UTMS 2011–15 August 23, 2011

Kobayashi hyperbolic imbeddings into toric varieties

by

Yusaku Tiba

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

KOBAYASHI HYPERBOLIC IMBEDDINGS INTO TORIC VARIETIES

YUSAKU TIBA

Abstract. Our main goal of this article is to give a characterization of an algebraic divisor on an algebraic torus whose complement is Kobayashi hyperbolically imbedded into a toric projective variety. As an application of our main theorem, we prove the following: the complement of the union of $n+1$ hyperplanes in the n-dimensional projective space $\mathbb{P}^n(\mathbb{C})$ in general position and a general hypersurface of degree n in $\mathbb{P}^n(\mathbb{C})$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$.

1. Introduction and main result

We fix a free module $N = \mathbb{Z}^r$ of rank r over the ring $\mathbb{Z}$ of rational integers. Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) = \mathbb{Z}^r$ be the dual $\mathbb{Z}$-module of N. Let

$$\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$$

be the canonical $\mathbb{Z}$-bilinear pairing. Let $T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = (\mathbb{C}^*)^r$ be the r-dimensional algebraic torus. Let S be a finite subset of M. Let D be a divisor on T_N which is defined by a Laurent polynomial

$$\sum_{I=(i_1,\dots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r},$$

where $a_I \in \mathbb{C}^*$.

By the main theorem of [6], every entire curve $f : \mathbb{C} \to T_N \setminus \mathrm{supp}\, D$ is *algebraically degenerate*, i.e., the image of f is contained in a proper subvariety of T_N. In this paper, we deal with Kobayashi hyperbolicity of $T_N \setminus \mathrm{supp}\, D$, where $f : \mathbb{C} \to T_N \setminus \mathrm{supp}\, D$ is *most degenerate* to a constant. Moreover, we give a characterization of D such that $T_N \setminus \mathrm{supp}\, D$ is Kobayashi hyperbolically imbedded into a toric variety.

Now, we recall some basic facts about Kobayashi hyperbolic imbedding. The concept of Kobayashi hyperbolic imbedding was introduced in Kobayashi [3] to obtain a generalization of the big Picard theorem. The classical big Picard theorem is stated as follows:

If a function f holomorphic on the punctured disk in $\mathbb{C}$ omits $\{0,1\} \subset \mathbb{C}$, then f can be extended to a meromorphic function on the full disk.

Recall that $\mathbb{C} \setminus \{0,1\}$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^1(\mathbb{C})$. The following generalization of the big Picard theorem obtained in [2]:

Let X be an m-dimensional complex manifold and let A be a closed complex subspace of X consisting of hypersurfaces with normal crossing singularities. Let Z be a complex space and Y be a complex subspace of Z. If Y is Kobayashi hyperbolically imbedded into Z, then every holomorphic map $h : X \setminus A \to Y$ extends to a holomorphic map $\tilde{h} : X \to Z$.

Kobayashi hyperbolic imbedding is also closely related to the structure of a family of holomorphic mappings (see, e.g., [4] Chap. 6, [5]).

It is a famous conjecture proposed by S. Kobayashi that $\mathbb{P}^n(\mathbb{C}) \setminus Y$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$ if Y is a generic hypersurface of degree $d \geq 2n+1$. H. Fujimoto proved in [1] that Kobayashi conjecture is true if Y is a union of hyperplanes in $\mathbb{P}^n(\mathbb{C})$, i.e., $\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^d H_i$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$ if $H_1, \dots, H_d$ are hyperplanes in general position and $d \geq 2n+1$. As a special case of our main theorem, we obtain the following:

Corollary 1. *Let $H_1, \dots, H_{n+1}$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position, and let Y be a general hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$. If $d \geq n$, then*

$$\mathbb{P}^n(\mathbb{C}) \setminus \left(\bigcup_{i=1}^{n+1} H_i \cup Y \right)$$

is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$.

Before stating our main theorem, we give necessary definitions. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let A be a finite subset of M. Define

$$\mathcal{L}_A := \{a - b \in M_{\mathbb{R}} \mid a, b \in A\}.$$

Let V_A be an $\mathbb{R}$-vector subspace of $M_{\mathbb{R}}$ generated by all elements in $\mathcal{L}_A$. Define

$$\mathcal{H}_A := \{H \subset V_A \,|\, \text{hyperplane of } V_A \text{ generated by elements in } \mathcal{L}_A\},$$

where a hyperplane of V_A is an $\mathbb{R}$-vector subspace of codimension one in V_A.

Let P be an integral convex polytope in $M_{\mathbb{R}}$ such that $\dim P = r$. Here the dimenison of a convex polytope P is the dimension of a subspace of $M_{\mathbb{R}}$ which is generated by $\{a - b \,|\, a, b \in P\}$. Then there exists the toric projective variety X associated to P (see [7] Chap. 2), and there exists the imbedding $i : T_N \to X$.

Theorem 1 (Main Theorem)**.** *Let S be a finite subset of M such that $S \subset P$. Assume the following conditions for all positive dimensional faces τ of P:*

(i) *$\tau \cap S \neq \emptyset$, and the dimension of the convex hull of $\tau \cap S$ is equal to the dimension of τ.*

(ii) *Let $H \in \mathcal{H}_{\tau\cap S}$, and let $\phi_H : V_{\tau\cap S} \to V_{\tau\cap S}/H$ be the canonical morphism. Let $x \in \tau \cap S$. Then $\sharp(\phi_H(\tau \cap S - x)) \geq \dim \tau + 1$ for all $H \in \mathcal{H}_{\tau\cap S}$, where $\sharp(\phi_H(\tau \cap S - x))$ is the number of the elements in*

$$\{\phi_H(y - x) \in V_{\tau\cap S}/H \,|\, y \in \tau \cap S\}$$

(note that this condition is independent of a choice of x in $\tau \cap S$).

Then $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded into X for a general divisor D of the linear system $|\{z_1^{i_1} z_2^{i_2} \cdots z_r^{i_r}\}_{(i_1,i_2,\ldots,i_r)\in S}|$ in T_N.

If an algebraic divisor D on T_N is a union of translations of subtori in T_N, it is much more elementary to prove the existence of a toric variety into which $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded. Let D_i, $i = 1, \ldots, q$ be an algebraic divisor on T_N which is defined by

$$z_1^{a_{i,1}} \cdots z_r^{a_{i,r}} - c_i = 0,$$

where $(a_{i,1}, \ldots, a_{i,r}) \in M = \mathbb{Z}^r$ and $c_i \in \mathbb{C}^*$ for $i = 1, \ldots, q$. Put $a_i = (a_{i1}, \ldots, a_{ir}) \in M$. Assume that $M_{\mathbb{R}}$ is generated by $a_1, \ldots, a_q$, i.e.,

$$M_{\mathbb{R}} = \{k_1 a_1 + \cdots + k_q a_q \in M_{\mathbb{R}} \,|\, (k_1, \ldots, k_q) \in \mathbb{R}^q\}.$$

Then the following theorem holds.

Theorem 2. *There exists a toric projective variety X such that $T_N \setminus \operatorname{supp}\,(\sum_{i=1}^q D_i)$ is Kobayashi hyperbolically imbedded into X.*

The plan of this paper is as follows. Section 2 is devoted to the preparations, and we prove the Brody hyperbolicity of $T_N \setminus \operatorname{supp} D$, i.e., we prove that there exists no entire curve in $T_N \setminus \operatorname{supp} D$. In Section 3, we will prove the Main Theorem 1 and Corollary 1. We will also show a proposition which is useful to construct examples for the Main Theorem 1 (Proposition 1). In Section 4, we prove Theorem 2, and prove a generalization of the big Picard theorem (Corollary 2).

Acknowledgement. The author would like to express his deep gratitudes to his advisor Professor Junjiro Noguchi for his heartful helps and discussions. The author is also grateful to Mr. Makoto Miura for giving him many useful comments.

2. Brody hyperbolicity of $T_N \setminus \operatorname{supp} D$

In this section, we prove the Brody hyperbolicity of $T_N \setminus \operatorname{supp} D$. First, we show the following lemma.

Lemma 1. *Let $l \in \mathbb{N}$. Let $S_1, \ldots, S_{l+1}$ be subsets of $\mathbb{Z}^l$ such that $\sharp(S_j) < \infty$ for $j = 1, \ldots, l+1$. Let $Q_1(z_1, \ldots, z_l)$, $\ldots$, $Q_{l+1}(z_1, \ldots, z_l)$ be Laurent polynomials of $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_l, z_l^{-1}]$ such that*

$$Q_j(z_1, \ldots, z_l) = \sum_{I=(i_1,\ldots,i_l)\in S_j} a_{j,I} z_1^{i_1} \cdots z_l^{i_l},$$

for $j = 1, \ldots, l+1$. Let $d_j = \sharp(S_j)$, and let $N = \sum_{j=1}^{l+1} d_j$. Then $Q_1, \ldots, Q_{l+1}$ have no common zero point in $(\mathbb{C}^*)^l$ for general $[\ldots : a_{1,I} : \ldots : a_{2,I} : \ldots : a_{l+1,I} : \ldots] \in \mathbb{P}^{N-1}(\mathbb{C})$.

Proof. Let Z be the subvariety in $(\mathbb{C}^*)^l \times \mathbb{P}^{N-1}(\mathbb{C})$ defined by

$$\{\big((z_1, \ldots, z_l), [\ldots : a_{j,I} : \ldots]\big) \in (\mathbb{C}^*)^l \times \mathbb{P}^{N-1}(\mathbb{C}) \,|\, \sum_{I=(i_1,\ldots,i_l)\in S_j} a_{j,I} z_1^{i_1} \cdots z_l^{i_l} = 0 \text{ for } j = 1, \ldots, l+1\}.$$

For $x \in (\mathbb{C}^*)^l$, we denote the fiber of Z over x by Z_x. Then

$$\dim Z_x \leq \sum_{i=1}^{l+1} d_i - (l+1) - 1 = N - l - 2.$$

It follows that $\dim Z \leq (N-l-2) + l = N-2$. Let $p : (\mathbb{C}^*)^l \times \mathbb{P}^{N-1}(\mathbb{C}) \to \mathbb{P}^{N-1}(\mathbb{C})$ be the projection. Then $\dim p(Z) \leq N-2$, and $p(Z)$ is contained in a proper subvariety of $\mathbb{P}^{N-1}(\mathbb{C})$. If $[\ldots : a_{1,I} : \ldots : a_{2,I} : \ldots : a_{l+1,I} : \ldots] \in \mathbb{P}^{N-1}(\mathbb{C})$ is not contained in $p(Z)$, then $Q_1, \ldots, Q_{l+1}$ have no common zero point in $(\mathbb{C}^*)^l$. $\square$

Lemma 2. *Let S be a finite subset in M. Assume the following condition.*

> *Let $H \in \mathcal{H}_S$, and let $\phi_H : M_{\mathbb{R}} \to M_{\mathbb{R}}/H$ be the canonical morphism. Then $\sharp(\phi(S)) \geq r+1$ for all $H \in \mathcal{H}_S$, where $\sharp(\phi(S))$ is the number of the elements in $\{\phi(y) \in M_{\mathbb{R}}/H \mid y \in S\}$.*

Then $T_N \setminus \operatorname{supp} D$ and $\operatorname{supp} D$ contain no translation of positive dimensional subtorus in T_N for a general divisor D of the linear system $|\{z_1^{i_1} z_2^{i_2} \cdots z_r^{i_r}\}_{(i_1,i_2,\ldots,i_r)\in S}|$ on T_N.

Proof. Let $H \in \mathcal{H}_S$, and let $(h_1, \ldots, h_r) \in M^r$ be a $\mathbb{Z}$-basis of M such that $h_1, \ldots, h_{r-1}$ generate an $\mathbb{R}$-vector subspace H. We denote $h_i = (h_{i,1}, \ldots, h_{i,r}) \in M$ for $i = 1, \ldots, r$. Let $u_i := z_1^{h_{i,1}} \cdots z_r^{h_{i,r}}$. It follows that $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}] = \mathbb{C}[u_1, u_1^{-1}, \ldots, u_r, u_r^{-1}]$. Let $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$, and let

$$\sum_{I=(i_1,\ldots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r},$$

be a Laurent polynomial in $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}]$. Then there exist non-zero Laurent polynomials $Q_1(u_1, \ldots, u_{r-1}), \ldots, Q_t(u_1, \ldots, u_{r-1})$ in $\mathbb{C}[u_1, u_1^{-1}, \ldots, u_{r-1}, u_{r-1}^{-1}]$ and integers $d_1 < d_2 < \cdots < d_t$ such that

$$\sum_{I=(i_1,\ldots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r} = \sum_{i=1}^{t} Q_i(u_1, \ldots, u_{r-1}) u_r^{d_i}.$$

By the condition of the lemma, it follows that $t \geq r+1$. Because of Lemma 1, there exists a proper subvariety Y_H in $\mathbb{P}^{\sharp(S)-1}(\mathbb{C})$ which satisfies the following:

If $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$ *is not contained in* Y_H, *then* $Q_{j_1}, Q_{j_2}, \ldots, Q_{j_r}$ *have no common zero point in* $(\mathbb{C}^*)^{r-1}$ *for any* $1 \leq j_1 < j_2 < \ldots < j_r \leq t$.

Since the number of the elements in $\mathcal{H}_S$ is finite, $\bigcup_{H \in \mathcal{H}_S} Y_H$ is a subvariety of $\mathbb{P}^{\sharp(S)-1}(\mathbb{C})$.

Fix $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$ which is not contained in $\bigcup_{H \in \mathcal{H}_S} Y_H$. Let D be the divisor of T_N defined by the Laurent polynomial

$$\sum_{I=(i_1,\ldots,i_r) \in S} a_I z_1^{i_1} \cdots z_r^{i_r}.$$

Let Y be a translation of subtorus in T_N such that $1 \leq \dim Y \leq r-1$. Let k be the codimension of Y. There exist primitive elements $b_1 = (b_{1,1}, \ldots, b_{1,r}), \ldots, b_k = (b_{k,1}, \ldots, b_{k,r}) \in M$ and $c_1, \ldots, c_k \in \mathbb{C}^*$ such that

$$S = \{(z_1, \ldots, z_r) \in (\mathbb{C}^*)^r \mid z_1^{b_{j,1}} \cdots z_r^{b_{j,r}} = c_j \text{ for } j = 1, \ldots, k\}.$$

Let W be the subspace in $M_{\mathbb{R}}$ which is generated by $b_1, \ldots, b_k$. Let W' be the largest subspace of W generated by elements in $\mathcal{L}_S$. Define the canonical morphisms $\phi_W : M_{\mathbb{R}} \to M_{\mathbb{R}}/W$, $\phi_{W'} : M_{\mathbb{R}} \to M_{\mathbb{R}}/W'$, $\psi : M_{\mathbb{R}}/W' \to M_{\mathbb{R}}/W$. By the definition of W', ψ is injective on $\phi_{W'}(S)$. Without loss of generality, we may assume that $b_1, \ldots, b_l$ is a basis of W' where $l \leq k$. There exist $b_{k+1} = (b_{k+1,1}, \ldots, b_{k+1,r}), \ldots, b_r = (b_{r,1}, \ldots, b_{r,r}) \in M$ such that $b_1, \ldots, b_r$ be a basis of M. Put $u_1 = z_1^{b_{1,1}} \cdots z_r^{b_{1,r}}, \ldots, u_r = z_1^{b_{r,1}} \cdots z_r^{b_{r,r}}$. There exist the canonical isomorphisms

$$M/(W' \cap M) \simeq \mathbb{Z}b_{l+1} + \cdots + \mathbb{Z}b_r \simeq \mathbb{Z}^{r-l},$$
$$M/(W \cap M) \simeq \mathbb{Z}b_{k+1} + \cdots + \mathbb{Z}b_r \simeq \mathbb{Z}^{r-k},$$

where $\mathbb{Z}b_{l+1} + \cdots + \mathbb{Z}b_r$ (resp. $\mathbb{Z}b_{k+1} + \cdots + \mathbb{Z}b_r$) is the $\mathbb{Z}$-module generated by $b_{l+1}, \ldots, b_r$ (resp. $b_{k+1}, \ldots, b_r$). Therefore, we may assume that

$$\phi_{W'}(S) \subset \mathbb{Z}b_{l+1} + \cdots + \mathbb{Z}b_r \simeq \mathbb{Z}^{r-l},$$

and

$$\phi_W(S) \subset \mathbb{Z}b_{k+1} + \cdots \mathbb{Z}b_r \simeq \mathbb{Z}^{r-k}.$$

Let $Q_{J'}(u_1,\ldots,u_l)$ (resp. $R_J(u_1,\ldots,u_k)$) be a Laurent polynomial of $\mathbb{C}[u_1,u_1^{-1},\ldots,u_l,u_l^{-1}]$ (resp. $\mathbb{C}[u_1,u_1^{-1},\ldots,u_k,u_k^{-1}]$) such that

$$\begin{aligned}\sum_{I=(i_1,\ldots,i_r)\in S} a_I z_1^{i_1}\cdots z_r^{i_r} &= \sum_{J'=(j'_{l+1},\ldots,j'_r)\in\phi_{W'}(S)} Q_{J'}(u_1,\ldots,u_l)u_{l+1}^{j'_{l+1}}\cdots u_r^{j'_r}\\ &= \sum_{J=(j_{k+1},\ldots,j_r)\in\phi_W(S)} R_J(u_1,\ldots,u_k)u_{k+1}^{j_{k+1}}\cdots u_r^{j_r}.\end{aligned}$$

We take $H\in\mathcal{H}_S$ such that $W'\subset H$. Because $[\ldots : a_I : \ldots]\in\mathbb{P}^{\sharp(S)-1}(\mathbb{C})$ is not contained in Y_H, there exist at least two elements in $\{J'\}_{J'\in\phi_{W'}(S)}$ such that $Q_{J'}(c_1,\ldots,c_l)\neq 0$. Since ψ is a one-to-one correspondence between $\phi_{W'}(S)$ and $\phi_W(S)$, there exist at least two elements in $\{J\}_{J\in\phi_W(S)}$ such that $R_J(c_1,\ldots,c_k)\neq 0$. It follows that

$$D|_Y : \sum_{J=(j_{k+1},\ldots,j_r)\in\phi_W(S)} R_J(c_1,\ldots,c_k)u_{k+1}^{j_{k+1}}\cdots u_r^{j_r}=0,$$

since $Y=\{(u_1,\ldots,u_r)\in(\mathbb{C}^*)^r \mid u_1=c_1,\ldots,u_k=c_k\}$. Hence $Y\cap\operatorname{supp}D\neq\emptyset$ and $Y\not\subset\operatorname{supp}D$. This completes the proof. $\square$

The following theorem is proved in [6].

Theorem 3 ([6, Main Theorem, Proposition 1.8])**.** *Let D be an algebraic effective reduced divisor of a semi-Abelian variety A over the complex number field $\mathbb{C}$ (D may be the zero-divisor). Let $f:\mathbb{C}\to A\backslash\operatorname{supp}D$ be an arbitrary holomorphic mapping. Then the Zariski closure B of the image of f in A is a translate of a semi-Abelian subvariety of A, and $B\cap\operatorname{supp}D=\emptyset$.*

By Lemma 2 and Theorem 3, the following theorem holds.

Theorem 4. *Let S be a finite subset in M. Assume the following condition.*

> *Let $H\in\mathcal{H}_S$, and let $\phi_H: M_{\mathbb{R}}\to M_{\mathbb{R}}/H$ be the canonical morphism. Then $\sharp(\phi_H(S))\geq r+1$ for all $H\in\mathcal{H}_S$, where $\sharp(\phi_H(S))$ is the number of the elements in $\{\phi_H(y)\in M_{\mathbb{R}}/H \mid y\in S\}$.*

Then $T_N\setminus\operatorname{supp}D$ and $\operatorname{supp}D$ have no non-constant holomorphic map from $\mathbb{C}$ for a general divisor D of the linear system $|\{z_1^{i_1}z_2^{i_2}\cdots z_r^{i_r}\}_{(i_1,i_2,\ldots,i_r)\in S}|$ on T_N.

3. Proof of the Main Theorem 1

Let P be an integral convex polytope in $M_{\mathbb{R}}$ such that $\dim P = r$. Let D be an algebraic effective reduced divisor on T_N. There exists the toric projective variety X which is associated to P. We denote the closure of D in X by $\overline{D}$. There exist T_N-invariant irreducible (Weil) divisors $A_1, \dots, A_k$ in T_N such that $X \setminus \bigcup_{i=1}^{k} A_i = T_N$.

Lemma 3. *Assume that the following two conditions are satisfied.*

(a) *There exists neither non-constant holomorphic map*

$$f : \mathbb{C} \to T_N \setminus \operatorname{supp} D,$$

nor non-constant map

$$f : \mathbb{C} \to \operatorname{supp} D.$$

(b) *For any partition of indices $I \cup J = \{1, 2, \dots, k\}$, there exists neither non-constant holomorphic map*

$$f : \mathbb{C} \to \bigcap_{i \in I} A_i \setminus \big(\bigcup_{j \in J} A_j \cup \operatorname{supp} \overline{D}\big),$$

nor non-constant holomorphic map

$$f : \mathbb{C} \to \big(\bigcap_{i \in I} A_i \cap \operatorname{supp} \overline{D}\big) \setminus \bigcup_{j \in J} A_j.$$

Then $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded in X.

Proof. Assume $T_N \setminus \operatorname{supp} D$ is not Kobayashi hyperbolically imbedded in X. Then there exists a non-constant holomorphic map $f : \mathbb{C} \to X$ which satisfies the following condition (see Theorem (3.6.5) of Kobayashi [4]):

For any $R > 0$, there exists a sequence of holomorphic maps $f_i : D_R \to T_N \setminus \operatorname{supp} D$ for $i = 1, 2, \dots,$ such that $\{f_i\}_{i=1,2,\dots}$ converges uniformly on any compact sets in D_R to f. Here $D_R = \{z \in \mathbb{C} \mid |z| < R\}$.

Let Δ be the fan of the toric projective variety X. Assume that $f(z) \in A_i$ for some i and $z \in \mathbb{C}$. There exists an r-dimensional convex cone $\sigma \in \Delta$ such that

$$f(z) \in U_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M],$$

where $\sigma = \{m \in M_{\mathbb{R}} \mid \langle x, m\rangle \geq 0 \text{ for all } x \in \sigma\}$. There exist $h_1, \ldots, h_p \in \mathbb{C}[\sigma^{\vee} \cap M]$ such that

$$A_i \cap U_\sigma = \{h_1 = 0\} \cap \cdots \cap \{h_p = 0\},$$

and $\{h_j = 0\} \cap T_N = \emptyset$ for all $j = 1, \ldots, p$. Let B be a sufficiently small neighborhood of z. Because $f_j(B)$ is contained in $U_\sigma \cap T_N$ for large j, it follows that $h_l \circ f_j \neq 0$ on B for $l = 1, \ldots, p$ and large j. Then $h_l \circ f \equiv 0$ on B for $l = 1, \ldots, p$ by Hurwitz theorem. It follows that $f(\mathbb{C})$ is contained in A_i. Hence, $f(\mathbb{C}) \cap A_j = \emptyset$ or $f(\mathbb{C}) \subset A_j$ for all $j = 1, \ldots, k$. By the same argument, it follows that $f(\mathbb{C}) \cap \operatorname{supp} \overline{D} = \emptyset$ or $f(\mathbb{C}) \subset \operatorname{supp} \overline{D}$. This contradicts the assumption of the lemma. □

Proof of the Main Theorem 1. Let $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$, and let D be the divisor on T_N defined by the Laurent polynomial

$$\sum_{I=(i_1,\ldots,l_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r} = 0.$$

We show that X and D satisfy the conditions $(a), (b)$ of Lemma 3 for general $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$. By Theorem 4, the condition (a) of Lemma 3 holds for general $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$. Let I, J be a partition of $\{1, 2, \ldots, k\}$. Let Z be an irreducible component of $\bigcap_{i\in I} A_i$. Because there exists the one-to-one correspondence between the faces of P and the T_N-invariant irreducible subvarieties in X (see §2.3 of Oda [7]), there exists the face τ of P which correspondes to Z. Let l be the dimension of $V_{\tau\cap P}$. Fix a basis $b_1 = (b_{1,1}, \ldots, b_{1,r}), \ldots, b_l = (b_{l,1}, \ldots, b_{l,r}) \in M$ of $\mathbb{Z}$-module $V_{\tau\cap P} \cap M$. Then there is the canonical isomorphism $V_{\tau\cap P} \cap M \simeq \mathbb{Z}^l$. Let $u_1 = z_1^{b_{1,1}} \cdots z_r^{b_{1,r}}, \ldots, u_l = z_1^{b_{l,1}} \cdots z_r^{b_{l,r}}$. Then $Z \setminus \bigcup_{j\in J} A_j$ is biholomorphic to $\operatorname{Spec} \mathbb{C}[u_1, u_1^{-1}, \ldots, u_l, u_l^{-1}]$. Let $x \in \tau \cap S$. It follows that $\tau \cap S - x \in V_{\tau\cap S} \cap M \simeq \mathbb{Z}^r$. Hence $(\overline{D} \setminus \bigcup_{j\in J} A_j)|_Z$ is defined by the Laurent polynomial

$$\sum_{I'=(i_1,\ldots,i_l)\in\tau\cap S-x} c_{I'} u_1^{i_1} \cdots u_l^{i_l},$$

where $c_{I'}$ is equal to some element of $\{a_I\}_{I\in V}$. By the assumption of the Main Theorem 1 and Theorem 4, there exists neither non-constant holomorphic map

$$f : \mathbb{C} \to \bigcap_{i \in I} A_i \setminus \big(\bigcup_{j \in J} A_j \cup \operatorname{supp} \overline{D}\big),$$

nor non-constant holomorphic map

$$f : \mathbb{C} \to \big(\bigcap_{i \in I} A_i \cap \operatorname{supp} \overline{D}\big) \setminus \bigcup_{j \in J} A_j.$$

Hence the condition (b) of Lemma 3 holds. This completes the proof. □

The following proposition gives examples of P and S which satisfy the conditions of the Main Theorem 1.

Proposition 1. *Let S be a finite subset of M, and let P be an integral convex polytope in $M_{\mathbb{R}}$ such that $S \subset P$. Let ϱ be any one-dimensional face of P. If $\sharp(\varrho \cap S) \geq r + 1$, then P satisfies the conditions* (i), (ii) *of the Main Theorem 1.*

Proof. Let τ be a positive dimensional face of P. It is easy to see that τ satisfies the condition (i) of the Main Theorem 1. Let $H \in \mathcal{H}_{\tau \cap S}$. There exists a one-dimensional face ϱ of τ such that $\varrho - x \not\subset H$ for $x \in \tau \cap S$. Then it follows that

$$\sharp(\phi_H(\tau \cap S - x)) \geq \sharp(\phi_H(\varrho \cap S - x)) \geq r + 1 \geq \dim \tau + 1,$$

where $\phi_H : E_{\tau \cap S} \to E_{\tau \cap S}/H$ is the canonical morphism. This completes the proof. □

Now we prove Corollary 1. Let $d \geq r$. Let

$$P = \{(x_1, \dots, x_r) \in M_{\mathbb{R}} \mid \sum_{i=1}^{r} x_i \leq d,\ x_i \geq 0 \ \text{for} \ \ i = 1, \dots, r\},$$

and let

$$S = \{(x_1, \dots, x_r) \in M \mid \sum_{i=1}^{r} x_i \leq d,\ x_i \geq 0 \ \text{for} \ \ i = 1, \dots, r\}.$$

Then the toric variety X defined by P is r-dimensional complex projective space $\mathbb{P}^r(\mathbb{C})$, and elements in the linear system $|\{z_1^{i_1} z_2^{i_2} \cdots z_r^{i_r}\}_{(i_1, i_2, \dots, i_r) \in S}|$ are d-dimensional hypersurfaces of $\mathbb{P}^r(\mathbb{C})$. It is easy to verify that S and P satisfy the assumption of Proposition 1.

Example 1. Let $S = \{(0,0),(2,0),(0,2),(1,2),(2,1)\}$. Let D be a divisor on $\mathbb{C}^* \times \mathbb{C}^* = \operatorname{Spec} \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ defined by the following polynomial:

$$a_{00} + a_{20}z_1^2 + a_{02}z_2^2 + a_{12}z_1z_2^2 + a_{21}z_1^2z_2,$$

where $[a_{00} : a_{20} : a_{02} : a_{12} : a_{21}]$ is a generic point of $\mathbb{P}^4(\mathbb{C})$. The following cases satisfy the conditions of the Main Theorem 1.

(1) Take $P = \{(z_1, z_2) \in M_{\mathbb{R}} \mid z_1 + z_2 \le 3, z_1 \ge 0, z_2 \ge 0\}$. Then X is the two-dimensional complex projective space $\mathbb{P}^2(\mathbb{C})$.

(2) Take $P = \{(z_1, z_2) \in M_{\mathbb{R}} \mid z_1 + z_2 \le 3, z_1 \ge 0, z_2 \ge 0, z_2 \le 2\}$. Then X is the Hirzebruch surface F_1.

(3) Take $P = \{(z_1, z_2) \in M_{\mathbb{R}} \mid z_1 \ge 0, z_2 \ge 0, z_1 \le 2, z_2 \le 2\}$. Then X is the product space of the one-dimensional projective spaces $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

(4) Take $P = \{(z_1, z_2) \in M_{\mathbb{R}} \mid z_1 + z_2 \le 3, z_1 \ge 0, z_2 \ge 0, z_1 \le 2, z_2 \le 2\}$. Then X is a one-point blowing-up of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

4. Proof of Theorem 2

In this section, we deal with an algebraic divisor on T_N which is an union of translations of subtori. Let D_i, $i = 1, \ldots, q$ be the algebraic divisor on T_N which is defined by

$$z_1^{a_{i,1}} \cdots z_r^{a_{i,r}} - c_i = 0,$$

where $(a_{i,1}, \ldots, a_{i,r}) \in M = \mathbb{Z}^r$ and $c_i \in \mathbb{C}^*$ for $i = 1, \ldots, q$. Put $a_i = (a_{i,1}, \ldots, a_{i,r}) \in M$. Assume that $\mathbb{R}$-vector space $M_{\mathbb{R}}$ is generated by $a_1, \ldots, a_q$, i.e.,

$$M_{\mathbb{R}} = \{k_1a_1 + \cdots + k_qa_q \in M_{\mathbb{R}} \mid (k_1, \ldots, k_q) \in \mathbb{R}^q\}.$$

Let $I = (\delta_1, \ldots, \delta_q) \in \{-1, +1\}^q$ where $\delta_j = -1$ or $(+1)$. Let

$$C_I = \mathbb{R}_{\ge 0}(\delta_1 a_1) + \cdots + \mathbb{R}_{\ge 0}(\delta_q a_q),$$

where

$$\mathbb{R}_{\ge 0}(\delta_1 a_1) + \cdots + \mathbb{R}_{\ge 0}(\delta_q a_q) = \{r_1\delta_1a_1 + \cdots + r_q\delta_qa_q \in M_{\mathbb{R}} \mid r_1 \ge 0, \ldots, r_q \ge 0\}.$$

Then C_I is a convex rational polyhedral cone. We put

$$\Pi = \{C_I \subset M_{\mathbb{R}} \mid I \in \{-1, +1\}^q\},$$

and we put

$$\Pi' = \{C \in \Pi \,|\, C \text{ is strongly convex}\}.$$

The strong convexity of cone means that it contains no nonzero subspace of $M_{\mathbb{R}}$. Let

$$\Delta(r) = \{C^\vee \subset N_{\mathbb{R}} \,|\, C \in \Pi'\},$$

where

$$C^\vee = \{v \in N_{\mathbb{R}} \,|\, \langle v, m\rangle \geq 0 \text{ for all } m \in C\}.$$

Then $\Delta(r)$ is a finite set of r-dimensional strongly convex rational polyhedral cones in $N_{\mathbb{R}}$. Here the dimension of a cone σ is the dimension of the smallest $\mathbb{R}$-subspace of $N_{\mathbb{R}}$ containing σ. Let Δ be the collection of all faces of cones in $\Delta(r)$, i.e.,

$$\Delta = \{\sigma \subset N_{\mathbb{R}} \,|\, \text{ there exists } \tau \in \Delta(r) \text{ such that } \sigma \text{ is a face of } \tau \}.$$

Because elements in $\Delta(r)$ are strongly convex rational polyhedral cones, Δ is a collection of strongly convex rational polyhedral cones.

Lemma 4. *The collection Δ is a finite and complete fan in N, i.e., Δ satisfies the following conditions:*

(i) *Every face of any $\sigma \in \Delta$ is contained in Δ.*

(ii) *For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.*

(iii) *Δ is a finite set and the support $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma$ coincides with the entire $N_{\mathbb{R}}$.*

Proof. (i) is clear by the definition.

Let $\sigma \in \Delta(r)$, and let $\tau \in \Delta$. We show that $\sigma \cap \tau$ is a face of σ. By the definition, there exists $\sigma' \in \Delta(r)$ and $m \in \sigma'^\vee$ such that $\tau = \sigma' \cap \{m\}^\perp$, where $\{m\}^\perp = \{v \in N_{\mathbb{R}} \,|\, \langle v, m\rangle = 0\}$. There exist $1 \leq j_1 < \cdots < j_l \leq q$ such that

$$\sigma \cap \sigma' = \sigma \cap \bigcap_{k=1}^{l} \{a_{j_k}\}^\perp.$$

It follows that $\sigma \cap \sigma'$ is a face of σ. Because $m \in (\sigma \cap \sigma')^\vee$, it follows that $\sigma \cap \tau = (\sigma \cap \sigma') \cap \{m\}^\perp$ is a face of $\sigma \cap \sigma'$. Hence $\sigma \cap \tau$ is a face of σ.

Now we show the condition (ii) of the lemma. Let $\tau, \tau' \in \Delta$. There exists $\sigma \in \Delta(r)$ such that τ is a face of σ. Then $\sigma \cap \tau'$ is a face of σ by the above argument. It follows that $\tau \cap \tau' = \tau \cap (\sigma \cap \tau')$ is a face of σ. Hence $\tau \cap \tau'$ is a face of τ. In the same way, $\tau \cap \tau'$ is a face of τ'.

We show the condition (iii) of the lemma. The finiteness of Δ is obvious. For any $v \in N_{\mathbb{R}}$, there exists $(\delta_1, \dots, \delta_q) \in \{-1, +1\}^q$ such that $\langle v, \delta_i a_i \rangle \geq 0$ for $i = 1, \dots, q$, and $C := \{s_1 \delta_1 a_1 + \cdots + s_q \delta_q a_q \in M_{\mathbb{R}} \mid s_1 \geq 0, \dots, s_q \geq 0\}$ is strongly convex. Then $\sigma := C^\vee \in \Delta$ and $v \in \sigma$. □

Let X be a toric variety associated to the fan Δ. Then X is compact (see Theorem 1.11. of [7]).

A real valued function $h : |\Delta| \to \mathbb{R}$ is said to be a Δ-linear support function if it is $\mathbb{Z}$-valued on $N \cap |\Delta|$ and is linear on each $\sigma \in \Delta$. Namely, there exists $l_\sigma \in M$ for each $\sigma \in \Delta$ such that $h(n) = \langle l_\sigma, n \rangle$ for $n \in \sigma$ and that $\langle l_\sigma, n \rangle = \langle l_\tau, n \rangle$ holds for $n \in \tau < \sigma$. Here $\tau < \sigma$ means that τ is a face of σ. Assume that, for any $\sigma \in \Delta(r)$ and any $n \in N_{\mathbb{R}}$, we have $\langle l_\sigma, n \rangle \geq h(n)$ with the equality holding if and only if $n \in \sigma$. In this case, h is said to be strictly upper convex with respect to Δ.

Lemma 5. *X is projective.*

Proof. Define Δ-linear support function h by

$$h(n) = -\sum_{j=1}^{q} |\langle n, a_j \rangle|,$$

for $n \in N_{\mathbb{R}}$. Let $C \in \Pi'$, and let $(\delta_1, \dots, \delta_q) \in \{-1, +1\}^q$ such that $C = \{\mathbb{R}_{\geq 0} \delta_1 a_1 + \cdots + \mathbb{R}_{\geq 0} \delta_q a_q\}$. Then $l_\sigma = -(\delta_1 a_1 + \cdots + \delta_q a_q)$ for $\sigma = C^\vee$. Hence $\langle n, l_\sigma \rangle \geq h(n)$ for $n \in N_{\mathbb{R}}$ and the equality holds if and only if $n \in \sigma$. Therefore h is a strictly upper convex with respect to Δ. Then X is a toric projective variety (see Corollary 2.14. of [7]). □

Let $A_1, \dots, A_k$ be T_N-invariant irreducible (Weil) divisors of X such that $X \setminus \bigcup_{i=1}^k A_i = T_N$. Let $\overline{D}_i$ be the closure of D_i in X for $i = 1, \dots, q$. In the same way of the proof of Lemma 3, the following lemma holds.

Lemma 6. *Assume that the following two conditions are satisfied.*

(a') *There exists no non-constant holomorphic map*

$$f : \mathbb{C} \to T_N \setminus \bigcup_{i=1}^{q} \operatorname{supp} D_i.$$

(b') *Let* $I \subset \{1, \ldots, k\}$, $J \subset \{1, \ldots, q\}$ *such that* $I \neq \emptyset$ *or* $J \neq \emptyset$. *Let* $I' = \{1, \ldots, k\} \setminus I$, $J' = \{1, \ldots .q\} \setminus J$. *Then there exists no non-constant holomorphic map*

$$f : \mathbb{C} \to \left(\bigcap_{j \in I} \operatorname{supp} A_j \cap \bigcap_{j \in J} \operatorname{supp} \overline{D}_j \right) \setminus \left(\bigcup_{j \in I'} \operatorname{supp} A_j \cup \bigcup_{j \in J'} \operatorname{supp} \overline{D}_j \right).$$

Then $T_N \setminus \bigcup_{i=1}^{q} \operatorname{supp} D_i$ *is Kobayashi hyperbolically imbedded in* X.

Now we prove Theorem 2

Proof of Theorem 2. We show that X and $D_1, \ldots, D_q$ satisfy the condition (a'), (b') of Lemma 6.

Let

$$f : \mathbb{C} \to T_N \setminus \bigcup_{j=1}^{q} \operatorname{supp} D_j,$$

be a holomorphic map. There exist holomorphic functions $g_1, \ldots, g_r$ on $\mathbb{C}$ such that

$$f = (\exp g_1, \ldots, \exp g_r) : \mathbb{C} \to T_N \setminus \bigcup_{i=1}^{q} \operatorname{supp} D_i.$$

It holds that

$$\exp(a_{j,1} g_1 + \cdots + a_{j,r} g_r) - c_j \neq 0,$$

for all $j = 1, \ldots, q$ on $\mathbb{C}$. By the small Picard theorem, $\exp(a_{j,1} g_1 + \cdots + a_{j,r} g_r) - c_j$ is a constant function. Hence $a_{j,1} g_1 + \cdots + a_{j,r} g_r$ is constant. Since $a_1, \ldots, a_q$ generate $\mathbb{R}$-vector space $M_{\mathbb{R}} = \mathbb{R}^r$, it follows that $g_1, \ldots, g_r$ are constant functions. Therefore X and $D_1, \ldots, D_q$ satisfy the condition (a') of Lemma 6.

Let $I \subset \{1, \ldots, k\}$, $J \subset \{1, \ldots, q\}$ such that $I \neq \emptyset$ or $J \neq \emptyset$. Let $I' = \{1, \ldots, k\} \setminus I$, $J' = \{1, \ldots .q\} \setminus J$. Let

$$f : \mathbb{C} \to \left(\bigcap_{j \in I} \operatorname{supp} A_j \cap \bigcap_{j \in J} \operatorname{supp} \overline{D}_j \right) \setminus \left(\bigcup_{j \in I'} \operatorname{supp} A_j \cup \bigcup_{j \in J'} \operatorname{supp} \overline{D}_j \right),$$

be a holomorphic map. It follows that $f(\mathbb{C}) \subset A_i$ (resp. $f(\mathbb{C}) \subset \overline{D}_i$) or $f(\mathbb{C}) \cap A_i = \emptyset$ (resp. $f(\mathbb{C}) \cap \overline{D}_i = \emptyset$) for $i = 1, \dots, k$ (resp. for $i = 1, \dots, q$). We show that f is a constant map. There exists an element of $\sigma \in \mathit{\Delta}(r)$ such that $f(\mathbb{C}) \subset U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. There exist $(\delta_1, \dots, \delta_q) \in \{-1, +1\}^q$ such that

$$\sigma^\vee = \mathbb{R}_{\geq 0}\delta_1 a_1 + \cdots + \mathbb{R}_{\geq 0}\delta_q a_q.$$

We take primitive elements $b_1 = (b_{1,1}, \dots, b_{1,r}), \dots, b_q = (b_{q,1}, \dots, b_{q,r})$ of M such that $d_i b_i = \delta_i a_i$ where d_i is a positive integer, i.e., $\mathbb{R}a_i \cap M = \mathbb{Z}b_i$ and $\mathbb{R}_{\geq 0}\delta_i a_i = \mathbb{R}_{\geq 0} b_i$. There exist $b_{q+1} = (b_{q+1,1}, \dots, b_{q+1,r}), \dots, b_l = (b_{l,1}, \dots, b_{l,q}) \in M$ such that $\sigma^\vee \cap M = \mathbb{Z}_{\geq 0} b_1 + \cdots + \mathbb{Z}_{\geq 0} b_l$ where l is a positive integer. Let $u_i = z_1^{b_{i,1}} \cdots z_r^{b_{i,r}}$ for $i = 1, \dots, l$. Then $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[u_1, \dots, u_l]$. Since $f(\mathbb{C}) \subset A_i$ or $f(\mathbb{C}) \cap A_i = \emptyset$ for $i = 1, \dots, k$, it follows that $u_i \circ f \equiv 0$ or $u_i \circ f \neq 0$ on $\mathbb{C}$ for $i = 1, \dots, q$. Since $f(\mathbb{C}) \subset \overline{D}_i$ or $f(\mathbb{C}) \cap \overline{D}_i = \emptyset$ for $i = 1, \dots, q$, it follows that $u_i^{d_i} \circ f \equiv c_i^{\delta_i}$ or $u_i^{d_i} \circ f \neq c_i^{\delta_i}$ on $\mathbb{C}$ for $i = 1, \dots, q$. By the small Picard theorem, $u_i \circ f$ is a constant function for $i = 1, \dots, q$. Since $b_j \in \mathbb{Q}_{\geq 0} b_1 + \cdots + \mathbb{Q}_{\geq 0} b_q$ for $j > q$, there exist relations such that

$$u_j^{\rho_j} = u_1^{\mu_{j,1}} \cdots u_q^{\mu_{j,q}} \quad \text{for} \quad j > q,$$

where ρ_j is a positive integer and $\mu_{j,i}$ is a non-negative integer. Hence $u_j \circ f$ is constant function for $j > q$, and f is a constant map. Therefore $X, D_1, \dots, D_q$ satisfy the condition (b') of Lemma 6. $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded into X by Lemma 6. $\square$

Corollary 2. *Let $D(1) := \{x \in \mathbb{C} \mid |x| < 1\}$, and let $D(1)^* := D(1) \setminus \{0\}$. Let f, g be holomorphic functions on $D(1)^*$ such that $f \neq 0, g \neq 0$, $f \neq g$ and $f \neq g^{-1}$ on $D(1)^*$. Then f and g are extended to meromorphic functions on $\Delta(1)$.*

Proof. Let D and D' be the divisors on $(\mathbb{C}^*)^2 = \operatorname{Spec} \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ defined by $z_1 z_2 - 1 = 0$ and $z_1 z_2^{-1} - 1 = 0$. Then (f, g) is a holomorphic map from $D(1)^*$ to $(\mathbb{C}^*)^2 \setminus \operatorname{supp}(D + D')$. By Theorem 2, there exists toric projective variety X such that $(\mathbb{C}^*)^2 \setminus \operatorname{supp}(D + D')$ is Kobayashi hyperbolically imbedded into X. By a generalization of the big Picard theorem in [2], (f, g) are extended to a holomorphic map $F : D(1) \to$

X. Since X and $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ are birational, there exist meromorphic functions $\tilde{f}, \tilde{g}$ on $D(1)$ such that the holomorphic map $(\tilde{f}, \tilde{g}) : D(1) \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is a extension of (f, g). $\square$

Corollary 2 is the classical big Picard theorem if $g = 1$.

References

[1] H. Fujimoto, On holomorphic maps into a taut complex space. Nagoya Math. J. 46 (1972), 49–61.

[2] P. Kiernan, Hyperbolically imbedded spaces and the big Picard theorem. Math. Ann. 204 (1973), 203–209.

[3] S. Kobayashi, Hyperbolic manifolds and holomorphic mappings. Pure and Applied Mathematics, 2 Marcel Dekker, Inc., New York 1970.

[4] S. Kobayashi, Hyperbolic complex spaces, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 318, Springer-Verlag, Berlin, 1998.

[5] J. Noguchi, Moduli spaces of holomorphic mappings into hyperbolically imbedded complex spaces and locally symmetric spaces, Invent. Math. **93** (1988), 15-34.

[6] J. Noguchi, On holomorphic curves in semi-Abelian varieties. Math. Z. 228(1998), 713–721.

[7] T. Oda, Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties. Erg. Math. 3/15. Springer-Verlag, Berlin-Tokyo 1985.

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-9814, Japan.

chiba@ms.u-tokyo.ac.jp

Preprint Series, Graduate School of Mathematical Sciences, The University of Tokyo

UTMS

2011–4 Toshio Oshima and Nobukazu Shimeno: *Boundary value problems on Riemannian symmetric spaces of the noncompact type.*

2011–5 Toshio Oshima: *Fractional calculus of Weyl algebra and Fuchsian differential equations.*

2011–6 Junjiro Noguchi and Jörg Winkelmann: *Order of meromorphic maps and rationality of the image space.*

2011–7 Mourad Choulli, Oleg Yu. Imanuvilov, Jean-Pierre Puel and Masahiro Yamamoto: *Inverse source problem for the lineraized Navier-Stokes equations with interior data in arbitrary sub-domain.*

2011–8 Toshiyuki Kobayashi and Yoshiki Oshima: *Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs.*

2011–9 Shigeo Kusuoka and Song Liang: *A classical mechanical model of Brownian motion with one particle coupled to a random wave field.*

2011–10 Kenji Nakahara: *Uniform estimate for distributions of the sum of i.i.d. random variables with fat tail: Infinite variance case.*

2011–11 Wenyan Wang, Bo Han and Masahiro Yamamoto: *Inverse heat problem of determining time-dependent source parameter in reproducing kernel space.*

2011–12 Kazuki Hiroe: *Linear differential equations on $\mathbb{P}^1$ and root systems.*

2011–13 Shigeo Kusuoka and Takenobu Nakashima: *A remark on credit risk models and copula.*

2011–14 Tomohiko Ishida and Nariya Kawazumi: *The Lie algebra of rooted planar trees.*

2011–15 Yusaku Tiba: *Kobayashi hyperbolic imbeddings into toric varieties.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012