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THE SECOND MAIN THEOREM FOR ENTIRE CURVES INTO HILBERT MODULAR SURFACES

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Abstract. Our main goal of this article is to prove the second main theorem for entire curves into Hilbert modular surfaces. We show a condition such that entire curves in a Hilbert modular surface of general type are contained in the exceptional divisor of a Hilbert modular surface. We also show the second main theorem for compact leaves of holomorphic foliation on a smooth projective algebraic surface.

1. Introduction and main result

In this paper, we prove the second main theorem for entire curves into Hilbert modular surfaces which are generically non-tangent to a naturally defined holomorphic foliations.

First we recall the definition and basic properties of Hilbert modular surfaces. Let K be a real quadratic field of discriminant D and let $\mathfrak{o}$ be its ring of integers. A Hilbert modular group Γ is defined by $\mathbf{SL}_2(\mathfrak{o})/\{1,-1\}$ where

$$\mathbf{SL}_2(\mathfrak{o}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathfrak{o},\ ad - bc = 1 \right\}.$$

Then Γ acts on $\mathfrak{h} \times \mathfrak{h}$ where $\mathfrak{h}$ is the upper half plane. By adding finite number of cusps, the complex space $\mathfrak{h}^2/\Gamma$ can be compactified to a complex projective algebraic variety $\overline{\mathfrak{h}^2/\Gamma}$. In $\overline{\mathfrak{h}^2/\Gamma}$, there exist singular points which are quotient singular points and cusps. Quotient singular points arise from the points on $\mathfrak{h} \times \mathfrak{h}$ with non-trivial isotropy group in Γ. The minimal resolution of quotient singularity consists of the Hirzebruch-Jung string (a chain of rational curves intersecting transversally). The minimal resolution of cusp consists of a cycle of rational curves. By resolving the singularities of $\overline{\mathfrak{h}^2/\Gamma}$ in the canonical minimal way, we get a Hilbert modular surface X and X is a non-singular complex projective algebraic surface. In [5] and [6], the Enriques-Kodaira rough classification of Hilbert modular surfaces was done and it was proved that a Hilbert modular surface is of general type if the discriminant D is big.

On $\mathfrak{h}^2/\Gamma$, there are two natural foliations arising from the vertical and the horizontal ones on $\mathfrak{h} \times \mathfrak{h}$. These foliations induce two foliations on X which are called the Hilbert modular foliations. The Hilbert modular foliations are transverse each other outside the exceptional divisors of the minimal resolution $X \to \overline{\mathfrak{h}^2/\Gamma}$. These exceptional divisors are algebraic leaves of the two foliations and there does not exist othere compact leaf of the Hilbert modular foliations. For more properties of the Hilbert modular foliations, see [2] and Section 4 of [3].

Our main theorem is the following second main theorem:

Theorem 1. *Let E be the union of exceptional divisors of the minimal resolution $X \to \overline{\mathfrak{h}^2/\Gamma}$. Let K_X be the canonical line bundle of X. Let f be a non-constant holomorphic* *map from $\mathbb{C}$ to a Hilbert modular surface X which is generically non-tangent to the Hilbert modular foliations. Then it follows that*

$$T_f(r, K_X) + T_f(r, E) \leq 2N_1(r, f^*E) + S_f(r),$$

where $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$, and "$\|$" means that the inequality holds for all $r \in (0, +\infty)$ possibly except for a subset with finite Lebesgue measure.

In [1], the second main theorem for non-constant entire curves into a compactified locally symmetric space which are not contained in the exceptional divisor was proved. If one applies their second main theorem to Hilbert modular surface, the same inequality for non-constant entire curves as that in Theorem 1 follows. However, their second main theorem does not deal with the case where Γ has the torsion, and our proof of Theorem 1 is completely different from the proof of the second main theorem in [1].

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2. Preliminaries on Nevanlinna theory

We introduce some functions which play an important role in the Nevanlinna theory. In this section, we mention [8] as general reference. Let A be an effective divisor on $\mathbb{C}$. A is written as $\sum m_j P_j$ where $\{P_j\}$ is a set of discrete points in $\mathbb{C}$ and m_j are positive integers. Let k be a natural number or $+\infty$. Put $n_k(r, A) = \sum_{|P_j|<r} \min\{k, m_j\}$. We define the counting function of A by

$$N_k(r, A) = \int_1^r \frac{n_k(t, A)}{t} dt.$$

Let M be a complex projective algebraic manifold, and let L be a holomorphic line bundle on M. Let h be a hermitian metric of L and let $\omega = -dd^c \log h$ be its Chern form. Let $f : \mathbb{C} \to M$ be a non-constant holomorphic map. Then we define the characteristic function of L by

$$T_f(r, L) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*\omega,$$

where $\Delta(t) = \{z \in \mathbb{C} \,|\, |z| < t\}$. When L is an ample line bundle on X, we define $T_f(r) = T_f(r, L)$. Let B be an effective reduced divisor on M such that the associated line bundle $[B]$ is L. If $f(\mathbb{C})$ is not contained in B, then $N_\infty(r, f^*B) \leq T_f(r, L) + O(1)$ (see Chapter V of [8]).

If an effective reduced divisor B on M is simple normal crossing, then the logarithmic cotangent bundle $\Omega^1_M(\log B)$ is defined. Define

$$\mathbb{P}(\Omega^1_M(\log B)) = \mathbf{Proj} \bigoplus_{d \geq 0} S^d(\Omega^1_M(\log B)).$$

Because of the definition, a non-constant holomorphic map $f : \mathbb{C} \to M$ which is not contained in B lifts to the holomorphic map $f' : \mathbb{C} \to \mathbb{P}(\Omega^1_M(\log B))$. There exists a tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\Omega^1_M(\log B))$. The following theorem is called McQuillan's "Tautological inequality" (see [7], [9] for proof), and this is one of a geometric interpretation of the lemma of the logarithmic derivative.

Theorem 2 (McQuillan's "Tautological Inequality", [7])**.** *Let $f : \mathbb{C} \to M$ be a non-constant holomorphic map such that $f(\mathbb{C}) \not\subset B$. Let $f' : \mathbb{C} \to \mathbb{P}(\Omega^1_M(\log B))$ be the canonical lift of f. Then*

$$T_{f'}(r, \mathcal{O}(1)) \leq N_1(r, f^*B) + S_f(r).$$

3. Singular holomorphic foliation and the second main theorem

Let M be a smooth complex surface. A singular holomorphic foliation $\mathcal{F}$ on M is given by an open covering $\{U_j\}$ of M and holomorphic vector fields $v_j \in H^0(U_j, T_M)$ with isolated zeros such that $v_i = g_{ij}v_j$ on $U_i \cap U_j$ for some non-vanishing holomorphic functions $g_{ij} \in H^0(U_i \cap U_j, \mathcal{O}^*)$. The singular set $Sing(\mathcal{F})$ is the discrete subsets defined by the zero points of v_j. Then $\mathcal{F}$ induces a holomorphic line bundle $T_{\mathcal{F}}$ on M and a sheaf injection $T_{\mathcal{F}} \to T_M$ (we may assume that $T_{\mathcal{F}}$ is a subline bundle of T_M on $M \setminus Sing(\mathcal{F})$). There exists an exact sequence of sheaves

$$0 \to T_{\mathcal{F}} \to T_M \to \mathcal{I}_Z N_{\mathcal{F}} \to 0$$

for a suitable line bundle $N_{\mathcal{F}}$ and a ideal sheaf I_Z of a suitable subscheme Z of X. The support of Z is contained in $Sing(\mathcal{F})$. Taking the dual of this exact sequence, we obtain an exact sequence of sheaves

$$0 \to N^*_{\mathcal{F}} \to \Omega^1_M \to I_Z T^*_{\mathcal{F}} \to 0.$$

Let C be a hypersurface of M. If C is tangent to $T_{\mathcal{F}}$ on $C \setminus (Sing(\mathcal{F}) \cup Sing(C))$, we call C is a leaf of $\mathcal{F}$.

We show the second main theorem of compact leaves.

Theorem 3. *Let M be a smooth complex projective algebraic surface and $\mathcal{F}$ be a singular holomorphic foliation on M. Let C be an effective reduced divisor on M with simple normal crossing. Assume that C is a leaf of $\mathcal{F}$. Let $f : \mathbb{C} \to M$ be a non-constant holomorphic map which is generically non-tangent to $\mathcal{F}$. Then it follows that*

$$T_f(r, N^*_{\mathcal{F}}) + T_f(r, C) \leq N_1(r, f^*C) + S_f(r).$$

Proof. Let $\{U_j\}$ be an open covering of M. We take $\omega_j \in H^0(U_j, \Omega^1_M)$ which is a generator of the sheaf $N^*_{\mathcal{F}}$ on U_j. Here we consider that $N^*_{\mathcal{F}}$ is a subsheaf of Ω^1_M. Let $h_j \in H^0(U_j, \mathcal{O})$ be a holomorphic function which defines C on U_j, i.e., $(h_j) = C|_{U_j}$. Then there exist holomorphic function $g_j \in H^0(U_j, \mathcal{O})$ and a holomorphic one-form $\eta_j \in H^0(U_j, \Omega^1_M)$ such that

$$\omega_j = h_j \eta_j + g_j dh_j$$

(see [2]). Then it follows that

$$\frac{\omega_j}{h_j} = \eta_j + g_j \frac{dh_j}{h_j},$$

and we may consider $N^*_{\mathcal{F}}(C)$ as a subsheaf of $\Omega^1_M(\log C)$. There exists an epimorphism

$$S^d(\Omega^1_M(\log C)) \to S^d(\Omega^1_M(\log C)/N^*_{\mathcal{F}}(C)) \to 0,$$

and

$$\mathbf{Proj}\bigoplus_{d\geq 0} S^d(\Omega^1_M(\log C)/N^*_{\mathcal{F}}(C))$$

is a hypersurface of

$$\mathbf{Proj}\bigoplus_{d\geq 0} S^d(\Omega^1_M(\log C)) = \mathbb{P}(\Omega^1_M(\log C)).$$

Let $S = \mathbf{Proj}\bigoplus_{d\geq 0} S^d(\Omega^1_M(\log C)/N^*_{\mathcal{F}}(C))$. The line bundle $[S]$ on $\mathbb{P}(\Omega^1_M(\log C))$ which is defined by S is linearly equivalent to $\mathcal{O}(1)\otimes\pi^*N_{\mathcal{F}}(-C)$, where $\pi : \mathbb{P}(\Omega^1_M(\log C)) \to M$ is the canonical projection and $\mathcal{O}(1)$ is the tautological line bundle of $\mathbb{P}(\Omega^1_M(\log C))$. Since f is not tangent to $\mathcal{F}$ generically, the image of f' is not contained in S. It holds that

$$0 \leq N_\infty(r, f'^*S) \leq T_{f'}(r, \mathcal{O}(1)\otimes\pi^*N_{\mathcal{F}}(-C)) = T_{f'}(r,\mathcal{O}(1)) - T_f(r, N^*_{\mathcal{F}}(C)).$$

By Theorem 2, it follows that

$$T_f(r, N^*_{\mathcal{F}}) + T_f(r, C) \leq T_{f'}(r, \mathcal{O}(1)) \leq N_1(r, f^*C) + S_f(r).$$

□

4. PROOF OF THE MAIN THEOREM

Now we prove Theorem 1

(proof of Theorem 1). Let $\mathcal{F}$ and $\mathcal{G}$ be the Hilbert modular foliations on a Hilbert modular surface X. Then $\mathcal{F}$ and $\mathcal{G}$ are transverse each other outside the exceptional divisor E and they have a first order tangency along E. Let U be an open set of X and let $\omega_1 \in H^0(U, N^*_{\mathcal{F}})$ and $\omega_2 \in H^0(U, N^*_{\mathcal{G}})$. By taking $\omega_1\wedge\omega_2 \in K_X$, there exists isomorphism $N^*_{\mathcal{F}}\otimes N^*_{\mathcal{G}} \simeq K_X(-E)$. Then

$$K_X = N^*_{\mathcal{F}} + N^*_{\mathcal{G}} + E.$$

By Theorem 3, it follows that

$$\begin{aligned} T_f(r, K_X) + T_f(r, E) &= T_f(r, N^*_{\mathcal{F}}) + T_f(r, N^*_{\mathcal{G}}) + 2T_f(r, E) \\ &= T_f(r, N^*_{\mathcal{F}}(E)) + T_f(r, N^*_{\mathcal{G}}(E)) \\ &\leq 2N_1(r, f^*E) + S_f(r). \end{aligned}$$

□

We show one corollary about the degeneracy of entire curves.

Corollary 1. *Let X be a Hilbert modular surface of general type. Let f be a non-constant holomorphic map from $\mathbb{C}$ to X. Assume that f is ramified over E, i.e.,*

$$f^*E \geq 2\,\mathrm{supp} f^*E.$$

Then $f(\mathbb{C})$ is contained in E.

Proof. Assume that f is generically non-tangent to the Hilbert modular foliations. Since $f^*E \geq 2\,\mathrm{supp} f^*E$, it follows that

$$2N_1(r, f^*E) \leq N_\infty(r, f^*E) \leq T_f(r, E).$$

Using Theorem 1, it follows that

$$T_f(r, K_X) \leq S_f(r).$$

This is a contradiction since K_X is ample. Therefore f is contained in a leaf of the Hilbert modular foliation. By the Theorem of McQuillan [7], all parabolic leaves on a smooth projective algebraic surface of general type are compact. Then the parabolic leaf which contains $f(\mathbb{C})$ is compact. The compact leaf of the Hilbert modular foliations is the exceptional divisor E. Hence $f(\mathbb{C})$ is contained in E. $\square$

The condition that
$$f^*E \geq 2\,\mathrm{supp} f^*E$$
can not be removed. The diagonal set
$$\{(z_1, z_2) \in \mathfrak{h} \times \mathfrak{h} \mid z_1 = z_2\}$$
projects down to a curve in $\mathfrak{h}^2/\Gamma$. The closure of this curve in a Hilbert modular surface X is a rational curve which is not contained in E (see Section 4 of [4]).

References

[1] Y. Aihara and J. Noguchi, Value distribution of meromorphic mappings into compactified locally symmetric spaces, Kodai Math. J. **14** (1991), 320–334.

[2] M. Brunella, Birational geometry of foliations, First LatinAmerican Congress of Mathematicians, Notas de Curso, IMPA (2000).

[3] M. Brunella, Subharmonic variation of the leafwise Poincare metric, Invent. Math. **152** (2003), 119–148.

[4] F. Hirzebruch, Hilbert modular surfaces, Enseignement Math. **19** (1973), 183–281.

[5] F. Hirzebruch and A. Van de Ven, Hilbert modular surfaces and the classification of algebraic surfaces, Invent. Math. **23** (1974), 1–29.

[6] F. Hirzebruch and D. Zagier, Classification of Hilbert modular surfaces, Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, (1977), 43–77.

[7] M. McQuillan, Diophantine approximations and foliations, Inst. Hautes Etudes Sci. Publ. Math. **87** (1998), 121–174.

[8] J. Noguchi and T. Ochiai, Geometric function theory in several complex variables, Translated from the Japanese by Noguchi, Translations of Mathematical Monographs, **80**. American Mathematical Society, Providence, RI, 1990.

[9] P. Vojta, Diophantine approximation and Nevanlinna theory, Arithmetic geometry, 111–224, Lecture Notes in Math., 2009, Springer, Berlin, 2011.

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