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ON RECONSTRUCTION OF LAMÉ COEFFICIENTS FROM PARTIAL CAUCHY DATA

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ABSTRACT. For the isotropic Lamé system we prove that if the Lamé coefficient μ is a positive constant both Lamé coefficients can be recovered from the partial Cauchy data.

In a bounded domain $\Omega \subset \mathbf{R}^2$ with smooth boundary we consider the isotropic Lamé system:

$$\sum_{j,k,l=1}^{2} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right) = 0 \quad \text{in } \Omega,\ 1 \le i \le 2 \tag{0.1}$$

$$u|_{\Gamma_0} = 0, \quad u|_{\tilde{\Gamma}} = f, \tag{0.2}$$

where $\widetilde{\Gamma}$ is an arbitrary fixed subdomain of $\partial\Omega$, $\Gamma_0 = \partial\Omega \setminus \widetilde{\Gamma}$,

$$C_{ijkl} = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad 1 \le i,j,k,l \le 2$$

with the Kronecker delta δ_{ij}. The smooth functions λ and μ are called the Lamé coefficients, $u(x) = (u_1(x), u_2(x))$ is the displacement. Assume that

$$\mu(x) > 0 \quad \text{on } \overline{\Omega}, \quad (\lambda+\mu)(x) > 0 \quad \text{on } \overline{\Omega}. \tag{0.3}$$

We set

$$\Lambda_{\lambda,\mu}(f) = \left(\sum_{j,k,l=1}^{2} \nu_j C_{1jkl}\frac{\partial u_k}{\partial x_l}, \sum_{j,k,l=1}^{2} \nu_j C_{2jkl}\frac{\partial u_k}{\partial x_l}\right),$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal vector to $\partial\Omega$. Denote

$$\mathcal{L}_{\lambda,\mu}u = \left(\sum_{j,k,l=1}^{2} \frac{\partial}{\partial x_j}\left(C_{1jkl}\frac{\partial u_k}{\partial x_l}\right), \sum_{j,k,l=1}^{2} \frac{\partial}{\partial x_j}\left(C_{2jkl}\frac{\partial u_k}{\partial x_l}\right)\right).$$

The partial Cauchy data $\mathcal{C}_{\lambda,\mu}$ is defined by

$$\mathcal{C}_{\lambda,\mu} = \{(u, \Lambda_{\lambda,\mu}(f))|_{\tilde{\Gamma}};\ \mathcal{L}_{\lambda,\mu}u = 0 \quad \text{in } \Omega, u|_{\partial\Omega} = f,\ \operatorname{supp} f \subset \widetilde{\Gamma}\}.$$

In this paper, we consider the following inverse problem:*Suppose that the partial Cauchy data $\mathcal{C}_{\lambda,\mu}$ are given. Can we determine the Lamé coefficients λ and μ?*

This inverse problem has been studied since early 90's. In two dimensions Akamatsu, Nakamura and Steinberg [1] proved that for the case of full Cauchy data ($\widetilde{\Gamma} = \partial\Omega$) one can recover the Lamé coefficients and its normal derivatives of an arbitrary order on the

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boundary provided that Lamé coefficients are C^∞ functions. Later in [12] Nakamura and Uhlmann for the case of full Cauchy data established that the Lamé coefficients are uniquely determined, assuming that they are sufficiently close to a pair of positive constants. For the three dimensional case in [13], [14] these authors and independently in [6] Eskin and Ralston proved the uniqueness for both Lamé coefficients provided that μ is close to a positive constant. The proofs in the above papers rely on construction of complex geometric optics solutions. On the other hand, unlike the case of Schrödinger operator, for partial Cauchy data, the construction of such solutions for the isotropic Lamé system seems to be possible only for the dense set of Lamé coefficients. To our best knowledge there are no results on the unique recovery of the Lamé coefficients from the partial Cauchy data. Also we mention that a linearized version of this inverse problem was studied in [7].

Finally we mention that this inverse problem is closely related to the method known as Electrical Impedance Tomography (EIT). EIT method is widely used for detecting oil field and minerals beneath earth's surface, diagnosis of the breast cancer. For the mathematical treatment of this problem we refer to [2], [4], [5], [9], [10], [11], [15].

We state our main result as follows.

Theorem 0.1. *Let Ω be a simply connected domain with smooth boundary, (0.3) hold true, μ_1, μ_2 be some positive constants and $\lambda_1, \lambda_2 \in C^3(\overline{\Omega})$. If $\mathcal{C}_{\lambda_1,\mu_1} = \mathcal{C}_{\lambda_2,\mu_2}$ then $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.*

Throughout the paper we use following notations: $i = \sqrt{-1}$, $x_1, x_2 \in \mathbf{R}$, $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbf{C}$. We identify $x = (x_1, x_2) \in \mathbf{R}^2$ with $z = x_1 + ix_2 \in \mathbf{C}$. We set $\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$, $\partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$. We say that a function $a(x)$ is antiholomorphic in Ω if $\partial_z a(x)|_\Omega \equiv 0$. By ∂_z^{-1} we denote the operator

$$\partial_z^{-1} g = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1, \xi_2)}{\overline{\zeta} - \overline{z}} d\xi_2 d\xi_1. \tag{0.4}$$

It is known (see e.g [17]) that the operator ∂_z^{-1} is continuous from the space $C^{m+\alpha}(\overline{\Omega})$ into $C^{m+1+\alpha}(\overline{\Omega})$ for any integer non-negative m and positive α from interval $(0, 1)$.

For the proof, we need the following proposition:

Proposition 0.1. *If $\mathcal{C}_{\lambda_1,\mu_1} = \mathcal{C}_{\lambda_2,\mu_2}$, then*

$$(\mu_1 - \mu_2)|_{\tilde{\Gamma}} = (\lambda_1 - \lambda_2)|_{\tilde{\Gamma}} = 0. \tag{0.5}$$

Proof of Proposition 0.1. The proof of the proposition follows from [1]. The only difference is that we are using the calculus of pseudodifferential operators with symbols of limiting smoothness. Since the Lamé system is translationally invariant, it suffices to prove the statement of the proposition at point $x = 0$. Taking into account that the isotropic Lamé system is rotationally invariant, without loss of generality we may assume that $\nu(0) = (0, 1)$. Therefore locally near zero we may assume that the boundary of Ω is given by equation $x_2 - \ell(x_1) = 0$ and $x \in \Omega$ implies $x_2 - \ell(x_1) > 0$. Moreover $\ell'(0) = 0$. After change of variables $y_1 = x_1$ and $y_2 = x_2 - \ell(x_1)$ the domain Ω near 0 is transformed into some open set $\mathcal{G}$ in $\mathbb{R} \times (0, 1)$. Consider the Lamé system in the new coordinates. Let e be a function with compact support concentrated in a ball of a small radius centered at zero and $e \equiv 1$ in a small neighborhood of zero. We set $U = (U_1, U_2)$ where

$$U_1(y) = \int_{\mathbb{R}^1} (1 + |\xi_1|^2)^{\frac{1}{2}} (\hat{eu})(\xi_1, y_2) d\xi_1 = \Lambda(D)(eu)$$

and $U_2(y) = D_{y_2}(eu), D_{y_2} = \frac{1}{i}\frac{\partial}{\partial y_2}$. Here we used the notation

$$\hat{u}(\xi_1, y_2) = \int_{\mathbb{R}^1} u(y) e^{-iy_1\xi_1} dy_1.$$

In the new notations problem (0.1), (0.2) can be written in the form

$$D_{y_2} U = M(y, D_{y_1})U + F, \quad U_1|_{y_2=0} = \Lambda(D)(ef).$$

The function F satisfies the estimate

$$\|F\|_{L^2(\mathbb{R}^2)} \leq C\|f\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{0.6}$$

Here $M(y, D_{y_1})$ is a 2×2 pseudodifferential operator with the principal symbol $M_1(y, \xi_1)$ given by

$$M_1(y, \xi_1) = \begin{pmatrix} 0 & \Lambda_1 E \\ A^{-1} M_{21} \Lambda_1^{-1} & A^{-1} M_{22} \end{pmatrix},$$

where

$$M_{21}(y, \xi_1) = -\mu\xi_1^2 E - (\lambda + \mu)\widetilde{\xi}^T \widetilde{\xi}, \quad M_{22}(y, \xi_1) = -(\lambda + \mu)(\widetilde{\xi}^T G + G^T \widetilde{\xi}) - 2\mu(\widetilde{\xi}, G)E,$$

$$A(y, \xi_1) = (\lambda + \mu)G^T G + \mu|G|^2 E, \quad \Lambda_1 = |\xi_1|, \quad \widetilde{\xi} = (\xi_1, 0), \; G = (-\ell', 1)$$

and E is the unit matrix. It is well known that all the eigenvalues $\alpha(y, \xi_1)$ of the matrix $M_1(y, \xi_1)$ satisfy the equation $(\widetilde{\xi} + \widetilde{G}\alpha, \widetilde{\xi} + \widetilde{G}\alpha) = 0$. Hence we have two eigenvalues which depend smoothly on y and ξ_1

$$\alpha_\pm(y, \xi_1) = -\frac{(\widetilde{\xi}, G)}{|G|^2} \pm \left(-\frac{\xi_1^2}{|G|^2} + \frac{(\widetilde{\xi}, G)^2}{|G|^4} \right)^{\frac{1}{2}}.$$

The corresponding eigenvectors are

$$w_1^\pm(y, \xi_1) = \left(\frac{\widetilde{\xi} + \alpha_\pm G}{|\widetilde{\xi} + \alpha_\pm G|}, \frac{\alpha_\pm}{|\xi_1|} \frac{\widetilde{\xi} + \alpha_\pm G}{|\widetilde{\xi} + \alpha_\pm G|} \right).$$

The Jordan form of the matrix M_1 has two Jordan blocks of the size 2×2:

$$(M_1 - \alpha_\pm)\eta^\pm = |\xi_1| w_1^\pm,$$

where $\eta^\pm(y, \xi_1) = (\eta_1^\pm(y, \xi_1), \eta_2^\pm(y, \xi_1))$,

$$\eta_1^\pm(y, \xi_1) = -\frac{\lambda + 3\mu}{\lambda + \mu} \frac{|\xi_1|}{|\widetilde{\xi} + \alpha_\pm G|} G, \quad \eta_2^\pm(y, \xi_1) = \frac{1}{|\xi_1|} \left(-\alpha_\pm \frac{\lambda + 3\mu}{\lambda + \mu} \frac{|\xi_1|}{|\widetilde{\xi} + \alpha_\pm G|} G + |\xi_1| \frac{\widetilde{\xi} + \alpha_\pm G}{|\widetilde{\xi} + \alpha_\pm G|} \right).$$

Observe that $\alpha_\pm(0,\xi_1)=\pm i|\xi_1|$, $w_1^-(0,\xi_1)=\frac{1}{\sqrt{2}|\xi_1|}(\xi_1,-i|\xi_1|,-i\xi_1,\xi_1)$, $\eta^-=(0,-\frac{\lambda(0)+3\mu(0)}{\sqrt{2}(\lambda(0)+\mu(0))}$, $\frac{\xi_1}{\sqrt{2}|\xi_1|},\frac{i}{\sqrt{2}}\frac{\mu(0)}{\lambda(0)+\mu(0)})$. From (0.6) and the standard a priori estimates for the systems of elliptic equations see (e.g., [8]) one can show that

$$\|B_j(\lambda,\mu,y_1,D_{y_1})U_{\lambda,\mu}\|_{L^2(\mathbb{R}^1)}\le C\|f\|_{H^{\frac12}(\partial\Omega)},\quad \forall j\in\{1,2\},$$

where $B_1(\lambda,\mu,y_1,D_{y_1}),B_2(\lambda,\mu,y_1,D_{y_1})$ are pseudodifferential operators of the class C^3S^0 (for definition see e.g. [16]) and the principal symbols of these operators satisfy

$$B_1(\lambda,\mu,0,\xi_1)=w^-(0,\xi_1),\quad B_2(\lambda,\mu,0,\xi_1)=\eta^-(0,\xi_1).$$

Consider a matrix pseudodifferential operator

$$\mathcal{B}(\lambda,\mu,y_1,D_{y_1})=\begin{pmatrix}B_{1,3}(\lambda,\mu,y_1,D_{y_1}) & B_{1,4}(\lambda,\mu,y_1,D_{y_1})\\ B_{2,3}(\lambda,\mu,y_1,D_{y_1}) & B_{2,4}(\lambda,\mu,y_1,D_{y_1})\end{pmatrix}.$$

The corresponding principal symbol of this operator is invertible matrix at point $(0,\xi_1),\xi_1\neq 0$

$$\mathcal{B}_0(\lambda,\mu,0,\xi_1)=\begin{pmatrix}\frac{-i\xi_1}{\sqrt{2}|\xi_1|} & \frac{\xi_1}{\sqrt{2}|\xi_1|}\\ \frac{\xi_1}{\sqrt{2}|\xi_1|} & \frac{i}{\sqrt{2}}\frac{\mu(0)}{\lambda(0)+\mu(0)}\end{pmatrix}.$$

Then there exists a parametrix to the operator $\mathcal{B}$ (see e.g. [16]) which is a pseudodifferential operator with symbol C^3S^0. We denote this pseudodifferential operator as $\mathcal{B}^{-1}(\lambda,\mu,y_1,D_{y_1})$ with the principal symbol satisfying

$$\mathcal{B}_0^{-1}(\lambda,\mu,0,\xi_1)=\frac{1}{\det\mathcal{B}_0(\lambda,\mu,0,\xi_1)}\begin{pmatrix}\frac{i}{\sqrt{2}}\frac{\mu(0)}{\lambda(0)+\mu(0)} & -\frac{\xi_1}{\sqrt{2}|\xi_1|}\\ \frac{-\xi_1}{\sqrt{2}|\xi_1|} & -\frac{i\xi_1}{\sqrt{2}|\xi_1|}\end{pmatrix}.$$

Then

(0.7)
$$\|U_{2,\lambda,\mu}+\mathcal{B}^{-1}(\lambda,\mu,y,D_{y_1})\begin{pmatrix}B_{1,1}(\lambda,\mu,y_1,D_{y_1}) & B_{1,2}(\lambda,\mu,y_1,D_{y_1})\\ B_{2,1}(\lambda,\mu,y_1,D_{y_1}) & B_{2,2}(\lambda,\mu,y_1,D_{y_1})\end{pmatrix}U_{1,\lambda,\mu}\|_{L^2(\mathbb{R}^1)}\le C\|f\|_{H^{\frac12}(\partial\Omega)}.$$

The operator $\Lambda_{\lambda,\mu}(f)$ in the new coordinates can be written as

$$\Lambda_{\lambda,\mu}(f)=A_{\lambda,\mu}(y_1)U_{2,\lambda,\mu}+\mathcal{P}_{\lambda,\mu}(y_1,D_{y_1})f$$

where

$$A_{\lambda,\mu}(0)=\begin{pmatrix}0 & i\lambda(0)\\ i\mu(0) & 0\end{pmatrix},\quad \mathcal{P}_{\lambda,\mu}(0,D_{y_1})=\begin{pmatrix}i(\lambda(0)+\mu(0))\xi_1 & 0\\ 0 & i\mu(0)\xi_1\end{pmatrix}.$$

Since the partial Cauchy data are the same we have

$$A_{\lambda_1,\mu_1}(y_1)U_{2,\lambda_1,\mu_1}-A_{\lambda_2,\mu_2}(y_1)U_{2,\lambda_2,\mu_2}+(\mathcal{P}_{\lambda_1,\mu_1}(y_1,D_{y_1})-\mathcal{P}_{\lambda_2,\mu_2}(y_1,D_{y_1}))f=0.$$

Using (0.7) we obtain

$$\|K(y_1,D_{y_1})f\|_{L^2(\mathbb{R}^1)}\le C\|f\|_{H^{\frac12}(\partial\Omega)}. \tag{0.8}$$

where

$$K(y_1,D_{y_1}) = -A_{\lambda_1,\mu_1}(y_1)\mathcal{B}^{-1}(\lambda_1,\mu_1,y_1,D_{y_1})\begin{pmatrix} B_{1,1}(\lambda_1,\mu_1,y_1,D_{y_1}) & B_{1,2}(\lambda_1,\mu_1,y_1,D_{y_1}) \\ B_{2,1}(\lambda_1,\mu_1,y_1,D_{y_1}) & B_{2,2}(\lambda_1,\mu_1,y_1,D_{y_1}) \end{pmatrix}\Lambda(D)(e\cdot)$$
$$+A_{\lambda_2,\mu_2}(y_1)\mathcal{B}^{-1}(\lambda_2,\mu_2,y_1,D_{y_1})\begin{pmatrix} B_{1,1}(\lambda_2,\mu_2,y_1,D_{y_1}) & B_{1,2}(\lambda_2,\mu_2,y_1,D_{y_1}) \\ B_{2,1}(\lambda_2,\mu_2,y_1,D_{y_1}) & B_{2,2}(\lambda_2,\mu_2,y_1,D_{y_1}) \end{pmatrix}\Lambda(D)(e\cdot)$$
$$+(\mathcal{P}_{\lambda_1,\mu_1}(y_1,D_{y_1}) - \mathcal{P}_{\lambda_2,\mu_2}(y_1,D_{y_1})). \tag{0.9}$$

By (0.9) the operator $K(y_1,D_{y_1})$ belong to the class C^2S^1. On the other hand, by (0.8) the principal symbol $K_1(y,\xi_1)$ of the operator $K(y_1,D_{y_1})$ should be zero.(Otherwise we have the contradiction to the Gardings inequality.) The simple computations provide

$$K(0,1) = \begin{pmatrix} \frac{i\lambda_1(0)(\lambda_1+\mu_1)(0)}{\mu_1(0)} - \frac{i\lambda_2(0)(\lambda_2+\mu_2)(0)}{\mu_2(0)} & \lambda_1(0)-\lambda_2(0) \\ -\mu_1(0)+\mu_2(0) & -i(\lambda_1(0)+\mu_1(0))+i(\lambda_2(0)+\mu_2(0)) \end{pmatrix}.$$

The proof of the proposition is completed. $\square$

Now we proceed to the proof of theorem 0.1.

Proof. Since μ_1 and μ_2 are assumed to be some constants, Proposition 0.1 implies immediately that $\mu_1 = \mu_2$.

Instead of the vector function u, it is more convenient for us to work with the complex valued function $D = u_1 + iu_2$. The isotropic Lamé system for unknown function D, for the case $\mu = const$, can be written as

$$\partial_{\overline{z}}(2(\lambda+\mu)(\partial_z D + \overline{\partial_z D})) + 4\mu\partial_{\overline{z}}\partial_z D = 0 \quad \text{in}\,\Omega. \tag{0.10}$$

This equation can be solved explicitly. Indeed using the fact that μ is a constant and domain Ω is simply connected we have

$$2(\lambda+\mu)(\partial_z D + \overline{\partial_z D}) + 4\mu\partial_z D = \Theta(z), \tag{0.11}$$

where Θ is a holomorphic function in Ω. Then $\mathrm{Re}\,\partial_z D = \frac{1}{4\lambda+8\mu}\frac{\Theta(z)+\overline{\Theta(z)}}{2}$ and $\mathrm{Im}\partial_z D = \frac{1}{4\mu}\frac{\Theta(z)-\overline{\Theta(z)}}{2i}$. Since

$$\partial_z D = \frac{(\lambda+3\mu)}{8\mu(\lambda+2\mu)}\Theta - \frac{(\lambda+\mu)}{8\mu(\lambda+2\mu)}\overline{\Theta}, \tag{0.12}$$

we have

$$D = \Psi(\overline{z}) + \partial_z^{-1}\left\{\frac{\lambda+3\mu}{8\mu(\lambda+2\mu)}\Theta\right\} - \overline{\Theta}\partial_z^{-1}\left\{\frac{\lambda+\mu}{8\mu(\lambda+2\mu)}\right\}, \tag{0.13}$$

where $\Psi(\overline{z})$ is an arbitrary antiholomorphic function.

Using the fact that Lamé system is rotationally invariant from Proposition 0.1 one can immediately obtain that the following Cauchy data are the same

$$C_{\lambda_1,\mu_1} = C_{\lambda_2,\mu_2}, \tag{0.14}$$

where

$$C_{\lambda,\mu} = \left\{\left(u, \frac{\partial u}{\partial \nu}\right)|_{\tilde{\Gamma}};\, \mathcal{L}_{\lambda,\mu}u = 0 \quad \text{in}\ \Omega, u|_{\partial\Omega} = f,\ \mathrm{supp}\, f \subset \widetilde{\Gamma}\right\}.$$

Let a function D_1 be a solution to the Lamé system in Ω:

$$D_1 = \Psi_1(\overline{z}) + \partial_z^{-1}\left\{\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)}\Theta_1\right\} - \overline{\Theta_1}\partial_z^{-1}\left\{\frac{\lambda_1+\mu_1}{8\mu_1(\lambda_1+2\mu_1)}\right\}. \tag{0.15}$$

Let us fix some functions $\Theta_1(z)\in C^2(\overline{\Omega})$ and $\Psi_1(\overline{z})\in C^2(\overline{\Omega})$ such that $D_1|_{\Gamma_0}=0$. Since by (0.14) the partial Cauchy data C_{λ_j,μ_j} are the same, there exist functions $\Theta_2(z)\in C^2(\overline{\Omega})$ and $\Psi_2(\overline{z})\in C^2(\overline{\Omega})$ such that for the function D_2 given by formula

$$D_2 = \Psi_2(\overline{z}) + \partial_z^{-1}\left\{\frac{\lambda_2+3\mu_1}{8\mu_2(\lambda_2+2\mu_1)}\Theta_2\right\} + \overline{\Theta_2}\partial_z^{-1}\left\{\frac{\lambda_2+\mu_1}{8\mu_2(\lambda_2+2\mu_1)}\right\}, \tag{0.16}$$

we have

$$D_1 = D_2 \quad \text{on } \partial\Omega, \qquad \frac{\partial D_2}{\partial\nu} = \frac{\partial D_2}{\partial\nu} \quad \text{on } \widetilde{\Gamma}. \tag{0.17}$$

By (0.11), (0.14), (0.13) and Proposition 0.1, we have $\Theta_1=\Theta_2$ on $\widetilde{\Gamma}$. Since Θ_j are holomorphic we have

$$\Theta_1 = \Theta_2 \quad \text{on} \quad \Omega. \tag{0.18}$$

Now let us fix some smooth holomorphic function $\Theta_1\in C^3(\overline{\Omega})$. Then for any positive ϵ one can choose $\Psi^*_{1,\epsilon}(\overline{z})\in C^3(\overline{\Omega})$ such that

$$\psi_\epsilon = \Psi^*_{1,\epsilon}(\overline{z}) + \partial_z^{-1}\left\{\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)}\Theta_1\right\} + \overline{\Theta_1}\partial_z^{-1}\left\{\frac{\lambda_1+\mu_1}{8\mu_1(\lambda_1+2\mu_1)}\right\},$$

$$\psi_\epsilon \to 0 \quad \text{in } C^3(\Gamma_0) \quad \text{as } \epsilon\to 0. \tag{0.19}$$

Now let us construct the function R_ϵ as the solution to the boundary value problem

$$\mathcal{L}_{\lambda_1,\mu_1}R_\epsilon = 0 \quad \text{in } \Omega, \qquad R_\epsilon|_{\partial\Omega} = -\Upsilon\psi_\epsilon,$$

where Υ is some extension operator continuous from $C^3(\Gamma_0)$ to $C^3(\partial\Omega)$. By (0.19)

$$\|R_\epsilon\|_{C^2(\overline{\Omega})} \to 0 \quad \text{as } \epsilon\to +0. \tag{0.20}$$

Thus we have the sequence $D_{1,\epsilon}=\psi_\epsilon+R_\epsilon$ such that

$$D_{1,\epsilon} = \Psi_{1,\epsilon}(\overline{z}) + \partial_z^{-1}\left\{\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)}\Theta_{1,\epsilon}\right\} + \overline{\Theta_{1,\epsilon}}\partial_z^{-1}\left\{\frac{\lambda_1+\mu_1}{8\mu_1(\lambda_1+2\mu_1)}\right\} \text{ in } \Omega, \quad D_{1,\epsilon}|_{\Gamma_0}=0.$$

We claim that

$$\Theta_{1\epsilon} \to \Theta_1 \text{ in } C^2(\overline{\Omega}) \quad \text{as } \epsilon\to 0. \tag{0.21}$$

Indeed

$$2(\lambda_1+\mu_1)(\partial_z R_\epsilon + \overline{\partial_z R_\epsilon}) + 4\mu_1\partial_z R_\epsilon = \Theta_{1\epsilon}-\Theta_1.$$

Hence by (0.20) we obtain (0.21). Since the partial Cauchy data are the same, by (0.18) there exists a sequence $D_{2,\epsilon}$ such that

$$D_{2,\epsilon} = \Psi_{2,\epsilon}(\overline{z}) + \partial_z^{-1}\left\{\frac{\lambda_2+3\mu_1}{8\mu_2(\lambda_2+2\mu_1)}\Theta_{1,\epsilon}\right\} + \overline{\Theta_{1,\epsilon}}\partial_z^{-1}\left\{\frac{\lambda_2+\mu_1}{8\mu_2(\lambda_2+2\mu_1)}\right\} \text{ in } \Omega, \quad D_{2,\epsilon}|_{\Gamma_0}=0.$$

We set

$$D_\epsilon = D_{1,\epsilon} - D_{2,\epsilon} = \Psi_{1,\epsilon} - \Psi_{2\epsilon} + \partial_z^{-1}\left\{\left(\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+3\mu_1}{8\mu_2(\lambda_2+2\mu_1)}\right)\Theta_{1,\epsilon}\right\}$$

$$+\overline{\Theta_{1,\epsilon}}\left(\partial_z^{-1}\left\{\frac{\lambda_1+\mu_1}{8\mu_1(\lambda_1+2\mu_1)}\right\} - \partial_z^{-1}\left\{\frac{\lambda_2+\mu_1}{8\mu_2(\lambda_2+2\mu_1)}\right\}\right), \tag{0.22}$$

$$D_\epsilon|_{\partial\Omega} = 0, \quad D_\epsilon|_{\widetilde{\Gamma}} = \frac{\partial D_\epsilon}{\partial \nu}|_{\widetilde{\Gamma}} = 0.$$

Passing to the limit in (0.22) and using the standard a priori estimates for the Lamé system we have

$$D = \Psi + \partial_z^{-1}\left\{(\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+3\mu_1}{8\mu_1(\lambda_2+2\mu_1)})\Theta_1\right\}$$

$$+\overline{\Theta_1}\left(\partial_z^{-1}\left\{\frac{\lambda_1+\mu_1}{8\mu_1(\lambda_1+2\mu_1)}\right\} - \partial_z^{-1}\left\{\frac{\lambda_2+\mu_1}{8\mu_2(\lambda_2+2\mu_1)}\right\}\right) \quad \text{in } \Omega, \tag{0.23}$$

$$D|_{\partial\Omega} = 0, \quad \frac{\partial D}{\partial \nu}|_{\widetilde{\Gamma}} = 0.$$

Next we make a choice of the holomorphic function Θ_1. We set $\Theta_1 = e^{\tau\Phi}$, $\Phi(z) = (z-\widetilde{z})^2$ where τ is a positive parameter, $\widetilde{z} = \widetilde{x}_1 + i\widetilde{x}_2$ and $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$ is an arbitrary point from Ω.

Differentiating equation (0.23) by z we have

(0.24)

$$\partial_z D = \left(\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+3\mu_1}{8\mu_1(\lambda_2+2\mu_1)}\right)e^{\tau\Phi} + e^{\tau\overline{\Phi}}\left(\frac{\lambda_1+\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+\mu_1}{8\mu_1(\lambda_2+2\mu_1)}\right), D|_{\partial\Omega} = 0.$$

Multiplying (0.24) by $e^{-\tau\overline{\Phi}}$ and integrating by parts we obtain

$$\int_\Omega \left\{\left(\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+3\mu_1}{8\mu_1(\lambda_2+2\mu_1)}\right)e^{2\tau i \mathrm{Im}\Phi} + \left(\frac{\lambda_1+\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+\mu_1}{8\mu_1(\lambda_2+2\mu_1)}\right)\right\} dx = 0.$$

Using the stationary phase argument (see e.g. [3]), we write the above formula as

$$\frac{2\pi\left(\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+3\mu_1}{8\mu_1(\lambda_2+2\mu_1)}\right)(\widetilde{x})e^{2\tau i \mathrm{Im}\Phi(\widetilde{x})}}{\tau|\det \mathrm{Im}\,\Phi''(\widetilde{x})|^{\frac{1}{2}}}$$

$$+\int_\Omega \left(\frac{\lambda_1+\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+\mu_1}{8\mu_1(\lambda_2+2\mu_1)}\right) dx + o(\frac{1}{\tau}) = 0 \text{ as } \tau \to +\infty.$$

Hence

$$\left(\frac{\lambda_1+3\mu_1}{8\mu_1(\lambda_1+2\mu_1)} - \frac{\lambda_2+3\mu_1}{8\mu_1(\lambda_2+2\mu_1)}\right)(\widetilde{x}) = 0.$$

Hence, $\lambda_1(\widetilde{x}) = \lambda_2(\widetilde{x})$. The proof of theorem is completed. □

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