aligned}$$

and thus,

$$\frac{\partial}{\partial t} v_n^\varepsilon(t,x) + \mathscr{L}^\varepsilon v_n^\varepsilon(t,x) + g_n^\varepsilon(t,x) = 0.$$

This implies the assertion. ■

Set

$$\tilde{v}_n^\varepsilon(t,x) = \mathrm{E}\left[\int_0^{\tau_D(X^{\varepsilon,x}) \wedge (T-t)} \exp\left(-\int_0^r c(X_v^{\varepsilon,x}, \varepsilon) dv \right) g_n^\varepsilon(r+t, X_r^{\varepsilon,x}) dr \right].$$

By [F]–[I], we find that there are $C_n > 0$, $\tilde{m}_n \in \mathbb{N}$ which are independent of ε and the function $M_n \in C([0,T)) \cap L^p([0,T), dt)$ determined by $u^0, v_1^0, \ldots, v_{n-1}^0$ such that

$$|g_n^\varepsilon(t,x)| \le C_n M_n(t)(1 + |x|^{2\tilde{m}_n}). \tag{5.12}$$

The inequalities (2.2) and (5.12) imply

$$|\tilde{v}_n^\varepsilon(t,x)| \le C_n' \int_t^T M_n(r) dr (1 + |x|^{2\tilde{m}_n}) \tag{5.13}$$

for some $C_n' > 0$ which is also independent of ε.

Proposition 5. $v_n^\varepsilon = \tilde{v}_n^\varepsilon$.

Proof. The assertion is easily obtained by the similar argument to Theorem 5.1.9 in [13]. ■

Proof of Theorem 3. By (5.8) and Proposition 5, we have $u^\varepsilon(t,x) - (u^0(t,x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t,x)) = \varepsilon^n \tilde{v}_n^\varepsilon(t,x)$. Our assertion is now immediately obtained by the inequality (5.13). ■

5.4 Proof of Theorem 10

Let v_n^ε and $\tilde{v}_n^\varepsilon$ be as in Section 5.3. Thanks for the assumption $I \subset [0,\infty)$, the same argument as the proof of Proposition 4 tells us that v_n^ε is a viscosity solution of (5.7). Moreover, we have the next proposition.

Proposition 6. *The function $\tilde{v}_n^\varepsilon$ is a viscosity solution of* (5.7).

Proof. Until the end of the proof we suppress ε in the notation. First, we check the continuity. By the similar argument to the proof of Lemma 4.2 in [17], we see that v_n is continuous on $[0,T) \times \bar{D}$. Moreover, by (5.13), we get

$$\sup_{x \in K \cap \bar{D}} |\tilde{v}_n(t,x)| \leq C_n'(1 + \sup_{x \in K} |x|^{2m}) \left\{ \int_0^T M_n(r)dr - \int_0^t M_n(r)dr \right\} \longrightarrow 0, \quad t \to T$$

for any compact set $K \subset \mathbb{R}^d$. Thus, v_n is continuous on $[0,T] \times \bar{D}$.

Next, we show that v_n is a viscosity subsolution of (5.7). Take any $(t,x) \in [0,T) \times D$ and let φ be $C^{1,2}$-function such that $v_n - \psi$ has a maximum 0 at (t,x). We may assume that φ and its derivatives have polynomial growth rates in x uniformly in t. By the Markov property, we have

$$\mathrm{E}\left[J(h)\tilde{v}_n(t+h, X_h^x)1_{\{\tau_D(X^x) \geq h\}}\right] = \mathrm{E}\left[\int_0^{(T-t)\wedge(\tau_D(X^x_{\cdot+h})+h)} J(r)g_n(t+r, X_r^x)dr 1_{\{\tau_D(X^x) \geq h\}}\right],$$

where $J(r) = \exp\left(-\int_0^r c(X_v^x, \varepsilon)dv\right)$. Since $\tau_D(X^x_{\cdot+h}) = \tau_D(X^x) - h$ on $\{\tau_D(X^x) \geq h\}$, we obtain

$$\mathrm{E}\left[J(h)\tilde{v}_n(t+h, X_h^x)1_{\{\tau_D(X^x) \geq h\}}\right] = \tilde{v}_n(t,x) - \mathrm{E}\left[\int_0^h J(r)g_n(t+r, X_r^x)dr 1_{\{\tau_D(X^x) \geq h\}}\right] - A_h,$$

where

$$A_h = \mathrm{E}\left[\int_0^{(T-t)\wedge\tau_D(X^x)} J(r)g_n(t+r, X_r^x)dr 1_{\{\tau_D(X^x) < h\}}\right].$$

Therefore,

$$\varphi(t,x) = \tilde{v}_n(t,x) = \mathrm{E}\left[J(h)\tilde{v}_n(t+h, X_h^x)1_{\{\tau_D(X^x) \geq h\}}\right] + \mathrm{E}[\int_0^h g_n(t+r, X_r^x)dr 1_{\{\tau_D(X^x) \geq h\}}] + A_h$$

$$\leq \quad \mathrm{E}\left[J(h)\varphi(t+h,X_h^x)1_{\{\tau_D(X^x)\geq h\}}\right] + \mathrm{E}[\int_0^h g_n(t+r,X_r^x)dr 1_{\{\tau_D(X^x)\geq h\}}] + A_h.$$

Applying Ito's formula to $J(r)\varphi(t+r,X_r^x)$, we get

$$\begin{aligned} & -\frac{1}{h}\int_0^h \mathrm{E}\left[\left\{\left(\frac{\partial}{\partial t}+\mathscr{L}\right)\varphi(t+r,X_r^x)+g_n(t+r,X_r^x)\right\}1_{\{\tau_D(X^x)\geq h\}}\right]dr \\ \leq \quad & \frac{A_h-\varphi(t,x)P(\tau_D(X^x)<h)}{h}. \end{aligned} \tag{5.14}$$

By (5.12) and the Schwarz inequality, we have

$$|A_h| \leq C_n''(1+|\iota(x)|^{2m})\int_0^T M_r dt P(\tau_D(X^x)<h)^{1/2}$$

for some $C_n''>0$. Using (5.1) and the Chebyshev inequality, we obtain

$$\begin{aligned} P(\tau_D(X^x)<h) \quad & \leq \quad \mathrm{E}[\sup_{0\leq r\leq h}|X_r^x-x|\geq \mathrm{dist}(x,\partial D)] \leq \frac{C_n'''}{\mathrm{dist}(x,\partial D)^8}\mathrm{E}[\sup_{r\in[0,h]}|X_r^x-x|^8] \\ & \leq \quad \frac{C_n''''}{\mathrm{dist}(x,\partial D)^8}(1+|x|^{2\tilde{m}})h^3 \end{aligned}$$

for some $C_n''', C_n'''', \tilde{m}>0$. Thus, letting $h\to 0$ in (5.14), we see that

$$-\frac{\partial}{\partial t}\varphi(t,x)-\mathscr{L}\varphi(t,x)-g_n(t,x)\leq 0.$$

Hence, $\tilde{v}_n$ is a viscosity subsolution of (5.7). By the similar argument, we also find that $\tilde{v}_n$ is a viscosity supersolution. By the definition of $\tilde{v}_n$, we easily get $\tilde{v}_n(T,x)=0$ for $x\in D$ and $\tilde{v}_n(t,x)=0$ for $(t,x)\in[0,T]\times\partial D$. ■

To see the equivalence $v_n^\varepsilon=\tilde{v}_n^\varepsilon$, we need to give a new proof of Proposition 5 under the assumptions of Theorem 10.

Proof of Proposition 5. Set $\bar{u}_n^\varepsilon(t,x) = u^0(t,x)+\sum_{k=1}^{n-1}\varepsilon^k v_k^0(t,x)+\varepsilon^n\tilde{v}_n^\varepsilon(t,x)$. The analogous argument of the proof of Proposition 4 implies that $\bar{u}_n^\varepsilon$ is a viscosity solutions of (2.3). We easily see that $\bar{u}_n^\varepsilon$ has a polynomial growth rate in x uniformly in t. Then, Theorem 9 leads us to $\bar{u}_n^\varepsilon=u^\varepsilon$. This equality and (5.8) imply the assertion. ■

Now, we obtain the assertion of Theorem 10 by the same way as that of Theorem 3.

5.5 Proof of Proposition 1

First, we notice the following relation:

$$\tilde{\mathscr{L}}_1^0 P_t^D \bar{f}(x) = \nu\rho\sigma^3 t\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)P_t^D\bar{f}(x). \tag{5.15}$$

Then, using the relations $\mathscr{L}^0\tilde{\mathscr{L}}_1^0 P_t^D \bar{f}(x) = \tilde{\mathscr{L}}_1^0 \mathscr{L}^0 P_t^D \bar{f}(x)$ and

$$\left(\frac{\partial}{\partial t} + \mathscr{L}^0\right) P_{T-t}^D \bar{f}(x) = 0,$$

we get

$$\left(\frac{\partial}{\partial t} + \mathscr{L}^0\right) \frac{T-t}{2} \tilde{\mathscr{L}}_1^0 P_{T-t}^D \bar{f}(x) \quad = \quad -\tilde{\mathscr{L}}_1^0 P_{T-t}^D \bar{f}(x). \tag{5.16}$$

Also, we have

$$\left(\frac{\partial}{\partial t} + \mathscr{L}^0\right) \int_0^{T-t} P_{T-t-r}^D \left(\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_r^D \bar{f}\right)(x) dr = -\tilde{\mathscr{L}}_1^0 P_{T-t}^D \bar{f}(x), \quad x \in (l, \infty). \tag{5.17}$$

Therefore, the function

$$\eta(t,x) = \int_0^{T-t} P_{T-t-r}^D \left(\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_r^D \bar{f}\right)(x) dr - \frac{T-t}{2} \tilde{\mathscr{L}}_1^0 P_{T-t}^D \bar{f}(x) \tag{5.18}$$

satisfies the following PDE

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \mathscr{L}^0\right) \eta(t,x) = 0, & (t,x) \in [0,T) \times (l,\infty), \\ \eta(T,x) = 0, & x \in (l,\infty), \\ \eta(t,l) = -\dfrac{T-t}{2} \tilde{\mathscr{L}}_1^0 P_{T-t}^D \bar{f}(l), & t \in [0,T). \end{cases}$$

Then Theorem 6.5.2 in [6] implies

$$\eta(0,x) = -\frac{1}{2} \mathrm{E}[(T - \tau_D(X^{0,x}))\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X^{0,x})}^D \bar{f}(l) 1_{\{\tau_D(X^{0,x})<T\}}]. \tag{5.19}$$

By (5.18) and (5.19), we get the assertion. ■

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