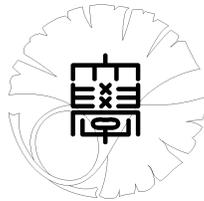# Inverse problem by Cauchy data
on arbitrary subboundary for
system of elliptic equations

by

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Inverse problem by Cauchy data on arbitrary subboundary for system of elliptic equations

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Abstract

We consider an inverse problem of determining coefficient matrices in an N-system of second-order elliptic equations in a bounded two dimensional domain by a set of Cauchy data on arbitrary subboundary. The main result of the article is as follows: If two systems of elliptic operators generate the same set of partial Cauchy data on an arbitrary subboundary, then the coefficient matrices of the first-order and zero-order terms satisfy the prescribed system of first-order partial differential equations. The main result implies the uniqueness of any two coefficient matrices provided that the one remaining matrix among the three coefficient matrices is known.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and let $\widetilde{\Gamma}$ be an open set on $\partial\Omega$ and $\Gamma_0 = \partial\Omega \setminus \overline{\overline{\widetilde{\Gamma}}}$, let ν be the unit outward normal vector to $\partial\Omega$. Consider the following boundary value problem:

$$L(x, D)u = \Delta u + 2A\partial_z u + 2B\partial_{\overline{z}} u + Qu = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0. \tag{1}$$

Here $u = (u_1, \ldots, u_N)$ is an unknown vector-valued function and A, B, Q be smooth $N \times N$ matrices, $i = \sqrt{-1}$, $x = (x_1, x_2) \in \mathbb{R}^2$, x is identified with $z = x_1 + ix_2 \in \mathbb{C}$, $\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$ and $\partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$.

Consider the following partial Cauchy data:

$$\mathcal{C}_{A,B,Q} = \{(u, \frac{\partial u}{\partial \nu})|_{\widetilde{\Gamma}};\ L(x, D)u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0,\ u \in H^1(\Omega)\}.$$

The paper is concerned with the following inverse problem: *Using the partial Cauchy data $\mathcal{C}_{A,B,Q}$, determine matrix coefficients A, B, Q.*

Note that we allowed freely choose Dirichlet data on $\widetilde{\Gamma}$ and measure the corresponding $\frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}$. In one special case of $N = 1$ and $A = B = 0$, this inverse boundary value problem is related to so called the Calderón's problem (see [5]), which is a mathematical realization of *Electrical Impedance Tomography*.

Similarly to the case of $N = 1$ in [12], the simultaneous determination of all three coefficients A, B, Q is impossible, but we can establish some equations for coefficient matrices (A, B, Q) which generate the same partial Cauchy data.

Our main result is

Theorem 1 *Let $A_j, B_j \in C^{5+\alpha}(\overline{\Omega})$ and $Q_j \in C^{4+\alpha}(\overline{\Omega})$ for $j = 1, 2$ and some $\alpha \in (0, 1)$. Suppose that $\mathcal{C}_{A_1,B_1,Q_1} = \mathcal{C}_{A_2,B_2,Q_2}$. Then*

$$A_1 = A_2 \quad and \quad B_1 = B_2 \quad on\ \widetilde{\Gamma}, \tag{2}$$

$$2\partial_z(A_1 - A_2) + B_2(A_1 - A_2) + (B_1 - B_2)A_1 - (Q_1 - Q_2) = 0 \quad in\ \Omega \tag{3}$$

and

$$2\partial_{\overline{z}}(B_1 - B_2) + A_2(B_1 - B_2) + (A_1 - A_2)B_1 - (Q_1 - Q_2) = 0 \quad in\ \Omega. \tag{4}$$

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In the case of $N = 1$ and two dimensions, there are many works and we refer to some of them, and here we do not intend to provide a complete list. In the case $\widetilde{\Gamma} = \partial\Omega$ of the full Cauchy data, the uniqueness in determining a potential q in the two dimensional case was proved for the conductivity equation by Nachman in [16] within C^4 conductivities, and later in [1] within L^∞ conductivities. For a convection equation see [6]. The case of the Schrödinger equation was solved by Bukhegim [3]. In the case of the partial Cauchy data on arbitrary subboundary, the uniqueness was obtained in [9] for potential $q \in C^{5+\alpha}(\overline{\Omega})$, and in [13], the regularity assumption was improved to $C^\alpha(\overline{\Omega})$ in the case of the full Cauchy data and up to $W_p^1(\Omega)$ with $p > 2$ in the case of partial Cauchy data on arbitrary subboundary. The case of general second-order elliptic equation was studied in the papers [12] and [10]. The results of [9] were extended to a Riemannian surface in [7]. The case where voltages are applied and currents are measured on disjoint subboundaries was discussed and the uniqueness is proved in [11]. Conditional stability estimates in determining a potential are obtained in [17]. For the Calderón problem for the Schrödinger equation in dimension three or more, we refer to the papers [4], [14], [15] and [18]. To the best knowledge of the authors, there are no publications for the uniqueness for weakly coupling system of second-order elliptic partial differential equations, and Theorem 1 is the affirmative answer.

Theorem 1 asserts that any two coefficient matrices among three are uniquely determined by partial Cauchy data on arbitrary subboundary $\widetilde{\Gamma}$ for the system of elliptic differential equations. That is,

Corollary 2 *Let* $(A_j, B_j, Q_j) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \times C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$ *for some* $\alpha \in (0, 1)$ *and be complex-valued. We assume that either* $A_1 \equiv A_2$ *or* $B_1 \equiv B_2$ *or* $Q_1 \equiv Q_2$ *in* Ω. *Then* $\mathcal{C}_{A_1, B_1, Q_1} = \mathcal{C}_{A_2, B_2, Q_2}$ *implies* $(A_1, B_1, Q_1) = (A_2, B_2, Q_2)$ *in* Ω.

Proof.

Case 1: $Q_1 = Q_2$.

Denote $R(x, D)(w_1, w_2) = (2\partial_z w_1 + B_2 w_1 + w_2 A_1, 2\partial_{\bar{z}} w_2 + A_2 w_2 + w_1 B_1)$. Therefore, applying Theorem 1, we obtain

$$R(x, D)(A_1 - A_2, B_1 - B_2) = 0 \quad \text{in } \Omega \tag{5}$$

and

$$(A_1 - A_2)|_{\widetilde{\Gamma}} = (B_1 - B_2)|_{\widetilde{\Gamma}} = 0. \tag{6}$$

Let a function $\psi \in C^2(\overline{\Omega})$ satisfy $|\nabla\psi| > 0$ on $\overline{\Omega}$, λ be a large positive parameter and $\phi = e^{\lambda\psi}$. Then there exist constants τ_0 and C independent of τ such that

$$|\tau|^{\frac{1}{2}}\|we^{\tau\phi}\|_{L^2(\Omega)} \le C\|(\partial_z w)e^{\tau\phi}\|_{L^2(\Omega)}, \quad \forall \tau \ge \tau_0 \text{ and } \forall w \in H_0^1(\Omega) \tag{7}$$

and

$$|\tau|^{\frac{1}{2}}\|we^{\tau\phi}\|_{L^2(\Omega)} \le C\|(\partial_{\bar{z}} w)e^{\tau\phi}\|_{L^2(\Omega)}, \quad \forall \tau \ge \tau_0 \text{ and } \forall w \in H_0^1(\Omega). \tag{8}$$

Consider the boundary value problem

$$R(x, D)(w_1, w_2) = (f_1, f_2) \quad \text{in } \Omega, \quad (w_1, w_2)|_{\partial\Omega} = 0. \tag{9}$$

Applying the Carleman estimates (7), (8) to each of N^2 equations in (9), we have

$$|\tau|^{\frac{1}{2}}\|(w_1, w_2)e^{\tau\phi}\|_{L^2(\Omega)} \le C\left(\sum_{j=1}^2 \|f_j e^{\tau\phi}\|_{L^2(\Omega)} + \|(w_1, w_2)e^{\tau\phi}\|_{L^2(\Omega)}\right), \quad \forall \tau \ge \tau_0. \tag{10}$$

The second term on the right-hand side of (10) can be absorbed into the left-hand side. Therefore we have

$$|\tau|^{\frac{1}{2}}\|(w_1, w_2)e^{\tau\phi}\|_{L^2(\Omega)} \le C\sum_{j=1}^2 \|f_j e^{\tau\phi}\|_{L^2(\Omega)}, \quad \forall \tau \ge \tau_0. \tag{11}$$

Using (11) and repeating the arguments in [8], we prove that a solution of the Cauchy problem (5), (6) is zero.

Case 2: $B_1 = B_2$.

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From equation (4), we have
$$(A_1 - A_2)B_1 = (Q_1 - Q_2) \quad \text{in } \Omega.$$

Hence equation (3) can be written as
$$2\partial_z(A_1 - A_2) + B_2(A_1 - A_2) - (A_1 - A_2)B_1 = 0 \quad \text{in } \Omega, \quad (A_1 - A_2)|_{\widetilde{\Gamma}} = 0. \tag{12}$$

Using (7), for the boundary value problem:
$$2\partial_z w + B_2 w - w B_1 = f \quad \text{in } \Omega, \quad w|_{\partial\Omega} = 0,$$

we obtain the estimate
$$|\tau|^{\frac{1}{2}} \|we^{\tau\phi}\|_{L^2(\Omega)} \le C\|fe^{\tau\phi}\|_{L^2(\Omega)} \quad \forall \tau \ge \tau_0. \tag{13}$$

Using Carleman estimate (13) and repeating the arguments in [8], we prove that solution of the Cauchy problem (12) is zero. Then equation (4) implies that $Q_1 = Q_2$.

The proof in the case $A_1 = A_2$ is the same.∎

Next we consider other form of elliptic systems:
$$\widetilde{L}(x, D)u = \Delta u + \mathcal{A}\partial_{x_1}u + \mathcal{B}\partial_{x_2}u + Qu. \tag{14}$$

Here $\mathcal{A}, \mathcal{B}, Q$ are complex-valued $N \times N$ matrices. Let us define the following set of partial Cauchy data:
$$\widetilde{C}_{\mathcal{A},\mathcal{B},Q} = \left\{ (u, \frac{\partial u}{\partial \nu})|_{\widetilde{\Gamma}}; \; \widetilde{L}(x, D)u = \Delta u + \mathcal{A}\partial_{x_1}u + \mathcal{B}\partial_{x_2}u + Qu = 0 \text{ in } \Omega, \; u|_{\Gamma_0} = 0, u \in H^1(\Omega) \right\}.$$

Then one can prove the following corollary.

Corollary 3 Let $Q_1, Q_2 \in C^{4+\alpha}(\overline{\Omega})$ and let two pairs of complex-valued coefficient matrices $(\mathcal{A}_1, \mathcal{B}_1), (\mathcal{A}_2, \mathcal{B}_2) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. We assume that $Q_1 \equiv Q_2$ in Ω. Then $(\mathcal{A}_1, \mathcal{B}_1) \equiv (\mathcal{A}_2, \mathcal{B}_2)$ in Ω.

Proof. Observe that $\widetilde{L}(x, D) = \Delta + A\partial_z + B\partial_{\bar{z}} + Q$ where $A = \mathcal{A} + i\mathcal{B}$ and $B = \mathcal{A} - i\mathcal{B}$. Therefore, applying Corollary 2, we complete the proof. ∎

Remark. Unlike Corollary 2, in the two cases of $\mathcal{A}_1 \equiv \mathcal{A}_2$ and $\mathcal{B}_1 \equiv \mathcal{B}_2$, we can not, in general, claim that $(\mathcal{A}_1, \mathcal{B}_1, Q_1) = (\mathcal{A}_2, \mathcal{B}_2, Q_2)$. By the same argument as Corollary 2, we can prove only
(i) $\frac{\partial \mathcal{B}_1}{\partial x_1} = \frac{\partial \mathcal{B}_2}{\partial x_1}$ in Ω if $\mathcal{A}_1 = \mathcal{A}_2$ in Ω.
(ii) $\frac{\partial \mathcal{A}_1}{\partial x_2} = \frac{\partial \mathcal{A}_2}{\partial x_2}$ in Ω if $\mathcal{B}_1 = \mathcal{B}_2$ in Ω.

Moreover consider the following example
$$\Omega = (0, 1) \times (0, 1),$$
$$\widetilde{\Gamma} = \{(x_1, x_2); \, x_2 = 0, \, 0 < x_1 < 1\} \cup \{(x_1, x_2)|x_2 = 1, \, 0 < x_1 < 1\},$$

and let us choose $\eta(x_2) \in C_0^\infty(0, 1)$. Then the operators $\widetilde{L}(x, D)$ and $e^{s\eta}\widetilde{L}(x, D)e^{-s\eta}$ generate the same partial Cauchy data, but the matrix coefficient matrices are not equal.

2 Preliminary results

Throughout the paper, we use the following notations.

Notations. $i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$, $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$, $\beta = (\beta_1, \beta_2), |\beta| = \beta_1 + \beta_2$. $D = (\frac{1}{i}\frac{\partial}{\partial x_1}, \frac{1}{i}\frac{\partial}{\partial x_2})$. Let χ_G be the characteristic function of the set G. The tangential derivative on the boundary is given by $\partial_{\vec{\tau}} = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to $\partial\Omega$, $B(\hat{x}, \delta) = \{x \in \mathbb{R}^2; |x - \hat{x}| < \delta\}$, $S(\hat{x}, \delta) = \{x \in \mathbb{R}^2; |x - \hat{x}| = \delta\}$. We set $(u, v)_{L^2(\Omega)} = \int_\Omega u\bar{v}dx$ for functions u, v, while by (a, b) we denote the scalar product in $\mathbb{R}^2$ if there is no fear of confusion. For

$f : \mathbb{R}^2 \to \mathbb{R}^1$, the symbol f'' denotes the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_k \partial x_j}$, $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to another Banach space Y. Let E be the $N \times N$ unit matrix. We set $\|u\|_{H^{1,\tau}(\Omega)} = (\|u\|^2_{H^1(\Omega)} + |\tau|^2 \|u\|^2_{L^2(\Omega)})^{\frac{1}{2}}$. Finally for any $\widetilde{x} \in \partial\Omega$, we introduce the left and the right tangential derivatives as follows:

$$\mathbf{D}_+(\widetilde{x})f = \lim_{s \to +0} \frac{f(\ell(s)) - f(\widetilde{x})}{s},$$

where $\ell(0) = \widetilde{x}$, $\ell(s)$ is a parametrization of $\partial\Omega$ near $\widetilde{x}$, s is the length of the curve, and we are moving clockwise as s increases;

$$\mathbf{D}_-(\widetilde{x})f = \lim_{s \to -0} \frac{f(\widetilde{\ell}(s)) - f(\widetilde{x})}{s},$$

where $\widetilde{\ell}(0) = \widetilde{x}$, $\widetilde{\ell}(s)$ is the parametrization of $\partial\Omega$ near $\widetilde{x}$, s is the length of the curve, and we are moving counterclockwise as s increases. By $o_X(\frac{1}{\tau^\kappa})$ we denote a function $f(\tau, \cdot)$ such that $\|f(\tau, \cdot)\|_X = o(\frac{1}{\tau^\kappa})$ as $|\tau| \to +\infty$.

For some $\alpha \in (0, 1)$, we consider a function $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^{6+\alpha}(\overline{\Omega})$ with real-valued φ and ψ such that

$$\partial_z \Phi(z) = 0 \quad \text{in } \Omega, \quad \operatorname{Im} \Phi|_{\Gamma_0^*} = 0, \tag{15}$$

where Γ_0^* is an open set on $\partial\Omega$ such that $\Gamma_0 \subset\subset \Gamma_0^*$. Denote by $\mathcal{H}$ the set of all the critical points of the function Φ:

$$\mathcal{H} = \{z \in \overline{\Omega}; \frac{\partial\Phi}{\partial z}(z) = 0\}.$$

Assume that Φ has no critical points on $\overline{\widetilde{\Gamma}}$, and that all critical points are nondegenerate:

$$\mathcal{H} \cap \partial\Omega \subset \Gamma_0, \quad \partial_z^2 \Phi(z) \neq 0, \quad \forall z \in \mathcal{H}. \tag{16}$$

Then Φ has only a finite number of critical points and we can set:

$$\mathcal{H} \setminus \Gamma_0 = \{\widetilde{x}_1, ..., \widetilde{x}_\ell\}, \quad \mathcal{H} \cap \Gamma_0 = \{\widetilde{x}_{\ell+1}, ..., \widetilde{x}_{\ell+\ell'}\}. \tag{17}$$

Let $\partial\Omega = \cup_{j=1}^{\mathcal{N}} \gamma_j$, where γ_j is a closed contour. The following proposition was proved in [9].

Proposition 1 *Let $\widetilde{x}$ be an arbitrary point in Ω. There exists a sequence of functions $\{\Phi_\epsilon\}_{\epsilon \in (0,1)}$ satisfying (15), (16) and there exists a sequence $\{\widetilde{x}_\epsilon\}, \epsilon \in (0, 1)$ such that*

$$\widetilde{x}_\epsilon \in \mathcal{H}_\epsilon = \{z \in \overline{\Omega}; \frac{\partial\Phi_\epsilon}{\partial z}(z) = 0\}, \quad \widetilde{x}_\epsilon \to \widetilde{x} \quad as \; \epsilon \to +0.$$

Moreover for any j from $\{1, \ldots, \mathcal{N}\}$, we have

$$\mathcal{H}_\epsilon \cap \gamma_j = \emptyset \quad if \; \gamma_j \cap \widetilde{\Gamma} \neq \emptyset,$$

$$\mathcal{H}_\epsilon \cap \gamma_j \subset \Gamma_0 \quad if \; \gamma_j \cap \widetilde{\Gamma} = \emptyset$$

and

$$\operatorname{Im} \Phi_\epsilon(\widetilde{x}_\epsilon) \notin \{\operatorname{Im} \Phi_\epsilon(x); x \in \mathcal{H}_\epsilon \setminus \{\widetilde{x}_\epsilon\}\} \; and \; \operatorname{Im} \Phi_\epsilon(\widetilde{x}_\epsilon) \neq 0.$$

The following proposition was proved in [12].

Proposition 2 *Let $\widehat{\Gamma}_* \subset\subset \widetilde{\Gamma}$ be an arc with the left endpoint x_- and the right endpoint x_+ oriented clockwise. For any $\widehat{x} \in \operatorname{Int} \widehat{\Gamma}_*$, there exists a function $\Phi(z)$ which satisfies (15), (16), $\operatorname{Im} \Phi|_{\partial\Omega \setminus \widehat{\Gamma}_*} = 0$,*

$$\widehat{x} \in \mathcal{G} = \{x \in \widehat{\Gamma}_*; \frac{\partial \operatorname{Im} \Phi}{\partial \vec{\tau}}(x) = 0\}, \quad card\, \mathcal{G} < \infty \tag{18}$$

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and

$$(\frac{\partial}{\partial \overline{\tau}})^2 Im\,\Phi(x) \neq 0 \quad \forall x \in \mathcal{G} \setminus \{x_-, x_+\}. \tag{19}$$

Moreover

$$Im\,\Phi(\widehat{x}) \neq Im\,\Phi(x), \quad \forall x \in \mathcal{G} \setminus \{\widehat{x}\} \ and \quad Im\,\Phi(\widehat{x}) \neq 0 \tag{20}$$

and

$$\mathbf{D}_+(x_-)(\frac{\partial}{\partial \overline{\tau}})^6 Im\,\Phi \neq 0, \quad \mathbf{D}_-(x_+)(\frac{\partial}{\partial \overline{\tau}})^6 Im\,\Phi \neq 0. \tag{21}$$

Later we use the following Proposition (see [9]) :

Proposition 3 *Let* Φ *satisfy (15) and (16). For every* $g \in L^1(\Omega)$, *we have*

$$\int_\Omega g e^{\tau(\Phi-\overline{\Phi})} dx \to 0 \quad as \quad \tau \to +\infty.$$

Moreover

Proposition 4 *Let* Φ *satisfy (15), (16),* $g \in W_p^1(\Omega)$ *with some* $p > 2$, $g|_{\mathcal{H}} = 0$ *and* $supp\,g \subset \Omega$. *Then*

$$\int_\Omega g e^{\tau(\Phi-\overline{\Phi})} dx = o(\frac{1}{\tau}) \quad as \quad \tau \to +\infty.$$

Proof. By the Sobolev imbedding theorem, the function g belongs to $C^\alpha(\overline{\Omega})$ for some positive α. Note that by (16) and the assumption on g, we have

$$\|g\frac{(\nabla\psi,\nu)e^{\tau(\Phi-\overline{\Phi})}}{2i|\nabla\psi|^2}\|_{C(\overline{S(\widetilde{x}_j,\delta)})} \leq \frac{C\|g\|_{C(\overline{S(\widetilde{x}_j,\delta)})}}{\delta} \leq \frac{C}{\delta^{1-\alpha}}. \tag{22}$$

Also

$$\mathrm{div}\,(g\frac{\nabla\psi}{2i\tau|\nabla\psi|^2}) = (\nabla g, \frac{\nabla\psi}{2i\tau|\nabla\psi|^2}) + g\mathrm{div}\left(\frac{\nabla\psi}{2i\tau|\nabla\psi|^2}\right).$$

Since

$$|(\nabla g, \frac{\nabla\psi}{2i\tau|\nabla\psi|^2})| \leq C\sum_{j=1}^{\ell+\ell'}\frac{|\nabla g(x)|}{|x-\widetilde{x}_j|},$$

by the Hölder inequality we conclude that $(\nabla g, \frac{\nabla\psi}{2i\tau|\nabla\psi|^2}) \in L^1(\Omega)$. By (16) and assumption that $g|_{\mathcal{H}} = 0$, we obtain

$$\left|g\mathrm{div}\left(\frac{\nabla\psi}{2i\tau|\nabla\psi|^2}\right)\right| \leq C\sum_{j=1}^{\ell+\ell'}\frac{|g(x)|}{|x-\widetilde{x}_j|^2} \leq C\sum_{j=1}^{\ell+\ell'}\frac{\|g\|_{C^\alpha(\overline{\Omega})}}{|x-\widetilde{x}_j|^{2-\alpha}}.$$

Therefore $\mathrm{div}\left(g\frac{\nabla\psi}{2i\tau|\nabla\psi|^2}\right) \in L^1(\Omega)$. By (22), passing to the limit as δ goes to zero, we have

$$J = \int_\Omega g e^{\tau(\Phi-\overline{\Phi})} dx = \lim_{\delta\to 0}\int_{\Omega\setminus\cup_{j=1}^{\ell+\ell'} S(\widetilde{x}_j,\delta)} g e^{\tau(\Phi-\overline{\Phi})} dx = \lim_{\delta\to 0}\int_{\Omega\setminus\cup_{j=1}^{\ell+\ell'} S(\widetilde{x}_j,\delta)} g\frac{(\nabla\psi,\nabla)e^{\tau(\Phi-\overline{\Phi})}}{2i\tau|\nabla\psi|^2} dx$$

$$= \lim_{\delta\to 0}\int_{\cup_{j=1}^{\ell+\ell'} S(\widetilde{x}_j,\delta)} g\frac{(\nabla\psi,\nu)e^{\tau(\Phi-\overline{\Phi})}}{2i\tau|\nabla\psi|^2} d\sigma - \lim_{\delta\to 0}\int_{\Omega\setminus\cup_{j=1}^{\ell+\ell'} S(\widetilde{x}_j,\delta)} \mathrm{div}\,(g\frac{\nabla\psi}{2i\tau|\nabla\psi|^2})e^{\tau(\Phi-\overline{\Phi})} dx$$

$$= -\int_\Omega \mathrm{div}\,(g\frac{\nabla\psi}{2i\tau|\nabla\psi|^2})e^{\tau(\Phi-\overline{\Phi})} dx.$$

Using Proposition 3, we finish the proof. ∎

Consider the boundary value problem

$$L(x,D)u = f \quad \mathrm{in} \quad \Omega, \quad u|_{\partial\Omega} = 0.$$

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Proposition 5 *Suppose that Φ satisfies (15), (16), $u \in H_0^1(\Omega)$ and $\|A\|_{L^\infty(\Omega)} + \|B\|_{L^\infty(\Omega)} + \|Q\|_{L^\infty(\Omega)} \leq K$. Then there exist $\tau_0 = \tau_0(K, \Phi)$ and $C = C(K, \Phi)$, independent of u and τ, such that*

$$|\tau|\|ue^{\tau\varphi}\|_{L^2(\Omega)}^2 + \|ue^{\tau\varphi}\|_{H^1(\Omega)}^2 + \|\frac{\partial u}{\partial \nu}e^{\tau\varphi}\|_{L^2(\Gamma_0)}^2 + \tau^2\|\frac{\partial \Phi}{\partial z}|ue^{\tau\varphi}\|_{L^2(\Omega)}^2$$

$$\leq C(\|(L(x,D)u)e^{\tau\varphi}\|_{L^2(\Omega)}^2 + |\tau|\int_{\widetilde{\Gamma}_*}|\frac{\partial u}{\partial \nu}|^2 e^{2\tau\varphi}d\sigma) \tag{23}$$

for all $|\tau| > \tau_0$.

For the scalar equation, the estimate is proved in [12] . In order to prove this estimate for the system, it is sufficient to apply the scalar estimate to each equation in the system and take an advantage of the second large parameter in order to absorb the right-hand side.

Using estimate (23), we obtain

Proposition 6 *There exists a constant τ_0 such that for $|\tau| \geq \tau_0$ and any $f \in L^2(\Omega)$, there exists a solution to the boundary value problem*

$$L(x,D)u = f \quad in \ \Omega, \quad u|_{\Gamma_0} = 0 \tag{24}$$

such that

$$\|u\|_{H^{1,\tau}(\Omega)}/\sqrt{|\tau|} \leq C\|f\|_{L^2(\Omega)}. \tag{25}$$

Moreover if $f/\partial_z\Phi \in L^2(\Omega)$, then for any $|\tau| \geq \tau_0$ there exists a solution to the boundary value problem (24) such that

$$\|u\|_{H^{1,\tau}(\Omega)} \leq C\|f/\partial_z\Phi\|_{L^2(\Omega)}. \tag{26}$$

The constants C in (25) and (26) are independent of τ.

The proof is exactly the same as the proof of Proposition 2.5 in [12] and relies on the Carleman estimate (23).

Let us introduce the operators:

$$\partial_{\overline{z}}^{-1}g = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1, \xi_2)}{\zeta - z}d\xi_1 d\xi_2, \quad \partial_z^{-1}g = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1, \xi_2)}{\overline{\zeta} - \overline{z}}d\xi_1 d\xi_2.$$

Then we have (e.g., p.47, 56, 72 in [19]):

Proposition 7 A) *Let $m \geq 0$ be an integer number and $\alpha \in (0,1)$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(C^{m+\alpha}(\overline{\Omega}), C^{m+\alpha+1}(\overline{\Omega}))$.*
B) *Let $1 \leq p \leq 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$.*
C) *Let $1 < p < \infty$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), W_p^1(\Omega))$.*

For any matrix $B \in C^{5+\alpha}(\overline{\Omega})$, consider the linear operators T_B and P_B such that

$$(2\partial_z + B)T_B g = g \quad in \ \Omega; \quad (2\partial_{\overline{z}} + B)P_B g = g \quad in \ \Omega \tag{27}$$

and

$$T_B, P_B \in \mathcal{L}(H^s(\Omega), H^{s+1}(\Omega)) \cap \mathcal{L}(C^{k+\alpha}(\Omega), C^{k+1+\alpha}(\Omega)) \quad \forall s \in [0,6], \forall k \in \{0,1,\ldots 6\}, \tag{28}$$

and

$$T_B, P_B \in \mathcal{L}((H^1(\Omega))', L^2(\Omega)). \tag{29}$$

The existence of the operators T_B, P_B with the above properties follows from the regularity theory of elliptic systems on the plane (see e.g., [20]).

Let $e \in C_0^\infty(\Omega)$ satisfy $|e(x)| \leq 1$, the support of e be concentrated in a small neighborhood of $\mathcal{H} \setminus \overline{\Gamma}_0$ and e be identically equal to one in an open set $\mathcal{O}$ which contains $\mathcal{H} \setminus \overline{\Gamma}_0$. We introduce the operators $\mathfrak{T}_B$ and $\mathfrak{P}_B$ by

$$\mathfrak{T}_B = \frac{1}{2}\sum_{j=0}^\infty (-1)^j (\frac{1}{2}\partial_z^{-1}eB)^j \partial_z^{-1}, \quad \mathfrak{P}_B = \frac{1}{2}\sum_{j=0}^\infty (-1)^j (\frac{1}{2}\partial_{\overline{z}}^{-1}eB)^j \partial_{\overline{z}}^{-1}. \tag{30}$$

Taking the function e such that $\int_{\text{supp}\,e} 1 dx$ is sufficiently small, we have

$$\|\partial_z^{-1} eB\|_{\mathcal{L}(L^p(\Omega), L^p(\Omega))} < 1 \quad \text{and} \quad \|\partial_{\bar{z}}^{-1} eB\|_{\mathcal{L}(L^p(\Omega), L^p(\Omega))} < 1. \tag{31}$$

Indeed, by Proposition 7 for any $p > 1$ there exists a number $q \in (1, p)$ such that the operators $\partial_z^{-1}, \partial_{\bar{z}}^{-1} : L^q(\Omega) \to L^p(\Omega)$ are continuous. Therefore

$$\|\partial_z^{-1} eBg\|_{L^p(\Omega)} \le \|\partial_z^{-1}\|_{\mathcal{L}(L^q(\Omega), L^p(\Omega))} \|B\|_{L^\infty(\Omega)} \|eg\|_{L^q(\Omega)}$$

$$\le \|\partial_z^{-1}\|_{\mathcal{L}(L^q(\Omega), L^p(\Omega))} \|B\|_{L^\infty(\Omega)} \left(\int_{supp\,e} 1 dx \right)^{(p-q)/p} \|g\|_{L^p(\Omega)},$$

and if $\int_{\text{supp}\,e} 1 dx$ is small, then we easily have (31).

Hence the operators $\mathfrak{T}_B$ and $\mathfrak{P}_B$ introduced in (30) are correctly defined.

We define two other operators:

$$\mathcal{R}_\tau g = \frac{1}{2} e^{\tau(\Phi - \overline{\Phi})} \partial_{\bar{z}}^{-1} (g e^{\tau(\overline{\Phi} - \Phi)}), \quad \widetilde{\mathcal{R}}_\tau g = \frac{1}{2} e^{\tau(\overline{\Phi} - \Phi)} \partial_z^{-1} (g e^{\tau(\Phi - \overline{\Phi})}). \tag{32}$$

For any $N \times N$ matrix B with elements from $C^1(\overline{\Omega})$, we set

$$\mathrm{T}_B = \mathfrak{T}_B - T_B(1 - e)B\mathfrak{T}_B, \quad \mathrm{P}_B = \mathfrak{P}_B - P_B(1 - e)B\mathfrak{P}_B,$$

$$\widetilde{\mathcal{R}}_{\tau, B} g = \mathfrak{T}_{B, \tau} g - e^{\tau(\overline{\Phi} - \Phi)} T_B(e^{\tau(\Phi - \overline{\Phi})}(1 - e)B\mathfrak{T}_{B, \tau} g), \tag{33}$$

$$\mathcal{R}_{\tau, B} g = \mathfrak{P}_{B, \tau} g - e^{\tau(\Phi - \overline{\Phi})} P_B(e^{\tau(\overline{\Phi} - \Phi)}(1 - e)B\mathfrak{P}_{B, \tau} g)$$

and

$$\mathfrak{T}_{B, \tau} = e^{\tau(\overline{\Phi} - \Phi)} \mathfrak{T}_B e^{\tau(\Phi - \overline{\Phi})}, \quad \mathfrak{P}_{B, \tau} = e^{\tau(\Phi - \overline{\Phi})} \mathfrak{P}_B e^{\tau(\overline{\Phi} - \Phi)}. \tag{34}$$

For any $g \in C^\alpha(\overline{\Omega})$, the functions $\mathcal{R}_{\tau, B} g$ and $\widetilde{\mathcal{R}}_{\tau, B} g$ solve the equations:

$$(2\partial_{\bar{z}} + 2\tau \partial_{\bar{z}}\overline{\Phi} + B)\mathcal{R}_{\tau, B} g = g \quad \text{in } \Omega, \quad (2\partial_z + 2\tau \partial_z \Phi + B)\widetilde{\mathcal{R}}_{\tau, B} g = g \quad \text{in } \Omega. \tag{35}$$

We have

Proposition 8 *Let* $B \in C^1(\overline{\Omega}), g \in C^2(\overline{\Omega}), supp\,g \subset\subset \{x | e(x) = 1\}$ *and* $g|_{\mathcal{H}} = 0$. *Then for* $p \in (1, \infty)$, *we have*

$$\|\widetilde{\mathcal{R}}_{\tau, B} g - \frac{g}{2\tau \partial_z \Phi}\|_{L^p(\Omega)} + \|\mathcal{R}_{\tau, B} g - \frac{g}{2\tau \partial_{\bar{z}}\overline{\Phi}}\|_{L^p(\Omega)} = o(\frac{1}{\tau}) \quad as\ |\tau| \to \infty. \tag{36}$$

Proof. By Proposition 3.4 of [12], for any $p > 1$, we have

$$\|\widetilde{\mathcal{R}}_\tau g - \frac{g}{2\tau \partial_z \Phi}\|_{L^p(\Omega)} + \|\mathcal{R}_\tau g - \frac{g}{2\tau \partial_{\bar{z}}\overline{\Phi}}\|_{L^p(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to \infty. \tag{37}$$

Propositions 4 and 7 yield

$$\|R_\tau \{\frac{g}{2\tau \partial_z \Phi}\}\|_{L^p(\Omega)} + \|\widetilde{R}_\tau \{\frac{g}{2\tau \partial_{\bar{z}}\overline{\Phi}}\}\|_{L^p(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to \infty. \tag{38}$$

Thanks to (38) and (37), we obtain

$$\|\mathfrak{T}_{B, \tau} g - \frac{g}{2\tau \partial_z \Phi}\|_{L^p(\Omega)} + \|\mathfrak{P}_{B, \tau} g - \frac{g}{2\tau \partial_{\bar{z}}\overline{\Phi}}\|_{L^p(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to \infty. \tag{39}$$

By $\text{supp}\,g \subset\subset \{x | e(x) = 1\}$ and (28), (39), we obtain the asymptotic formula:

$$\|e^{\tau(\overline{\Phi} - \Phi)} T_B e^{\tau(\Phi - \overline{\Phi})} \circ (1 - e)B\mathfrak{T}_{B, \tau} g\|_{L^p(\Omega)} + \|e^{\tau(\Phi - \overline{\Phi})} P_B e^{\tau(\overline{\Phi} - \Phi)} \circ (1 - e)B\mathfrak{P}_{B, \tau} g\|_{L^p(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to \infty.$$

The proof is completed. ∎

3 Proof of Theorem 1

Proof of Theorem 1.

Step 1: Construction of complex geometric optics solutions.

Let the function Φ satisfy (15), (16) and $\widetilde{x}$ be some point from $\mathcal{H} \setminus \Gamma_0$. Without loss of generality, we may assume that $\widetilde{\Gamma}$ is an arc with the endpoints $x_\pm$.

Consider the following operator:

$$
\begin{aligned}
L_1(x, D) &= 4\partial_z\partial_{\overline{z}} + 2A_1\partial_z + 2B_1\partial_{\overline{z}} + Q_1 \\
&= (2\partial_z + B_1)(2\partial_{\overline{z}} + A_1) + Q_1 - 2\partial_z A_1 - B_1 A_1 \\
&= (2\partial_{\overline{z}} + A_1)(2\partial_z + B_1) + Q_1 - 2\partial_{\overline{z}} B_1 - A_1 B_1.
\end{aligned}
\tag{40}
$$

Let $(w_0, \widetilde{w}_0) \in C^{6+\alpha}(\overline{\Omega})$ be a nontrivial solution to the boundary value problem:

$$
\mathcal{K}(x, D)(w_0, \widetilde{w}_0) = (2\partial_{\overline{z}} w_0 + A_1 w_0, 2\partial_z \widetilde{w}_0 + B_1 \widetilde{w}_0) = 0 \quad \text{in } \Omega, \quad w_0 + \widetilde{w}_0 = 0 \quad \text{on } \Gamma_0.
\tag{41}
$$

We have

Proposition 9 *Let $\widetilde{x}$ be an arbitrary point from $\mathcal{H} \setminus \overline{\Gamma}_0$ and $\vec{z} \in \mathbb{C}^N$ be an arbitrary vector. There exists a solution $(w_0, \widetilde{w}_0) \in C^{6+\alpha}(\overline{\Omega})$ to problem (41) such that*

$$
w_0(\widetilde{x}) = \vec{z},
\tag{42}
$$

$$
\lim_{x \to x_\pm} \frac{|w_0(x)|}{|x - x_\pm|^{98}} = \lim_{x \to x_\pm} \frac{|\widetilde{w}_0(x)|}{|x - x_\pm|^{98}} = 0
\tag{43}
$$

and

$$
\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} w_0(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} \widetilde{w}_0(x) \quad \forall x \in \mathcal{H} \setminus \{\widetilde{x}\} \text{ and } \forall \alpha_1 + \alpha_2 \leq 6.
\tag{44}
$$

Proof. Let us fix a point $\widetilde{x}$ from $\mathcal{H} \setminus \{\widetilde{x}\}$. By Proposition 4.2 of [12] there exists a holomorphic function $a(z) \in C^7(\overline{\Omega})$ such that $Im\, a|_{\Gamma_0} = 0$, $a(\widetilde{x}) = 1$ and a vanishes at each point of the set $\{x_\pm\} \cup \mathcal{H} \setminus \{\widetilde{x}\}$. Let $(w_{0,0}, \widetilde{w}_{0,0}) \in C^{6+\alpha}(\overline{\Omega})$ be a solution to problem (41) such that $w_{0,0}(\widetilde{x}) = \vec{z}$. Since $(w_0, \widetilde{w}_0) = (a^{10} w_{0,0}, \overline{a}^{10} \widetilde{w}_{0,0})$ solves problem (41) and satisfies (44)-(42), the proof of the proposition is completed. ∎

Now we start the construction of complex geometric optics solution. Let the pair $(w_0, \widetilde{w}_0)$ be defined by Proposition 9. Short computations and (40) yield

$$
L_1(x, D)(w_0 e^{\tau\Phi}) = (Q_1 - 2\partial_z A_1 - B_1 A_1) w_0 e^{\tau\Phi}, \quad L_1(x, D)(\widetilde{w}_0 e^{\tau\overline{\Phi}}) = (Q_1 - 2\partial_{\overline{z}} B_1 - A_1 B_1) \widetilde{w}_0 e^{\tau\overline{\Phi}}.
\tag{45}
$$

Let e_1, e_2 be smooth functions such that

$$
\text{supp}\, e_1 \subset\subset \text{supp}\, e = 1, \quad e_1 + e_2 = 1 \quad \text{on } \Omega,
\tag{46}
$$

and e_1 vanishes in a neighborhood of $\partial\Omega$ and e_2 vanishes in a neighborhood of the set $\mathcal{H} \setminus \overline{\Gamma}_0$.

Denote $G_\epsilon = \{x \in \Omega | dist(\text{supp}\, e_1, x) > \epsilon\}$. We have

Proposition 10 *Let $B, q \in C^{5+\alpha}(\overline{\Omega})$ for some positive α and $\widetilde{q} \in W_p^1(\overline{\Omega})$ for some $p > 2$. Suppose that $q|_{\mathcal{H}} = \widetilde{q}|_{\mathcal{H}} = 0$. There exist smooth functions $m_\pm \in C^2(\overline{G}_\epsilon)$ which is independent of τ such that for any $\overline{G}_\epsilon \cap \text{supp}\, e = \emptyset$, the asymptotic formulae hold true:*

$$
\widetilde{\mathcal{R}}_{\tau, B_1}(e_1(q + \frac{\widetilde{q}}{\tau})) = e^{\tau(\overline{\Phi} - \Phi)} \left(\frac{m_+ e^{2i\tau\psi(\widetilde{x})}}{\tau^2} + o_{C^2(\overline{G}_\epsilon)}(\frac{1}{\tau^2}) \right) \quad as\, |\tau| \to +\infty,
\tag{47}
$$

$$
\mathcal{R}_{\tau, A_1}(e_1(q + \frac{\widetilde{q}}{\tau})) = e^{\tau(\Phi - \overline{\Phi})} \left(\frac{m_- e^{-2i\tau\psi(\widetilde{x})}}{\tau^2} + o_{C^2(\overline{G}_\epsilon)}(\frac{1}{\tau^2}) \right) \quad as\, |\tau| \to +\infty.
\tag{48}
$$

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Proof. By the Sobolev imbedding theorem the function $\tilde{q}$ belong to the space $C^\alpha(\bar{\Omega})$ with some positive α. Therefore the trace of $\tilde{q}$ on $\mathcal{H}$ defined correctly. For all N and for any domain G_{ϵ_0} with $\epsilon_0 > 0$ there exists a function $m_{+,N} \in C^2(\overline{G}_{\epsilon_0})$ such that

$$\frac{e^{-2\tau i\psi}}{2}(-1)^N(\frac{1}{2}\partial_z^{-1}eB)^N\partial_z^{-1}\left(e^{\tau(\Phi-\overline{\Phi})}e_1(q+\frac{\tilde{q}}{\tau})\right)|_{G_{\epsilon_0}} = e^{-2i\tau\psi}(\frac{m_{+,N}e^{2i\tau\psi(\tilde{x})}}{\tau^2} + o_{C^2(\overline{G}_{\epsilon_0})}(\frac{1}{\tau^2})). \quad (49)$$

This formula follows immediately from the stationary phase argument, the assumption that functions $q,\tilde{q}$ equal zero on $\mathcal{H}$, Proposition 4 and the representation of the operator $(-1)^N(\frac{1}{2}\partial_z^{-1}eB)^N\partial_z^{-1}e_1$ in the form:

$$(-1)^N(\frac{1}{2}\partial_z^{-1}eB)^N\partial_z^{-1}e_1g = \int_\Omega \tilde{K}(x,\xi)e_1(\xi)g(\xi)d\xi,$$

where

$$\tilde{K}(x,\xi) = \frac{\tilde{K}_*(x,\xi)}{x_1 - ix_2 - (\xi_1 - i\xi_2)}, \quad \tilde{K}_*(x,\xi) \in C^5(\bar{\Omega}) \times C^5(\bar{\Omega}).$$

Next let $x^0 = (x_1^0, x_2^0)$ be an arbitrary fixed point in Ω, $\partial^\beta = \partial_{x_1^0}^{\beta_1}\partial_{x_1^0}^{\beta_1}$, and $z^0 = x_1^0 + ix_2^0$. Let $CV = -\frac{1}{2}\partial_z^{-1}eVB$ for any matrix valued function $V(x)$. By Proposition 7 there exists $\hat{N}$ such that the operator

$$C^N \in \mathcal{L}(L^{\frac{4}{3}}(\Omega), C^5(\bar{\Omega})) \quad \forall N \geq \hat{N}. \quad (50)$$

We write the operator $\frac{(-1)^N}{2}(\frac{1}{2}\partial_z^{-1}eB)^N\partial_z^{-1}$ in the form of the integral operator

$$\frac{(-1)^N}{2}(\frac{1}{2}\partial_z^{-1}eB)^N\partial_z^{-1}e_1g = \frac{1}{\pi}\int_\Omega \frac{\mathcal{K}_N(x,\xi)e_1(\xi)g(\xi_1,\xi_2)}{x_1 - ix_2 - (\xi_1 - i\xi_2)}d\xi_1d\xi_2.$$

Let us estimate the kernel $\mathcal{K}_N$. Observe that

$$\mathcal{K}_N(x_1^0, x_2^0, \xi) = (-1)^N C^N \frac{E}{2\pi(\bar{z}^0 - \bar{z})}. \quad (51)$$

Since $\sup_{z^0 \in G_\epsilon, |\beta| \leq 5}\|\partial^\beta \frac{e}{\bar{z}^0 - \bar{z}}\|_{L^{\frac{4}{3}}(\Omega)} + \sup_{z^0 \in \Omega}\|\frac{1}{\bar{z}^0 - \bar{z}}\|_{L^{\frac{4}{3}}(\Omega)} < \infty$ there exists $r \in (0,1)$ independent of N such that

$$\sup_{z^0 \in \Omega}\|\partial_z^{-1}C^{N-1}\frac{E}{2\pi(\bar{z}^0 - \bar{z})}\|_{L^{\frac{4}{3}}(\Omega)} \leq r^{N-\hat{N}}. \quad (52)$$

By (52), (50) we obtain

$$\|\mathcal{K}_N(x,\cdot)\|_{(C^5(\overline{G}_\epsilon)\cap L^\infty(\Omega))\times C^5(\bar{\Omega})} \leq Cr^{N-\hat{N}}. \quad (53)$$

By (53) there exist a function $\mathcal{K}(x,\xi) \in (C^5(\overline{G}_\epsilon) \cap L^\infty(\Omega)) \times C^5(\bar{\Omega})$ such that

$$\sum_{j=\hat{N}+2}^\infty \frac{e^{-2\tau i\psi}}{2}(-1)^j(\frac{1}{2}\partial_z^{-1}eB)^j\partial_z^{-1}e_1g = e^{-2\tau i\psi}\frac{1}{\pi}\int_\Omega \frac{\mathcal{K}(x,\xi)e_1(\xi)g(\xi)}{x_1 - ix_2 - (\xi_1 - i\xi_2)}d\xi.$$

So, by the stationary phase argument there exists a function $m \in (C^2(\overline{G}_\epsilon) \cap L^\infty(\Omega)) \times C^5(\bar{\Omega})$ such that

$$\sum_{j=\hat{N}+2}^\infty e^{-2\tau i\psi}(-1)^j(\frac{1}{2}\partial_z^{-1}eB)^j\partial_z^{-1}(e^{2i\tau\psi}e_1(q+\frac{\tilde{q}}{\tau})) = e^{-2\tau i\psi}(\frac{m(x)e^{2i\tau\psi(\tilde{x})}}{\tau^2} + o_{L^\infty(G_\epsilon)}(\frac{1}{\tau^2})) \, \forall\tilde{\epsilon} > 0, \quad (54)$$

$$\sum_{j=\hat{N}+2}^\infty e^{-2\tau i\psi}(-1)^j(\frac{1}{2}\partial_z^{-1}eB)^j\partial_z^{-1}(e^{2i\tau\psi}e_1(q+\frac{\tilde{q}}{\tau})) = e^{-2\tau i\psi}(\frac{m(x)e^{2i\tau\psi(\tilde{x})}}{\tau^2} + o_{C^2(\overline{G}_\epsilon)}(\frac{1}{\tau^2})). \quad (55)$$

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By (54), (55), (49) for any positive $\widetilde{\epsilon}$ we have :

$$\mathfrak{T}_{B_1,\tau}(e_1(q+\frac{\widetilde{q}}{\tau}))|_{G_{\widetilde{\epsilon}}} = e^{\tau(\overline{\Phi}-\Phi)}\left(\frac{m_+e^{2i\tau\psi(\widetilde{x})}}{\tau^2}+o_{L^\infty(\overline{G}_{\widetilde{\epsilon}})}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty, \tag{56}$$

$$\mathfrak{P}_{B_1,\tau}(e_1(q+\frac{\widetilde{q}}{\tau}))|_{G_{\widetilde{\epsilon}}} = e^{\tau(\Phi-\overline{\Phi})}\left(\frac{m_-e^{-2i\tau\psi(\widetilde{x})}}{\tau^2}+o_{L^\infty(\overline{G}_{\widetilde{\epsilon}})}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty. \tag{57}$$

and

$$\mathfrak{T}_{B_1,\tau}(e_1(q+\frac{\widetilde{q}}{\tau}))|_{G_\epsilon} = e^{\tau(\overline{\Phi}-\Phi)}\left(\frac{m_+e^{2i\tau\psi(\widetilde{x})}}{\tau^2}+o_{C^2(\overline{G}_\epsilon)}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty, \tag{58}$$

$$\mathfrak{P}_{B_1,\tau}(e_1(q+\frac{\widetilde{q}}{\tau}))|_{G_\epsilon} = e^{\tau(\Phi-\overline{\Phi})}\left(\frac{m_-e^{-2i\tau\psi(\widetilde{x})}}{\tau^2}+o_{C^2(\overline{G}_\epsilon)}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty. \tag{59}$$

Let positive $\hat{\epsilon}$ be such that $\text{supp}\,(1-e) \subset G_{\hat{\epsilon}}$ and $\hat{\epsilon} < \epsilon, \epsilon'' \in (\hat{\epsilon},\epsilon)$ Then using (56) we have

$$e^{-2\tau i\psi}T_{B_1}(e^{\tau(\Phi-\overline{\Phi})}(1-e)B_1\mathfrak{T}_{B_1,\tau}e_1(q+\frac{\widetilde{q}}{\tau})) = e^{-2\tau i\psi}T_{B_1}((1-e)B_1\mathfrak{T}_{B_1}(e^{\tau(\Phi-\overline{\Phi})}e_1(q+\frac{\widetilde{q}}{\tau})))$$

$$= e^{-2\tau i\psi+2i\tau\psi(\widetilde{x})}T_{B_1}((1-e)\chi_{G_{\epsilon''}}B_1\frac{m_+}{\tau^2}+(1-e)\chi_{G_{\epsilon''}}o_{C^2(\overline{G}_{\epsilon''})}(\frac{1}{\tau^2}))+$$

$$e^{-2\tau i\psi+2i\tau\psi(\widetilde{x})}T_{B_1}((1-e)(1-\chi_{G_{\epsilon''}})B_1\frac{m_+}{\tau^2}+(1-e)(1-\chi_{G_{\epsilon''}})o_{L^\infty(G_{\widetilde{\epsilon}})}(\frac{1}{\tau^2})). \tag{60}$$

Here in order to obtain the last equality we used (56) and (46). Using (58), (60), (28) and Proposition 7 we obtain (47). $\blacksquare$

Denote $q_1 = P_{A_1}((Q_1-2\partial_z A_1-B_1 A_1)w_0)-M_1, q_2 = T_{B_1}((Q_1-2\partial_{\overline{z}}B_1-A_1 B_1)\widetilde{w}_0)-M_2 \in C^{5+\alpha}(\overline{\Omega})$, where the functions $M_1 \in Ker(\partial_{\overline{z}}+A_1)$ and $M_2 \in Ker(\partial_z+B_1)$ are taken such that

$$q_1(\widetilde{x}) = q_2(\widetilde{x}) = 0, \quad \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}q_1(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}q_2(x) \quad \forall x \in \mathcal{H} \setminus \{\widetilde{x}\} \text{ and } \forall \alpha_1+\alpha_2 \le 5. \tag{61}$$

By Proposition 10, there exist functions $m_\pm \in C^2(\partial\Omega)$ such that

$$\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\frac{\widetilde{q}_1}{\tau})) = e^{\tau(\overline{\Phi}-\Phi)}\left(\frac{m_+e^{2i\tau\psi(\widetilde{x})}}{\tau^2}+o_{H^1(\partial\Omega)}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty \tag{62}$$

and

$$\mathcal{R}_{\tau,A_1}(e_1(q_2+\frac{\widetilde{q}_2}{\tau})) = e^{\tau(\Phi-\overline{\Phi})}\left(\frac{m_-e^{-2i\tau\psi(\widetilde{x})}}{\tau^2}+o_{H^1(\partial\Omega)}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty. \tag{63}$$

Next we introduce the functions $w_{-1}, \widetilde{w}_{-1}, a_\pm, b_\pm \in C^2(\overline{\Omega})$ as a solutions to the following boundary value problems:

$$\mathcal{K}(x,D)(w_{-1},\widetilde{w}_{-1}) = 0 \quad \text{in } \Omega, \quad (w_{-1}+\widetilde{w}_{-1})|_{\Gamma_0} = \frac{q_1}{2\partial_z\Phi}+\frac{q_2}{2\partial_{\overline{z}}\overline{\Phi}}, \tag{64}$$

$$\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}w_{-1}(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\widetilde{w}_{-1}(x) \quad \forall x \in \mathcal{H} \text{ and } \forall \alpha_1+\alpha_2 \le 2,$$

$$\mathcal{K}(x,D)(a_\pm,b_\pm) = 0 \quad \text{in } \Omega, \quad (a_\pm+b_\pm)|_{\Gamma_0} = m_\pm. \tag{65}$$

We set $p_1 = -(Q_1-2\partial_{\overline{z}}B_1-A_1 B_1)(\frac{e_1 q_1}{2\partial_z\Phi}+w_{-1})+L_1(x,D)(\frac{e_2 q_1}{2\partial_z\Phi})$, $p_2 = -(Q_1-2\partial_z A_1-B_1 A_1)(\frac{e_1 q_2}{2\partial_{\overline{z}}\overline{\Phi}}+\widetilde{w}_{-1})+L_1(x,D)(\frac{e_2 q_2}{2\partial_{\overline{z}}\overline{\Phi}})$, $\widetilde{q}_2 = T_{B_1}p_2-\widetilde{M}_2, \widetilde{q}_1 = P_{A_1}p_1-\widetilde{M}_1$, where $\widetilde{M}_1 \in Ker(\partial_{\overline{z}}+A_1)$ and $\widetilde{M}_2 \in Ker(\partial_z+B_1)$ are taken such that

$$\widetilde{q}_1(\widetilde{x}) = \widetilde{q}_2(\widetilde{x}) = 0, \quad \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\widetilde{q}_1(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\widetilde{q}_2(x) \quad \forall x \in \mathcal{H} \setminus \{\widetilde{x}\} \text{ and } \forall \alpha_1+\alpha_2 \le 2. \tag{66}$$

Since $\frac{\widetilde{q}_1}{2\partial_z\Phi}, \frac{\widetilde{q}_2}{2\partial_{\overline{z}}\overline{\Phi}} \in H^1(\partial\Omega)$ by (66), there exists a solution $(w_{-2},\widetilde{w}_{-2}) \in H^1(\overline{\Omega})$ to the boundary value problem

$$\mathcal{K}(x,D)(w_{-2},\widetilde{w}_{-2}) = 0 \quad \text{in } \Omega, \quad (w_{-2}+\widetilde{w}_{-2})|_{\Gamma_0} = \frac{\widetilde{q}_1}{2\partial_z\Phi}+\frac{\widetilde{q}_2}{2\partial_{\overline{z}}\overline{\Phi}}. \tag{67}$$

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We introduce the functions $w_{0,\tau}, \widetilde{w}_{0,\tau} \in H^1(\Omega)$ by

$$w_{0,\tau} = w_0 + \frac{w_{-1} - e_2 q_1/2\partial_z\Phi}{\tau} + \frac{1}{\tau^2}(e^{2i\tau\psi(\widetilde{x})}a_+ + e^{-2i\tau\psi(\widetilde{x})}a_- + w_{-2} - \frac{\widetilde{q}_1}{2\partial_z\Phi}) \tag{68}$$

and

$$\widetilde{w}_{0,\tau} = \widetilde{w}_0 + \frac{\widetilde{w}_{-1} - e_2 q_2/2\partial_{\overline{z}}\overline{\Phi}}{\tau} + \frac{1}{\tau^2}(e^{2i\tau\psi(\widetilde{x})}b_+ + e^{-2i\tau\psi(\widetilde{x})}b_- + \widetilde{w}_{-2} - \frac{\widetilde{q}_2}{2\partial_{\overline{z}}\overline{\Phi}}). \tag{69}$$

Simple computations and Proposition 8 for any $p \in (1, \infty)$ imply the asymptotic formula:

$$L_1(x,D)(-e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - \frac{e_2(q_1 + \widetilde{q}_1/\tau)e^{\tau\Phi}}{2\tau\partial_z\Phi} - e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau))$$

$$-\frac{e_2(q_2 + \widetilde{q}_2/\tau)e^{\tau\overline{\Phi}}}{2\tau\partial_{\overline{z}}\overline{\Phi}}) = -L_1(x,D)(e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) + \frac{e_2(q_1 + \widetilde{q}_1/\tau)e^{\tau\Phi}}{2\tau\partial_z\Phi})$$

$$-L_1(x,D)(e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau)) + \frac{e_2(q_2 + \widetilde{q}_2/\tau)e^{\tau\overline{\Phi}}}{2\tau\partial_{\overline{z}}\overline{\Phi}})$$

$$= -(Q_1 - 2\partial_{\overline{z}}B_1 - A_1 B_1)e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - (Q_1 - 2\partial_z A_1 - B_1 A_1)e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau))$$

$$-e^{\tau\Phi}L_1(x,D)(\frac{e_2(q_1 + \widetilde{q}_1/\tau)}{2\tau\partial_z\Phi}) - e^{\tau\overline{\Phi}}L_1(x,D)(\frac{e_2(q_2 + \widetilde{q}_2/\tau)}{2\tau\partial_{\overline{z}}\overline{\Phi}})$$

$$-(Q_1 - 2\partial_{\overline{z}}B_1 - A_1 B_1)\widetilde{w}_0 e^{\tau\overline{\Phi}} - (Q_1 - 2\partial_z A_1 - B_1 A_1)w_0 e^{\tau\Phi}$$

$$-(Q_1 - 2\partial_{\overline{z}}B_1 - A_1 B_1)e^{\tau\Phi}\frac{e_1 q_1}{2\tau\partial_z\Phi} - (Q_1 - 2\partial_z A_1 - B_1 A_1)e^{\tau\overline{\Phi}}\frac{e_1 q_2}{2\tau\partial_{\overline{z}}\overline{\Phi}}$$

$$+\frac{1}{\tau}((Q_1 - 2\partial_{\overline{z}}B_1 - A_1 B_1)\frac{e_1 q_1}{2\partial_z\Phi} + L_1(x,D)(\frac{e_2 q_1}{2\partial_z\Phi}))e^{\tau\Phi}$$

$$+\frac{1}{\tau}((Q_1 - 2\partial_z A_1 - B_1 A_1)\frac{e_2 q_2}{2\partial_{\overline{z}}\overline{\Phi}} + L_1(x,D)(\frac{e_2 q_2}{2\partial_{\overline{z}}\overline{\Phi}}))e^{\tau\overline{\Phi}}$$

$$= -\frac{1}{\tau}(Q_1 - 2\partial_{\overline{z}}B_1 - A_1 B_1)w_{-1}e^{\tau\Phi} - \frac{1}{\tau}(Q_1 - 2\partial_z A_1 - B_1 A_1)\widetilde{w}_{-1}e^{\tau\overline{\Phi}}$$

$$-(Q_1 - 2\partial_{\overline{z}}B_1 - A_1 B_1)\widetilde{w}_0 e^{\tau\overline{\Phi}} - (Q_1 - 2\partial_z A_1 - B_1 A_1)w_0 e^{\tau\Phi} + e^{\tau\varphi}o_{L^p(\Omega)}(\frac{1}{\tau}). \tag{70}$$

Using this formula, we prove the following proposition.

Proposition 11 *For any $p > 1$, we have the asymptotic formula:*

$$L_1(x,D)(w_{0,\tau}e^{\tau\Phi} + \widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}} - e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau))) = e^{\tau\varphi}o_{L^p(\Omega)}(\frac{1}{\tau}), \tag{71}$$

$$(w_{0,\tau}e^{\tau\Phi} + \widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}} - e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau)))|_{\Gamma_0} = e^{\tau\varphi}o_{H^1(\Gamma_0)}(\frac{1}{\tau^2}). \tag{72}$$

Proof. By (15), (62), (63), (67) and (67)-(69), we have

$$(w_{0,\tau}e^{\tau\Phi} + \widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}} - e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau)))|_{\Gamma_0}$$

$$= (w_{0,\tau}e^{\tau\varphi} + \widetilde{w}_{0,\tau}e^{\tau\varphi} - e^{\tau\varphi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - e^{\tau\varphi}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau)))|_{\Gamma_0}$$

$$= e^{\tau\varphi}(w_0 + \frac{w_{-1} - e_2 q_1/2\partial_z\Phi}{\tau} + \frac{1}{\tau^2}(e^{2i\tau\psi(\widetilde{x})}a_+ + e^{-2i\tau\psi(\widetilde{x})}a_- + w_{-2} - \frac{\widetilde{q}_1}{2\partial_z\Phi})$$

$$+\widetilde{w}_0 + \frac{\widetilde{w}_{-1} - e_2 q_2/2\partial_{\overline{z}}\overline{\Phi}}{\tau} + \frac{1}{\tau^2}(e^{2i\tau\psi(\widetilde{x})}b_+ + e^{-2i\tau\psi(\widetilde{x})}b_- + \widetilde{w}_{-2} - \frac{\widetilde{q}_2}{2\partial_{\overline{z}}\overline{\Phi}})$$

$$-e^{\tau\varphi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - e^{\tau\varphi}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau))))|_{\Gamma_0}$$

$$= e^{\tau\varphi}\{\frac{1}{\tau^2}(e^{2i\tau\psi(\widetilde{x})}a_+ + e^{-2i\tau\psi(\widetilde{x})}a_- + e^{2i\tau\psi(\widetilde{x})}b_+ + e^{-2i\tau\psi(\widetilde{x})}b_-)$$

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$$-e^{\tau\varphi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde{q}_1/\tau)) - e^{\tau\varphi}\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau))\}|_{\Gamma_0} = e^{\tau\varphi}o_{H^1(\Gamma_0)}(\frac{1}{\tau^2}).$$

Here in order to obtain the final equality, we used Proposition 10. Similarly to (45) we obtain

$$L_1(x,D)(w_{0,\tau}e^{\tau\Phi} + \widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}} - \frac{e_2(q_1+\widetilde{q}_1/\tau)e^{\tau\Phi}}{2\tau\partial_z\Phi} - \frac{e_2(q_2+\widetilde{q}_2/\tau)e^{\tau\overline{\Phi}}}{2\tau\partial_{\overline{z}}\overline{\Phi}})$$
$$= (Q_1 - 2\partial_{\overline{z}}B_1 - A_1B_1)(w_{0,\tau} - \frac{e_2(q_1+\widetilde{q}_1/\tau)}{2\tau\partial_z\Phi})e^{\tau\Phi}$$
$$+ (Q_1 - 2\partial_z A_1 - B_1A_1)(\widetilde{w}_{0,\tau} - \frac{e_2(q_2+\widetilde{q}_2/\tau)}{2\tau\partial_{\overline{z}}\overline{\Phi}})e^{\tau\overline{\Phi}}. \qquad (73)$$

By (73) and (70), we obtain (71). ∎

We set $\mathcal{O}_\epsilon = \{x \in \Omega; dist(x,\partial\Omega) \le \epsilon\}$. In order to construct the last term in complex geometric optics solution, we need the following proposition:

Proposition 12 *Let $A,B \in C^{5+\alpha}(\overline{\Omega})$ and $Q \in C^{4+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, $f \in L^p(\Omega)$ for some $p > 2$, $dist(\overline{\Gamma}_0, supp f) > 0$, $q \in H^{\frac{1}{2}}(\Gamma_0)$, and ϵ be a small positive number such that $\overline{\mathcal{O}_\epsilon} \cap (\mathcal{H} \setminus \Gamma_0) = \emptyset$. Then there exists C independent of τ and τ_0 such that for all $|\tau| > \tau_0$, there exists a solution to the boundary value problem*

$$L(x,D)w = fe^{\tau\Phi} \quad in \ \Omega, \quad w|_{\Gamma_0} = qe^{\tau\varphi}/\tau \qquad (74)$$

such that

$$\sqrt{|\tau|}\|we^{-\tau\varphi}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|(\nabla w)e^{-\tau\varphi}\|_{L^2(\Omega)} + \|we^{-\tau\varphi}\|_{H^{1,\tau}(\mathcal{O}_\epsilon)} \le C(\|f\|_{L^p(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma_0)}). \qquad (75)$$

Proof. First let us assume that f is identically equal to zero. Let $(d,\widetilde{d}) \in H^1(\Omega) \times H^1(\overline{\Omega})$ satisfy

$$\mathcal{K}(x,D)(d,\widetilde{d}) = 0 \quad in \ \Omega, \quad (d+\widetilde{d})|_{\Gamma_0} = q. \qquad (76)$$

For exitance of such a solution see e.g. [20]. By (45) and (76), we have

$$L(x,D)(\frac{d}{\tau}e^{\tau\Phi} + \frac{\widetilde{d}}{\tau}e^{\tau\overline{\Phi}}) = \frac{1}{\tau}(Q - 2\partial_z A - BA)de^{\tau\Phi} + \frac{1}{\tau}(Q - 2\partial_{\overline{z}}B - AB)\widetilde{d}e^{\tau\overline{\Phi}}.$$

By Proposition 6, there exists a solution w to the boundary value problem

$$L(x,D)\tilde{w} = -\frac{1}{\tau}(Q - 2\partial_z A - BA)de^{\tau\Phi} - \frac{1}{\tau}(Q - 2\partial_{\overline{z}}B - AB)\widetilde{d}e^{\tau\overline{\Phi}}, \quad \tilde{w}|_{\Gamma_0} = 0$$

such that there exists a constant $C > 0$ such that

$$\|\tilde{w}e^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)} \le \frac{C}{\sqrt{|\tau|}}\|(Q - 2\partial_z A - BA)de^{i\tau\psi} + (Q - 2\partial_{\overline{z}}B - AB)\widetilde{d}e^{-\tau\psi})\|_{L^2(\Omega)} \le \frac{C}{\sqrt{|\tau|}}\|q\|_{H^{\frac{1}{2}}(\Gamma_0)}$$

for all large $\tau > 0$.

Then the function $(\frac{d}{\tau}e^{\tau\Phi} + \frac{\widetilde{d}}{\tau}e^{\tau\overline{\Phi}}) + \tilde{w}$ is a solution to (74) which satisfies (75) if $f \equiv 0$.

If f is not identically equal zero, then we consider the function $\tilde{w} = \widetilde{e}e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B}(e_1q_0)$, where $\widetilde{e} \in C_0^\infty(\Omega)$, $\widetilde{e}|_{supp e_1} = 1$ and $q_0 = P_A f - M$, where a function $M \in C^5(\overline{\Omega})$ belongs to $Ker\,(2\partial_z + B)$ and chosen such that $q_0|_{\mathcal{H}} = 0$. Then $L(x,D)\tilde{w} = (Q - 2\partial_{\overline{z}}B - AB)\tilde{w} + \widetilde{e}e_1 fe^{\tau\Phi} + 2\widetilde{e}e^{\tau\Phi}q_0\partial_{\overline{z}}e_1 + e^{\tau\Phi}(2\partial_{\overline{z}} + A)(\partial_z e\widetilde{\mathcal{R}}_{\tau,B}(e_1q_0))$. Since, by Proposition 8, the function $\tilde{f}(\tau,\cdot) = e^{-\tau\Phi}L(x,D)\tilde{w} - f$ can be represented as a sum of two functions, where the first one equal to zero in a neighborhood of $\mathcal{H}$ and is bounded uniformly in τ in $L^2(\Omega)$ norm, the second one is $O_{L^2(\Omega)}(\frac{1}{\tau})$. Applying Proposition 6 to the boundary value problem

$$L(x,D)w_* = \tilde{f}e^{\tau\Phi} \quad in \ \Omega, \quad w_*|_{\Gamma_0} = 0,$$

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we construct a solution such that

$$\|w_* e^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)} \le C\|f\|_{L^p(\Omega)}.$$

The function $w^* - \tilde{w}$ solves the boundary value problem (74) and satisfies estimate (75). $\blacksquare$

Using Propositions 12 and 11, we construct the last term u_{-1} in complex geometric optics solution which satisfies

$$\sqrt{|\tau|}\|u_{-1}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|(\nabla u_{-1})\|_{L^2(\Omega)} + \|u_{-1}\|_{H^{1,\tau}(\mathcal{O}_\epsilon)} = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \tag{77}$$

Finally we obtain a complex geometric optics solution in the form:

$$u_1(x) = w_{0,\tau}e^{\tau\Phi} + \widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}} - e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(q_1 + \widetilde{q}_1/\tau) - e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(q_2 + \widetilde{q}_2/\tau) + e^{\tau\varphi}u_{-1}. \tag{78}$$

Obviously

$$L_1(x,D)u_1 = 0 \quad \text{in } \Omega, \quad u_1|_{\Gamma_0} = 0. \tag{79}$$

Let u_1 be a complex geometrical optics solution as in (78).

Let $\mathbf{e} \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\mathbf{e}$ is equal to one in a ball of small radius centered at 0. We set

$$\eta(x,s) = \mathbf{e}((x - \widetilde{x})e^{s^2}). \tag{80}$$

Then the operator

$$L_2(x,s,D) = e^{-s\eta}L_2(x,D)e^{s\eta} = \Delta + 2(A_2 + 2s\eta_{\overline{z}})\partial_z + 2(B_2 + 2s\eta_z)\partial_{\overline{z}} + Q_2$$
$$+ \quad (s\Delta\eta + s^2(\nabla\eta,\nabla\eta))E + 2s\eta_z A_2 + 2s\eta_{\overline{z}}B_2$$

is of the form (1) and has the same partial Cauchy data as the operator $L_2(x,D)$. Also for the operator $L_2(x,s,D)$, one can construct a similar complex geometric optics solution.

Consider the operator

$$L_2(x,s,D)^* = 4\partial_z\partial_{\overline{z}} - 2A_{2,s}^*\partial_{\overline{z}} - 2B_{2,s}^*\partial_z + Q_{2,s}^* - 2\partial_{\overline{z}}A_{2,s}^* - 2\partial_z B_{2,s}^*$$
$$= (2\partial_z - A_{2,s}^*)(2\partial_{\overline{z}} - B_{2,s}^*) + Q_2^* - 2\partial_{\overline{z}}A_2^* - A_2^*B_2^*$$
$$= (2\partial_{\overline{z}} - B_{2,s}^*)(2\partial_z - A_{2,s}^*) + Q_2^* - 2\partial_z B_2^* - B_2^*A_2^*.$$

Similarly we construct the complex geometric optics solutions to the operator $L_2(x,s,D)^*$. Let $(w_1, \widetilde{w}_1) \in C^{6+\alpha}(\overline{\Omega})$ be a solutions to the following boundary value problem:

$$\mathcal{M}(x,D)(w_1, \widetilde{w}_1) = ((2\partial_{\overline{z}} - B_2^*)w_1, (2\partial_z - A_2^*)\widetilde{w}_1) = 0 \quad \text{in } \Omega, \quad (w_1 + \widetilde{w}_1)|_{\Gamma_0} = 0, \tag{81}$$

$$\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}w_1(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\widetilde{w}_1(x) \quad \forall x \in \mathcal{H} \text{ and } \forall\alpha_1 + \alpha_2 \le 2,$$

$$\lim_{x \to x_\pm}\frac{|w_1(x)|}{|x - x_\pm|^{98}} = \lim_{x \to x_\pm}\frac{|\widetilde{w}_1(x)|}{|x - x_\pm|^{98}} = 0. \tag{82}$$

Such a pair $(w_1, \widetilde{w}_1)$ exists due to Proposition 9. We set $(w_{1,s}, \widetilde{w}_{1,s}) = e^{s\eta}(w_1, \widetilde{w}_1)$. Observe that

$$L_2(x,s,D)^*(w_{1,s}e^{-\tau\Phi}) = (Q_2^* - 2\partial_{\overline{z}}A_2^* - A_2^*B_2^*)w_{1,s}e^{-\tau\Phi},$$

$$L_2(x,s,D)^*(\widetilde{w}_{1,s}e^{-\tau\overline{\Phi}}) = (Q_2^* - 2\partial_z B_2^* - B_2^*A_2^*)\widetilde{w}_{1,s}e^{-\tau\overline{\Phi}}.$$

We set

$$P_{-B_{2,s}^*} = e^{s\eta}P_{-B_2^*}e^{-s\eta}, T_{-A_{2,s}^*} = e^{s\eta}T_{-A_2^*}e^{-s\eta}, \widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*} = e^{s\eta}\widetilde{\mathcal{R}}_{-\tau,-A_2^*}e^{-s\eta}, \quad \mathcal{R}_{-\tau,-B_{2,s}^*} = e^{s\eta}\mathcal{R}_{-\tau,-B_2^*}e^{-s\eta}, \tag{83}$$

$$q_3 = P_{-B_2^*}((Q_2^* - 2\partial_{\overline{z}}A_2^* - A_2^*B_2^*)w_1) - M_3, \quad q_4 = T_{-A_2^*}((Q_2^* - 2\partial_z B_2^* - B_2^*A_2^*)\widetilde{w}_1) - M_4. \tag{84}$$

Denote $q_{3,s} = P_{-B_{2,s}^*}((Q_2^* - 2\partial_{\overline{z}}A_2^* - A_2^*B_2^*)w_{1,s}) - M_{3,s} = e^{s\eta}q_3, q_{4,s} = T_{-A_{2,s}^*}((Q_2^* - 2\partial_z B_2^* - B_2^*A_2^*)\widetilde{w}_{1,s}) - M_{4,s} = e^{s\eta}q_4$ where the functions $M_{j,s} = e^{s\eta}M_j$, $M_3 \in Ker(2\partial_{\overline{z}} - B_2^*)$ and $M_4 \in Ker(2\partial_z - A_2^*)$ are chosen such that

$$q_3(\widetilde{x}) = q_4(\widetilde{x}) = 0, \quad \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}q_3(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}q_4(x) \quad \forall x \in \mathcal{H} \setminus \{\widetilde{x}\} \text{ and } \forall\alpha_1 + \alpha_2 \le 5. \tag{85}$$

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By (85) the functions $\frac{q_3}{2\partial_z \Phi}, \frac{q_4}{2\partial_{\overline{z}}\overline{\Phi}}$ belong to the space $C^2(\overline{\Gamma}_0)$. Therefore we can introduce the functions $w_{-3}, \widetilde{w}_{-3}, \widetilde{a}_\pm, \widetilde{b}_\pm \in C^2(\overline{\Omega})$ as a solutions to the following boundary value problems:

$$\mathcal{M}(x, D)(w_{-3}, \widetilde{w}_{-3}) = 0 \quad \text{in } \Omega, \quad (w_{-3} + \widetilde{w}_{-3})|_{\Gamma_0} = \frac{q_3}{2\partial_z \Phi} + \frac{q_4}{2\partial_{\overline{z}}\overline{\Phi}}, \tag{86}$$

$$\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} w_{-3}(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\widetilde{w}_{-3}(x) \quad \forall x \in \mathcal{H} \text{ and } \forall \alpha_1 + \alpha_2 \leq 2,$$

$$\mathcal{M}(x, D)(\widetilde{a}_\pm, \widetilde{b}_\pm) = 0 \quad \text{in } \Omega, \quad (\widetilde{a}_\pm + \widetilde{b}_\pm)|_{\Gamma_0} = \widetilde{m}_\pm. \tag{87}$$

Let

$$p_3 = -(Q_2^* - 2\partial_{\overline{z}} A_2^* - A_2^* B_2^*)(\frac{e_1 q_{3,s}}{2\partial_z \Phi} + w_{-3,s}) - L_2(x, s, D)^*(\frac{q_{3,s} e_2}{2\partial_z \Phi}),$$

$$p_4 = -(Q_2^* - 2\partial_z B_2^* - B_2^* A_2^*)(\frac{e_1 q_{4,s}}{2\partial_{\overline{z}}\overline{\Phi}} + \widetilde{w}_{-3,s}) - L_2(x, s, D)^*(\frac{q_{4,s} e_2}{2\partial_{\overline{z}}\overline{\Phi}})$$

and

$$\widetilde{q}_3 = e^{-s\eta}(P_{-B_{2,s}^*} p_3 - \widetilde{M}_{3,s}), \quad \widetilde{q}_4 = e^{-s\eta}(T_{-A_{2,s}^*} p_4 - \widetilde{M}_{4,s}),$$

where $\widetilde{M}_{3,s} \in Ker(2\partial_{\overline{z}} - B_{2,s}^*), \widetilde{M}_{4,s} \in Ker(2\partial_z - A_{2,s}^*)$, and $(\widetilde{q}_{3,s}, \widetilde{q}_{4,s}) = e^{s\eta}(\widetilde{q}_3, \widetilde{q}_4)$ are chosen such that

$$\widetilde{q}_{3,s}(\widetilde{x}) = \widetilde{q}_{4,s}(\widetilde{x}) = 0, \quad \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\widetilde{q}_{3,s}(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\widetilde{q}_{4,s}(x) \quad \forall x \in \mathcal{H} \setminus \{\widetilde{x}\} \text{ and } \forall \alpha_1 + \alpha_2 \leq 2. \tag{88}$$

The following asymptotic formula holds true:

Proposition 13 *There exist smooth functions $\widetilde{m}_\pm \in C^2(\partial\Omega)$, independent of τ and s, such that*

$$\widetilde{\mathcal{R}}_{-\tau, -A_{2,s}^*}(e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)) = \frac{\widetilde{m}_+ e^{2i\tau(\psi + \psi(\widetilde{x}))}}{\tau^2} + e^{2i\tau\psi} o_{H^1(\partial\Omega)}(\frac{1}{\tau^2}) \quad as \ |\tau| \to +\infty \tag{89}$$

and

$$\mathcal{R}_{-\tau, -B_{2,s}^*}(e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)) = \frac{\widetilde{m}_- e^{-2i\tau(\psi + \psi(\widetilde{x}))}}{\tau^2} + e^{-2i\tau\psi} o_{H^1(\partial\Omega)}(\frac{1}{\tau^2}) \quad as \ |\tau| \to +\infty. \tag{90}$$

Proof. The functions $q_{3,s}, q_{4,s}$ belong to the space $C^{5+\alpha}(\overline{\Omega})$ $q_{3,s}, q_{4,s}$ belong to the space $W_p^1(\Omega)$ for any $p > 1$. By (85) and (88), we have $q_{3,s} = q_{4,s} = \widetilde{q}_{3,s} = \widetilde{q}_{4,s} = 0$ on $\mathcal{H}$. By (83) and (84), we have

$$\widetilde{\mathcal{R}}_{-\tau, -A_{2,s}^*}(e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)) = e^{s\eta}\widetilde{\mathcal{R}}_{-\tau, -A_2^*}(e_1(q_3 + \widetilde{q}_3/\tau))$$

and

$$\mathcal{R}_{-\tau, -B_{2,s}^*}(e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)) = e^{s\eta}\mathcal{R}_{-\tau, -B_2^*}(e_1(q_4 + \widetilde{q}_4/\tau)).$$

Then applying Proposition 10 and taking into account (80), we obtain Proposition 13. ∎

By (88), there exists a pair $(w_{-4}, \widetilde{w}_{-4}) \in H^1(\Omega)$ which solves the boundary value problem

$$\mathcal{M}(x, D)(w_{-4}, \widetilde{w}_{-4}) = 0 \quad \text{in } \Omega, \quad (w_{-4} + \widetilde{w}_{-4})|_{\Gamma_0} = \frac{\widetilde{q}_3}{2\partial_z \Phi} + \frac{\widetilde{q}_4}{2\partial_{\overline{z}}\overline{\Phi}}. \tag{91}$$

We set $(w_{-3,s}, \widetilde{w}_{-3,s}) = e^{s\eta}(w_{-3}, \widetilde{w}_{-3}), (\widetilde{a}_{\pm,s}, \widetilde{b}_{\pm,s}) = e^{s\eta}(\widetilde{a}_\pm, \widetilde{b}_\pm)$. We introduce the function $w_{1,s,\tau}, \widetilde{w}_{1,s,\tau}$ by formulas

$$w_{1,s,\tau} = w_{1,s} + \frac{w_{-3,s} + e_2 q_{3,s}/2\partial_z \Phi}{\tau} + \frac{1}{\tau^2}(e^{2i\tau\psi(\widetilde{x})}\widetilde{a}_{+,s} + e^{-2i\tau\psi(\widetilde{x})}\widetilde{a}_{-,s} + w_{-4,s} - \frac{e_2 \widetilde{q}_{3,s}}{2\partial_z \Phi}) \tag{92}$$

and

$$\widetilde{w}_{1,s,\tau} = \widetilde{w}_{1,s} + \frac{\widetilde{w}_{-3,s} + e_2 q_{4,s}/2\partial_{\overline{z}}\overline{\Phi}}{\tau} + \frac{1}{\tau^2}(e^{2i\tau\psi(\widetilde{x})}\widetilde{b}_{+,s} + e^{-2i\tau\psi(\widetilde{x})}\widetilde{b}_{-,s} + \widetilde{w}_{-4,s} - \frac{e_2 \widetilde{q}_{4,s}}{2\partial_{\overline{z}}\overline{\Phi}}). \tag{93}$$

By (85) and (88), the functions $w_{1,s,\tau}, \widetilde{w}_{1,s,\tau}$ belong to $H^1(\Omega)$. Using (38), for any $p \in (1, +\infty)$ we have

$$
L_2(x,s,D)^* \left(-e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}(e_1(q_{3,s} + \tfrac{\widetilde{q}_{3,s}}{\tau})) + \frac{e^{-\tau\Phi}e_2(q_{3,s} + \tfrac{\widetilde{q}_{3,s}}{\tau})}{2\tau\partial_z\Phi} \right.
$$

$$
\left. -e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}(e_1(q_{4,s} + \tfrac{\widetilde{q}_{4,s}}{\tau})) + \frac{e^{-\tau\overline{\Phi}}e_2(q_{4,s} + \tfrac{\widetilde{q}_{4,s}}{\tau})}{2\tau\partial_{\overline{z}}\overline{\Phi}} \right)
$$

$$
= -L_2(x,s,D)^* \left(e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}(e_1(q_{3,s} + \tfrac{\widetilde{q}_{3,s}}{\tau})) - \frac{e^{-\tau\Phi}e_1(q_{3,s} + \tfrac{\widetilde{q}_{3,s}}{\tau})}{2\tau\partial_z\Phi} \right)
$$

$$
-L_2(x,s,D)^* \left(e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}(e_1(q_{4,s} + \tfrac{\widetilde{q}_{4,s}}{\tau})) - \frac{e^{-\tau\overline{\Phi}}e_2(q_{4,s} + \tfrac{\widetilde{q}_{4,s}}{\tau})}{2\tau\partial_{\overline{z}}\overline{\Phi}} \right)
$$

$$
= -e^{\tau\Phi}(Q_2^* - 2\partial_z B_2^* - B_2^* A_2^*)\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}(e_1(q_{3,s} + \tfrac{\widetilde{q}_{3,s}}{\tau})) + e^{\tau\Phi}L_2(x,s,D)^*(\frac{e_2(q_{3,s} + \tfrac{\widetilde{q}_{3,s}}{\tau})}{2\tau\partial_z\Phi})
$$

$$
-e^{-\tau\overline{\Phi}}(Q_2^* - 2\partial_{\overline{z}}A_2^* - A_2^* B_2^*)\mathcal{R}_{-\tau,-B_{2,s}^*}(e_1(q_{4,s} + \tfrac{\widetilde{q}_{4,s}}{\tau})) + e^{\tau\overline{\Phi}}L_2(x,s,D)^*(\frac{e_2(q_{4,s} + \tfrac{\widetilde{q}_{4,s}}{\tau})}{2\tau\partial_{\overline{z}}\overline{\Phi}})
$$

$$
-(Q_2^* - 2\partial_{\overline{z}}A_2^* - A_2^* B_2^*)(w_{1,s} + \frac{w_{-3,s}}{\tau})e^{-\tau\Phi}
$$

$$
-(Q_2^* - 2\partial_z B_2^* - B_2^* A_2^*)(\widetilde{w}_{1,s} + \frac{w_{-3,s}}{\tau})e^{-\tau\overline{\Phi}} + o_{L^p(\Omega)}(\tfrac{1}{\tau}). \tag{94}
$$

Setting $v^* = w_{1,s,\tau}e^{-\tau\Phi} + \widetilde{w}_{1,s,\tau}e^{-\tau\overline{\Phi}} - e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}(e_1(q_{3,s} + \tfrac{\widetilde{q}_{3,s}}{\tau})) - e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}(e_1(q_{4,s} + \tfrac{\widetilde{q}_{4,s}}{\tau}))$ for any $p \in (1, \infty)$, we obtain that

$$
L_2(x,s,D)v^* = e^{-\tau\varphi}o_{L^p(\Omega)}(\tfrac{1}{\tau}) \quad \text{in } \Omega, \quad v^*|_{\Gamma_0} = e^{-\tau\varphi}o_{H^1(\Gamma_0)}(\tfrac{1}{\tau}). \tag{95}
$$

Using (95) and Proposition 12 and 11, we construct the last term v_{-1} in complex geometric optics solution which solves the boundary value problem

$$
L_2(x,s,D)v_{-1} = L_2(x,s,D)v^* \quad \text{in } \Omega, \quad v_{-1}|_{\Gamma_0} = v^* \tag{96}
$$

and we obtain

$$
\sqrt{|\tau|}\|v_{-1}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|(\nabla v_{-1})\|_{L^2(\Omega)} + \|v_{-1}\|_{H^{1,\tau}(\mathcal{O}_\epsilon)} = o(\tfrac{1}{\tau}). \tag{97}
$$

Finally we have a complex geometric optics solution for Schrödinger operator $L_2(x,s,D)$ in a form:

$$
v = w_{1,s,\tau}e^{-\tau\Phi} + \widetilde{w}_{1,s,\tau}e^{-\tau\overline{\Phi}} - e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}(e_1(q_{3,s} + \tfrac{\widetilde{q}_{3,s}}{\tau})) \tag{98}
$$

$$
-e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}(e_1(q_{4,s} + \tfrac{\widetilde{q}_{4,s}}{\tau})) + v_{-1}e^{-\tau\varphi}.
$$

By (98), (95) and (96), we have

$$
L_2(x,s,D)v = 0 \quad \text{in } \Omega, \quad v|_{\Gamma_0} = 0. \tag{99}
$$

Step 2:Asymptotic formula.
Let $u_2 = u_2(s,x)$ be a solution to the following boundary value problem:

$$
L_2(x,s,D)u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}, \quad \frac{\partial u_2}{\partial\nu}|_{\widetilde{\Gamma}} = \frac{\partial u_1}{\partial\nu}|_{\widetilde{\Gamma}}. \tag{100}
$$

Setting $u = u_1 - u_2$, we have

$$
L_2(x,s,D)u + 2(A_1 - A_{2,s})\partial_z u_1 + 2(B_1 - B_{2,s})\partial_{\overline{z}}u_1 + (Q_1 - Q_{2,s})u_1 = 0 \quad \text{in } \Omega \tag{101}
$$

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and

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\tilde{\Gamma}} = 0. \tag{102}$$

Let v be a function given by (98). Taking the scalar product of (101) with v in $L^2(\Omega)$ and using (99) and (102), we obtain

$$0 = \mathfrak{G}(u_1, v) = \int_{\Omega} (2(A_1 - A_{2,s})\partial_z u_1 + 2(B_1 - B_{2,s})\partial_{\bar{z}} u_1 + (Q_1 - Q_{2,s})u_1, \overline{v})dx. \tag{103}$$

Our goal is to obtain the asymptotic formula for the right-hand side of (103). We have

Proposition 14 *There exists a constant $\mathcal{C}_0$, independent of τ, such that the following asymptotic formula is valid as $|\tau| \to +\infty$:*

$$I_0 = ((Q_1 - Q_{2,s})u_1, v)_{L^2(\Omega)} \tag{104}$$

$$= \int_{\Omega} (((Q_1 - Q_{2,s})w_0, \overline{\widetilde{w}_{1,s}}) + ((Q_1 - Q_{2,s})\widetilde{w}_0, \overline{w_{1,s}}))dx$$

$$+ \frac{\mathcal{C}_0}{\tau} + 2\pi \frac{((Q_1 - Q_{2,s})w_0, \overline{w_{1,s}})(\tilde{x})e^{2\tau i\psi(\tilde{x})} + ((Q_1 - Q_{2,s})\widetilde{w}_0, \overline{\widetilde{w}_{1,s}})(\tilde{x})e^{-2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}}$$

$$+ \frac{1}{2\tau i}\int_{\partial\Omega} ((Q_1 - Q_{2,s})w_0, \overline{w_{1,s}})e^{2\tau i\psi} \frac{(\nu, \nabla\psi)}{|\nabla\psi|^2}d\sigma - \frac{1}{2\tau i}\int_{\partial\Omega} ((Q_1 - Q_{2,s})\widetilde{w}_0, \overline{\widetilde{w}_{1,s}})e^{-2\tau i\psi} \frac{(\nu, \nabla\psi)}{|\nabla\psi|^2}d\sigma + o(\frac{1}{\tau}).$$

Proof. By (68), (69), (61), (66), (78) and Propositions 8 and 4, we have

$$u_1(x) = (w_0 + \frac{w_{-1}}{\tau})e^{\tau\Phi} + (\widetilde{w}_0 + \frac{\widetilde{w}_{-1}}{\tau})e^{\tau\overline{\Phi}} - \frac{q_2 e^{\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}} - \frac{q_1 e^{\tau\Phi}}{2\tau\partial_z\Phi} + e^{\tau\varphi}o_{L^2(\Omega)}(\frac{1}{\tau}) \text{ as } \tau \to +\infty. \tag{105}$$

Using (92), (93), (85), (88), (98) and Propositions 8 and 4, we obtain

$$v(x) = (w_{1,s} + \frac{w_{-2,s}}{\tau})e^{-\tau\Phi} + (\widetilde{w}_{1,s} + \frac{\widetilde{w}_{-2,s}}{\tau})e^{-\tau\overline{\Phi}} + \frac{q_{4,s}e^{-\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}} + \frac{q_{3,s}e^{-\tau\Phi}}{2\tau\partial_z\Phi} + e^{-\tau\varphi}o_{L^2(\Omega)}(\frac{1}{\tau}) \text{ as } \tau \to +\infty. \tag{106}$$

By (105) and (106), we obtain the following asymptotic formula:

$$((Q_1 - Q_{2,s})u_1, v)_{L^2(\Omega)} = ((Q_1 - Q_{2,s})((w_0 + \frac{w_{-1}}{\tau})e^{\tau\Phi} + (\widetilde{w}_0 + \frac{\widetilde{w}_{-1}}{\tau})e^{\tau\overline{\Phi}} - \frac{q_2 e^{\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}} - \frac{q_1 e^{\tau\Phi}}{2\tau\partial_z\Phi} + e^{\tau\varphi}o_{L^2(\Omega)}(\frac{1}{\tau})),$$

$$(w_{1,s} + \frac{w_{-2,s}}{\tau})e^{-\tau\Phi} + (\widetilde{w}_{1,s} + \frac{\widetilde{w}_{-2,s}}{\tau})e^{-\tau\overline{\Phi}} + \frac{q_{4,s}e^{-\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}} + \frac{q_{3,s}e^{-\tau\Phi}}{2\tau\partial_z\Phi} + e^{-\tau\varphi}o_{L^2(\Omega)}(\frac{1}{\tau}))_{L^2(\Omega)}$$

$$= \int_{\Omega} (((Q_1 - Q_{2,s})\widetilde{w}_0, \overline{w_{1,s}}) + \frac{1}{\tau}((Q_1 - Q_{2,s})\widetilde{w}_0, \overline{w_{-2,s}}) + \frac{1}{\tau}((Q_1 - Q_{2,s})\widetilde{w}_{-1}, \overline{w_{1,s}})$$

$$+ ((Q_1 - Q_{2,s})w_0, \overline{\widetilde{w}_{1,s}}) + \frac{1}{\tau}((Q_1 - Q_{2,s})w_{-1}, \overline{\widetilde{w}_{1,s}}) + ((Q_1 - Q_{2,s})w_0, \overline{\widetilde{w}_{-2,s}}))dx$$

$$+ \frac{1}{\tau}\int_{\Omega} ((Q_1 - Q_{2,s})w_0, \frac{\overline{q_{4,s}}}{2\overline{\partial_z\Phi}}) - ((Q_1 - Q_{2,s})\frac{q_2}{2\overline{\partial_z\Phi}}, \overline{w_{1,s}})$$

$$- ((Q_1 - Q_{2,s})\frac{q_1}{2\overline{\partial_{\bar{z}}\Phi}}, \overline{\widetilde{w}_{1,s}}) + ((Q_1 - Q_{2,s})\widetilde{w}_0, \frac{\overline{q_{3,s}}}{2\partial_z\Phi}))dx$$

$$+ \int_{\Omega} (((Q_1 - Q_{2,s})w_0, \overline{w_{1,s}})e^{2i\tau\psi} + ((Q_1 - Q_{2,s})\widetilde{w}_0, \overline{\widetilde{w}_{1,s}})e^{-2i\tau\psi})dx + o(\frac{1}{\tau}).$$

Applying the stationary phase argument (see e.g., [2]) to the last integral on the right-hand side of this formula, we complete the proof of Proposition 14. ∎

We set

$$\mathcal{U} = w_{0,\tau}e^{\tau\Phi} + \widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}}, \quad \mathcal{V} = w_{1,s,\tau}e^{-\tau\Phi} + \widetilde{w}_{1,s,\tau}e^{-\tau\overline{\Phi}}.$$

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By the stationary phase argument and formulae (41), (81), (68), (69), (92) and (93), short calculations yield that there exist constants $\kappa_k, \widetilde{\kappa}_k$, independent of τ, such that

$$I_1 \equiv 2((A_1 - A_{2,s})\partial_z \mathcal{U}, \mathcal{V})_{L^2(\Omega)}$$

$$= (2(A_1 - A_{2,s})(\partial_z(w_{0,\tau}e^{\tau\Phi}) + \partial_z \widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}}), w_{1,s,\tau}e^{-\tau\Phi} + \widetilde{w}_{1,s,\tau}e^{-\tau\overline{\Phi}})_{L^2(\Omega)}$$

$$= \sum_{k=1}^{3} \tau^{2-k}\kappa_k + e^{2i\tau\psi(\widetilde{x})}(((A_1 - A_{2,s})\partial_z \Phi a_+, \widetilde{w}_{1,s})_{L^2(\Omega)} + ((A_1 - A_{2,s})\partial_z \Phi w_0, \widetilde{b}_{-,s})_{L^2(\Omega)})$$

$$+ e^{-2i\tau\psi(\widetilde{x})}(((A_1 - A_{2,s})\partial_z \Phi a_-, \widetilde{w}_{1,s})_{L^2(\Omega)} + ((A_1 - A_{2,s})\partial_z \Phi w_0, \widetilde{b}_{+,s})_{L^2(\Omega)})$$

$$+ 2\int_\Omega ((A_1 - A_{2,s})\partial_z \widetilde{w}_0, \overline{\widetilde{w}_{1,s}})e^{-2i\tau\psi}dx - \int_\Omega (2\partial_z(A_1 - A_{2,s})w_0, \overline{w_{1,s}})e^{2\tau i\psi}dx$$

$$- \int_\Omega 2((A_1 - A_{2,s})w_0, \overline{\partial_{\overline{z}}w_{1,s}})e^{2\tau i\psi}dx + \int_{\partial\Omega} (\nu_1 - i\nu_2)((A_1 - A_{2,s})w_0, \overline{w_{1,s}})e^{2i\tau\psi}d\sigma$$

$$+ o\left(\frac{1}{\tau}\right)$$

$$= \sum_{k=1}^{3} \tau^{2-k}\kappa_k + e^{2i\tau\psi(\widetilde{x})}(((A_1 - A_{2,s})\partial_z \Phi a_+, \widetilde{w}_{1,s})_{L^2(\Omega)} + ((A_1 - A_{2,s})\partial_z \Phi w_0, \widetilde{b}_{-,s})_{L^2(\Omega)})$$

$$+ e^{-2i\tau\psi(\widetilde{x})}(((A_1 - A_{2,s})\partial_z \Phi a_-, \widetilde{w}_{1,s})_{L^2(\Omega)} + ((A_1 - A_{2,s})\partial_z \Phi w_0, \widetilde{b}_{+,s})_{L^2(\Omega)})$$

$$- \int_\Omega ((A_1 - A_{2,s})B_1 \widetilde{w}_0, \overline{\widetilde{w}_{1,s}})e^{-2i\tau\psi}dx - \int_\Omega (2\partial_z(A_1 - A_{2,s})w_0, \overline{w_{1,s}})e^{2\tau i\psi}dx$$

$$- \int_\Omega ((A_1 - A_{2,s})w_0, \overline{B_{2,s}^* w_{1,s}})e^{2\tau i\psi}dx + \int_{\partial\Omega} (\nu_1 - i\nu_2)((A_1 - A_{2,s})w_0, \overline{w_{1,s}})e^{2i\tau\psi}d\sigma$$

$$+ o\left(\frac{1}{\tau}\right) \qquad (107)$$

and

$$I_2 \equiv ((B_1 - B_{2,s})\partial_{\overline{z}} \mathcal{U}, \mathcal{V})_{L^2(\Omega)}$$

$$= (2(B_1 - B_{2,s})(e^{\tau\Phi}\partial_{\overline{z}}w_{0,\tau} + \partial_{\overline{z}}(\widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}})), w_{1,s,\tau}e^{-\tau\Phi} + \widetilde{w}_{1,s,\tau}e^{-\tau\overline{\Phi}})_{L^2(\Omega)}$$

$$= \sum_{k=1}^{3} \tau^{2-k}\widetilde{\kappa}_k + e^{2i\tau\psi(\widetilde{x})}(((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}b_+, w_{1,s})_{L^2(\Omega)} + ((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \widetilde{a}_{-,s})_{L^2(\Omega)})$$

$$+ e^{-2i\tau\psi(\widetilde{x})}(((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}b_-, w_{1,s})_{L^2(\Omega)} + ((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \widetilde{a}_{+,s})_{L^2(\Omega)})$$

$$+ \int_\Omega 2((B_1 - B_{2,s})\partial_{\overline{z}}w_0, \overline{w_{1,s}})e^{2\tau i\psi}dx - \int_\Omega (2\partial_{\overline{z}}(B_1 - B_{2,s})\widetilde{w}_0, \overline{\widetilde{w}_{1,s}})e^{-2\tau i\psi}dx$$

$$- \int_\Omega (2(B_1 - B_{2,s})\widetilde{w}_0, \overline{\partial_z \widetilde{w}_{1,s}})e^{-2\tau\psi}dx + \int_{\partial\Omega} (\nu_1 + i\nu_2)((B_1 - B_{2,s})\widetilde{w}_0, \overline{\widetilde{w}_{1,s}})e^{-2i\tau\psi}d\sigma + o\left(\frac{1}{\tau}\right)$$

$$= \sum_{k=1}^{3} \tau^{2-k}\widetilde{\kappa}_k + e^{2i\tau\psi(\widetilde{x})}(((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}b_+, w_{1,s})_{L^2(\Omega)} + ((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, a_{-,s})_{L^2(\Omega)})$$

$$+ e^{-2i\tau\psi(\widetilde{x})}(((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}b_-, w_{1,s})_{L^2(\Omega)} + ((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, a_{+,s})_{L^2(\Omega)})$$

$$- \int_\Omega ((B_1 - B_{2,s})A_1 w_0, \overline{w_{1,s}})e^{2\tau i\psi}dx - \int_\Omega (2\partial_{\overline{z}}(B_1 - B_{2,s})\widetilde{w}_0, \overline{\widetilde{w}_{1,s}})e^{-2\tau i\psi}dx$$

$$- \int_\Omega ((B_1 - B_{2,s})\widetilde{w}_0, \overline{A_{2,s}^* \widetilde{w}_{1,s}})e^{-2\tau i\psi}dx + \int_{\partial\Omega} (\nu_1 + i\nu_2)((B_1 - B_{2,s})\widetilde{w}_0, \overline{\widetilde{w}_{1,s}})e^{-2i\tau\psi}d\sigma + o\left(\frac{1}{\tau}\right). \qquad (108)$$

Using (35) and integrating by parts, we obtain

$$I_3 = -\int_\Omega (2(A_1 - A_{2,s})\partial_z(e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\} + e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\})$$

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$$+2(B_1 - B_{2,s})\partial_{\overline{z}}(e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\} + e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\}), \overline{\mathcal{V}})dx$$

$$= -\int_\Omega (2(A_1 - A_{2,s})(e^{\tau\Phi}(-B_1\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\} + e_1(q_1 + \widetilde{q}_1/\tau)) + e^{\tau\overline{\Phi}}\partial_z\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\})$$

$$+2(B_1 - B_{2,s})(e^{\tau\Phi}\partial_{\overline{z}}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\} + e^{\tau\overline{\Phi}}(-A_1\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\} + e_1(q_2 + \widetilde{q}_2/\tau))), \overline{\mathcal{V}})dx$$

$$= -\int_\Omega (2(A_1 - A_{2,s})e^{\tau\Phi}(-B_1\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\} + e_1(q_1 + \widetilde{q}_1/\tau))$$

$$+2(B_1 - B_{2,s})e^{\tau\overline{\Phi}}(-A_1\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\} + e_1(q_2 + \widetilde{q}_2/\tau)), \overline{\mathcal{V}})dx$$

$$+\int_\Omega (2\partial_z(A_1 - A_{2,s})e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\} + 2\partial_{\overline{z}}(B_1 - B_{2,s})e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\}, \overline{\mathcal{V}})dx$$

$$+\int_\Omega (2(A_1 - A_{2,s})e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\}, \partial_z\overline{\mathcal{V}}) + (2(B_1 - B_{2,s})e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\}, \partial_{\overline{z}}\overline{\mathcal{V}})dx$$

$$-\int_{\partial\Omega} \{(\nu_1 - i\nu_2)((A_1 - A_{2,s})e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\}, \overline{\mathcal{V}})$$

$$+(\nu_1 + i\nu_2)((B_1 - B_{2,s})e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\}, \overline{\mathcal{V}})\}d\sigma$$

$$= -\int_\Omega (2(A_1 - A_{2,s})e^{\tau\Phi}(-B_1\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\} + e_1(q_1 + \widetilde{q}_1/\tau))$$

$$+2(B_1 - B_{2,s})e^{\tau\overline{\Phi}}(-A_1\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\} + e_1(q_2 + \widetilde{q}_2/\tau)), \overline{\mathcal{V}})dx$$

$$+\int_\Omega (2\partial_z(A_1 - A_{2,s})e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\} + 2\partial_{\overline{z}}(B_1 - B_{2,s})e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\}, \overline{\mathcal{V}})dx$$

$$+\int_\Omega (2(A_1 - A_{2,s})\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\}, \partial_z\overline{w_{1,s}}) + (2(B_1 - B_{2,s})\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\}, \partial_{\overline{z}}\overline{\widetilde{w}_{1,s}})dx$$

$$+2\int_\Omega e^{\tau(\overline{\Phi}-\Phi)}(e_1(q_2 + \widetilde{q}_2/\tau), \overline{\mathbf{P}_{A_1}^*((A_1 - A_{2,s})^*(\partial_{\overline{z}}\widetilde{w}_{1,s} - \tau\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_{1,s}))})dx$$

$$+2\int_\Omega e^{\tau(\Phi-\overline{\Phi})}(e_1(q_1 + \widetilde{q}_1/\tau), \overline{\mathbf{T}_{B_1}^*((B_1 - B_{2,s})^*(\partial_z w_{1,s} - \tau\partial_z \Phi w_{1,s}))})dx$$

$$-\int_{\partial\Omega} \{(\nu_1 - i\nu_2)((A_1 - A_{2,s})e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}\{e_1(q_2 + \widetilde{q}_2/\tau)\}, \overline{\mathcal{V}})$$

$$+(\nu_1 + i\nu_2)((B_1 - B_{2,s})e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1 + \widetilde{q}_1/\tau)\}, \overline{\mathcal{V}})\}d\sigma. \quad (109)$$

By (62) and (63), the boundary integrals in (109) are $O(\frac{1}{\tau^2})$. By (66) and Proposition 4, we have

$$2\int_\Omega e^{\tau(\overline{\Phi}-\Phi)}(e_1\widetilde{q}_2/\tau, \overline{\mathbf{P}_{A_1}^*((A_1 - A_{2,s})^*(\partial_{\overline{z}}\widetilde{w}_{1,s} - \tau\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_{1,s}))})dx$$

$$+2\int_\Omega e^{\tau(\Phi-\overline{\Phi})}(e_1\widetilde{q}_1/\tau, \overline{\mathbf{T}_{B_1}^*((B_1 - B_{2,s})^*(\partial_z w_{1,s} - \tau\partial_z \Phi w_{1,s}))})dx = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \quad (110)$$

Applying the stationary phase argument, (110), Propositions 8 and 3, we obtain from (109) that there exists a constant $\mathcal{C}_1$ independent of τ such that

$$I_3 = \frac{\mathcal{C}_1}{\tau} + 2\int_\Omega e^{\tau(\overline{\Phi}-\Phi)}(e_1 q_2, \overline{\mathbf{P}_{A_1}^*((A_1 - A_{2,s})^*(-\tau\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_{1,s}))})dx$$

$$+2\int_\Omega e^{\tau(\Phi-\overline{\Phi})}(e_1 q_1, \overline{\mathbf{T}_{B_1}^*((B_1 - B_{2,s})^*(-\tau\partial_z \Phi w_{1,s}))})dx + o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \quad (111)$$

Using (35) and integrating by parts, we obtain

$$I_4 = \int_\Omega (2(A_1 - A_{2,s})\partial_z\mathcal{U} + 2(B_1 - B_{2,s})\partial_{\overline{z}}\mathcal{U},$$

$$\overline{-e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\} - e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\}})dx$$

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$$= -\int_\Omega (2(A_1 - A_{2,s})\partial_z \widetilde{w}_0 e^{\tau\overline{\Phi}} + 2(B_1 - B_{2,s})\partial_{\overline{z}} w_0 e^{\tau\Phi},$$

$$\overline{e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_2^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\} + e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\}})dx$$

$$- \int_\Omega (2(A_1 - A_{2,s})(\partial_z w_0 + \tau\partial_z \Phi w_0)e^{\tau\Phi}, \overline{e^{-\tau\overline{\Phi}}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\}})dx$$

$$- \int_\Omega (2(B_1 - B_{2,s})(\partial_{\overline{z}} \widetilde{w}_0 + \tau\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0)e^{\tau\overline{\Phi}}, \overline{e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\}})dx$$

$$+ \int_\Omega ((2\partial_z(A_1 - A_{2,s})w_0 e^{\tau\Phi}, \overline{e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_2^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\}})$$

$$+ (2\partial_{\overline{z}}(B_1 - B_{2,s})\widetilde{w}_0 e^{\tau\overline{\Phi}}, \overline{e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\}}))dx$$

$$- \int_{\partial\Omega} \{(\nu_1 - i\nu_2)((A_1 - A_{2,s})w_0 e^{\tau\Phi}, \overline{e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\}})$$

$$+ (\nu_1 + i\nu_2)((B_1 - B_{2,s})\widetilde{w}_0 e^{\tau\overline{\Phi}}, \overline{e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\}})\}d\sigma$$

$$+ \int_\Omega (2(A_1 - A_{2,s})w_0 e^{\tau\Phi}, \overline{\partial_{\overline{z}}(e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\})})dx$$

$$+ \int_\Omega (2(B_1 - B_{2,s})\widetilde{w}_0 e^{\tau\overline{\Phi}}, \overline{\partial_z(e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\})})dx$$

$$= -\int_\Omega (2(A_1 - A_{2,s})\partial_z \widetilde{w}_0 e^{\tau\overline{\Phi}} + 2(B_1 - B_{2,s})\partial_{\overline{z}} w_0 e^{\tau\Phi},$$

$$\overline{e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\} + e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\}})dx$$

$$- \int_\Omega (2\mathbf{T}_{-A_{2,s}^*}^*((A_1 - A_{2,s})(\partial_z w_0 + \tau\partial_z \Phi w_0)), \overline{e^{\tau(\overline{\Phi}-\Phi)}e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)})dx$$

$$- \int_\Omega (2\mathbf{P}_{-B_{2,s}^*}^*((B_1 - B_{2,s})(\partial_{\overline{z}} \widetilde{w}_0 + \tau\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0)), \overline{e^{\tau(\Phi-\overline{\Phi})}e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)})dx$$

$$+ \int_\Omega (2\partial_z(A_1 - A_{2,s})w_0 e^{\tau\Phi}, \overline{e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\}})$$

$$+ (2\partial_{\overline{z}}(B_1 - B_{2,s})\widetilde{w}_0 e^{\tau\overline{\Phi}}, \overline{e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\}}))dx$$

$$- \int_{\partial\Omega} \{(\nu_1 - i\nu_2)((A_1 - A_{2,s})w_0 e^{\tau\Phi}, \overline{e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\}})$$

$$+ (\nu_1 + i\nu_2)((B_1 - B_{2,s})\widetilde{w}_0 e^{\tau\overline{\Phi}}, \overline{e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\}})\}d\sigma$$

$$+ \int_\Omega (2(A_1 - A_{2,s})w_0 e^{\tau\Phi}, \overline{e^{-\tau\overline{\Phi}}B_{2,s}^*\mathcal{R}_{-\tau,-B_{2,s}^*}\{e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)\} + e^{-\tau\overline{\Phi}}e_1(q_{4,s} + \widetilde{q}_{4,s}/\tau)})dx$$

$$+ \int_\Omega (2(B_1 - B_{2,s})\widetilde{w}_0 e^{\tau\overline{\Phi}}, \overline{e^{-\tau\Phi}A_{2,s}^*\widetilde{\mathcal{R}}_{-\tau,-A_{2,s}^*}\{e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)\} + e^{-\tau\Phi}e_1(q_{3,s} + \widetilde{q}_{3,s}/\tau)})dx. \tag{112}$$

By Proposition 13, the boundary integral in (112) is $O(\frac{1}{\tau^2})$. By (88) and Proposition 4, we have

$$\frac{1}{\tau}\int_\Omega (2\mathbf{T}_{-A_{2,s}^*}^*((A_1 - A_{2,s})(\partial_z w_0 + \tau\partial_z \Phi w_0)), \overline{e^{\tau(\overline{\Phi}-\Phi)}e_1\widetilde{q}_{3,s}})dx$$

$$- \int_\Omega (2\mathbf{P}_{-B_{2,s}^*}^*((B_1 - B_{2,s})(\partial_{\overline{z}} \widetilde{w}_0 + \tau\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0)), \overline{e^{\tau(\Phi-\overline{\Phi})}e_1\widetilde{q}_{4,s}})dx = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \tag{113}$$

Applying the stationary phase argument, Propositions 8 and 3, and (113), we obtain from (112) that there exists a constant $\mathcal{C}_2$, independent of τ, such that

$$I_4 = \frac{\mathcal{C}_2}{\tau} - \int_\Omega (2\mathbf{T}_{-A_{2,s}^*}^*((A_1 - A_{2,s})\tau\partial_z \Phi w_0), \overline{e^{\tau(\overline{\Phi}-\Phi)}e_1 q_{3,s}})dx$$

$$-\int_{\Omega}(2\mathbf{P}^*_{-B^*_{2,s}}((B_1-B_{2,s})\tau\partial_{\bar{z}}\overline{\Phi}\widetilde{w}_0),\overline{e^{\tau(\Phi-\overline{\Phi})}e_1 q_{4,s}})dx+o(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{114}$$

Step 3:derivation of equations (2)-(4).
We set

$$\mathcal{U}_1(x)=w_{0,\tau}e^{\tau\Phi}+\widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}}-e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(q_1+\widetilde{q}_1/\tau)\}-e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}\{e_1(q_2+\widetilde{q}_2/\tau)\},$$

$$\mathcal{V}_1(x)=w_{1,s,\tau}e^{-\tau\Phi}+\widetilde{w}_{1,s,\tau}e^{-\tau\overline{\Phi}}-e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-A^*_{2,s}}\{e_1(q_{3,s}+\widetilde{q}_{3,s}/\tau)\}$$

$$-e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-B^*_{2,s}}\{e_1(q_{4,s}+\widetilde{q}_{4,s}/\tau)\}.$$

By (77), (98) and Proposition 8, we have

$$\mathfrak{G}(u_{-1}e^{\tau\varphi},v-(w_{1,s,\tau}e^{-\tau\Phi}+\widetilde{w}_{1,s,\tau}e^{-\tau\overline{\Phi}}))=\mathfrak{G}(u_1-(w_{0,\tau}e^{\tau\Phi}+\widetilde{w}_{0,\tau}e^{\tau\overline{\Phi}}),v_{-1}e^{-\tau\varphi}) \tag{115}$$

$$=o(\frac{1}{\sqrt{\tau}}) \quad \text{as } \tau\to+\infty.$$

Then

$$\mathfrak{G}(u_1,v)=\int_{\widetilde{\Gamma}}(\nu_1-i\nu_2)((A_1-A_{2,s})w_0,\overline{w_{1,s}})e^{2i\tau\psi}d\sigma+\int_{\widetilde{\Gamma}}(\nu_1+i\nu_2)((B_1-B_{2,s})\widetilde{w}_0,\overline{\widetilde{w}_{1,s}})e^{-2i\tau\psi}d\sigma \tag{116}$$

$$+o(\frac{1}{\sqrt{\tau}}).$$

Let Φ be given in Proposition 2. Then by (43),(82) and the stationary phase argument, the asymptotic formula holds:

$$\mathfrak{G}(u_1,v)=\frac{1}{\sqrt{\tau}}\sum_{x\in\mathcal{G}}\{(\nu_1-i\nu_2)((A_1-A_{2,s})w_0,\overline{w_{1,s}})e^{2i\tau\psi}+(\nu_1+i\nu_2)((B_1-B_{2,s})\widetilde{w}_0,\overline{\widetilde{w}_{1,s}})e^{-2i\tau\psi}\}(x) \tag{117}$$

$$+o(\frac{1}{\sqrt{\tau}}) \quad \text{as } \tau\to+\infty.$$

Since for any $\widehat{x}$ one can find Φ such that $\widehat{x}\in\mathcal{G}$ and $Im\Phi(\widehat{x})\neq Im\Phi(x)$ for any $x\in\mathcal{G}\setminus\{\widehat{x}\}$, we have

$$((A_1-A_{2,s})w_0,\overline{w_{1,s}})=((B_1-B_{2,s})\widetilde{w}_0,\overline{\widetilde{w}_{1,s}})=0 \quad \text{on } \Gamma_0.$$

These equalities and Proposition 9 imply (2).
Next we claim that

$$\mathfrak{G}(e^{\tau\varphi}u_{-1},v)=\mathfrak{G}(u_1,e^{-\tau\varphi}v_{-1})=o(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{118}$$

Obviously, by (77) and Proposition 8, we see that

$$\mathfrak{G}(e^{\tau\varphi}u_{-1},v-\mathcal{V})=o(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{119}$$

Let $\chi\in C^\infty_0(\Omega)$ satisfy $\chi|_{\Omega\setminus\mathcal{O}_{\frac{\epsilon}{2}}}=1$. By (77), we have

$$\mathfrak{G}(e^{\tau\varphi}u_{-1},\mathcal{V})=\mathfrak{G}(e^{\tau\varphi}u_{-1},\chi\mathcal{V})+o(\frac{1}{\tau})$$

$$=\int_{\Omega}(2(A_1-A_{2,s})\partial_z(e^{\tau\varphi}u_{-1})+2(B_1-B_{2,s})\partial_{\bar{z}}(e^{\tau\varphi}u_{-1}),\chi\overline{\mathcal{V}})dx+o(\frac{1}{\tau})$$

$$=\int_{\Omega}(2(A_1-A_{2,s})\partial_z(e^{\tau\varphi}u_{-1}),\overline{\chi\widetilde{w}_{1,s}e^{\tau\overline{\Phi}}})+(2(B_1-B_{2,s})\partial_{\bar{z}}(e^{\tau\varphi}u_{-1}),\overline{\chi w_{1,s}e^{\tau\Phi}})dx+o(\frac{1}{\tau}). \tag{120}$$

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Let functions w_4, w_5 solve the equations $(-\partial_{\overline{z}} + B_1^*)w_4 = 2(A_1 - A_{2,s})^*\widetilde{w}_{1,s}$ and $(-\partial_z + A_1^*)w_5 = 2(B_1 - B_{2,s})^*w_{1,s}$.

Taking the scalar product of equation (69) and the function $w_5 e^{\tau\Phi} + w_4 e^{\tau\overline{\Phi}}$, after integration by parts we obtain

$$\int_\Omega (2\partial_z(e^{\tau\varphi}u_{-1}) + A_1(e^{\tau\varphi}u_{-1}), 2(A_1 - A_{2,s})^*\overline{\widetilde{w}_{1,s}e^{\tau\overline{\Phi}}}) \tag{121}$$

$$+(2\partial_{\overline{z}}(e^{\tau\varphi}u_{-1}) + B_1(e^{\tau\varphi}u_{-1}), 2(B_1 - B_{2,s})^*\overline{w_{1,s}e^{\tau\Phi}})dx = o(\frac{1}{\tau}).$$

By (120) and (121), we obtain the first equality in (118). The proof of the second equality in (118) is the same.

By (15), (2), (80), (112), (109), (108), (107) and (118), we have the asymptotic formula:

$$e^{2i\tau\psi(\widetilde{x})}(((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}b_+, w_{1,s})_{L^2(\Omega)} + ((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \widetilde{a}_-)_{L^2(\Omega)}) \tag{122}$$

$$+e^{-2i\tau\psi(\widetilde{x})}(((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}b_-, w_{1,s})_{L^2(\Omega)} + ((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \widetilde{a}_+)_{L^2(\Omega)})$$

$$+e^{2i\tau\psi(\widetilde{x})}(((A_1 - A_{2,s})\partial_z\Phi a_+, \widetilde{w}_{1,s})_{L^2(\Omega)} + ((A_1 - A_{2,s})\partial_z\Phi w_0, \widetilde{b}_-)_{L^2(\Omega)})$$

$$+e^{-2i\tau\psi(\widetilde{x})}(((A_1 - A_{2,s})\partial_z\Phi a_-, \widetilde{w}_{1,s})_{L^2(\Omega)} + ((A_1 - A_{2,s})\partial_z\Phi w_0, \widetilde{b}_+)_{L^2(\Omega)})$$

$$-\pi\frac{(\mathcal{Q}_+ w_0, \overline{w}_1)e^{2i\tau\psi(\widetilde{x})+s}}{\tau|\det\psi''(\widetilde{x})|^{\frac{1}{2}}} - \pi\frac{(\mathcal{Q}_- \widetilde{w}_0, \overline{\widetilde{w}_1})e^{-2i\tau\psi(\widetilde{x})+s}}{\tau|\det\psi''(\widetilde{x})|^{\frac{1}{2}}} + \mathcal{P}(\tau) + o(\frac{1}{\tau}),$$

where $\mathcal{Q}_+ = 2\partial_z(A_1 - A_2) + B_2(A_1 - A_2) + (B_1 - B_2)A_1 - (Q_1 - Q_2)$ and $\mathcal{Q}_- = 2\partial_{\overline{z}}(B_1 - B_2) + A_1(B_1 - B_2) + (A_1 - A_2)B_1 - (Q_1 - Q_2)$ and

$$\mathcal{P}(\tau) = -2\tau\int_\Omega (\mathbf{T}^*_{-A_2^*}((A_1 - A_{2,s})\partial_z\Phi e^{s\eta}w_0), \overline{q_3 + \widetilde{q}_3/\tau})e^{2i\tau\psi}dx$$

$$-\tau\int_\Omega (2\mathbf{P}^*_{-B_2^*}((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}e^{s\eta}\widetilde{w}_0), \overline{q_4 + \widetilde{q}_4/\tau})e^{-2i\tau\psi}dx$$

$$-2\tau\int_\Omega e^{-2i\tau\psi}(q_2 + \widetilde{q}_2/\tau, \overline{\mathbf{P}^*_{A_1}((A_1 - A_{2,s})^*(\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_{1,s}))})dx$$

$$-2\tau\int_\Omega (q_1 + \widetilde{q}_1/\tau, \overline{\mathbf{T}^*_{B_1}((B_1 - B_{2,s})^*(\partial_z\Phi w_{1,s}))})e^{2i\tau\psi}dx$$

Observe that

$$\mathbf{T}^*_{-A_2^*}((A_1 - A_{2,s})e^{s\eta}w_0) + e^{s\eta}w_0 \in Ker\,\mathbf{T}^*_{-A_2^*}, \quad \mathbf{P}^*_{-B_2^*}((B_1 - B_{2,s})e^{s\eta}\widetilde{w}_0) + e^{s\eta}\widetilde{w}_0 \in Ker\,\mathbf{P}^*_{-B_2^*}$$

and

$$\mathbf{P}^*_{A_1}((A_1 - A_{2,s})^*\widetilde{w}_{1,s}) + \widetilde{w}_{1,s} \in Ker\,\mathbf{P}^*_{A_1}, \quad \mathbf{T}^*_{B_1}((B_1 - B_{2,s})^*w_{1,s}) + w_{1,s} \in Ker\,\mathbf{T}^*_{B_1}.$$

Thanks to Proposition 4 and above relations, there exist functions $r_{1,s} \in Ker\,\mathbf{T}^*_{-A_2^*}, r_{2,s} \in Ker\,\mathbf{P}^*_{-B_2^*}, r_{3,s} \in Ker\,\mathbf{P}^*_{A_1}, r_{4,s} \in Ker\,\mathbf{T}^*_{B_1}$ such that

$$\mathcal{P}(\tau) = 2\tau\int_\Omega (\partial_z\Phi w_0, \overline{e^{\tau(\overline{\Phi}-\Phi)}q_{3,s}})dx + \tau\int_\Omega (r_{1,s}, \overline{e^{\tau(\overline{\Phi}-\Phi)}q_3})dx$$

$$+2\tau\int_\Omega (\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \overline{e^{\tau(\Phi-\overline{\Phi})}q_{4,s}})dx + \tau\int_\Omega (r_{2,s}, \overline{e^{\tau(\Phi-\overline{\Phi})}q_4})dx$$

$$+2\tau\int_\Omega e^{\tau(\overline{\Phi}-\Phi)}\partial_z\Phi(q_2 + \widetilde{q}_2/\tau, \overline{\widetilde{w}_{1,s}})dx + \tau\int_\Omega e^{\tau(\overline{\Phi}-\Phi)}(q_2, \overline{r}_{3,s})dx$$

$$+2\tau\int_\Omega e^{\tau(\Phi-\overline{\Phi})}q_1, \overline{\partial_z\Phi\widetilde{w}_{1,s}})dx + \tau\int_\Omega e^{\tau(\Phi-\overline{\Phi})}(q_1, \overline{r}_{4,s})dx + o(\frac{1}{\tau}) \quad as\ \tau \to +\infty. \tag{123}$$

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Integrating by parts in the above equality, we have

$$\mathcal{P}(\tau) = -2 \int_\Omega (w_0, \overline{e^{\tau(\overline{\Phi}-\Phi)}\partial_z q_{3,s}})dx + \tau \int_\Omega (r_{1,s}, \overline{e^{\tau(\overline{\Phi}-\Phi)}q_3})dx$$

$$-2 \int_\Omega (\widetilde{w}_0, \overline{e^{\tau(\Phi-\overline{\Phi})}\partial_{\overline{z}} q_{4,s}})dx + \tau \int_\Omega (r_{2,s}, \overline{e^{\tau(\Phi-\overline{\Phi})}q_4})dx$$

$$+2 \int_\Omega e^{\tau(\overline{\Phi}-\Phi)}(\partial_z q_2, \overline{\widetilde{w}_{1,s}})dx + \tau \int_\Omega e^{\tau(\overline{\Phi}-\Phi)}(q_2, \overline{r}_{3,s})dx$$

$$+2 \int_\Omega e^{\tau(\Phi-\overline{\Phi})}(\partial_{\overline{z}} q_1, \overline{w}_{1,s})dx + \tau \int_\Omega e^{\tau(\Phi-\overline{\Phi})}(q_1, \overline{r}_{4,s})dx + o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \tag{124}$$

Applying the stationary phase argument, we obtain that there exists a constant $\mathcal{C}_3$, independent of τ, such that

$$\mathcal{P}(\tau) = \frac{\mathcal{C}_3}{\tau} - 2\pi \frac{(\mathcal{Q}_+ w_0, \overline{w}_1)e^{2i\tau\psi(\widetilde{x})+s}}{\tau|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} - 2\pi \frac{(\mathcal{Q}_- \widetilde{w}_0, \overline{\widetilde{w}_1})e^{-2i\tau\psi(\widetilde{x})+s}}{\tau|\det \psi''(\widetilde{x})|^{\frac{1}{2}}}$$

$$+ \frac{e^{2i\tau\psi(\widetilde{x})}}{\tau|\det \psi''(\widetilde{x})|^{\frac{1}{2}}}(\mathfrak{D}(\ell_4) + \mathfrak{D}(\ell_2)) + \frac{e^{-2i\tau\psi(\widetilde{x})}}{\tau|\det \psi''(\widetilde{x})|^{\frac{1}{2}}}(\mathfrak{D}(\ell_1) + \mathfrak{D}(\ell_3)), \tag{125}$$

where $\ell_1 = (q_1, \overline{r_{4,s}}), \ell_2 = (q_2, \overline{r_{3,s}}), \ell_3 = (r_{2,s}, \overline{q_4}), \ell_4 = (r_{1,s}, \overline{q_3})$ and for any smooth function $\ell(x)$ we set

$$\mathfrak{D}(\ell) = \left(\partial_z \left(\frac{\ell_z(\widetilde{x})(z-\widetilde{z})}{\partial_z \Phi}\right) - \partial_{\overline{z}}\left(\frac{\ell_{\overline{z}}(\widetilde{x})(\overline{z}-\overline{\widetilde{z}})}{\partial_{\overline{z}}\overline{\Phi}}\right) + \frac{1}{2}\partial_z\left(\frac{\ell_{zz}(\widetilde{x})(z-\widetilde{z})^2}{\partial_z \Phi}\right) - \frac{1}{2}\partial_{\overline{z}}\left(\frac{\ell_{\overline{zz}}(\widetilde{x})(\overline{z}-\overline{\widetilde{z}})^2}{\partial_{\overline{z}}\overline{\Phi}}\right) \right)(\widetilde{x}).$$

Since $\psi(\widetilde{x}) \neq 0$, we obtain from (125) and (122):

$$((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}b_+, w_{1,s})_{L^2(\Omega)} + ((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \widetilde{a}_{-,s})_{L^2(\Omega)} \tag{126}$$

$$+((A_1 - A_{2,s})\partial_z\Phi a_+, \widetilde{w}_{1,s})_{L^2(\Omega)} + ((A_1 - A_2)\partial_z\Phi w_0, \widetilde{b}_{-,s})_{L^2(\Omega)}$$

$$-\pi \frac{(\mathcal{Q}_+ w_0, \overline{w}_1)e^s}{|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} + \frac{\mathfrak{D}(\ell_4) + \mathfrak{D}(\ell_2)}{|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} = 0$$

and

$$((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}b_-, w_{1,s})_{L^2(\Omega)} + ((B_1 - B_{2,s})\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \widetilde{a}_{+,s})_{L^2(\Omega)}$$

$$+((A_1 - A_{2,s})\partial_z\Phi a_-, \widetilde{w}_{1,s})_{L^2(\Omega)} + ((A_1 - A_{2,s})\partial_z\Phi w_0, \widetilde{b}_{+,s})_{L^2(\Omega)}$$

$$-\pi \frac{(\mathcal{Q}_- \widetilde{w}_0, \overline{\widetilde{w}_1})e^s}{|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} + \frac{\mathfrak{D}(\ell_1) + \mathfrak{D}(\ell_3)}{|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} = 0. \tag{127}$$

Integrating by parts in (126) and (127), we obtain

$$((\nu_1 - i\nu_2)\partial_{\overline{z}}\overline{\Phi}b_+, w_1)_{L^2(\partial\Omega)} + ((\nu_1 - i\nu_2)\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \widetilde{a}_-)_{L^2(\partial\Omega)} \tag{128}$$

$$+((\nu_1 + i\nu_2)\partial_z\Phi a_+, \widetilde{w}_1)_{L^2(\partial\Omega)} + ((\nu_1 + i\nu_2)\partial_z\Phi w_0, \widetilde{b}_-)_{L^2(\partial\Omega)}$$

$$-\pi \frac{(\mathcal{Q}_+ w_0, \overline{w}_1)e^s}{|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} + \frac{\mathfrak{D}(\ell_4) + \mathfrak{D}(\ell_2)}{|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} = 0$$

and

$$((\nu_1 - i\nu_2)\partial_{\overline{z}}\overline{\Phi}b_-, w_1)_{L^2(\partial\Omega)} + ((\nu_1 - i\nu_2)\partial_{\overline{z}}\overline{\Phi}\widetilde{w}_0, \widetilde{a}_+)_{L^2(\partial\Omega)}$$

$$+((\nu_1 + i\nu_2)\partial_z\Phi a_-, \widetilde{w}_1)_{L^2(\partial\Omega)} + ((\nu_1 + i\nu_2)\partial_z\Phi w_0, \widetilde{b}_+)_{L^2(\partial\Omega)}$$

$$-\pi \frac{(\mathcal{Q}_- \widetilde{w}_0, \overline{\widetilde{w}_1})e^s}{|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} + \frac{\mathfrak{D}(\ell_1) + \mathfrak{D}(\ell_3)}{|\det \psi''(\widetilde{x})|^{\frac{1}{2}}} = 0. \tag{129}$$

Observe that

$$\sum_{k=1}^{4} |\mathfrak{D}(\ell_k)| \le C_4 \tag{130}$$

with the constant C_4 independent of s. We prove this inequality for $\mathfrak{D}(\ell_4)$. The proof for remaining terms is similar. By (80) the functions $(A_1 - A_{2,s})e^{s\eta}w_0$ are bounded uniformly in the space $H^1(\Omega)'$. Hence, by (29,the functions $\mathbf{T}^*_{-A_2}(\partial_z\Phi(A_1 - A_{2,s})e^{s\eta}w_0)$ are uniformly bounded in $L^2(\Omega)$. Then the functions $r_{1,s}$ are uniformly bounded in $L^2(\Omega)$ and $Ker\, \mathbf{T}^*_{-A_2}$. Therefore the functions $r_{1,s}$ are uniformly bounded in $C^5(K)$ for any compact $K \subset\subset \Omega$. Since $\ell_1 = (r_{1,s}, \overline{q_3})$, the proof of (130) is completed.

Passing to the limit in (128) and (129) as s goes to infinity, we obtain $(\mathcal{Q}_+ w_0, \overline{w}_1)(\widetilde{x}) = (\mathcal{Q}_- \widetilde{w}_0, \overline{\widetilde{w}_1})(\widetilde{x}) = 0$. These equalities and (42) imply the equalities (3) and (4) at point $\widetilde{x}$. According to Proposition 1, a point $\widetilde{x}$ can be chosen arbitrarily close to any point of domain Ω after an appropriate choice of the function Φ. The proof of the theorem is completed.

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