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Abstract

We consider stationary solutions to the three-dimensional Navier-Stokes equations for viscous incompressible flows in the presence of a linear strain. For certain class of strains we prove a Liouville type theorem under suitable decay conditions on vorticity fields.

1 Introduction

In this paper we consider stationary solutions to the three-dimensional Navier-Stokes equations for viscous incompressible flows with a linear strain:

$$\begin{cases} -\Delta U + Mx\cdot\nabla U + MU + U\cdot\nabla U + \nabla P = 0 & x\in\mathbb{R}^3, \\ \nabla\cdot U = 0 & x\in\mathbb{R}^3, \end{cases} \tag{NS$_{\rm M}$}$$

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_i \in \mathbb{R}. \tag{1.1}$$

Here $U(x) = (U_1(x), U_2(x), U_3(x))$ represents the velocity field, $P(x)$ is the pressure field, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, and each λ_i is a given real number.

The system (NS$_{\rm M}$) is closely related with the original Navier-Stokes equations. For example, the first equation of (NS$_{\rm M}$) is formally obtained by considering the stationary solution to the Navier-Stokes equations of the form $U(x) + Mx$. If the trace of M, denoted by $\mathrm{Tr}(M)$ in the sequel, is equal to zero then the second equation of ($\mathrm{NS_M}$) is also recovered. Even in the case $\mathrm{Tr}(M) \neq 0$, ($\mathrm{NS_M}$) is derived from the Navier-Stokes equations through self-similar solutions. To formulate this relation in a more precise way, let us recall the three-dimensional Navier-Stokes equations for viscous incompressible flows:

$$\begin{cases} v_t - \Delta v + v \cdot \nabla v + \nabla p &= 0 \qquad t > 0, \quad x \in \mathbb{R}^3, \\ \nabla \cdot v &= 0 \qquad t > 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{NS}$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ and $p = p(x,t)$. As stated above, when $\mathrm{Tr}(M) = 0$ the system ($\mathrm{NS_M}$) describes the stationary solutions to (NS) of the form $v(x) = U(x) + Mx$ and $p(x) = P(x) - \frac{1}{2}|Mx|^2$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. The reader is referred to [8] for the analysis of the nonstationary problem (NS) with a linear strain, where more general matrices M are treated. If $\mathrm{Tr}(M) < 0$ then ($\mathrm{NS_M}$) is related with the forward self-similar solutions to (NS) with a linear strain, i.e., the solutions to (NS) of the form

$$v(x,t) = \frac{1}{\sqrt{2\alpha t}}(U + S_1)(\frac{x}{\sqrt{2\alpha t}}), \qquad p(x,t) = \frac{1}{2\alpha t}(P + S_2)(\frac{x}{\sqrt{2\alpha t}}), \tag{1.2}$$

where $\alpha = |\mathrm{Tr}(M)|/3$, $S_1(x) = (M - \alpha I)x$, $S_2(x) = (\alpha^2|x|^2 - |Mx|^2)/2$. Finally, if $\mathrm{Tr}(M) > 0$ then ($\mathrm{NS_M}$) describes the backward self-similar solutions to (NS) with a linear strain,

$$v(x,t) = \frac{1}{\sqrt{2\alpha(T-t)}}(U+S_1)(\frac{x}{\sqrt{2\alpha(T-t)}}), \qquad p(x,t) = \frac{1}{2\alpha(T-t)}(P+S_2)(\frac{x}{\sqrt{2\alpha(T-t)}}), \tag{1.3}$$

where $T > 0$, and S_1, S_2, and α are the same as above.

Despite of the simple structure of the matrix M in (1.1), the above observation shows that ($\mathrm{NS_M}$) describes three important classes of solutions to (NS) depending on the eigenvalues λ_i of M. However, it is still not clear whether ($\mathrm{NS_M}$) admits nontrivial solutions or not, except for the following cases:

$$\text{(i) } \lambda_i > 0, \quad i = 1,2,3 \quad \text{(ii) } \lambda_1 < 0, \ \lambda_2 < 0, \ \sum_{i=1}^{3} \lambda_i = 0, \quad \text{(iii) } \lambda_1 = \lambda_2 = \lambda_3 < 0.$$

We note that the sign of the eigenvalues λ_i plays a critical role for the existence of nontrivial solutions to ($\mathrm{NS_M}$). Indeed, if λ_i is positive then the transport term $Mx \cdot \nabla$ possesses an expanding effect in x_i direction, which tends to trivialize solutions. Conversely, if λ_i is negative then the term $Mx \cdot \nabla$ induces a localization in x_i direction, bringing an effect to keep solutions nontrivial.

In this paper we study the case when one of λ_i is negative and the other two are positive, for this case is essentially open in the literature and is also important as an intermediate case between (i) and (ii). By suitable scaling and coordinate transformation we may assume without loss of generality that

$$\lambda_1 = -\lambda < 0, \qquad \lambda_2 = 1, \qquad \lambda_3 = \mu \geq 1. \tag{1.4}$$

Before stating our results, we briefly recall the known results on the cases (i)-(iii).

(i) $\lambda_i > 0,\ i = 1, 2, 3$: The most important example is $\lambda_1 = \lambda_2 = \lambda_3 > 0$. In this case ($\mathrm{NS_M}$) is called “Leray’s equation”, for it was suggested by [10] to prove the existence of blow-up solutions to (NS) by constructing backward self-similar solutions. For this particular case it was proved by [14] that any weak solution to Leray’s equation in $L^3(\mathbb{R}^3)$ must be trivial. This result declared that Leray’s idea does not give the construction of blow-up solutions to (NS). A simpler proof of the same conclusion was obtained by [15] under a slightly stronger assumption. The result of [14, 15] was extended by [17], where the condition of the spatial decay on U was completely removed. The expanding effect of $Mx \cdot \nabla$ in all directions was essentially used in [17]. Although the eigenvalues λ_i in [14, 15, 17] are assumed to be positive and identical, one can apply the method especially in [17] for proving the nonexistence of nontrivial solutions to ($\mathrm{NS_M}$) even when the eigenvalues are all positive but does not coincide with each other. We also refer to [3] for a related problem on the Euler equations.

(ii) $\lambda_1 < 0, \lambda_2 < 0, \displaystyle\sum_{i=1}^{3} \lambda_i = 0$: When $\lambda_1 = \lambda_2$ ($\mathrm{NS_M}$) has an explicit two-dimensional solution, called the Burgers vortex [1]. Even in the case $\lambda_1 \neq \lambda_2$ the analog of the Burgers vortex is known to exist; see [4, 5, 12, 13]. For stability of the Burgers vortex the reader is referred to a recent book [6, Chapter 2] and references cited there.

(iii) $\lambda_1 = \lambda_2 = \lambda_3 < 0$: In this case ($\mathrm{NS_M}$) describes the forward self-similar solutions to (NS), and their existence is already well known. For example, see [2, 7, 9, 16].

For more references about forward and backward self-similar solutions to (NS) the reader is referred to [6].

Now let us go back to the case (1.4) treated in the present paper. In this case the solutions are more likely to be trivial due to the expanding effect of $Mx \cdot \nabla$ in two directions. However, the presence of the negative eigenvalue λ_1 gives rise to the interaction of the localization and the expansion through the diffusion and the nonlinearity, which makes the problem rather complicated. The aim of this paper is to give sufficient conditions for (U, P) so that U must be a constant vector, by overcoming this difficulty. The key idea is to focus on the vorticity field $\Omega = \nabla \times U$. The assumptions and the main result of this paper are stated as follows.

(C0) $\displaystyle |U(x)| + \frac{|P(x)|}{1 + |x|} \in L^\infty(\mathbb{R}^3)$;

(C1) either (i) there is $\{x^{(n)}\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty} |x_1^{(n)}| = \infty, \quad \sup_n (|x_2^{(n)}| + |x_3^{(n)}|) < \infty, \quad \lim_{n\to\infty} \frac{P(x^{(n)})}{x_1^{(n)}} = 0$$

or (ii) there is $\{x^{(n)}\} \subset \mathbb{R}^3$ such that $\displaystyle\lim_{n\to\infty} |x^{(n)}| = \infty, \quad \lim_{n\to\infty} U_1(x^{(n)}) = 0$;

(C2) $(1 + |x|)|\Omega(x)| \in L^{p_0}(\mathbb{R}^3) \quad$ for some $p_0 \in (1, 3)$;

(C3) there is $\theta_0 > \lambda$ such that

either (i) $(1 + |x_2|)^{\theta_0 + 1}|\Omega(x)| \in L^\infty(\mathbb{R}^3)$ or (ii) $(1 + |x_3|)^{\frac{\theta_0}{\mu} + 1}|\Omega(x)| \in L^\infty(\mathbb{R}^3)$ holds.

Theorem 1.1 *Let* $(U, P) \in (C^2(\mathbb{R}^3))^3 \times C^1(\mathbb{R}^3)$ *be a solution to* $(\mathrm{NS_M})$*. Assume that* **(C0)**-**(C3)** *hold. Then* $U \equiv \mathrm{const}$*.*

Remark 1.2 Under the conditions **(C0)** and **(C2)** it is not difficult to deduce $\nabla^k U \in L^\infty(\mathbb{R}^3)$ for each $k \in \mathbb{N}$. We will freely use this fact in the rest of the paper.

This theorem implies that when the vorticity field decays sufficiently fast there are only trivial solutions to $(\mathrm{NS_M})$. We note that the absolute value of each eigenvalue represents the intensity of its straining effect, and it crucially acts on the structure of $(\mathrm{NS_M})$. In particular, the ratios of $|\lambda_1| = \lambda$ (localizing effect) and $\lambda_2 = 1, \lambda_3 = \mu$ (expanding effect) are important and they appear in the condition **(C3)**.

As in the previous papers [14, 15, 17], the key of our proof is to estimate the generalized pressure

$$\Pi(x) = \frac{1}{2}|U(x)|^2 + Mx \cdot U(x) + P(x). \tag{1.5}$$

However, the arguments in [14, 15, 17] rely on the positivity of each λ_i in the core part of the proof. So another new idea is needed to deal with the negative eigenvalue in our case. Under the conditions **(C0)** and **(C2)** the generalized pressure Π is written as $\Pi = a + \Pi_0$, where a is a constant and Π_0 decays uniformly at $|x| \to \infty$. The basic strategy is to investigate the spatial decay of Π_0 in details. In particular, we establish the pointwise estimates of $|\Pi_0(x)|$ from above and below that cannot be compatible to hold at the same time when Π_0 is not trivial. Theorem 1.1 is an immediate consequence of this result. As for the lower bound, we observe that Π_0 satisfies the inequality $\Delta\Pi_0 - Mx\cdot\nabla\Pi_0 - U\cdot\nabla\Pi_0 \geq 0$ and then apply the argument in [11] to get

$$|\Pi_0(x)| \geq C_{x_1}(1 + x_2^2 + (1 + x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} \qquad \text{if} \quad \Pi_0(x) \not\equiv 0, \tag{1.6}$$

where C_{x_1} is a positive constant independent of x_2 and x_3; see Proposition 3.5. In fact, when Π_0 decays at spatial infinity the estimate (1.6) is proved only under the conditions **(C0)** and **(C1')**: $\lim_{|x|\to\infty} |U_1(x)| = 0$. Especially, it is possible to derive the conclusion in Theorem 1.1 by alternatively assuming **(C0)**, **(C1')**, and suitable decay conditions on Π (or on Π_0) so as to contradict with (1.6). Although we do not need to pay much attention on vorticity fields in this alternative result, instead, there we are forced to assume strong spatial decay conditions on Π if $|\lambda|$ is large. But these are not so "realistic" assumptions because Π includes the pressure term P for which we cannot expect fast spatial decay in general even if U decays rapidly. On the other hand, the flows with localized vorticity fields are considered to be natural objects, and Theorem 1.1 excludes the possibility of the realization of such flows.

From mathematical point of view it is essential that Π_0 solves the Poisson equation with the inhomogeneous terms which are written in terms of the vorticity field Ω. Then under the assumptions in Theorem 1.1 the lower bound (1.6) is improved by

$$|\Pi_0(0, x_2, 0)| \geq C_l(1 + x_2^2)^{-l} \quad \text{or} \quad |\Pi_0(0, 0, x_3)| \geq C_l(1 + x_3^2)^{-l} \qquad \text{if} \quad \Pi_0(x) \not\equiv 0, \tag{1.7}$$

for all $l > 0$; see Proposition 3.8. Since $l > 0$ in (1.7) is arbitrary it is not difficult to obtain the upper bound of $|\Pi_0(x)|$ such that a contradiction arises. Indeed, after establishing several estimates of Ω by using the vorticity equations, we can deduce some polynomial decay of Π_0 from the analysis of the Poisson equation.

The plan of this paper is as follows. In Section 2.1 we recall some equations which Π or Ω satisfies. In Section 2.2 we prove some estimates of Ω by using the vorticity equations. In this step we use the weighted estimates of the Ornstein-Uhlenbeck semigroup which are given in the appendix. In Section 2.3 we give the estimates of the velocity field from the Biot-Savart law. Section 3 is devoted to establish the pointwise estimates of Π_0. Then Theorem 1.1 is proved in Section 4.

2 Preliminaries

2.1 Fundamental equality

In this section we state several equalities which are fundamental in this paper. Set

$$\Pi(x) = \frac{1}{2}|U(x)|^2 + P(x) + Mx \cdot U(x). \tag{2.1}$$

Let $\mathcal{L}$ be the differential operator defined by

$$\mathcal{L}f = \Delta f - Mx \cdot \nabla f. \tag{2.2}$$

Proposition 2.1 *Let (U, P) be a smooth solution to* $(\mathrm{NS_M})$. *Then the following equalities hold.*

$$\begin{aligned} \mathcal{L}\Pi - U \cdot \nabla \Pi &= |\Omega|^2, & (2.3)\\ -\Delta U_j - (U \times \Omega)_j + \partial_j \Pi &= -Mx \cdot (\nabla U_j - \partial_j U), & (2.4)\\ \mathcal{L}\Omega + (M - \mathrm{Tr}(M)I)\Omega &= U \cdot \nabla \Omega - \Omega \cdot \nabla U. & (2.5) \end{aligned}$$

Proof. Since each equality is derived from a direct computation without difficulty we omit the details here.

2.2 Estimates for vorticity

In this section we prove some estimates of Ω from the vorticity equations (2.5).

Proposition 2.2 *Assume that* **(C0)** *and* **(C2)** *hold. Let $k = 0, 1, 2$. Then*

$$(1 + |x|)|\nabla^k \Omega(x)| \in L^p(\mathbb{R}^3) \qquad \text{for all} \quad p \in [p_0, \infty]. \tag{2.6}$$

Moreover, we have

$$\begin{aligned} (1 + |x_2|)^{\theta_0+1}|\nabla^k \Omega(x)| &\in L^\infty(\mathbb{R}^3) \qquad \text{if (i) of } (\mathbf{C3}) \text{ holds}, & (2.7)\\ (1 + |x_3|)^{\frac{\theta_0}{\mu}+1}|\nabla^k \Omega(x)| &\in L^\infty(\mathbb{R}^3) \qquad \text{if (ii) of } (\mathbf{C3}) \text{ holds}. & (2.8) \end{aligned}$$

To prove Proposition 2.2 we introduce the semigroup $e^{t\mathcal{L}}f$ associated with the operator $\mathcal{L}$ defined by

$$(e^{t\mathcal{L}}f)(x) = (2\pi)^{-\frac{3}{2}}(\det Q_t)^{-\frac{1}{2}}e^{-t\mathrm{Tr}(M)}\int_{\mathbb{R}^3} e^{-\frac{1}{2}\left\{\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}y_1^2+\frac{1}{e^{2t}-1}y_2^2+\frac{\mu}{e^{2\mu t}-1}y_3^2\right\}} f(e^{-tM}(x-y))\,\mathrm{d}y. \tag{2.9}$$

Here

$$\det Q_t = \lambda^{-1}\mu^{-1}(e^{2t\lambda}-1)(1-e^{-2t})(1-e^{-2\mu t}). \tag{2.10}$$

The operator like $\mathcal{L}$ is well known as the Ornstein-Uhlenbeck operator. The representation (2.9) is easily obtained through the Fourier transform, so we proceed by admitting (2.9).

Lemma 2.3 *Let $\theta_1, \theta_2, \theta_3 \geq 0$ and $1 \leq q \leq p \leq \infty$. Set*

$$b(x) = (1+x_1^2)^{\theta_1} + (1+x_2^2)^{\theta_2} + (1+x_3^2)^{\theta_3}. \tag{2.11}$$

Then for each $k \in \mathbb{N}\cup\{0\}$ there are positive constants C and c such that

$$\|b\nabla^k e^{t\mathcal{L}}f\|_{L^p} \leq Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}}e^{ct}\|bf\|_{L^q}. \tag{2.12}$$

The proof of Lemma 2.3 will be stated in the appendix. The L^p-L^q estimates for $e^{t\mathcal{L}}$ without weight functions are obtained by [8] for a general class of M.

Proof of Proposition 2.2. We give the proof only for (2.6), since (2.7) and (2.8) are obtained in the similar manner. By taking (2.5) and the Laplace transform into account we set

$$\Phi(F) = \int_0^\infty e^{t\mathcal{L}}e^{t\left(M-(\mathrm{Tr}(M)+c')I\right)}\left(c'\Omega - U\cdot\nabla F + F\cdot\nabla U\right)\mathrm{d}t. \tag{2.13}$$

Here F satisfies $bF \in (L^{p_0}(\mathbb{R}^3)\cap L^{p_1}(\mathbb{R}^3))^3$ and $b\partial_j F \in (L^{p_0}(\mathbb{R}^3)\cap L^{p_2}(\mathbb{R}^3))^3$ for some $p_1, p_2 \in (p_0,\infty]$ satisfying $1/p_1 > 1/p_0 - 2/3$ and $1/p_2 > 1/p_0 - 1/3$, and $c' > 0$ is taken sufficiently large. Then by Lemma 2.3 and by using the L^∞ bound of U and ∇U, it is not difficult to see

$$\begin{aligned}
&\|b\Phi(F)\|_{L^{p_0}\cap L^{p_1}} \leq C\|b\Omega\|_{L^{p_0}} + \delta(c')\big(\|bF\|_{L^{p_0}} + \|b\nabla F\|_{L^{p_0}}\big),\\
&\|b\nabla\Phi(F)\|_{L^{p_0}\cap L^{p_2}} \leq C\|b\Omega\|_{L^{p_0}} + \delta(c')\big(\|bF\|_{L^{p_0}} + \|b\nabla F\|_{L^{p_0}}\big),\\
&\|b\Phi(F_1) - b\Phi(F_2)\|_{L^{p_0}\cap L^{p_1}} \leq \delta(c')\big(\|bF_1 - bF_2\|_{L^{p_0}} + \|b\nabla F_1 - b\nabla F_2\|_{L^{p_0}}\big),\\
&\|b\nabla\Phi(F_1) - b\nabla\Phi(F_2)\|_{L^{p_0}\cap L^{p_2}} \leq \delta(c')\big(\|bF_1 - bF_2\|_{L^{p_0}} + \|b\nabla F_1 - b\nabla F_2\|_{L^{p_0}}\big).
\end{aligned}$$

Here the constant $\delta(c')$ satisfies $\delta(c') \to 0$ as $c' \to \infty$. Hence by taking c' large enough we find a fixed point F_* of Φ from the contraction mapping theorem in the natural weighted Sobolev space. Since $\nabla^k U$ is bounded we can also show that F_* is smooth and bounded, and satisfies the equation

$$\mathcal{L}F_* + (M - (\mathrm{Tr}(M)+c')I)F_* = -c'\Omega + U\cdot\nabla F_* - F_*\cdot\nabla U. \tag{2.14}$$

Moreover, solving the adjoint equation of (2.14), we can show the uniqueness of solutions to (2.14) in $(L^{p_0}(\mathbb{R}^3))^3$; the details are omitted here since the argument is standard. Thus we have $\Omega = F_*$, i.e., $b\Omega \in (L^{p_1}(\mathbb{R}^3))^3$ and $b\partial_j\Omega \in (L^{p_0}(\mathbb{R}^3)\cap L^{p_2}(\mathbb{R}^3))^3$. Repeating this argument at most finite times, we conclude that $b\Omega \in (L^\infty(\mathbb{R}^3))^3$ and $b\partial_j\Omega \in (L^\infty(\mathbb{R}^3))^3$. The property $b\partial_{ij}^2\Omega \in (L^p(\mathbb{R}^3))^3$ for $p \in [p_0,\infty]$ is then proved by the same argument as above, if one uses the equality $\nabla e^{t\mathcal{L}}f = e^{t\mathcal{L}}e^{-tM}\nabla f$. This completes the proof of Proposition 2.2.

2.3 Estimates for velocity

Let V be the velocity field recovered from Ω via the Biot-Savart law, i.e.,

$$V(x) = (-\Delta)^{-1}\nabla\times\Omega = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)}{|x-y|^3}\times\Omega(y)\,\mathrm{d}y. \tag{2.15}$$

Then by **(C0)** we have

$$U = u_c + V \qquad\qquad u_c:\ \text{a constant vector}. \tag{2.16}$$

Proposition 2.4 *Assume that* **(C0)** *and* **(C2)** *hold. Then*

$$|V(x)| \le C(1+|x|)^{-1}. \tag{2.17}$$

Proof. We first note the inequality

$$(1+|x|)|V(x)| \le C\Big(\int_{\mathbb{R}^3}\frac{|\Omega(y)|}{|x-y|}\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{(1+|y|)|\Omega(y)|}{|x-y|^2}\,\mathrm{d}y\Big) =: C(I_1+I_2).$$

Then for $1/p_0'+1/p_0=1$, the term I_1 is estimated as

$$\begin{aligned} I_1 &\le \int_{|x-y|\le 1}\frac{|\Omega(y)|}{|x-y|}\,\mathrm{d}y + \int_{|x-y|\ge 1}\frac{|\Omega(y)|}{|x-y|}\,\mathrm{d}y \\ &\le C\|\Omega\|_{L^\infty} + \Big(\int_{|x-y|\ge 1}|x-y|^{-p_0'}(1+|y|)^{-p_0'}\,\mathrm{d}y\Big)^{\frac{1}{p_0'}}\|(1+|\cdot|)\Omega\|_{L^{p_0}} < \infty, \end{aligned}$$

since $p_0\in(1,3)$. By Proposition 2.2 we have $(1+|x|)|\Omega(x)|\in L^{p_0}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Then by applying the Hardy-Littlewood-Sobolev inequality and the Calderón-Zygmund inequality, we get $I_2\in L^\infty(\mathbb{R}^3)$. This completes the proof.

3 Estimates for Π

In this section we establish the estimates for Π, which is the core of the proof of Theorem 1.1. From (2.4) we have

$$-\Delta\Pi = -\nabla\cdot(U\times\Omega) + \sum_j \partial_j\big(Mx\cdot(\nabla U_j - \partial_j U)\big). \tag{3.1}$$

Taking (3.1) into account, we set

$$\begin{aligned} \Pi_0(x) &:= -(-\Delta)^{-1}\nabla\cdot(U\times\Omega) + \sum_j(-\Delta)^{-1}\partial_j\big(M(\cdot)\cdot(\nabla U_j-\partial_j U)\big) \\ &= C\sum_j\int_{\mathbb{R}^3}\frac{x_j-y_j}{|x-y|^3}\Big((U(y)\times\Omega(y))_j - My\cdot(\nabla U_j(y)-\partial_j U(y))\Big)\,\mathrm{d}y. \end{aligned} \tag{3.2}$$

3.1 Upper bound of $-\Pi_0$

Proposition 3.1 *Assume that* **(C0)** *and* **(C2)** *hold. Set* $\langle x\rangle = (1+|x|^2)^{1/2}$. *Then*

$$\|\Pi_0\|_{L^{q_0}} \le C(1+\|U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^{p_0}}, \tag{3.3}$$

$$\|\nabla\Pi_0\|_{L^p} \le C(1+\|U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^p}, \tag{3.4}$$

$$\|\nabla^2\Pi_0\|_{L^p} \le C\big((1+\|\nabla U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^p} + (1+\|U\|_{L^\infty})\|\langle\cdot\rangle\nabla\Omega\|_{L^p}\big), \tag{3.5}$$

for $1/q_0 = 1/p_0 - 1/3$ *and for all* $p\in[p_0,\infty)$. *In particular,* $\Pi_0, \nabla\Pi_0 \in L^\infty(\mathbb{R}^3)$ *and*

$$\lim_{R\to\infty}\sup_{|x|\ge R}(|\Pi_0(x)| + |\nabla\Pi_0(x)|) = 0. \tag{3.6}$$

Moreover, if **(C3)** *holds in addition, then there is* $\delta>0$ *such that*

$$|\Pi_0(0,x_2,0)| \le C(1+|x_2|)^{-\delta} \qquad \text{if (i) of } \textbf{(C3)} \text{ holds,} \tag{3.7}$$

$$|\Pi_0(0,0,x_3)| \le C(1+|x_3|)^{-\delta} \qquad \text{if (ii) of } \textbf{(C3)} \text{ holds.} \tag{3.8}$$

Proof. It is easy to see that

$$|\Pi_0(x)| \le C(1+\|U\|_{L^\infty})\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}\langle y\rangle|\Omega(y)|\,\mathrm{d}y. \tag{3.9}$$

Hence by the Hardy-Littlewood-Sobolev inequality we have

$$\|\Pi_0\|_{L^{q_0}} \le C(1+\|U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^{p_0}} \qquad \text{for } \frac{1}{q_0} = \frac{1}{p_0} - \frac{1}{3}. \tag{3.10}$$

Moreover, the Calderón-Zygmund inequality implies

$$\|\nabla\Pi_0\|_{L^p} \le C(1+\|U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^p} < \infty \qquad \text{for all } p\in[p_0,\infty). \tag{3.11}$$

by Proposition 2.2. The estimate for $\|\nabla^2\Pi_0\|_{L^p}$ is obtained in the similar manner. To prove (3.7) we use the inequality (3.9) and observe that

$$\begin{aligned}
&(1+|x_2|)^\delta|\Pi_0(x)| \\
\le\ & C\Big(\int_{\mathbb{R}^3}\frac{1}{|x-y|^{2-\delta}}(1+|y|)|\Omega(y)|\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{1}{|x-y|^2}(1+|y|)(1+|y_2|)^\delta|\Omega(y)|\,\mathrm{d}y\Big) \\
=\ & C(I_1(x)+I_2(x)).
\end{aligned} \tag{3.12}$$

Since $(1+|x|)|\Omega(x)| \in L^{p_0}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ and $p_0\in(1,3)$, if $\delta\in(0,\theta_0)$ is small enough, then it is not difficult to see $I_1\in L^\infty(\mathbb{R}^3)$ by dividing the integral into $\int_{|x-y|\le1}$ and $\int_{|x-y|\ge1}$. As for I_2, we observe that

$$\begin{aligned}
I_2(0,x_2,0) &= \int_{\mathbb{R}^3}\frac{1}{(x_2-y_2)^2+y_1^2+y_3^2}(1+|y|)(1+|y_2|)^\delta|\Omega(y)|\,\mathrm{d}y \\
&\le C\int_{|y_1|+|y_3|\le1}\frac{1}{(x_2-y_2)^2+y_1^2+y_3^2}(1+|y_2|)^{1+\delta}|\Omega(y)|\,\mathrm{d}y \\
&\quad + C\int_{|y_1|+|y_3|\ge1}\frac{1}{(x_2-y_2)^2+y_1^2+y_3^2}(1+|y_2|)^{1+\delta}|\Omega(y)|\,\mathrm{d}y \\
&\quad + C\int_{|y_1|+|y_3|\ge1}\frac{1}{|x_2-y_2|+|y_1|+|y_3|}(1+|y_2|)^\delta|\Omega(y)|\,\mathrm{d}y \\
&= I_{2,1}(x_2)+I_{2,2}(x_2)+I_{2,3}(x_2).
\end{aligned}$$

Then $I_{2,1} \in L^\infty(\mathbb{R})$ if $\delta \in (0, \theta_0)$. As for $I_{2,2}$, we note that for any $\epsilon > 0$ if $\delta < \epsilon\theta_0$ then $(1+|y_2|)^{1+\delta}|\Omega(y)| \le C\{(1+|y_2|)|\Omega(y)|\}^{1-\epsilon}$ by (i) of **(C3)**. Since $\{(1+|y|)|\Omega(y)|\}^{1-\epsilon} \in L^p(\mathbb{R}^3)$ for some $p \in (1,3)$ if $\epsilon > 0$ is sufficiently small due to **(C2)**, we have $I_{2,2} \in L^\infty(\mathbb{R}^3)$ by the Hölder inequality. Similarly, from $(1+|y_2|)^\delta|\Omega(y)| \le C|\Omega(y)|^{1-\epsilon}$ for any $\epsilon \in (0,1)$ with $\delta < \epsilon(1+\theta_0)$, we have

$$|I_{2,3}(x_2)| \le C\Big(\int_{|y_1|+|y_3|\ge 1} \frac{1}{(|x_2-y_2|+|y_1|+|y_3|)^{q'}(1+|y|)^{(1-\epsilon)q'}}\,\mathrm{d}y\Big)^{\frac{1}{q'}} \|\langle\cdot\rangle\Omega\|^{1-\epsilon}_{L^{(1-\epsilon)q}},$$

where $1/q' + 1/q = 1$. We choose $\epsilon > 0$ sufficiently small so that both $p_0 \le (1-\epsilon)q$ and $q < 3/(1+2\epsilon)$ are satisfied. Then the right-hand side of the above inequality is uniformly bounded with respect to x_2, since $(1-\epsilon)q' > 3/2$ in such case. The estimate (3.8) is proved in the same way. This completes the proof.

The condition **(C0)** implies $|\Pi(x)| \le C(1+|x|)$, and hence, we have from (3.1) and the definition of Π_0,

$$\Pi(x) = \sum_i a_i x_i + a_0 + \Pi_0(x), \tag{3.13}$$

for some $a_i \in \mathbb{R}$, $i = 0,1,2,3$. Then (2.3) yields

$$(U + Mx)\cdot a = -|\Omega|^2 + \Delta\Pi_0 - (U+Mx)\cdot\nabla\Pi_0, \qquad a = (a_1,a_2,a_3). \tag{3.14}$$

By Proposition 3.1 the right-hand side of (3.14) has the order $o(|x|)$ at $|x| \to \infty$, so a must be the zero vector. Hence we have $\Pi = a_0 + \Pi_0$ and

$$\mathcal{L}\Pi_0 - U\cdot\nabla\Pi_0 = |\Omega|^2. \tag{3.15}$$

Since $|\Pi_0(x)| \to 0$ as $|x| \to \infty$ by Proposition 3.1, the strong maximum principle implies

Corollary 3.2 *Assume that* **(C0)** *and* **(C2)** *hold. Then either* $\Pi_0 \equiv 0$ *or* $\Pi_0(x) < 0$ *for all* $x \in \mathbb{R}^3$.

By using (2.4) we can derive the estimates for the derivatives of Π_0, which are different from the ones in Proposition 3.1.

Proposition 3.3 *Assume that* **(C0)**, **(C2)**, **(C3)** *hold. Let* $k = 1,2$. *Then it follows that*

$$|\nabla^k\Pi_0(x)| \le C(1+|x_1|+|x_3|)(1+|x_2|)^{-\theta_0} \qquad \text{if (i) of } \mathbf{(C3)} \text{ holds}, \tag{3.16}$$
$$|\nabla^k\Pi_0(x)| \le C(1+|x_1|+|x_2|)(1+|x_3|)^{-\frac{\theta_0}{\mu}} \qquad \text{if (ii) of } \mathbf{(C3)} \text{ holds}. \tag{3.17}$$

Proof. It suffices to consider the case when (i) of **(C3)** holds. By (2.4) and $\Pi = a_0 + \Pi_0$ we have

$$\begin{aligned} \partial_j\Pi_0 = \partial_j\Pi &= \Delta U_j + (U\times\Omega)_j - Mx\cdot(\nabla U_j - \partial_j U) \\ &= -(\nabla\times\Omega)_j + (U\times\Omega)_j - Mx\cdot(\nabla U_j - \partial_j U) \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.18}$$

Here we have used $\Delta U = -\nabla \times \Omega$. From Propositions 2.2, 2.4 we have

$$|I_1(x)| + |I_2(x)| \le C(1 + |x_2|)^{-\theta_0 - 1}. \tag{3.19}$$

As for I_3, we have from **(C3)**,

$$|I_3(x)| \le C|x||\Omega(x)| \ \ \le \ \ C(1 + |x_1| + |x_3|)(1 + |x_2|)^{-\theta_0}. \tag{3.20}$$

The estimate for $\nabla^2 \Pi_0$ is proved in the same way, due to Proposition 2.2. This completes the proof.

3.2 Lower bound of $-\Pi_0$

For the moment we consider a smooth nontrivial function f which satisfies

$$\mathcal{L}f - B \cdot \nabla f \ge 0, \qquad \lim_{R\to\infty} \sup_{|x|\ge R} |f(x)| = 0. \tag{3.21}$$

In this section B is always assumed to be a smooth vector function satisfying $\nabla \cdot B = 0$. The strong maximum principle implies that $f(x) < 0$ for all $x \in \mathbb{R}^3$. The aim of this section is to derive a lower bound on the spatial decay of $-f$. We start from the "rough" lower bound.

Proposition 3.4 *Let $f \in BC^2(\mathbb{R}^3)$ be a nontrivial solution to (3.21). Assume that*

$$\lim_{R\to\infty} \sup_{|x|\ge R} \frac{|B(x)|}{|x|} = 0. \tag{3.22}$$

Then for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$-f(x) \ge C_\epsilon e^{-\frac{\lambda(1+\epsilon)}{2}x_1^2 - \frac{\epsilon}{2}(x_2^2 + \mu x_3^2)}, \qquad x \in \mathbb{R}^3. \tag{3.23}$$

Proof. We set

$$\tilde{f}(x) = -f(x)e^{-\frac{1}{2}({x_2}^2 + \mu {x_3}^2)} = -f(x)e^{-\frac{1}{2}x^t M_0 x}, \tag{3.24}$$

where

$$M_\gamma = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad \text{for} \quad \gamma \in \mathbb{R}. \tag{3.25}$$

Then the direct calculations yield

$$\Delta \tilde{f} = e^{-\frac{1}{2}x^t M_0 x}\big(- \Delta f + 2M_0 x \cdot \nabla f - f|M_0 x|^2 + f\mathrm{Tr}(M_0)\big),$$
$$(-B + M_\lambda x) \cdot \nabla \tilde{f} = e^{-\frac{1}{2}x^t M_0 x}\big(B \cdot \nabla f - M_\lambda x \cdot \nabla f - fM_0 x \cdot B + fM_\lambda x \cdot M_0 x\big).$$

Thus we see

$$\begin{aligned} \tilde{\mathcal{L}}\tilde{f} \ &:= \ \Delta \tilde{f} + (-B + M_\lambda x) \cdot \nabla \tilde{f} + (\mathrm{Tr}(M_0) - M_0 x \cdot B)\tilde{f} \\ &= \ e^{-\frac{1}{2}x^t M_0 x}\big(- \Delta f + B \cdot \nabla f + Mx \cdot \nabla f\big) \\ &= \ e^{-\frac{1}{2}x^t M_0 x}(-\mathcal{L}f + B \cdot \nabla f) \le 0. \end{aligned} \tag{3.26}$$

Now we set $N = 2\|\tilde{f}\|_{L^\infty} > 0$, and let $\delta \in (0, 1/4)$ and $K > 1$. Then we define the function F_δ by

$$F_\delta(x) = \frac{1}{w(x)} \log(\frac{\tilde{f}(x)}{N} + \delta) < 0,$$

where

$$w(x) = K + \frac{1}{2}(\lambda {x_1}^2 + {x_2}^2 + \mu {x_3}^2) = K + \frac{1}{2} x^t M_\lambda x.$$

Since

$$\nabla F_\delta = \frac{\nabla \tilde{f}}{w(\tilde{f} + N\delta)} - \frac{\nabla w}{w} F_\delta,$$

and

$$\begin{aligned}
\Delta F_\delta &= \frac{\Delta \tilde{f}}{w(\tilde{f} + N\delta)} - 2\frac{\nabla w \cdot \nabla F_\delta}{w} - \frac{\Delta w}{w} F_\delta - \frac{|\nabla \tilde{f}|^2}{w(\tilde{f} + N\delta)^2} \\
&= \frac{\Delta \tilde{f}}{w(\tilde{f} + N\delta)} - 2\frac{\nabla w \cdot \nabla F_\delta}{w} - \frac{\Delta w}{w} F_\delta - w|\nabla F_\delta|^2 - \frac{|\nabla w|^2}{w} {F_\delta}^2 - 2F_\delta \nabla w \cdot \nabla F_\delta,
\end{aligned}$$

we get from (3.26) the equation for F_δ such as

$$\begin{aligned}
-\Delta F_\delta \geq &\big(- B + M_\lambda x + 2\frac{\nabla w}{w} + 2F_\delta \nabla w\big) \cdot \nabla F_\delta \\
&+ \big((-B + M_\lambda x) \cdot \frac{\nabla w}{w} + \frac{\Delta w}{w} + \frac{|\nabla w|^2}{w} F_\delta\big) F_\delta + w|\nabla F_\delta|^2 + \frac{(\mathrm{Tr}(M_0) - M_0 x \cdot B)\tilde{f}}{w(\tilde{f} + N\delta)}.
\end{aligned}$$

Since $F_\delta < 0$, we have for large $p \in \mathbb{N}$,

$$\begin{aligned}
(2p-1)\int_{\mathbb{R}^3} |\nabla F_\delta|^2 F_\delta^{2(p-1)} \, \mathrm{d}x &= \int_{\mathbb{R}^3} -\Delta F_\delta {F_\delta}^{2p-1} \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \big(- B + M_\lambda x + 2\frac{\nabla w}{w} + 2F_\delta \nabla w\big) \cdot \nabla F_\delta {F_\delta}^{2p-1} \, \mathrm{d}x \\
&+ \int_{\mathbb{R}^3} \{(-B + M_\lambda x) \cdot \frac{\nabla w}{w} + \frac{\Delta w}{w} + \frac{|\nabla w|^2}{w} F_\delta\} {F_\delta}^{2p} \, \mathrm{d}x \\
&+ \int_{\mathbb{R}^3} w|\nabla F_\delta|^2 {F_\delta}^{2p-1} \, \mathrm{d}x + \int_{\mathbb{R}^3} \frac{(\mathrm{Tr}(M_0) - M_0 x \cdot B)\tilde{f}}{w(\tilde{f} + N\delta)} {F_\delta}^{2p-1} \, \mathrm{d}x.
\end{aligned} \tag{3.27}$$

By the integration by parts and $\nabla \cdot B = 0$ the first term of right hand side of (3.27) equals

$$\frac{1}{2p} \int_{\mathbb{R}^3} \nabla \cdot \big(- M_\lambda x - 2\frac{\nabla w}{w} - 2F_\delta \nabla w\big) {F_\delta}^{2p} \, \mathrm{d}x.$$

Since the third term of the right hand sider of (3.27) is nonpositive and $\mathrm{Tr}(M_0) > 0$, we get

$$\begin{aligned}(2p-1)\int_{\mathbb{R}^3} |\nabla F_\delta|^2 {F_\delta}^{2(p-1)}\,\mathrm{d}x \quad &\leq \quad \frac{1}{p}\int_{\mathbb{R}^3}\big(-\frac{1}{2}\mathrm{Tr}(M_\lambda) - \nabla\cdot\frac{\nabla w}{w} - \nabla\cdot(F_\delta\nabla w)\big){F_\delta}^{2p}\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\big((-B+M_\lambda x)\cdot\nabla w + \Delta w + |\nabla w|^2 F_\delta\big)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\frac{|M_0 x\cdot B|}{w|F_\delta|}{F_\delta}^{2p}\,\mathrm{d}x.\end{aligned}$$

By the integration by parts we have

$$\int_{\mathbb{R}^3}\nabla\cdot(F_\delta\nabla w){F_\delta}^{2p}dx = \frac{2p}{2p+1}\int_{\mathbb{R}^3}\Delta w {F_\delta}^{2p+1}\,\mathrm{d}x,$$

and observe that $\nabla w = M_\lambda x$ and $\Delta w = \mathrm{Tr}(M_\lambda) > 0$. Thus we obtain

$$\begin{aligned}(2p-1)\int_{\mathbb{R}^3}|\nabla F_\delta|^2{F_\delta}^{2(p-1)}\,\mathrm{d}x &\leq \int_{\mathbb{R}^3}\bigg((-B+M_\lambda x)\cdot\nabla w - \frac{\mathrm{Tr}(M_\lambda)w}{2p} + \big(1-\frac{1}{p}-\frac{2wF_\delta}{2p+1}\big)\Delta w \\ &\qquad + (F_\delta + \frac{1}{pw})|\nabla w|^2 + \frac{|M_0 x\cdot B|}{|F_\delta|}\bigg)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3}\bigg((-B+M_\lambda x)\cdot M_\lambda x + \big(1-\frac{2wF_\delta}{2p+1}\big)\mathrm{Tr}(M_\lambda) \\ &\qquad + (F_\delta+\frac{1}{pw})|M_\lambda x|^2 + \frac{|M_0x\cdot B|}{|F_\delta|}\bigg)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\ &= I_1 + I_2. \end{aligned} \tag{3.28}$$

Here

$$\begin{aligned} I_1 &= \int_{F_\delta > -1-\epsilon}\bigg((-B+M_\lambda x)\cdot M_\lambda x + \big(1-\frac{2wF_\delta}{2p+1}\big)\mathrm{Tr}(M_\lambda) \\ &\qquad + (F_\delta+\frac{1}{pw})|M_\lambda x|^2 + \frac{|M_0x\cdot B|}{|F_\delta|}\bigg)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\ I_2 &= \int_{F_\delta \leq -1-\epsilon}\bigg((-B+M_\lambda x)\cdot M_\lambda x + \big(1-\frac{2wF_\delta}{2p+1}\big)\mathrm{Tr}(M_\lambda) \\ &\qquad + (F_\delta+\frac{1}{pw})|M_\lambda x|^2 + \frac{|M_0x\cdot B|}{|F_\delta|}\bigg)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x. \end{aligned} \tag{3.29}$$

We claim that if $p \gg (\|F_\delta\|_{L^\infty}+1)(K+1)$ then there are positive constants C' and R' which are independent of p and δ such that

$$I_1 \leq C'\|F_\delta\chi_{\{F_\delta > -1-\epsilon\}}\|_{L^{2p-1}}^{2p-1}, \qquad I_2 \leq C'\|F_\delta\chi_{\{|x|\leq R'\}}\|_{L^{2p}}^{2p}.$$

Indeed, we have

$$
\begin{aligned}
I_1 \le \int_{F_\delta > -1-\epsilon} \Big(& \frac{|B\cdot M_\lambda x|}{w} + \frac{M_\lambda x\cdot M_\lambda x}{w} + \frac{\mathrm{Tr}(M_\lambda)}{w} - 2\mathrm{Tr}(M_\lambda)\frac{F_\delta}{2p+1} \\
& + \frac{|M_\lambda x|^2}{pw^2} + \frac{|M_0 x\cdot B|}{w|F_\delta|}\Big) {F_\delta}^{2p}\,\mathrm{d}x \\
\le C\big(1 + \| & \frac{B\cdot M_\lambda x}{w}\|_{L^\infty} + \|\frac{B\cdot M_0 x}{w}\|_{L^\infty}\big)\|F_\delta \chi_{\{F_\delta>-1-\epsilon\}}\|^{2p-1}_{L^{2p-1}}.
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 \ \le & \int_{F_\delta\le -1-\epsilon} \Big(|B\cdot M_\lambda x| + M_\lambda x\cdot M_\lambda x + \mathrm{Tr}(M_\lambda)\big(1 - \frac{2wF_\delta}{2p+1}\big)\Big)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\
& + \int_{F_\delta\le -1-\epsilon} \big(-(1+\epsilon)|M_\lambda x|^2 + \frac{|M_\lambda x|^2}{pw} + |M_0 x\cdot B|\big)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\
\le & \int_{F_\delta\le -1-\epsilon} \Big(\frac{|B\cdot M_\lambda x|}{w} + \frac{|B\cdot M_0 x|}{w} + \frac{\mathrm{Tr}(M_\lambda)}{w} - \mathrm{Tr}(M_\lambda)\frac{2F_\delta}{2p+1} \\
& + \frac{|M_\lambda x|^2}{pw^2} - \epsilon\frac{|M_\lambda x|^2}{w}\Big){F_\delta}^{2p}\,\mathrm{d}x.
\end{aligned}
$$

We observe that if R' and p are sufficiently large and $|x|\ge R'$ then

$$
\frac{|B\cdot M_\lambda x|}{w} + \frac{|B\cdot M_0 x|}{w} + \frac{\mathrm{Tr}(M_\lambda)}{w} - \mathrm{Tr}(M_\lambda)\frac{2F_\delta}{2p+1} + \frac{|M_\lambda x|^2}{pw^2} - \epsilon\frac{|M_\lambda x|^2}{w} \le 0.
$$

Therefore

$$
I_2 \le C\big(1 + \|\frac{B\cdot M_\lambda x}{w}\|_{L^\infty} + \|\frac{B\cdot M_0 x}{w}\|_{L^\infty}\big)\|F_\delta\chi_{\{|x|\le R'\}}\|^{2p}_{L^{2p}}.
$$

So the claim holds by taking

$$
C' = C\big(1 + \|\frac{B\cdot M_\lambda x}{w}\|_{L^\infty} + \|\frac{B\cdot M_0 x}{w}\|_{L^\infty}\big).
$$

We have from the Sobolev inequality

$$
\|F_\delta\|^{2p}_{L^{6p}} = \|{F_\delta}^p\|^2_{L^6} \le C\|\nabla({F_\delta}^p)\|^2_{L^2} = C\int_{\mathbb{R}^3} p^2 {F_\delta}^{2(p-1)}|\nabla F_\delta|^2\,\mathrm{d}x.
$$

Then by the claim and (3.28) we get

$$
\|F_\delta\|^{2p}_{L^{6p}} \le C\frac{p^2}{2p-1}\big(\|F_\delta\chi_{\{F_\delta>-1-\epsilon\}}\|^{2p-1}_{L^{2p-1}} + \|F_\delta\chi_{\{|x|\le R'\}}\|^{2p}_{L^{2p}}\big).
$$

Hence by letting $p\to\infty$ we have

$$
\|F_\delta\|_{L^\infty} \le \|F_\delta\chi_{\{F_\delta>-1-\epsilon\}}\|_{L^\infty} + \|F_\delta\chi_{\{|x|\le R'\}}\|_{L^\infty} \le 1+\epsilon + \|F_\delta\chi_{\{|x|\le R'\}}\|_{L^\infty}.
$$

Since R' does not depend on δ and K, we have for $|x|\le R'$,

$$
|F_\delta(x)| \le -\frac{1}{K+\frac{1}{2}(x^t M_\lambda x)}\log\big(\frac{\inf_{|x|\le R'}\tilde{f}(x)}{N} + \delta\big) \le -\frac{1}{K}\log\big(\frac{\inf_{|x|\le R'}\tilde{f}(x)}{N}\big) \le \epsilon
$$

if K is sufficiently large but independent of δ. So we have $\|F_\delta\|_{L^\infty} \leq 1+2\epsilon$, that is, $\log(\frac{\tilde{f}(x)}{N}+\delta) \geq -(1+2\epsilon)\big(K+\frac{1}{2}(x^tM_\lambda x)\big)$, which implies

$$\frac{\tilde{f}(x)}{N}+\delta \geq e^{-(1+2\epsilon)K}e^{-\frac{(1+2\epsilon)}{2}x^tM_\lambda x}.$$

Hence the proof is complete by letting $\delta \to 0$ and from the definition of $\tilde{f}(x)$.

Next we show a more precise lower bound of $-f$ under the additional condition on B.

Proposition 3.5 *Let $f \in BC^2(\mathbb{R}^3)$ be a nontrivial solution to (3.21). Assume that $B \in (L^\infty(\mathbb{R}^3))^3$ and*

$$\lim_{R\to\infty} \sup_{|x_1|\leq R_0, |x_2|+|x_3|\geq R} |B_1(x)| = 0 \qquad \text{for all} \quad R_0 > 0. \tag{3.30}$$

Then for all $\theta > \lambda$ and $\epsilon > 0$ there is $C_{\theta,\epsilon} > 0$ such that

$$-f(x) \geq C_{\theta,\epsilon}(1+x_2^2+(1+x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}}e^{-\frac{1+\epsilon}{2}\lambda x_1^2}. \qquad x\in\mathbb{R}^3. \tag{3.31}$$

Proof. For $\epsilon, \epsilon' > 0$ we set

$$W_{\epsilon,\epsilon'}(x) := (1+x_2^2+(1+x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}}e^{\frac{-\epsilon'}{2}(x_2{}^2+\mu x_3{}^2)-\frac{1+\epsilon}{2}\lambda x_1{}^2}, \qquad H_{\epsilon,\epsilon'}(x) := \frac{W_{\epsilon,\epsilon'}(x)}{-f(x)} \geq 0.$$

Note that by Proposition 3.4 the function $H_{\epsilon,\epsilon'}(x)$ rapidly decays at spatial infinity for each $\epsilon, \epsilon' > 0$. The direct calculation shows

$$\begin{aligned}
\nabla H_{\epsilon,\epsilon'} &= -H_{\epsilon,\epsilon'}\frac{\nabla f}{f} - \frac{\nabla W_{\epsilon,\epsilon'}}{f} = -H_{\epsilon,\epsilon'}\frac{\nabla f}{f} + \frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}H_{\epsilon,\epsilon'},\\
\Delta H_{\epsilon,\epsilon'} &= -\frac{\Delta f}{f}H_{\epsilon,\epsilon'} - 2\frac{\nabla f}{f}\cdot\nabla H_{\epsilon,\epsilon'} - \frac{\Delta W_{\epsilon,\epsilon'}}{f}\\
&= -\frac{\Delta f}{f}H_{\epsilon,\epsilon'} + 2(\frac{\nabla H_{\epsilon,\epsilon'}}{H_{\epsilon,\epsilon'}} - \frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}})\cdot\nabla H_{\epsilon,\epsilon'} + \frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}H_{\epsilon,\epsilon'}.
\end{aligned}$$

Thus by (3.21) we have

$$\begin{aligned}
-\Delta H_{\epsilon,\epsilon'} &\leq (B+Mx)\cdot\frac{\nabla f}{f}H_{\epsilon,\epsilon'} - 2(\frac{\nabla H_{\epsilon,\epsilon'}}{H_{\epsilon,\epsilon'}} - \frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}})\cdot\nabla H_{\epsilon,\epsilon'} - \frac{\Delta W_{\epsilon,\epsilon'}}{W}H_{\epsilon,\epsilon'}\\
&\leq \big(-B-Mx+\frac{2\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\big)\cdot\nabla H_{\epsilon,\epsilon'} - \big((-B-Mx)\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} + \frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\big)H_{\epsilon,\epsilon'}.
\end{aligned}$$

Then the integration by parts yields

$$\begin{aligned}
&(2p-1)\int_{\mathbb{R}^3}|\nabla H_{\epsilon,\epsilon'}|^2H_{\epsilon,\epsilon'}^{2(p-1)}\,\mathrm{d}x\\
&\leq -\frac{1}{2p}\int_{\mathbb{R}^3}\nabla\cdot\big(-B-Mx+2\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\big)H_{\epsilon,\epsilon'}^{2p}\,\mathrm{d}x\\
&\qquad\qquad - \int_{\mathbb{R}^3}\big((-B-Mx)\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} + \frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\big)H_{\epsilon,\epsilon'}^{2p}\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^3}\{-\frac{\mathrm{Tr}(M)}{2p} + \frac{1}{p}\nabla\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} + \frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} + (-B-Mx)\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\}H_{\epsilon,\epsilon'}^{2p}dx
\end{aligned} \tag{3.32}$$

We observe that

$$\frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} - Mx\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} = (\theta+2\epsilon'\theta)\frac{{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}-1}{x_3}^2}{1+{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}}} - \lambda + \lambda^2\epsilon(1+\epsilon){x_1}^2 + \epsilon' x^t M_0 x$$
$$+ (\epsilon')^2 x^t M_0 x - \epsilon\lambda - \epsilon'\mathrm{Tr}(M_0) + O(\frac{1}{1+x_2^2+x_3^2}), \qquad (3.33)$$

$$-B\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} = \lambda(1+\epsilon)B_1x_1 + \epsilon' B\cdot M_0 x + \frac{\theta}{\mu}\frac{\mu B_2 x_2 + B_3 x_3(1+{x_3}^2)^{\frac{1}{\mu}-1}}{1+{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}}}, \qquad (3.34)$$

and

$$\nabla\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} = -(\lambda+\epsilon\lambda+\epsilon'+\epsilon'\mu) + O(\frac{1}{1+x_2^2+x_3^2}). \qquad (3.35)$$

From the assumption on B and the condition $\theta > \lambda$, if ϵ and ϵ' are small enough and p is sufficiently large then there exists $R > 0$ independent of ϵ' (but depending on ϵ) such that the integrand of the right hand side of (3.32) is nonnegative when $|x| \geq R$. Indeed, it suffices to consider each case of (i) $|x_1| \geq R/2$ and (ii) $|x_1| \leq R/2$ and $(x_2^2+x_3^2)^{1/2} \geq R/2$; when $|x_1| \geq R/2$ the term $\lambda^2\epsilon(1+\epsilon){x_1}^2 + \epsilon' x^t M_0 x$ is dominant, and when $|x_1| \leq R/2$ and $(x_2^2+x_3^2)^{1/2} \geq R/2$ the term

$$(\theta+2\epsilon'\theta)\frac{{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}-1}{x_3}^2}{1+{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}}} + \epsilon' x^t M_0 x$$

becomes dominant by the assumptions. Therefore we have

$$(2p-1)\int_{\mathbb{R}^3} |\nabla H_{\epsilon,\epsilon'}|^2 H_{\epsilon,\epsilon'}^{2(p-1)}\,\mathrm{d}x \leq C\|H_{\epsilon,\epsilon'}\chi_{\{|x|\leq R\}}\|_{L^{2p}}^{2p},$$

and then $\|H_{\epsilon,\epsilon'}\|_{L^{6p}}^{2p} \leq Cp^2(2p-1)^{-1}\|H_{\epsilon,\epsilon'}\chi_{\{|x|\leq R\}}\|_{L^{2p}}^{2p}$. By taking $p\to\infty$, we have $\|H_{\epsilon,\epsilon'}\|_{L^\infty} \leq \|H_{\epsilon,\epsilon'}\chi_{\{|x|\leq R\}}\|_{L^\infty}$ for all small $\epsilon' > 0$, and thus $\|H_{\epsilon,0}\|_{L^\infty} \leq \|H_{\epsilon,0}\chi_{\{|x|\leq R\}}\|_{L^\infty}$. Since $\inf_{|x|\leq R}(-f(x)) \neq 0$ for each $R > 0$, we have

$$0 < H_{\epsilon,0}(x) = \frac{(1+x_2^2+(1+x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} e^{-\frac{1+\epsilon}{2}\lambda {x_1}^2}}{-f(x)} \leq C_{\theta,\epsilon} \qquad \text{if} \quad |x| \leq R.$$

So we conclude that $|H_{\epsilon,0}(x)| \leq \|H_{\epsilon,0}\chi_{\{|x|\leq R\}}\|_{L^\infty} \leq C_{\theta,\epsilon}$, which gives

$$-f(x) \geq C_{\theta,\epsilon}(1+x_2^2+(1+x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} e^{-\frac{1+\epsilon}{2}\lambda {x_1}^2}.$$

This completes the proof of Proposition 3.5.

Remark 3.6 The function $f(x) = -(1+(x_2^2+x_3^2)/2)^{-1}e^{-x_1^2}$ satisfies (3.21) with $B = 0$, $\lambda = 2$, and $\mu = 1$. Hence (3.31) is considered to be rather optimal under the conditions in Proposition 3.5.

Corollary 3.7 *Assume that* **(C0)**-**(C2)** *hold and that* $\Pi_0 \not\equiv 0$. *Then for all* $\theta > \lambda$ *and* $\epsilon > 0$ *there is* $C_{\theta,\epsilon} > 0$ *such that*

$$-\Pi_0(x) \geq C_{\theta,\epsilon}(1 + x_2^2 + (1 + x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} e^{-\frac{1+\epsilon}{2}\lambda x_1^2}. \qquad x \in \mathbb{R}^3. \tag{3.36}$$

Proof. From (2.16) and Proposition 2.4 it suffices to show $U_1 = V_1$; then the assumptions in Proposition 3.5 are satisfied. Assume that (i) of **(C1)** holds. Then by the relation $\Pi(x) = |U(x)|^2/2 + P(x) + Mx \cdot (u_c + V(x))$ we must have $u_c = (0, u_{c,2}, u_{c,3})$ since $\Pi(x) = a_0 + \Pi_0(x)$ is bounded function. Thus $U_1 = V_1$ follows. When (ii) of **(C1)** holds $u_c = (0, u_{c,2}, u_{c,3})$ is trivial due to Proposition 2.4. This completes the proof.

3.3 Lower bound of $-\Pi_0$ in (x_2, x_3) direction

Proposition 3.8 *Assume that* **(C0)**-**(C3)** *hold and that* $\Pi_0 \not\equiv 0$. *Then for any* $l > 0$ *there is* $C > 0$ *such that*

$$-\Pi_0(0, x_2, 0) \geq C(1 + |x_2|)^{-l} \qquad \text{if (i) of } (\mathbf{C3}) \text{ holds,} \tag{3.37}$$

$$-\Pi_0(0, 0, x_3) \geq C(1 + |x_3|)^{-l} \qquad \text{if (ii) of } (\mathbf{C3}) \text{ holds.} \tag{3.38}$$

Proof. We give the proof only for the case when (i) of **(C3)** holds, since the other case is proved in the same way. Set $g(x_2) = -\Pi_0(0, x_2, 0) > 0$. From (2.3), g satisfies

$$\partial_2^2 g - x_2\partial_2 g = (\partial_1^2\Pi_0)(0, x_2, 0) + (\partial_3^2\Pi_0)(0, x_2, 0) - U(0, x_2, 0)\cdot(\nabla\Pi_0)(0, x_2, 0) - |\Omega(0, x_2, 0)|^2,$$

and hence, by Proposition 3.3 and **(C0)**,

$$\partial_2^2 g - x_2\partial_2 g \leq C(1 + |x_2|)^{-\theta_0}. \tag{3.39}$$

Now we use the same argument as in Proposition 3.5 to establish the lower bound of g. Set

$$h_{l,\epsilon}(x_2) = \frac{w_{l,\epsilon}(x_2)}{g(x_2)}, \qquad w_{l,\epsilon}(x_2) = (1 + x_2^2)^{-l} e^{-\epsilon x_2^2}, \qquad l, \epsilon > 0. \tag{3.40}$$

Then $h \in W^{2,p}(\mathbb{R}^3)$ for all $p \gg 1$, and we have the inequality

$$\begin{aligned}
&(2p-1)\int_{\mathbb{R}} |\partial_2 h_{l,\epsilon}(x_2)|^2 |h_{l,\epsilon}(x_2)|^{2(p-1)}\,\mathrm{d}x_2 \\
&\leq \frac{1}{2p}\int_{\mathbb{R}} \big(1 - 2\partial_2(\frac{\partial_2 w_{l,\epsilon}}{w_{l,\epsilon}})\big)|h_{l,\epsilon}(x_2)|^{2p}\,\mathrm{d}x_2 \\
&\qquad - \int_{\mathbb{R}} \big(-\frac{x_2\partial_2 w_{l,\epsilon}}{w_{l,\epsilon}} + \frac{\partial_2^2 w_{l,\epsilon}}{w_{l,\epsilon}} - C\frac{(1+|x_2|)^{-\theta_0}}{g}\big)|h_{l,\epsilon}(x_2)|^{2p}\,\mathrm{d}x_2.
\end{aligned} \tag{3.41}$$

Since $l > 0$, $\theta_0 > \lambda$, and $g(x_2) \geq C(1 + |x_2|)^{-\theta}$ for all $\theta > \lambda$ by Corollary 3.7, there is $R \geq 1$ independent of $\epsilon > 0$ such that

$$(2p-1)\int_{\mathbb{R}} |\partial_2 h_{l,\epsilon}(x_2)|^2 |h_{l,\epsilon}(x_2)|^{2(p-1)}\,\mathrm{d}x_2 \leq C\|h_{l,\epsilon}\|_{L^{2p}(B_R)}^{2p}.$$

Then the Gagliardo-Nirenberg inequality yields

$$\|h^p_{l,\epsilon}\|_{L^\infty} \le C\|h^p_{l,\epsilon}\|^{\frac12}_{L^2}\|\partial_2(h^p)\|^{\frac12}_{L^2} \le Cp^{\frac14}\|h^p\|^{\frac12}_{L^2}\|h\|^{\frac{p}{2}}_{L^{2p}(B_R)},$$

that is, $\|h_{l,\epsilon}\|_{L^\infty} \le (Cp)^{1/(4p)}\|h_{l,\epsilon}\|^{\frac12}_{L^{2p}}\|h_{l,\epsilon}\|^{\frac12}_{L^{2p}(B_R)}$. Tending $p\to\infty$, we get $\|h_{l,\epsilon}\|_{L^\infty} \le \|h_{l,\epsilon}\|_{L^\infty(B_R)} < \infty$. Since R is independent of $\epsilon > 0$, we have $g(x_2) \ge C(1+|x_2|)^{-l}$ for all $l > 0$. This completes the proof.

4 Proof of Theorem 1.1

Proof of Theorem 1.1. If $\Pi_0 \not\equiv 0$ then the lower bound for Π_0 in Proposition 3.8 contradicts with the decay estimate of Π_0 in (3.7) or (3.8). Hence $\Pi_0 \equiv 0$, i.e., $\Pi \equiv$ const. Thus we have $\Omega \equiv 0$ from (2.3), which implies $U = u_c =$ const.

5 Appendix

Proof of Lemma 2.3. We first give the proof for $k = 0$. For simplicity of notations we set

$$h(t,x) = e^{-\frac12\left(\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}x_1^2+\frac{1}{e^{2t}-1}x_2^2+\frac{\mu}{e^{2\mu t}-1}x_3^2\right)},\quad G(t) = (2\pi)^{-\frac32}(\det Q_t)^{-\frac12}e^{-t\mathrm{Tr}(M)},\quad F(t,x) = f(e^{-tM}x).$$

Then we have

$$b(x)(e^{t\mathcal{L}}f)(x) = G(t)b(x)\int_{\mathbb{R}^3} h(t,y)F(t,x-y)\,\mathrm{d}y = G(t)\int_{\mathbb{R}^3} b(x)h(t,x-y)F(t,y)\,\mathrm{d}y,$$

and by the definition of $b(x)$ we obtain

$$|b(x)(e^{t\mathcal{L}}f)(x)| \le CG(t)\big(\int_{\mathbb{R}^3} b(x-y)h(t,x-y)|F(t,y)|\,\mathrm{d}y + \int_{\mathbb{R}^3} h(t,x-y)b(y)|F(t,y)|\,\mathrm{d}y\big). \tag{5.1}$$

For $1 \le q \le p \le \infty$ and $1 \le r < \infty$ satisfying $1/p = 1/r + 1/q - 1$ we get by the Young inequality

$$\|be^{t\mathcal{L}}f\|_{L^p} \le CG(t)\big(\|bh(t)\|_{L^r}\|F(t)\|_{L^q} + \|h(t)\|_{L^r}\|bF(t)\|_{L^q}\big). \tag{5.2}$$

We observe that

$$\|F(t)\|^q_{L^q} = e^{t\mathrm{Tr}(M)}\int_{\mathbb{R}^3}|f(z)|^q\,\mathrm{d}z \le e^{t\mathrm{Tr}(M)}\int_{\mathbb{R}^3}|b(z)|^q|f(z)|^q\,\mathrm{d}z = e^{t\mathrm{Tr}(M)}\|bf\|^q_{L^q},$$

and

$$\|bF(t)\|^q_{L^q} = \int_{\mathbb{R}^3}|b(y)|^q|f(e^{-tM}y)|^q\,\mathrm{d}y \le Ce^{ct}e^{t\mathrm{Tr}(M)}\int_{\mathbb{R}^3}|b(z)|^q|f(z)|^q\,\mathrm{d}z \le Ce^{ct}\|bf\|^q_{L^q},$$

where C and c depend on θ_i and λ_i. So we have

$$\|be^{t\mathcal{L}}f\|_{L^p} \le C(\det Q_t)^{-\frac12}e^{ct}\|bf\|_{L^q}\big(\|bh(t)\|_{L^r} + \|h(t)\|_{L^r}\big). \tag{5.3}$$

The direct calculation implies

$$\|h(t)\|_{L^r} = \Big(\int_{\mathbb{R}^3} e^{-\frac{r}{2}\big\{\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}y_1^2+\frac{1}{e^{2t}-1}y_2^2+\frac{\mu}{e^{2\mu t}-1}y_3^2\big\}}\,\mathrm{d}y\Big)^{\frac{1}{r}} = \Big(\int_{\mathbb{R}^3} e^{-z^2}\,\mathrm{d}z\Big)^{\frac{1}{r}} G_r(t) \le C G_r(t),$$

where

$$G_r(t) = \big(\frac{2}{r}\big)^{\frac{3}{2r}}\big(\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}\big)^{\frac{-1}{2r}}\big(\frac{1}{e^{2t}-1}\big)^{\frac{-1}{2r}}\big(\frac{\mu}{e^{2\mu t}-1}\big)^{\frac{-1}{2r}}.$$

Next we compute

$$\begin{aligned}
&\|bh(t)\|_{L^r}\\
&= \Big(\int_{\mathbb{R}^3} |b(y)|^r e^{-\frac{r}{2}\big(\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}y_1^2+\frac{1}{e^{2t}-1}y_2^2+\frac{\mu}{e^{2\mu t}-1}y_3^2\big)}\,\mathrm{d}y\Big)^{\frac{1}{r}}\\
&\le C\bigg(\int_{\mathbb{R}^3}\big(1+\frac{2}{r}\frac{e^{2\lambda t}-1}{\lambda e^{2\lambda t}}{z_1}^2\big)^{\theta_1 r}+\big(1+\frac{2}{r}(e^{2t}-1){z_2}^2\big)^{\theta_2 r}+\big(1+\frac{2}{r}\frac{e^{2\mu t}-1}{\mu}{z_3}^2\big)^{\theta_3 r}\,\mathrm{d}y\bigg)^{\frac{1}{r}} G_r(t).
\end{aligned}$$

Since $\displaystyle\int_{\mathbb{R}} |z_j|^{2\theta_j r} e^{-{z_j}^2}\,\mathrm{d}z_j < C$ for $1 \le r < \infty$ we have

$$\|bh(t)\|_{L^r} \le C\big(1+\big(\frac{2}{r}\frac{e^{2\lambda t}-1}{\lambda e^{2\lambda t}}\big)^{\theta_1}+\big(\frac{2}{r}(e^{2t}-1)\big)^{\theta_2}+\big(\frac{2}{r}\frac{e^{2\mu t}-1}{\mu}\big)^{\theta_3}\big)G_r(t).$$

Then by combining the estimates of $\|h(t)\|_{L^r}$ and $\|bh(t)\|_{L^r}$ with (5.3) we obtain

$$\|be^{t\mathcal{L}}f\|_{L^p} \le C(\det Q_t)^{-\frac{1}{2}}e^{ct}\|bf\|_{L^q}G_r(t)\big(1+\big(\frac{2}{r}\frac{e^{2\lambda t}-1}{\lambda e^{2\lambda t}}\big)^{\theta_1}+\big(\frac{2}{r}(e^{2t}-1)\big)^{\theta_2}+\big(\frac{2}{r}\frac{e^{2\mu t}-1}{\mu}\big)^{\theta_3}\big).$$

Observing that

$$(\det Q_t)^{-\frac{1}{2}}G_r(t) \le Ce^{(\frac{1+\mu}{r}-\lambda)t}\Big\{\frac{1}{(1-e^{-2t\lambda})(1-e^{-2t})(1-e^{-2t\mu})}\Big\}^{\frac{1}{2}(1-\frac{1}{r})},$$

we finally obtain

$$\|be^{t\mathcal{L}}f\|_{L^p} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}e^{ct}\|bf\|_{L^q},$$

where the constants C and c depend only on θ_i, λ_i, p, and q. As for the case $r=\infty$, the only possibility is $p=\infty$ and $q=1$. Then the similar argument shows

$$\|be^{t\mathcal{L}}f\|_{L^\infty} \le C(\det Q_t)^{-\frac{1}{2}}e^{ct}\|bf\|_{L^1}\big(\|bh(t)\|_{L^\infty}+\|h(t)\|_{L^\infty}\big).$$

Since h and bh are bounded functions in time and space we complete the proof for $k=0$. For $k=1$ it will be sufficient to show that

$$\|b\partial_1 e^{t\mathcal{L}}f\|_{L^p} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}e^{ct}\|bf\|_{L^q}.$$

But as in the case of $k=0$ it is not difficult to derive the inequality

$$\begin{aligned}
\|b\partial_1 e^{t\mathcal{L}}f\|_{L^p} &\le Ce^{ct}\|bf\|_{L^q}\Big\{\frac{1}{(1-e^{-2t\lambda})(1-e^{-2t})(1-e^{-2t\mu})}\Big\}^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\big(\frac{1}{1-e^{-2t\lambda}}\big)^{\frac{1}{2}}\\
&\le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}e^{ct}\|bf\|_{L^q}.
\end{aligned}$$

The estimates (2.12) for higher order derivatives are proved in the same manner. This completes the proof.

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