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A system of fifth-order partial differential equations describing a surface which contains many circles

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A SYSTEM OF FIFTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS DESCRIBING A SURFACE WHICH CONTAINS MANY CIRCLES

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Abstract. Let $z = f(x, y)$ be a germ of a C^5-surface at the origin in $\mathbb{R}^3$ containing several continuous families of circular arcs. For examples, we have a usual torus with 4 such families and R. Blum's cyclide with 6 such families. We introduce a system of fifth-order nonlinear partial differential equations for f, and prove that this system of equations describes such a surface germ completely. As applications, we obtain the analyticity of f, the finite dimensionality of the solution space of such a system of differential equations with an upper estimate 21 for the dimension. Further we prove the non-integrability of the systems corresponding the surfaces including six continuous families of circular arcs; this result implies a local characterization of Darboux cyclides.

1. Introduction

In 1848, Yvon Villarceau [12] found that a usual torus includes 4 continuous families of circles passing through every point of the surface; of course, only two of them are new. These new circles, so called Villarceau circles, are slanted against the rotation axis and are not perpendicular to this axis (see Figure 1). Further in 1980, Richard Blum [1] found that some special cyclides include 4∼6 continuous families of circles passing through every point of them (see Figure 2). Here, a general cyclide is defined by a quartic equation

$$\alpha(x_1^2 + x_2^2 + x_3^2)^2 + 2(x_1^2 + x_2^2 + x_3^2)\sum_{i=1}^{3}\beta_i x_i + \sum_{i,j=1}^{3}\gamma_{ij}x_i x_j + 2\sum_{i=1}^{3}\delta_i x_i + \epsilon = 0 \quad (1.1)$$

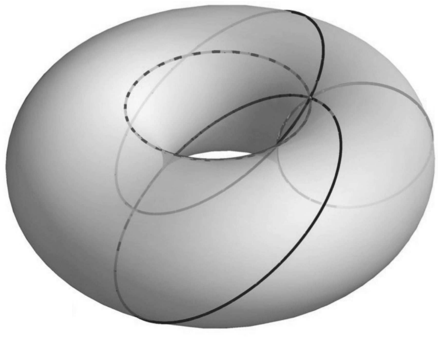

Figure 1. A torus and Villarsceau circles

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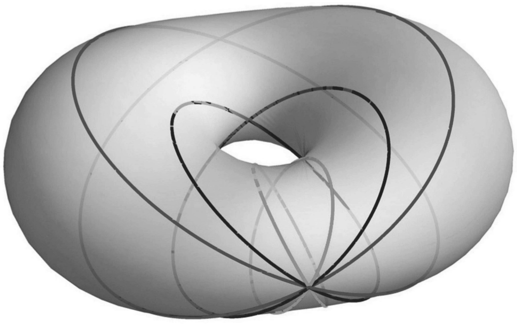

FIGURE 2. An example of Blum cyclides:
$(x^2 + y^2 + z^2)^2 - 6x^2 - 4y^2 + 4z^2 + 1 = 0.$

with real numbers $\alpha \neq 0, \beta_i, \gamma_{ij}, \delta_i, \epsilon$ (Darboux [2]). Then a usual torus and a 6-circle Blum cyclide correspond to the case $\alpha = 1, \beta_* = 0, \delta_* = 0, \gamma_{ij} = -2a_i\delta_{ij}, \epsilon = a_4^2$ with $0 < a_4 < a_1 = a_2$, $a_3 = -a_4$, and to that with $0 < a_4 < a_2 < a_1$, $-a_4 \neq a_3 < a_4$, respectively. At the same time, R. Blum gave the following conjecture in [1]:

Conjecture 1. A closed C^∞-surface in $\mathbb{R}^3$ which contains seven circles through each point is a sphere.

N. Takeuchi [9] (and [11]:the survey of Takeuchi's results) solved this conjecture affirmatively for closed surfaces with genus $g \leq 1$ by using the intersection number theory for 1-dimensional homotopy groups. Further, replacing 1-dimensional homotopy groups by 1-dimensional homology groups with $\mathbb{Z}_2$-coefficients, we obtained the following extension in [3]:

Theorem 1.1. *We have some positive integer N_g ($\leq 2^{2g+1} - 1$) for any $g = 1, 2, 3, \dots$ such that, for $\forall g \geq 1$, there is no closed surface with genus g in E^3 which contains N_g circles through each point. In particular, we can take $N_2 = 11$.*

Moreover, this is strengthened to the following theorem, which is a direct corollary of Theorem 2.4 in the present paper:

Theorem 1.2. *There is no C^4-surface in E^3 other than totally umbilical surfaces, which contains* 11 *circular arcs or line-segments through each point. Here, a totally umbilical surface means an open subset of a sphere or a plane in E^3.*

Proof. If a surface $M \subset E^3$ satisfying all the conditions in the statement is not totally umbilical, then there is a non-umbilical point $p \in M$. Apply Theorem 2.4 to M at p. Then we have a contradiction concerning the upper estimate 10 for the number of circular arcs or line segments on M passing through p. This completes the proof. $\square$

On the other hand, the geometric key tool (Theorem 1 in [8]) in the proof in [9] can be generalized to the following; that is, the condition "any two of three circles satisfy some conditions" can be replaced by "either one of three circles satisfies some conditions with any of other circles".

Proposition 1.3. *Let M be a C^∞ surface in E^3. Suppose that, through each point of M, there exist three circles of E^3 contained in M, and that either one circle of them is tangent to others or have two points in common. Then M is a totally umbilical surface.*

Proof. Let p be any point of M. Suppose that there exist three circles C, C_1, C_2 passing through p included in M such that, for each $i = 1, 2$, C is tangent to C_i at p, or have two points in common with C_i. Let S_i be the sphere (or the plane) including $C \cup C_i$ for each $i = 1, 2$. If C is tangent to C_i at p, then by Meusnier's theorem we have that

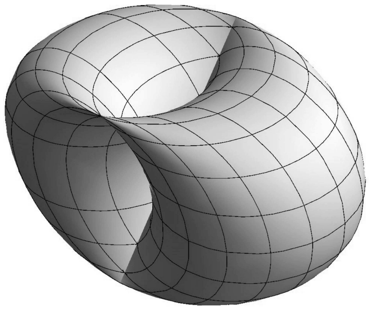

FIGURE 3. The example of a non-cyclide:
$(x^2 + y^2 + z^2)^2 - 4y^2z^2 - 4x^2 = 0$.
This surface has the singularities along $x = 0, y = \pm z$.

S_i is the curvature sphere for M at p concerning the common tangent line to C and C_i. Note that the tangent plane at p to M coincides with the one to S_i, and that S_1 and S_2 include C, respectively. Thus we have $S_1 = S_2$. Hence, any two of C, C_1, C_2 are tangent to each other at p, or have two points in common. Therefore, M satisfies the conditions in Theorem 1 of [8]. This completes the proof. □

In October 2011, we found in the internet arXiv (110.2338v1) with title:
A surface containing a line and a circle through each point is a quadric
by Fedor Nilov and Mikhail Skopenkov concerning surfaces including several circular arcs. They found a surface which is not a cyclide, but includes 2 families of circles (see Figure 3):

$$(x^2 + y^2 + z^2)^2 - 4y^2z^2 - 4x^2 = 0.$$

Indeed, since we can rewrite this as

$$x = \sqrt{1 - y^2} + \sqrt{1 - z^2},$$

$y = y_0$, or $z = z_0$ becomes a circle.

Further they proved the following (Theorem 4.3):

Theorem 1.4. *Let Φ be a smooth closed surface in $\mathbb{R}^3$ homeomorphic to either a sphere or a torus. If through each point of the surface one can draw at least* **4 distinct circles fully contained in the surface** *(and continuously depending on the point) then the surface is* **a cyclide***.*

They extended Takeuchi's idea on intersection numbers of fundamental groups and used a classical theorem on the relationship between cospherical circles and cyclides. So the proof relies on the global information of the surface. On the other hand their counter example is not a closed surface, but a surface with singularities. At the same time, they gave a conjecture (also see [7]):

Conjecture 2. 3 distinct continuous families of circles $\Longrightarrow$ cyclides

Our aim is to solve such conjectures or to find all the surface germs containing several continuous families of circular arcs only by using elementary calculus and differential equations. In particular we do not use any topological information of the surfaces; closeness, genus e.t.c..

The plan of this paper is the following: In Section 2, we give the main results Theorems 2.2, 2.3, 2.4, 2.6, 2.7, 2.10 with the necessary definition Definition 2.1. Section 3 is devoted to the study of fundamental properties concerning circles and surfaces including the proofs of Theorems 2.2, 2.3, 2.4. Further, Section 4 is devoted to the study

concerning our fifth order partial differential equations including the proofs of Theorems 2.6, 2.7. In Section 5, we give some important propositions concerning general cyclides and the proof of Theorem 2.10. Some of our proofs require heavy calculations, and so for the readers to check the calculations we prepare three Mathematica source files and their pdf versions "check-fifth1$\sim$3" in the following websites for download:
`http://www.u-gakugei.ac.jp/~nobuko/manycircles.html`
`http://agusta.ms.u-tokyo.ac.jp/microlocal/manycircles.html`

2. Main results

Definition 2.1. (The key polynomial $Z(T)$). Let $z = f(x, y)$ be a C^4-class function defined in a neighborhood of $(0, 0) \in \mathbb{R}^2$. Put the Taylor coefficients of f at (x, y) as follows:

$$\begin{cases} a := f_x(x, y),\ b := f_y(x, y), \\ c_0 := f_{xx}(x, y)/2,\ c_1 := f_{xy}(x, y),\ c_2 := f_{yy}(x, y)/2, \\ d_0 := f_{xxx}(x, y)/3!,\ d_1 := f_{xxy}(x, y)/2!, \\ d_2 := f_{xyy}(x, y)/2!,\ d_3 := f_{yyy}(x, y)/3!, \\ e_0 := f_{xxxx}(x, y)/4!,\ e_1 := f_{xxxy}(x, y)/3!,\ e_2 := f_{xxyy}(x, y)/2!^2, \\ e_3 := f_{xyyy}(x, y)/3!,\ e_4 := f_{yyyy}(x, y)/4!. \end{cases}$$

We define some polynomials $C(T), D(T), E(T), R(T), S(T), K(T), W(T)$ and the key polynomial $Z(T)$ in T as follows:

$$\begin{aligned} C(T) &:= c_0 + c_1T + c_2T^2, \quad D(T) := d_0 + d_1T + d_2T^2 + d_3T^3, \\ E(T) &:= e_0 + e_1T + e_2T^2 + e_3T^3 + e_4T^4, \\ R(T) &:= (b^2 + 1)T^2 + 2abT + a^2 + 1, \\ S(T) &:= D(T)R(T) - 2(bT + a)C(T)^2, \\ K(T) &:= R'(T)C(T) - R(T)C'(T), \\ W(T) &:= bS(T) + C(T)K(T) = 2TC(T)^2 + (bD(T) - C'(T)C(T))R(T), \end{aligned}$$

where $C'(T) = \partial_T C(T), R'(T) = \partial_T R(T)$,...etc..

$$\begin{aligned} &Z(T) \equiv Z(T; x, y) := \\ &K(T)^2\big(R(T)E(T) - C(T)^3\big) + R(T)K(T)D(T)\big(D'(T)R(T) \\ &- 3(b^2 + 1)TD(T)\big) + D(T)^2R(T)[-ab\big(2K(T) + TK'(T)\big) \\ &- 2(a^2 + 1)(b^2 + 1)C(T) + \big((a^2 + 1)c_2 + (b^2 + 1)c_0\big)R(T)] \\ &+ 2R(T)C(T)[(bT + a)\{D(T)K'(T)C(T) + D(T)K(T)C'(T) \\ &- D'(T)K(T)C(T)\} - bD(T)C(T)K(T)] \\ &+ 4C(T)^4(bT + a)\{\big((a^2 - 1)c_2 + (b^2 + 1)c_0\big)(bT + a) \\ &- \frac{1}{2}ac_1R'(T) + 2a(c_2 - c_0) - bc_1\}. \end{aligned}$$

It is easy to verify that (the degree of $Z(T)$ in T) ≤ 10. Consider a C^4-surface germ $M = \{z = f(x, y)\}$ at $(0, 0, f(0, 0))$. We call

$$P(t) := Z(t; 0, 0)/(c_0(0, 0) - c_2(0, 0))$$

the characteristic polynomial of the surface germ M if

$$f(0, 0) = f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = 0, \tag{2.1}$$

and

$$f_{xx}(0, 0) - f_{yy}(0, 0) \neq 0. \tag{2.2}$$

Indeed, we can always get such conditions (2.1) after some suitable translation and rotation of $\mathbb{R}^3$ for M. Further condition (2.2) means that the origin is not an umbilical point of M under this expression of M. As we see in Proposition 3.6, we have a more simplified form:

$$\begin{aligned} P(t) =&(t^2+1)\overset{\circ}{D}(t)\{2t(t^2+1)\overset{\circ}{D}{}'(t)-(5t^2+1)\overset{\circ}{D}(t)\} \\ &+4(\overset{\circ}{c}_0-\overset{\circ}{c}_2)t^2\{(t^2+1)\overset{\circ}{E}(t)-\overset{\circ}{C}(t)^3\}, \end{aligned} \tag{2.3}$$

where $\overset{\circ}{c}_j := c_j(0,0), \overset{\circ}{C}(t) := C(t;0,0)$, etc.. There are 16 choices of such Euclidean coordinate systems for a non-umbilical surface germ, and all the variations of the characteristic polynomials are

$$P(t), P(-t), -t^{10}P(1/t), -t^{10}P(-1/t)$$

(see Remark 3.7). In this sense, the set $\{t \in \mathbb{R}\,; P(t) = 0, t \neq 0\}$ is meaningful, and it plays a decisive role in our theory. Under the additional condition

$$d_0(0,0) = d_1(0,0) = d_2(0,0) = d_3(0,0) = 0,$$

we have $P(t) = 4(\overset{\circ}{c}_0-\overset{\circ}{c}_2)t^2Q(t)$ with a polynomial $Q(t)$ of degree 6, which we call *the reduced characteristic polynomial*:

$$Q(t) := (t^2+1)\overset{\circ}{E}(t)-\overset{\circ}{C}(t)^3. \tag{2.4}$$

In Proposition 3.6, we will give some examples of characteristic polynomials which have 10 non-zero real distinct roots. On the other hand the characteristic polynomial of any general cyclide germ have at most 6 non-zero real distinct roots (Proposition 5.2).

Then we have our main theorem:

Theorem 2.2. *Let $z = f(x,y)$ be a C^4-function defined in $U_{\delta_0} = \{x^2+y^2 < \delta_0^2\}$ ($\delta_0 > 0$) satisfying* (2.1). *Assume that the origin is not an umbilical point of $M := \{z = f(x,y), (x,y) \in U_{\delta_0}\}$, that is,*

$$c_0(0,0) - c_2(0,0) = \frac{1}{2}(f_{xx}(0,0) - f_{yy}(0,0)) \neq 0.$$

Then we have the following (i), (ii), (iii).

(i) Let $t_0, s_0 \in \mathbb{R}$. If

$$M \cap \{y = t_0x + s_0z\} \tag{2.5}$$

is a circular arc or a line segment in a neighborhood of the origin, then

$$Z(t_0;0,0) = 0.$$

Further, if it is a circular arc, then $C(t_0;0,0) \neq 0$ and under an additional condition $t_0 \neq 0$ we have

$$s_0 = \frac{(t_0^2+1)D(t_0;0,0)}{2(c_0(0,0)-c_2(0,0))t_0C(t_0;0,0)}.$$

If it is a line segment, then $C(t_0;0,0) = D(t_0;0,0) = E(t_0;0,0) = 0$.

(ii) Let $t(x,y), s(x,y)$ be real-valued continuous functions defined in a neighborhood of $(0,0)$ such that, for some $\delta > 0$ and any $(x_0,y_0) \in U_\delta$, the set

$$M \cap \{y - y_0 = t(x_0,y_0)(x-x_0) + s(x_0,y_0)(z - f(x_0,y_0))\} \tag{2.6}$$

coincides with a circle in a neighborhood of $(x_0, y_0, f(x_0,y_0))$. Assume that $t(0,0) \neq 0$. Consider a continuous function

$$T(x,y) := \frac{t(x,y) + f_x(x,y)s(x,y)}{1 - f_y(x,y)s(x,y)} \tag{2.7}$$

defined in a neighborhood of $(0,0)$. *Then,* $T(x,y), s(x,y), t(x,y)$ *satisfy the following equations:*

$$Z(T(x,y);x,y) = 0, \tag{2.8}$$

$$s(x,y) = \frac{S(T)}{W(T)}, \quad t(x,y) = \frac{TK(T)C(T) - aS(T)}{W(T)}. \tag{2.9}$$

Moreover, if $t(x,y), s(x,y)$ *are constant on each circular arc* (2.6), f *is a* C^5*-function in* U_{δ_0} *and* $Z'(t(0,0);0,0) \neq 0$, *then* $T(x,y)$ *is a* C^1*-function in a neighborhood of the origin satisfying the following equation:*

$$(\partial_x + T(x,y)\partial_y)T(x,y) = \frac{2S(T)}{K(T)}. \tag{2.10}$$

(*iii*) *Conversely, let* $f(x,y) \in C^5(U_{\delta_0})$, *and let* $T(x,y)$ *be a real-valued* C^1*-function defined in a neighborhood of* $(0,0)$ *satisfying* $T(0,0) \neq 0$, *equations* (2.8),(2.10). *Then,* $t(x,y)$, $s(x,y)$ *defined by* (2.9) *belong to* $C^1(U_\delta)$ *for a small* $\delta > 0$, *and satisfy that, for any* $(x_0,y_0) \in U_\delta$, *the set*

$$M \cap \{y - y_0 = t(x_0,y_0)(x - x_0) + s(x_0,y_0)(z - f(x_0,y_0))\}$$

coincides with a circle in a neighborhood of $(x_0, y_0, f(x_0,y_0))$, *and that* $t(x,y), s(x,y)$ *are constant on this circular arc.*

We also get a similar characterization for ruled surfaces.

Theorem 2.3. *Let* $z = f(x,y)$ *be a* C^3*-function defined in* $U_{\delta_0} = \{x^2+y^2 < \delta_0^2\}$ $(\delta_0 > 0)$ *satisfying* (2.1). *Then we have the following* (*i*), (*ii*).

(*i*) *Let* $t(x,y), s(x,y)$ *be real-valued* C^1*-class functions defined in a neighborhood of* $(0,0)$ *such that, for some* $\delta > 0$ *and any* $(x_0,y_0) \in U_\delta$, *the set*

$$M \cap \{y - y_0 = t(x_0,y_0)(x - x_0) + s(x_0,y_0)(z - f(x_0,y_0))\} \tag{2.11}$$

coincides with a line in a neighborhood of $(x_0, y_0, f(x_0,y_0))$. *Consider a* C^1*-function*

$$T(x,y) := \frac{t(x,y) + f_x(x,y)s(x,y)}{1 - f_y(x,y)s(x,y)} \tag{2.12}$$

defined in a neighborhood of $(0,0)$. *Then, line segment* (2.11) *is included in*

$$\{y - y_0 = T(x_0,y_0)(x - x_0)\}, \tag{2.13}$$

and we have the following equations:

$$\begin{cases} C(T(x,y);x,y) = 0, \\ (\partial_x + T(x,y)\partial_y)T(x,y) = 0. \end{cases} \tag{2.14}$$

(*ii*) *Conversely, let* $f(x,y) \in C^3(U_{\delta_0})$, *and let* $T(x,y)$ *be a real-valued* C^1*-function defined in a neighborhood of* $(0,0)$ *satisfying equations* (2.14). *Then, for some* $\delta > 0$ *and any* $(x_0,y_0) \in U_\delta$, *the set*

$$M \cap \{y - y_0 = T(x_0,y_0)(x - x_0)\}$$

coincides with a line in a neighborhood of $(x_0, y_0, f(x_0,y_0))$, *and that* $T(x,y)$ *is constant on this line segment.*

As applications of Theorem 2.2, we have the following theorems:

Theorem 2.4. *Let* $M \subset \mathbb{R}^3$ *be a* C^4*-surface germ at the origin of* $\mathbb{R}^3$. *Assume that the origin is not an umbilical point of* M. *Then, the total number of circular arcs and line segments* $(\subset M)$ *passing through the origin is less than or equal to* 10. *More precisely, put*

$$\mathcal{C} := \{C;\ C \text{ is a circle or a line in } \mathbb{R}^3$$

$$\textit{such that } (0,0,0) \in C \cap U_\delta \subset M \textit{ for a } \delta > 0\},$$

where $U_\delta := \{(x,y,z) \in \mathbb{R}^3\,; x^2+y^2+z^2 < \delta^2\}$. *Then, we have* $\#\mathcal{C} \le 10$.

Remark 2.5. It should be noted that our result is independent of any global structure of M; for example, free from closeness and genus of the surface. As we see in Proposition 3.6, there is a surface germ at the origin whose characteristic polynomial has 10 non-zero real distinct roots. This means that Theorem 2.4 gives the best possible estimate in the sense of the infinitesimal analysis because by Lemma 3.2 such a surface includes infinitesimally 10 circles passing through the origin. Further we remark here that Montaldi [5] has also gotten 10 as the upper limit of the number of circles for generic surface germs by some categorical arguments. On the other hand, our result cover all the non-umbilical surface germs.

Though equation (2.10)

$$(\partial_x + T(x,y)\partial_y)T(x,y) = \frac{2S(T)}{K(T)}$$

looks like a first order PDE, this is a fifth-order PDE for $f(x,y)$:

$$\sum_{j=0}^{5} \binom{5}{j} T^j \partial_x^{5-j}\partial_y^j f(x,y) = \frac{24N(T)}{R(T)K(T)^3}. \tag{2.15}$$

This is because T is an analytic function of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f$ through $Z(T) = 0$. Here $N(T)$ is a polynomial in T of degree 14 defined by

$$\begin{aligned} N(T) :=& -K(T)\left((\partial_x + T\partial_y)Z(T) - \frac{K(T)^2R(T)}{24}\sum_{j=0}^{5}\binom{5}{j}T^j\partial_x^{5-j}\partial_y^j f\right) \\ & - 2S(T)Z'(T), \end{aligned} \tag{2.16}$$

where $(\partial_x + T\partial_y)Z(T) := (\partial_x + T\partial_y)Z(T;x,y)$ means a differentiation for each coefficient of $Z(T)$. It is easy to see that the degree of $N(T)$ in T is at most 14. As we give an explicit form of $N(T)$ in Theorem 2.6, all the coefficients of $N(T)$ are polynomials of derivatives a, b, c_*, d_*, e_* of $f(x,y)$ introduced in Definition 2.1.

Theorem 2.6. *Let* $z = f(x,y)$ *be a* C^5*-class function defined in a neighborhood of the origin satisfying* (2.1). *Assume that the origin is not an umbilical point of* $M := \{z = f(x,y)\}$. *Let* $P(t)$ *be the characteristic polynomial at the origin. For an integer* ℓ $(1 \le \ell \le 10)$, *we suppose that there exist* ℓ *non-zero real numbers* $\{t_k\}_{k=1}^{\ell}$ *satisfying* $P(t_k) = 0,\ P'(t_k) \neq 0$ *for each* $k = 1,\dots,\ell$, *and that* M *includes* ℓ *continuous families of circular arcs associated with* $\{t_k\}_{k=1}^{\ell}$. *Let* $T_k(x,y)$ *be the function* T *corresponding to non-zero simple root* t_k*; that is,* $T_k(0,0) = t_k$ $(k = 1,...,\ell)$. *Then* f *is a solution of the following system of fifth-order partial differential equations:*

$$\begin{cases} Z(T_k(x,y)) = 0, \quad T_k(0,0) = t_k, \\ \displaystyle\sum_{j=0}^{5}\binom{5}{j}T_k(x,y)^j\partial_x^{5-j}\partial_y^j f(x,y) = \frac{24N(T_k(x,y))}{R(T_k(x,y))K(T_k(x,y))^3} \\ (1 \le k \le \ell), \end{cases} \tag{2.17}$$

where $N(T)$ *defined at* (2.16) *is a polynomial in* T *of degree* 14 *with the following explicit form:*

$$\begin{aligned} N(T) =& \\ & -5R(T)K(T)^2E'(T)\big[R(T)D(T) - 2(bT+a)C(T)^2\big] \\ & + D(T)^3R(T)B_1(T) + 2D(T)^2D'(T)R(T)^2B_2(T) \end{aligned}$$

$$
\begin{aligned}
&- D(T)^2R(T)^2K(T)\big[(3d_3T + d_2)\big(5R(T) - (b^2+1)T^2\big)\\
&+ (d_1T + 3d_0)(b^2+1)\big] + D(T)^2B_3(T)\\
&+ 2(bT+a)D(T)D'(T)R(T)C(T)B_4(T)\\
&+ 10(bT+a)D(T)D''(T)R(T)^2K(T)C(T)^2 + D(T)B_5(T)\\
&- 4(bT+a)D'(T)R(T)K(T)C(T)^3\big[5(bT+a)C'(T) + 2bC(T)\big]\\
&+ 4(bT+a)C(T)^4K(T)\Big[3d_0B_6(T) + d_1B_7(T) + d_2B_8(T)\\
&- 3d_3TB_9(T)\Big] + 4C(T)^4B_{10}(T).
\end{aligned}
$$

Here, a, b, c_*, d_*, e_* are the higher-order derivatives of $f(x, y)$, and $C(T), C'(T), D(T)$, $D'(T), D''(T), E(T), E'(T), R(T), R'(T), K(T), K'(T)$ are the polynomials in T (or their derivatives in T) with coefficients in polynomials in a, b, c_*, d_*, e_*, which are introduced in Definition 2.1. Further, $B_1(T), ..., B_{10}(T)$ are the polynomials in T, a, b, c_* given by the following:

$$
\begin{aligned}
B_1(T) :=\ & K(T)\big\{42(b^2+1)R(T) - 20(a^2+b^2+1)\big\}\\
&- 4c_2R(T)\big\{3abR(T) - 2(a^2+b^2+1-a^2b^2)T + 2ab(a^2+1)\big\}\\
&+ 2(b^2+1)c_1R(T)\{3R(T) + 4abT + 4a^2 + 4\}\\
&- 4(b^2+1)R(T)R'(T)c_0,\\
B_2(T) :=\ & -\big(4(b^2+1)T + 5ab\big)K(T) - \big((b^2+1)T^2 + a^2 + 1\big)K'(T)\\
&+ 4(a^2+1)(b^2+1)C(T) - 2\big((a^2+1)c_2 + (b^2+1)c_0\big)R(T),\\
B_3(T) :=\ & K(T)^2\big\{-4aR(T)C'(T) + 12(b^2+1)(bT+a)TC(T)\\
&+ 8ab(bT+a)C(T) + 12bR(T)C(T) + 2bR(T)(2c_0 + c_1T)\big\}\\
&+ K(T)K'(T)\big\{4abT(bT+a)C(T) - 12(bT+a)R(T)C(T)\\
&+ bTR(T)(2c_0 + c_1T) + aTR(T)C'(T)\big\}\\
&+ K(T)\big\{-4a\big(b^2+6\big)R(T)C(T)^2 - \big(18(b^2+1)T(bT+a)\\
&+ 16ab(bT+a) - 4b(a^2+1)\big)R(T)C(T)C'(T)\\
&+ 8(a^2+1)(b^2+1)(bT+a)C(T)^2 - 8(bT+a)\big((a^2+1)c_2\\
&+ (b^2+1)c_0\big)R(T)C(T) + 4abT(bc_1 - 2ac_2)R(T)C(T)\\
&+ 4a(b^2+1)(2c_0 + c_1T)R(T)C(T) - 2ac_2(2c_0 + c_1T)R(T)^2\\
&- 2bc_0C'(T)R(T)^2 - 12(b^2+1)(bT+a)TR'(T)C(T)^2\\
&- 8ab(bT+a)R'(T)C(T)^2 - 4(bT+a)R(T)R'(T)C(T)C'(T)\\
&+ 4bR(T)R'(T)C(T)^2 - 4(bT+a)R(T)^2C'(T)^2\\
&- 8c_2(bT+a)R(T)^2C(T) + 4bR(T)^2C(T)C'(T)\big\}\\
&- 2(bT+a)K'(T)C(T)\big\{2abTR'(T)C(T) + 5R'(T)R(T)C(T)\\
&+ 6R(T)^2C'(T)\big\} - 4(bT+a)K''(T)R(T)C(T)^2\big(R(T) + abT\big)\\
&- 8(a^2+1)(b^2+1)(bT+a)R'(T)C(T)^3 + 8(bT+a)\big((a^2+1)c_2\\
&+ (b^2+1)c_0\big)R(T)R'(T)C(T)^2\\
&- 8(a^2+1)(b^2+1)(bT+a)R(T)C(T)^2C'(T),\\
B_4(T) :=\ & 2\big(3(b^2+1)T + 5ab\big)K(T)C(T) + 5R(T)K(T)C'(T)\\
&+ 2\big((b^2+1)T^2 + a^2 + 1\big)K'(T)C(T) - 8(a^2+1)(b^2+1)C(T)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 4R(T)C(T)\big((a^2+1)c_2+(b^2+1)c_0\big),\\
B_5(T) &:= 9C(T)^2K(T)^3 + K(T)^2C(T)^2\big\{6R'(T)C(T)+4R(T)C'(T)\\
&+ 8b(bT+a)C(T) - 8(bT+a)^2C'(T)\big\}\\
&- 8(bT+a)^2K(T)K'(T)C(T)^3 + K(T)C(T)^2\big\{-8(bT+a)(bc_1\\
&- 2ac_2)R(T)C(T) - 8b(bT+a)R(T)C'(T)C(T) + 8b^2R(T)C(T)^2\\
&- 4b(bT+a)R(T)C(T)C'(T) - 48(bT+a)^2C(T)\big((a^2-1)c_2\\
&+ (b^2+1)c_0\big) + 24ac_1(bT+a)R'(T)C(T) - 48(bT+a)C(T)\big(2a(c_2\\
&- c_0) - bc_1\big) + 12(bT+a)^2R(T)C'(T)^2 - 8b(bT+a)R'(T)C(T)^2\\
&+ 8(bT+a)^2R'(T)C(T)C'(T) + 16c_2(bT+a)^2R(T)C(T)\\
&- 8b(bT+a)R(T)C(T)C'(T)\big\}\\
&+ 8(bT+a)^2K'(T)C(T)^3\big\{R'(T)C(T)+3R(T)C'(T)\big\}\\
&+ 8(bT+a)^2R(T)C(T)^4K''(T)\\
&- 16b(bT+a)R(T)C(T)^4\big((a^2-1)c_2+(b^2+1)c_0\big)\\
&- 32(bT+a)^2R(T)C(T)^3C'(T)\big((a^2-1)c_2+(b^2+1)c_0\big)\\
&+ 16a(bT+a)R(T)R'(T)c_1C(T)^3C'(T) + 4abR(T)R'(T)c_1C(T)^4\\
&+ 8a(b^2+1)(bT+a)R(T)c_1C(T)^4 - 8bR(T)C(T)^4\big(2a(c_2-c_0)\\
&- bc_1\big) - 32(bT+a)R(T)C(T)^3C'(T)\big(2a(c_2-c_0)-bc_1\big),\\
B_6(T) &:= -(b^2+1)(bT+a)+2a,\\
B_7(T) &:= -(b^2+1)T(bT+a)+aR'(T)+2(aT+b),\\
B_8(T) &:= -(a^2-3)(bT+a)+aTR'(T)-4a-4(bT+a)R(T),\\
B_9(T) &:= (a^2-1)(bT+a)+2a+4(bT+a)R(T),\\
B_{10}(T) &:= K(T)^2\big\{2bC(T)-4(bT+a)C'(T)\big\}\\
&- 2(bT+a)C(T)K(T)K'(T) + K(T)\big\{aR'(T)c_1C(T)\\
&- 4(bT+a)C(T)\big((a^2-1)c_2+(b^2+1)c_0\big) - 4a(c_2-c_0)C(T)\\
&+ 2bc_1C(T) - 2a(bT+a)^2c_2(2c_0+c_1T) - 2b(bT+a)^2c_0C'(T)\\
&+ 2ab(bT+a)c_1C(T) + a(bT+a)^2c_1C'(T)\\
&+ \frac{1}{2}(bT+a)R'(T)c_1(2c_0+c_1T) - 2(bT+a)(c_2-c_0)(2c_0+c_1T)\\
&+ (bT+a)c_1C'(T)\big\} + 8b(bT+a)^2\big((a^2-1)c_2+(b^2+1)c_0\big)C(T)^2\\
&+ 16(bT+a)^3C(T)C'(T)\big((a^2-1)c_2+(b^2+1)c_0\big)\\
&- 8a(bT+a)^2R'(T)c_1C(T)C'(T) - 2ab(bT+a)R'(T)c_1C(T)^2\\
&- 4a(b^2+1)(bT+a)^2c_1C(T)^2 + 4b(bT+a)C(T)^2\big(2a(c_2-c_0)\\
&- bc_1\big) + 16(bT+a)^2C(T)C'(T)\big(2a(c_2-c_0)-bc_1\big).
\end{aligned}
$$

Further the converse statement also holds in the sense of (iii) of Theorem 2.2.

Theorem 2.7. *Let $M : z = f(x,y)$ be a $C^{5+\theta}$-class surface germ at the origin satisfying condition (2.1), where θ $(0 < \theta < 1)$ is an exponent for Hölder continuity. Assume that the origin is not an umbilical point of M. Let $P(t)$ be its characteristic polynomial at $(0,0)$. Suppose that M contains two continuous families of circular arcs in the sense of (ii) of Theorem 2.2, where these families correspond to two distinct non-zero real simple roots t_1, t_2 of $P(t) = 0$, respectively. Then, f is an analytic function which* *is uniquely determined only by the partial derivatives at* $(0,0)$ *up to 8th-order. More precisely, such surface-germs are classified by at most* 21 *real parameters.*

Definition 2.8. A conformal transformation in $\mathbb{R}^3$ is a finite composition of translations, rotations, and the inversions $\vec{x} = \lambda\vec{y}/|\vec{y}|^2$ $(\lambda > 0)$ in $\mathbb{R}^3$. As a result, reflections and dilations are conformal transformations. Two surface germs M, M' at $p \in \mathbb{R}^3$ are said to be conformally equivalent to each other if there is a conformal transformation F with $F(p) = p$ such that $F(M) = M'$ as a surface germ.

Remark 2.9. A conformal transformation maps a sphere (or (a plane)$\cup\{\infty\}$) onto a sphere or (a plane)$\cup\{\infty\}$: As a result, a circle (or (a line)$\cup\{\infty\}$) onto a circle or (a line)$\cup\{\infty\}$. Hence the total number of circles and lines passing through a point on a surface is preserved under a conformal transformation. Further it is well-known that a general cyclide is transformed into another general cyclide by any conformal transformation. N. Takeuchi [10] proved any general cyclide is conformally equivalent to a cyclide of the following type:

$$\alpha(x_1^2 + x_2^2 + x_3^2)^2 + \sum_{i}^{3} \gamma_{ii} x_i^2 + \epsilon = 0. \tag{2.18}$$

Theorem 2.10. *Let* $z = f(x, y)$ *be a* C^5*-class function defined in a neighborhood of* $x = y = 0$ *satisfying conditions* (2.1), (2.2). *Let* $P(t)$ *be the characteristic polynomial at the origin for the surface germ* $M : z = f(x, y)$. *Suppose that* $P(t) = 0$ *has 6 distinct non-zero real simple roots* $\{t_k\}_{k=1}^6$, *and that* M *includes 6 continuous families of circular arcs corresponding to* $\{t_k\}_{k=1}^6$. *Then,* f *is analytic at* $x = y = 0$, *and* f *is completely determined by the* 11 *Taylor expansion coefficients* c_0, c_2, d_*, e_* *at* $(0,0)$. *Furthermore, suppose the following additional conditions*

$$d_0 = d_1 = d_2 = d_3 = 0 \quad at\ (0,0)$$

with some generic conditions on c_0, c_2, e_0, e_2, e_4. *Then we obtain that* $e_1 = e_3 = 0$ *at* $(0,0)$, *and that the surface germ* $z = f(x,y)$ *at* $x = y = 0$ *is a general cyclide. More precisely,* M *is conformally equivalent to a germ at* $(0,0,*)$ *of the following 6-circle Blum cyclide:*

$$(x^2 + y^2 + z^2)^2 - 2a_1x^2 - 2a_2y^2 - 2a_3z^2 + a_4^2 = 0, \tag{2.19}$$

where $a_1 > a_3 > a_4 > 0$, $-a_2 > a_4$. *Further this surface* (2.19) *has the same characteristic roots* $\{t_k\}_{k=1}^6$.

3. Circles and surfaces –the proofs of Theorems 2.2, 2.3, 2.4–

First of all, we prepare some elementary lemmas concerning a circle and a surface in $\mathbb{R}^3$.

Lemma 3.1. *Let* C *be a circle in* $\mathbb{R}^3$. *We assume that* $C \subset \{y = tx + sz\}$ *and that* C *is tangent to a plane* $\{z = ax + by\}$ *at the origin, where* a, b, t, s *are real numbers. Then, if* $4(1 - bs) > a^2s^2$, *we have the following expression for any point* $(x, y(x), z(x)) \in C$ *sufficiently close to the origin:*

$$z(x) = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + O(x^5),$$

where

$$\begin{aligned} A_1 &= \frac{a+bt}{1-bs}, \\ A_2 &= \frac{(t^2 + ast - bs + 1)}{2(s^2 + t^2 + 1)(1 - bs)^3 v} \\ &\quad \times \{(a^2 + b^2)s^2 + 2(at - b)s + (b^2 + 1)t^2 + 2abt + a^2 + 1\}, \end{aligned}$$

$$\begin{aligned}A_3 =& \frac{(t^2+ast-bs+1)^2(as^2+ts+bt+a)}{2(s^2+t^2+1)^2(1-bs)^5v^2}\\ &\times\{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1\},\\ A_4 =& \frac{(t^2+ast-bs+1)^3}{8(s^2+t^2+1)^3(1-bs)^7v^3}\\ &\times\{(s^2+t^2+1)(1-bs)^2+5(as^2+ts+bt+a)^2\}\\ &\times\{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1\}.\end{aligned}$$

Here, v is the z-coordinate of the center of C. In particular, we obtain the following equations:

$$\begin{aligned}A_3 &= \frac{2(1-bs)(as^2+ts+bt+a)}{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1}A_2^2,\\ A_4 &= \frac{(1-bs)^2\{(s^2+t^2+1)(1-bs)^2+5(as^2+ts+bt+a)^2\}}{\{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1\}^2}A_2^3.\end{aligned}$$

Proof. (For the check of calculations, the readers can employ a Mathematica file "check-fifth1" in our website written in Section 1.) The tangent line to C at the origin is $\{y = tx + s(ax+by), z = ax+by\}$; that is,

$$\left\{\left(x, \frac{t+as}{1-bs}x, \frac{a+bt}{1-bs}x\right) \; ; \; x \in \mathbb{R}\right\}.$$

Therefore, putting $(u, tu+sv, v)$ be a center of C, we have

$$u + (tu+sv)\frac{t+as}{1-bs} + v\frac{a+bt}{1-bs} = 0.$$

Since $t^2+ast-bs+1 > 0$ by $(as)^2 - 4(1-bs) < 0$, we get

$$u = -\frac{as^2+ts+bt+a}{t^2+ast-bs+1}v.$$

From the equation of C :

$$(x-u)^2 + (y-tu-sv)^2 + (z-v)^2 = u^2 + (tu+sv)^2 + v^2$$

and $y = tx + sz$, we get

$$\begin{aligned}0 =&(s^2+1)z^2 + 2\{tsx - s(tu+sv) - v\}z\\ &+ (1+t^2)x^2 - 2\{u + t(tu+sv)\}x.\end{aligned}$$

Putting $k := s(tu+sv)+v = (s^2+t^2+1)(1-bs)(t^2+ast-bs+1)^{-1}v \neq 0$, we have

$$\begin{aligned}z(x) =& \frac{1}{s^2+1}\Big(-tsx + k\\ &-k\sqrt{1-\frac{2kts-2(s^2+1)\{u+t(tu+sv)\}+(s^2+t^2+1)x}{k^2}x}\Big)\\ =&\frac{-tsx + k\left(\frac{\tau(x)x}{2k^2} + \frac{\tau(x)^2x^2}{8k^4} + \frac{\tau(x)^3x^3}{16k^6} + \frac{5\tau(x)^4x^4}{128k^8} + O(x^5)\right)}{s^2+1}.\end{aligned}$$

Here

$$\tau(x) = C + Dx := 2kts - 2(s^2+1)\{u+t(tu+sv)\} + (s^2+t^2+1)x,$$

and so we have

$$C := \frac{2(as^2+ts+bt+a)(s^2+t^2+1)}{t^2+ast-bs+1}v.$$

Therefore for $z(x) = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + O(x^5)$,

$$\begin{aligned}
A_1 &= -\frac{u + t(tu + sv)}{k} = \frac{(a + bt)(s^2 + t^2 + 1)}{k(t^2 + ast - bs + 1)}v = \frac{a + bt}{1 - bs},\\
A_2 &= \frac{D}{2k(s^2 + 1)} + \frac{C^2}{8k^3(s^2 + 1)}\\
&= \frac{(t^2 + ast - bs + 1)\{(s^2 + t^2 + 1)(1 - bs)^2 + (as^2 + ts + bt + a)^2\}}{2(s^2 + 1)(s^2 + t^2 + 1)(1 - bs)^3 v},\\
A_3 &= \frac{2CD}{8k^3(s^2 + 1)} + \frac{C^3}{16k^5(s^2 + 1)}\\
&= \frac{(t^2 + ast - bs + 1)^2(as^2 + ts + bt + a)}{2(s^2 + 1)(s^2 + t^2 + 1)^2(1 - bs)^5 v^2}\\
&\quad \times \{(s^2 + t^2 + 1)(1 - bs)^2 + (as^2 + ts + bt + a)^2\},\\
A_4 &= \frac{D^2}{8k^3(s^2 + 1)} + \frac{3C^2D}{16k^5(s^2 + 1)} + \frac{5C^4}{128k^7(s^2 + 1)}\\
&= \frac{(t^2 + ast - bs + 1)^3}{8(s^2 + 1)(s^2 + t^2 + 1)^3(1 - bs)^7 v^3}\\
&\quad \times \{(s^2 + t^2 + 1)(1 - bs)^2 + 5(as^2 + ts + bt + a)^2\}\\
&\quad \times \{(s^2 + t^2 + 1)(1 - bs)^2 + (as^2 + ts + bt + a)^2\}.
\end{aligned}$$

Since

$$\begin{aligned}
&(s^2 + t^2 + 1)(1 - bs)^2 + (as^2 + ts + bt + a)^2\\
&= (s^2 + 1)\{(a^2 + b^2)s^2 + 2(at - b)s + (b^2 + 1)t^2 + 2abt + a^2 + 1\}, \qquad (3.1)
\end{aligned}$$

we get our conclusions. □

Lemma 3.2. *Let $z = f(x, y)$ be a C^4-function which has a Taylor expansion at $(0, 0)$:*

$$ax + by + \sum_{j=0}^{2} c_j x^{2-j} y^j + \sum_{j=0}^{3} d_j x^{3-j} y^j + \sum_{j=0}^{4} e_j x^{4-j} y^j + o((x^2 + y^2)^2).$$

Let M and H be surface germs at $(0, 0, 0)$ such that

$$M = \{z = f(x, y)\}, \quad H = \{y = tx + sz\}$$

with some $t, s \in \mathbb{R}$. Assume that $4(1 - bs) > a^2s^2$. Then $M \cap H$ has an expression in a neighborhood of $(0, 0, 0)$ such that $M \cap H = \{(x, y(x), z(x));\ x \in (-\varepsilon, \varepsilon)\}$ with

$$\begin{aligned}
z(x) &= A_1x + A_2x^2 + A_3x^3 + A_4x^4 + o(x^4),\\
y(x) &= (t + sA_1)x + sA_2x^2 + sA_3x^3 + sA_4x^4 + o(x^4),\\
A_1 &:= \frac{a + bt}{1 - bs}, \quad T := \frac{t + as}{1 - bs}, \quad A_2 := \frac{1}{1 - bs}\left(c_0 + c_1T + c_2T^2\right),\\
A_3 &:= \frac{1}{1 - bs}\left(c_1sA_2 + d_0 + (2c_2sA_2 + d_1)T + d_2T^2 + d_3T^3\right),\\
A_4 &:= \frac{1}{1 - bs}\left(c_1sA_3 + c_2s^2A_2^2 + d_1sA_2 + e_0\right.\\
&\quad \left.+ (2c_2sA_3 + 2d_2sA_2 + e_1)T + (3d_3sA_2 + e_2)T^2 + e_3T^3 + e_4T^4\right).
\end{aligned}$$

In particular, if $M \cap H$ is a circular arc in a neighborhood of the origin, then $A_2 \neq 0$ and we have the following equations:

$$A_3 = \frac{2(1 - bs)(as^2 + ts + bt + a)}{(a^2 + b^2)s^2 + 2(at - b)s + (b^2 + 1)t^2 + 2abt + a^2 + 1}A_2^2, \qquad (3.2)$$

$$A_4 = \frac{(1-bs)^2\{(s^2+t^2+1)(1-bs)^2 + 5(as^2+ts+bt+a)^2\}}{\{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1\}^2} A_2^3, \tag{3.3}$$

where the denominator $(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1$ *is always positive. Conversely, if* $A_2 \neq 0$*, and* a, b, c_j, d_j, e_j *satisfy* (3.2)*,* (3.3)*, then* $M \cap H$ *coincides with a circle at the origin up to 4-th order in* x*.*

Proof. Since $M \cap H = \{(x, tx+sz, z)\ ;\ z = f(x, tx+sz)\}$ and

$$z(x) = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + o(x^4),$$

we have

$$\begin{aligned}
&A_1x + A_2x^2 + A_3x^3 + A_4x^4 + o(x^4)\\
&= ax + b(tx + s(A_1x + A_2x^2 + A_3x^3 + A_4x^4)) + c_0x^2\\
&+ c_1x(tx + s(A_1x + A_2x^2 + A_3x^3)) + c_2(tx + s(A_1x + A_2x^2 + A_3x^3))^2\\
&+ d_0x^3 + d_1x^2(tx + s(A_1x + A_2x^2)) + d_2x(tx + s(A_1x + A_2x^2))^2\\
&+ d_3(tx + s(A_1x + A_2x^2))^3 + e_0x^4 + e_1x^3(tx + s(A_1x))\\
&+ e_2x^2(tx + s(A_1x))^2 + e_3x(tx + s(A_1x))^3 + e_4(tx + s(A_1x))^4.
\end{aligned}$$

By picking up the coefficients of x in both sides, we get $A_1 = a + b(t + sA_1)$, therefore

$$A_1 = \frac{a+bt}{1-bs}, \text{ or } \quad t + sA_1 = \frac{t+as}{1-bs} \equiv T.$$

By picking up the coefficients of x^2, we get

$$A_2 = bsA_2 + c_0 + c_1(t+sA_1) + c_2(t+sA_1)^2.$$

Hence, $A_2 = (c_0 + c_1T + c_2T^2)/(1-bs)$. By picking up the coefficients of x^3, we get

$$\begin{aligned}
A_3 &= bsA_3 + c_1sA_2 + 2c_2(t+sA_1)sA_2 + d_0 + d_1(t+sA_1)\\
&\quad + d_2(t+sA_1)^2 + d_3(t+sA_1)^3.
\end{aligned}$$

Therefore,

$$A_3 = \frac{1}{1-bs}\left(c_1sA_2 + d_0 + (2c_2sA_2 + d_1)T + d_2T^2 + d_3T^3\right).$$

By picking up the coefficients of x^4, we get

$$\begin{aligned}
A_4 &= bsA_4 + c_1sA_3 + c_2\{s^2A_2^2 + 2(t+sA_1)sA_3\} + d_1sA_2\\
&\quad + 2d_2(t+sA_1)sA_2 + 3d_3(t+sA_1)^2sA_2 + e_0 + e_1(t+sA_1)\\
&\quad + e_2(t+sA_1)^2 + e_3(t+sA_1)^3 + e_4(t+sA_1)^4.
\end{aligned}$$

Hence,

$$\begin{aligned}
A_4 =&\frac{1}{1-bs}\left(c_1sA_3 + c_2s^2A_2^2 + d_1sA_2 + e_0\right.\\
&\left. + (2c_2sA_3 + 2d_2sA_2 + e_1)T + (3d_3sA_2 + e_2)T^2 + e_3T^3 + e_4T^4\right).
\end{aligned}$$

Combining these results with Lemma 3.1, we have (3.2), (3.3). The positivity of the denominator of (3.2) follows from the expression (3.1). Conversely, if (3.2) and (3.3) hold, we set a circle

$$\begin{aligned}
C := H \cap \{(x,y,z)\ ;\ &(x-u)^2 + (y-tu-sv)^2 + (z-v)^2\\
&= u^2 + (tu+sv)^2 + v^2\}
\end{aligned}$$

with

$$v = \frac{(t^2+ast-bs+1)}{2(s^2+t^2+1)(1-bs)^3A_2}$$

$$\times \{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1\},$$

$$u = -\frac{as^2+ts+bt+a}{2(s^2+t^2+1)(1-bs)^3 A_2} \times \{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1\}.$$

Then, by Lemma 3.1 we obtain the same Taylor expansion $A_1x+A_2x^2+A_3x^3+A_4x^4+o(x^4)$ at $x=0$ for $\tilde{z}(x)$ satisfying $(x,\tilde{y}(x),\tilde{z}(x)) \in C$. Thus the proof is completed. $\square$

Proposition 3.3. *Let $a, b, c_*, d_*, e_*, s, t$ be the coefficients introduced in Lemma 3.2. Put $T = (t+as)/(1-bs)$, and let $C(T), D(T), E(T), R(T), S(T), K(T), W(T), Z(T)$ be the polynomials in T introduced in Definition 2.1, whose coefficients are polynomials in a, b, c_*, d_*, e_*. Assume that $4(1-bs) > a^2s^2, C(T) \neq 0$. Then, the system of equations (3.2), (3.3) under $W(T) \neq 0$ is equivalent to the following system of equations:*

$$Z(T) = 0, \quad s = S(T)/W(T). \tag{3.4}$$

Further, the system of equations (3.2), (3.3) under $W(T) = 0$ is equivalent to the following system:

$$\begin{cases} W(T) = S(T) = 0, \\ \Big(R(T)^2\big(b^2E(T) - bC'(T)D(T) - bC(T)D'(T) \\ +C'(T)^2C(T) + c_2C(T)^2\big) - C(T)^3\big(1+5T^2+(a+bT)^2\big)\Big)s^2 \\ +\Big(R(T)^2\big(-2bE(T)+C'(T)D(T)+C(T)D'(T)\big) \\ -8(a+bT)TC(T)^3\Big)s \\ +R(T)^2E(T) - C(T)^3\Big(1+5(a+bT)^2+T^2\Big) = 0. \end{cases} \tag{3.5}$$

In both cases, we have $Z(T) = 0$.

Proof. (For the check of calculations, the readers can employ a mathematica file "check-fifth1" in our website written in Section 1.) First, we rewrite A_2, A_3, A_4 in Lemma 3.2 as follows:

$$\begin{aligned} A_2 &= \frac{C(T)}{1-bs}, \\ A_3 &= \frac{D(T)+sC'(T)A_2}{1-bs} = \frac{(1-bs)D(T)+sC'(T)C(T)}{(1-bs)^2}, \\ A_4 &= \frac{E(T)+sC'(T)A_3+sD'(T)A_2+s^2c_2A_2^2}{1-bs} \\ &= \frac{(1-bs)E(T)+sC'(T)D(T)+sC(T)D'(T)}{(1-bs)^2} \\ &\quad + \frac{s^2C'(T)^2C(T)+s^2c_2C(T)^2}{(1-bs)^3}. \end{aligned}$$

Hence, the equation (3.2) is equivalent to the following:

$$\begin{aligned} &(1-bs)D(T)+sC'(T)C(T) \\ &= \frac{2(1-bs)(as^2+ts+bt+a)C(T)^2}{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1}. \end{aligned}$$

Since $t = (1-bs)T - as$, we can rewrite the right side as

$$\frac{2\big(a+(b+s)T\big)C(T)^2}{R(T)}.$$

Hence, (3.2) is equivalent to

$$(1-bs)D(T)R(T)+sC'(T)C(T)R(T)-2\big(a+(b+s)T\big)C(T)^2=0,$$

that is, $S(T)-sW(T)=0$. Therefore, if $W(T)\neq 0$,

$$s=\frac{D(T)R(T)-2(bT+a)C(T)^2}{2TC(T)^2-R(T)C'(T)C(T)+bR(T)D(T)}=\frac{S(T)}{W(T)}.$$

On the other hand, as for the coefficient of the right side of (3.3) we have

$$\begin{aligned}&\frac{(1-bs)^2\{(s^2+t^2+1)(1-bs)^2+5(as^2+ts+bt+a)^2\}}{\{(a^2+b^2)s^2+2(at-b)s+(b^2+1)t^2+2abt+a^2+1\}^2}\\&=\Big(1+s^2+5\big(a+(b+s)T\big)^2+\big(as-(1-bs)T\big)^2\Big)/R(T)^2.\end{aligned}$$

Hence the equation (3.3) is equivalent to

$$\begin{aligned}&(1-bs)^2E(T)+s(1-bs)\big(C'(T)D(T)+C(T)D'(T)\big)\\&\quad+s^2\big(C'(T)^2C(T)+c_2C(T)^2\big)\\&=\frac{\Big(1+s^2+5\big(a+(b+s)T\big)^2+\big(as-(1-bs)T\big)^2\Big)C(T)^3}{R(T)^2}.\end{aligned}$$

Since $R(T)=(b^2+1)T^2+2abT+a^2+1>0$, this is equivalent to the following:

$$\begin{aligned}&R(T)^2\Big((1-bs)^2E(T)+s(1-bs)\big(C'(T)D(T)+C(T)D'(T)\big)\\&\quad+s^2\big(C'(T)^2C(T)+c_2C(T)^2\big)\Big)\\&-\Big(1+s^2+5\big(a+(b+s)T\big)^2+\big(as-(1-bs)T\big)^2\Big)C(T)^3=0.\end{aligned}\tag{3.6}$$

Multiply (3.6) by $W(T)^2$ and replace $sW(T)$ by $S(T)$. Then we obtain the following equation for T:

$$\begin{aligned}&R(T)^2\Big((W(T)-bS(T))^2E(T)+S(T)(W(T)-bS(T))\big(C'(T)D(T)\\&+C(T)D'(T)\big)+S(T)^2\big(C'(T)^2C(T)+c_2C(T)^2\big)\Big)\\&-\Big(W(T)^2+S(T)^2+5\big(aW(T)+(bW(T)+S(T))T\big)^2\\&+\big(aS(T)-(W(T)-bS(T))T\big)^2\Big)C(T)^3=0.\end{aligned}\tag{3.7}$$

Using equalities $W(T)=bS(T)+C(T)K(T), S(T)=D(T)R(T)-2(bT+a)C(T)^2$ and the expressions of $R(T),K(T),K'(T),C(T),C'(T)$ by T, we get

$$(\text{ the left side of (3.7) })=R(T)C(T)^2Z(T).$$

Thus we obtain that the system (3.2), (3.3) is equivalent to (3.4) under $W(T)\neq 0$. Further, if $W(T)=0$, the system of (3.2) and (3.3) is equivalent to the system $S(T)=W(T)=0$ and (3.6). In particular we have $Z(T)=0$. Further, (3.6) is a quadratic equation in s as in (3.5). This completes the proof. □

Lemma 3.4. *Let $f(x,y)$ be a C^4-class function defined in a neighborhood of the origin of $\mathbb{R}^2$, and T be a real variable independent from x,y. We denote by a,b,c_*,d_*,e_* the higher-order derivatives of $f(x,y)$ defined in Definition* 2.1. *Further $C(T),C'(T),D(T),D'(T),E(T),E'(T),\ R(T),R'(T),K(T),K'(T),S(T),W(T),Z(T)$ are the polynomials in T with coefficients in these derivatives, which are introduced in Definition* 2.1. *For any polynomial $G(T)=g_0(x,y)+g_1(x,y)T+\cdots+g_N(x,y)T^N$ in T with coefficients in*

C^1-class functions defined in a neighborhood of the origin of $\mathbb{R}^2$, we define a polynomial $L_T[G(T)]$ in T by

$$L_T[G(T)] := \sum_{j=0}^{N} T^j(\partial_x + T\partial_y)g_j(x,y). \tag{3.8}$$

Then we have the following equalities:

$$\begin{cases} L_T[D(T)] & = 4E(T), \\ L_T[D'(T)] & = 3E'(T), \\ L_T[C(T)] & = 3D(T), \\ L_T[C'(T)] & = 2D'(T), \\ L_T[R(T)] & = 4(bT+a)C(T), \\ L_T[a] & = 2c_0 + c_1 T, \\ L_T[b] & = C'(T), \\ L_T[bT+a] & = 2C(T), \\ L_T[R'(T)] & = 4bC(T) + 2(bT+a)C'(T), \\ L_T[K(T)] & = 3R'(T)D(T) - 2R(T)D'(T) \\ & \quad +2C(T)\big(2bC(T) - (bT+a)C'(T)\big), \\ L_T[K'(T)] & = 6(b^2+1)D(T) - 2(d_2 + 3Td_3)R(T) \\ & \quad +4(bc_1 - 2ac_2)C(T), \\ L_T[c_0] & = 3d_0 + d_1 T, \\ L_T[c_1] & = 2d_1 + 2d_2 T, \\ L_T[c_2] & = d_2 + 3d_3 T. \end{cases} \tag{3.9}$$

Moreover, if $T = T(x,y)$ is a C^1-class function defined in a neighborhood of the origin of $\mathbb{R}^2$ satisfying equation (2.10)*:*

$$(\partial_x + T(x,y)\partial_y)T(x,y) = \frac{2S(T(x,y))}{K(T(x,y))}$$

with $W(T(x,y)) \neq 0, K(T(x,y)) \neq 0$, then we have

$$(\partial_x + T(x,y)\partial_y)\left(\frac{S(T(x,y))}{W(T(x,y))}\right) = \left.\frac{4C(T)Z(T)}{K(T)W(T)^2}\right|_{T=T(x,y)}. \tag{3.10}$$

Proof. (For the check of calculations, the readers can employ a mathematica file "check-fifth1" in our website written in Section 1.) We have the following direct calculations:

$$\begin{aligned} L_T[D(T)] &= \sum_{j=0}^{3} T^j(\partial_x d_j + T\partial_y d_j) \\ &= (4e_0 + Te_1) + T(3e_1 + 2Te_2) + T^2(2e_2 + 3Te_3) + T^3(e_3 + 4Te_4) \\ &= 4E(T), \\ L_T[D'(T)] &= \sum_{j=1}^{3} jT^{j-1}(\partial_x d_j + T\partial_y d_j) \\ &= (3e_1 + 2Te_2) + 2T(2e_2 + 3Te_3) + 3T^2(e_3 + 4Te_4) = 3E'(T), \\ L_T[C(T)] &= \sum_{j=0}^{2} T^j(\partial_x c_j + T\partial_y c_j) \\ &= (3d_0 + Td_1) + T(2d_1 + 2Td_2) + T^2(d_2 + 3Td_3) = 3D(T), \\ L_T[C'(T)] &= (\partial_x c_1 + T\partial_y c_1) + 2T(\partial_x c_2 + T\partial_y c_2) \end{aligned}$$

$$
\begin{aligned}
&= (2d_1 + 2Td_2) + 2T(d_2 + 3Td_3) = 2D'(T),\\
&L_T[a] = f_{xx} + Tf_{xy} = 2c_0 + c_1T,\\
&L_T[b] = f_{xy} + Tf_{yy} = c_1 + 2Tc_2 = C'(T),\\
&L_T[R(T)] = 2bT^2L_T[b] + 2T(bL[a] + aL[b]) + 2aL[a]\\
&= (2bT^2 + 2aT)(c_1 + 2Tc_2) + 2(bT + a)(2c_0 + Tc_1)\\
&= 4(bT + a)C(T),\\
&L_T[bT + a] = T(c_1 + 2c_2T) + 2c_0 + c_1T = 2C(T),\\
&L_T[R'(T)] = 2T2bL_T[b] + 2bL_T[a] + 2aL_T[b]\\
&= 2(2bT + a)C'(T) + 2b(c_1T + 2c_0) = 4bC(T) + 2(bT + a)C'(T),\\
&L_T[K(T)] = L_T[R'(T)C(T) - C'(T)R(T)]\\
&= \big(4bC(T) + 2(bT + a)C'(T)\big)C(T) + R'(T)3D(T) - 2D'(T)R(T)\\
&- C'(T)4(bT + a)C(T)\\
&= 3R'(T)D(T) - 2R(T)D'(T) + 2C(T)\big(2bC(T) - (bT + a)C'(T)\big),\\
&L_T[K'(T)] = L_T[2(b^2 + 1)C(T) - 2c_2R(T)]\\
&= 4bC'(T)C(T) + 2(b^2 + 1)\cdot 3D(T) - 2(d_2 + 3Td_3)R(T)\\
&- 8c_2(bT + a)C(T)\\
&= 6(b^2 + 1)D(T) - 2R(T)(d_2 + 3Td_3) + 4(bc_1 - 2ac_2)C(T),\\
&L_T[c_0] = \partial_x c_0 + T\partial_y c_0 = 3d_0 + d_1T,\\
&L_T[c_1] = \partial_x c_1 + T\partial_y c_1 = 2d_1 + 2d_2T,\\
&L_T[c_2] = \partial_x c_2 + T\partial_y c_2 = d_2 + 3d_3T.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&(\partial_x + T(x,y)\partial_y)\left(\frac{S(T(x,y))}{W(T(x,y))}\right)\\
&= \left.\frac{W(T)L_T[S(T)] - S(T)L_T[W(T)]}{W(T)^2}\right|_{T=T(x,y)}\\
&+ \left.\frac{\big(W(T)S'(T) - S(T)W'(T)\big)\cdot(\partial_x + T(x,y)\partial_y)T(x,y)}{W(T)^2}\right|_{T=T(x,y)}.
\end{aligned}
$$

Hence, to show (3.10), we have only to prove that

$$
\begin{aligned}
X(T) &:= K(T)\big(W(T)L_T[S(T)] - S(T)L_T[W(T)]\big)\\
&\quad + 2S(T)\big(W(T)S'(T) - S(T)W'(T)\big) - 4C(T)Z(T) = 0
\end{aligned}
$$

as a polynomial in T. Since

$$
W(T) = bS(T) + C(T)K(T),\ S(T) = D(T)R(T) - 2(bT + a)C(T)^2,
$$

we have the following:

$$
\begin{aligned}
S'(T) &= D'(T)R(T) + D(T)R'(T) - 2bC(T)^2 - 4(bT + a)C(T)C'(T),\\
W'(T) &= bS'(T) + C'(T)K(T) + C(T)K'(T).
\end{aligned}
$$

Further,

$$
\begin{aligned}
L_T[S(T)] &= R(T)L_T[D(T)] + D(T)L_T[R(T)] - 2L_T[bT + a]C(T)^2\\
&\quad - 4(bT + a)C(T)L_T[C(T)],\\
L_T[W(T)] &= S(T)L_T[b] + bL_T[S(T)] + K(T)L_T[C(T)]
\end{aligned}
$$

$$+ C(T)L_T[K(T)].$$

Therefore we conclude $X(T) = 0$. This completes the proof. $\square$

The proof of Theorem 2.2.
(i) Let $t_0, s_0 \in \mathbb{R}$. Assume that

$$M \cap \{y = t_0 x + s_0 z\}$$

is a circular arc or a line segment in a neighborhood of the origin. Since M is tangent to $z = 0$ at the origin, we can apply Lemma 3.2 to $M \cap \{y = t_0 x + s_0 z\}$. Therefore, if $M \cap \{y = t_0 x + s_0 z\}$ is a line segment, then we have $A_2 = A_3 = A_4 = 0$. Hence $Z(t_0; 0, 0) = 0$ since $T = t_0, C(T) = D(T) = E(T) = 0$ at $x = y = 0$. Further, if $M \cap \{y = t_0 x + s_0 z\}$ is a circular arc, we have $A_2 \neq 0$. Hence $C(T) = c_0 + c_2 t_0^2 \neq 0$ at $x = y = 0$. Therefore we can apply Proposition 3.3, and get $Z(t_0; 0, 0) = 0$. When $t_0 \neq 0$, noting that

$$\begin{aligned} W(T)|_{x=y=0} &= [2TC(T)^2 + (bD(T) - C'(T)C(T))R(T)]_{x=y=0} \\ &= \big(2t_0(c_0 + c_2 t_0^2) - 2c_2 t_0(t_0^2 + 1)\big)(c_0 + c_2 t_0^2) \\ &= 2(c_0 - c_2)t_0(c_0 + c_2 t_0^2) \neq 0, \end{aligned}$$

we have $s_0 = [S(T)/W(T)]_{x=y=0} = \dfrac{(t_0^2 + 1)D(t_0; 0, 0)}{2(c_0 - c_2)t_0(c_0 + c_2 t_0^2)}$.

(ii) Let $t(x, y), s(x, y)$ be real-valued continuous functions defined in a neighborhood of $(0, 0)$ such that, for some $\delta > 0$ and any $(x_0, y_0) \in U_\delta$, the set

$$M \cap \{y - y_0 = t(x_0, y_0)(x - x_0) + s(x_0, y_0)(z - f(x_0, y_0))\}$$

coincides with a circle in a neighborhood of $(x_0, y_0, f(x_0, y_0))$. Assume that $t(0, 0) \neq 0$ and that $Z'(t(0, 0); 0, 0) \neq 0$. Since f is a C^4-function with $f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = 0$, we may assume $4\big(1 - f_y(x_0, y_0)s(x_0, y_0)\big) > f_x(x_0, y_0)^2 s(x_0, y_0)^2$ for any $(x_0, y_0) \in U_\delta$. Further, setting a continuous function

$$T(x, y) := \frac{t(x, y) + f_x(x, y)s(x, y)}{1 - f_y(x, y)s(x, y)},$$

we have $W(T)|_{x=y=0} = 2(c_0 - c_2)t(0, 0)(c_0 + c_2 t(0, 0)^2) \neq 0$. Hence we may also assume that $W(T(x_0, y_0); x_0, y_0) \neq 0$ for any $(x_0, y_0) \in U_\delta$. Therefore we can apply Lemma 3.2 and Proposition 3.3 to $f^*(x, y) := f(x_0 + x, y_0 + y) - f(x_0, y_0)$ at $x = y = 0$. Indeed, since $f^*(0, 0) = 0, a = f^*_x(0, 0) = f_x(x_0, y_0), b = f^*_y(0, 0) = f_y(x_0, y_0), c_0 = (1/2)f^*_{xx}(0, 0) = (1/2)f_{xx}(x_0, y_0)$, ...etc., f^* satisfies $4(1 - bs(x_0, y_0)) > a^2 s(x_0, y_0)^2$, $W(T) \neq 0$ with

$$T = (t(x_0, y_0) + as(x_0, y_0))/(1 - bs(x_0, y_0)) = T(x_0, y_0).$$

Hence we have

$$Z(T(x_0, y_0); x_0, y_0) = 0, s(x_0, y_0) = \left.\frac{S(T(x, y); x, y)}{W(T(x, y); x, y)}\right|_{x=x_0, y=y_0}.$$

The other equation concerning $t(x_0, y_0)$ at (2.9) follows directly from $s(x_0, y_0) = (S/W)_{x=x_0, y=y_0}$ and $T(x_0, y_0) = (t(x_0, y_0) + as(x_0, y_0))/(1 - bs(x_0, y_0))$. Moreover, we assume that $t(x, y), s(x, y)$ are constant on each circular arc (2.6), f is a C^5-function in U_{δ_0} and that $Z'(t(0, 0); 0, 0) \neq 0$. Then $T(x, y)$ is a C^1-function in a neighborhood of the origin by the implicit function theorem which is applied to a C^1-function $Z(T; x, y)$ of T, x, y with $\partial_T Z(t(0, 0); 0, 0) \neq 0$. In particular, $t(x, y), s(x, y)$ are also C^1-functions because we have the expressions (2.9). The (x, y)-component of a tangent vector to

$$\{z = f(x, y)\} \cap \{y - y_0 = t(x_0, y_0)(x - x_0) + s(x_0, y_0)(z - f(x_0, y_0)\}$$

at (x_0, y_0, z_0) is given by

$$\big(1 - s(x_0, y_0)f_y(x_0, y_0), t(x_0, y_0) + s(x_0, y_0)f_x(x_0, y_0)\big).$$

Hence we obtain two equations from our assumptions:

$$(\partial_x + T(x, y)\partial_y)t(x, y) = 0, \quad (\partial_x + T(x, y)\partial_y)s(x, y) = 0.$$

Therefore

$$\begin{aligned}
&(\partial_x + T(x, y)\partial_y)T(x, y) \\
&= \frac{(1 - bs)s(\partial_x + T\partial_y)a + (t + as)s(\partial_x + T\partial_y)b}{(1 - bs)^2} \\
&= s\frac{(1 - bs)(2c_0 + Tc_1) + (t + as)(c_1 + 2Tc_2)}{(1 - bs)^2} = s\frac{2c_0 + Tc_1 + TC'(T)}{1 - bs} \\
&= S(T)\frac{2C(T)}{W(T) - bS(T)} = \frac{2S(T)}{K(T)}.
\end{aligned}$$

(iii) Conversely, let $f(x, y) \in C^5(U_{\delta_0})$, and let $T(x, y)$ be a real-valued C^1-function defined in a neighborhood of $(0, 0)$ satisfying $T(0, 0) \neq 0$ and equations (2.8),(2.10). Let $t(x, y)$, $s(x, y)$ be functions defined by (2.9) which are belonging to $C^1(U_\delta)$ for a small $\delta > 0$. Then by Lemma 3.4 we have

$$\begin{aligned}
&(\partial_x + T(x, y)\partial_y)s(x, y) = (\partial_x + T(x, y)\partial_y)\left(\frac{S(T(x, y))}{W(T(x, y))}\right) \\
&= \left.\frac{4C(T)Z(T)}{K(T)W(T)^2}\right|_{T=T(x,y)} = 0.
\end{aligned}$$

Further we have

$$\begin{aligned}
&(\partial_x + T(x, y)\partial_y)t(x, y) = (\partial_x + T(x, y)\partial_y)((1 - bs)T - as) \\
&= (1 - bs)(2S(T)/K(T)) - sT(\partial_x + T\partial_y)b - s(\partial_x + T\partial_y)a \\
&= \frac{2S(T)C(T) - S(T)T(c_1 + 2Tc_2) - S(T)(2c_0 + Tc_1)}{W(T)} = 0.
\end{aligned}$$

Take any point (x_0, y_0) close to the origin, and consider an integral curve $y = \varphi(x)$ passing through (x_0, y_0) for the vector field $\partial_x + T(x, y)\partial_y$; that is, $\varphi(x)$ is a solution to the following initial value problem for an ordinary differential equation:

$$\frac{d\varphi(x)}{dx} = T(x, \varphi(x)), \quad \varphi(x_0) = y_0.$$

Indeed, since $T(x, y)$ is of C^1-class, such a solution uniquely exists. By the arguments above, we know that functions $t(x, \varphi(x)), s(x, \varphi(x))$ are constant; that is, $t(x, \varphi(x)) = t(x_0, y_0), s(x, \varphi(x)) = s(x_0, y_0)$. Put

$$C := \{\big(x, \varphi(x), f(x, \varphi(x))\big) \ ; \ x \in (x_0 - \varepsilon, x_0 + \varepsilon)\}$$

for a sufficiently small $\varepsilon > 0$. Then C is equivalent to the following C' as a curve germ at $(x_0, y_0, f(x_0, y_0))$:

$$C' := M \cap \{y - y_0 = t(x_0, y_0)(x - x_0) + s(x_0, y_0)(z - f(x_0, y_0))\}.$$

Indeed, since

$$\begin{aligned}
&\frac{d}{dx}\big(\varphi(x) - t(x_0, y_0)x - s(x_0, y_0)f(x, \varphi(x))\big) \\
&= T(x, \varphi(x)) - t(x_0, y_0) \\
&\quad - s(x_0, y_0)\{f_x(x, \varphi(x)) + f_y(x, \varphi(x))T(x, \varphi(x))\} \\
&= T(x, \varphi(x))\{1 - s(x, \varphi(x))f_y(x, \varphi(x))\} \\
&\quad - \{t(x, \varphi(x)) + s(x, \varphi(x))f_x(x, \varphi(x))\} = 0,
\end{aligned}$$

we have $C \subset C'$. Hence $C = C'$ as curve-germs at $(x_0, y_0, f(x_0, y_0))$. Therefore, for any point $(x_1, y_1, f(x_1, y_1)) \in C$, C is expressed as

$$M \cap \{y - y_1 = t(x_1, y_1)(x - x_1) + s(x_1, y_1)(z - f(x_1, y_1))\}.$$

in a small neighborhood of $(x_1, y_1, f(x_1, y_1))$. Since $Z(T(x_1, y_1); x_1, y_1) = 0, s(x_1, y_1) = S(T(x_1, y_1); x_1, y_1)/W(T(x_1, y_1); x_1, y_1)$, by Proposition 3.3 and Lemma 3.2 we conclude that C coincides with a circle up to 4th-order at $(x_1, y_1, f(x_1, y_1))$. Note that C is a plane-curve ($C \subset \{y - y_0 = t(x_0, y_0)(x - x_0) + s(x_0, y_0)(z - f(x_0, y_0))\}$). Therefore we conclude that C is a circular arc by the next lemma (Lemma 3.5) concerning a characterization of a circle. Since $t(x, y), s(x, y)$ are constant on C, this completes the proof of Theorem 2.2 except for the proof of Lemma 3.5.

Lemma 3.5. *Let $g(x)$ be a real valued C^3-function on an interval $(\alpha, \beta) \subset \mathbb{R}$. Suppose the following condition for $C = \{(x, g(x)) \in \mathbb{R}^2\,;\, x \in (\alpha, \beta)\}$: For any $x_0 \in (\alpha, \beta)$, C coincides with a circle up to 3rd-order in $x - x_0$ as curve germs at $(x_0, g(x_0))$. Then C is a circular arc.*

Proof. We fix a $x_0 \in (\alpha, \beta)$. Let $C' = \{(x, h(x)) \in \mathbb{R}^2\,;\, |x - x_0| < \varepsilon\}$ with a small $\varepsilon > 0$ be the circular arc which coincides with C up to 3rd-order in $x - x_0$ at $x = x_0$. Let $(m, n) \in \mathbb{R}^2, R > 0$ be the center and the radius of C', respectively. Therefore for some $\sigma = \pm 1$, we have

$$\begin{aligned} h(x_0 + t) =& n + \sigma\sqrt{R^2 - (x_0 - m)^2}\left\{1 - \frac{2(x_0 - m)t + t^2}{R^2 - (x_0 - m)^2}\right\}^{1/2} \\ =& n + \sigma\sqrt{R^2 - (x_0 - m)^2} - \frac{\sigma(2(x_0 - m)t + t^2)}{2\sqrt{R^2 - (x_0 - m)^2}} \\ & - \frac{\sigma(2(x_0 - m)t + t^2)^2}{8\sqrt{R^2 - (x_0 - m)^2}^3} - \frac{\sigma(2(x_0 - m)t + t^2)^3}{16\sqrt{R^2 - (x_0 - m)^2}^5} + o(t^3). \end{aligned}$$

Hence we get the following:

$$\begin{aligned} g'(x_0) &= -\frac{\sigma(x_0 - m)}{\sqrt{R^2 - (x_0 - m)^2}}, \\ g''(x_0) &= -\frac{\sigma}{\sqrt{R^2 - (x_0 - m)^2}} - \frac{\sigma(x_0 - m)^2}{\sqrt{R^2 - (x_0 - m)^2}^3} \\ &= -\frac{\sigma R^2}{\sqrt{R^2 - (x_0 - m)^2}^3}, \\ g'''(x_0) &= -\frac{3\sigma(x_0 - m)}{\sqrt{R^2 - (x_0 - m)^2}^3} - \frac{3\sigma(x_0 - m)^3}{\sqrt{R^2 - (x_0 - m)^2}^5} \\ &= -\frac{3\sigma(x_0 - m)R^2}{\sqrt{R^2 - (x_0 - m)^2}^5} = \frac{3g'(x_0)R^2}{(R^2 - (x_0 - m)^2)^2}. \end{aligned}$$

Thus we obtain a differential equation for $g(x)$:

$$g'''(x) = \frac{3g'(x)g''(x)^2}{1 + g'(x)^2}.$$

Hence we have

$$\int \frac{g'''(x)}{g''(x)}dx = \int \frac{3g'(x)g''(x)}{g'(x)^2 + 1}dx,$$

and get

$$g''(x) = C_0(g'(x)^2 + 1)^{3/2}$$

with some constant C_0. Further

$$C_0x + C_1 = \int \frac{g''(x)}{(g'(x)^2+1)^{3/2}}dx = \frac{g'(x)}{\sqrt{g'(x)^2+1}}$$

with another constant C_1. Finally we have

$$g(x) = \int \frac{C_0x + C_1}{\sqrt{1-(C_0x+C_1)^2}}dx + C_2 = -\frac{\sqrt{1-(C_0x+C_1)^2}}{C_0} + C_2.$$

Thus we get an equation of a circle:

$$(g(x) - C_2)^2 + (x + C_1C_0^{-1})^2 = C_0^{-2}.$$

□

The proof of Theorem 2.3. Let $z = f(x, y)$ be a C^3-function defined in $U_{\delta_0} = \{x^2 + y^2 < \delta_0^2\}$ ($\delta_0 > 0$) satisfying (2.1).
(i) Let $t(x, y), s(x, y), T(x, y)$ be the real-valued C^1-class functions introduced in the statement of Theorem 2.3. Since f is a C^3-function with $f_x(0,0) = f_y(0,0) = f_{xy}(0,0) = 0$, we may assume

$$4(1 - f_y(x_0, y_0)s(x_0, y_0)) > f_x(x_0, y_0)^2 s(x_0, y_0)^2$$

for any $(x_0, y_0) \in U_\delta$. Hence we can apply Lemma 3.2 to $f^*(x, y) := f(x_0 + x, y_0 + y) - f(x_0, y_0)$ at $x = y = 0$. Therefore we have

$$0 = A_2 = \frac{c_0 + c_1T + c_2T^2}{1 - bs(x_0, y_0)} = \frac{C(T(x_0, y_0); x_0, y_0)}{1 - f_y(x_0, y_0)s(x_0, y_0)},$$

and the inclusion

$$(2.11) \subset \{y - y_0 = T(x_0, y_0)(x - x_0)\}$$

because $t(x_0, y_0) + s(x_0, y_0)A_1 = T(x_0, y_0)$. Further, since $T(x, y)$ is constant on line segment (2.11), we obtain

$$(\partial_x + T(x_0, y_0)\partial_y)T(x, y) = 0 \quad \text{at } (x_0, y_0).$$

Thus we obtain equations (2.14).
(ii) Conversely, let $T(x, y)$ be a real-valued C^1-function defined in a neighborhood of $(0, 0)$ satisfying equations (2.14). Take any point (x_0, y_0) close to the origin, and consider an integral curve $y = \varphi(x)$ passing through (x_0, y_0) for the vector field $\partial_x + T(x, y)\partial_y$; that is, $\varphi(x)$ is a solution to the following initial value problem for an ordinary differential equation:

$$\frac{d\varphi(x)}{dx} = T(x, \varphi(x)), \quad \varphi(x_0) = y_0.$$

Indeed, since $T(x, y)$ is of C^1-class, such a solution uniquely exists. Then, because

$$\frac{dT(x, \varphi(x))}{dx} = T_x(x, \varphi(x)) + \varphi'(x)T_y(x, \varphi(x)) = (T_x + TT_y)_{y=\varphi(x)} = 0,$$

we have $T(x, \varphi(x)) = T(x_0, y_0)$ and $\varphi(x) = y_0 + T(x_0, y_0)(x - x_0)$. Therefore we get

$$\begin{aligned}\frac{d^2}{dx^2}f(x, \varphi(x)) &= \frac{d}{dx}\big(f_x(x, \varphi(x)) + f_y(x, \varphi(x))T(x_0, y_0)\big)\\ &= 2C(T(x_0, y_0); x, \varphi(x)) = 2C(T(x, y); x, y)|_{y=\varphi(x)} = 0.\end{aligned}$$

Hence,

$$\begin{aligned}L :&= M \cap \{y - y_0 = T(x_0, y_0)(x - x_0)\}\\ &= \{\big(x, y_0 + T(x_0, y_0)(x - x_0), f(x, \varphi(x))\big)\}\end{aligned}$$

is a line segment, and $T(x, y)$ is constant on this line segment. Thus the proof of Theorem 2.3 is completed.

Before going to the proof of Theorem 2.4, we prepare some basic results on characteristic polynomials and exceptional circles.

Proposition 3.6. *Let $z = f(x, y)$ be a C^4-class function defined in a neighborhood of $(0, 0) \in \mathbb{R}^2$. We assume conditions (2.1), (2.2) for $f(x, y)$ at the origin. Then we have the following results (i) $\sim$ (iv) on the characteristic polynomial $P(t)$ for the surface germ $M = \{z = f(x, y)\}$ at the origin :*

(*i*) *$P(t)$ has the following expressions:*

$$
\begin{aligned}
P(t) &:= Z(t; 0, 0)/(c_0 - c_2) \\
&= (t^2+1)D(t)\{2t(t^2+1)D'(t) - (5t^2+1)D(t)\} \\
&\quad + 4(c_0 - c_2)t^2\{(t^2+1)E(t) - C(t)^3\} \\
&= -d_0^2 + \big(-4c_0^3(c_0 - c_2) - 6d_0^2 + d_1^2 + 2d_0d_2 + 4(c_0 - c_2)e_0\big)t^2 \\
&+ \big(-8d_0d_1 + 4d_1d_2 + 4d_0d_3 + 4(c_0 - c_2)e_1\big)t^3 \\
&+ \big(-12c_0^2(c_0 - c_2)c_2 - 5d_0^2 - 2d_1^2 - 4d_0d_2 + 3d_2^2 + 6d_1d_3 \\
&+ 4(c_0 - c_2)e_0 + 4(c_0 - c_2)e_2\big)t^4 \\
&+ \big(-8d_0d_1 + 8d_2d_3 + 4(c_0 - c_2)e_1 + 4(c_0 - c_2)e_3\big)t^5 \\
&+ \big(-12c_0(c_0 - c_2)c_2^2 - 3d_1^2 - 6d_0d_2 + 2d_2^2 + 4d_1d_3 + 5d_3^2 \\
&+ 4(c_0 - c_2)e_2 + 4(c_0 - c_2)e_4\big)t^6 \\
&+ \big(-4d_1d_2 - 4d_0d_3 + 8d_2d_3 + 4(c_0 - c_2)e_3\big)t^7 \\
&+ \big(-4(c_0 - c_2)c_2^3 - d_2^2 - 2d_1d_3 + 6d_3^2 + 4(c_0 - c_2)e_4\big)t^8 + d_3^2t^{10}.
\end{aligned}
$$

where $c_j := c_j(0, 0), C(t) := C(t; 0, 0)$, etc..

(*ii*) *Conversely, a real polynomial $P(t) = \sum_{j=0}^{10} a_jt^j$ is a characteristic polynomial of some non-umbilical surface germ $z = f(x, y)$ satisfying (2.1), (2.2) if and only if $P(t)$ satisfies the following conditions:*

$$
\begin{cases}
a_1 = a_9 = 0, \ a_5 = a_3 + a_7, \\
a_0 \le 0, \ a_{10} \ge 0, \\
a_0 - a_2 + a_4 - a_6 + a_8 - a_{10} > 0.
\end{cases}
\tag{3.11}
$$

(*iii*) *$P(t)$ is not identically zero, and so $\#\{t \in \mathbb{R}\,; P(t) = 0\} \le 10$. When $a_3 = a_7 = 0$ (and so all the odd-degree coefficients vanish), we have $\#\{t \in \mathbb{R}\,; P(t) = 0, t \ne 0\} \le 6$.*

(*iv*) *There are some examples of characteristic polynomials for non-umbilical surface germs which have 10 non-zero real distinct roots.*

Proof. (For the check of calculations, the readers can employ a mathematica file "check-fifth1" in our website written in Section 1.)

(*i*) Since $a = b = c_1 = 0$, we have

$$
\begin{aligned}
&R(t) = 1 + t^2, \quad C(t) = c_0 + c_2t^2, \\
&K(t) = 2t(c_0 + c_2t^2) - 2c_2t(t^2+1) = 2(c_0 - c_2)t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{Z(t; 0, 0)}{c_0 - c_2} &= 4(c_0 - c_2)t^2\big((t^2+1)E(t) - (c_0 + c_2t^2)^3\big) \\
&+ 2t(t^2+1)D(t)\big((t^2+1)D'(t) - 3tD(t)\big) \\
&+ \frac{1}{c_0 - c_2}(t^2+1)D(t)^2\big(-2(c_0 + c_2t^2) + (c_2 + c_0)(t^2+1)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= 4(c_0 - c_2)t^2\big((t^2+1)E(t) - (c_0 + c_2t^2)^3\big) \\
&\quad + 2t(t^2+1)D(t)\big((t^2+1)D'(t) - 3tD(t)\big) \\
&\quad + (t^2-1)(t^2+1)D(t)^2 \\
&= 4(c_0 - c_2)t^2\big((t^2+1)E(t) - (c_0 + c_2t^2)^3\big) \\
&\quad + (t^2+1)D(t)\big(2t(t^2+1)D'(t) - (5t^2+1)D(t)\big).
\end{aligned}
$$

Thus we obtain expression (2.3) of $P(t)$. Further by a Mathematica program we get the explicit form of $P(t)$ in the statement.

(ii) From the simplified expression of $P(t) = \sum_{j=0}^{10} a_j t^j$ we obtain

$$
\sum_{k=0}^{5}(-1)^k a_{2k} + \sqrt{-1}\sum_{k=0}^{4}(-1)^k a_{2k+1} = P(\sqrt{-1}) = 4(c_0 - c_2)^4 > 0.
$$

Since $a_1 = a_9 = 0, a_0 \leq 0, a_{10} \geq 0$ by the explicit form of $P(t)$, we get conditions (3.11). Conversely, assume that real numbers $a_0, ..., a_{10}$ satisfy (3.11). Put $c_0 = \big((a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})/4\big)^{1/4} > 0, c_2 = 0, d_0 = \sqrt{-a_0}, d_1 = d_2 = 0, d_3 = \sqrt{a_{10}}$. Then, by using the explicit form of $P(t)$ we can determine e_0, e_1, e_2, e_3, e_4 in turn so that the j-th degree coefficient of $P(t)$ coincides with a_j for $j = 2, 3, 4, 5, 6$. Therefore, since $a_7 = a_5 - a_3$ and $a_8 = 4(c_0 - c_2)^4 - (a_0 - a_2 + a_4 - a_6 - a_{10})$, we get $P(t) = \sum_{j=0}^{10} a_j t^j$. Thus, the characterization by conditions (3.11) is proved.

(iii) The first statement is a direct consequence of (3.11). Suppose that $a_3 = a_7 = 0$ (and so all the odd degree coefficients vanish). Put $Q(z) := a_0 + a_2z + a_4z^2 + a_6z^3 + a_8z^4 + a_{10}z^5$. Note that $a_{10} \geq 0, Q(0) \leq 0, Q(-1) > 0$. We have only to consider the following 4 cases; i) $a_{10} > 0, Q(0) < 0$, ii) $a_{10} > 0, Q(0) = 0$, iii) $a_{10} = 0, Q(0) < 0$, iv) $a_{10} = Q(0) = 0$. In case i), since $Q(-\infty) = -\infty, Q(-1) > 0, Q(0) < 0$, we have $\#\{z \in \mathbb{R};\ z < 0, Q(z) = 0\} \geq 2$. Therefore the equation $P(t) = Q(t^2) = 0$ has at least 4 non-real roots. Hence $\#\{t \in \mathbb{R};\ P(t) = 0, t \neq 0\} \leq 6$. In case ii), we have $\#\{z \in \mathbb{R};\ z < 0, Q(z) = 0\} \geq 1$ and $Q(0) = 0$. Therefore we conclude that $\#\{z \in \mathbb{R};\ z > 0, Q(z) = 0\} \leq 3$. Hence $\#\{t \in \mathbb{R};\ P(t) = 0, t \neq 0\} \leq 6$. In case iii),we have $Q(-1) > 0 > Q(0)$. Therefore, $\#\{z \in \mathbb{R};\ z < 0, Q(z) = 0\} \geq 1$. Hence, $\#\{t \in \mathbb{R};\ P(t) = 0, t \neq 0\} \leq 6$ because $P(t) = Q(t^2)$ and $\mathrm{degree}_z(Q(z)) \leq 4$. In case iv), we have $P(t) = t^2(a_2 + a_4t^2 + a_6t^4 + a_8t^6)$. Therefore our statement is clear. This completes the proof of (iii).

(iv) Thanks to (ii), we have only to find a real polynomial $P(t)$ satisfying (3.11) with $a_0 < 0, a_{10} = 1$ such that $P(t)$ has 10 non-zero real roots $x_1, ..., x_7, y_1, y_2, y_3$. Put

$$
s_1 := y_1 + y_2 + y_3, \quad s_2 := y_1y_2 + y_2y_3 + y_3y_1, \quad s_3 := y_1y_2y_3,
$$

and

$$
g_j := \sum_{1 \leq k_1 < k_2 < \cdots < k_j \leq 7} x_{k_1} \cdots x_{k_j}
$$

for $j = 1, ..., 7$. Therefore $a_0, ..., a_9$ are written as follows:

$$
\begin{aligned}
a_0 &= s_3g_7, \\
a_1 &= -(s_2g_7 + s_3g_6), \\
a_2 &= s_1g_7 + s_2g_6 + s_3g_5, \\
a_k &= (-1)^k(g_{10-k} + s_1g_{9-k} + s_2g_{8-k} + s_3g_{7-k}) \quad (3 \leq k \leq 6), \\
a_7 &= -(g_3 + s_1g_2 + s_2g_1 + s_3), \\
a_8 &= g_2 + s_1g_1 + s_2, \\
a_9 &= -(g_1 + s_1).
\end{aligned}
$$

Consequently by the three equations $0 = a_1 = a_9 = a_5 - a_3 - a_7$ we get

$$s_1 = -g_1, \quad s_2 = g_6 G_2(g)/G_1(g), \quad s_3 = -g_7 G_2(g)/G_1(g),$$

where

$$\begin{aligned}
G_1(g) &:= g_1 g_6 - g_3 g_6 + g_5 g_6 - g_7 + g_2 g_7 - g_4 g_7, \\
G_2(g) &:= g_1 g_2 - g_3 - g_1 g_4 + g_5 + g_1 g_6 - g_7.
\end{aligned}$$

Hence the two inequalities $a_0 < 0, a_0 - a_2 + a_4 - a_6 + a_8 - 1 > 0$ are written as follows:

$$\begin{aligned}
&- g_7^2 G_2(g)/G_1(g) < 0, \\
&- (g_1 g_6 - g_7)\big((g_1 - g_3 + g_5 - g_7)^2 + (1 - g_2 + g_4 - g_6)^2\big)/G_1(g) > 0.
\end{aligned}$$

Therefore conditions (3.11) are all satisfied if

$$G_1(g)G_2(g) > 0, \quad G_1(g)G_3(g) > 0$$

with $G_3(g) := g_7 - g_1 g_6$. On the other hand, y_1, y_2, y_3 are the roots of

$$0 = y^3 - s_1 y^2 + s_2 y - s_3 = \left(y - \frac{s_1}{3}\right)^3 - 3p\left(y - \frac{s_1}{3}\right) + q$$

with $p = (s_1^2 - 3s_2)/9,\ q = (9s_1 s_2 - 2s_1^3 - 27s_3)/27$. Hence y_1, y_2, y_3 are all real, and distinct from each other if $4p^3 - q^2 > 0$; that is,

$$-G_2(g)G_4(g)/(27G_1(g)^3) > 0,$$

where

$$\begin{aligned}
G_4(g) := &-g_1^4 g_2 g_6^3 + 4g_1^2 g_2^2 g_6^3 + g_1^3 g_3 g_6^3 - 8g_1 g_2 g_3 g_6^3 + g_1^3 g_2 g_3 g_6^3 \\
&+ 4g_3^2 g_6^3 - g_1^2 g_3^2 g_6^3 + g_1^4 g_4 g_6^3 - 8g_1^2 g_2 g_4 g_6^3 + 8g_1 g_3 g_4 g_6^3 - g_1^3 g_3 g_4 g_6^3 \\
&+ 4g_1^2 g_4^2 g_6^3 - g_1^3 g_5 g_6^3 + 8g_1 g_2 g_5 g_6^3 - g_1^3 g_2 g_5 g_6^3 - 8g_3 g_5 g_6^3 \\
&+ 2g_1^2 g_3 g_5 g_6^3 - 8g_1 g_4 g_5 g_6^3 + g_1^3 g_4 g_5 g_6^3 + 4g_5^2 g_6^3 - g_1^2 g_5^2 g_6^3 - g_1^4 g_6^4 \\
&+ 8g_1^2 g_2 g_6^4 - 8g_1 g_3 g_6^4 + g_1^3 g_3 g_6^4 - 8g_1^2 g_4 g_6^4 + 8g_1 g_5 g_6^4 - g_1^3 g_5 g_6^4 \\
&+ 4g_1^2 g_6^5 + 4g_1^5 g_6^2 g_7 - 17g_1^3 g_2 g_6^2 g_7 - g_1^3 g_2^2 g_6^2 g_7 + 17g_1^2 g_3 g_6^2 g_7 \\
&- 8g_1^4 g_3 g_6^2 g_7 + 19g_1^2 g_2 g_3 g_6^2 g_7 - 18g_1 g_3^2 g_6^2 g_7 + 4g_1^3 g_3^2 g_6^2 g_7 \\
&+ 17g_1^3 g_4 g_6^2 g_7 + 2g_1^3 g_2 g_4 g_6^2 g_7 - 19g_1^2 g_3 g_4 g_6^2 g_7 - g_1^3 g_4^2 g_6^2 g_7 \\
&- 17g_1^2 g_5 g_6^2 g_7 + 8g_1^4 g_5 g_6^2 g_7 - 19g_1^2 g_2 g_5 g_6^2 g_7 + 36g_1 g_3 g_5 g_6^2 g_7 \\
&- 8g_1^3 g_3 g_5 g_6^2 g_7 + 19g_1^2 g_4 g_5 g_6^2 g_7 - 18g_1 g_5^2 g_6^2 g_7 + 4g_1^3 g_5^2 g_6^2 g_7 \\
&- 16g_1^3 g_6^3 g_7 - 8g_1 g_2 g_6^3 g_7 - g_1^3 g_2 g_6^3 g_7 + 8g_3 g_6^3 g_7 + 17g_1^2 g_3 g_6^3 g_7 \\
&+ 8g_1 g_4 g_6^3 g_7 + g_1^3 g_4 g_6^3 g_7 - 8g_5 g_6^3 g_7 - 17g_1^2 g_5 g_6^3 g_7 - 8g_1 g_6^4 g_7 \\
&- 8g_1^4 g_6 g_7^2 + 45g_1^2 g_2 g_6 g_7^2 + 8g_1^4 g_2 g_6 g_7^2 - 18g_1^2 g_2^2 g_6 g_7^2 - 45g_1 g_3 g_6 g_7^2 \\
&+ 8g_1^3 g_3 g_6 g_7^2 - 9g_1 g_2 g_3 g_6 g_7^2 - 8g_1^3 g_2 g_3 g_6 g_7^2 + 27g_3^2 g_6 g_7^2 \\
&- 45g_1^2 g_4 g_6 g_7^2 - 8g_1^4 g_4 g_6 g_7^2 + 36g_1^2 g_2 g_4 g_6 g_7^2 + 9g_1 g_3 g_4 g_6 g_7^2 \\
&+ 8g_1^3 g_3 g_4 g_6 g_7^2 - 18g_1^2 g_4^2 g_6 g_7^2 + 45g_1 g_5 g_6 g_7^2 - 8g_1^3 g_5 g_6 g_7^2 \\
&+ 9g_1 g_2 g_5 g_6 g_7^2 + 8g_1^3 g_2 g_5 g_6 g_7^2 - 54g_3 g_5 g_6 g_7^2 - 9g_1 g_4 g_5 g_6 g_7^2 \\
&- 8g_1^3 g_4 g_5 g_6 g_7^2 + 27g_5^2 g_6 g_7^2 + 62g_1^2 g_6^2 g_7^2 - 17g_1^2 g_2 g_6^2 g_7^2 \\
&- 45g_1 g_3 g_6^2 g_7^2 + 17g_1^2 g_4 g_6^2 g_7^2 + 45g_1 g_5 g_6^2 g_7^2 + 4g_6^3 g_7^2 + 4g_1^3 g_7^3 \\
&- 27g_1 g_2 g_7^3 - 8g_1^3 g_2 g_7^3 + 27g_1 g_2^2 g_7^3 + 4g_1^3 g_2^2 g_7^3 + 27g_3 g_7^3 - 27g_2 g_3 g_7^3 \\
&+ 27g_1 g_4 g_7^3 + 8g_1^3 g_4 g_7^3 - 54g_1 g_2 g_4 g_7^3 - 8g_1^3 g_2 g_4 g_7^3 + 27g_3 g_4 g_7^3
\end{aligned}$$

$$
\begin{aligned}
&+27g_1g_4^2g_7^3+4g_1^3g_4^2g_7^3-27g_5g_7^3+27g_2g_5g_7^3-27g_4g_5g_7^3\\
&-72g_1g_6g_7^3+45g_1g_2g_6g_7^3+27g_3g_6g_7^3-45g_1g_4g_6g_7^3-27g_5g_6g_7^3\\
&+27g_7^4-27g_2g_7^4+27g_4g_7^4.
\end{aligned}
$$

Consequently, we have only to find some real numbers $x_1, ..., x_7$ such that

$$G_1(g)G_2(g)>0, \quad G_1(g)G_3(g)>0, \quad G_4(g)<0.$$

Indeed, we can find such examples x_j's by using "FindInstance" command in Mathematica program, though we must input at least the four values $x_1, ..., x_4$ by trial and errors (otherwise, the procedure never stopped for the seven values). In this way, we found that

$$x_1=1,\ x_2=-2, x_3=2,\ x_4=3,\ x_5=4,\ x_6=-10,\ x_7=1/10$$

satisfy $G_1(g)>0,\ G_2(g)>0,\ G_3(g)>0,\ G_4(g)<0$. Thus we obtain the corresponding values of $a_0, ..., a_{10}$:

$$
\begin{aligned}
&a_0=-\frac{382464}{609019}, \quad a_1=0, \quad a_2=\frac{2505121816}{15225475}, \quad a_3=-\frac{3662378874}{3045095},\\
&a_4=\frac{29627935751}{15225475}, \quad a_5=-\frac{5493568311}{6090190}, \quad a_6=-\frac{14502009729}{60901900},\\
&a_7=\frac{1831189437}{6090190}, \quad a_8=-\frac{4181509819}{60901900}, \quad a_9=0, \quad a_{10}=1,
\end{aligned}
$$

which satisfy $a_5-(a_3+a_7)=0,\ a_0-a_2+a_4-a_6+a_8-1>0$. Further the roots of $\sum_{j=0}^{10} a_jX^j=0$ are

$$-10,\ -2,\ 1/10,\ 1,\ 2,\ 3,\ 4,\ -0.0519793,\ 0.13882,\ 1.81316,$$

where the last three numbers are the approximation values.

□

Remark 3.7. As we stated in Definition 2.1, there are 16 choices of Euclidean coordinate systems for a non-umbilical surface germ $z=f(x,y)$ satisfying conditions (2.1), (2.2). Let (x,y,z) be such a coordinate system. Then $x'=\sigma_1x, y'=\sigma_2y, z'=\sigma_3z$ or $x'=\sigma_1y, y'=\sigma_2x, z'=\sigma_3z$ for any $\sigma_1,\sigma_2,\sigma_3\in\{\pm1\}$ are also such a system. To get all the variations of $P(t)$ for a non-umbilical surface germ, it is sufficient to see the change of $P(t)$ under 4 fundamental transformations; $(x,y,z)\to(-x,y,z), (x,y,z)\to(x,-y,z), (x,y,z)\to(x,y,-z)$, and $(x,y,z)\to(y,x,z)$. Under $x\to-x$, we have $C(t)\to C(t), D(t)\to-D(-t), D'(t)\to D'(-t), E(t)\to E(-t)$. Hence, $P(t)\to P(-t)$. Under $y\to-y$, we have $C(t)\to C(t), D(t)\to D(-t), D'(t)\to-D'(-t), E(t)\to E(-t)$. Hence, $P(t)\to P(-t)$. Under $z\to-z$, clearly $P(t)\to P(t)$. Under $(x,y)\to(y,x)$, we have $C(t)\to t^2C(1/t), D(t)\to t^3D(1/t), D'(t)=d_1+2d_2t+3d_3t^2\to d_2+2d_1t+3d_0t^2=3t^2D(1/t)-tD'(1/t), E(t)\to t^4E(1/t)$. Therefore,

$$
\begin{aligned}
P(t)\to&(t^2+1)t^3D(1/t)\big(2t(t^2+1)(3t^2D(1/t)-tD'(1/t))\\
&-(5t^2+1)t^3D(1/t)\big)+4(c_2-c_0)t^2\big((t^2+1)t^4E(1/t)-t^6C(1/t)^3\big)\\
=&-t^{10}P(1/t).
\end{aligned}
$$

Thus, all the variations of $P(t)$ are $\{P(\pm t), -t^{10}P(\pm1/t)\}$. In this sense, $P(t)$ is essentially unique for a non-umbilical surface germ.

The following lemma is concerning circles contained in the tangent plane of the surface; for example, M is a torus defined by $M:=\{(\sqrt{x^2+y^2}-R)^2+z^2=r^2\}$ and circles $C_\pm=\{z=\pm r, x^2+y^2=R^2\}$ for $0<r<R$.

Lemma 3.8. *Let $f(x,y)$ be a C^4-function defined in a neighborhood of the origin satisfying conditions* (2.1), (2.2). *Let*

$$f(x,y) = c_0x^2 + c_2y^2 + \sum_{j=0}^{3} d_j x^{3-j} y^j + \sum_{j=0}^{4} e_j x^{4-j} y^j + o(\sqrt{x^2+y^2}^4)$$

be the Taylor expansion of f at the origin. Set a surface germ $M = \{z = f(x,y)\}$ at the origin, and the tangent plane $H = \{z = 0\}$ of M at the origin. Suppose that $M \cap H$ includes a circular arc $C = \{z = 0, y = \alpha x + \beta(x^2+y^2), x^2+y^2 < \delta^2\}$ for some $\alpha, \beta, \delta \in \mathbb{R}$ with $\beta \neq 0, \delta > 0$. Then we have

$$\begin{cases} c_0 + c_2\alpha^2 = 0, \\ 2c_2\alpha\beta(1+\alpha^2) + D(\alpha) = 0, \\ c_2(1+\alpha^2)(1+5\alpha^2)\beta^2 + (1+\alpha^2)\beta D'(\alpha) + E(\alpha) = 0. \end{cases} \tag{3.12}$$

In particular we have $P(\alpha) = 0$. Further, if $\alpha \neq 0$, then β is uniquely determined by α and $D(\alpha) \neq 0, c_2 \neq 0$. If $\alpha = 0$, then we have $d_0 = c_0 = c_2\beta^2 + d_1\beta + e_0 = 0$ and $c_2 \neq 0$. Here, $D(t), D'(t), E(t), P(t)$ are the polynomials introduced in Definition 2.1.

Proof. (For the check of calculations, the readers can employ a mathematica file "check-fifth1" in our website written in Section 1.) We solve $y = \alpha x + \beta(x^2+y^2)$ with respect to y in a neighborhood of the origin as follows:

$$\begin{aligned} y(x) &= \frac{1}{2\beta}\left\{1 - \sqrt{1 - 4(\alpha\beta + \beta^2 x)x}\right\} \\ &= \alpha x + (\beta + \alpha^2\beta)x^2 + (2\alpha\beta^2 + 2\alpha^3\beta^2)x^3 \\ &\quad + (\beta^3 + 6\alpha^2\beta^3 + 5\alpha^4\beta^3)x^4 + o(x^4). \end{aligned}$$

Hence we have

$$\begin{aligned} 0 &= f(x, y(x)) \\ &= c_0x^2 + c_2\{\alpha + (\beta+\alpha^2\beta)x + (2\alpha\beta^2 + 2\alpha^3\beta^2)x^2\}^2 x^2 + d_0x^3 \\ &\quad + d_1\{\alpha + (\beta+\alpha^2\beta)x\}x^3 + d_2\{\alpha + (\beta+\alpha^2\beta)x\}^2x^3 \\ &\quad + d_3\{\alpha + (\beta+\alpha^2\beta)x\}^3x^3 \\ &\quad + (e_0 + e_1\alpha + e_2\alpha^2 + e_3\alpha^3 + e_4\alpha^4)x^4 + o(x^4) \\ &= (c_0 + c_2\alpha^2)x^2 + \big(2c_2\alpha\beta(1+\alpha^2) + D(\alpha)\big)x^3 \\ &\quad + \big(c_2(1+\alpha^2)(1+5\alpha^2)\beta^2 + (1+\alpha^2)\beta D'(\alpha) + E(\alpha)\big)x^4 + o(x^4). \end{aligned}$$

Therefore by picking up the coefficients of x^j in both sides for $j = 2,3,4$ we obtain (3.12). Multiplying the third equation of (3.12) by $4\alpha^2(1+\alpha^2)c_2$, we get

$$\begin{aligned} 0 =& (1+5\alpha^2)\big(2c_2\alpha(1+\alpha^2)\beta\big)^2 + 2\alpha(1+\alpha^2)D'(\alpha)\big(2c_2\alpha(1+\alpha^2)\beta\big) \\ &+ 4\alpha^2(c_2 + c_2\alpha^2)E(\alpha) \\ =& (1+5\alpha^2)D(\alpha)^2 - 2\alpha(1+\alpha^2)D(\alpha)D'(\alpha) - 4(c_0 - c_2)\alpha^2E(\alpha) \\ =& -P(\alpha)/(1+\alpha^2). \end{aligned}$$

Hence we have $P(\alpha) = 0$. Further, the other statements directly follow from (3.12) and $c_0 - c_2 \neq 0$. Thus the proof is completed. ☐

The proof of Theorem 2.4. After a suitable rotation, we can write $M = \{z = f(x,y)\}$ with a C^4-function $f(x,y)$ defined in a neighborhood of the origin of $\mathbb{R}^2$ satisfying conditions (2.1) and (2.2). Put

$$\mathcal{C} := \{C;\ C \text{ is a circle or a line in } \mathbb{R}^3$$

$$\text{such that } (0,0,0) \in C \cap U_\delta \subset M \text{ for a } \delta > 0\},$$

where $U_\delta := \{(x,y,z) \in \mathbb{R}^3\ ; x^2+y^2+z^2 < \delta^2\}$. We choose a tangent vector $(u_C, v_C, 0) \in \mathbb{R}^3 \setminus 0$ to C for each $C \in \mathcal{C}$. Further we choose a non-zero vector $(x_C, y_C, z_C) \in \mathbb{R}^3$ satisfying

$$\{x_C x + y_C y + z_C z = 0\} \supset C$$

for each circle $C \in \mathcal{C}$. In particular, $x_C u_C + y_C v_C = 0$. Note that $\mathcal{C}$ is decomposed into a direct sum $\bigcup_{j=1}^{9} \mathcal{C}_j$:

$$\begin{aligned}
\mathcal{C}_1 &:= \{C \in \mathcal{C}; C \text{ is a circle with } x_C \neq 0, y_C \neq 0\},\\
\mathcal{C}_2 &:= \{C \in \mathcal{C}; C \text{ is a circle with } x_C = 0, y_C \neq 0\},\\
\mathcal{C}_3 &:= \{C \in \mathcal{C}; C \text{ is a circle with } x_C \neq 0, y_C = 0\},\\
\mathcal{C}_4 &:= \{C \in \mathcal{C}; C \text{ is a circle with } x_C = y_C = 0, u_C v_C \neq 0\},\\
\mathcal{C}_5 &:= \{C \in \mathcal{C}; C \text{ is a circle with } x_C = y_C = v_C = 0\},\\
\mathcal{C}_6 &:= \{C \in \mathcal{C}; C \text{ is a circle with } x_C = y_C = u_C = 0\},\\
\mathcal{C}_7 &:= \{C \in \mathcal{C}; C \text{ is a line with } u_C v_C \neq 0\},\\
\mathcal{C}_8 &:= \{C \in \mathcal{C}; C \text{ is a line with } v_C = 0\},\\
\mathcal{C}_9 &:= \{C \in \mathcal{C}; C \text{ is a line with } u_C = 0\}.
\end{aligned}$$

Let $P(t)$ be the characteristic polynomial for the non-umbilical surface germ M at the origin. Then, we can define a mapping

$$\Phi : \mathcal{C} \ni C \mapsto v_C/u_C \in K := \{t \in \mathbb{R};\ P(t) = 0\} \cup \{\infty\},$$

where we define $\Phi(C) := \infty$ if $u_C = 0$. Indeed, for a circle $C \in \mathcal{C}_1 \cup \mathcal{C}_2$ we obtain $\Phi(C) = -x_C/y_C \in K$ by (i) of Theorem 2.2, and for a circle $C \in \mathcal{C}_4 \cup \mathcal{C}_5$ we obtain $\Phi(C) = v_C/u_C \in K$ by Lemma 3.8. Further, for a line $C \in \mathcal{C}$, $\Phi(C) = v_C/u_C \in K$ by (i) of Theorem 2.2 because $C \subset M \cap \{y = \Phi(C)x\}$. We give the proofs corresponding to the values of $d_0, d_3, c_0, c_2, e_0, e_4$. Before going to each case, we note the following facts (a)$\sim$(f):

(a) If $\mathcal{C}_2 \cup \mathcal{C}_5 \cup \mathcal{C}_8 \neq \emptyset$, then $d_0 = 0$ because $0 \in K$ and so $0 = P(0) = -d_0^2$ by Proposition 3.6. In the same way (after exchanging the roles of x and y), we have $d_3 = 0$ if $\mathcal{C}_3 \cup \mathcal{C}_6 \cup \mathcal{C}_9 \neq \emptyset$.

(b) By (i) of Theorem 2.2 we know that

$$\Phi : \mathcal{C}_1 \to K, \quad \Phi : \mathcal{C}_7 \to K$$

are injective, respectively, and that

$$\begin{aligned}
\Phi(\mathcal{C}_1) &\subset K_1 := \{t \in \mathbb{R};\ P(t) = 0, C(t) \neq 0, t \neq 0\},\\
\Phi(\mathcal{C}_7) &\subset K_2 := \{t \in \mathbb{R};\ P(t) = C(t) = D(t) = E(t) = 0, t \neq 0\}.
\end{aligned}$$

(c) By Lemma 3.8, we know that

$$\Phi : \mathcal{C}_4 \to K$$

is injective, and that

$$\Phi(\mathcal{C}_4) \subset K_3 := \{t \in \mathbb{R};\ P(t) = C(t) = 0, D(t) \neq 0, t \neq 0\}.$$

(d) If $\mathcal{C}_2 \neq \emptyset$, then $c_0 \neq 0$ and $\#(\mathcal{C}_2) \leq 2$. Indeed, let C be a circle belonging to $\mathcal{C}_2$. Then, by (i) of Theorem 2.2, $c_0 = C(0;0,0) \neq 0$, and such a C is locally written as $C = M \cap \{y = sz\}$ with some $s \in \mathbb{R}$ in a neighborhood of the origin. Therefore by (3.5) of Proposition 3.3 we have

$$c_0^2(c_2 - c_0)s^2 + c_0 d_1 s + e_0 - c_0^3 = 0$$

because $t_0 = T = a = b = c_1 = 0$ in our case. Since $c_0 \neq 0, c_0 - c_2 \neq 0$, we have $\#(\mathcal{C}_2) \leq 2$. In the same way (after changing the roles of x and y) we have that, if $\mathcal{C}_3 \neq \emptyset$, then $c_2 \neq 0$ and $\#(\mathcal{C}_3) \leq 2$.

(e) If $\mathcal{C}_5 \neq \emptyset$, then by Lemma 3.8 we know that $c_0 = 0, c_2 \neq 0$ and $\#(\mathcal{C}_5) \leq \#\{\beta \in \mathbb{R};\ c_2\beta^2 + d_1\beta + e_0 = 0, \beta \neq 0\}$. In the same way (after changing the roles of x and y) we have that, if $\mathcal{C}_6 \neq \emptyset$, then we know that $c_2 = 0, c_0 \neq 0$ and $\#(\mathcal{C}_6) \leq \#\{\beta \in \mathbb{R};\ c_0\beta^2 + d_2\beta + e_4 = 0, \beta \neq 0\}$.

(f) If $\mathcal{C}_8 \neq \emptyset$, then by (i) of Theorem 2.2 we have that $c_0 = d_0 = e_0 = 0$, and that $\#(\mathcal{C}_8) = 1$. Indeed $\#(\mathcal{C}_8) = 1$ because $C \in \mathcal{C}_8$ is just the line $\{y = z = 0\}$. In the same way (after changing the roles of x and y) we have that, if $\mathcal{C}_9 \neq \emptyset$, then we have that $c_2 = d_3 = e_4 = 0$, and that $\#(\mathcal{C}_9) = 1$.

Hereafter we enter the case-wise proofs (i)$\sim$(xii).

(i) ($d_0 d_3 \neq 0$). By (a), we have $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_8 \cup \mathcal{C}_9 = \emptyset$. Further, by (b) and (c) we have

$$\#(\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7) \leq \sum_{j=1}^{3} \#(K_j) = \#(K_1 \cup K_2 \cup K_3) \leq 10$$

because $K_1 \cup K_2 \cup K_3 \subset \{t \in \mathbb{R};\ P(t) = 0, t \neq 0\}$ and K_1, K_2, K_3 are mutually disjoint from each other. Hence $\#(\mathcal{C}) \leq 10$.

(ii) ($d_0 = 0, d_3 \neq 0, c_0 \neq 0$). By (a), we have $\mathcal{C}_3 \cup \mathcal{C}_6 \cup \mathcal{C}_9 = \emptyset$. Further, by (b) and (c) we have an injection $\Phi : \mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7 \to K_1 \cup K_2 \cup K_3$. Since $d_0 = 0$, by Proposition 3.6 we know that $P(t)$ has the form $P(t) = t^2 Q(t)$ with some polynomial $Q(t)$ of degree at most 8. Therefore,

$$\#(\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7) \leq \#(K_1 \cup K_2 \cup K_3) \leq \#\{t \in \mathbb{R};\ Q(t) = 0\} \leq 8$$

Since $c_0 \neq 0$, we have $\mathcal{C}_5 \cup \mathcal{C}_8 = \emptyset$ by (e) and (f). By (d) we have $\#(\mathcal{C}_2) \leq 2$. Hence we have $\#(\mathcal{C}) \leq 8 + 2 = 10$.

(iii) ($d_0 = 0, d_3 \neq 0, c_0 = 0, e_0 \neq 0$). By the same argument as in (ii), we have $\mathcal{C}_3 \cup \mathcal{C}_6 \cup \mathcal{C}_9 = \emptyset$ and $\#(\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7) \leq 8$. Since $c_0 = 0, e_0 \neq 0$, we have $\mathcal{C}_2 \cup \mathcal{C}_8 = \emptyset$ by (d) and (f). On the other hand, by (e) we have $\#(\mathcal{C}_5) \leq 2$. Hence we have $\#(\mathcal{C}) \leq 8 + 2 = 10$.

(iv) ($d_0 = 0, d_3 \neq 0, c_0 = e_0 = 0$). By the same argument as in (ii), we have $\mathcal{C}_3 \cup \mathcal{C}_6 \cup \mathcal{C}_9 = \emptyset$ and $\#(\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7) \leq 8$. Since $c_0 = 0$, we have $\mathcal{C}_2 = \emptyset$ by (d). On the other hand $\#(\mathcal{C}_8) \leq 1$ by (f). Further $\#(\mathcal{C}_5) \leq \#\{\beta \in \mathbb{R};\ c_2\beta^2 + d_1\beta + e_0 = 0, \beta \neq 0\} \leq 1$ by (e) because $c_2 \neq 0, e_0 = 0$. Hence we have $\#(\mathcal{C}) \leq 8 + 1 + 1 = 10$.

(v) ($d_0 \neq 0, d_3 = 0, c_2 \neq 0$). By the same argument as in (ii) (after exchanging the roles of x and y) we have $\#(\mathcal{C}) \leq 10$.

(vi) ($d_0 \neq 0, d_3 = 0, c_2 = 0, e_4 \neq 0$). By the same argument as in (iii) (after exchanging the roles of x and y) we have $\#(\mathcal{C}) \leq 10$.

(vii) ($d_0 \neq 0, d_3 = 0, c_2 = e_4 = 0$). By the same argument as in (iv) (after exchanging the roles of x and y) we have $\#(\mathcal{C}) \leq 10$.

(viii) ($d_0 = d_3 = 0, c_0 c_2 \neq 0$). By (b) and (c) we have an injection $\Phi : \mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7 \to K_1 \cup K_2 \cup K_3$. Since $d_0 = d_3 = 0$, by Proposition 3.6 we know that $P(t)$ has the form $P(t) = t^2 Q(t)$ with some polynomial $Q(t)$ of degree at most 6. Therefore,

$$\#(\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7) \leq \#(K_1 \cup K_2 \cup K_3) \leq \#\{t \in \mathbb{R};\ Q(t) = 0\} \leq 6$$

because $K_1 \cup K_2 \cup K_3 \subset \{t \in \mathbb{R};\ P(t) = 0, t \neq 0\}$ and K_1, K_2, K_3 are mutually disjoint from each other. Since $c_0 c_2 \neq 0$, we have $\mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_8 \cup \mathcal{C}_9 = \emptyset$ by (e) and (f). Further, by (d) we have $\#(\mathcal{C}_2) \leq 2, \#(\mathcal{C}_3) \leq 2$. Hence we have $\#(\mathcal{C}) \leq 6 + 2 + 2 = 10$.

(ix) ($d_0 = d_3 = 0, c_0 = 0, c_2 \neq 0, e_0 \neq 0$). By the same argument as in (viii) we have $\#(\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7) \leq 6$. Since $c_0 = 0, c_2 e_0 \neq 0$, we have $\mathcal{C}_2 \cup \mathcal{C}_6 \cup \mathcal{C}_8 \cup \mathcal{C}_9 = \emptyset$ by (d),

(e) and (f). On the other hand $\#(\mathcal{C}_3) \leq 2, \#(\mathcal{C}_5) \leq 2$ by (d) and (e). Thus we have $\#(\mathcal{C}) \leq 6+2+2 = 10$.

(x) ($d_0 = d_3 = 0, c_0 = 0, c_2 \neq 0, e_0 = 0$). By the same argument as in (viii) we have $\#(\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_7) \leq 6$. Since $c_0 = 0, c_2 \neq 0, e_0 = 0$, we have $\mathcal{C}_2 \cup \mathcal{C}_6 \cup \mathcal{C}_9 = \emptyset$ by (d), (e) and (f). On the other hand $\#(\mathcal{C}_3) \leq 2, \#(\mathcal{C}_5) = \{\beta \in \mathbb{R};\ c_2\beta^2 + d_1\beta + e_0 = 0, \beta \neq 0\} \leq 1$ by (d), (e) and $e_0 = 0$. Since $\#(\mathcal{C}_8) \leq 1$ by (f), we have $\#(\mathcal{C}) \leq 6+2+1+1 = 10$.

(xi) ($d_0 = d_3 = 0, c_0 \neq 0, c_2 = 0, e_4 \neq 0$). By the same argument as in (ix) (after exchanging the roles of x and y) we have $\#(\mathcal{C}) \leq 10$.

(xii) ($d_0 = d_3 = 0, c_0 \neq 0, c_2 = 0, e_4 = 0$). By the same argument as in (x) (after exchanging the roles of x and y) we have $\#(\mathcal{C}) \leq 10$.

There are no cases with $c_0 = c_2 = 0$ because $c_0 - c_2 \neq 0$. Thus the proof of Theorem 2.4 is completed.

4. The System of fifth order PDE's –The proofs of Theorems 2.6 and 2.7–

The proof of Theorem 2.6. (For the check of calculations, the readers can employ a mathematica file "check-fifth1" in our website written in Section 1.) To prove the former part of the statement, we have only to prove the equivalency between the fifth-order equation for f:

$$\sum_{j=0}^{5} \binom{5}{j} T^j \partial_x^{5-j}\partial_y^j f(x,y) = \frac{24N(T)}{R(T)K(T)^3}.$$

and the first order equation for T:

$$(\partial_x + T(x,y)\partial_y)T(x,y) = \frac{2S(T)}{K(T)}$$

under conditions

$$Z(T(x,y);x,y) = 0,\ T(0,0) \neq 0,\ Z'(T(0,0);0,0) \neq 0.$$

In fact, applying $(\partial_x + T(x,y)\partial_y)$ to $Z(T(x,y);x,y) = 0$, by the definition (2.16) of $N(T)$ we get

$$\begin{aligned}
0 &= Z'(T(x,y);x,y)\cdot((\partial_x + T(x,y)\partial_y))T(x,y) \\
&\quad + (\partial_x + T\partial_y)Z(T;x,y)|_{T=T(x,y)} \\
&= Z'(T(x,y);x,y)\frac{2S(T(x,y))}{K(T(x,y))} + (\partial_x + T\partial_y)Z(T;x,y)|_{T=T(x,y)} \\
&\quad + Z'(T(x,y);x,y)\left((\partial_x + T(x,y)\partial_y))T(x,y) - \frac{2S(T(x,y))}{K(T(x,y))}\right) \\
&= \frac{1}{K(T(x,y))}\left(-N(T) + \frac{K(T)^3R(T)}{24}\sum_{j=0}^{5}\binom{5}{j}T^j\partial_x^{5-j}\partial_y^j f\right)\Bigg|_{T=T(x,y)} \\
&\quad + Z'(T(x,y);x,y)\left((\partial_x + T(x,y)\partial_y))T(x,y) - \frac{2S(T(x,y))}{K(T(x,y))}\right).
\end{aligned}$$

Therefore, since $K(T(0,0);0,0) = 2(c_0(0,0) - c_2(0,0))T(0,0) \neq 0$, we get the equivalency of two equations. In order to get the explicit form of $N(T)$, from the definition of $N(T)$, we have

$$N(T) = -K(T)\left((\partial_x + T\partial_y)Z(T) - \frac{K(T)^2R(T)}{24}\sum_{j=0}^{5}\binom{5}{j}T^j\partial_x^{5-j}\partial_y^j f\right)$$

$$
\begin{aligned}
&- 2S(T)Z'(T) \\
&= - K(T)\left(L_T[Z(T)] - K(T)^2 R(T) L_T[E(T)]\right) - 2S(T)Z'(T).
\end{aligned}
$$

Hence we have only to show that

$$
N(T) + K(T)\left(L_T[Z(T)] - K(T)^2 R(T) L_T[E(T)]\right) + 2S(T)Z'(T) = 0,
$$

where $N(T)$ is the polynomial given in the statement of this theorem. We can check this equality by using a Mathematica program. Indeed, since $L_T[*]$ satisfies the Leibnitz rule

$$
L_T[F(T)G(T)] = L_T[F(T)]G(T) + F(T)L_T[G(T)],
$$

we can calculate $L_T[Z(T)]$ as a derivative in a fictional variable u; that is, after replacing $a, b, RT, RT1, CT, DT, DT1, ET, \ldots$ by $a[u], b[u], RT[u], CT[u], DT[u], DT1[u], ET[u], \ldots$ in the expression of ZT, we take the derivative of ZT in u. Here $RT, RT1, CT, DT, DT1, ET, ZT, \ldots$ are the notations for $R(T), R'(T), C(T), D(T), D'(T), E(T), Z(T), \ldots$ in the mathematica program, respectively. Then, in the expression of the derivative of ZT in u, we replace $a'[u], a[u], b'[u], b[u], RT'[u], RT[u], CT'[u], CT[u], \ldots$ by $L_T[a], a, L_T[b], b, L_T[R(T)], RT, L_T[C(T)], CT, \ldots$, respectively. Further, we can calculate $Z'(T)$ in a similar way. At the last step of the proof, we use the explicit forms of $R(T), R'(T), R''(T), C(T), C'(T), C''(T), K(T), K'(T), K''(T), D(T), D'(T), D''(T)$ as polynomials in T. In the expression of $N(T)$, it is clear that the first term $-5R(T)K(T)^2E'(T)\left[R(T)D(T) - 2(bT+a)C(T)^2\right]$ is independent from other terms. Further this term has degree 14 in T. Hence the degree in T of $N(T)$ is 14. This completes the proof.

The proof of Theorem 2.7. By Theorem 2.6 we have the following system of differential equations for $f(x,y)$:

$$
\left\{
\begin{array}{l}
Z(T_k(x,y)) = 0, \quad T_k(0,0) = t_k, \\
\displaystyle\sum_{j=0}^{5} \binom{5}{j} T_k(x,y)^j \partial_x^{5-j}\partial_y^j f(x,y) = \frac{24N(T_k(x,y))}{R(T_k(x,y))K(T_k(x,y))^3}, \\
(k = 1, 2).
\end{array}
\right.
\tag{4.1}
$$

First, we claim that this system is an analytic elliptic system of fifth-order equations for f. To do so, we note that the k-th differential equation is an analytic quasi-linear equation with the fifth order principal symbol

$$
(\xi + T_k(x,y)\eta)^5,
$$

where ξ, η are the symbols for ∂_x, ∂_y, respectively. Indeed, $T_k(x,y)$ is an analytic function of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f$ through the equation $Z(T_k(x,y); x, y) = 0$ with $Z(T_k(0,0); 0, 0) = 0$, $Z'(T_k(0,0); 0, 0) \neq 0$. Further, under $T_1(0,0) \neq T_2(0,0)$ we find that system (4.1) is elliptic because

$$
\{(\xi,\eta) \in \mathbb{R}^2;\ (\xi + T_1(x,y)\eta)^5 = (\xi + T_2(x,y)\eta)^5 = 0\} = \{(0,0)\}
$$

in a neighborhood of $x = y = 0$. Therefore, by the well-known regularity theory for nonlinear elliptic equations we have the analyticity of $f(x,y)$ at the origin. For example, we can apply the regularity theorem due to C. B. Morrey Jr. [6] to the following single (but complex-valued) quasi-linear elliptic equation for f:

$$
\begin{aligned}
&\left(\sum_{j=0}^{5} \binom{5}{j} T_1(x,y)^j \partial_x^{5-j}\partial_y^j + \sqrt{-1} \sum_{j=0}^{5} \binom{5}{j} T_2(x,y)^j \partial_x^{5-j}\partial_y^j \right) f(x,y) \\
&= \frac{24N(T_1(x,y))}{R(T_1(x,y))K(T_1(x,y))^3} + \sqrt{-1}\frac{24N(T_2(x,y))}{R(T_2(x,y))K(T_2(x,y))^3},
\end{aligned}
$$

where each $T_k(x,y)$ is written as $\tau_k(\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f)$ with some analytic function τ_k of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f$ satisfying $Z(\tau_k; x, y) = 0$. In order to show the finite dimensionality of the solution space, we introduce some differential operators $L_{k\ell}(\partial_x, \partial_y; f)$ $(k = 0, 1, ..., \ell,\ \ell = 9, 10, \ldots)$ of order ℓ as follows:

$$L_{k\ell}(\partial_x, \partial_y; f) := (\partial_x + t_1\partial_y)^{\ell-5-k}(\partial_x + t_2\partial_y)^k \sum_{j=0}^{5} \binom{5}{j} T_1(x,y)^j \partial_x^{5-j}\partial_y^j$$

for $0 \le k \le \ell - 5$, and

$$L_{k\ell}(\partial_x, \partial_y; f) := (\partial_x + t_1\partial_y)^{\ell-k}(\partial_x + t_2\partial_y)^{k-5} \sum_{j=0}^{5} \binom{5}{j} T_2(x,y)^j \partial_x^{5-j}\partial_y^j$$

for $\ell - 4 \le k \le \ell$. Here we note that the principal symbol of $L_{k\ell}(\partial_x, \partial_y; f)$ at $x = y = 0$ is

$$(\xi + t_1\eta)^{\ell-k}(\xi + t_2\eta)^k. \tag{4.2}$$

Then the solution f to (4.1) satisfies the following system of $\ell + 1$ quasilinear analytic differential equations of order ℓ:

$$\begin{cases} L_{k\ell}(\partial_x, \partial_y; f)f = \\ \quad (\partial_x + t_1\partial_y)^{\ell-5-k}(\partial_x + t_2\partial_y)^k \left(\dfrac{24N(T_1(x,y))}{R(T_1(x,y))K(T_1(x,y))^3}\right) \\ (0 \le k \le \ell - 5), \\ L_{k\ell}(\partial_x, \partial_y; f)f = \\ \quad (\partial_x + t_1\partial_y)^{\ell-k}(\partial_x + t_2\partial_y)^{k-5} \left(\dfrac{24N(T_2(x,y))}{R(T_2(x,y))K(T_2(x,y))^3}\right) \\ (\ell - 4 \le k \le \ell). \end{cases} \tag{4.3}$$

Recalling (4.2), for any $\ell \ge 9$ we can find all the derivatives

$$(\partial_x + t_1\partial_y)^k(\partial_x + t_2\partial_y)^{\ell-k} f(0,0) \quad (0 \le k \le \ell)$$

from $(\partial_x^p\partial_y^q f(0,0);\ p + q \le \ell - 1)$. Therefore, since $t_1 \ne t_2$, we can find inductively all the derivatives of f at the origin only from $(\partial_x^p\partial_y^q f(0,0);\ p + q \le 8)$. Hence we know the finite dimensionality of the solution space f. Further, by more precise arguments, we know that $(\partial_x^p\partial_y^q f(0,0); p + q \le 8)$ are determined only by A and B, where

$$\begin{aligned} A &:= (\partial_x^p\partial_y^q f(0,0);\ p + q \le 4), \\ B &:= \big((\partial_x + t_1\partial_y)^p(\partial_x + t_2\partial_y)^q f(0,0);\ p, q \le 4,\ 5 \le p + q \le 8\big). \end{aligned}$$

Indeed, since (4.3) hold for $(\ell, k) = (5,0), (5,5)$ (that is, the original equations), we can find

$$(\partial_x + t_1\partial_y)^k(\partial_x + t_2\partial_y)^{5-k} f(0,0) \quad (0 \le k \le 5)$$

from the values A and B (see Figure 4). Therefore we get the values

$$C_5 := (\partial_x^p\partial_y^q f(0,0);\ 0 \le p + q \le 5).$$

Then, since (4.3) hold for $(\ell, k) = (6,0), (6,1), (6,5), (6,6)$, we can find

$$(\partial_x + t_1\partial_y)^k(\partial_x + t_2\partial_y)^{5-k} f(0,0) \quad (0 \le k \le 6)$$

from the values A, B, C_5. Repeating this argument up to $\ell = 8$, we can determine finally

$$\partial_x^p\partial_y^q f(0,0) \quad (0 \le p + q \le 8)$$

only by A and B. Since $f(0,0) = f_x(0,0) = f_y(0,0) = f_{xy}(0,0) = 0$, and t_1, t_2 are functions of A, we know that the solution space f is classified by $(15 - 4) + 10 = 21$ real parameters.

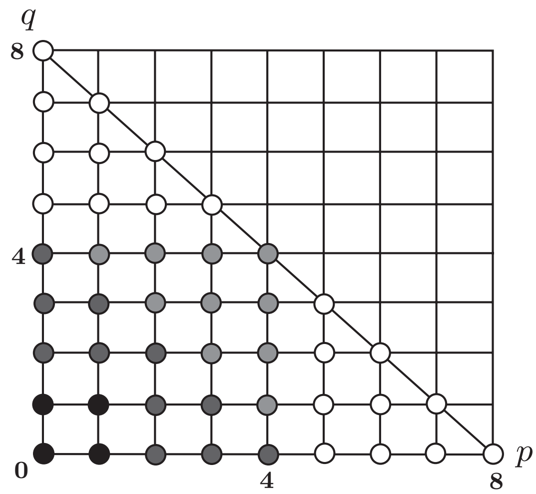

FIGURE 4. The lattice of (p, q).

5. General cyclides and the proof of Theorem 2.10

Before giving a proof of Theorem 2.10, we introduce some results on general cyclides.

Proposition 5.1. *Let $z = f(x, y)$ be a C^4-function defined in a neighborhood of the origin satisfying conditions* (2.1), (2.2). *Assume that $M = \{z = f(x, y)\}$ coincides with a general cyclide as a surface germ at the origin. Then we have*

$$e_1 = \frac{2d_0d_1 - d_1d_2 - d_0d_3}{c_0 - c_2}, \quad e_3 = \frac{-2d_2d_3 + d_1d_2 + d_0d_3}{c_0 - c_2} \tag{5.1}$$

at $(0, 0)$. *Here c_*, d_*, e_* are the derivatives of $f(x, y)$ introduced in Definition* 2.1. *Conversely, if f satisfies conditions* (5.1), *then there is a unique germ of a general cyclide M' such that the local defining function $z = g(x, y)$ of M' coincides with $z = f(x, y)$ up to the fourth-order derivatives at* $(0, 0)$.

Proof. (For the check of calculations, the readers can employ a mathematica file "check-fifth2" in our website written in Section 1.) Since the expression (1.1) is invariant under translations and rotations, we can assume under (2.1) that $z = f(x, y)$ is equivalent to the following:

$$\begin{aligned} z &= \gamma_{11}x^2 + \gamma_{22}y^2 + \gamma_{33}z^2 + 2\gamma_{31}xz + 2\gamma_{32}yz \\ &\quad + 2(x^2 + y^2 + z^2)(\beta_1 x + \beta_2 y + \beta_3 z) + \alpha(x^2 + y^2 + z^2)^2, \end{aligned} \tag{5.2}$$

where $\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{31}, \gamma_{32}, \beta_1, \beta_2, \beta_3, \alpha$ are some 9 constants. We write the Taylor expansion of $f(x, y)$ at $(0, 0)$ as follows:

$$\begin{aligned} f(x, y) &= c_0x^2 + c_2y^2 + d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3 \\ &\quad + e_0x^4 + e_1x^3y + e_2x^2y^2 + e_3xy^3 + e_4y^4 + o((x^2 + y^2)^2). \end{aligned}$$

Therefore we have

$$\begin{aligned} &c_0x^2 + c_2y^2 + d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3 \\ &+ e_0x^4 + e_1x^3y + e_2x^2y^2 + e_3xy^3 + e_4y^4 \\ &= \gamma_{11}x^2 + \gamma_{22}y^2 + \gamma_{33}(c_0x^2 + c_2y^2)^2 + 2\gamma_{31}x(c_0x^2 + c_2y^2 + d_0x^3 \\ &\quad + d_1x^2y + d_2xy^2 + d_3y^3) + 2\gamma_{32}y(c_0x^2 + c_2y^2 + d_0x^3 + d_1x^2y \\ &\quad + d_2xy^2 + d_3y^3) \end{aligned}$$

$$
\begin{aligned}
&+ 2\big(x^2 + y^2 + (c_0x^2 + c_2y^2)^2\big)\big(\beta_1 x + \beta_2 y + \beta_3(c_0x^2 + c_2y^2)\big) \\
&+ \alpha\big(x^2 + y^2 + (c_0x^2 + c_2y^2)^2\big)^2 + o((x^2 + y^2)^2).
\end{aligned}
$$

Hence by comparing the coefficients in both sides we obtain the following equations:

$$
\begin{aligned}
&c_0 = \gamma_{11}, \quad c_2 = \gamma_{22}, \\
&d_0 = 2\gamma_{31}c_0 + 2\beta_1, \quad d_1 = 2\gamma_{32}c_0 + 2\beta_2, \\
&d_2 = 2\gamma_{31}c_2 + 2\beta_1, \quad d_3 = 2\gamma_{32}c_2 + 2\beta_2, \\
&e_0 = \gamma_{33}c_0^2 + 2\gamma_{31}d_0 + 2\beta_3 c_0 + \alpha, \\
&e_1 = 2\gamma_{31}d_1 + 2\gamma_{32}d_0, \\
&e_2 = 2\gamma_{33}c_0c_2 + 2\gamma_{31}d_2 + 2\gamma_{32}d_1 + 2\beta_3(c_0 + c_2) + 2\alpha, \\
&e_3 = 2\gamma_{31}d_3 + 2\gamma_{32}d_2, \\
&e_4 = \gamma_{33}c_2^2 + 2\gamma_{32}d_3 + 2\beta_3 c_2 + \alpha.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
\gamma_{31} &= \frac{d_0 - d_2}{2(c_0 - c_2)}, \quad \beta_1 = \frac{c_0d_2 - c_2d_0}{2(c_0 - c_2)}, \\
\gamma_{32} &= \frac{d_1 - d_3}{2(c_0 - c_2)}, \quad \beta_2 = \frac{c_0d_3 - c_2d_1}{2(c_0 - c_2)}.
\end{aligned}
$$

and equations (5.1). Conversely, under (5.1) we put $\beta_1, \beta_2, \gamma_{31}, \gamma_{32}$ as above, and

$$
\begin{aligned}
\gamma_{11} &= c_0, \quad \gamma_{22} = c_2, \\
\gamma_{33} &= \frac{(c_0 - c_2)(e_0 - e_2 + e_4) - (d_0 - d_2)^2 + (d_1 - d_3)^2}{(c_0 - c_2)^3}, \\
\beta_3 &= \frac{(c_0 + c_2)e_2 - 2c_2e_0 - 2c_0e_4}{2(c_0 - c_2)^2} + \frac{2c_2d_0^2 + (c_0 + c_2)(d_2^2 - d_1^2) - 2c_0d_3^2}{2(c_0 - c_2)^3} \\
&\quad + \frac{(3c_0 + c_2)d_1d_3 - (c_0 + 3c_2)d_0d_2}{2(c_0 - c_2)^3}, \\
\alpha &= \frac{c_2^2e_0 - c_0c_2e_2 + c_0^2e_4}{(c_0 - c_2)^2} + \frac{c_0c_2(d_1^2 - d_2^2) - c_2^2d_0^2 + c_0^2d_3^2}{(c_0 - c_2)^3} \\
&\quad + \frac{(c_0 + c_2)(-c_0d_1d_3 + c_2d_0d_2)}{(c_0 - c_2)^3}.
\end{aligned}
$$

Then (5.2) is the unique cyclide which coincides with M up to the fourth order in (x, y) at $x = y = 0$. □

Proposition 5.2. *We inherit the notation $M, f(x, y)$ with the assumptions from Proposition 5.1. Then the characteristic polynomial $P(t)$ at the origin for M has the following form:*

$$
\begin{aligned}
P(t) &= -d_0^2 + (-4c_0^4 + 4c_0^3c_2 - 6d_0^2 + d_1^2 + 2d_0d_2 + 4c_0e_0 - 4c_2e_0)t^2 \\
&\quad + \big(-12c_0^3c_2 + 12c_0^2c_2^2 - 5d_0^2 - 2d_1^2 - 4d_0d_2 + 3d_2^2 + 6d_1d_3 - 4c_2e_0 \\
&\quad - 4c_2e_2 + 4c_0(e_0 + e_2)\big)t^4 + \big(-12c_0^2c_2^2 - 3d_1^2 - 6d_0d_2 + 2d_2^2 \\
&\quad + 4d_1d_3 + 5d_3^2 - 4c_2e_2 - 4c_2e_4 + 4c_0(3c_2^3 + e_2 + e_4)\big)t^6 \\
&\quad + (-4c_0c_2^3 + 4c_2^4 - d_2^2 - 2d_1d_3 + 6d_3^2 + 4c_0e_4 - 4c_2e_4)t^8 + d_3^2t^{10},
\end{aligned}
$$

where c_0, c_2, d_*, e_* mean their values at $x = y = 0$. In particular, we have $\#\{t \in \mathbb{R};\ P(t) = 0, t \neq 0\} \leq 6$.

Proof. (For the check of calculations, the readers can employ a mathematica file "check-fifth2" in our website written in Section 1.) This form is directly obtained from the explicit one in Proposition 3.6 and conditions (5.1). Further, since all the odd degree terms of $P(t)$ vanish, we have $\#\{t \in \mathbb{R};\ P(t) = 0, t \neq 0\} \leq 6$ by Proposition 3.6. $\square$

Example 5.3. Let M be a six-circle Blum cyclide

$$(x^2 + y^2 + z^2)^2 - 2ax^2 - 2by^2 - 2cz^2 + d^2 = 0, \tag{5.3}$$

such that a, b, c, d are real numbers satisfying $a > c > d > 0$, $b < -d$. Then the characteristic polynomial at $(0, 0, \sigma_2\sqrt{c + \sigma_1\sqrt{c^2 - d^2}})$ with $\sigma_1 = \pm 1, \sigma_2 = \pm 1$ is given by

$$\begin{aligned} P(t) = &\frac{(a-b)(b-c)(d-b)(d+b)}{4(c^2-d^2)^2(c+\sigma_1\ell)^2} t^2 \\ &\times \left(t^2 - \frac{a-c}{c-b}\right)\left(t^2 - \frac{a-d}{d-b}\right)\left(t^2 - \frac{a+d}{-b-d}\right). \end{aligned}$$

Indeed, since z^2 satisfies

$$z^2 = c - x^2 - y^2 + \sigma_1\sqrt{c^2 - d^2 + 2(a-c)x^2 + 2(b-c)y^2}$$

in a neighborhood of $(0, 0)$, we have

$$\begin{aligned} z &= \sigma_2\sqrt{c - x^2 - y^2 + \sigma_1\sqrt{c^2 - d^2 + 2(a-c)x^2 + 2(b-c)y^2}} \\ &= \sigma_2\Big(c - x^2 - y^2 + \sigma_1\ell\Big(1 + \frac{(a-c)x^2 + (b-c)y^2}{\ell^2} \\ &\quad - \frac{((a-c)x^2 + (b-c)y^2)^2}{2\ell^4} + o((x^2+y^2)^2)\Big)\Big)^{1/2} \\ &= \sigma_2\sqrt{c + \sigma_1\ell}\Big(1 + \frac{(a - c - \sigma_1\ell)x^2 + (b - c - \sigma_1\ell)y^2}{\sigma_1\ell(c + \sigma_1\ell)} \\ &\quad - \frac{((a-c)x^2 + (b-c)y^2)^2}{2\sigma_1\ell^3(c + \sigma_1\ell)} + o((x^2+y^2)^2)\Big)^{1/2} \\ &= \sigma_2\sqrt{c + \sigma_1\ell}\Big(1 + \frac{(a - c - \sigma_1\ell)x^2 + (b - c - \sigma_1\ell)y^2}{2\sigma_1\ell(c + \sigma_1\ell)} \\ &\quad - \frac{((a-c)x^2 + (b-c)y^2)^2}{4\sigma_1\ell^3(c + \sigma_1\ell)} \\ &\quad - \frac{\big((a - c - \sigma_1\ell)x^2 + (b - c - \sigma_1\ell)y^2\big)^2}{8\ell^2(c + \sigma_1\ell)^2} + o((x^2+y^2)^2)\Big), \end{aligned}$$

where $\ell = \sqrt{c^2 - d^2}$. Therefore $f(x, y) := z(x, y) - \sigma_2\sqrt{c + \sigma_1\ell}$ satisfies conditions (2.1), (2.2), and so we have the following Taylor coefficients at the origin:

$$\begin{aligned} c_0 &= \frac{\sigma_2\left(\sigma_1(a-c) - \ell\right)}{2\ell\sqrt{c + \sigma_1\ell}}, \quad c_2 = \frac{\sigma_2\left(\sigma_1(b-c) - \ell\right)}{2\ell\sqrt{c + \sigma_1\ell}}, \\ e_0 &= -\sigma_2\,\frac{2\sigma_1(a-c)^2(c + \sigma_1\ell) + \ell(a - c - \sigma_1\ell)^2}{8\ell^3\sqrt{c + \sigma_1\ell}^3}, \\ e_2 &= -\sigma_2\,\frac{2\sigma_1(a-c)(b-c)(c + \sigma_1\ell) + \ell(a - c - \sigma_1\ell)(b - c - \sigma_1\ell)}{4\ell^3\sqrt{c + \sigma_1\ell}^3}, \\ e_4 &= -\sigma_2\,\frac{2\sigma_1(b-c)^2(c + \sigma_1\ell) + \ell(b - c - \sigma_1\ell)^2}{8\ell^3\sqrt{c + \sigma_1\ell}^3}, \end{aligned}$$

and all the other coefficients up to the fourth order vanish at the origin. Hence the characteristic polynomial at $(0,0,\sigma_2\sqrt{c+\sigma_1\sqrt{c^2-d^2}})$ is

$$\begin{aligned}P(t) &= 4(c_0-c_2)t^2\Big((t^2+1)(e_0+e_2t^2+e_4t^4)-(c_0+c_2t^2)^3\Big)\\ &= \frac{(a-b)(b-c)(d-b)(d+b)}{4(c^2-d^2)^2(c+\sigma_1\ell)^2}t^2\\ &\quad\times\left(t^2-\frac{a-c}{c-b}\right)\left(t^2-\frac{a-d}{d-b}\right)\left(t^2-\frac{a+d}{-b-d}\right).\end{aligned}$$

In particular we have the 6 non-zero real roots for $P(t)=0$. (For the check of calculations, the readers can employ a mathematica file "check-fifth2" in our website written in Section 1.)

In K-T [4] (also see Blumcircles-parameter.pdf, Blumcircles-parameter.nb in our website written in Section 1), we obtained the concrete expressions of the six continuous families of circles on some Blum 6-circle cyclides with the explicit forms of $T_j(x,y)$ $(j=1,\ldots,6)$, which are given as follows:

Proposition 5.4. *Let M be the six-circle Blum cyclide given at* (5.3). *Then for each non-zero real root of $P(t)=0$, we have a continuous family of circles in a neighborhood of $(0,0,\sigma_2\sqrt{c+\sigma_1\sqrt{c^2-d^2}})$ with $\sigma_1,\sigma_2=\pm1$. In fact, we have the six non-zero real roots:*

$$\pm\sqrt{\frac{a-c}{c-b}},\quad \pm\sqrt{\frac{a-d}{d-b}},\quad \pm\sqrt{\frac{a+d}{-b-d}}.$$

(i) *For the characteristic roots $\pm\sqrt{(a-c)/(c-b)}$, with $v_1=\sqrt{(a-c)(c-b)(c^2-d^2)}$ we have*

$$t_{1,\pm}(x,y)=T_{1,\pm}(x,y),\quad s_{1,\pm}(x,y)=0,$$

and

$$T_{1,\pm}(x,y):= \frac{-2(a-c)(c-b)xy\mp v_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}}{(b-c)\big(c^2-d^2+2(a-c)x^2\big)}.$$

(ii) *For the characteristic roots $\pm\sqrt{(a-d)/(d-b)}$, with $v_2=\sqrt{2(a-d)(d-b)(c-d)}$ we have*

$$\begin{aligned}&t_{2,\pm}(x,y):=\\ &\Big\{2(b-c)(a-d)xyz\mp\sigma_1\sigma_2v_2\big(-by^2-cz^2\\ &+(x^2+y^2+z^2)(y^2+z^2)\big)\Big\}\Big/\Big\{(d-b)z\big((d-c)(d+x^2+y^2\\ &+z^2)-2(a-c)x^2\big)\mp\sigma_1\sigma_2v_2xy(x^2+y^2+z^2-b)\Big\},\\ &s_{2,\pm}(x,y):=\\ &\Big\{(c-d)y\big((b-d)(d+x^2+y^2+z^2)+2(a-b)x^2\big)\\ &\pm\sigma_1\sigma_2v_2xz(x^2+y^2+z^2-c)\Big\}\Big/\Big\{(d-b)z\big((d-c)(d+x^2+y^2\\ &+z^2)-2(a-c)x^2\big)\mp\sigma_1\sigma_2v_2xy(x^2+y^2+z^2-b)\Big\}.\end{aligned}$$

Further $T_{2,\pm}(x,y)=P_{2,\pm}(x,y)/Q_2(x,y)$ with

$$P_{2,\pm}(x,y):=$$

$$
\begin{aligned}
&xy\Big\{ -(c^2-d^2)\big(3ab-2ac-2bc-(a+b-4c)d-d^2\big)\\
&+2(a-c)\big(-2ab+ac+2bc+(a-3c)d+d^2\big)x^2\\
&+2(b-c)\big(-2ab+2ac+bc+(b-3c)d+d^2\big)y^2\\
&+\sigma_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}\,\big(2c^2(a+b)-3abc\\
&+(ab+ac+bc-4c^2)d\\
&-(a+b-c)d^2+d^3+2(a-c)(b-d)x^2+2(b-c)(a-d)y^2\big)\Big\}\\
&\pm\sigma_1 v_2\sqrt{c-x^2-y^2+\sigma_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}}\\
&\times\Big\{\sigma_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}\,\big(c^2-d^2+(a-c)x^2\\
&+(b-c)y^2\big)+c\big(c^2-d^2+2(a-c)x^2+2(b-c)y^2\big)\Big\},\\
&Q_2(x,y):=\\
&-(b-d)(c-d)^2(2c+d)(c+d)\\
&+(b-d)(d-c)\big(6ac-7c^2+4(a-c)d+d^2\big)x^2\\
&-(b-c)(b-d)(c-d)(3c+d)y^2-2(a-c)(b-d)(2a-3c+d)x^4\\
&+2(b-c)\big(-2ab+ac+2bc+(a-3c)d+d^2\big)x^2y^2\\
&-\sigma_1\sqrt{c^2-d^2+2(a-c)x^2+2(b-c)y^2}\\
&\times\big\{(b-d)(c-d)(2c-d)(c+d)\\
&+(b-d)\big(4ac-5c^2-2(a-c)d+d^2\big)x^2+(b-c)(b-d)(c-d)y^2\\
&-2(a-c)(b-d)x^4-2(b-c)(a-d)x^2y^2\big\}.
\end{aligned}
$$

(iii) *For the characteristic roots* $\pm\sqrt{(a+d)/(-d-b)}$, *setting* $v_3=\sqrt{2(a+d)(-b-d)(c+d)}$ *we have a similar expression to* (ii) (only replace d in (ii) by $-d$). We omit the detailed forms of $T_{3,\pm}(x,y)$.

Lemma 5.5. *Let* $Q(t)=\sum_{k=0}^n q_k t^k$ *be a polynomial in* t *of degree* $n(>0)$ *with coefficients* $q_0,...,q_n\in\mathbb{C}$ $(q_n\neq 0)$. *Suppose that* $Q(t)=0$ *has* n *separate roots* $t_1,...,t_n\in\mathbb{C}$. *Let* $A(t)$ *be a polynomial in* t *with complex coefficients, and* $s_1,...,s_m$ *be* $m(>0)$ *different complex numbers such that* $\{s_p;p=1,...,m\}\cap\{t_j;j=1,...,n\}=\emptyset$. *Then the solution* $(g_0,...,g_{n-1})\in\mathbb{C}^n$ *to the system*

$$
\sum_{k=0}^{n-1}(t_j)^k g_k=\left.\frac{A(t)}{\prod_{p=1}^m(t-s_p)}\right|_{t=t_j}\qquad (j=1,...,n)\tag{5.4}
$$

is given by

$$
g_k=\alpha_k+\sum_{p=1}^m\beta_p\gamma_{k,p}\tag{5.5}
$$

for $k=0,...,n-1$. *Here* β_p *is the residue of* $A(t)/(\prod_{r=1}^m(t-s_p))$ *at* $t=s_p$ *given by*

$$
\beta_p:=\frac{A(s_p)}{\prod_{r\neq p}(s_p-s_r)},
$$

and α_k*'s are the coefficients of the remainder* $\sum_{k=0}^{n-1}\alpha_k t^k$ *of the division* $B(t)/Q(t)$ *for the polynomial*

$$
B(t):=\frac{A(t)}{\prod_{p=1}^m(t-s_p)}-\sum_{p=1}^m\frac{\beta_p}{t-s_p}.
$$

Further, the coefficient $\gamma_{k,p}$ *at* (5.5) *is given by*

$$\gamma_{k,p} := -\sum_{\ell=0}^{n-k-1} q_{\ell+k+1}(s_p)^{\ell} \Big/ Q(s_p),$$

which satisfies

$$1/(t_j - s_p) = \sum_{k=0}^{n-1} \gamma_{k,p}(t_j)^k$$

for any $j = 1, ..., n$.

Proof. For any p $(1 \leq p \leq m)$ and any j $(1 \leq j \leq n)$ we have

$$\frac{1}{t_j - s_p} = -\frac{Q(t_j) - Q(s_p)}{(t_j - s_p)Q(s_p)} = \frac{1}{Q(s_p)} R(t_j; s_p)$$

with $R(t; s) := -\sum_{k=0}^{n-1}(\sum_{\ell=0}^{n-k-1} q_{\ell+k+1}s^{\ell})t^k$, and so

$$\gamma_{k,p} = -\sum_{\ell=0}^{n-k-1} q_{\ell+k+1}(s_p)^{\ell} \Big/ Q(s_p).$$

Further we have the following unique expression:

$$\frac{A(t)}{\prod_{p=1}^{m}(t - s_p)} = U(t)Q(t) + \sum_{k=0}^{n-1} \alpha_k t^k + \sum_{p=1}^{m} \frac{\beta_p}{t - s_p}$$

with some polynomial $U(t)$ and some $\alpha_0, ..., \alpha_{n-1}, \beta_1, ..., \beta_m \in \mathbb{C}$. Indeed we have

$$\beta_p = A(s_p) \Big/ \left(\prod_{r \neq p} (s_p - s_r) \right),$$

and so $\sum_{k=0}^{n-1} \alpha_k t^k$ is the remainder of the division

$$\left(\frac{A(t)}{\prod_{p=1}^{m}(t - s_p)} - \sum_{p=1}^{m} \frac{\beta_p}{t - s_p} \right) \Big/ Q(t).$$

Consequently for $t = t_j$ we get the following:

$$\begin{aligned} \frac{A(t_j)}{\prod_{p=1}^{m}(t_j - s_p)} &= \sum_{k=0}^{n-1} \alpha_k (t_j)^k + \sum_{p=1}^{m} \frac{\beta_p}{t_j - s_p} \\ &= \sum_{k=0}^{n-1} \alpha_k (t_j)^k + \sum_{p=1}^{m} \beta_p \left(\sum_{k=0}^{n-1} \gamma_{k,p}(t_j)^k \right) \\ &= \sum_{k=0}^{n-1} (t_j)^k \left(\alpha_k + \sum_{p=1}^{m} \beta_p \gamma_{k,p} \right). \end{aligned}$$

Thus we have a solution $(g_0, ..., g_{n-1})$ to (5.4) given by

$$g_k = \alpha_k + \sum_{p=1}^{m} \beta_p \gamma_{k,p}.$$

The uniqueness follows from

$$\det\left((t_j)^{k-1}; j, k = 1, ..., n\right) = \prod_{j>k} (t_j - t_k) \neq 0.$$

□

The proof of Theorem 2.10. By Theorem 2.6 we have the following system of 6 differential equations for $f(x,y)$:

$$\left\{\begin{array}{l} Z(T_k(x,y))=0, \quad T_k(0,0)=t_k, \\ \displaystyle\sum_{j=0}^{5} T_k(x,y)^j \left(\binom{5}{j}\partial_x^{5-j}\partial_y^j f(x,y)\right) = \frac{24N(T_k(x,y))}{R(T_k(x,y))K(T_k(x,y))^3} \\ (1\le k\le 6). \end{array}\right. \tag{5.6}$$

Since

$$\det\left(T_j(0,0)^{k-1}; j,k=1,...,6\right) = \prod_{j>k}(t_j-t_k)\neq 0,$$

the coefficient matrix $\left(T_j(x,y)^{k-1}; j,k=1,...,6\right)$ is invertible in a neighborhood of $x=y=0$. Therefore we can rewrite (5.6) as follows:

$$\left\{\begin{array}{l} Z(T_{j+1}(x,y))=0, \quad T_{j+1}(0,0)=t_{j+1}, \\ \partial_x^{5-j}\partial_y^j f(x,y) = G_j(\nabla f,\nabla^2 f,\nabla^3 f,\nabla^4 f,T_1,...,T_6), \\ (0\le j\le 5), \end{array}\right. \tag{5.7}$$

where G_j $(j=0,...,5)$ are analytic functions of $\nabla f,\nabla^2 f,\nabla^3 f,\nabla^4 f,T_1,...,T_6$. In particular, we know that f is a C^6-class function in a neighborhood of the origin. Hence f satisfies all the assumptions in Theorem 2.7, and so we obtain the analyticity of $f(x,y)$ at $x=y=0$. On the other hand, it is easy to see that all the Taylor coefficients at the origin of $f(x,y)$ are determined successively by the Taylor coefficients c_0,c_2,d_*,e_* at the origin. Thus the proof of the former part of Theorem 2.10 is completed. Now we suppose the additional conditions

$$d_0=d_1=d_2=d_3=0$$

at $(0,0)$. Since $a=b=c_1=d_0=d_1=d_2=d_3=0$ at $(0,0)$, by the explicit forms of $N(T),B_{10}(T)$ in Theorem 2.6 we get $N(T;0,0)=0$. Hence by equations (5.6) we obtain

$$\partial_x^{5-j}\partial_y^j f(0,0)=0 \quad (0\le j\le 5).$$

To get some necessary conditions on the values $c_0,c_2,e_0,...,e_4$ at the origin for a solution $f(x,y)$, we find sixth-order derivatives of f at the origin by using the differential equations (5.6). Precisely speaking, we find the following coefficients g_j,h_j $(j=0,1,...,5)$

$$\frac{\partial_x^{5-j}\partial_y^j f(x,y)}{(5-j)!j!} = g_j x + h_j y + O(x^2+y^2)$$

as $x,y\to 0$. Since all the fifth order derivatives of f vanish at the origin, concerning a,b,c_*,d_*,e_* we have

$$\begin{aligned}
a &= f_x(x,y)\equiv f_{xx}(0,0)x+f_{xy}(0,0)y\equiv 2c_0^*x,\\
b &= f_y(x,y)\equiv f_{xy}(0,0)x+f_{yy}(0,0)y\equiv 2c_2^*y,\\
c_0 &= f_{xx}(x,y)/2\equiv c_0^*+(f_{xxx}(0,0)x+f_{xxy}(0,0)y)/2\equiv c_0^*,\\
c_1 &= f_{xy}(x,y)\equiv c_1^*+f_{xxy}(0,0)x+f_{xyy}(0,0)y\equiv 0,\\
c_2 &= f_{yy}(x,y)/2\equiv c_2^*+(f_{xyy}(0,0)x+f_{yyy}(0,0)y)/2\equiv c_2^*,\\
d_0 &= f_{xxx}(x,y)/6\equiv (f_{xxxx}(0,0)x+f_{xxxy}(0,0)y)/6\equiv 4e_0^*x+e_1^*y,\\
d_1 &= f_{xxy}(x,y)/2\equiv (f_{xxxy}(0,0)x+f_{xxyy}(0,0)y)/2\equiv 3e_1^*x+2e_2^*y,\\
d_2 &= f_{xyy}(x,y)/2\equiv (f_{xxyy}(0,0)x+f_{xyyy}(0,0)y)/2\equiv 2e_2^*x+3e_3^*y,\\
d_3 &= f_{yyy}(x,y)/6\equiv (f_{xyyy}(0,0)x+f_{yyyy}(0,0)y)/6\equiv e_3^*x+4e_4^*y,\\
e_0 &= f_{xxxx}(x,y)/24\equiv e_0^*+(f_{xxxxx}(0,0)x+f_{xxxxy}(0,0)y)/24\equiv e_0^*,\\
e_1 &= f_{xxxy}(x,y)/6\equiv e_1^*+(f_{xxxxy}(0,0)x+f_{xxxyy}(0,0)y)/6\equiv e_1^*,
\end{aligned}$$

$$e_2 = f_{xxyy}(x,y)/4 \equiv e_2^* + (f_{xxxyy}(0,0)x + f_{xxyyy}(0,0)y)/4 \equiv e_2^*,$$
$$e_3 = f_{xyyy}(x,y)/6 \equiv e_3^* + (f_{xxyyy}(0,0)x + f_{xyyyy}(0,0)y)/6 \equiv e_3^*,$$
$$e_4 = f_{yyyy}(x,y)/24 \equiv e_4^* + (f_{xyyyy}(0,0)x + f_{yyyyy}(0,0)y)/24 \equiv e_4^*.$$

Here, c_j^*, e_j^* mean their values at the origin, and "$A \equiv 0$" means "$A = O(x^2+y^2)$ as $x, y \to 0$". Concerning $T(x,y)$, noting $K(T) \equiv 2T(c_0^* + c_2^*T^2) - 2c_2^*T(T^2+1) \equiv 2T(c_0^* - c_2^*)$, we have

$$\begin{aligned} Z(T) &\equiv K(T)^2\big(R(T)E(T) - C(T)^3\big) \\ &\equiv 4(c_0^* - c_2^*)^2T^2\Big((T^2+1)\sum_{j=0}^{4} e_j^*T^j - (c_2^*T^2 + c_0^*)^3\Big). \end{aligned}$$

Therefore, since $Z'(T) \neq 0$, we obtain

$$T_j(x,y) \equiv t_j \ (j = 1, ..., 6).$$

From now on, c_*, e_* mean their values at $x = y = 0$. By Theorem 2.6 we have the following: (For the check of calculations, the readers can employ a mathematica file "check-fifth2" in our website written in Section 1.)

$$\begin{aligned} &N(T) \equiv \\ &-5R(T)K(T)^2E'(T)\big[R(T)D(T) - 2(bT+a)C(T)^2\big] \\ &+ D(T)B_5(T) + 4C(T)^4B_{10}(T) \\ &\equiv -5R(T)K(T)^2E'(T)\big[R(T)D(T) - 2(bT+a)C(T)^2\big] \\ &+ D(T)\big(9C(T)^2K(T)^3 + K(T)^2C(T)^2(6R'(T)C(T) + 4R(T)C'(T))\big) \\ &+ 4C(T)^4\Big[K(T)^2\big\{2bC(T) - 4(bT+a)C'(T)\big\} \\ &- 2(bT+a)C(T)K(T)K'(T) + K(T)\big\{-4(bT+a)C(T)\big(-c_2+c_0\big) \\ &- 4a(c_2-c_0)C(T) - 2(bT+a)(c_2-c_0)(2c_0)\big\}\Big] \\ &\equiv -20T^2(T^2+1)(c_0-c_2)^2E'(T)[(T^2+1)D(T) \\ &- 2(bT+a)(c_0+c_2T^2)^2] + 4T^2(c_0-c_2)^2D(T)(c_0+c_2T^2)^2 \\ &\times \big(18T(c_0-c_2) + 12T(c_0+c_2T^2) + 8c_2T(T^2+1)\big) \\ &+ 4(c_0+c_2T^2)^4\Big[4T^2(c_0-c_2)^2(2b(c_0+c_2T^2) - 8c_2T(bT+a)) \\ &- 8T(c_0-c_2)^2(bT+a)(c_0+c_2T^2) + 2T(c_0-c_2)^2(-4bT(c_0+c_2T^2) \\ &+ 4c_0(bT+a))\Big] \\ &\equiv -20T^2(T^2+1)(c_0-c_2)^2E'(T)[(T^2+1)D(T) \\ &- 2(bT+a)(c_0+c_2T^2)^2] + 40T^3(c_0-c_2)^2D(T)(c_0+c_2T^2)^2 \\ &\times \big(3c_0 + c_2(2T^2-1)\big) - 160T^3c_2(c_0-c_2)^2(c_0+c_2T^2)^4(bT+a) \\ &\equiv 20(c_0-c_2)^2\Big[T^2\Big\{-(T^2+1)^2E'(T) + 2TC(T)^2\big(2C(T) + c_0 - c_2\big)\Big\} \\ &\times \Big(\big(4E(T) - TE'(T)\big)x + E'(T)y\Big) \\ &+ 4T^2C(T)^2\Big\{(T^2+1)E'(T) - 4c_2TC(T)^2\Big\}(c_2Ty + c_0x)\Big] \\ &\equiv 20(c_0-c_2)^2T^2\big(A_1(T)x + A_2(T)y\big), \end{aligned}$$

where we used $D(T) \equiv (4E(T) - TE'(T))x + E'(T)y,\ bT + a \equiv 2(c_2Ty + c_0x)$, and

$$A_1(T) := \Big\{ -(T^2+1)^2E'(T) + 2TC(T)^2\big(2C(T) + c_0 - c_2\big)\Big\}$$
$$\times \big(4E(T) - TE'(T)\big) + 4c_0C(T)^2\Big\{(T^2+1)E'(T) - 4c_2TC(T)^2\Big\},$$
$$A_2(T) := E'(T)\Big\{ -(T^2+1)^2E'(T) + 2TC(T)^2\big(2C(T) + c_0 - c_2\big)\Big\}$$
$$+ 4c_2TC(T)^2\Big\{(T^2+1)E'(T) - 4c_2TC(T)^2\Big\}.$$

Since we have the equations

$$\sum_{k=0}^{5}(t_j)^k(g_kx + h_ky) \equiv \frac{N(t_j)}{5R(t_j)K(t_j)^3} \equiv \frac{N(t_j)}{40(c_0 - c_2)^3t_j^3(t_j^2+1)}$$

for $j = 1, 2, ..., 6$, we get the following equations for g_j, h_j:

$$\sum_{k=0}^{5}(t_j)^kg_k = \frac{A_1(t_j)}{2(c_0 - c_2)t_j(t_j^2+1)}, \quad \sum_{k=0}^{5}(t_j)^kh_k = \frac{A_2(t_j)}{2(c_0 - c_2)t_j(t_j^2+1)}$$

for $j = 1, 2, ..., 6$. On the other hand, since $\{t_j\}$ are the non-zero real roots of the characteristic polynomial $P(t)$, we have $Q(t_j) = 0$ for $j = 1, ..., 6$ with

$$Q(t) = \sum_{j=0}^{6} q_jt^j := (t^2+1)E(t) - C(t)^3$$
$$= (e_4 - c_2^3)t^6 + e_3t^5 + (e_4 + e_2 - 3c_0c_2^2)t^4 + (e_3 + e_1)t^3$$
$$+ (e_2 + e_0 - 3c_0^2c_2)t^2 + e_1t + e_0 - c_0^3.$$

Therefore we can find $(g_0, ..., g_5), (h_0, ..., h_5)$ by using Lemma 5.5 and some Mathematica program for $n = 6, m = 3, s_1 = 0, s_2 = i(= \sqrt{-1}), s_3 = -i$,

$$q_0 = e_0 - c_0^3, \quad q_1 = e_1, \quad q_2 = e_2 + e_0 - 3c_0^2c_2,$$
$$q_3 = e_3 + e_1, \quad q_4 = e_4 + e_2 - 3c_0c_2^2, \quad q_5 = e_3, \quad q_6 = e_4 - c_2^3,$$

and $A(t) = A_1(t)/(2(c_0 - c_2))$, or $A_2(t)/(2(c_0 - c_2))$, respectively. As for $\gamma_{k,p}$, we have

$$\gamma_{k,1} = -\sum_{\ell=0}^{5-k}\frac{q_{\ell+k+1}0^\ell}{Q(0)} = -q_{k+1}/q_0,$$
$$\gamma_{k,2} = -\sum_{\ell=0}^{5-k}\frac{q_{\ell+k+1}i^\ell}{Q(i)} = -\frac{q_{k+1} + q_{k+2}i + \cdots + q_6i^{5-k}}{q_0 - q_2 + q_4 - q_6 + (q_1 - q_3 + q_5)i},$$
$$\gamma_{k,3} = -\sum_{\ell=0}^{5-k}\frac{q_{\ell+k+1}(-i)^\ell}{Q(-i)} = -\frac{q_{k+1} - q_{k+2}i + \cdots + q_6(-i)^{5-k}}{q_0 - q_2 + q_4 - q_6 - (q_1 - q_3 + q_5)i}$$

for $k = 0, 1, ..., 5$.

Case $A(t) = A_1(t)/(2(c_0 - c_2))$**.**

$$\beta_1 = \frac{A_1(0)}{2(c_0 - c_2)(-i)(i)} = \frac{A_1(0)}{2(c_0 - c_2)},$$
$$\beta_2 = \frac{A_1(i)}{2(c_0 - c_2)(i)(2i)} = -\frac{A_1(i)}{4(c_0 - c_2)},$$
$$\beta_3 = \frac{A_1(-i)}{2(c_0 - c_2)(-i)(-2i)} = -\frac{A_1(-i)}{4(c_0 - c_2)}.$$

We can calculate α_k $(k = 0, ..., 5)$, by Mathematica's polynomial remainder program for

$$\frac{A_1(t)}{2(c_0 - c_2)t(t^2+1)} - \frac{\beta_1}{t} - \frac{\beta_2}{t-i} - \frac{\beta_3}{t+i}.$$

Consequently we have

$$
\begin{aligned}
g_0 &= \alpha_0 + \beta_1\gamma_{0,1} + \beta_2\gamma_{0,2} + \beta_3\gamma_{0,3} \\
&= \frac{1}{2(c_0 - c_2)(c_2^3 - e_4)}\Big[c_2^3(24e_0^2 + e_1^2 - 12e_0e_2) + e_0e_3^2 - 24c_0^2c_2e_0(c_2^3 - e_4) \\
&- 24e_0^2e_4 - e_1^2e_4 + 12e_0e_2e_4 + c_0^3(12c_2^3e_2 - e_3^2 - 12e_2e_4)\Big] \\
g_1 &= \alpha_1 + \beta_1\gamma_{1,1} + \beta_2\gamma_{1,2} + \beta_3\gamma_{1,3} \\
&= \frac{1}{2(c_0 - c_2)(c_2^3 - e_4)}\Big[e_1e_3^2 + 2c_2^3(21e_0e_1 - 4e_1e_2 - 7e_0e_3) \\
&- 26c_0^2c_2e_1(c_2^3 - e_4) + 14c_0^3e_3(c_2^3 - e_4) - 42e_0e_1e_4 + 8e_1e_2e_4 + 14e_0e_3e_4\Big], \\
g_2 &= \alpha_2 + \beta_1\gamma_{2,1} + \beta_2\gamma_{2,2} + \beta_3\gamma_{2,3} \\
&= \frac{1}{2(c_0 - c_2)(c_2^3 - e_4)}\Big[e_0e_3^2 + e_2e_3^2 - 32c_0c_2^2e_0(c_2^3 - e_4) - 19e_1^2e_4 \\
&- 24e_0e_2e_4 + 8e_2^2e_4 + 8e_1e_3e_4 + 16c_0^3(c_2^3 - e_4)e_4 + 16e_0e_4^2 + c_2^3(19e_1^2 \\
&+ 24e_0e_2 - 8e_2^2 - 8e_1e_3 - 16e_0e_4) + c_0^2c_2(8c_2^3e_2 - 3e_3^2 - 8e_2e_4)\Big], \\
g_3 &= \alpha_3 + \beta_1\gamma_{3,1} + \beta_2\gamma_{3,2} + \beta_3\gamma_{3,3} \\
&= \frac{1}{2(c_0 - c_2)(c_2^3 - e_4)}\Big[e_1e_3^2 + e_3^3 - 36c_0c_2^2e_1(c_2^3 - e_4) + 12c_0^2c_2e_3(c_2^3 - e_4) \\
&- 22e_1e_2e_4 - 16e_0e_3e_4 + 14e_2e_3e_4 + 8e_1e_4^2 + 2c_2^3(11e_1e_2 + 8e_0e_3 - 7e_2e_3 \\
&- 4e_1e_4)\Big], \\
g_4 &= \alpha_4 + \beta_1\gamma_{4,1} + \beta_2\gamma_{4,2} + \beta_3\gamma_{4,3} \\
&= \frac{1}{2(c_0 - c_2)(c_2^3 - e_4)}\Big[-8c_2^6e_0 - 4c_0c_2^5e_2 + 16c_0^2c_2^4e_4 - 4e_2^2e_4 - 16c_0^2c_2e_4^2 \\
&- 2e_4(8e_1e_3 - 3e_3^2 + 4e_0e_4) + c_2^3(4e_2^2 + 16e_1e_3 - 5e_3^2 + 16e_0e_4 - 12e_2e_4) \\
&+ c_0c_2^2(-3e_3^2 + 4e_2e_4) + e_2(e_3^2 + 12e_4^2)\Big], \\
g_5 &= \alpha_5 + \beta_1\gamma_{5,1} + \beta_2\gamma_{5,2} + \beta_3\gamma_{5,3} \\
&= \frac{1}{2(c_0 - c_2)(c_2^3 - e_4)}\Big[-10c_2^6e_1 - 2c_0c_2^5e_3 + e_3^3 + 2c_0c_2^2e_3e_4 - 4e_2e_3e_4 \\
&- 10e_1e_4^2 + 6e_3e_4^2 + c_2^3(4e_2e_3 + 20e_1e_4 - 6e_3e_4)\Big].
\end{aligned}
$$

Case $A(t) = A_2(t)/(2(c_0 - c_2))$.

$$
\begin{aligned}
\beta_1 &= \frac{A_2(0)}{2(c_0 - c_2)(-i)(i)} = \frac{A_2(0)}{2(c_0 - c_2)}, \\
\beta_2 &= \frac{A_2(i)}{2(c_0 - c_2)(i)(2i)} = -\frac{A_2(i)}{4(c_0 - c_2)}, \\
\beta_3 &= \frac{A_2(-i)}{2(c_0 - c_2)(-i)(-2i)} = -\frac{A_2(-i)}{4(c_0 - c_2)}.
\end{aligned}
$$

We can calculate α_k $(k = 0, ..., 5)$, by Mathematica's polynomial remainder program for

$$\frac{A_2(t)}{2(c_0 - c_2)t(t^2+1)} - \frac{\beta_1}{t} - \frac{\beta_2}{t-i} - \frac{\beta_3}{t+i}.$$

Consequently we have

$$\begin{aligned}
h_0 &= \alpha_0 + \beta_1\gamma_{0,1} + \beta_2\gamma_{0,2} + \beta_3\gamma_{0,3} \\
&= \frac{1}{2(c_0-c_2)(c_0^3-e_0)}\Big[2c_0^5c_2e_1 - 2c_0^2c_2e_0e_1 - e_1^3 + 4e_0e_1e_2 + 10c_0^6e_3 \\
&+ e_0^2(-6e_1 + 10e_3) + c_0^3(6e_0e_1 - 4e_1e_2 - 20e_0e_3)\Big] \\
h_1 &= \alpha_1 + \beta_1\gamma_{1,1} + \beta_2\gamma_{1,2} + \beta_3\gamma_{1,3} \\
&= \frac{1}{2(c_0-c_2)(c_0^3-e_0)}\Big[-16c_0^4c_2^2e_0 + 16c_0c_2^2e_0^2 + 4c_0^5c_2e_2 - e_1^2e_2 \\
&+ c_0^2c_2(3e_1^2 - 4e_0e_2) + e_0(-6e_1^2 + 4e_2^2 + 16e_1e_3) + 8c_0^6e_4 + e_0^2(-12e_2 + 8e_4) \\
&+ c_0^3(5e_1^2 + 12e_0e_2 - 4e_2^2 - 16e_1e_3 - 16e_0e_4)\Big], \\
h_2 &= \alpha_2 + \beta_1\gamma_{2,1} + \beta_2\gamma_{2,2} + \beta_3\gamma_{2,3} \\
&= \frac{-1}{2(c_0-c_2)(c_0^3-e_0)}\Big[12c_0^4c_2^2e_1 - 12c_0c_2^2e_0e_1 + e_1^3 - 36c_0^5c_2e_3 + 36c_0^2c_2e_0e_3 \\
&+ e_1^2e_3 + 2e_0(4e_0 - 11e_2)e_3 + 2e_0e_1(7e_2 - 8e_4) - 2c_0^3(7e_1e_2 + 4e_0e_3 \\
&- 11e_2e_3 - 8e_1e_4)\Big], \\
h_3 &= \alpha_3 + \beta_1\gamma_{3,1} + \beta_2\gamma_{3,2} + \beta_3\gamma_{3,3} \\
&= \frac{1}{2(c_0-c_2)(c_0^3-e_0)}\Big[16c_2^3e_0^2 - 8c_0^4c_2^2e_2 - e_1^2e_2 - 8e_0e_2^2 + c_0c_2^2(3e_1^2 + 8e_0e_2) \\
&- 8e_0e_1e_3 + 19e_0e_3^2 + 32c_0^5c_2e_4 - 32c_0^2c_2e_0e_4 - 16e_0^2e_4 - e_1^2e_4 + 24e_0e_2e_4 \\
&+ c_0^3(-16c_2^3e_0 + 8e_2^2 + 8e_1e_3 - 19e_3^2 + 16e_0e_4 - 24e_2e_4)\Big], \\
h_4 &= \alpha_4 + \beta_1\gamma_{4,1} + \beta_2\gamma_{4,2} + \beta_3\gamma_{4,3} \\
&= \frac{-1}{2(c_0-c_2)(c_0^3-e_0)}\Big[-14c_2^3e_0e_1 - 26c_0^4c_2^2e_3 + 26c_0c_2^2e_0e_3 + e_1^2e_3 + 8e_0e_2e_3 \\
&+ 14e_0e_1e_4 - 42e_0e_3e_4 + 2c_0^3(7c_2^3e_1 - 4e_2e_3 - 7e_1e_4 + 21e_3e_4)\Big], \\
h_5 &= \alpha_5 + \beta_1\gamma_{5,1} + \beta_2\gamma_{5,2} + \beta_3\gamma_{5,3} \\
&= \frac{1}{2(c_0-c_2)(c_0^3-e_0)}\Big[c_2^3(e_1^2 + 12(-c_0^3 + e_0)e_2) + e_0e_3^2 + 24c_0c_2^2(c_0^3 - e_0)e_4 \\
&- e_1^2e_4 - 12e_0e_2e_4 + 24e_0e_4^2 - c_0^3(e_3^2 - 12e_2e_4 + 24e_4^2)\Big].
\end{aligned}$$

Since $g_j = \partial_x^{6-j}\partial_y^j f(0,0)/(j!(5-j)!), h_j = \partial_x^{5-j}\partial_y^{j+1} f(0,0)/(j!(5-j)!)$, we have 5 compatibility conditions:

$$\begin{aligned}
&f_1 := 5h_0 - g_1 = 0, \quad f_2 := 2h_1 - g_2 = 0, \quad f_3 := h_2 - g_3 = 0, \\
&f_4 := h_3 - 2g_4 = 0, \quad f_5 := h_4 - 5g_5 = 0.
\end{aligned}$$

We put $f_j^* = 2(c_0 - c_2)(c_0^3 - e_0)(c_2^3 - e_4)f_j$. Then we obtain the following expressions:

$$\begin{aligned}
f_1^* &= -5(c_2^3 - e_4)e_1^3 + 36(c_0^3 - e_0)^2(c_2^3 - e_4)e_3 - (c_0^3 - e_0)e_1e_3^2 \\
&+ 12(c_0^3 - e_0)(3c_0^2c_2 - e_0 - e_2)(c_2^3 - e_4)e_1, \\
f_2^* &= (c_0^3 - e_0)(3c_0^2c_2 - e_0 - e_2)e_3^2 - 24(c_0^3 - e_0)(c_2^3 - e_4)e_1e_3
\end{aligned}$$

$$
\begin{aligned}
&- (9c_0^3 - 6c_0^2c_2 - 7e_0 + 2e_2)(c_2^3 - e_4)e_1^2,\\
f_3^* &= -(c_0^3 - e_0)e_3^3 - (c_2^3 - e_4)e_1^3 - (c_2^3 - e_4)e_1^2e_3 - (c_0^3 - e_0)e_1e_3^2\\
&\quad + 8(c_0^3 - e_0)(c_2^3 - e_4)(3c_0c_2^2 - e_2 - e_4)e_1\\
&\quad + 8(c_0^3 - e_0)(c_2^3 - e_4)(3c_0^2c_2 - e_0 - e_2)e_3,\\
f_4^* &= -24(c_0^3 - e_0)(c_2^3 - e_4)e_1e_3 + (c_0^3 - e_0)(6c_0c_2^2 - 9c_2^3 - 2e_2 + 7e_4)e_3^2\\
&\quad + (c_2^3 - e_4)(3c_0c_2^2 - e_2 - e_4)e_1^2,\\
f_5^* &= -5(c_0^3 - e_0)e_3^3 - (c_2^3 - e_4)e_1^2e_3 + 36(c_0^3 - e_0)(c_2^3 - e_4)^2e_1\\
&\quad + 12(c_0^3 - e_0)(c_2^3 - e_4)(3c_0c_2^2 - e_2 - e_4)e_3.
\end{aligned}
$$

In particular we have

$$
\begin{aligned}
0 = f_2^* - f_4^* &= (c_0^3 - e_0)(3c_0^2c_2 - 6c_0c_2^2 + 9c_2^3 - e_0 + e_2 - 7e_4)e_3^2\\
&\quad - (c_2^3 - e_4)(9c_0^3 - 6c_0^2c_2 + 3c_0c_2^2 - 7e_0 + e_2 - e_4)e_1^2.
\end{aligned}
$$

Hence if $(c_0^3 - e_0)(3c_0^2c_2 - 6c_0c_2^2 + 9c_2^3 - e_0 + e_2 - 7e_4) \neq 0$, we get $e_3 = te_1$ with

$$
t = \pm\sqrt{\frac{(c_2^3 - e_4)(9c_0^3 - 6c_0^2c_2 + 3c_0c_2^2 - 7e_0 + e_2 - e_4)}{(c_0^3 - e_0)(3c_0^2c_2 - 6c_0c_2^2 + 9c_2^3 - e_0 + e_2 - 7e_4)}},
$$

and so from $f_2^* = 0$ we have

$$
\begin{aligned}
&\Big((c_0^3 - e_0)(3c_0^2c_2 - e_0 - e_2)t^2 - 24(c_0^3 - e_0)(c_2^3 - e_4)t\\
&- (9c_0^3 - 6c_0^2c_2 - 7e_0 + 2e_2)(c_2^3 - e_4)\Big)e_1^2 = 0.
\end{aligned}
$$

Therefore we conclude that, if $e_1 = e_3 = 0$ does not hold, then $e_0 = c_0^3$ or $e_0 = 3c_0^2c_2 - 6c_0c_2^2 + 9c_2^3 + e_2 - 7e_4$ or (a quadratic equation in e_0):

$$
\begin{aligned}
0 &= \Big(15c_0c_2^2 - 21c_2^3 - 5e_2 + 16e_4\Big)^2 e_0^2 - 2\Big(-378c_0^2c_2^7 + 5e_2^3\\
&- 153c_0c_2^5(6c_0^3 + e_2) + 18c_2^6(57c_0^3 + 7e_2) - 3c_0c_2^2(3c_0^3(20e_2 - 67e_4)\\
&+ e_2(10e_2 - 41e_4)) + 224e_4 - 41e_2^2e_4 + 80e_2e_4^2 + 9c_0^2c_2^4(30c_0^3 - 12e_2 + 67e_4)\\
&+ c_2^3(-224 + 51e_2^2 + 27c_0^3(13e_2 - 46e_4) - 201e_2e_4) - 3c_0^2c_2(5e_2^2 - 41e_2e_4\\
&+ 80e_4^2) + 3c_0^3(10e_2^2 - 67e_2e_4 + 112e_4^2)\Big)e_0 + 324c_0^8c_2^4 + 36c_2^6e_2^2 + e_2^4\\
&+ 9c_0^6(216c_2^6 + 12c_2^3(5e_2 - 19e_4) + (2e_2 - 7e_4)^2) - 108c_0^7c_2^2(12c_2^3\\
&+ 2e_2 - 7e_4) + 64e_2e_4 - 10e_2^3e_4 - 64e_4^2 + 25e_2^2e_4^2 + 4c_2^3(-16e_2\\
&+ 3e_2^3 + 16e_4 - 15e_2^2e_4) - 6c_0c_2^2(e_2^3 + c_2^3(32 + 6e_2^2) - 32e_4 - 5e_2^2e_4)\\
&- 18c_0^5c_2(72c_2^6 + 2e_2^2 + 6c_2^3(3e_2 - 17e_4) - 17e_2e_4 + 35e_4^2) + 9c_0^4c_2^2(36c_2^6\\
&- 7e_2^2 + 24e_2e_4 + 25e_4^2 - 12c_2^3(3e_2 + 5e_4)) + 6c_0^3(90c_2^6e_2 + 2e_2^3 + 96e_4\\
&- 17e_2^2e_4 + 35e_2e_4^2 + 3c_2^3(-32 + 9e_2^2 - 39e_2e_4)) - 3c_0^2c_2(72c_2^6e_2\\
&+ c_2^3(-128 + 21e_2^2 - 120e_2e_4) + 2(e_2^3 + 64e_4 - 10e_2^2e_4 + 25e_2e_4^2)).
\end{aligned}
$$

In other words, for generic c_0, c_2, e_0, e_2, e_4 we have $e_1 = e_3 = 0$. On the other hand, under $c_1 = d_* = e_1 = e_3 = 0$ we can apply Proposition 5.1. Hence there exists a unique germ M' of a general cyclide at the origin with the same data a, b, c_*, d_*, e_* at the origin. Since the characteristic polynomial of M' at the origin coincides with $P(t)$, it has the six distinct non-zero characteristic roots $t_1, ..., t_6$ with $C(t_j; 0, 0) \neq 0$ $(\forall j)$. Indeed $C(t_j; 0, 0) \neq 0$ follows from the assumptions on M at $(0, 0)$ and (i) of Theorem 2.2. Therefore by Lemma 5.6 we know that M' includes 6 continuous families of circular arcs corresponding to characteristic roots $t_1, ..., t_6$, and it is conformally equivalent to a general cyclide of type (2.18). Hence by the former part of Theorem 2.10 we conclude that M' coincides with our surface germ $z = f(x, y)$. This completes the proof except for the proof of Lemma 5.6, which is given independently of Theorem 2.10.

Lemma 5.6. *Let $M = \{z = f(x, y)\}$ be a C^4-class surface germ at $(0, 0, 0)$ with the following Taylor expansion at $(0, 0)$:*

$$f(x, y) = c_0x^2 + c_2y^2 + e_0x^4 + e_2x^2y^2 + e_4y^4 + o((x^2 + y^2)^2),$$

where c_0, c_2, e_0, e_2, e_4 are real coefficients with $c_0 - c_2 \neq 0$. We suppose that M is a general cyclide as a germ at the origin, and that the characteristic polynomial $P(t)$ at $(0, 0)$ of M has 6 distinct non-zero real roots $t_1, ..., t_6$ with $C(t_j; 0, 0) \neq 0$ $(\forall j)$. Then M is defined by the following equation in a neighborhood of the origin:

$$\begin{aligned} 0 = &- z + c_0x^2 + c_2y^2 + \frac{e_0 - e_2 + e_4}{(c_0 - c_2)^2}z^2 + 2\beta z(x^2 + y^2 + z^2) \\ &+ \alpha(x^2 + y^2 + z^2)^2, \end{aligned} \tag{5.8}$$

where

$$\alpha := \frac{c_2^2e_0 - c_0c_2e_2 + c_0^2e_4}{(c_0 - c_2)^2}, \quad \beta := \frac{(c_0 + c_2)e_2 - 2c_2e_0 - 2c_0e_4}{2(c_0 - c_2)^2}.$$

Further by some conformal transformation Φ, M is transformed into a germ at $\tau = (0, 0, *)$ of the following 6-circle Blum cyclide:

$$(x^2 + y^2 + z^2)^2 - 2a_1x^2 - 2a_2y^2 - 2a_3z^2 + a_4^2 = 0, \tag{5.9}$$

where $a_1 > a_3 > a_4 > 0$, $-a_2 > a_4$. In particular this surface (5.9) has the same characteristic roots $\{t_k\}_{k=1}^6$ at τ, and for every $j = 1, ..., 6$ the continuous family of circular arcs corresponding to t_j is transformed by Φ^{-1} into the continuous family of circular arcs on M corresponding to t_j.

Remark 5.7. As we mentioned at Remark 2.9, Takeuchi [10] proved that a general cyclide can be transformed into (2.18) by a conformal transformation. The arguments there are geometrically very interesting, but they are not germ-fixing arguments. Indeed, it is not so easy to construct a similar conformal transformation fixing the reference point.

Proof. (For the check of calculations, the readers can employ a mathematica file "check-fifth3" in our website written in Section 1.) By Proposition 5.1 and its proof we get (5.8). Since $a = b = c_1 = d_0 = d_1 = d_2 = d_3 = e_1 = e_3 = 0$ at the origin, the characteristic polynomial $P(t)$ has the form $P(t) = 4(c_0 - c_2)t^2H(t^2)$, where

$$H(h) := (e_4 - c_2^3)h^3 + (e_4 + e_2 - 3c_0c_2^2)h^2 + (e_2 + e_0 - 3c_0^2c_2)h + e_0 - c_0^3.$$

Then, by the assumption, the equation $H(h) = 0$ has 3 distinct positive solutions $h = h_1, h_2, h_3$ with $h_1 > h_2 > h_3 > 0$. Therefore

$$\begin{aligned} &(e_4 - c_2^3)h^3 + (e_4 + e_2 - 3c_0c_2^2)h^2 + (e_2 + e_0 - 3c_0^2c_2)h + e_0 - c_0^3 \\ &= (e_4 - c_2^3)(h - h_1)(h - h_2)(h - h_3), \end{aligned}$$

and so we can write e_0, e_2, e_4 by c_0, c_2, h_1, h_2, h_3 as follows:

$$\begin{aligned} e_0 = &\frac{1}{(1 + h_1)(1 + h_2)(1 + h_3)}\Big\{c_0^3 + c_0^3h_1 + c_0^3h_2 + c_0^3h_1h_2 + c_0^3h_3 \\ &+ c_0^3h_1h_3 + c_0^3h_2h_3 + 3c_0^2c_2h_1h_2h_3 - 3c_0c_2^2h_1h_2h_3 + c_2^3h_1h_2h_3\Big\}, \\ e_2 = &\frac{-1}{(1 + h_1)(1 + h_2)(1 + h_3)}\Big\{c_0^3 - 3c_0^2c_2 + c_0^3h_1 - 3c_0^2c_2h_1 + c_0^3h_2 \\ &- 3c_0^2c_2h_2 - 3c_0c_2^2h_1h_2 + c_2^3h_1h_2 + c_0^3h_3 - 3c_0^2c_2h_3 - 3c_0c_2^2h_1h_3 \end{aligned}$$

$$+ c_2^3h_1h_3 - 3c_0c_2^2h_2h_3 + c_2^3h_2h_3 - 3c_0c_2^2h_1h_2h_3 + c_2^3h_1h_2h_3\Big\},$$

$$e_4 = \frac{1}{(1+h_1)(1+h_2)(1+h_3)}\Big\{c_0^3 - 3c_0^2c_2 + 3c_0c_2^2 + c_2^3h_1 + c_2^3h_2$$

$$+ c_2^3h_1h_2 + c_2^3h_3 + c_2^3h_1h_3 + c_2^3h_2h_3 + c_2^3h_1h_2h_3\Big\}.$$

On the other hand we consider the surface N with real parameters $a, b, k_1, k_2, k_3, d, e$:

$$N: \quad 0 = a + bz + k_1x^2 + k_2y^2 + k_3z^2$$
$$+ dz\big(x^2 + y^2 + z^2\big) + e\big(x^2 + y^2 + z^2\big)^2.$$

We construct Φ as a composition of 3 conformal transformations; a translation in z-direction, an inversion with center at the origin, and another translation in z-direction:

$$\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1 : (x, y, z) \stackrel{\Phi_1}{\to} (x', y', z') \stackrel{\Phi_2}{\to} (x'', y'', z'') \stackrel{\Phi_3}{\to} (x''', y''', z''').$$

Then after the translation $\Phi_1 : x = x', y = y', z = z' + t$, we have a similar surface $N_1 = \Phi_1(N)$ with parameters $a', b', k_1', k_2', k_3', d', e'$, where

$$a' = a + bt + k_3t^2 + dt^3 + et^4, \quad b' = b + 2k_3t + 3dt^2 + 4et^3,$$
$$k_1' = k_1 + dt + 2et^2, \quad k_2' = k_2 + dt + 2et^2, \quad k_3' = k_3 + 3dt + 6et^2$$
$$d' = d + 4et, \quad e' = e.$$

Further, after the inversion:

$$x' = \frac{x''}{(x'')^2 + (y'')^2 + (z'')^2}, \ y' = \frac{y''}{(x'')^2 + (y'')^2 + (z'')^2},$$
$$z' = \frac{z''}{(x'')^2 + (y'')^2 + (z'')^2},$$

we have a similar surface $N_2 = \Phi_2(N_1)$ with parameters $a'', b'', k_1'', k_2'', k_3'', d'', e''$, where

$$a'' = e', \quad b'' = d', \quad k_1'' = k_1', \quad k_2'' = k_2', \quad k_3'' = k_3', \quad d'' = b', \quad e'' = a'.$$

Again after a translation $x'' = x''', y'' = y''', z'' = z''' + s$, we have a similar surface $N_3 = \Phi(N_2)$ with parameters $a''', b''', k_1''', k_2''', k_3''', d''', e'''$, where

$$a''' = a'' + b''s + k_3''s^2 + d''s^3 + e''s^4, \quad b''' = b'' + 2k_3''s + 3d''s^2 + 4e''s^3,$$
$$k_1''' = k_1'' + d''s + 2e''s^2, \quad k_2''' = k_2'' + d''s + 2e''s^2, \quad k_3''' = k_3'' + 3d''s + 6e''s^2,$$
$$d''' = d'' + 4e''s, \quad e''' = e''.$$

We suppose that the last surface is a Blum cyclide, then we get the equations $b''' = d''' = 0$. That is,

$$0 = d' + 2k_3's + 3b's^2 + 4a's^3, \quad 0 = b' + 4a's.$$

Further they are written by using the original parameters as follows (under $a = 0$):

$$0 = d + 4et + 2s(k_3 + 3dt + 6et^2) + 3s^2(b + 2k_3t + 3dt^2 + 4et^3)$$
$$+ 4s^3(bt + k_3t^2 + dt^3 + et^4),$$
$$0 = b + 2k_3t + 3dt^2 + 4et^3 + 4s(bt + k_3t^2 + dt^3 + et^4).$$

Hence we have the following equivalent conditions:

$$0 = d + 4et + 2s(k_3 + 3dt + 6et^2) + 2s^2(b + 2k_3t + 3dt^2 + 4et^3),$$
$$0 = b + 2k_3t + 3dt^2 + 4et^3 + 4s(bt + k_3t^2 + dt^3 + et^4).$$

Since the second equation above is solvable with respect to s:

$$s = -\frac{b + 2k_3t + 3dt^2 + 4et^3}{4(bt + k_3t^2 + dt^3 + et^4)},$$

we finally obtain an equation for t:

$$\begin{aligned}0 =& b^3 + 2b^2k_3t + 5b^2dt^2 + 20b^2et^3 + (-5bd^2 + 20bk_3e)t^4\\ &+ (-2k_3d^2 + 8k_3^2e - 4bde)t^5 + (-d^3 + 4k_3de - 8be^2)t^6.\end{aligned} \tag{5.10}$$

In our case, the parameters for the original surface $N = M$ are given by the following:

$$\begin{aligned}&a = 0,\ b = -1,\ k_1 = c_0,\ k_2 = c_2,\ k_3 = \frac{e_0 - e_2 + e_4}{(c_0 - c_2)^2},\\ &d = \frac{(c_0 + c_2)e_2 - 2c_2e_0 - 2c_0e_4}{(c_0 - c_2)^2},\ e = \frac{c_2^2e_0 - c_0c_2e_2 + c_0^2e_4}{(c_0 - c_2)^2}.\end{aligned}$$

Therefore equation (5.10) for t is written by using c_0, c_2, h_1, h_2, h_3 as follows:

$$\begin{aligned}0 = &\Big(1 + h_1 + h_2 + h_3 + h_1h_2 + h_2h_3 + h_3h_1 + h_1h_2h_3\\ &- 2(c_0 + c_2h_1 + c_0h_2 + c_0h_3 + c_2h_1h_2 + c_2h_1h_3 + c_0h_2h_3 + c_2h_1h_2h_3)t\\ &+ (c_0^2 - c_0^2h_1 + c_0^2h_2 + c_0^2h_3 + 2c_0c_2h_1 + c_2^2h_1h_2 + c_2^2h_1h_3 + 2c_0c_2h_2h_3\\ &- c_2^2h_2h_3 + c_2^2h_1h_2h_3)t^2\Big) \times \Big(1 + h_1 + h_2 + h_3 + h_1h_2 + h_2h_3 + h_1h_3\\ &+ h_1h_2h_3 - 2(c_0 + c_2h_2 + c_0h_1 + c_0h_3 + c_2h_1h_2 + c_2h_2h_3 + c_0h_1h_3\\ &+ c_2h_1h_2h_3)t + (c_0^2 - c_0^2h_2 + c_0^2h_1 + c_0^2h_3 + 2c_0c_2h_2 + c_2^2h_1h_2 + c_2^2h_2h_3\\ &+ 2c_0c_2h_1h_3 - c_2^2h_1h_3 + c_2^2h_1h_2h_3)t^2\Big) \times \Big(1 + h_1 + h_2 + h_3 + h_1h_2\\ &+ h_2h_3 + h_1h_3 + h_1h_2h_3 - 2(c_0 + c_2h_3 + c_0h_1 + c_0h_2 + c_2h_1h_3\\ &+ c_2h_2h_3 + c_0h_1h_2 + c_2h_1h_2h_3)t + (c_0^2 - c_0^2h_3 + c_0^2h_1 + c_0^2h_2\\ &+ 2c_0c_2h_3 + c_2^2h_1h_3 + c_2^2h_2h_3 + 2c_0c_2h_1h_2 - c_2^2h_1h_2 + c_2^2h_1h_2h_3)t^2\Big).\end{aligned}$$

Easily to see, the first and the second factors are some permutations of the third factor in h_1, h_2, h_3. The two roots corresponding to the third factor are given by

$$\begin{aligned}t_\pm = &\Big((c_0 + c_2h_3)(1 + h_1)(1 + h_2)\\ &\pm (c_0 - c_2)\sqrt{(1 + h_1)(1 + h_2)(h_1 - h_3)(h_2 - h_3)}\Big)\Big/\\ &\Big((1 + h_1 + h_2 - h_3)c_0^2 + 2(h_1h_2 + h_3)c_0c_2\\ &\quad + (h_1h_3 + h_2h_3 - h_1h_2 + h_1h_2h_3)c_2^2\Big).\end{aligned} \tag{5.11}$$

Since $h_1 > h_2 > h_3 > 0$, we know that the roots for the other two factors are not real. Therefore if the denominator of (5.11) does not vanish, we have two real solutions $t_\pm$ as above. Putting

$$k := \sqrt{(1 + h_1)(1 + h_2)(h_1 - h_3)(h_2 - h_3)}\ (> 0), \quad w := c_0/c_2,$$

we get the following expressions for $t_\pm$ by using c_2, w, h_1, h_2, h_3, k:

$$\begin{aligned}t_\pm =& \Big((w + h_3)(1 + h_1)(1 + h_2) \pm (w - 1)k\Big)\Big/\\ &\Big(c_2\big((1 + h_1 + h_2 - h_3)w^2 + 2(h_1h_2 + h_3)w + h_1h_3 + h_2h_3 - h_1h_2 + h_1h_2h_3\big)\Big).\end{aligned}$$

By using $t = t_+$, we find s as follows:

$$\begin{aligned}s =& -\frac{b + 2k_3t + 3dt^2 + 4et^3}{4(bt + k_3t^2 + dt^3 + et^4)}\\ =& \Big\{(c_0 - c_2)^2 - 2(e_0 - e_2 + e_4)t + 3(2c_2e_0 - c_0e_2 - c_2e_2 + 2c_0e_4)t^2\end{aligned}$$

$$- 4(c_2^2e_0 - c_0c_2e_2 + c_0^2e_4)t^3\Big\}\Big/\Big\{4t\big(-(c_0-c_2)^2 + (e_0 - e_2 + e_4)t$$
$$- (2c_2e_0 - c_0e_2 - c_2e_2 + 2c_0e_4)t^2 + (c_2^2e_0 - c_0c_2e_2 + c_0^2e_4)t^3\big)\Big\}$$
$$= -\Big\{c_2\Big(-h_1h_2 + h_1h_3 + h_2h_3 + h_1h_2h_3 + 2(h_1h_2+h_3)w$$
$$+ (1+h_1+h_2-h_3)w^2\Big)k\Big\}\Big/\Big\{2(1+h_1)(1+h_2)(h_1-h_3)(h_2-h_3)(w-1)\Big\}.$$

Thus we can find $(e''')^{-1}, k_3''', k_2''', k_1''', a'''$. First, we get

$$1/e''' = 1/e'' = 1/a' = 1/(bt + k_3t^2 + dt^3 + et^4)$$
$$= \frac{-c_2(u+vk)}{(w-1)^3(h_1-h_2)^2(h_1-h_3)(h_2-h_3)(1+h_3)}$$
$$\times \left(w + \frac{h_1h_2 + h_3 + k}{1+h_1+h_2-h_3}\right)^4,$$

where

$$u := h_1 + 2h_1^2 + h_1^3 + h_2 - 2h_1h_2 - h_1^2h_2 + 2h_1^3h_2 + 2h_2^2 - h_1h_2^2$$
$$- 4h_1^2h_2^2 + h_2^3 + 2h_1h_2^3 - 2h_3 + h_1h_3 + 2h_1^2h_3 - h_1^3h_3 + h_2h_3$$
$$+ h_1^2h_2h_3 + 2h_2^2h_3 + h_1h_2^2h_3 - h_2^3h_3 - 4h_3^2 - h_1h_3^2 + 2h_1^2h_3^2$$
$$- h_2h_3^2 - 2h_1h_2h_3^2 + 2h_2^2h_3^2 - 2h_3^3 - h_1h_3^3 - h_2h_3^3,$$
$$v := 2 - 2h_1^2 + 4h_1h_2 - 2h_2^2 + 4h_3 + 2h_3^2.$$

We prove here that $u + vk > 0$ under $h_1 > h_2 > h_3 > 0$. In fact putting positive numbers $p := h_1 - h_2$, $q := h_2 - h_3$, $r := h_3$, we get

$$u + vk = p^3(1+2q+r) + p(1+r)^2(1+2q+r)$$
$$+ 2p^2\big(1 + q + q^2 + 2r + qr + r^2$$
$$- \sqrt{q(p+q)(1+q+r)(1+p+q+r)}\big)$$
$$+ 2(1+r)^2\big(q + q^2 + qr + \sqrt{q(p+q)(1+q+r)(1+p+q+r)}\big)$$
$$> p^2\Big(p(1+2q+r) + 2(1+q+q^2+2r+qr+r^2)$$
$$- 2\sqrt{q(p+q)(1+q+r)(1+p+q+r)}\Big) > 0.$$

This is because concerning the last term we have

$$\Big(p(1+2q+r) + 2(1+q+q^2+2r+qr+r^2)\Big)^2$$
$$- 4q(p+q)(1+q+r)(1+p+q+r)$$
$$= 4 + 4p + p^2 + 8q + 8pq + 8q^2 + 16r + 12pr + 2p^2r + 24qr$$
$$+ 16pqr + 16q^2r + 24r^2 + 12pr^2 + p^2r^2 + 24qr^2 + 8pqr^2$$
$$+ 8q^2r^2 + 16r^3 + 4pr^3 + 8qr^3 + 4r^4 > 0.$$

Further, about $k_3''', k_2''', k_1''', a'''$ we have the following:

$$k_3''' = k_3'' + 3d''s + 6e''s^2 = k_3' + 3b's + 6a's^2$$
$$= (k_3 + 3dt + 6et^2) + 3(b + 2k_3t + 3dt^2 + 4et^3)s$$
$$+ 6(bt + k_3t^2 + dt^3 + et^4)s^2$$
$$= \frac{c_2(h_1 + h_2 + 2h_1h_2 - 2h_3 - h_1h_3 - h_2h_3)(w-1)}{2(1+h_1)(1+h_2)(1+h_3)},$$
$$k_2''' = k_2'' + d''s + 2e''s^2 = k_2' + b's + 2a's^2$$

$$
\begin{aligned}
&= k_2 + dt + 2et^2 + (b + 2k_3 t + 3dt^2 + 4et^3)s \\
&+ 2(bt + k_3 t^2 + dt^3 + et^4)s^2 \\
&= -\frac{c_2(2 + h_1 + h_2)(w - 1)}{2(1 + h_1)(1 + h_2)}, \\
k_1''' &= k_1'' + d''s + 2e''s^2 = k_1' + b's + 2a's^2 \\
&= k_1 + dt + 2et^2 + (b + 2k_3 t + 3dt^2 + 4et^3)s \\
&+ 2(bt + k_3 t^2 + dt^3 + et^4)s^2 \\
&= \frac{c_2(h_1 + h_2 + 2h_1 h_2)(w - 1)}{2(1 + h_1)(1 + h_2)}, \\
a''' &= a'' + b''s + k_3''s^2 + d''s^3 + e''s^4 = e' + d's + k_3's^2 + b's^3 + a's^4 \\
&= e + (d + 4et)s + (k_3 + 3dt + 6et^2)s^2 \\
&+ (b + 2k_3 t + 3dt^2 + 4et^3)s^3 + (bt + k_3 t^2 + dt^3 + et^4)s^4 \\
&= \frac{-c_2^3(u + vk)}{16(1 + h_1)^2(1 + h_2)^2(h_1 - h_3)(h_2 - h_3)(1 + h_3)(w - 1)} \\
&\times \left(w + \frac{h_1 h_2 + h_3 + k}{1 + h_1 + h_2 - h_3}\right)^4.
\end{aligned}
$$

Therefore the transformed surface $M' = \Phi(M)$ is written as

$$
(x^2 + y^2 + z^2)^2 - 2Ax^2 - 2By^2 - 2Cz^2 + D^2 = 0,
$$

where

$$
\begin{aligned}
A &= -k_1'''/(2e''') \\
&= \frac{c_2^2(h_1 + h_2 + 2h_1 h_2)(u + vk)\left(w + \frac{h_1 h_2 + h_3 + k}{1 + h_1 + h_2 - h_3}\right)^4}{4(w - 1)^2(h_1 - h_2)^2(h_1 - h_3)(h_2 - h_3)(1 + h_1)(1 + h_2)(1 + h_3)}, \\
B &= -k_2'''/(2e''') \\
&= -\frac{c_2^2(2 + h_1 + h_2)(u + vk)\left(w + \frac{h_1 h_2 + h_3 + k}{1 + h_1 + h_2 - h_3}\right)^4}{4(w - 1)^2(h_1 - h_2)^2(h_1 - h_3)(h_2 - h_3)(1 + h_1)(1 + h_2)(1 + h_3)}, \\
C &= -k_3'''/(2e''') \\
&= \frac{c_2^2(h_1 + h_2 + 2h_1 h_2 - 2h_3 - h_1 h_3 - h_2 h_3)(u + vk)}{4(w - 1)^2(h_1 - h_2)^2(h_1 - h_3)(h_2 - h_3)(1 + h_1)(1 + h_2)(1 + h_3)^2} \\
&\times \left(w + \frac{h_1 h_2 + h_3 + k}{1 + h_1 + h_2 - h_3}\right)^4, \\
D &= \sqrt{a'''/e'''} \\
&= \frac{c_2^2(u + vk)\left(w + \frac{h_1 h_2 + h_3 + k}{1 + h_1 + h_2 - h_3}\right)^4}{4(w - 1)^2(h_1 - h_2)(h_1 - h_3)(h_2 - h_3)(1 + h_1)(1 + h_2)(1 + h_3)}.
\end{aligned}
$$

Since $A > 0, D > 0, A > C$ and

$$
\begin{aligned}
\frac{C - D}{A} &= \frac{2(1 + h_1)(h_2 - h_3)}{(h_1 + h_2 + 2h_1 h_2)(1 + h_3)} > 0, \\
\frac{-B - D}{A} &= \frac{2(1 + h_2)}{h_1 + h_2 + 2h_1 h_2} > 0,
\end{aligned}
$$

we have $A > C > D > 0, -B > D$. Hence M' satisfies Blum's six circle conditions: $A > C > D > 0, D > B, B \neq -D$. Further, $\Phi((0,0,0)) = (0,0,-s-t^{-1}) =: \tau$ because

$$(0,0,0) \to (0,0,-t) \to (0,0,-1/t) \to (0,0,-s-t^{-1}).$$

Therefore the surface germ M at the origin is transformed into a six-circle Blum cyclide M' at τ by a conformal transformation Φ. On the other hand by Example 5.3 and Proposition 5.4 we have the six characteristic roots at $\tau \in M'$

$$\pm\sqrt{\frac{A-C}{C-B}}, \quad \pm\sqrt{\frac{A-D}{D-B}}, \quad \pm\sqrt{\frac{A+D}{-B-D}}$$

and the corresponding continuous families of circles. Since

$$\frac{A-C}{C-B} = h_3, \quad \frac{A-D}{D-B} = h_2, \quad \frac{A+D}{-B-D} = h_1,$$

we have the same characteristic roots $\pm\sqrt{h_3}, \pm\sqrt{h_2}, \pm\sqrt{h_1}$ for M' at τ with those for M at the origin, that is, $t_1, ..., t_6$. Further the continuous family of circular arcs on M' corresponding to t_j is transformed by Φ^{-1} into a continuous family of circular arcs on M corresponding to t_j. Indeed, the conformal image of a circular arc is a circular arc or a line segment. Let C_j be the circle on M' passing through $\tau = (0,0,-s-t^{-1})$ corresponding to the root t_j. Then the tangent vector to C_j at τ is given by $(1, t_j, 0)$. Since

$$\begin{aligned}
\Phi(x,y,z) &= \Phi_3(\Phi_2(x,y,z-t)) \\
&= \left(\frac{x}{x^2+y^2+(z-t)^2}, \frac{y}{x^2+y^2+(z-t)^2}, \frac{z-t}{x^2+y^2+(z-t)^2} - s\right) \\
&= \left(\frac{x}{t^2} + O(r^2), \frac{y}{t^2} + O(r^2), -s - \frac{1}{t} - \frac{z}{t^2} + O(r^2)\right),
\end{aligned}$$

where $r = \sqrt{x^2+y^2+z^2}$, the tangent vector to the curve $\Phi^{-1}(C_j)$ at the origin is the same vector $(1, t_j, 0)$. Therefore $\Phi^{-1}(C_j)$ cannot be a line-segment because of the assumption $C(t_j; 0, 0) \neq 0$. Hence the conformal image of the continuous family of circular arcs on M' corresponding to t_j is transformed under Φ^{-1} into a continuous family of circular arcs on M corresponding to t_j. This completes the proof. □

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