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A discrete isoperimetric inequality on lattices

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Abstract. We establish an isoperimetric inequality with constraint by n-dimensional lattices. We prove that, among all domains which consist of rectangular parallelepipeds with the common side-lengths, a cube is the best shape to minimize the ratio involving its perimeter and volume as long as the cube is realizable by the lattice. For its proof a solvability of finite difference Poisson-Neumann problems is verified. Our approach to the isoperimetric inequality is based on the technique used in a proof of the Aleksandrov-Bakelman-Pucci maximum principle, which was originally proposed by Cabré in 2000 to prove the classical isoperimetric inequality.

1. Introduction

The classical isoperimetric inequality asserts that for any bounded $E \subset \mathbf{R}^n$ we have

$$\frac{|\partial E|^n}{|E|^{n-1}} \geqq \frac{|\partial \mathbf{B}_1|^n}{|\mathbf{B}_1|^{n-1}}, \tag{1.1}$$

where $|E|$ and $|\partial E|$ denote, respectively, the volume of E and the perimeter of E, and $\mathbf{B}_r := \{x \in \mathbf{R}^n \mid |x| < r\}$ is a ball. This inequality says that among all domains a ball is the best shape to minimize the ratio given as the left hand side of (1.1). Related topics to the classical isoperimetric problem or arguments on its generalization can be found in the book [5] and the survey paper [22]. See also the recent book [27] for connections with Sobolev inequalities and optimal transportation.

In this paper we are concerned with the case where E is a collection of rectangular parallelepipeds with the common shape. To describe the situation more precisely we first define a weighted lattice. For each $i \in \{1, \dots, n\}$ we fix a positive constant $h_i > 0$ as a step size in the direction of x_i. Then the resulting lattice is

$$h\mathbf{Z}^n := (h_1\mathbf{Z}) \times \cdots \times (h_n\mathbf{Z}) = \{(h_1x_1, \dots, h_nx_n) \in \mathbf{R}^n \mid (x_1, \dots, x_n) \in \mathbf{Z}^n\}.$$

Consider a subset $\Omega \subset h\mathbf{Z}^n$. We define $\overline{\Omega}$, a closure of Ω, as

$$\overline{\Omega} := \left\{ x + \sum_{i=1}^n \sigma_i h_i e_i \;\middle|\; x \in \Omega,\ \sigma_1, \dots, \sigma_n \in \{-1, 0, 1\} \right\},$$

where $\{e_i\}_{i=1}^n \subset \mathbf{R}^n$ is the standard orthogonal basis of $\mathbf{R}^n$, e.g., $e_1 = (1, 0, \dots, 0)$. Note that this is not a closure in $\mathbf{R}^n$. We also set $\partial\Omega := \overline{\Omega} \setminus \Omega$, a boundary of

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Ω. Given a bounded $\Omega \subset h\mathbf{Z}^n$, we define a volume of Ω and a perimeter of Ω as, respectively,

$$\mathrm{Vol}(\Omega) := h^n \times (\#\Omega), \quad \mathrm{Per}(\Omega) := h^n \times \left(\sum_{i=1}^{n} \frac{\omega_i}{h_i} \right)$$

with

$$\omega_i = \omega_i[\Omega] = \sum_{x \in \Omega} \#(\{x \pm h_i e_i\} \cap \partial\Omega),$$

where $h^n := h_1 \times \cdots \times h_n$ and $\#A$ stands for the number of elements of a set A. The number ω_i counts the edges that are parallel to the x_i-direction and are connecting points of Ω with points of $\partial\Omega$. Our definitions of the volume and the perimeter are natural in that if we let

$$E = E[\Omega] := \bigcup_{(x_1,\dots,x_n)\in\Omega} \left[x_1 - \frac{h_1}{2}, x_1 + \frac{h_1}{2} \right] \times \cdots \times \left[x_n - \frac{h_n}{2}, x_n + \frac{h_n}{2} \right] \tag{1.2}$$

for a given $\Omega \subset h\mathbf{Z}^n$, we then have $\mathrm{Vol}(\Omega) = \mathcal{L}^n(E)$, the n-dimensional Lebesgue measure of E, and $\mathrm{Per}(\Omega) = \mathcal{H}^{n-1}(\partial E)$, the $(n-1)$-dimensional Hausdorff measure of ∂E (the boundary of E in $\mathbf{R}^n$).

We denote by $\mathbf{Q}_r$ and $\bar{\mathbf{Q}}_r$, respectively, the open and closed cube in $\mathbf{R}^n$ with center 0 and side-length $2r > 0$, i.e., $\mathbf{Q}_r := (-r, r)^n \subset \mathbf{R}^n$ and $\bar{\mathbf{Q}}_r := [-r, r]^n \subset \mathbf{R}^n$. Let $\bar{\mathbf{Q}}_r(a) := a + \bar{\mathbf{Q}}_r$ for $a \in \mathbf{R}^n$. The volume and perimeter of $\mathbf{Q}_r$ are, respectively, $|\mathbf{Q}_r| = (2r)^n$ and $|\partial\mathbf{Q}_r| = 2n(2r)^{n-1}$. We are now in a position to state our main result.

Theorem 1.1 (Discrete isoperimetric inequality)**.** *For any nonempty, bounded and connected $\Omega \subset h\mathbf{Z}^n$ we have*

$$\frac{\mathrm{Per}(\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} \geqq \frac{|\partial\mathbf{Q}_1|^n}{|\mathbf{Q}_1|^{n-1}}. \tag{1.3}$$

Moreover, the equality in (1.3) *holds if and only if $E[\Omega]$ is a cube, i.e, $E[\Omega] = \bar{\mathbf{Q}}_r(a)$ for some $r > 0$ and $a \in \mathbf{R}^n$.*

Here we say $\Omega \subset h\mathbf{Z}^n$ is *connected* if for all $x, y \in \Omega$ there exist $m \in \mathbf{N} = \{1, 2, \dots\}$ and $z_1, \dots, z_m \in \Omega$ such that $z_1 \in \overline{\{x\}}$, $z_{k+1} \in \overline{\{z_k\}}$ $(k = 1, \dots, m-1)$ and $y \in \overline{\{z_m\}}$. Although (1.3) can be regarded as a "continuous" isoperimetric inequality if we identify Ω with $E[\Omega]$ in (1.2), we call (1.3) a "discrete" isoperimetric inequality since our approach to Theorem 1.1 uses numerical techniques which study functions defined on the lattice $h\mathbf{Z}^n$. Note that our result is different from the classical one in that the minimizer of the left hand side of (1.3) is a cube. This is a consequence of the constraint by lattices. We also remark that the equality in (1.3) does not necessarily hold; consider the two dimensional case where $h_1 = 1$ and $h_2 = \sqrt{2}$.

The fact that round-shaped subsets are not optimal is observed in the following way when $n = 2$. Let $h_1 = h_2 = 1$ and $\Omega \subset \mathbf{Z}^2$ be nonempty, bounded and connected. We choose $R = \{a, a+1, \dots, a+M\} \times \{b, b+1, \dots, b+N\} \subset \mathbf{Z}^2$ as the minimal rectangle such that $\Omega \subset R$. Obviously, $\mathrm{Vol}(\Omega) < \mathrm{Vol}(R)$ if $\Omega \neq R$. We next consider their perimeters. Since Ω is connected, for each $x \in \{a, a+1, \dots, a+M\}$ there exist $(x, y_-), (x, y_+) \in \Omega$ such that $(x, y_- - 1), (x, y_+ + 1) \notin \Omega$. This implies $\omega_1[\Omega] \geqq 2(M+1) = \omega_1[R]$. Similarly, we obtain $\omega_2[\Omega] \geqq 2(N+1) = \omega_2[R]$, and therefore $\mathrm{Per}(\Omega) \geqq \mathrm{Per}(R)$. We thus conclude that $\mathrm{Per}(\Omega)^2/\mathrm{Vol}(\Omega) >$

$\mathrm{Per}(R)^2/\mathrm{Vol}(R)$, i.e., Ω is not optimal. Note that this argument is not valid if $n \geqq 3$.

On the contrary, if we define a volume and a perimeter of Ω as $\#\Omega$ and $\#(\partial\Omega)$, respectively, then a cube is not an optimal shape. This can be seen in the following simple example. Let $n = 2$, $h_1 = h_2 = 1$ again and consider planar subsets $\Omega_1 = \{(x, y) \in \mathbf{Z}^2 \mid |x| \leqq 1,\ |y| \leqq 1\}$ and $\Omega_2 = \{(x, y) \in \mathbf{Z}^2 \mid |x|+|y| \leqq 2\}$. We then have $\#\Omega_1 = 9$, $\#\Omega_2 = 13$ and $\#(\partial\Omega_1) = \#(\partial\Omega_2) = 12$. Thus the square Ω_1 is not a minimizer of the functional $(\#(\partial\Omega))^2/(\#\Omega)$. In the article [10] the author asserts that if Ω has a minimal $\#\partial\Omega$, then Ω is roughly diamond-shaped. The author of [10] also observes inequalities $(\#(\partial\Omega))^2/(\#\Omega) > 8$ for the two dimensional case and $(\#(\partial\Omega))^3/(\#\Omega)^2 > 36$ for the three dimensional case without detailed argument. We do not discuss such problems concerning the functional $(\#(\partial\Omega))^n/(\#\Omega)^{n-1}$ in the present paper.

Isoperimetric problems on discrete spaces are studied by many authors. The recent book [13] gives a survey of isoperimetric problems on graphs (networks) in Chapter 8. See also [9] for various results including discrete Sovolev inequalities on finite graphs. The paper [1] is a survey on several discrete isoperimetric inequalities. The main problem discussed in [1] is to determine an optimal subset of $\{0, 1\}^n$ equipped with the Hamming metric. Isoperimetric problems concerning lattices can be found in several previous work; however, their settings and problems are different from ours. The authors of [15] consider compact and convex subsets in $\mathbf{R}^2$ whose interior contain zero or one lattice point. Under the constraint some inequalities for geometrical functionals such as an area or a perimeter are derived. The paper [15] also summarizes known inequalities for geometrical functionals concerning subsets with the same constraint. See tables in [15] or references therein. Planar convex subsets are also considered in [2]. The author of [2] gives inequalities involving the number of interior or boundary lattice points lying in a convex region in the plane. In [4] isoperimetric problems for lattice-periodic sets are discussed. The reader is referred to its related work [14, 3, 24] for similar problems concerning multi-dimensional spaces or non-periodic sets. Properties of planar subsets with constraint by a triangular lattice are discussed in [11].

For the proof of our discrete isoperimetric inequality we employ the idea by Cabré. As an application of the technique used in a proof of the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, Cabré pointed out in [8] (and the original paper [7] in Catalan) that the ABP method gives a simple proof of the classical isoperimetric inequality (1.1). We refer the reader, if interested in the ABP maximum principle, to [12, Theorem 9.1] for $W^{2,n}$ solutions and [6, Theorem 3.2] for viscosity solutions. Discrete versions of the ABP estimate are established in a series of studies by Kuo and Trudinger; see [16, 21] for linear equations, [17] for nonlinear operators, [18, 20] for parabolic cases and [19, 20] for general meshes.

Unfortunately, the result in [8] does not cover subsets having corners such as (1.2) since domains E in [8] is assumed to be smooth in order to solve Neumann problems on E. To be more precise, the author of [8] takes a function u which solves the Poisson-Neumann problem

$$\begin{cases} -\Delta u = \dfrac{|\partial E|}{|E|} & \text{in } E, \\ \dfrac{\partial u}{\partial \nu} = -1 & \text{on } \partial E, \end{cases} \tag{1.4}$$

and proves (1.1) by studying the n-dimensional Lebesgue measure of the image of an upper contact set of u under its gradient ∇u. Here ν is the outward unit normal vector to ∂E. In this paper we solve a finite difference version of (1.4) instead of the continuous equation. Considering such discrete equations and their discrete solutions enables us to deal with non smooth domains.

Our proof is similar to that in [8] except that minimizers are not balls but cubes and that a superdifferential of u is used instead of its gradient ([16]). However, there are some extra difficulties in our case. One is a solvability of the discrete Poisson-Neumann problem. Such problems are discussed in the previous work [26, 23, 25, 28], but domains are restricted to rectangles ([26, 23, 28]) or their collections ([25]). For the proof of our discrete isoperimetric inequality, fortunately, it is enough to require u to be a subsolution of the Poisson equation in (1.4) and to satisfy the Neumann condition in (1.4) with some direction ν. For this reason we are able to construct such solutions on general subsets of $h\mathbf{Z}^n$. Solutions of (1.4) are not unique in our discrete case as well as the continuous case since adding a constant gives another solution. Accordingly, the resulting coefficient matrix of a linear system which corresponds to the discrete (1.4) is not invertible. Thus an existence of solutions to the problem will be established by determining the kernel of the matrix. We will give a proof of this existence result separately from that of the isoperimetric inequality to increase readability. Another difficulty is to study a necessary and sufficient condition which leads the equality in (1.3). This is not discussed in [8].

This paper is organized as follows. In Section 2 we give a proof of the discrete isoperimetric inequality. Since we use a discrete solution of the Poisson-Neumann problem in the proof, we show the existence of such solutions in Section 3. In Appendix we present two results on maximum principles; one is an ABP maximum principle shown by a similar method to the isoperimetric inequality, and the other is a strong maximum principle which is used in Section 3.

2. A proof of the discrete isoperimetric inequality

Throughout this paper we always assume

$$\Omega \subset h\mathbf{Z}^n \text{ is nonempty, bounded and connected.}$$

We first introduce a notion of *superdifferentials* and *upper contact sets*, and then study their properties. Let $u : \overline{\Omega} \to \mathbf{R}$. We denote by $\partial^+ u(z)$ a superdifferential of u on Ω at $z \in \Omega$, which is given as

$$\partial^+ u(z) := \{p \in \mathbf{R}^n \mid u(x) \leqq \langle p, x - z\rangle + u(z),\ \forall x \in \overline{\Omega}\},$$

where $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product in $\mathbf{R}^n$. It is easy to see that $\partial^+ u(z)$ is a closed set in $\mathbf{R}^n$. We next define $\Gamma[u]$, an upper contact set of u on Ω, as

$$\begin{aligned}\Gamma[u] :=& \{z \in \Omega \mid \partial^+ u(z) \neq \emptyset\}\\ =& \{z \in \Omega \mid \exists p \in \mathbf{R}^n \text{ such that } u(x) \leqq \langle p, x - z\rangle + u(z),\ \forall x \in \overline{\Omega}\}.\end{aligned}$$

For $x \in \Omega$ and $i \in \{1, \ldots, n\}$ we define discrete differential operators as follows:

$$\delta_i^+ u(x) := \frac{u(x + h_i e_i) - u(x)}{h_i}, \qquad \delta_i^- u(x) := -\frac{u(x - h_i e_i) - u(x)}{h_i},$$

$$\delta_i^2 u(x) := \frac{\delta_i^+ u(x) - \delta_i^- u(x)}{h_i} = \frac{u(x + h_i e_i) + u(x - h_i e_i) - 2u(x)}{h_i^2},$$

$$\Delta' u(x) := \sum_{j=1}^n \delta_j^2 u(x) = \sum_{j=1}^n \frac{u(x + h_j e_j) + u(x - h_j e_j)}{h_j^2} - \left(2 \sum_{j=1}^n \frac{1}{h_j^2}\right) u(x).$$

Lemma 2.1. *Let $u : \overline{\Omega} \to \mathbf{R}$. For all $z \in \Gamma[u]$ we have $\delta_i^+ u(z) \leqq \delta_i^- u(z)$ for every $i \in \{1, \ldots, n\}$ and*

$$\partial^+ u(z) \subset [\delta_1^+ u(z), \delta_1^- u(z)] \times \cdots \times [\delta_n^+ u(z), \delta_n^- u(z)]. \tag{2.1}$$

Remark 2.2. Since $\delta_i^+ u(z) \leqq \delta_i^- u(z)$ at $z \in \Gamma[u]$, we see $\delta_i^2 u(z) \leqq 0$ for all $i \in \{1, \ldots, n\}$.

Proof. Let $p = (p_1, \ldots, p_n) \in \partial^+ u(z)$. From the definition of the superdifferential it follows that $u(x) \leqq \langle p, x - z \rangle + u(z)$ for all $x \in \overline{\Omega}$. In particular, taking $x = z \pm h_i e_i \in \overline{\Omega}$, we have

$$u(z \pm h_i e_i) \leqq \langle p, \pm h_i e_i \rangle + u(z);$$

that is,

$$\frac{u(z + h_i e_i) - u(z)}{h_i} \leqq p_i \leqq -\frac{u(z - h_i e_i) - u(z)}{h_i}.$$

This implies $\delta_i^+ u(z) \leqq \delta_i^- u(z)$ and (2.1). □

Proof of Theorem 1.1. 1. Take $u : \overline{\Omega} \to \mathbf{R}$ as a discrete solution of the Neumann problem

$$(\mathrm{NP}) \begin{cases} -\Delta u \leqq \dfrac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} & \text{in } \ \Omega, \quad (2.2) \\ \dfrac{\partial u}{\partial \nu} = -1 & \text{on } \ \partial\Omega. \quad (2.3) \end{cases}$$

Here we say u is a discrete solution of (NP) if $-\Delta' u(x) \leqq \mathrm{Per}(\Omega)/\mathrm{Vol}(\Omega)$ for all $x \in \Omega$, and if, as for the Neumann boundary condition (2.3), for all $x \in \partial\Omega$ there exist some $i \in \{1, \ldots, n\}$ and $\sigma \in \{-1, 1\}$ such that $x + \sigma h_i e_i \in \Omega$ and

$$\frac{u(x) - u(x + \sigma h_i e_i)}{h_i} = -1.$$

We will prove the existence of such solutions u in the next section (Proposition 3.2).

2. Consider $\Gamma[u]$, the upper contact set of u on Ω. We claim

$$\mathbf{Q}_1 \subset \bigcup_{z \in \Gamma[u]} \partial^+ u(z). \tag{2.4}$$

Let $p \in \mathbf{Q}_1$. We take a maximum point $\hat{x} \in \overline{\Omega}$ of $u(x) - \langle p, x \rangle$ over $\overline{\Omega}$. To show (2.4) it is enough to prove that $\hat{x} \in \Omega$ since we then have $\hat{x} \in \Gamma[u]$ and $p \in \partial^+ u(\hat{x})$. Suppose by contradiction that $\hat{x} \in \partial\Omega$. Take any $i \in \{1, \ldots, n\}$ and $\sigma \in \{-1, 1\}$

such that $y := \hat{x} + \sigma h_i e_i \in \Omega$. Since $u(x) - \langle p, x\rangle$ attains its maximum at $\hat{x}$, we compute

$$\frac{u(\hat{x}) - u(y)}{h_i} \geqq \frac{\langle p, \hat{x}\rangle - \langle p, y\rangle}{h_i} = \frac{\langle p, -\sigma h_i e_i\rangle}{h_i} \geqq -|p_i| > -1.$$

This implies that u does not satisfy the boundary condition (2.3) at $\hat{x} \in \partial\Omega$, a contradiction.

3. By (2.4) we see

$$|\mathbf{Q}_1| = \mathcal{L}^n(\mathbf{Q}_1) \leqq \mathcal{L}^n\left(\bigcup_{z\in\Gamma[u]} \partial^+ u(z)\right) \leqq \sum_{z\in\Gamma[u]} \mathcal{L}^n(\partial^+ u(z)). \tag{2.5}$$

Also, for each $z \in \Gamma[u]$ Lemma 2.1 implies

$$\begin{aligned}\mathcal{L}^n(\partial^+ u(z)) &\leqq \mathcal{L}^n([\delta_1^+ u(z), \delta_1^- u(z)] \times \cdots \times [\delta_n^+ u(z), \delta_n^- u(z)])\\ &= (\delta_1^- u(z) - \delta_1^+ u(z)) \times \cdots \times (\delta_n^- u(z) - \delta_n^+ u(z))\\ &= h^n(-\delta_1^2 u(z)) \times \cdots \times (-\delta_n^2 u(z)).\end{aligned} \tag{2.6}$$

We next apply the arithmetic-geometric mean inequality to obtain

$$(-\delta_1^2 u(z)) \times \cdots \times (-\delta_n^2 u(z)) \leqq \left(\frac{-\delta_1^2 u(z) - \cdots - \delta_n^2 u(z)}{n}\right)^n = \left(\frac{-\Delta' u(z)}{n}\right)^n. \tag{2.7}$$

Consequently, combining (2.5)–(2.7) yields

$$|\mathbf{Q}_1| \leqq \sum_{z\in\Gamma[u]} h^n \left(\frac{-\Delta' u(z)}{n}\right)^n \leqq \sum_{z\in\Gamma[u]} h^n \frac{\mathrm{Per}(\Omega)^n}{n^n \mathrm{Vol}(\Omega)^n} \leqq \frac{\mathrm{Per}(\Omega)^n}{n^n \mathrm{Vol}(\Omega)^{n-1}}. \tag{2.8}$$

Since $n = |\partial\mathbf{Q}_1|/|\mathbf{Q}_1|$, it follows that

$$\frac{\mathrm{Per}(\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} \geqq n^n|\mathbf{Q}_1| = \frac{|\partial\mathbf{Q}_1|^n}{|\mathbf{Q}_1|^n}|\mathbf{Q}_1| = \frac{|\partial\mathbf{Q}_1|^n}{|\mathbf{Q}_1|^{n-1}}.$$

4. We next assume that the equality in (1.3) holds. In view of Steps 3, we then have $\Gamma[u] = \Omega$ by (2.8) and

$$\mathcal{L}^n(\mathbf{Q}_1) = \mathcal{L}^n\left(\bigcup_{x\in\Omega} \partial^+ u(x)\right), \tag{2.9}$$

$$\mathcal{L}^n(\partial^+ u(x)) = \mathcal{L}^n([\delta_1^+ u(x), \delta_1^- u(x)] \times \cdots \times [\delta_n^+ u(x), \delta_n^- u(x)]) \quad \text{for all } x \in \Omega, \tag{2.10}$$

$$\delta_1^2 u(x) = \cdots = \delta_n^2 u(x) =: \mu(x)\ (\leqq 0) \quad \text{for all } x \in \Omega \tag{2.11}$$

by (2.5), (2.6) and (2.7), respectively. We claim

$$\partial^+ u(x) = [\delta_1^+ u(x), \delta_1^- u(x)] \times \cdots \times [\delta_n^+ u(x), \delta_n^- u(x)] \quad \text{for all } x \in \Omega. \tag{2.12}$$

Since we have (2.1) and $\partial^+ u(x)$ is closed, it is enough to show

$$\partial^+ u(x) \supset (\delta_1^+ u(x), \delta_1^- u(x)) \times \cdots \times (\delta_n^+ u(x), \delta_n^- u(x)) \quad \text{for all } x \in \Omega.$$

Suppose that there were some $p \in (\delta_1^+ u(x), \delta_1^- u(x)) \times \cdots \times (\delta_n^+ u(x), \delta_n^- u(x))$ such that $p \notin \partial^+ u(x)$. Since $\partial^+ u(x)$ is closed, we would have $(p + \mathbf{B}_r) \cap \partial^+ u(x) = \emptyset$ for sufficiently small $r > 0$. Then, however, (2.10) would be violated. Thus (2.12) is proved. Also, as a consequence of (2.12), it easily follows from (2.4) and (2.9) that

$$\bar{\mathbf{Q}}_1 = \bigcup_{x \in \Omega} \partial^+ u(x). \tag{2.13}$$

5. We next prove that $\mu(x) = \mu(y) =: \mu$ and

$$\partial^+ u(y) = \partial^+ u(x) + h_i \mu e_i \tag{2.14}$$

for all $x, y \in \Omega$ such that $y = x + h_i e_i$ for some $i \in \{1, \dots, n\}$, where $\mu(\cdot)$ is the function in (2.11). Without loss of generality we may assume $x = 0$, $y = h_1 e_1$ and $u(x) = 0$. We then notice that $u(y) = h_1 \delta_1^+ u(0)$. Fix $i \in \{2, \dots, n\}$ and set $p^\pm := \delta_1^+ u(0) e_1 + \delta_i^\pm u(0) e_i \in \partial^+ u(0)$. Then, since $x = 0 \in \Gamma[u]$, we observe that $u(z) \leqq \langle p^\pm, z \rangle$ for all $z \in \overline{\Omega}$. In particular, letting $z = h_1 e_1 \pm h_i e_i$, we deduce $u(z) \leqq h_1 \delta_1^+ u(0) \pm h_i \delta_i^\pm u(0) = u(y) \pm h_i \delta_i^\pm u(0)$, i.e., $\delta_i^+ u(y) \leqq \delta_i^+ u(0)$ and $\delta_i^- u(0) \leqq \delta_i^- u(y)$. Changing the role of x and y we also have $\delta_i^+ u(y) \geqq \delta_i^+ u(0)$ and $\delta_i^- u(0) \geqq \delta_i^- u(y)$. Thus

$$\delta_i^+ u(y) = \delta_i^+ u(0) \quad \text{and} \quad \delta_i^- u(0) = \delta_i^- u(y) \tag{2.15}$$

for all $i \in \{2, \dots, n\}$. These equalities imply $\mu(x) = \mu(y) = \mu$, and then $\delta_1^\pm u(y)$ are computed as

$$\delta_1^- u(y) = \delta_1^+ u(x) = \delta_1^- u(x) + h_1 \mu, \quad \delta_1^+ u(y) = \delta_1^- u(y) + h_1 \mu = \delta_1^+ u(x) + h_1 \mu.$$

Namely, we have $[\delta_1^- u(y), \delta_1^+ u(y)] = [\delta_1^- u(x), \delta_1^+ u(x)] + h_1 \mu$, which together with (2.15) shows (2.14).

6. By translation we may let $0 \in \Omega$. Set $R := [-h_1/2, h_1/2] \times \cdots \times [-h_n/2, h_n/2]$ and choose $z \in \mathbf{R}^n$ so that $\partial^+ u(0) = z + \mu R$. Since Ω is now connected, as a consequence of Step 5 we see $\mu(x) \equiv \mu$ and $\partial^+ u(x) = \partial^+ u(0) + \mu x = z + \mu x + \mu R$ for all $x \in \Omega$. Therefore (2.13) implies

$$\bar{\mathbf{Q}}_1 = \bigcup_{x \in \Omega} (z + \mu x + \mu R).$$

Finally, from translation and rescaling it follows that

$$\bar{\mathbf{Q}}_{1/|\mu|}(-z/\mu) = \bigcup_{x \in \Omega} (x + R) = E[\Omega],$$

which is the desired conclusion. □

3. An existence result for the Poisson-Neumann problem

We shall prove the solvability of (NP), the Poisson equation with the Neumann boundary condition which appeared in the proof of the discrete isoperimetric inequality. Before starting the proof, using a simple example, we explain how to construct the solutions.

Example 3.1. Consider $\Omega \subset h\mathbf{Z}^2$ which consists of three points P_1, P_2 and P_3 in the left lattice of Figure 1. We also denote by $S_1, \dots, S_7$ all points on $\partial\Omega$ as in the same figure. In order to determine values of u on $\overline{\Omega}$ we solve a system of linear equations of the matrix form $L\vec{a} = \vec{b}$ which corresponds to the finite difference equation (NP). However, if we require u to satisfy the Neumann condition (2.3) at S_1 toward the both adjacent points P_1 and P_3, the linear system may not be solvable since the number of the unknowns is less than that of equations; in the present example they are 10 and 11, respectively. Thus we are tempted to consider the Neumann condition toward either P_1 or P_3 since we are now allowed to relax (2.3) in this way by the meaning of solutions. Then the number of equations decreases to 10, but, unfortunately, it becomes difficult to study the linear system since the new matrix L is not symmetric. In addition, we do not know a priori how to choose the adjacent point toward which the Neumann condition is satisfied.

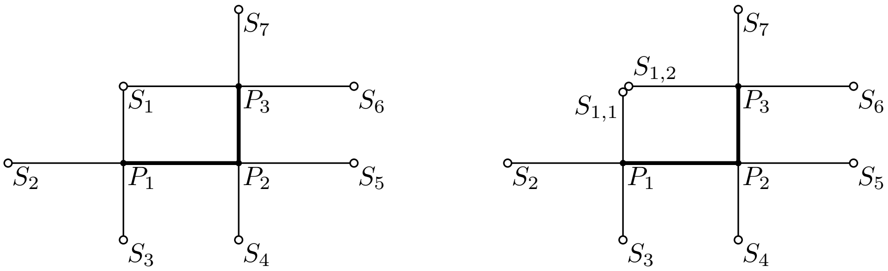

FIGURE 1. $\Omega = \{P_i\}_{i=1}^3$ and $\partial\Omega = \{S_i\}_{i=1}^7$. We solve a system of linear equations for the right lattice, and then define $u(S_1) := \max\{u(S_{1,1}),\ u(S_{1,2})\}$.

To avoid these situations we regard S_1 as two different points $S_{1,1}$ and $S_{1,2}$ which are connected to P_1 and P_3, respectively, and consider a modified system with new unknowns $u(S_{1,1})$ and $u(S_{1,2})$ instead of $u(S_1)$; see the right lattice in Figure 1. Then the number of the unknowns in our example becomes 11. Thanks to this increase of the unknowns, it turns out that the modified linear system admits at least one solution $(u(P_1), u(P_2), u(P_3), u(S_{1,1}), u(S_{1,2}), u(S_2), \dots, u(S_7))$. (In the notation of the proof below we write $u(S_{1,1}) = \beta(1,1)$ and $u(S_{1,2}) = \beta(1,2)$.) In the process of proving the solvability we find that the right hand side of (2.2) should be $\mathrm{Per}(\Omega)/\mathrm{Vol}(\Omega)$. Also, for its proof we employ the strong maximum principle for the discrete Laplace equation.

The remaining problem is how to define $u(S_1)$. We define $u(S_1)$ as the maximum of $u(S_{1,1})$ and $u(S_{1,2})$, so that, if $u(S_{1,1}) \geqq u(S_{1,2})$, we have $-\Delta' u(P_3) \leqq \mathrm{Per}(\Omega)/\mathrm{Vol}(\Omega)$ since $u(S_1) \geqq u(S_{1,2})$ and $\{u(S_1) - u(P_1)\}/h_2 = -1$ since $u(S_1) = u(S_{1,1})$. In this way we obtain a solution of (NP).

Proposition 3.2. *The problem* (NP) *admits at least one discrete solution.*

Proof. 1. We first introduce notations. Let $\Omega = \{P_1, \dots, P_M\}$ and $\partial\Omega = \{S_1, \dots, S_{N_0}\}$, where $M := \#\Omega$ and $N_0 := \#(\partial\Omega)$. For each $i \in \{1, \dots, M\}$ we define subsets $\mathcal{M}(i) \subset \{1, \dots, M\}$ and $\mathcal{N}(i) \subset \{1, \dots, N_0\}$ so that $\overline{\{P_i\}} \setminus \{P_i\} = \{P_j\}_{j \in \mathcal{M}(i)} \cup \{S_j\}_{j \in \mathcal{N}(i)}$. We also set $s_i := \#(\overline{\{S_i\}} \cap \Omega)$ for $i \in \{1, \dots, N_0\}$, which stands for the number of points of Ω adjacent to S_i, and $N := \sum_{j=1}^{N_0} s_j$. Next, for $i \in \{1, \dots, N_0\}$ we define a map $n_i : \{1, \dots, s_i\} \to \{1, \dots, M\}$ such that $n_i(1) < n_i(2) < \cdots$ and $\overline{\{S_i\}} \cap \Omega = \{P_{n_i(j)}\}_{j=1}^{s_i}$. (Such maps are unique.) We denote by n_i^{-1} the inverse map of n_i; that is, P_j is the $n_i^{-1}(j)$-th point of $(P_{n_i(1)}, \dots, P_{n_i(s_i)})$ if $P_j \in \overline{\{S_i\}} \cap \Omega$.

For $x, y \in \overline{\Omega}$ such that $y = x + \sigma h_i e_i$ with $\sigma = \pm 1$ and $i \in \{1, \dots, n\}$ we set $h(x,y) := h_i$. Obviously, we then have $h(x,y) = h(y,x)$. We denote by $E(i,j)$ the

$(M+N)\times(M+N)$ matrix with 1 in the (i,j) entry and 0 elsewhere. Given a vector

$$\vec{a} = {}^t(\alpha(1),\dots,\alpha(M),\beta(1,1),\dots,\beta(1,s_1),\dots,\beta(N_0,1),\dots,\beta(N_0,s_{N_0})) \in \mathbf{R}^{M+N}, \tag{3.1}$$

where ${}^t\vec{v}$ means the transpose of a vector $\vec{v}$, we define $u = u[\vec{a}] : \overline{\Omega} \to \mathbf{R}$ as

$$u(x) := \begin{cases} \alpha(i) & (x = P_i \in \Omega,\ i \in \{1,\dots,M\}), \\ \max\{\beta(i,j) \mid 1 \leqq j \leqq s_i\} & (x = S_i \in \partial\Omega,\ i \in \{1,\dots,N_0\}). \end{cases}$$

2. We consider the following system of linear equations

$$L\vec{a} = \vec{b}, \tag{3.2}$$

where $\vec{a} \in \mathbf{R}^{M+N}$ is the unknown vector and $\vec{b} = (b_k)_{k=1}^{M+N} \in \mathbf{R}^{M+N}$ is given as

$$b_k = \begin{cases} \dfrac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} & (k = 1,\dots,M), \\ \dfrac{-1}{h(S_j, P_{n_j(i)})} & (k = M + \sum_{l=0}^{j-1} s_l + i \text{ with } j \in \{1,\dots,N_0\},\ i \in \{1,\dots,s_j\}). \end{cases}$$

Here $s_0 = 0$. Also, the $(M+N)\times(M+N)$ matrix L is defined by

$$L := \begin{pmatrix} \theta I_M & 0 \\ 0 & 0 \end{pmatrix} - \sum_{i=1}^{M} \left\{ \sum_{j\in\mathcal{M}(i)} \frac{E(i,j)}{h(P_i,P_j)^2} + \sum_{j\in\mathcal{N}(i)} \frac{E\left(i, M + \sum_{l=0}^{j-1} s_l + n_j^{-1}(i)\right)}{h(P_i,S_j)^2} \right\}$$
$$+ \sum_{j=1}^{N_0} \sum_{i=1}^{s_j} \frac{E\left(M + \sum_{l=0}^{j-1} s_l + i, M + \sum_{l=0}^{j-1} s_l + i\right) - E\left(M + \sum_{l=0}^{j-1} s_l + i, n_j(i)\right)}{h(S_j, P_{n_j(i)})^2},$$

where I_M is the identity matrix of dimension M and $\theta := 2\sum_{i=1}^{n}(1/h_i^2)$. By definition L is symmetric. To check the symmetricity we first take $i \in \{1,\dots,M\}$ and $j \in \mathcal{M}(i)$. Then the (i,j) entry of L is $-1/h(P_i,P_j)^2$. Since $j \in \mathcal{M}(i)$, we see $P_j \in \overline{\{P_i\}}$. Thus $P_i \in \overline{\{P_j\}}$ and this implies $i \in \mathcal{M}(j)$. As a result, it follows that the (j,i) entry of L is $-1/h(P_j,P_i)^2$. We next let $i \in \{1,\dots,M\}$ and $j \in \mathcal{N}(i)$, so that the $(i, M + \sum_{l=0}^{j-1} s_l + n_j^{-1}(i))$ entry of L is $-1/h(P_i,S_j)^2$. In this case we have $S_j \in \overline{\{P_i\}}$, and so $P_i \in \overline{\{S_j\}}$. Then from the definition of n_j it follows that $n_j(t) = i$ for some $t \in \{1,\dots,s_j\}$, i.e., $t = n_j^{-1}(i)$. Since $(M + \sum_{l=0}^{j-1} s_l + n_j^{-1}(i), i) = (M + \sum_{l=0}^{j-1} s_l + t, n_j(t))$, we conclude that the $(M + \sum_{l=0}^{j-1} s_l + n_j^{-1}(i), i)$ entry of L is $-1/h(S_j, P_{n_j(t)})^2 = -1/h(S_j,P_i)^2$. Hence the symmetricity of L is proved.

3. We claim that if $\vec{a} \in \mathbf{R}^{M+N}$ is a solution of (3.2), then $u = u[\vec{a}]$ is a discrete solution of (NP). Let $x \in \Omega$, i.e., $x = P_i$ for some i. Without loss of generality we may assume $x = P_1$. Since $\vec{a}$ satisfies (3.2), comparing the first coordinates of the both sides in (3.2), we observe

$$\begin{aligned} \frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} &= \theta\alpha(1) - \sum_{j\in\mathcal{M}(1)} \frac{\alpha(j)}{h(P_1,P_j)^2} - \sum_{j\in\mathcal{N}(1)} \frac{\beta(j, n_j^{-1}(1))}{h(P_1,S_j)^2} \\ &\geqq \theta u(P_1) - \sum_{j\in\mathcal{M}(1)} \frac{u(P_j)}{h(P_1,P_j)^2} - \sum_{j\in\mathcal{N}(1)} \frac{u(S_j)}{h(P_1,S_j)^2} \\ &= -\Delta' u(P_1). \end{aligned}$$

We next let $x \in \partial\Omega$. Again we may assume $x = S_1$. We also let $\beta(1, j_0) = \max\{\beta(1, j) \mid 1 \leqq j \leqq s_1\}$. Then the $(M + j_0)$-th coordinates in (3.2) implies

$$\frac{\beta(1, j_0) - \alpha(n_1(j_0))}{h(S_1, P_{n_1(j_0)})^2} = \frac{-1}{h(S_1, P_{n_1(j_0)})},$$

that is,

$$\frac{u(S_1) - u(P_{n_1(j_0)})}{h(S_1, P_{n_1(j_0)})} = -1.$$

Consequently, we see that u is a discrete solution of (NP) in our sense.

4. We shall show that (3.2) is solvable. For this purpose, we first assert that $\mathrm{Ker}L = \mathbf{R}\vec{\xi}$, where $\mathrm{Ker}L$ is the kernel of L and

$$\vec{\xi} = {}^t(1, \ldots, 1) \in \mathbf{R}^{M+N}.$$

By the definition of L we see that the sum of each row of L is zero. This implies $\mathrm{Ker}L \supset \mathbf{R}\vec{\xi}$. We next let $\vec{a} \in \mathrm{Ker}L$, i.e., $L\vec{a} = 0$. We represent each component of $\vec{a}$ as in (3.1). Now, by the same argument as in Step 3 we see that $u = u[\vec{a}]$ is a discrete solution of

$$\text{(NP0)} \begin{cases} -\Delta u \leqq 0 & \text{in } \Omega, \quad (3.3) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \quad (3.4) \end{cases}$$

where the notion of a discrete solution of (NP0) is the same as that of (NP). We take a maximum point $z \in \overline{\Omega}$ of u over $\overline{\Omega}$. If $z \in \partial\Omega$, there exists some $y \in \overline{\{z\}} \cap \Omega$ such that $u(y) = u(z)$ since u satisfies the Neumann boundary condition (3.4) at z. Thus u attains its maximum at some point in Ω. Since Ω is now bounded and connected, the strong maximum principle (Theorem A.3) for the Laplace equation ensures that u must be some constant $c \in \mathbf{R}$ on $\overline{\Omega}$. From this it follows that $\alpha(1) = \cdots = \alpha(M) = c$. Also, since $L\vec{a} = 0$, we have $\beta(i, j) = \alpha(n_i(j))$ for all $i \in \{1, \ldots, N_0\}$ and $j \in \{1, \ldots, s_i\}$. As a result, we see $\vec{a} = c\vec{\xi} \in \mathbf{R}\vec{\xi}$. We thus conclude that $\mathrm{Ker}L = \mathbf{R}\vec{\xi}$.

5. Since L is symmetric and $\mathrm{Ker}L = \mathbf{R}\vec{\xi}$, we see that $(\mathrm{Im}L)^\perp = \mathbf{R}\vec{\xi}$, where $(\mathrm{Im}L)^\perp$ stands for the orthogonal complement of $\mathrm{Im}L$, the image of L. Thus, for $\vec{b}' \in \mathbf{R}^{M+N}$ it follows that $\vec{b}' \in \mathrm{Im}L$ if and only if $\langle \vec{\xi}, \vec{b}' \rangle = 0$. Noting that $-1/h_i$ appears ω_i times in a sequence $\{b_k\}_{k=M+1}^{M+N}$ for each $i \in \{1, \ldots, n\}$, we compute

$$\langle \vec{\xi}, \vec{b} \rangle = \frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} \times M + \sum_{i=1}^{n} \left(\frac{-1}{h_i} \times \omega_i \right) = \frac{\mathrm{Per}(\Omega)}{h^n} - \sum_{i=1}^{n} \frac{\omega_i}{h_i} = 0.$$

Consequently $\vec{b} \in \mathrm{Im}L$, and therefore the problem (3.2) has at least one solution $\vec{a} \in \mathbf{R}^{M+N}$. Hence by Step 3 the corresponding $u = u[\vec{a}]$ solves (NP). □

Remark 3.3. We have actually proved that u, which we constructed as a subsolution, is a solution of (2.2) in $\Omega \setminus \overline{\partial\Omega}$. Namely, we have $-\Delta' u(x) = \mathrm{Per}(\Omega)/\mathrm{Vol}(\Omega)$ for all $x \in \Omega \setminus \overline{\partial\Omega}$. This is clear from the construction of u.

Appendix A. Maximum principles

A.1. **An ABP maximum principle.** In Appendix we consider the second order fully nonlinear elliptic equations of the form

$$F(\partial_{x_1}^2 u, \dots, \partial_{x_n}^2 u) = f(x) \quad \text{in } \Omega, \tag{A.1}$$

where $F : \mathbf{R}^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given function such that $F(0, \dots, 0) = 0$. Let $\vec{\delta}^2 u(x) := (\delta_1^2 u(x), \dots, \delta_n^2 u(x))$. We say $u : \overline{\Omega} \to \mathbf{R}$ is a *discrete subsolution* of (A.1) if $F(\vec{\delta}^2 u(x)) \leqq f(x)$ for all $x \in \Omega$. As an ellipticity condition on F for our ABP estimate, we use the following:

(F1) $-\lambda \sum \vec{X} \leqq F(\vec{X})$ for all $\vec{X} \in \mathbf{R}^n$ with $\vec{X} \leqq 0$.

Here $\lambda > 0$. Also, $\sum \vec{X} := \sum_{i=1}^n X_i$ for $\vec{X} = (X_1, \dots, X_n) \in \mathbf{R}^n$ and the inequality $\vec{X} \leqq 0$ means that $X_i \leqq 0$ for every $i \in \{1, \dots, n\}$. For $K \subset h\mathbf{Z}^n$ and $g : K \to \mathbf{R}$ the n-norm of g over K is given as $\|g\|_{\ell^n(K)} := \left(\sum_{x \in K} h^n |g(x)|^n\right)^{1/n}$. We also set $\mathrm{diam}(\Omega) := \max_{x \in \Omega, y \in \partial\Omega} |x - y|$ and $|\mathbf{B}_r| := \mathcal{L}^n(\mathbf{B}_r)$.

Theorem A.1 (ABP maximum principle)**.** *Assume* (F1)*. Let $u : \overline{\Omega} \to \mathbf{R}$ be a discrete subsolution of* (A.1)*. Then the estimate*

$$\max_{\overline{\Omega}} u \leqq \max_{\partial\Omega} u + C_A \mathrm{diam}(\Omega) \|f\|_{\ell^n(\Gamma[u])} \tag{A.2}$$

holds, where $C_A = C_A(\lambda, n)$ is given as $C_A = (\lambda n |\mathbf{B}_1|^{1/n})^{-1}$.

A crucial estimate to prove Theorem A.1 is

Proposition A.2. *For all $u : \overline{\Omega} \to \mathbf{R}$ we have*

$$\max_{\overline{\Omega}} u \leqq \max_{\partial\Omega} u + \frac{\mathrm{diam}(\Omega)}{n|\mathbf{B}_1|^{1/n}} \| - \Delta' u\|_{\ell^n(\Gamma[u])}. \tag{A.3}$$

Proof. 1. We first prove $\mathbf{B}_d \subset \bigcup_{z \in \Gamma[u]} \partial^+ u(z)$, where d is a constant given as $d = (\max_{\overline{\Omega}} u - \max_{\partial\Omega} u)/\mathrm{diam}(\Omega)$. If $d = 0$, the assertion is obvious. We assume $d > 0$, i.e., $u(\hat{x}) = \max_{\overline{\Omega}} u > \max_{\partial\Omega} u$ for some $\hat{x} \in \Omega$. Let $p \in \mathbf{B}_d$ and set $\phi(x) := \langle p, x - \hat{x} \rangle$. We take a maximum point z of $u - \phi$ over $\overline{\Omega}$. Then we have $z \in \Omega$. Indeed, for all $x \in \partial\Omega$ we observe

$$u(x) - \phi(x) \leqq \max_{\partial\Omega} u + |p| \cdot |x - \hat{x}| < \max_{\partial\Omega} u + d \cdot \mathrm{diam}(\Omega) = \max_{\overline{\Omega}} u = u(\hat{x}) - \phi(\hat{x}).$$

Thus $z \in \Omega$, and so we conclude that $z \in \Gamma[u]$ and $p \in \partial^+ u(z)$.

2. By Step 1 the estimate (2.5) with $\mathbf{B}_d$ instead of $\mathbf{Q}_1$ holds. Thus the same argument as in the proof of Theorem 1.1 yields

$$|\mathbf{B}_d| \leqq \sum_{z \in \Gamma[u]} h^n \left(\frac{-\Delta' u(z)}{n} \right)^n = \frac{1}{n^n} \| - \Delta' u\|_{\ell^n(\Gamma[u])}^n.$$

Applying $|\mathbf{B}_d| = d^n |\mathbf{B}_1|$ to the above inequality, we obtain (A.3) by the choice of d. □

Proof of Theorem A.1. By Remark 2.2 we have $\vec{\delta}^2 u(z) \leqq 0$ for $z \in \Gamma[u]$, and therefore the condition (F1) yields $-\lambda \Delta' u(z) = -\lambda \sum \vec{\delta}^2 u(z) \leqq F(\vec{\delta}^2 u(z))$. Since u is a discrete subsolution of (A.1), we also have $F(\vec{\delta}^2 u(z)) \leqq f(z)$. Applying these two inequalities to (A.3), we obtain (A.2). □

A.2. **A strong maximum principle.** Although the strong maximum principle for the Laplace equation is enough for the proof of Proposition 3.2, we consider a wider class of equations in this subsection. We study homogeneous equations of the form

$$F(\partial_{x_1}^2 u, \ldots, \partial_{x_n}^2 u) = 0 \quad \text{in } \Omega. \tag{A.4}$$

From the ABP maximum principle (A.2) we learn that all discrete subsolutions u of (A.4) satisfy

$$\max_{\overline{\Omega}} u \leqq \max_{\Omega} u$$

if (F1) holds. This is the so-called weak maximum principle. Our aim in this subsection is to prove that a certain weaker condition on F actually leads to the strong maximum principle and conversely the weaker condition is necessary for it. Here the rigorous meaning of the strong maximum principle is

(SMP) If $u : \overline{\Omega} \to \mathbf{R}$ is a discrete subsolution of (A.4) such that $\max_{\overline{\Omega}} u = \max_{\Omega} u$, then u must be constant on $\overline{\Omega}$.

Following the classical theory of partial differential equations, we consider only bounded and connected subsets $\Omega \subset h\mathbf{Z}^n$ for (SMP). It turns out that the strong maximum principle holds if and only if F satisfies the following weak ellipticity condition (F2). It is easily seen that (F1) implies (F2).

(F2) If $\vec{X} \leqq 0$ and $F(\vec{X}) \leqq 0$ for $\vec{X} \in \mathbf{R}^n$, then $\vec{X}$ must be zero, i.e., $\vec{X} \equiv 0$.

Theorem A.3 (Strong maximum principle)**.** *The two conditions* (SMP) *and* (F2) *are equivalent.*

To show this theorem we first study discrete quadratic functions. They will be used when we prove that (SMP) implies (F2).

Example A.4. Let $(A_1, \ldots, A_n) \in \mathbf{R}^n$. We define a quadratic function $q : h\mathbf{Z}^n \to \mathbf{R}$ as

$$q(x) := \sum_{j=1}^{n} (h_j x_j)^2 A_j \quad \text{for } x = (h_1 x_1, \ldots, h_n x_n) \in h\mathbf{Z}^n.$$

Then $\delta_i^2 q$ is a constant for each $i \in \{1, \ldots, n\}$. Indeed, we observe

$$\begin{aligned} \delta_i^2 q(x) &= \frac{q(x + h_i e_i) + q(x - h_i e_i) - 2q(x)}{h_i^2} \\ &= \frac{h_i^2 (x_i + 1)^2 A_i + h_i^2 (x_i - 1)^2 A_i - 2h_i^2 x_i^2 A_i}{h_i^2} = 2A_i \end{aligned}$$

for all $x = (h_1 x_1, \ldots, h_n x_n) \in h\mathbf{Z}^n$.

Proof of Theorem A.3. 1. We first assume (F2). Let $u : \overline{\Omega} \to \mathbf{R}$ is a discrete subsolution of (A.4) such that $u(\hat{x}) = \max_{\overline{\Omega}} u$ for some $\hat{x} \in \Omega$. This maximality implies that for each $i \in \{1, \ldots, n\}$

$$\delta_i^2 u(\hat{x}) = \frac{u(\hat{x} + h_i e_i) + u(\hat{x} - h_i e_i) - 2u(\hat{x})}{h_i^2} \leqq \frac{u(\hat{x}) + u(\hat{x}) - 2u(\hat{x})}{h_i^2} = 0.$$

Thus $\vec{\delta}^2 u(\hat{x}) \leqq 0$. Since u is a discrete subsolution, we also have $F(\vec{\delta}^2 u(\hat{x})) \leqq 0$. It now follows from (F2) that $\vec{\delta}^2 u(\hat{x}) \equiv 0$, and hence we see that $u(\hat{x}) = u(\hat{x} \pm h_i e_i)$ for all i. We next apply the above argument with the new central point $\hat{x} \pm h_i e_i$ if the point is in Ω. Iterating this procedure, we finally conclude that $u \equiv u(\hat{x})$ on $\overline{\Omega}$ since Ω is now connected.

2. We next assume (SMP). Take any $\vec{X} = (X_1, \ldots, X_n) \in \mathbf{R}^n$ such that $\vec{X} \leqq 0$ and $F(\vec{X}) \leqq 0$. We may assume $0 \in \Omega$. Now, we take the quadratic function q in Example A.4 with $A_i = X_i/2 \leqq 0$. By the calculation in Example A.4 we then have $\delta_i^2 q(x) = X_i$ for all i, i.e., $\vec{\delta}^2 q(x) = \vec{X}$. Thus $F(\vec{\delta}^2 q(x)) = F(\vec{X}) \leqq 0$, which means that q is a discrete subsolution of (A.4). Next, we deduce from the nonpositivity of each A_i that q attains its maximum over $\overline{\Omega}$ at $0 \in \Omega$. Therefore (SMP) ensures that $q \equiv q(0) = 0$ on $\overline{\Omega}$, which implies that $A_i = 0$ for all $i \in \{1, \ldots, n\}$. Consequently, we find $\vec{X} \equiv 0$. $\square$

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References

[1] S. L. Bezrukov, Isoperimetric problems in discrete spaces, Extremal problems for finite sets, Bolyai Soc. Math. Stud., vol. 3 (1994), 59–91.
[2] H. D. Block, Discrete isoperimetric-type inequalities, Proc. Amer. Math. Soc. 8 (1957), 860–862.
[3] J. Bokowski, H. Hadwiger, J. M. Wills, Eine Ungleichung zwischen Volumen, Oberfläche und Gitterpunktanzahl konvexer Körper im n-dimensionalen euklidischen Raum (German), Math. Z. 127 (1972), 363–364.
[4] P. Brass, Isoperimetric inequalities for densities of lattice-periodic sets, Monatsh. Math. 127 (1999), no. 3, 177–181.
[5] Y. D. Burago, V. A. Zalgaller, Geometric inequalities, Grundlehren der Mathematischen Wissenschaften, 285, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1988.
[6] L. A. Caffarelli, X. Cabré, Fully nonlinear elliptic equations, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995.
[7] X. Cabré, Partial differential equations, geometry and stochastic control (Catalan), Butl. Soc. Catalana Mat. 15 (2000), no. 1, 7–27.
[8] X. Cabré, Elliptic PDE's in probability and geometry: symmetry and regularity of solutions, Discrete Contin. Dyn. Syst. 20 (2008), no. 3, 425–457.
[9] F. Chung, Discrete isoperimetric inequalities, Surv. Differ. Geom. IX (2004), 53–82.
[10] D. E. Daykin, An Isoperimetric Problem on a Lattice, Math. Mag. 46 (1973), no. 4, 217–219.
[11] J. A. M. S. Duarte, M. C. A. M. Marques, An approach to the isoperimetric problem on some lattices, Nuovo Cimento B (11) 54 (1979), no. 2, 508–516.
[12] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[13] L. J. Grady, J. R. Polimeni, Discrete calculus, Applied analysis on graphs for computational science, Springer-Verlag London, Ltd., London, 2010.
[14] H. Hadwiger, Gitterperiodische Punktmengen und Isoperimetrie (German), Monatsh. Math. 76 (1972), 410–418.
[15] P. W. Hillock, P. R. Scott, Inequalities for lattice constrained planar convex sets, J. Inequal. Pure Appl. Math. 3 (2002), no. 2, Article 23, 10 pp.
[16] H.-J. Kuo, N. S. Trudinger, Linear elliptic difference inequalities with random coefficients, Math. Comp. 55 (1990), no. 191, 37–53.
[17] H.-J. Kuo, N. S. Trudinger, Discrete methods for fully nonlinear elliptic equations, SIAM J. Numer. Anal. 29 (1992), no. 1, 123–135.

[18] H.-J. Kuo, N. S. Trudinger, On the discrete maximum principle for parabolic difference operators, RAIRO Modél. Math. Anal. Numér. 27 (1993), no. 6, 719–737.

[19] H.-J. Kuo, N. S. Trudinger, Positive difference operators on general meshes, Duke Math. J. 83 (1996), no. 2, 415–433.

[20] H.-J. Kuo, N. S. Trudinger, Evolving monotone difference operators on general space-time meshes, Duke Math. J. 91 (1998), no. 3, 587–607.

[21] H.-J. Kuo, N. S. Trudinger, A note on the discrete Aleksandrov-Bakelman maximum principle, Proceedings of 1999 International Conference on Nonlinear Analysis (Taipei), Taiwanese J. Math. 4 (2000), no. 1, 55–64.

[22] R. Osserman, The isoperimetric inequality, Bull. Amer. Math. Soc. 84 (1978), no. 6, 1182–1238.

[23] J. B. Rosser, Finite-difference solution of Poisson's equation in rectangles of arbitrary proportions, Z. Angew. Math. Phys. 28 (1977), no. 2, 186–196.

[24] U. Schnell, J. M. Wills, Two isoperimetric inequalities with lattice constraints, Monatsh. Math. 112 (1991), no. 3, 227–233.

[25] U. Schumann, J. Benner, Direct solution of the discretized Poisson-Neumann problem on a domain composed of rectangles, J. Comput. Phys. 46 (1982), no. 1, 1–14.

[26] U. Schumann, R. A. Sweet, A direct method for the solution of Poisson's equation with Neumann boundary conditions on a staggered grid of arbitrary size, J. Comput. Phys. 20 (1976), no. 2, 171–182.

[27] C. Villani, Optimal transport, Old and new, Grundlehren der Mathematischen Wissenschaften, 338. Springer-Verlag, Berlin, 2009.

[28] Y. Zhuang and X.-H. Sun, A high-order fast direct solver for singular Poisson equations, J. Comput. Phys. 171 (2001), no. 1, 79–94.

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