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Abstract

We consider the problem of deriving Brownian motions from classical mechanical systems in this paper. Precisely, we consider a system with one massive particle coupling to an ideal random wave field, evolved according to classical mechanical principles. We prove the almost sure existence and uniqueness of the solution of the considered dynamics, prove the convergence of the solution under a certain scaling limit, and give the precise expression of the limiting process, a diffusion process.

Keywords: wave field, classical mechanics, diffusions, convergence, Brownian motion

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1 Introduction

We consider a system with massive particle(s) interacting with an ideal environment. The dynamics is fully deterministic, Newtonian, as long as the initial condition is given, which means that the only source of randomness is from the initial condition of the environment. We are interested in the behavior of the massive particle(s) in an appropriate limit such that the environment becomes more and more "fast" (see below for the precise meaning of this description) in such a manner that the variance of momentum transfer stays of order 1.

In this paper, we consider the mentioned problem in the framework of wave field environment, we discuss the limit behavior of the massive particle when the speed of propagation of the wave is very fast (see (1.3) below for the precise expression). We

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study here the simplest model which consists of only one massive particle interacting with a scalar wave field, and the whole system is in dimension 1.

This paper is along the same line as [11]: we derive Brownian motion as a Brownian limit of a classical mechanical system consists of massive particle(s) and ideal environment. This type of model, called a mechanical model of Brownian motion, was first introduced and studied by Holley [8], and extended by, *e.g.*, Dürr-Goldstein-Lebowitz [5], [6], [7], Calderoni-Dürr-Kusuoka [3], Kusuoka-Liang [11] and others. In all these papers, the environment is given by an ideal gas, *i.e.*, a system consists of infinite "light" particles with its initial distribution given by Poisson point process.

In the present paper, we consider the similar problem with ideal "wave" environment (see (1.1) or equivalently (1.2)). The same dynamics is also discussed, from different aspect, by Komech-Kunze-Spohn [10]. This model, by [10], is also related to hydrodynamics [12], homogenization in periodic and random environments [1] [4], interface and vortex dynamics in GinGbureg-Landau theories [9], *etc.* See [10] and the references therein for more related topics.

Let us now give the precise description of our model. Write the mass of the massive particle as M. We use $q(t)$ and $p(t)$ to denote the position and the momentum of the massive particle at time t, respectively, and use $(\phi(x,t), u(x,t))$ to describe the state of the wave at position x and time t. We consider a Hamiltonian system with its Hamiltonian functional given by

$$H(\phi,u,q,p) = a_4^2(1+\frac{p^2}{a_4^2 M})^{1/2}+\frac{1}{2}\int_{\mathbf{R}}(a_1|u(x)|^2+a_3|\nabla\phi(x)|^2)dx+a_2\int_{\mathbf{R}}dx\phi(x)\rho(x-q), \tag{1.1}$$

where $a_1, \cdots, a_4$ are positive numbers. So our system is given by the following standard differential equations:

$$\begin{cases} \frac{d}{dt}q(t) = \frac{1}{M}\frac{p(t)}{\sqrt{1+a_4^{-2}M^{-1}p(t)^2}} \\ \dfrac{d}{dt}p(t) = a_2\displaystyle\int_{\mathbf{R}}\phi(x,t)\nabla\rho(x-q(t))dx \\ \dfrac{d}{dt}\phi(x,t) = a_1 u(x,t) \\ \dfrac{d}{dt}u(x,t) = a_3\Delta\phi(x,t) - a_2\rho(x-q(t)) \\ (q(0),p(0)) = (q_0,p_0) \end{cases} \tag{1.2}$$

Here " ∇ " and " Δ " denote the first and the second partial derivatives with respect to x (or derivatives if only one variable). The initial conditions $\phi(x,0)$ and $u(x,0)$ will be given later (see (1.3) below).

Notice that a_4 stands for the velocity of light, and $\sqrt{a_1 a_3}$ is the propagation of the wave. Also, the integral $\int_{\mathbf{R}} dx\phi(x)\rho(x-q)$ is a smoothen of $\phi(q)$, which is introduced by [10], to keep energey bounded. The smoothing function ρ is called "charge distribution" in [10], as an anology to Maxwell-Lorentz equations. For the sake of simplicity, we assume that $\rho \in C_0^\infty(\mathbf{R})$.

We next make some simple observation in order to give the initial conditions $\phi(x,0)$ and $u(x,0)$.

Notice that the corresponding Gibbs measure is given by $e^{-\beta H(\phi,u,q,p)}$. Take $\beta = 1$. The part $e^{-\frac{1}{2}a_1\int |u(x)|^2 dx}$ in Gibbs measure suggests the following initial condition with respect to $u(x)$: $u(x,0)$ is a Gaussian white noise with mean 0 and variance a_1^{-1}.

For the initial condition $\phi(x,0)$ with respect to ϕ, first notice that

$$\begin{aligned}
& \int_{\mathbf{R}} dx\phi(x)\rho(x-q) \\
= & \int_{\mathbf{R}} dx(\phi(x)-\phi(0))\rho(x-q) + \phi(0)\int_{\mathbf{R}} \rho(x-q)dx \\
= & \int_0^\infty dy\Big(\int_y^\infty \rho(x-q)dx\Big)\nabla\phi(y) - \int_{-\infty}^0 dy\Big(\int_{-\infty}^y \rho(x-q)dx\Big)\nabla\phi(y) + \phi(0)\int_{\mathbf{R}} \rho(u)du.
\end{aligned}$$

Define

$$m(y) = \begin{cases} \int_y^\infty \rho(x-q_0)dx, & y \geq 0, \\ -\int_{-\infty}^y \rho(x-q_0)dx, & y < 0. \end{cases}$$

If we assume that $\phi(0)(=\phi(x,0))$ is a constant, the calculation above gives us that

$$\int_{\mathbf{R}} dx\phi(x)\rho(x-q_0) = \int_{-\infty}^\infty \nabla\phi(y)m(y)dy + \text{constant}.$$

Therefore, the part $e^{-\frac{1}{2}a_3\int |\nabla\phi(x)|^2 dx - a_2\int_{\mathbf{R}} dx\phi(x)\rho(x-q_0)}$ in Gibbs measure is equal to a constant multiple of $e^{-\frac{1}{2}a_3\int\left(\nabla\phi(y)+a_3^{-1}a_2 m(y)\right)^2 dy}$. When we take normalization, the constant term disappears. Therefore, this calculation suggests that the initial condition with respect to ϕ is: $\nabla\phi(x) + a_3^{-1}a_2 m(x)$ is a Gaussian white noise with mean 0 and variance a_3^{-1}.

In conclusion, we get the following initial condition with respect to the wave:

$$\begin{cases} u(x,0) = a_1^{-1/2}\dot{B}_1(x), \\ \phi(x,0) + a_3^{-1}a_2\int_0^x m(y)dy = c + a_3^{-1/2}B_2(x), \end{cases} \tag{1.3}$$

where c is a constant, $\{B_1(x); x \in \mathbf{R}\}$ and $\{B_2(x); x \in \mathbf{R}\}$ are two independent standard Brownian motions, and $\{\dot{B}_1(x)\}$ means the white noise corresponding to $\{B_1(x)\}$.

We are interested in the following limit: assume that $a_i = a_i(\lambda) \in [1,\infty)$, $i = 1,\cdots,4$, are parameters with same index $\lambda \in [1,\infty)$. Our assumption that the propagation of the wave goes to infinity implies that $a_1a_3 \to \infty$ and as $\lambda \to \infty$. See Theorem 1.1 and the explanation following it for the other conditions. For a_4, the velocity of light, it can either be fixed or $\to \infty$. In the present paper, we consider the solution of (1.2) + (1.3), with the rigorous meaning of it given in Section 2. In particular, we are interested in its limit behavior when $\lambda \to \infty$.

The main result of the present paper is the following.

THEOREM 1.1 *1. For any fixed λ, (1.2) + (1.3) has a unique solution for P-almost every initial condition.*

2. Assume that a_1, a_2 and a_3 satisfy the following:

(A1) $a_2 = a_1^{1/4} a_3^{3/4}$,

(A2) $\lim_{\lambda\to\infty} a_1 a_3 = \infty$,

(A3) $\lim_{\lambda\to\infty} a_1 = \infty$.

Then when $\lambda \to \infty$, the distribution of $\{(q(t), p(t))\}_{t\geq 0}$, the solution of (1.2) + (1.3), converges to the distribution of the diffusion process with generator

$$L = \frac{1}{2}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \frac{\partial^2}{\partial p^2} - \frac{1}{2}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \widetilde{p}\frac{\partial}{\partial p} + \frac{1}{M}\widetilde{p}\frac{\partial}{\partial q}, \tag{1.4}$$

where $\widetilde{p} = \frac{p}{\sqrt{1+a_4^{-1}M^{-1}p^2}}$ if a_4 is a constant and $\widetilde{p} = p$ if $\lim_{\lambda\to\infty} a_4 = \infty$. Here the convergence means the weak convergence of the distributions on $D([0,\infty), \mathbf{R})$ equipped with the Skorohod metric.

Let us explain a little bit about the condition of Theorem 1.1 (2). As claimed, $\sqrt{a_1 a_3}$ is the propagation of the wave, so (A2) corresponds to our setting that "the propagation of the wave is very fast". The other two conditions are chosen such that our limit as $\lambda \to \infty$ is meanful, *i.e.*, the limit process exists and has its coefficients of both $\frac{\partial^2}{\partial p^2}$ and $\frac{\partial}{\partial p}$ not 0. Indeed, we have by Section 6 that the drift term has order $a_1^{-1/2} a_2^2 a_3^{-3/2}$, this suggests our condition (A1). Also, the coefficient of the diffusion term caused by B_2 has order $a_1^{-1/4} a_2 a_3^{-3/4}$, which is equal to 1 by (A1); and the coefficient of the diffusion term caused by B_1 has order $a_1^{-1} \cdot a_1^{-1/4} a_2 a_3^{-3/4}$, which is the same as that of a_1^{-1}. The condition (A3) is to ensure that this does not diverge, and in this case, the effect of B_1 disappears in the limit. We could have also assumed that a_1 is a constant instead of $a_1 \to \infty$, and in this case, we will get a limit resulted by both B_2 and B_1, by exactly the same method of the present paper. We focus on the case $a_1 \to \infty$ for the sake of simplicity of the expressions.

There are certainly infinitely many concrete examples of (a_1, a_2, a_3) that satisfy our conditions (A1) $\sim$ (A3), for example, $(a_1, a_2, a_3) = (\lambda, \lambda, \lambda)$ or $(a_1, a_2, a_3) = (\lambda^3, 1, \lambda^{-1})$, *etc.*. Especially the latter case corresponds to the model that the interaction keeps order 1.

One of the main ideas of this paper is the induction of the two approximations (3.1) and (4.3). Both of them are essentially necessary in our proof: (3.1), a translation of s, is used as a "measurable approximation", such that many of our calculations including the formula of integration by parts (*e.g.*, (3.4)) are valid; (4.3) is an approximation that does not include s explicitly, this is essentially used, *e.g.*, in (4.4).

The rest of this paper is orginazed as follows. In Section 2, we first give the rigorous definition of the solution of (1.2) + (1.3), which is suggested natually by the case with smooth initial conditions (Subsection 2.1), and then give the proof of the first assertion of Theorem 1.1, the unique existence of the solution. In particular, this gives us our basic decomposition for our proof (see (2.16)). The term $I_1(t)$ in (2.16) gives us approximately the diffusion term of L (see Lemma 5.4), and the proof of this fact is given in Sections 3 $\sim$ 5. In Section 6, we show that the term $I_2(t)$ in (2.16) gives us approximately the drift term of L (see Lemma 6.1). The proof of the second half of Theorem 1.1 is given in Section 7, with the help of "martingale theory".

2 Definition and unique existence of the solution

In this section, we give the rigorous definition of the solution of (1.2) + (1.3), and prove the unique existence of it.

2.1 Observation for the case with smooth initial condition

This subsection is dedicated to the observation for the case with smooth initial condition. This suggests our rigorous definition of the solution of (1.2) + (1.3), which will be given in the next subsection.

Let $h_1, h_2 \in C^1(\mathbf{R})$, and we consider the standard differential equation (1.2) combined with initial condition

$$\begin{cases} u(x,0) = h_1'(x), \\ \phi(x,0) = h_2(x). \end{cases} \tag{2.1}$$

Lemma 2.1 *The solution of (1.2) + (2.1) satisfies*

$$\begin{aligned} \frac{d}{dt}p(t) &= a_2 \int_{\mathbf{R}} \overline{\phi}(x,t) \nabla \rho(x-q(t))dx \\ &\quad + \frac{1}{2} a_3^{-1} a_2^2 \int_{\mathbf{R}} dx \rho(x-q(t)) \int_0^{\sqrt{a_1 a_3}t} dr \\ &\qquad \Big(\rho(x+r-q(t-\frac{1}{\sqrt{a_1a_3}}r)) - \rho(x-r-q(t-\frac{1}{\sqrt{a_1a_3}}r))\Big), \end{aligned} \tag{2.2}$$

with $\overline{\phi}(x,t)$ given by

$$\overline{\phi}(x,t) = \frac{1}{2}h_2(x+\sqrt{a_1a_3}t) + \frac{1}{2}h_2(x-\sqrt{a_1a_3}t) + \frac{1}{2\sqrt{a_1a_3}}h_1(x+\sqrt{a_1a_3}t) - \frac{1}{2\sqrt{a_1a_3}}h_1(x-\sqrt{a_1a_3}t). \tag{2.3}$$

We prove Lemma 2.1 in the rest of this subsection. Let $\overline{\phi}(x,t)$ be the solution of the following heat equation:

$$\begin{cases} \frac{d^2}{dt^2}\overline{\phi}(x,t) = a_1a_3\Delta\overline{\phi}(x,t) \\ \overline{\phi}(x,0) = h_2(x) \\ \frac{d}{dt}\overline{\phi}(x,0) = h_1'(x), \end{cases} \tag{2.4}$$

and let

$$y(x,t) = \phi(x,t) - \overline{\phi}(x,t).$$

Then by the definition of $\phi(x,t)$, we get that $y(x,t)$ satisfies

$$\begin{cases} \frac{d^2}{dt^2}y(x,t) = a_1a_3\Delta y(x,t) - a_1a_2\rho(x-q(t)) \\ y(x,0) = 0 \\ \frac{d}{dt}y(x,0) = 0. \end{cases} \tag{2.5}$$

We first have the following result with respect to $\overline{\phi}(x,t)$.

Lemma 2.2 *The solution $\overline{\phi}(x,t)$ of (2.4) is given by (2.3).*

Proof. By general result of heat equation, there exist functions f and g such that

$$\overline{\phi}(x,t) = f(x-\sqrt{a_1a_3}t) + g(x+\sqrt{a_1a_3}t). \tag{2.6}$$

This combined with our initial conditions gives us our assertion. ∎

We next deal with $y(x,t)$. We first prepare the following general result.

Lemma 2.3 *For any function $f(x,t)$, if $y(x,t)$ satisfies*

$$\frac{d^2}{dt^2}y(x,t) = a_1a_3\Delta y(x,t) + a_1a_2f(x,t), \tag{2.7}$$

and the initial condition $y(x,0) = y_t(x,0) = 0$, then

$$y(x,t) = a_1a_2\int_0^t dr\int_0^r ds f(x-\sqrt{a_1a_3}t+2\sqrt{a_1a_3}r-\sqrt{a_1a_3}s, s). \tag{2.8}$$

Proof. We have by (2.7) and a simple calculation that

$$\begin{aligned} &\frac{d^2}{dt^2}\Big(y(x+\sqrt{a_1a_3}t,t)\Big) \\ =\ & a_1a_3y_{xx}(x+\sqrt{a_1a_3}t,t) + 2\sqrt{a_1a_3}y_{xt}(x+\sqrt{a_1a_3}t,t) + y_{tt}(x+\sqrt{a_1a_3}t,t) \\ =\ & a_1a_3y_{xx}(x+\sqrt{a_1a_3}t,t) + 2\sqrt{a_1a_3}y_{xt}(x+\sqrt{a_1a_3}t,t) \\ & \qquad +a_1a_3y_{xx}(x+\sqrt{a_1a_3}t,t) + a_1a_2f(x+\sqrt{a_1a_3}t,t) \\ =\ & 2\sqrt{a_1a_3}\frac{d}{dx}\Big(\frac{d}{dt}\Big(y(x+\sqrt{a_1a_3}t,t)\Big)\Big) + a_1a_2f(x+\sqrt{a_1a_3}t,t). \end{aligned}$$

Therefore, with $z(x,t) = \frac{d}{dt}\Big(y(x+\sqrt{a_1a_3}t,t)\Big)$ we have $\frac{d}{dt}z(x,t) = 2\sqrt{a_1a_3}\frac{d}{dx}z(x,t) + a_1a_2f(x+\sqrt{a_1a_3}t,t)$, hence

$$\frac{d}{dt}\Big(z(x-2\sqrt{a_1a_3}t,t)\Big) = a_1a_2f(x-\sqrt{a_1a_3}t,t).$$

So

$$z(x-2\sqrt{a_1a_3}t,t) = z(x,0) + a_1a_2\int_0^t f(x-\sqrt{a_1a_3}s,s)ds.$$

This is true for any $x \in \mathbf{R}$ and $t \geq 0$. Therefore,

$$\begin{aligned}\frac{d}{dt}\Big(y(x+\sqrt{a_1a_3}t,t)\Big) &= z(x,t)\\ &= z(x+2\sqrt{a_1a_3}t,0) + a_1a_2\int_0^t f(x+2\sqrt{a_1a_3}t-\sqrt{a_1a_3}s,s)ds.\end{aligned}$$

So

$$y(x+\sqrt{a_1a_3}t,t) = y(x,0) + \int_0^t dr\Big(z(x+2\sqrt{a_1a_3}r,0) + a_1a_2\int_0^r f(x+2\sqrt{a_1a_3}r-\sqrt{a_1a_3}s,s)ds\Big)$$

for any $x \in \mathbf{R}$ and $t \geq 0$. Substituting x by $x-\sqrt{a_1a_3}t$ in the equation above, with the help of the initial condition, we get our assertion. ∎

Lemma 2.4 *If $y(x,t)$ satisfies (2.5), then*

$$\begin{aligned}& a_2\int_{\mathbf{R}} y(x,t)\nabla\rho(x-q(t))dx\\ =\; & \frac{1}{2}a_3^{-1}a_2^2\int_{\mathbf{R}} dx\rho(x-q(t))\int_0^{\sqrt{a_1a_3}t} dr\\ & \Big(\rho(x+r-q(t-\frac{1}{\sqrt{a_1a_3}}r)) - \rho(x-r-q(t-\frac{1}{\sqrt{a_1a_3}}r))\Big).\end{aligned}$$

Proof. Since $\rho \in C_0^\infty$, we have

$$\int_{\mathbf{R}} y(x,t)\nabla\rho(x-q(t))dx = -\int_{\mathbf{R}}\nabla y(x,t)\rho(x-q(t))dx. \tag{2.9}$$

By Lemma 2.3 applied to $f(x,t) = -\rho(x-q(t))$, we get

$$y(x,t) = -a_1a_2\int_0^t dr\int_0^r ds\rho(x-\sqrt{a_1a_3}t+2\sqrt{a_1a_3}r-\sqrt{a_1a_3}s-q(s)).$$

So

$$\begin{aligned}&\nabla y(x,t)\\ =\; & -a_1a_2\int_0^t dr\int_0^r ds\nabla\rho(x-\sqrt{a_1a_3}t+2\sqrt{a_1a_3}r-\sqrt{a_1a_3}s-q(s))\end{aligned}$$

$$
\begin{aligned}
&= -a_3^{-1}a_2\int_0^{\sqrt{a_1a_3}t}ds\int_s^{\sqrt{a_1a_3}t}dr\nabla\rho(x-\sqrt{a_1a_3}t+2r-s-q(\sqrt{a_1a_3}^{-1}s))\\
&= -\frac{1}{2}a_3^{-1}a_2\int_0^{\sqrt{a_1a_3}t}ds\Big(\rho(x+\sqrt{a_1a_3}t-s-q(\sqrt{a_1a_3}^{-1}s))\\
&\qquad -\rho(x-\sqrt{a_1a_3}t+s-q(\sqrt{a_1a_3}^{-1}s))\Big)\\
&= -\frac{1}{2}a_3^{-1}a_2\int_0^{\sqrt{a_1a_3}t}dr\Big(\rho(x+r-q(t-\sqrt{a_1a_3}^{-1}r))-\rho(x-r-q(t-\sqrt{a_1a_3}^{-1}r))\Big),
\end{aligned}
$$

where in the last equality, we used change of variable $r=\sqrt{a_1a_3}t-s$.

Combining this with (2.9), we get our assertion. ∎

Lemma 2.1 is an easy result of (1.2) and Lemmas 2.2, 2.4. ∎

2.2 Case with non-smooth initial condition

Lemma 2.1 suggests the following.

DEFINITION 2.5 *We say that $(p(t),q(t))$ is a (weak) solution of (1.2) + (1.3), if it satisfies*

$$
\begin{aligned}
\frac{d}{dt}p(t) &= a_2\int_{\mathbf{R}}\widetilde{\phi}(x,t)\nabla\rho(x-q(t))dx\\
&\quad +\frac{1}{2}a_3^{-1}a_2^2\int_{\mathbf{R}}dx\rho(x-q(t))\int_0^{\sqrt{a_1a_3}t}dr\\
&\quad \Big(\rho(x+r-q(t-\frac{1}{\sqrt{a_1a_3}}r))-\rho(x-r-q(t-\frac{1}{\sqrt{a_1a_3}}r))\Big)
\end{aligned}
\tag{2.10}
$$

with $\widetilde{\phi}(x,t)$ given by

$$
\begin{aligned}
\widetilde{\phi}(x,t) &= \frac{1}{2}a_3^{-1/2}B_2(x+\sqrt{a_1a_3}t)+\frac{1}{2}a_3^{-1/2}B_2(x-\sqrt{a_1a_3}t)\\
&\quad +\frac{1}{2}a_1^{-1}a_3^{-1/2}B_1(x+\sqrt{a_1a_3}t)-\frac{1}{2}a_1^{-1}a_3^{-1/2}B_1(x-\sqrt{a_1a_3}t)\\
&\quad -\frac{1}{2}a_3^{-1}a_2\int_0^{x+\sqrt{a_1a_3}t}m(y)dy-\frac{1}{2}a_3^{-1}a_2\int_0^{x-\sqrt{a_1a_3}t}m(y)dy+c.
\end{aligned}
\tag{2.11}
$$

Before going further, we first notice that in order to prove Theorem 1.1, it suffices to prove the unique existence and the convergence of the distribution of $\{(q(t),p(t))\}_{t\in[0,T]}$ for any given $T>0$. Choose an arbitrary $T>0$ and fix it throughout this paper. Notice that in this section, we are considering the existence for every fixed λ, so a_4 is also fixed and finite. Since $\left|\frac{1}{M}\frac{p(t)}{\sqrt{1+a_4^{-2}M^{-1}p(t)^2}}\right|\le\frac{a_4}{\sqrt{M}}$, it is clear that $|q(t)|\le|q_0|+\frac{a_4}{\sqrt{M}}T$ for any $t\in[0,T]$. Also, by assumption, there exists a constant r_ρ such that $\rho(x)=0$ for any $|x|\ge r_\rho$. Let $R_0=r_\rho+|q_0|+\frac{a_4}{\sqrt{M}}T$.

In the rest of this section, we show the unique existence of the solution as defined by Definition 2.5, for any fixed $a_1,a_2,a_3,a_4>0$ and P-almost every $\omega\in\Omega$.

The proof is based on the routine Picard iteration approximating method, so we give only the sketch. We prove the existence here. The uniqueness of the solution can be gotten in exactly the same way, and we omit it here.

Let $q_0(t) = q_0, p_0(t) = p_0$, and for any $n \geq 0$, let

$$q_{n+1}(t) = q_0 + \int_0^t \frac{p_{n+1}(s)}{M\sqrt{1+\frac{p_{n+1}(s)^2}{a_4^2 M}}} ds, \qquad p_{n+1}(t) = p_0 + \int_0^t \dot{p}_{n+1}(s) ds,$$

with $\dot{p}_{n+1}(s)$ given by

$$\begin{aligned}\dot{p}_{n+1}(s) &= a_2 \int_{\mathbf{R}} \widetilde{\phi}(x,s) \nabla\rho(x - q_n(s)) dx \\ &\quad + \frac{1}{2} a_3^{-1} a_2^2 \int_{\mathbf{R}} dx \rho(x - q_n(s)) \int_0^{\sqrt{a_1 a_3} s} dr \\ &\quad \Big(\rho(x + r - q_n(s - \frac{1}{\sqrt{a_1 a_3}} r)) - \rho(x - r - q_n(s - \frac{1}{\sqrt{a_1 a_3}} r))\Big),\end{aligned}$$

where $\widetilde{\phi}(x,s)$ is as given by (2.11).

Let $A_1 = a_2 \|\nabla^2 \rho\|_\infty$ and $A_2 = 8 a_3^{-1} a_2^2 R_0^2 \|\nabla\rho\|_\infty \|\rho\|_\infty$. Then by definition and a simple calculation,

$$\begin{aligned}&|\dot{p}_{n+1}(u) - \dot{p}_n(u)| \\ \leq\ & a_2 \int_{|x|\leq R_0} |\widetilde{\phi}(x,u)| \|\nabla^2\rho\|_\infty |q_n(u) - q_{n-1}(u)| dx \\ &+ \frac{1}{2} a_3^{-1} a_2^2 \int_{|x|\leq R_0} dx \Big(\|\nabla\rho\|_\infty |q_n(u) - q_{n-1}(u)| 2 \|\rho\|_\infty 2R_0 \\ &\qquad + 2\|\rho\|_\infty \int_0^{(\sqrt{a_1 a_3} u)\wedge(2R_0)} \|\nabla\rho\|_\infty |q_n(u - \frac{r}{\sqrt{a_1 a_3}}) - q_{n-1}(u - \frac{r}{\sqrt{a_1 a_3}})| dr \Big) \\ \leq\ & \sup_{\theta\in[0,u]} |q_n(\theta) - q_{n-1}(\theta)| \Big(A_1 \int_{|x|\leq R_0} |\widetilde{\phi}(x,u)| dx + A_2 \Big).\end{aligned}$$

So

$$\begin{aligned}&|p_{n+1}(s) - p_n(s)| \\ \leq\ & \int_0^s |\dot{p}_{n+1}(u) - \dot{p}_n(u)| du \\ \leq\ & \sup_{\theta\in[0,s]} |q_n(\theta) - q_{n-1}(\theta)| \times s (A_1 2R_0 \sup_{|x|\leq R_0, u\in[0,s]} |\widetilde{\phi}(x,u)| + A_2),\end{aligned}$$

hence

$$\begin{aligned}&|q_{n+1}(t) - q_n(t)| \\ \leq\ & \int_0^t \Big(\frac{1}{M} |p_{n+1}(s) - p_n(s)| \wedge \frac{2a_4}{\sqrt{M}}\Big) ds \\ \leq\ & \int_0^t \frac{1}{M} \sup_{\theta\in[0,s]} |q_n(\theta) - q_{n-1}(\theta)| ds \times t \Big(A_1 2R_0 \sup_{|x|\leq R_0, u\in[0,t]} |\widetilde{\phi}(x,u)| + A_2 \Big) \quad (2.12)\end{aligned}$$

Let

$$b_n(t) = \sup_{0 \le \eta \le t} |q_{n+1}(\eta) - q_n(\eta)|.$$

Then $b_0(t) = \sup_{0 \le \eta \le t} |q_1(\eta) - q_0(\eta)| \le \frac{a_4}{\sqrt{M}} t \le \frac{a_4}{\sqrt{M}} T$. By induction, this combined with (2.12) gives us that

$$b_n(t) \le \frac{a_4}{M^{n+\frac{1}{2}} n!} T^{2n+1} \Big(A_1 2R_0 \sup_{|x| \le R_0, |u| \le T} |\widetilde{\phi}(x,u)| + A_2 \Big)^n$$

for any $n \ge 1$ and $t \in [0,T]$. By the definition of $\widetilde{\phi}$ and property of Brownian motion, we have that $\sup_{|x| \le R_0, |u| \le T} |\widetilde{\phi}(x,u)|$ is P-almost surely finite. Therefore, $\sum_{n=0}^{\infty} b_n(T) < \infty$, P-almost surely, hence $q_n(t)$ converges P-almost surely as $n \to \infty$, uniformly with respect to $t \in [0,T]$.

Write the limit as $q(t)$, $t \in [0,T]$, and let $p(t) = p_0 + \int_0^t \dot{p}(s) ds$ with $\frac{d}{dt} p(s)$ given by (2.10).

By a calculation similar to the one we just used in the construction of $q(t)$ and $p(t)$, we have that the defined $q(t)$ and $p(t)$ satisfy $\dot{q}(t) = \frac{p(t)}{M\sqrt{1+a_4^{-2} M^{-1} p(t)^2}}$, or equivalently,

$$q(t) - q_0 = \int_0^t \frac{p(s)}{M\sqrt{1 + a_4^{-2} M^{-1} p(s)^2}} ds. \tag{2.13}$$

This completes the proof of the existence. ∎

In the following sections, we will take $\lambda \to \infty$, so a_4 might not be fixed, which means that the velocity of the massive particle might be very fast. To solve this problem, we define $\tau_n = \inf\{t > 0; \Big| \frac{1}{M} \frac{p(t)}{\sqrt{1+a_4^{-2} M^{-1} p(t)^2}} \Big| \ge n\}$ for any $n \in \mathbf{N}$. (This is essential in the case that $a_4 \to \infty$. In the case that a_4 does not depend on λ, we have $\tau_n = \infty$ for any $n > a_4 M^{-1/2}$). Notice that in order to prove Theorem 1.1 (2), it suffices to prove the assertion for $t \in [0, T \wedge \tau_n]$ for any $n \in \mathbf{N}$. Choose any $n \in \mathbf{N}$ and fix it from now on. We have that $|q(t)| \le |q_0| + nt$ for any $t \in [0, T \wedge \tau_n]$. Let $R_1 = |q_0| + nT + r_\rho$, where r_ρ is the constant chosen such that $\rho(x) = 0$ for any $|x| \ge r_\rho$.

Let $\mathcal{F}_t = \sigma\{B_1(u), B_2(u); |u| \le t\}$ for any $t \ge 0$. Then the following is an easy consequence of (2.10) and (2.11).

Lemma 2.6 *$(q(t \wedge \tau_n), p(t \wedge \tau_n))$ is $\mathcal{F}_{\sqrt{a_1 a_3} t + R_1}$-measurable for any $t \in [0,T]$.*

Let $I_1(t)$ and $I_2(t)$ denote the integrals of the first and the second term on the right hand side of (2.10), respectively, *i.e.*, we let

$$\begin{aligned} I_1(t) &= a_2 \int_0^t ds \int_{\mathbf{R}} \nabla\rho(x - q(s)) \widetilde{\phi}(x,s) dx, \qquad (2.14) \\ I_2(t) &= \frac{1}{2} a_3^{-1} a_2^2 \int_0^t ds \int_{\mathbf{R}} dx \rho(x - q(s)) \int_0^{\sqrt{a_1 a_3} s} dr \\ &\qquad \Big(\rho(x + r - q(s - \frac{1}{\sqrt{a_1 a_3}} r)) - \rho(x - r - q(s - \frac{1}{\sqrt{a_1 a_3}} r)) \Big). \qquad (2.15) \end{aligned}$$

Notice that the constant c in (2.11) disappears after taking integral, since $\int_{\mathbf{R}} \nabla\rho(x-q(s))dx = 0$.

Then we have the following basic decomposition of $p(t)$:

$$p(t) = p_0 + I_1(t) + I_2(t). \tag{2.16}$$

In Sections $3 \sim 7$, we will show that, after taking limit $\lambda \to \infty$, $I_1(t)$ gives us the diffusion term of L (Lemma 5.4), and $I_2(t)$ gives us the drift term (Lemma 6.1).

3 First approximation of the term $I_1(t)$

In this and the following two sections, we show that the term $I_1(t)$ in (2.14) gives us approximately the diffusion term of our generator L in (1.4) (see Lemma 5.4 for the precise statement).

We define

$$\widetilde{s} = \Big(\Big(|s| - \frac{2R_1}{\sqrt{a_1 a_3}}\Big) \vee 0\Big) \wedge T \wedge \tau_n, \qquad s \in \mathbf{R}. \tag{3.1}$$

This is one of the two important approximations of s we induce in the present paper. (The other one is s_z, which will be given in (4.3)). We have the following as a result of Lemma 2.6.

Lemma 3.1 *For any $s \in [-T, T]$, we have that $(q(\widetilde{s}), p(\widetilde{s}))$ is $\mathcal{F}_{|y|}$-measurable for any $y \in \mathbf{R}$ satisfying $|y - \sqrt{a_1 a_3} s| \le R_1$.*

Proof. If $s \le \frac{2R_1}{\sqrt{a_1 a_3}}$, then $\widetilde{s} = 0$, hence $q(\widetilde{s})$ and $p(\widetilde{s})$ are constant. If $|s| \ge \frac{2R_1}{\sqrt{a_1 a_3}}$, by Lemma 2.6, we have that $q(\widetilde{s})$ is $\mathcal{F}_{\sqrt{a_1 a_3}|s| - R_1}$-measurable. Our assertion is now trivial since $|y - \sqrt{a_1 a_3} s| \le R_1$ implies $|y| \ge \sqrt{a_1 a_3}|s| - R_1$. ∎

The following decomposition is easy:

$$I_1(t \wedge \tau_n) = I_{11}(t) + I_{12}(t) + \cdots + I_{18}(t),$$

where

$$\begin{aligned}
I_{11}(t) &= \frac{1}{2} a_2 a_3^{-1/2} \int_0^{t\wedge\tau_n} 1_{[\frac{2R_1}{\sqrt{a_1 a_3}}, \infty)}(s) ds \int_{\mathbf{R}} \nabla\rho(x - q(\widetilde{s})) B_2(x + \sqrt{a_1 a_3} s) dx, \\
I_{12}(t) &= \frac{1}{2} a_2 a_3^{-1/2} \int_0^{t\wedge\tau_n} 1_{[\frac{2R_1}{\sqrt{a_1 a_3}}, \infty)}(s) ds \int_{\mathbf{R}} \nabla\rho(x - q(\widetilde{s})) B_2(x - \sqrt{a_1 a_3} s) dx, \\
I_{13}(t) &= \frac{1}{2} a_2 a_3^{-1/2} \int_0^{t\wedge\tau_n} ds \int_{\mathbf{R}} \Big(\nabla\rho(x - q(s)) - \nabla\rho(x - q(\widetilde{s})\Big) \times \\
&\qquad \times \Big(B_2(x + \sqrt{a_1 a_3} s) + B_2(x - \sqrt{a_1 a_3} s)\Big) dx, \\
I_{14}(t) &= \frac{1}{2} a_2 a_1^{-1} a_3^{-1/2} \int_0^{t\wedge\tau_n} ds \int_{\mathbf{R}} \Big(\nabla\rho(x - q(s)) - \nabla\rho(x - q(\widetilde{s})\Big) \times \\
&\qquad \times \Big(B_1(x + \sqrt{a_1 a_3} s) - B_1(x - \sqrt{a_1 a_3} s)\Big) dx,
\end{aligned}$$

$$
\begin{aligned}
I_{15}(t) &= \frac{1}{2}a_2a_3^{-1/2}\int_0^{t\wedge\tau_n} 1_{[0,\frac{2R_1}{\sqrt{a_1a_3}})}(s)ds\int_{\mathbf{R}}\nabla\rho(x-q(\tilde{s}))\times\\
&\qquad\times\Big(B_2(x+\sqrt{a_1a_3}s)+B_2(x-\sqrt{a_1a_3}s)\Big)dx,\\
I_{16}(t) &= \frac{1}{2}a_2a_1^{-1}a_3^{-1/2}\int_0^{t\wedge\tau_n} 1_{[0,\frac{2R_1}{\sqrt{a_1a_3}})}(s)ds\int_{\mathbf{R}}\nabla\rho(x-q(\tilde{s}))\times\\
&\qquad\times\Big(B_1(x+\sqrt{a_1a_3}s)-B_1(x-\sqrt{a_1a_3}s)\Big)dx,\\
I_{17}(t) &= -\frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}}\Big(\int_0^{x+\sqrt{a_1a_3}s} m(y)dy+\int_0^{x-\sqrt{a_1a_3}s} m(y)dy\Big)\nabla\rho(x-q(s))dx,\\
I_{18}(t) &= \frac{1}{2}a_2a_1^{-1}a_3^{-1/2}\int_0^{t\wedge\tau_n} 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)ds\int_{\mathbf{R}}\nabla\rho(x-q(\tilde{s}))\times\\
&\qquad\times\Big(B_1(x+\sqrt{a_1a_3}s)-B_1(x-\sqrt{a_1a_3}s)\Big)dx.
\end{aligned}
$$

In the rest of this section, we show that $I_{13}\sim I_{18}$ are negligible when $\lambda\to\infty$ (see Lemma 3.6).

Lemma 3.2 *We have* $\lim_{\lambda\to\infty}E\Big[\sup_{0\le t\le T}|I_{1i}(t)|\Big]=0$ *for* $i=3,4$.

Proof. We prove the assertion for $i=3$ here. The one for $i=4$ can be gotten in exactly the same way with the help of (A3).

First make the decomposition

$$I_{13}(t)=I_{131}(t)+I_{132}(t)$$

with

$$
\begin{aligned}
I_{131}(t) &= \frac{1}{2}a_2a_3^{-1/2}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}}\Big[\nabla\rho(x-q(s))-\nabla\rho(x-q(\tilde{s}))+\nabla^2\rho(x-q(\tilde{s}))(q(s)-q(\tilde{s}))\Big]\\
&\qquad\times\Big(B_2(x+\sqrt{a_1a_3}s)+B_2(x-\sqrt{a_1a_3}s)\Big)dx, \qquad (3.2)\\
I_{132}(t) &= -\frac{1}{2}a_2a_3^{-1/2}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}}\nabla^2\rho(x-q(\tilde{s}))(q(s)-q(\tilde{s}))\\
&\qquad\times\Big(B_2(x+\sqrt{a_1a_3}s)+B_2(x-\sqrt{a_1a_3}s)\Big)dx. \qquad (3.3)
\end{aligned}
$$

For any $s\in[0,t\wedge\tau_n]$, we have $|q(s)-q(\tilde{s})|\le n|s-\tilde{s}|\le\frac{2R_1n}{\sqrt{a_1a_3}}$, so

$$
\begin{aligned}
&\Big|\nabla\rho(x-q(s))-\nabla\rho(x-q(\tilde{s}))+\nabla^2\rho(x-q(\tilde{s}))(q(s)-q(\tilde{s}))\Big|\\
\le\ &\|\nabla^3\rho\|_\infty|q(s)-q(\tilde{s})|^2\\
\le\ &\|\nabla^3\rho\|_\infty\Big(\frac{2R_1n}{\sqrt{a_1a_3}}\Big)^2.
\end{aligned}
$$

Also, the integrand in (3.2) is equal to 0 if $|x|\ge R_1$, so the integral domain $\{x\in\mathbf{R}\}$ in (3.2) can be converted to $\{|x|\le R_1\}$, and in this domain, we have for any $s\in[0,T]$,

$$
\begin{aligned}
E[|B_2(x\pm\sqrt{a_1a_3}s)|] &\le E[|B_2(x\pm\sqrt{a_1a_3}s)|^2]^{1/2}\le(R_1+\sqrt{a_1a_3}T)^{1/2}\\
&\le R_1^{1/2}+(a_1a_3)^{1/4}T^{1/2}\le(1+(a_1a_3)^{1/4})(R_1^{1/2}+T^{1/2}).
\end{aligned}
$$

Therefore, with $C_1 := 8n^2TR_1^3\|\nabla^3\rho\|_\infty(R_1^{1/2}+T^{1/2})$, we have

$$\begin{aligned}
& E\Big[\sup_{0\le t\le T}|I_{131}(t)|\Big] \\
\le\ & \frac{1}{2}a_2a_3^{-1/2}\int_0^T ds\int_{|x|\le R_1}\|\nabla^3\rho\|_\infty\Big(\frac{2R_1n}{\sqrt{a_1a_3}}\Big)^2 E\Big[|B_2(x+\sqrt{a_1a_3}s)|+|B_2(x-\sqrt{a_1a_3}s)|\Big]dx \\
\le\ & \frac{1}{2}a_2a_3^{-1/2}T2R_1\|\nabla^3\rho\|_\infty\Big(\frac{2R_1n}{\sqrt{a_1a_3}}\Big)^2 2(1+(a_1a_3)^{1/4})(R_1^{1/2}+T^{1/2}) \\
=\ & C_1a_2a_3^{-1/2}\frac{1}{a_1a_3}(1+(a_1a_3)^{1/4}),
\end{aligned}$$

which converges to 0 as $\lambda\to\infty$ by (A1) and (A2).

For the term $I_{132}(t)$, we have by Lemma 3.1 that

$$\begin{aligned}
\int_{\mathbf{R}}\nabla^2\rho(x-q(\tilde{s}))B_2(x+\sqrt{a_1a_3}s)dx &= \int_{\mathbf{R}}\nabla^2\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))B_2(y)dy \\
&= -\int_{\mathbf{R}}\nabla\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y) \qquad (3.4)
\end{aligned}$$

Similarly,

$$\int_{\mathbf{R}}\nabla^2\rho(x-q(\tilde{s}))B_2(x-\sqrt{a_1a_3}s)dx = -\int_{\mathbf{R}}\nabla\rho(y+\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y).$$

So

$$\begin{aligned}
I_{132}(t) &= \frac{1}{2}a_2a_3^{-1/2}\int_0^{t\wedge\tau_n} ds(q(s)-q(\tilde{s})) \\
&\quad\times\Big(\int_{\mathbf{R}}\nabla\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y)+\int_{\mathbf{R}}\nabla\rho(y+\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y)\Big).
\end{aligned}$$

We have $|q(s)-q(\tilde{s})|\le\frac{2R_1n}{\sqrt{a_1a_3}}$. Also,

$$\begin{aligned}
& E\Big[\Big|\int_{\mathbf{R}}\nabla\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y)\Big|\Big] \\
\le\ & E\Big[\Big|\int_{|y-\sqrt{a_1a_3}s|\le R_1}\nabla\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y)\Big|^2\Big]^{1/2} \\
\le\ & \Big(\int_{|y-\sqrt{a_1a_3}s|\le R_1}\|\nabla\rho\|_\infty^2dy\Big)^{1/2} \\
=\ & \|\nabla\rho\|_\infty\sqrt{2R_1},
\end{aligned}$$

and similarly,

$$E\Big[\Big|\int_{\mathbf{R}}\nabla\rho(y+\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y)\Big|\Big]\le\|\nabla\rho\|_\infty\sqrt{2R_1}.$$

So with $C_2 := 2nTR_1\|\nabla\rho\|_\infty\sqrt{2R_1}$, we have

$$\begin{aligned}
& E\Big[\sup_{0\le t\le T}|I_{132}(t)|\Big] \\
\le\ & \frac{1}{2}a_2a_3^{-1/2}\int_0^T ds\frac{2R_1n}{\sqrt{a_1a_3}}\Big(E\Big[\Big|\int_{|y-\sqrt{a_1a_3}s|\le R_1}\nabla\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y)\Big|\Big] \\
& \qquad\qquad +E\Big[\Big|\int_{|y+\sqrt{a_1a_3}s|\le R_1}\nabla\rho(y+\sqrt{a_1a_3}s-q(\tilde{s}))dB_2(y)\Big|\Big]\Big) \\
\le\ & \frac{1}{2}a_2a_3^{-1/2}T\frac{2R_1n}{\sqrt{a_1a_3}}2\|\nabla\rho\|_\infty\sqrt{2R_1} \\
=\ & C_2a_2a_3^{-1/2}\cdot\frac{1}{\sqrt{a_1a_3}},
\end{aligned}$$

which converges to 0 as $\lambda\to\infty$ by (A1) and (A2).

Combining the above, we get our assertion for I_{13}. ∎

Lemma 3.3 *We have* $\lim_{\lambda\to\infty}E\Big[\sup_{0\le t\le T}|I_{1i}(t)|\Big]=0$ *for* $i=5,6$.

Proof. For any $x\in\mathbf{R}$ with $|x|\le R_1$ and any $s\in[0,\frac{2R_1}{\sqrt{a_1a_3}})$, we have

$$E[|B_2(x\pm\sqrt{a_1a_3}s)|]\le E[|B_2(x\pm\sqrt{a_1a_3}s)|^2]^{1/2}\le(|x|+\sqrt{a_1a_3}s)^{1/2}\le\sqrt{3R_1}.$$

Therefore, with $C_3:=4R_1^2\|\nabla\rho\|_\infty\sqrt{3R_1}$, we have

$$\begin{aligned}
& E\Big[\sup_{0\le t\le T}|I_{15}(t)|\Big] \\
\le\ & \frac{1}{2}a_2a_3^{-1/2}\int_0^T 1_{[0,\frac{2R_1}{\sqrt{a_1a_3}})}(s)ds \\
& \quad\times\int_{|x|\le R_1}\|\nabla\rho\|_\infty\Big(E[|B_2(x+\sqrt{a_1a_3}s)|]+E[|B_2(x-\sqrt{a_1a_3}s)|]\Big)dx \\
\le\ & \frac{1}{2}a_2a_3^{-1/2}\frac{2R_1}{\sqrt{a_1a_3}}2R_1\|\nabla\rho\|_\infty 2\sqrt{3R_1} \\
=\ & C_3a_2a_3^{-1/2}\cdot\frac{1}{\sqrt{a_1a_3}},
\end{aligned}$$

which converges to 0 as $\lambda\to\infty$ by (A1) and (A2).

The assertion for $I_{16}(t)$ can be gotten in exactly the same way by (A3). ∎

Lemma 3.4 $\sup_{\omega\in\Omega,t\in[0,T]}|I_{17}(t)|\to 0$ *as* $\lambda\to\infty$.

Proof. By using change of variables and the formula of integration by parts, we have

$$\begin{aligned}
I_{17} =\ & -\frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n}ds\int_{\mathbf{R}}\Big(\int_0^x m(y)dy\Big)\times \\
& \quad\times\Big(\nabla\rho(x-\sqrt{a_1a_3}s-q(s))+\nabla\rho(x+\sqrt{a_1a_3}s-q(s))\Big)dx \\
=\ & \frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n}ds\int_{\mathbf{R}}\Big(\rho(x-\sqrt{a_1a_3}s-q(s))+\rho(x+\sqrt{a_1a_3}s-q(s))\Big)m(x)dx.
\end{aligned}$$

So we can rewrite

$$I_{17}(t) = I_{171}(t) + I_{172}(t) + I_{173}(t)$$

with

$$\begin{aligned}
I_{171}(t) &= \frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}}\Big(\rho(x-\sqrt{a_1a_3}s-q(s))-\rho(x-\sqrt{a_1a_3}s-q_0)\Big)m(x)dx \\
&\quad+\frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}}\Big(\rho(x+\sqrt{a_1a_3}s-q(s))-\rho(x+\sqrt{a_1a_3}s-q_0)\Big)m(x)dx \\
I_{172}(t) &= \frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}}\rho(x-\sqrt{a_1a_3}s-q_0)m(x)dx \\
I_{173}(t) &= \frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}}\rho(x+\sqrt{a_1a_3}s-q_0)m(x)dx.
\end{aligned}$$

Notice that by definition and assumption, $m(x) = 0$ if $|x| \geq R_1$. So in all of the integrals above, the integral domains $\{x \in \mathbf{R}\}$ can be rewritten as $\{|x| \leq R_1\}$, hence the integral domain $\{s \in [0, t\wedge\tau_n]\}$ can be rewritten as $\{s \in [0, t\wedge\tau_n]\}\cap\{|s| \leq \frac{2R_1}{\sqrt{a_1a_3}}\}$.

Therefore, with $C_4 := 2nR_1^2\|\nabla\rho\|_\infty\int_{\{|x|\leq R_1\}}|m(x)|dx$, we have

$$\begin{aligned}
|I_{171}(t)| &\leq \frac{1}{2}a_2^2a_3^{-1}\int_{\{|x|\leq R_1\}}|m(x)|dx\int_{\{0\leq s\leq\frac{2R_1}{\sqrt{a_1a_3}}\}}\|\nabla\rho\|_\infty snds\times 2 \\
&= C_4a_2^2a_3^{-1}\Big(\frac{1}{\sqrt{a_1a_3}}\Big)^2.
\end{aligned}$$

The last expression does not depend on $\omega \in \Omega$ or $t \in [0, T]$, and converges to 0 as $\lambda \to \infty$ by (A1) and (A2).

For $I_{172}(t) + I_{173}(t)$, notice that in general,

$$\int_{\mathbf{R}} f'(x-c_0)f(x)dx = -\int_{\mathbf{R}} f(x-c_0)f'(x)dx = -\int_{\mathbf{R}} f(x)f'(x+c_0)dx$$

for any $f \in C_0^1$ and $c_0 \in \mathbf{R}$. Therefore, since $m'(x) = -\rho(x-q_0)$ by definition, we have

$$\begin{aligned}
I_{172}(t) &= -\frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}} m'(x-\sqrt{a_1a_3}s)m(x)dx \\
&= \frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}} m'(x+\sqrt{a_1a_3}s)m(x)dx \\
&= -\frac{1}{2}a_2^2a_3^{-1}\int_0^{t\wedge\tau_n} ds\int_{\mathbf{R}}\rho(x+\sqrt{a_1a_3}s-q_0)m(x)dx \\
&= -I_{173}(t),
\end{aligned}$$

hence

$$I_{172}(t) + I_{173}(t) = 0.$$

This completes our proof. ∎

Lemma 3.5 $E\Big[\sup_{0\leq t\leq T}|I_{18}(t)|\Big] \to 0$ *as* $\lambda \to \infty$.

Proof. By exactly the same method as in Lemma 4.2 below, we get that $\sup_{\lambda\in[1,\infty)} E\Big[\sup_{0\le t\le T} a_1|I_{18}(t)|\Big] < \infty$. Since $a_1 \to \infty$ by (A3), this implies our assertion. ∎

By Lemmas 3.2 $\sim$ 3.5, we have that $I_{13}(t) \sim I_{18}(t)$ are negligible, hence $I_1(t)$ is approximately equal to $I_{11}(t) + I_{12}(t)$. Precisely, we have the following.

Lemma 3.6 $\lim_{\lambda\to\infty} E[\sup_{0\le t\le T} |I_1(t) - I_{11}(t) - I_{12}(t)|] = 0$.

The discussion with respect to $I_{11}(t)$ and $I_{12}(t)$ will be given in the next two sections.

4 Tightness of $I_{11}(t)$ and $I_{12}(t)$

We deal with the terms $I_{11}(t)$ and $I_{12}(t)$ in this and the next section. The discussion is divided into two steps. First in this section, we give their decompositions and show that {the distribution of $\{I_{1i}(t)\}_{0\le t\le T}; \lambda \ge 1$} is tight for $i = 1, 2$; and in the next section, with the help of the result of this section, we find the expressions of their limits as $\lambda \to \infty$.

As in Kusuoka-Liang [11], we are considering the problem in space D given by

$$\begin{aligned} D &= D[0,T] \\ &= \Big\{w : [0,T] \to \mathbf{R};\quad w(t) = w(t+) := \lim_{s\downarrow t} w(s), t \in [0,T), \\ &\qquad\qquad \text{and } w(t-) := \lim_{s\uparrow t} w(s) \text{ exists}, t \in (0,T]\Big\}, \end{aligned}$$

with Skorohod metric which is given as follows. Let

$$\Lambda = \Big\{\lambda : [0,T] \to [0,T];\ \text{continuous, non-decreasing}, \lambda(0) = 0, \lambda(T) = T\Big\}.$$

For any $\lambda \in \Lambda$, we define

$$\|\lambda\|^0 = \sup_{0\le s<t\le T} \Big|\log \frac{\lambda(t) - \lambda(s)}{t - s}\Big|.$$

Also, for any $w, \widetilde{w} \in D$, we define

$$d^0(w, \widetilde{w}) = \inf_{\lambda\in\Lambda} \Big\{\|\lambda\|^0 \vee \|w - \widetilde{w}\circ\lambda\|_\infty\Big\},$$

where $\|w\|_\infty = \sup_{0\le t\le T} |w(t)|$. Finally, let $\mathcal{P}(D)$ denote the set of probability measures on D. (See Billingsley [2] for more details about the space D and the tightness of the probability measures on it).

The main result of this section is the following.

Lemma 4.1 {*The distribution of* $\{I_{1i}(t)\}_{t\in[0,T]}; \lambda \ge 1$} *is tight in* $\mathcal{P}(D)$ *for* $i = 1, 2$.

We prove Lemma 4.1 in the rest of this section. Since the proofs are exactly the same, we give here the proof for $i = 1$ only.

First, by Lemma 3.1, we can rewrite $I_{11}(t)$ as $I_{11}(t) = \overline{I_{11}}(t \wedge \tau_n)$ with $\overline{I_{11}}(\cdot)$ given by

$$\overline{I_{11}}(t) = -\frac{1}{2}a_2 a_3^{-1/2} \int_{\mathbf{R}} dB_2(y) \int_0^t 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y - \sqrt{a_1a_3}s - q(\tilde{s}))ds. \qquad (4.1)$$

It suffices to prove the tightness for $\overline{I_{11}}(t)$. By [11, Theorem 5.1.7], this is a result of Lemmas 4.2, 4.3 and 4.4.

Lemma 4.2 $\sup_{\lambda\geq 1} E[\sup_{0\leq t\leq T} |\overline{I_{11}}(t)|^2] < \infty$. *In particular,* $\sup_{\lambda\geq 1} E[\sup_{0\leq t\leq T} |\overline{I_{11}}(t)|] < \infty$.

Lemma 4.3 *There exists a contant $C > 0$ such that*

$$E[|\overline{I_{11}}(t_2) - \overline{I_{11}}(t_1)|^2 \cdot |\overline{I_{11}}(t_3) - \overline{I_{11}}(t_2)|^2] \leq C(t_3 - t_1)^2$$

for any $0 \leq t_1 \leq t_2 \leq t_3 \leq T$.

Lemma 4.4 *There exists a contant $C > 0$ such that*

$$E[|\overline{I_{11}}(t_2) - \overline{I_{11}}(t_1)|] \leq C(t_2 - t_1)^{1/2}$$

for any $0 \leq t_1 \leq t_2 \leq T$.

We give a decomposition of $\overline{I_{11}}$ before proving Lemma 4.2. This decomposition is also used in Section 5.

Notice that by definition, $\overline{I_{11}}(t) = 0$ if $t \leq \frac{2R_1}{\sqrt{a_1a_3}}$. Also, for any $t > \frac{2R_1}{\sqrt{a_1a_3}}$, if $y < 0$, then $\int_0^t 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y - \sqrt{a_1a_3}s - q(\tilde{s}))ds = 0$; and if $0 < y < \sqrt{a_1a_3}t - R_1$, then $\rho(y - \sqrt{a_1a_3}s - q(\tilde{s})) = 0$ for any $s \in (-\infty, -\frac{R_1}{\sqrt{a_1a_3}}] \cup [t, \infty)$, hence

$$\begin{aligned}
&\int_0^t 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y - \sqrt{a_1a_3}s - q(\tilde{s}))ds - \int_{-\infty}^{\infty} \rho(y - \sqrt{a_1a_3}s - q(\tilde{s}))ds \\
= &-\int_{-\frac{R_1}{\sqrt{a_1a_3}}}^{\frac{2R_1}{\sqrt{a_1a_3}}} \rho(y - \sqrt{a_1a_3}s - q(\tilde{s}))ds, \qquad t \in [\frac{2R_1}{\sqrt{a_1a_3}}, T]. \qquad (4.2)
\end{aligned}$$

Now, we induce a new approximation of $|s|$, given as follows: Let

$$s_z = \Big(\frac{|z| - R_1}{\sqrt{a_1a_3}} \wedge T \wedge \tau_n\Big) \vee 0, \qquad z \in \mathbf{R}. \qquad (4.3)$$

We say that this is an approximation of $|s|$ because of the following: For any $|s| \leq T\wedge\tau_n$, whenever $\rho(y - \sqrt{a_1a_3}s - q(s_y)) \neq 0$ or $\rho(y - \sqrt{a_1a_3}s - q(\tilde{s})) \neq 0$, we have that $|s_y - |s|| \leq \frac{2R_1}{\sqrt{a_1a_3}}$. Indeed, we get by assumption $|y - \sqrt{a_1a_3}s| \leq R_1$. Since $|s| \leq T\wedge\tau_n$, this implies that $s_y = \frac{|y|-R_1}{\sqrt{a_1a_3}} \vee 0$. If $|y| \leq R_1$, then $|s_y - |s|| = |s| \leq \frac{2R_1}{\sqrt{a_1a_3}}$; if $|y| \geq R_1$, then $|s_y - |s|| = |\frac{|y|-R_1}{\sqrt{a_1a_3}} - |s|| \leq |\frac{|y|}{\sqrt{a_1a_3}} - |s|| + \frac{R_1}{\sqrt{a_1a_3}} \leq \frac{2R_1}{\sqrt{a_1a_3}}$. This completes the proof.

Also, similarly to the case for $\tilde{s}$, we have the following.

Lemma 4.5 *For any $y \in \mathbf{R}$, we have that $(q(s_y), p(s_y))$ is $\mathcal{F}_{|y|}$-measurable.*

Notice that

$$\int_{-\infty}^{\infty} \rho(y - \sqrt{a_1 a_3} s - q(s_y)) ds = \frac{1}{\sqrt{a_1 a_3}} \int_{-\infty}^{\infty} \rho(u) du \tag{4.4}$$

for any $y \in \mathbf{R}$. This combined with (4.2) and the paragraph prior to it gives us the following decomposition of $\overline{I_{11}}(t)$ for $t \geq \frac{2R_1}{\sqrt{a_1 a_3}}$.

$$\overline{I_{11}}(t) = K_1(t) + K_2(t) + \cdots + K_5(t), \qquad t \in [\frac{2R_1}{\sqrt{a_1 a_3}}, T],$$

with

$$\begin{aligned}
K_1(t) &= -\frac{1}{2} a_2 a_3^{-1/2} \frac{1}{\sqrt{a_1 a_3}} \Big(\int_{-\infty}^{\infty} \rho(u) du \Big) B_2(\sqrt{a_1 a_3} t), \qquad (4.5)\\
K_2(t) &= -\frac{1}{2} a_2 a_3^{-1/2} \frac{1}{\sqrt{a_1 a_3}} \Big(\int_{-\infty}^{\infty} \rho(u) du \Big) \Big(B_2(\sqrt{a_1 a_3} t - R_1) - B_2(\sqrt{a_1 a_3} t) \Big),\\
K_3(t) &= -\frac{1}{2} a_2 a_3^{-1/2} \int_{0 < y < \sqrt{a_1 a_3} t - R_1} dB_2(y) \int_{-\infty}^{\infty}\\
&\qquad \Big\{ \rho(y - \sqrt{a_1 a_3} s - q(\tilde{s})) - \rho(y - \sqrt{a_1 a_3} s - q(s_y)) \Big\} ds,\\
K_4(t) &= \frac{1}{2} a_2 a_3^{-1/2} \int_{0 < y < \sqrt{a_1 a_3} t - R_1} dB_2(y) \int_{-\frac{R_1}{\sqrt{a_1 a_3}}}^{\frac{2R_1}{\sqrt{a_1 a_3}}} \rho(y - \sqrt{a_1 a_3} s - q(\tilde{s})) ds,\\
K_5(t) &= -\frac{1}{2} a_2 a_3^{-1/2} \int_{y > \sqrt{a_1 a_3} t - R_1} dB_2(y) \int_0^t 1_{[\frac{2R_1}{\sqrt{a_1 a_3}}, \infty)}(s) \rho(y - \sqrt{a_1 a_3} s - q(\tilde{s})) ds.
\end{aligned}$$

By (A1), the term $K_1(t)$ is nothing but a constant multiple of Brownian motion, so we have the following.

Lemma 4.6 $\sup_{\lambda \geq 1} E[\sup_{\frac{2R_1}{\sqrt{a_1 a_3}} \leq t \leq T} |K_1(t)|^2] < \infty$.

The fact that $K_3(t)$ and $K_4(t)$ are negligible is easy:

Lemma 4.7 *We have $\lim_{\lambda \to \infty} E[\sup_{\frac{2R_1}{\sqrt{a_1 a_3}} \leq t \leq T} |K_i(t)|^2] = 0$ for $i = 3, 4$.*

Proof. For $i = 3$, notice that

$$\begin{aligned}
|\rho(y - \sqrt{a_1 a_3} s - q(\tilde{s})) - \rho(y - \sqrt{a_1 a_3} s - q(s_y))| &\leq \|\nabla \rho\|_\infty |\tilde{s} - s_y| 1_{\{|y - \sqrt{a_1 a_3} s| \leq R_1\}}\\
&\leq \|\nabla \rho\|_\infty \frac{4 R_1 n}{\sqrt{a_1 a_3}} 1_{\{|y - \sqrt{a_1 a_3} s| \leq R_1\}},
\end{aligned}$$

so by Lemma 3.1 and Lemma 4.5, with $C_6 := (8nR_1^2\|\nabla\rho\|_\infty)^2T$, we have

$$
\begin{aligned}
& E[\sup_{\frac{2R_1}{\sqrt{a_1a_3}}\le t\le T}|K_3(t)|^2] \\
\le\ & \Big(-\frac{1}{2}a_2a_3^{-1/2}\Big)^2 4\int_{0<y<\sqrt{a_1a_3}T-R_1}dy \\
& \quad\times E\Big[\Big(\int_{-\infty}^{\infty}\Big\{\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))-\rho(y-\sqrt{a_1a_3}s-q(s_y))\Big\}ds\Big)^2\Big] \\
\le\ & \Big(-\frac{1}{2}a_2a_3^{-1/2}\Big)^2 4(\sqrt{a_1a_3}T-R_1)\Big(\frac{2R_1}{\sqrt{a_1a_3}}\cdot\|\nabla\rho\|_\infty\frac{4R_1n}{\sqrt{a_1a_3}}\Big)^2 \\
\le\ & (8nR_1^2\|\nabla\rho\|_\infty)^2T\Big(a_2a_3^{-1/2}\Big)^2\Big(\frac{1}{\sqrt{a_1a_3}}\Big)^3 = C_6\Big(a_2a_3^{-1/2}\Big)^2\Big(\frac{1}{\sqrt{a_1a_3}}\Big)^3,
\end{aligned}
$$

which converges to 0 as $\lambda\to\infty$ by (A1) and (A2).

The proof for $i=4$ is similar. Since

$$
\Big|\int_{-\frac{R_1}{\sqrt{a_1a_3}}}^{\frac{2R_1}{\sqrt{a_1a_3}}}\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))ds\Big|\le\frac{3R_1}{\sqrt{a_1a_3}}\|\rho\|_\infty 1_{\{y\le 3R_1\}},
$$

with $C_7 := (3R_1)^3\|\rho\|_\infty^2$, we have

$$
\begin{aligned}
& E[\sup_{\frac{2R_1}{\sqrt{a_1a_3}}\le t\le T}|K_4(t)|^2]\le 4E[|K_4(T)|^2] \\
=\ & \Big(\frac{1}{2}a_2a_3^{-1/2}\Big)^2 4\int_{0<y<\sqrt{a_1a_3}T-R_1}dyE\Big[\Big(\int_{-\frac{R_1}{\sqrt{a_1a_3}}}^{\frac{2R_1}{\sqrt{a_1a_3}}}\rho(y-\sqrt{a_1a_3}s-q(\tilde{s}))ds\Big)^2\Big] \\
\le\ & \Big(\frac{1}{2}a_2a_3^{-1/2}\Big)^2 4\cdot 3R_1\cdot\Big(\frac{3R_1}{\sqrt{a_1a_3}}\|\rho\|_\infty\Big)^2 = C_7(a_2a_3^{-1/2})^2\frac{1}{a_1a_3},
\end{aligned}
$$

which converges to 0 as $\lambda\to\infty$ by (A1) and (A2). ∎

The discussion with respect to $K_2(t)$ and $K_5(t)$ is more complicated. We show in the present section that they are L^2 bounded (see Lemma 4.8), which is enough for the proof of Lemma 4.2. The fact that they are also negligible will be proved in the next section.

Lemma 4.8 *We have* $\sup_{\lambda\ge1}E[\sup_{\frac{2R_1}{\sqrt{a_1a_3}}\le t\le T}|K_i(t)|^2]<\infty$ *for* $i=2,5$.

Proof. The assertion for $i=2$ is a result of (A1), (A2) and the following calculation.

$$
\begin{aligned}
& E\Big[\sup_{\frac{2R_1}{\sqrt{a_1a_3}}\le t\le T}|B_2(\sqrt{a_1a_3}t-R_1)-B_2(\sqrt{a_1a_3}t)|^2\Big] \\
\le\ & 2E\Big[\sup_{\frac{2R_1}{\sqrt{a_1a_3}}\le t\le T}|B_2(\sqrt{a_1a_3}t-R_1)|^2\Big]+2E\Big[\sup_{\frac{2R_1}{\sqrt{a_1a_3}}\le t\le T}|B_2(\sqrt{a_1a_3}t)|^2\Big] \\
\le\ & 16\sqrt{a_1a_3}T.
\end{aligned}
$$

For $i=5$, first notice that by the formula of integration by parts,

$$
\begin{aligned}
K_5(t) = -\frac{1}{2}a_2a_3^{-1/2}\int_0^t 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)ds \\
\Big(-\int_{\sqrt{a_1a_3}t-R_1<y<\sqrt{a_1a_3}t+R_1} B_2(y)\nabla\rho(y-\sqrt{a_1a_3}s-q(\widetilde{s}))dy \\
-B_2(\sqrt{a_1a_3}t-R_1)\rho(\sqrt{a_1a_3}t-R_1-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\Big|-\int_{\sqrt{a_1a_3}t-R_1<y<\sqrt{a_1a_3}t+R_1} B_2(y)\nabla\rho(y-\sqrt{a_1a_3}s-q(\widetilde{s}))dy \\
&\quad -B_2(\sqrt{a_1a_3}t-R_1)\rho(\sqrt{a_1a_3}t-R_1-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big| \\
\leq\ & \sup_{u\in[\sqrt{a_1a_3}t-R_1,\sqrt{a_1a_3}t+R_1]}|B_2(u)|(2R_1\|\nabla\rho\|_\infty+\|\rho\|_\infty)1_{\{s>t-\frac{2R_1}{\sqrt{a_1a_3}}\}},
\end{aligned}
$$

we get

$$
|K_5(t)| \leq \frac{1}{2}a_2a_3^{-1/2}\frac{2R_1}{\sqrt{a_1a_3}}(2R_1\|\nabla\rho\|_\infty+\|\rho\|_\infty)\sup_{0\leq u\leq\sqrt{a_1a_3}T+R_1}|B_2(u)|.
$$

We have

$$
\begin{aligned}
E\Big[\sup_{0\leq u\leq\sqrt{a_1a_3}T+R_1}|B_2(u)|^2\Big] &\leq 4E\Big[|B_2(\sqrt{a_1a_3}T+R_1)|^2\Big] \\
&= 4\ \ (\sqrt{a_1a_3}T+R_1) \leq 4(\sqrt{a_1a_3}+1)(T+R_1).
\end{aligned}
$$

Therefore, with $C_8 := R_1^2(2R_1\|\nabla\rho\|_\infty+\|\rho\|_\infty)^2 4(T+R_1)$, we have

$$
\begin{aligned}
&E[\sup_{\frac{2R_1}{\sqrt{a_1a_3}}\leq t\leq T}|K_5(t)|^2] \\
\leq\ & \Big(\frac{1}{2}a_2a_3^{-1/2}\frac{2R_1}{\sqrt{a_1a_3}}(2R_1\|\nabla\rho\|_\infty+\|\rho\|_\infty)\Big)^2 E\Big[\sup_{0\leq u\leq\sqrt{a_1a_3}T}|B_2(u)|^2\Big] \\
\leq\ & \Big(\frac{1}{2}a_2a_3^{-1/2}\frac{2R_1}{\sqrt{a_1a_3}}(2R_1\|\nabla\rho\|_\infty+\|\rho\|_\infty)\Big)^2 4(\sqrt{a_1a_3}+1)(T+R_1) \\
=\ & C_8\Big(a_2a_3^{-1/2}\frac{1}{\sqrt{a_1a_3}}\Big)^2(\sqrt{a_1a_3}+1),
\end{aligned}
$$

which is bounded for $\lambda\geq 1$ by (A1) and (A2). ∎

Proof of Lemma 4.2. It is just a combination of Lemmas 4.6, 4.7 and 4.8. ∎

Proof of Lemma 4.3. For any $0\leq t_1\leq t_2\leq t_3\leq T$, we have by (4.1) that

$$
\begin{aligned}
&\overline{I_{11}}(t_2)-\overline{I_{11}}(t_1) \\
=\ & -\frac{1}{2}a_2a_3^{-1/2}\int_{\mathbf{R}} dB_2(y)\int_{t_1}^{t_2} 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y-\sqrt{a_1a_3}s-q(\widetilde{s}))ds \\
=\ & -\frac{1}{2}a_2a_3^{-1/2}\int_{(\sqrt{a_1a_3}t_1-R_1,\sqrt{a_1a_3}t_2+R_1)} dB_2(y)\int_{t_1}^{t_2} 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y-\sqrt{a_1a_3}s-q(\widetilde{s}))ds,
\end{aligned}
$$

In the same way, the similar re-expression for $\overline{I_{11}}(t_3) - \overline{I_{11}}(t_2)$ holds, too.

Rewrite the integral domains of y as $(\sqrt{a_1a_3}t_1 - R_1, \sqrt{a_1a_3}t_2 - R_1) \cup (\sqrt{a_1a_3}t_2 - R_1, \sqrt{a_1a_3}t_2 + R_1)$ and $(\sqrt{a_1a_3}t_2 - R_1, \sqrt{a_1a_3}t_2 + R_1) \cup (\sqrt{a_1a_3}t_2 + R_1, \sqrt{a_1a_3}t_3 + R_1)$. Since

$$(a+b)^2(c+d)^2 \leq 2a^2(c+d)^2 + 4b^2c^2 + 4b^2d^2, \qquad \text{for all } a,b,c,d \in \mathbf{R},$$

in order to show Lemma 4.3, it sufficient to show the estimates for the four corresponding terms.

First, taking conditional expectation with respect to $\mathcal{F}_{\sqrt{a_1a_3}t_2 - R_1}$, we get

$$\begin{aligned}
&\Big(\frac{1}{2}a_2 a_3^{-1/2}\Big)^4 \\
&\qquad E\Big[\Big(\int_{(\sqrt{a_1a_3}t_1 - R_1, \sqrt{a_1a_3}t_2 - R_1)} dB_2(y_1) \int_{t_1}^{t_2} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}}, \infty)}(s)\rho(y_1 - \sqrt{a_1a_3}s - q(\tilde{s}))\Big)^2 \times \\
&\qquad \times \Big(\int_{(\sqrt{a_1a_3}t_2 - R_1, \sqrt{a_1a_3}t_3 + R_1)} dB_2(y_2) \int_{t_2}^{t_3} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}}, \infty)}(s)\rho(y_2 - \sqrt{a_1a_3}s - q(\tilde{s}))\Big)^2\Big] \\
= \;&\Big(\frac{1}{2}a_2 a_3^{-1/2}\Big)^4 \\
&\qquad E\Big[\Big(\int_{(\sqrt{a_1a_3}t_1 - R_1, \sqrt{a_1a_3}t_2 - R_1)} dB_2(y_1) \int_{t_1}^{t_2} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}}, \infty)}(s)\rho(y_1 - \sqrt{a_1a_3}s - q(\tilde{s}))\Big)^2 \times \\
&\qquad \times E\Big[\Big(\int_{(\sqrt{a_1a_3}t_2 - R_1, \sqrt{a_1a_3}t_3 + R_1)} dB_2(y_2) \int_{t_2}^{t_3} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}}, \infty)}(s)\rho(y_2 - \sqrt{a_1a_3}s - q(\tilde{s}))\Big)^2 \\
&\qquad\qquad \Big| \mathcal{F}_{\sqrt{a_1a_3}t_2 - R_1}\Big]\Big].
\end{aligned} \tag{4.6}$$

For the conditional expectation above, notice that $\{B_2(y_2); y_2 \in (\sqrt{a_1a_3}t_2 - R_1, \sqrt{a_1a_3}t_3 + R_1)\}$ is independent to $\mathcal{F}_{\sqrt{a_1a_3}t_2 - R_1}$, so

$$\begin{aligned}
&E\Big[\Big(\int_{(\sqrt{a_1a_3}t_2 - R_1, \sqrt{a_1a_3}t_3 + R_1)} dB_2(y_2) \\
&\qquad \int_{t_2}^{t_3} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}}, \infty)}(s)\rho(y_2 - \sqrt{a_1a_3}s - q(\tilde{s}))\Big)^2 \Big| \mathcal{F}_{\sqrt{a_1a_3}t_2 - R_1}\Big] \\
= \;& \int_{(\sqrt{a_1a_3}t_2 - R_1, \sqrt{a_1a_3}t_3 + R_1)} dy_2 E\Big[\Big(\int_{t_2}^{t_3} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}}, \infty)}(s)\rho(y_2 - \sqrt{a_1a_3}s - q(\tilde{s}))\Big)^2 \Big| \mathcal{F}_{\sqrt{a_1a_3}t_2 - R_1}\Big] \\
\leq \;& \Big(\sqrt{a_1a_3}(t_3 - t_2) + 2R_1\Big)\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}} \wedge (t_3 - t_2)\|\rho\|_\infty\Big)^2,
\end{aligned}$$

where when passing to the last line, we used the obvious fact

$$\Big| \int_{t_2}^{t_3} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}}, \infty)}(s)\rho(y_2 - \sqrt{a_1a_3}s - q(\tilde{s}))\Big| \leq \frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}} \wedge \Big((t_3 - t_2)\|\rho\|_\infty\Big). \tag{4.7}$$

In the same way, we have

$$E\Big[\Big(\int_{(\sqrt{a_1a_3}t_1 - R_1, \sqrt{a_1a_3}t_2 - R_1)} dB_2(y_1) \int_{t_1}^{t_2} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}}, \infty)}(s)\rho(y_1 - \sqrt{a_1a_3}s - q(\tilde{s}))\Big)^2\Big]$$

$$
\begin{aligned}
&= \int_{(\sqrt{a_1a_3}t_1-R_1,\sqrt{a_1a_3}t_2-R_1)} dy_1 E\Big[\Big(\int_{t_1}^{t_2} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y_1-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)^2\Big] \\
&\le \sqrt{a_1a_3}(t_2-t_1)\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big)^2. \qquad (4.8)
\end{aligned}
$$

So with $C_9 := 2R_1^4\|\rho\|_\infty^4$, we have

$$
\begin{aligned}
&(4.6) \\
&\le \Big(\frac{1}{2}a_2a_3^{-1/2}\Big)^4 \cdot \sqrt{a_1a_3}(t_2-t_1)\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big)^2 \times \\
&\qquad \times\Big(\sqrt{a_1a_3}(t_3-t_2)+2R_1\Big)\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}} \wedge \Big((t_3-t_2)\|\rho\|_\infty\Big)\Big)^2 \\
&\le \Big(\frac{1}{2}a_2a_3^{-1/2}\Big)^4 \cdot \sqrt{a_1a_3}(t_2-t_1)\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big)^2 \times \\
&\qquad \times\Big(\sqrt{a_1a_3}(t_3-t_2)\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big)^2+2R_1\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big)(t_3-t_2)\|\rho\|_\infty\Big) \\
&= C_9\Big(a_2a_3^{-1/2}\Big)^4\frac{1}{a_1a_3}(t_3-t_2)(t_2-t_1) \\
&= C_9(t_3-t_2)(t_2-t_1)
\end{aligned}
$$

by (A1).

Similarly, by (A1), there exists a constant $C_{10}>0$ such that

$$
\begin{aligned}
&\Big(\frac{1}{2}a_2a_3^{-1/2}\Big)^4 \\
&\quad E\Big[\Big(\int_{(\sqrt{a_1a_3}t_2-R_1,\sqrt{a_1a_3}t_2+R_1)} dB_2(y_1)\int_{t_1}^{t_2} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y_1-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)^2 \times \\
&\quad \times\Big(\int_{(\sqrt{a_1a_3}t_2+R_1,\sqrt{a_1a_3}t_3+R_1)} dB_2(y_2)\int_{t_2}^{t_3} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y_2-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)^2\Big] \\
&\le C_{10}(t_3-t_2)(t_2-t_1).
\end{aligned}
$$

Finally, let us deal with the "crossing term". We fisrt confirm the following general fact, which is not difficult to be gotten by a careful calculation with respect to Gaussion distribution:

Claim. For any non-random bounded functions f_1 and f_2, we have

$$
\begin{aligned}
&E\Big[\Big(\int_{(u_1,u_2)} dB_2(y)f_1(y)\Big)^2\Big(\int_{(u_1,u_2)} dB_2(y)f_2(y)\Big)^2\Big] \\
&= \Big(\int_{(u_1,u_2)} f_1(y)^2dy\Big)\Big(\int_{(u_1,u_2)} f_2(y)^2dy\Big)+2\Big(\int_{(u_1,u_2)} f_1(y)f_2(y)dy\Big)^2. \qquad (4.9)
\end{aligned}
$$

Proof of the Claim. For any $\alpha,\beta>0$, we have that

$$
E\Big[\exp\Big(\alpha\int f_1(y)dB_2(y)+\beta\int f_2(y)dB_2(y)-\frac{1}{2}\int(\alpha f_1(y)+\beta f_2(y))^2dy\Big)\Big]=1,
$$

so

$$E\Big[e^{\alpha\int f_1(y)dB_2(y)}e^{\beta\int f_2(y)dB_2(y)}\Big] = \exp\Big(\frac{1}{2}\int(\alpha f_1(y)+\beta f_2(y))^2 dy\Big).$$

This is true for any $\alpha,\beta>0$, so the coefficients of the term $\alpha^2\beta^2$ on the both sides are equal to each other. The coefficient of $\alpha^2\beta^2$ of the left hand side is $E\Big[\frac{1}{2}\Big(\int f_1(y)dB_1(y)\Big)^2\frac{1}{2}\Big(\int f_2(y)dB_1(y)\Big)^2\Big]$. The coefficient of $\alpha^2\beta^2$ of the right hand side is that of $\frac{1}{8}\Big(\alpha^2\int f_1(y)^2dy+\beta^2\int f_2(y)^2dy+2\alpha\beta\int f_1(y)f_2(y)dy\Big)^2$, which is equal to $\frac{1}{4}\Big(\int f_1(y)^2dy\Big)\Big(\int f_2(y)^2dy\Big)+\frac{1}{2}\Big(\int f_1(y)f_2(y)dy\Big)^2$. This gives us (4.9).

Now, since $q(\widetilde{s})$ is $\mathcal{F}_{\sqrt{a_1a_3}t_2-R_1}$-measurable for any $s\in[t_1,t_2]$, and $\{B_2(y);y\in(\sqrt{a_1a_3}t_2-R_1,\sqrt{a_1a_3}t_2+R_1)\}$ is independent to $\mathcal{F}_{\sqrt{a_1a_3}t_2-R_1}$, by taking conditional expectation with respect to $\mathcal{F}_{\sqrt{a_1a_3}t_2-R_1}$, we get the first equality of the following formula by (4.9). So with $C_{11}:=3R_1^4\|\rho\|_\infty^4$, we have

$$\begin{aligned}
&\Big(\frac{1}{2}a_2a_3^{-1/2}\Big)^4\\
&\qquad E\Big[\Big(\int_{(\sqrt{a_1a_3}t_2-R_1,\sqrt{a_1a_3}t_2+R_1)}dB_2(y_1)\int_{t_1}^{t_2}ds1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y_1-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)^2\times\\
&\qquad\times\Big(\int_{(\sqrt{a_1a_3}t_2-R_1,\sqrt{a_1a_3}t_2+R_1)}dB_2(y_2)\int_{t_2}^{t_3}ds1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y_2-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)^2\Big]\\
=\ &\Big(\frac{1}{2}a_2a_3^{-1/2}\Big)^4\\
&\qquad E\Big[\Big\{\int_{(\sqrt{a_1a_3}t_2-R_1,\sqrt{a_1a_3}t_2+R_1)}dy_1\Big(\int_{t_1}^{t_2}ds1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y_1-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)^2\Big\}\times\\
&\qquad\quad\times\Big\{\int_{(\sqrt{a_1a_3}t_2-R_1,\sqrt{a_1a_3}t_2+R_1)}dy_2\Big(\int_{t_2}^{t_3}ds1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y_2-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)^2\Big\}\\
&\qquad+2\Big\{\int_{(\sqrt{a_1a_3}t_2-R_1,\sqrt{a_1a_3}t_2+R_1)}dy\Big(\int_{t_1}^{t_2}ds1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)\\
&\qquad\qquad\times\Big(\int_{t_2}^{t_3}ds1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)\Big\}^2\Big]\\
\le\ &\Big(\frac{1}{2}a_2a_3^{-1/2}\Big)^4\Big[\Big\{2R_1\cdot(t_2-t_1)\|\rho\|_\infty\cdot\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big\}\times\Big\{2R_1\cdot(t_3-t_2)\|\rho\|_\infty\cdot\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big\}\\
&\qquad\qquad+2\Big(2R_1\cdot(t_2-t_1)\|\rho\|_\infty\cdot(t_3-t_2)\|\rho\|_\infty\Big)\Big(2R_1\cdot\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big)^2\Big)\Big]\\
=\ &C_{11}\Big(a_2a_3^{-1/2}\Big)^4\frac{1}{a_1a_3}(t_2-t_1)(t_3-t_2)\\
=\ &C_{11}(t_2-t_1)(t_3-t_2)
\end{aligned}$$

by (A1). Here when passing the first inequality, we used (4.7) and the similar one with t_2 and t_3 substituted by t_1 and t_2, respectively.

Combinging the above, we get our assertion. ∎

Proof of Lemma 4.4. The calculation is similar to the one in (4.8).

Let $C_{12} := 2R_1^2\|\rho\|_\infty^2$, then we have

$$
\begin{aligned}
& E[|\overline{I_{11}}(t_2) - \overline{I_{11}}(t_1)|^2] \\
= & \Big(\frac{1}{2}a_2{a_3}^{-1/2}\Big)^2 E\Big[\Big|\int_{(\sqrt{a_1a_3}t_1-R_1,\sqrt{a_1a_3}t_2+R_1)} dB_2(y)\int_{t_1}^{t_2} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big|^2\Big] \\
= & \Big(\frac{1}{2}a_2{a_3}^{-1/2}\Big)^2 \int_{(\sqrt{a_1a_3}t_1-R_1,\sqrt{a_1a_3}t_2+R_1)} dy E\Big[\Big(\int_{t_1}^{t_2} ds 1_{[\frac{2R_1}{\sqrt{a_1a_3}},\infty)}(s)\rho(y-\sqrt{a_1a_3}s-q(\widetilde{s}))\Big)^2\Big] \\
\le & \Big(\frac{1}{2}a_2{a_3}^{-1/2}\Big)^2 \Big(\sqrt{a_1a_3}(t_2-t_1)+2R_1\Big)\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}} \wedge \Big((t_2-t_1)\|\rho\|_\infty\Big)\Big)^2 \\
\le & \Big(\frac{1}{2}a_2{a_3}^{-1/2}\Big)^2 \Big(\sqrt{a_1a_3}(t_2-t_1)\cdot\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}} + 2R_1(t_2-t_1)\|\rho\|_\infty\Big)\times\Big(\frac{2R_1\|\rho\|_\infty}{\sqrt{a_1a_3}}\Big) \\
= & C_{12}\Big(a_2{a_3}^{-1/2}\Big)^2\frac{1}{\sqrt{a_1a_3}}(t_2-t_1) \\
= & C_{12}(t_2-t_1)
\end{aligned}
$$

by (A1). ∎

This completes the proof of Lemma 4.1.

5 The limits for $I_{11}(t)$ and $I_{12}(t)$

We showed in Section 4 that {the distribution of $\{I_{1i}(t); t \in [0,T]\}; \lambda \ge 1\}$ is tight in $\mathcal{P}(D)$ for $i = 1, 2$ (Lemma 4.1). In this section, we find their limits when $\lambda \to \infty$. This combined with Lemma 3.6 gives us the limit distribution of $I_1(t)$ as $\lambda \to \infty$ (see Lemma 5.4).

Again, since the methods are exactly the same, we give here the proof for $I_{11}(t)$ only. Use the same notations as in Section 4. It suffices to consider $\overline{I_{11}}$. We first notice the following.

Lemma 5.1 *For any $t \in [\frac{2R_1}{\sqrt{a_1a_3}}, T]$, we have that $\lim_{\lambda\to\infty} E[|K_i(t)|^2] = 0$ for $i = 2, 5$.*

Proof.

$$
\begin{aligned}
& E[|K_2(t)|^2] \\
= & \Big(-\frac{1}{2}a_2a_3^{-1/2}\frac{1}{\sqrt{a_1a_3}}\int_{-\infty}^{\infty}\rho(u)du\Big)^2 E\Big[\Big|B_2(\sqrt{a_1a_3}t-R_1)-B_2(\sqrt{a_1a_3}t)\Big|^2\Big] \\
= & \Big(-\frac{1}{2}a_2a_3^{-1/2}\frac{1}{\sqrt{a_1a_3}}\int_{-\infty}^{\infty}\rho(u)du\Big)^2 R_1,
\end{aligned}
$$

which converges to 0 as $\lambda \to \infty$ by (A1) and (A2).

For $i = 5$, since $\Big| \int_0^t 1_{[\frac{2R_1}{\sqrt{a_1 a_3}},\infty)}(s)\rho(y - \sqrt{a_1 a_3}s - q(\tilde{s}))ds \Big| \leq \|\rho\|_\infty \frac{2R_1}{\sqrt{a_1 a_3}}$, we have

$$
\begin{aligned}
& E[|K_5(t)|^2] \\
= \; & \Big(-\frac{1}{2}a_2 a_3^{-1/2}\Big)^2 \int_{\sqrt{a_1 a_3}t - R_1 < y < \sqrt{a_1 a_3}t + R_1} dy E\Big[\Big(\int_0^t 1_{[\frac{2R_1}{\sqrt{a_1 a_3}},\infty)}(s)\rho(y - \sqrt{a_1 a_3}s - q(\tilde{s}))ds\Big)^2\Big] \\
\leq \; & \Big(-\frac{1}{2}a_2 a_3^{-1/2}\Big)^2 2R_1 \Big(\|\rho\|_\infty \frac{2R_1}{\sqrt{a_1 a_3}}\Big)^2,
\end{aligned}
$$

which converges to 0 as $\lambda \to \infty$ by (A1) and (A2). ∎

Lemma 5.2 *There exists a process $\widetilde{K_2}(t)$ such that*

$$
\overline{I_{11}}(t) = K_1(t) + \widetilde{K_2}(t), \qquad t \in [0, T], \tag{5.1}
$$

with $K_1(t)$ as defined in (4.5), and $\widetilde{K_2}(t)$ satisfies

$$
\begin{aligned}
& \lim_{\lambda \to \infty} E[|\widetilde{K_2}(t)|^2] = 0 \textit{ for any } t > 0, \\
& \sup_{\lambda \geq 1} E[\sup_{0 \leq t \leq T} |\widetilde{K_2}(t)|^2] < \infty.
\end{aligned}
$$

Proof. Just define

$$
\widetilde{K_2}(t) = -1_{\{0 \leq t < \frac{2R_1}{\sqrt{a_1 a_3}}\}} K_1(t) + 1_{\{t \geq \frac{2R_1}{\sqrt{a_1 a_3}}\}} \sum_{i=2}^{5} K_i(t).
$$

Now our assertion is a result of Lemmas 4.6, 4.7, 4.8 and 5.1 combined with the following calculation.

$$
\begin{aligned}
& E[\sup_{0 \leq t \leq \frac{2R_1}{\sqrt{a_1 a_3}}} |K_1(t)|^2] \\
= \; & \Big(-\frac{1}{2}a_2 a_3^{-1/2} \frac{1}{\sqrt{a_1 a_3}} \int_{-\infty}^{\infty} \rho(u)du\Big)^2 E[\sup_{0 \leq t \leq \frac{2R_1}{\sqrt{a_1 a_3}}} |B_2(\sqrt{a_1 a_3}t)|^2] \\
\leq \; & 4\Big(-\frac{1}{2}a_2 a_3^{-1/2} \frac{1}{\sqrt{a_1 a_3}} \int_{-\infty}^{\infty} \rho(u)du\Big)^2 2R_1,
\end{aligned}
$$

which converges to 0 as $\lambda \to \infty$ by (A1) and (A2). ∎

We are now ready to show that $\widetilde{K_2}(t)$ is negligible.

Lemma 5.3 $\lim_{\lambda \to \infty} E[\sup_{t \in [0,T]} |\widetilde{K_2}(t)|] = 0$.

Proof. Since the distribution of $\{K_1(t); t \in [0, T]\}$ does not depend on $\lambda \geq 1$, and by Lemma 4.1, the distribution of $\{I_{11}(t); t \in [0, T]\}$ is tight for $\lambda \geq 1$, we get that the distribution of $\{\widetilde{K_2}(t) = I_{11}(t) - K_1(t); t \in [0, T]\}$ is also tight for $\lambda \geq 1$.

Let Q be any cluster point of it as $\lambda \to \infty$, and let $\{w(t)\}_{t\in[0,T]}$ be the canonical process under it. For any $t \in [0,T]$ and any $r > 0$, we have by Lemma 5.2 that

$$P\Big(\Big\{|\widetilde{K_2}(t)| > r\Big\}\Big) \le \frac{1}{r^2} E\Big[|\widetilde{K_2}(t)|^2\Big] \to 0, \qquad \text{as } \lambda \to \infty.$$

On the other hand, the left hand side above converges to $Q\Big(\Big\{|\omega(t)| > r\Big\}\Big)$ as $\lambda \to \infty$. So

$$Q\Big(\Big\{|\omega(t)| > r\Big\}\Big) = 0$$

for any $r > 0$. Hence

$$Q\Big(\Big\{|\omega(t)| = 0\Big\}\Big) = 1.$$

Since $\widetilde{K_2}(t) = I_{11}(t) - K_1(t)$ is continuous with respect to t (for any fixed λ), we have that the canonical process of Q is also continuous with respect to t. Therefore,

$$Q\Big(\Big\{|\omega(t)| = 0, \text{ for all } t \in [0,T]\Big\}\Big) = 1. \tag{5.2}$$

Now we are ready to prove our lemma. We have for any $\varepsilon > 0$ that

$$\begin{aligned} & E[\sup_{t\in[0,T]} |\widetilde{K_2}(t)|] \\ \le\ & E[\sup_{t\in[0,T]} |\widetilde{K_2}(t)|, \sup_{t\in[0,T]} |\widetilde{K_2}(t)| > \varepsilon] + \varepsilon \\ \le\ & E[\sup_{t\in[0,T]} |\widetilde{K_2}(t)|^2]^{1/2} P(\sup_{t\in[0,T]} |\widetilde{K_2}(t)| > \varepsilon)^{1/2} + \varepsilon. \end{aligned}$$

$E[\sup_{t\in[0,T]} |\widetilde{K_2}(t)|^2]^{1/2}$ is bounded for $\lambda \ge 1$ by Lemma 5.2. Therefore, in order to show that $E[\sup_{t\in[0,T]} |\widetilde{K_2}(t)|] \to 0$ as $\lambda \to \infty$, it suffices to show that for any $\varepsilon > 0$, $P(\sup_{t\in[0,T]} |\widetilde{K_2}(t)| > \varepsilon) \to 0$ as $\lambda \to \infty$. We show it in the following.

For any $a > 0$, let

$$A = \{\sup_{t\in[0,T]} |\omega(t)| > 2a\}, \qquad B = \{\sup_{t\in[0,T]} |\omega(t)| > a\}.$$

Then it is easy to see that for any $\omega_0 \in A$ and ω with $d^0(\omega, \omega_0) < a$, we have $\omega \in B$. So $A \subset \overline{A} \subset B^o \subset B$. Therefore, since $\overline{A}$ is closed, we have

$$\limsup_{\lambda\to\infty}(P \circ \widetilde{K_2}^{-1})(A) \le \limsup_{\lambda\to\infty}(P \circ \widetilde{K_2}^{-1})(\overline{A}) \le Q(\overline{A}) \le Q(B),$$

which is equal to 0 by (5.2). Therefore,

$$\lim_{\lambda\to\infty} P(\sup_{t\in[0,T]} |\widetilde{K_2}(t)| > \varepsilon) = 0.$$

∎

Repeating the argument from Section 4 up to now with $\{I_{11}(t); 0 \le t \le T\}$ substituted by $\{I_{12}(t); 0 \le t \le T\}$, we get that under (A1) and (A2), $I_{12}(t)$ can also be decomposed as $\frac{1}{2}a_2 a_3^{-1/2} \frac{1}{\sqrt{a_1 a_3}} \int_{-\infty}^{\infty} \rho(u) du B_2(-\sqrt{a_1 a_3}(t \wedge \tau_n))$ plus a remainder, which satisfies $\lim_{\lambda\to\infty} E[\sup_{0\le t\le T} |\ \cdot\ (t)|] = 0$. Combining it with (5.1), Lemma 5.3 and Lemma 3.6, we get the following.

Lemma 5.4 *Let*

$$M(t) = -\frac{1}{2}a_2 a_3^{-1/2} \cdot \frac{1}{\sqrt{a_1 a_3}}\Big(\int_{-\infty}^{\infty} \rho(u)du\Big)\Big(B_2(\sqrt{a_1 a_3}t) - B_2(-\sqrt{a_1 a_3}t)\Big). \qquad (5.3)$$

Then $\{M(t)\}_t$ *has the same distribution as* $\{\frac{1}{\sqrt{2}}\Big(\int_{-\infty}^{\infty} \rho(u)du\Big)\overline{B}(t)\}_t$, *where* $\{\overline{B}(t)\}$ *is a standard Brownian motion, and we have*

$$I_1(t \wedge \tau_n) = M(t \wedge \tau_n) + \eta_1(t),$$

with $\eta_1(t)$ *satisfying*

$$\lim_{\lambda\to\infty} E[\sup_{0\le t\le T} |\eta_1(t)|] = 0.$$

6 The term $I_2(t)$

We deal with the term $I_2(t)$ in this section, and show that it gives us the drift term in L after taking $\lambda \to \infty$. This is done in two steps: We first show that it is tight for $\lambda \ge 1$, which is, after combined with Lemma 5.4, expressed as Lemma 6.4. We then use it to find the limit as $\lambda \to \infty$. Our main result of this section is the following, which is also our key lemma to prove Theorem 1.1.

Lemma 6.1 *There exists a process* $\eta(t)$ *such that*

$$p(t \wedge \tau_n) = p_0 + M(t \wedge \tau_n) - \frac{1}{2}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \int_0^{t\wedge\tau_n} \frac{p(s)}{M\sqrt{1 + a_4^{-2}M^{-1}p(s)^2}}ds + \eta(t),$$

where $M(t)$ *is as defined in (5.3), and*

$$\lim_{\lambda\to\infty} E[\sup_{0\le t\le T} |\eta(t)|] \to 0.$$

We show Lemma 6.1 in the rest of this section. We first have

$$\begin{aligned}
& I_2(t \wedge \tau_n) \\
= \; & \frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_{\mathbf{R}} dx \rho(x - q(s)) \int_0^{\sqrt{a_1 a_3}s} dr \\
& \qquad \Big(\rho(x + r - q(s - \frac{1}{\sqrt{a_1 a_3}}r)) - \rho(x - r - q(s - \frac{1}{\sqrt{a_1 a_3}}r))\Big) \\
= \; & \frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{\sqrt{a_1 a_3}s} dr \int_{\mathbf{R}} dx \rho(x - q(s))\rho(x + r - q(s - \frac{1}{\sqrt{a_1 a_3}}r)) \\
& -\frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{\sqrt{a_1 a_3}s} dr \int_{\mathbf{R}} dx \rho(x - q(s))\rho(x - r - q(s - \frac{1}{\sqrt{a_1 a_3}}r)) \\
= \; & \frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{(\sqrt{a_1 a_3}s)\wedge(2R_1)} dr \int_{|x|\le R_1} dx \times \\
& \Big[\rho(x - q(s))\rho(x + r - q(s - \frac{1}{\sqrt{a_1 a_3}}r)) - \rho(x + r - q(s))\rho(x - q(s - \frac{1}{\sqrt{a_1 a_3}}r))\Big],
\end{aligned}$$

where in the last equality, we used the change of variable $x - r \to x$ for the second integral. Also, we were able to rewrite the integral domains for r and x because $\rho(x - q(s))$ and $\rho(x - q(s - \frac{1}{\sqrt{a_1 a_3}} r))$ are not 0 only if $|x| \le R_1$, and in this case, $\rho(x + r - q(s - \frac{1}{\sqrt{a_1 a_3}} r))$ and $\rho(x + r - q(s))$ are not 0 only if $|r| \le 2R_1$.

We can decompose the integrand in the last expression as

$$\begin{aligned}
& \rho(x - q(s))\rho(x + r - q(s - \frac{1}{\sqrt{a_1 a_3}} r)) - \rho(x + r - q(s))\rho(x - q(s - \frac{1}{\sqrt{a_1 a_3}} r)) \\
= \;& J_1 + J_2 + J_3 + J_4,
\end{aligned}$$

with

$$\begin{aligned}
J_1 &= \rho(x - q(s))\Big\{\rho(x + r - q(s - \frac{1}{\sqrt{a_1 a_3}} r)) - \rho(x + r - q(s)) \\
&\qquad + \nabla\rho(x + r - q(s))\Big(q(s - \frac{1}{\sqrt{a_1 a_3}} r) - q(s)\Big)\Big\}, \\
J_2 &= -\rho(x + r - q(s))\Big\{\rho(x - q(s - \frac{1}{\sqrt{a_1 a_3}} r)) - \rho(x - q(s)) \\
&\qquad + \nabla\rho(x - q(s))\Big(q(s - \frac{1}{\sqrt{a_1 a_3}} r) - q(s)\Big)\Big\}, \\
J_3 &= -\rho(x - q(s))\nabla\rho(x + r - q(s))\Big(q(s - \frac{1}{\sqrt{a_1 a_3}} r) - q(s)\Big), \\
J_4 &= \rho(x + r - q(s))\nabla\rho(x - q(s))\Big(q(s - \frac{1}{\sqrt{a_1 a_3}} r) - q(s)\Big).
\end{aligned}$$

Let

$$I_{2i}(t) = \frac{1}{2} a_3^{-1} a_2^2 \int_0^{t \wedge \tau_n} ds \int_0^{(\sqrt{a_1 a_3} s) \wedge (2R_1)} dr \int_{|x| \le R_1} dx J_i, \quad i = 1, \cdots, 4.$$

Then

$$I_2(t \wedge \tau_n) = I_{21}(t) + I_{22}(t) + I_{23}(t) + I_{24}(t). \tag{6.1}$$

Lemma 6.2 *For $i = 1, 2$, we have $\lim_{\lambda \to \infty} \sup_{\omega \in \Omega} \sup_{t \in [0,T]} |I_{2i}(t)| = 0$.*

Proof. Since the proofs for $i = 1, 2$ are similar, we give the one for $i = 1$ only. For any $r \in [0, (\sqrt{a_1 a_3} s) \wedge (2R_1)]$, we have

$$\begin{aligned}
& \Big|\rho(x + r - q(s - \frac{1}{\sqrt{a_1 a_3}} r)) - \rho(x + r - q(s)) \\
&\qquad + \nabla\rho(x + r - q(s))\Big(q(s - \frac{1}{\sqrt{a_1 a_3}} r) - q(s)\Big)\Big| \\
\le\;& \|\nabla^2\rho\|_\infty \Big(q(s - \frac{1}{\sqrt{a_1 a_3}} r) - q(s)\Big)^2 \\
\le\;& \|\nabla^2\rho\|_\infty \Big(\frac{n}{\sqrt{a_1 a_3}} r\Big)^2 \\
\le\;& \|\nabla^2\rho\|_\infty \Big(\frac{n}{\sqrt{a_1 a_3}} 2R_1\Big)^2, \qquad s \in [0, T \wedge \tau_n].
\end{aligned}$$

So with $C_{13} := 8n^2TR_1^4\|\rho\|_\infty\|\nabla^2\rho\|_\infty$, we have for any $t\in[0,T]$,

$$\begin{aligned}
|I_{21}(t)| &\le \frac{1}{2}a_3^{-1}a_2^2\int_0^{t\wedge\tau_n}ds\int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)}dr\int_{|x|\le R_1}dx|\rho(x-q(s))|\times\\
&\qquad\times\Big|\rho(x+r-q(s-\frac{1}{\sqrt{a_1a_3}}r))-\rho(x+r-q(s))\\
&\qquad\qquad+\nabla\rho(x+r-q(s))\Big(q(s-\frac{1}{\sqrt{a_1a_3}}r)-q(s)\Big)\Big|\\
&\le \frac{1}{2}a_3^{-1}a_2^2T2R_1\cdot 2R_1\cdot\|\rho\|_\infty\|\nabla^2\rho\|_\infty\Big(\frac{n}{\sqrt{a_1a_3}}2R_1\Big)^2\\
&= C_{13}a_3^{-1}a_2^2\Big(\frac{1}{\sqrt{a_1a_3}}\Big)^2,
\end{aligned}$$

which converges to 0 as $\lambda\to\infty$ by (A1) and (A2). ∎

Lemma 6.3 *For $i=3,4$, we have that $\sup_{\lambda\ge1}E\Big[\sup_{0\le t\le T}\Big|\frac{d}{dt}I_{2i}(t)\Big|^2\Big]<\infty$, in particular, {the distribution of $\{I_{2i}(t)\}_{t\in[0,T]}\}_{\lambda\ge1}$ is tight in $\mathcal{P}(D)$.*

Proof. As in Kusuoka-Liang [11], the second half of the lemma is a simple result of the first half.

We proof the first half for $i=3$ in the following. The assertion for $i=4$ is done in the same way, and we omit it here. By definition,

$$\begin{aligned}
I_{23}(t) = &-\frac{1}{2}a_3^{-1}a_2^2\int_0^{t\wedge\tau_n}ds\int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)}dr\\
&\times\int_{|x|\le R_1}\rho(x-q(s))\nabla\rho(x+r-q(s))\Big(q(s-\frac{1}{\sqrt{a_1a_3}}r)-q(s)\Big)dx.
\end{aligned}$$

Notice that for any $0\le r\le(\sqrt{a_1a_3}t)\wedge(2R_1)$ and $t\in[0,T\wedge\tau_n]$, we have $\Big|q(t-\frac{1}{\sqrt{a_1a_3}}r)-q(t)\Big|\le\frac{n}{\sqrt{a_1a_3}}r\le\frac{2R_1n}{\sqrt{a_1a_3}}$. So with $C_{14} := \Big(4nR_1^3\|\rho\|_\infty\|\nabla\rho\|_\infty\Big)^2$, we have

$$\begin{aligned}
&E\Big[\sup_{0\le t\le T}\Big|\frac{d}{dt}I_{23}(t)\Big|^2\Big]\\
\le\ &\Big(\frac{1}{2}a_3^{-1}a_2^2\Big)^2E\Big[\sup_{0\le t\le T\wedge\tau_n}\Big|\int_0^{(\sqrt{a_1a_3}t)\wedge(2R_1)}dr\int_{|x|\le R_1}\\
&\qquad\rho(x-q(t))\nabla\rho(x+r-q(t))\Big(q(t-\frac{1}{\sqrt{a_1a_3}}r)-q(t)\Big)dx\Big|^2\Big]\\
\le\ &\Big(\frac{1}{2}a_3^{-1}a_2^2\Big)^2\Big(2R_12R_1\|\rho\|_\infty\|\nabla\rho\|_\infty\frac{2R_1n}{\sqrt{a_1a_3}}\Big)^2 = C_{14}
\end{aligned}$$

by (A1). ∎

Combining (2.16), Lemma 5.4, (6.1), Lemma 6.2 and Lemma 6.3, we get the following.

Lemma 6.4 *Let $M(t)$ be as given in Lemma 5.4, and let $u(t) = I_{23}(t) + I_{24}(t)$. Then there exists a process $\eta_2(t)$ such that*

$$p(t \wedge \tau_n) = p_0 + M(t \wedge \tau_n) + u(t) + \eta_2(t),$$

and $u(t)$ is differentible with respect to t, with

$$A_3 := \sup_{\lambda \geq 1} E\Big[\sup_{0 \leq t \leq T} \Big|\frac{d}{dt}u(t)\Big|^2\Big] < \infty,$$
$$A_4(\lambda) := E[\sup_{0 \leq t \leq T} |\eta_2(t)|] \to 0, \ \text{as } \lambda \to \infty.$$

Proof. Just let

$$\eta(t) = \eta_1(t) + I_{21}(t) + I_{22}(t),$$

where $\eta_1(t)$ is as given by Lemma 5.4, and we get our assertion. ∎

In order to get Lemma 6.1, we need to study $u(t)$ in more detail.

Notice that in the expressions of I_{23} and I_{24}, the integral domain $\{|x| \leq R_1\}$ can be converted to $\{x \in \mathbf{R}\}$, so by using change of variable $x - q(s) \to x$, we get that

$$\begin{aligned} u(t) &= -\frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)} dr \\ &\quad \times \int_{\mathbf{R}} \rho(x)\nabla\rho(x+r)\Big(q(s - \frac{1}{\sqrt{a_1a_3}}r) - q(s)\Big)dx \\ &\quad +\frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)} dr \\ &\quad \times \int_{\mathbf{R}} \rho(x+r)\nabla\rho(x)\Big(q(s - \frac{1}{\sqrt{a_1a_3}}r) - q(s)\Big)dx. \end{aligned}$$

Decompose it as

$$u(t) = u_1(t) + u_2(t) + u_3(t) \tag{6.2}$$

with

$$\begin{aligned} u_1(t) &= -\frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)} dr \int_{\mathbf{R}} dx \\ &\quad \times\rho(x)\nabla\rho(x+r)\Big(q(s - \frac{1}{\sqrt{a_1a_3}}r) - q(s) + \frac{p(s)}{M\sqrt{1 + a_4^{-2}M^{-1}p(s)^2}}\frac{1}{\sqrt{a_1a_3}}r\Big) \\ u_2(t) &= \frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)} dr \int_{\mathbf{R}} dx \\ &\quad \times\rho(x+r)\nabla\rho(x)\Big(q(s - \frac{1}{\sqrt{a_1a_3}}r) - q(s) + \frac{p(s)}{M\sqrt{1 + a_4^{-2}M^{-1}p(s)^2}}\frac{1}{\sqrt{a_1a_3}}r\Big) \\ u_3(t) &= \frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)} dr \\ &\quad \times \int_{\mathbf{R}} \rho(x)\nabla\rho(x+r)\frac{p(s)}{M\sqrt{1 + a_4^{-2}M^{-1}p(s)^2}}\frac{1}{\sqrt{a_1a_3}}rdx \end{aligned}$$

$$-\frac{1}{2}a_3^{-1}a_2^2\int_0^{t\wedge\tau_n}ds\int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)}dr$$
$$\times\int_{\mathbf{R}}\rho(x+r)\nabla\rho(x)\frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}}\frac{1}{\sqrt{a_1a_3}}rdx.$$

Lemma 6.5 *For $i=1,2$, we have $E\Big[\sup_{0\le t\le T}|u_i(t)|\Big]\to 0$ as $\lambda\to\infty$.*

Proof. Use the same notations as in Lemma 6.4. Let $A_5:=\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\rho(u)du$. Then for any $0<t_1<t_2\le T$, we have

$$E\Big[|M(t_2)-M(t_1)|\Big]=A_5E\Big[|\overline{B}(t_2)-\overline{B}(t_1)|\Big]\le A_5E\Big[|\overline{B}(t_2)-\overline{B}(t_1)|^2\Big]^{1/2}=A_5(t_2-t_1)^{1/2}.$$

By Lemma 6.4,

$$p(t_2\wedge\tau_n)-p(t_1\wedge\tau_n)=(M(t_2\wedge\tau_n)-M(t_1\wedge\tau_n))+(u(t_2)-u(t_1))+(\eta_2(t_2)-\eta_2(t_1)),$$

so

$$\begin{aligned}
&E\Big[|p(t_2\wedge\tau_n)-p(t_1\wedge\tau_n)|\Big]\\
\le\;& E\Big[|M(t_2\wedge\tau_n)-M(t_1\wedge\tau_n)|\Big]+E\Big[|u(t_2)-u(t_1)|\Big]+E\Big[|\eta_2(t_2)-\eta_2(t_1)|\Big]\\
\le\;& A_5(t_2-t_1)^{1/2}+A_3^{1/2}(t_2-t_1)+2A_4(\lambda). \qquad (6.3)
\end{aligned}$$

For any $s,u\in[0,T\wedge\tau_n]$ with $|s-u|\le\frac{2R_1}{\sqrt{a_1a_3}}$, we have by (6.3) that

$$\begin{aligned}
&E\Big[\Big|\frac{p(s\wedge\tau_n)}{M\sqrt{1+a_4^{-2}M^{-1}p(s\wedge\tau_n)^2}}-\frac{p(u\wedge\tau_n)}{M\sqrt{1+a_4^{-2}M^{-1}p(u\wedge\tau_n)^2}}\Big|\Big]\\
\le\;&\frac{1}{M}E\Big[\Big|p(s\wedge\tau_n)-p(u\wedge\tau_n)\Big|\Big]\\
\le\;&\frac{1}{M}\Big\{A_5\Big(\frac{2R_1}{\sqrt{a_1a_3}}\Big)^{1/2}+A_3^{1/2}\Big(\frac{2R_1}{\sqrt{a_1a_3}}\Big)+2A_4(\lambda)\Big\}.
\end{aligned}$$

Notice that

$$\begin{aligned}
u_1(t\wedge\tau_n)\;=\;&-\frac{1}{2}a_3^{-1}a_2^2\int_0^t ds\int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)}dr\int_{|x|\le R_1}dx\rho(x)\nabla\rho(x+r)\\
&\times\int_{s-\frac{1}{\sqrt{a_1a_3}}r}^s\Big(\frac{p(s\wedge\tau_n)}{M\sqrt{1+a_4^{-2}M^{-1}p(s\wedge\tau_n)^2}}-\frac{p(u\wedge\tau_n)}{M\sqrt{1+a_4^{-2}M^{-1}p(u\wedge\tau_n)^2}}\Big)du.
\end{aligned}$$

So

$$\begin{aligned}
&E\Big[\sup_{0\le t\le T}|u_1(t)|\Big]\\
\le\;&\frac{1}{2}a_3^{-1}a_2^2E\Big[\int_0^{T\wedge\tau_n}ds\Big|\int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)}dr\int_{|x|\le R_1}dx\rho(x)\nabla\rho(x+r)\\
&\times\int_{s-\frac{1}{\sqrt{a_1a_3}}r}^s\Big(\frac{p(s\wedge\tau_n)}{M\sqrt{1+a_4^{-2}M^{-1}p(s\wedge\tau_n)^2}}-\frac{p(u\wedge\tau_n)}{M\sqrt{1+a_4^{-2}M^{-1}p(u\wedge\tau_n)^2}}\Big)du\Big|\Big]\\
\le\;&\frac{1}{2}a_3^{-1}a_2^2T2R_1\cdot 2R_1\|\rho\|_\infty\|\nabla\rho\|_\infty\frac{2R_1}{\sqrt{a_1a_3}}\frac{1}{M}\Big(A_5\Big(\frac{2R_1}{\sqrt{a_1a_3}}\Big)^{1/2}+A_3^{1/2}\Big(\frac{2R_1}{\sqrt{a_1a_3}}\Big)+2A_4(\lambda)\Big),
\end{aligned}$$

which converges to 0 as $\lambda \to \infty$ by (A1) and (A2), since $A_4(\lambda) \to 0$.

The assertion for $i = 2$ is proved in exactly the same way, and we omit it here. ∎

Finally, for the term $u_3(\cdot)$, we have the following.

Lemma 6.6 $\lim_{\lambda\to\infty} \sup_{\omega\in\Omega, 0\le t\le T} \left| u_3(t) + \frac{1}{2M}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \int_0^{t\wedge\tau_n} \frac{p(s)}{\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} ds \right| = 0.$

Proof. The first term of $u_3(t)$ is, by changing of variable $x + r \to x$, equal to

$$\begin{aligned} &\frac{1}{2}a_3^{-1}a_2^2 \int_0^{t\wedge\tau_n} ds \int_0^{(\sqrt{a_1a_3}s)\wedge(2R_1)} dr \int_{\mathbf{R}} \rho(x-r)\nabla\rho(x) \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} \frac{1}{\sqrt{a_1a_3}} r dx \\ = &-\frac{1}{2}a_3^{-1}a_2^2 \frac{1}{\sqrt{a_1a_3}} \int_0^{t\wedge\tau_n} ds \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} \int_{-\left((\sqrt{a_1a_3}s)\wedge(2R_1)\right)}^0 r dr \int_{\mathbf{R}} \rho(x+r)\nabla\rho(x)dx. \end{aligned}$$

So

$$u_3(t) = -\frac{1}{2}a_3^{-1}a_2^2 \frac{1}{\sqrt{a_1a_3}} \int_0^{t\wedge\tau_n} ds \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} \int_{-\left((\sqrt{a_1a_3}s)\wedge(2R_1)\right)}^{(\sqrt{a_1a_3}s)\wedge(2R_1)} r dr \int_{\mathbf{R}} \rho(x+r)\nabla\rho(x)dx.$$

Notice that if $s > \frac{2R_1}{\sqrt{a_1a_3}}$, then the integral $\int_{-\left((\sqrt{a_1a_3}s)\wedge(2R_1)\right)}^{(\sqrt{a_1a_3}s)\wedge(2R_1)}$ above is equal to $\int_{-2R_1}^{2R_1}$, which is in turn equal to $\int_{-\infty}^{\infty}$. Therefore,

$$u_3(t) = u_{31}(t) + u_{32}(t),$$

with

$$\begin{aligned} u_{31}(t) &= -\frac{1}{2}a_3^{-1}a_2^2 \frac{1}{\sqrt{a_1a_3}} \int_0^{t\wedge\tau_n} ds \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} \int_{-\infty}^{\infty} r dr \int_{\mathbf{R}} \rho(x+r)\nabla\rho(x)dx, \\ u_{32}(t) &= \frac{1}{2}a_3^{-1}a_2^2 \frac{1}{\sqrt{a_1a_3}} \int_0^{t\wedge\tau_n} 1_{[0,\frac{2R_1}{\sqrt{a_1a_3}})}(s) ds \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} \\ &\qquad \times \int_{[-2R_1,2R_1]\setminus[-\sqrt{a_1a_3}s,\sqrt{a_1a_3}s]} r dr \int_{\mathbf{R}} \rho(x+r)\nabla\rho(x)dx. \end{aligned}$$

Notice that $s \le \tau_n$ implies $\left|\frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}}\right| \le n$, so for any $t \in [0,T]$ and $\omega \in \Omega$, we have that

$$|u_{32}(t)| \le \frac{1}{2}a_3^{-1}a_2^2 \frac{1}{\sqrt{a_1a_3}} \cdot \frac{2R_1}{\sqrt{a_1a_3}} \cdot n(2R_1)^2 \cdot 2R_1 \|\rho\|_\infty \|\nabla\rho\|_\infty,$$

which converges to 0 as $\lambda \to \infty$ by (A1) and (A2).

For the term $u_{31}(t)$, notice that

$$\begin{aligned}
&\int_{-\infty}^{\infty} r dr \int_{\mathbf{R}} \rho(x+r)\nabla\rho(x)dx = \int_{\mathbf{R}} r dr \int_{\mathbf{R}} \rho(x)\nabla\rho(x-r)dx \\
= &\int_{\mathbf{R}} \rho(x)dx \int_{\mathbf{R}} r\nabla\rho(x-r)dr = \int_{\mathbf{R}} \rho(x)dx \int_{\mathbf{R}} \rho(x-r)dr \\
= &\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2.
\end{aligned}$$

So by (A1),

$$u_{31}(t) = -\frac{1}{2}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \int_0^{t\wedge\tau_n} \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} ds.$$

∎

Proof of Lemma 6.1 This is just a combination of Lemma 6.4, (6.2), Lemma 6.5 and Lemma 6.6. ∎

7 Proof of the main result

Now, we are ready to prove Theorem 1.1.

Use the same notations as in Section 6. Let

$$Y(t) := p(t\wedge\tau_n) - \eta(t) = p_0 + M(t\wedge\tau_n) - \frac{1}{2}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \int_0^{t\wedge\tau_n} \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} ds.$$

Then for any $g \in C_0^\infty(\mathbf{R}^2)$, since

$$|g(q(t\wedge\tau_n), p(t\wedge\tau_n)) - g(q(t\wedge\tau_n), Y(t))| \le \|g_p\|_\infty |\eta(t)|,$$

we have by Lemma 6.1 that when $\lambda \to \infty$, $\{g(q(t\wedge\tau_n), p(t\wedge\tau_n)); t \in [0,T]\}$ and $\{g(q(t\wedge\tau_n), Y(t)); t \in [0,T]\}$ have the same limit.

Also, as in Theorem 1.1 (2), we define $\widetilde{p}(\cdot)$ as follows:

$$\widetilde{p}(t) = \begin{cases} p(t), & \text{if } \lim_{\lambda\to\infty} a_4 = \infty, \\ \frac{p(t)}{\sqrt{1+a_4^{-2}M^{-1}p(t)^2}}, & \text{if } a_4 \text{ is a constant }. \end{cases}$$

This is the limit of $\frac{p(t)}{\sqrt{1+a_4^{-2}M^{-1}p(t)^2}}$ when $\lambda \to \infty$.

By definition, we have for any $f \in C_0^\infty(\mathbf{R}^2)$,

$$\begin{aligned}
&f(q(t\wedge\tau_n), Y(t)) - f(q_0, Y(0)) \\
= &\int_0^{t\wedge\tau_n} f_q(q(s), Y(s)) \cdot \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} ds + \int_0^{t\wedge\tau_n} f_p(q(s), Y(s)) dM(s) \\
&- \int_0^{t\wedge\tau_n} f_p(q(s), Y(s)) \cdot \frac{1}{2}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}} ds \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_n} f_{pp}(q(s), Y(s)) d[M,M]_s.
\end{aligned}$$

Since $\int_0^{t\wedge\tau_n} f_p(q(s),Y(s))dM(s)$ is a martingale, and

$$d[M,M]_s = \frac{1}{2}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 ds,$$

this gives us that

$$f(q(t\wedge\tau_n),Y(t)) - f(q_0,Y(0)) - \int_0^{t\wedge\tau_n} f_q(q(s),Y(s))\cdot \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}}ds$$
$$+\int_0^{t\wedge\tau_n} f_p(q(s),Y(s))\cdot \frac{1}{2}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}}ds$$
$$-\frac{1}{4}\Big(\int_{\mathbf{R}} \rho(u)du\Big)^2 \int_0^{t\wedge\tau_n} f_{pp}(q(s),Y(s))ds$$

is a martingale for any $f \in C_0^\infty(\mathbf{R}^2)$. When $\lambda \to \infty$, since $f(q(t\wedge\tau_n),Y(t))$, $f_q(q(s\wedge\tau_n),Y(s))$, $f_p(q(s\wedge\tau_n),Y(s))$ and $f_{pp}(q(s\wedge\tau_n),Y(s))$ have the same limits as $f(q(t\wedge\tau_n),p(t\wedge\tau_n))$, $f_q(q(s\wedge\tau_n),p(s\wedge\tau_n))$, $f_p(q(s\wedge\tau_n),p(s\wedge\tau_n))$ and $f_{pp}(q(s\wedge\tau_n),p(s\wedge\tau_n))$, respectively, and $\frac{p(s)}{M\sqrt{1+a_4^{-2}M^{-1}p(s)^2}}$ converges to $\frac{1}{M}\widetilde{p}(s)$, this implies that the limit of the distributions of $\{(q(t\wedge\tau_n),p(t\wedge\tau_n)); t\in[0,T]\}$ is a solution of the martingale problem L stopped at τ_n. ∎

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