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Abstract

We consider a linear differential operators on $\mathbb{P}^1$ having unramified irregular singular points. For this operator, we attach the root lattice of a Kac-Moody Lie algebra and the certain element in this lattice. Then we study the Euler transform for this differential operator and show that this translation by the Euler transform can be understand as the Weyl group action on the root lattice. Moreover we show that if the differential operator is irreducible, then the corresponding element becomes a root of this root system.

Introduction

For a function $f(x)$, the following integral

$$I_a^\lambda f(x) = \frac{1}{\Gamma(\lambda)} \int_a^x (x-t)^{\lambda-1} f(t)\, dt$$

is called the Euler transform (or Riemann-Liouville integral) of $f(x)$ for $a, \lambda \in \mathbb{C}$. If we take a function $f(x) = (x-a)^\alpha \phi(x)$ where $\alpha \in \mathbb{C}\backslash\mathbb{Z}_{<0}$ and $\phi(x)$ is a holomorphic function on a neighborhood of $x = a$ and $\phi(a) \neq 0$, then it is known that

$$I_a^{-n} f(x) = \frac{d^n}{dx^n} f(x).$$

Hence one can consider the Euler transform to be a fractional or complex powers of the derivation $\partial = \frac{d}{dx}$. This may allow us to write $\partial^\lambda f(x) = I_a^{-\lambda} f(x)$ formally.

Moreover one can show a generalization of the Leibniz rule,

$$\partial^\lambda p(x)\psi(x) = \sum_{i=0}^{n} \binom{\lambda}{i} p^{(i)}(x) \partial^{\lambda-i} \phi(x),$$

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if $p(x)$ is a polynomial of degree equal to or less than n. Now let us consider a differential operator with polynomial coefficients,

$$P(x, \partial) = \sum_{i=0}^{n} a_i(x)\partial^i.$$

The above Leibniz rule assures that $\partial^{\lambda+m}P(x,\partial)\partial^{-\lambda}$ is also the new differential operator with polynomial coefficients if we choose a suitable $m \in \mathbb{Z}$. Moreover if $f(x)$ satisfies $P(x,\partial)f(x) = 0$ and $I_a^{-\lambda}f(x)$ is well-defined for some $a, \lambda \in \mathbb{C}$, then we can see that

$$\begin{aligned}\partial^{\lambda+m}P(x,\partial)\partial^{-\lambda}I_a^{-\lambda}f(x) &= \partial^{\lambda+m}P(x,\partial)\partial^{-\lambda+\lambda}f(x)\\ &= \partial^{-\lambda+m}P(x,\partial)f(x)\\ &= 0.\end{aligned}$$

Hence ∂^{λ} turns a differential equation with polynomial coefficients $P(x,\partial)u = 0$ into a new differential equation with polynomial coefficients $Q(x,\partial)u = 0$, and moreover a solution of $Q(x,\partial)u = 0$ is given by a solution of $P(x,\partial)u = 0$ if the Riemann-Liouville integral is well-defined. For this reason, it is improtant to study what diffetential equation can be obtained by the Euler transform from the known equation or when we can reduce the difficult equation to an easier one.

Let K be an algebraically closed field of characteristic zero and $W(x)$ the ring of differential operators with coefficients in $K(x)$, the field of rational functions. In [11], T. Oshima gave an algebraic definition of the Euler transform on $W(x)$ as an analogue of the middle convolution defined by N. Katz in [8]. The purpose of this paper is to investigate the Euler transform (or the middle convolution) by the theory of root systems of Kac-Moody Lie algebras.

In [5], W. Crawley-Boevey clarified the correspondence between Fuchsian differential equations and certain representations of quivers (deformed preprojective algebras). Here we call a differential equation Fuchsian if all singular points are regular singular points. This suggests that the theory of representations of quivers can be applied to the theory of differential equations. In fact, he solved the additive Deligne-Simpson problem of Fuchsian differential equations by his theory of representation of quivers. The middle convolution (or the Euler transform) gives the reflection functor of a quiver, i.e., the Weyl group action of this root system.

In this paper, we deal with differential equations with unramified irregular singular points and consider the correspondence between the Euler transform and the action of the Weyl group of a Kac-Moody Lie algebra as a generalization of Crawley-Boevey's result.

Our result can be roughly explained as follows. Let us take $P \in W(x)$ with unramified irregular singular points. We impose some generic conditions on P (see Section 3 for precise conditions). From local structures around singular points, we can define a $\mathbb{Z}$-lattice $L(P)$ and an element $\mathfrak{m}(P) \in L(P)$ which describes multiplicities of local exponents of formal solutions of $Pu = 0$. Then the translations of $\mathfrak{m}(P)$ by the Euler transform gives a group action on $L(P)$. We write this group by $\tilde{W}(P)$.

Then this $\tilde{W}(P)$-module $L(P)$ can be seen as a root lattice of a Kac-Moody Lie algebra. That is to say, there exists the root lattice $Q(P)$ and the Weyl group $W(P)$ of a symmetric Kac-Moody Lie algebra such that $L(P)$ is isomorphic to a quotient of $W(P)$-module $Q(P)$ (see Theorem 3.3).

In [4], P. Boalch considers meromorphic connections which have finitely many regular singular points and one unramified irregular singular point. He gives a correspondence between these connections and representations of quivers as a generalization of the result of Crawley-Boevey. If we restrict our case to this Boalch's case, we can obtain the root lattices whose Dynkin diagrams agree with Boalch's quivers.

Furthermore, we can define a generalization of the root system in $L(P)$ as an analogue of the root system of $Q(P)$. Then we show that if P is irreducible, then $\mathbf{m}(P) \in L(P)$ is the root of this generalized root system (see Theorem 3.6).

As a corollary, we can show that the irregular Katz algorithm shown by D. Arinkin and D. Yamakawa independently ([1], [17]).

Moreover as examples of our correspondence with root system, let us consider confluent equations of Heun's differential equations. Then we can obtain extended Dynkin diagrams of affine Lie algebras, $D_4^{(1)}$, $A_3^{(1)}$, $A_2^{(1)}$, $A_1^{(1)}$ and $A_1^{(1)} \oplus A_1^{(1)}$. These agree with symmetries of Bäcklund transforms of Painlevé equations which are obtained from these Heun's equations with an apparent singular point (see Section 3.4).

1 Local structures

Let K be an algebraically closed field of characteristic zero. Let us write the ring of polynomials with one variable x by $K[x]$. We also write the field of rational functions (resp. formal Laurent series) by $K(x)$ (resp. $K((x))$). These are differential ring and fields, i.e., they have the action of $\partial = \frac{d}{dx}$ in the usual sense. The Weyl algebra $W[x]$ is the polynomial ring of ∂ with the coefficients in $K[x]$, or equivalently to say, the K-algebra generated by x and $\partial = \frac{d}{dx}$ with the relation

$$[x, \partial] = x\partial - \partial x = -1.$$

We also consider the algebra of differential operators $W(x)$ (resp. $W((x))$) which is the polynomial ring of ∂ with the coefficient field $K(x)$ (resp. $K((x))$).

For $P \in W(x)$, the degree as the polynomial of ∂ is called rank of P and written by $\operatorname{rank} P$. For example, the rank of $P = \sum_{i=0}^{n} a_i(x)\partial^i$ is n. We define the degree of $P = \sum_{i=0}^{n} a_i(x)\partial^i \in W[x]$ by

$$\deg P = \max{}_{i=1,\dots,n}\{\deg_{K[x]} a_i(x)\}.$$

We shall give a review of local structures of elements of $W(x)$ around singular points. The materials of this section are well-known and found in standard references (for example [9],[10],[13],[16]).

1.1 Singular points, local decompositions and characteristic exponents

For $c \in K$ and a monomial $(x-c)^a\partial^b$, we introduce the weight

$$\mathrm{wt}_c((x-c)^a\partial^b) = a - b.$$

The weight of $P \in W((x-c))$ is defined as follows,

$$\mathrm{wt}_c(P) = \min\{\mathrm{wt}_c((x-c)^i\partial^j) \mid P = \sum_{i,j} a_{i,j}(x-c)^i\partial^j\}.$$

For $f(x) \in K((x-c))$, we can define the weight $\mathrm{wt}_c(f(x))$ by regarding $f(x)$ as an element in $W((x-c))$.

For an integer k, the k-homogeneous part of $P \in W((x-c))$ is

$$P_{(k)} = \sum_{i-j=k} a_{i,j}(x-c)^i\partial^j,$$

if we write $P = \sum_{i,j} a_{i,j}(x-c)^i\partial^j$.

Similarly we can define wt_∞ by

$$\mathrm{wt}_\infty(x^a\partial^b) = b - a.$$

Let us consider finitely generated $W(x)$-modules. We usually deal with left $W(x)$-modules and call them $W(x)$-module simply if there is no confusion. Let M be a finitely generated $W(x)$-module. Then it is known that M is a cyclic $W(x)$-module, that is, there exists $P \in W(x)$ such that

$$M \simeq M_P = W(x)/W(x)P$$

as $W(x)$-modules. Hence we sometimes identify these P and M_P.

Let us consider $P = \sum_{i=0}^{n} a_i(x)\partial^i \in W(x)$. We call n the rank of P and M_P. The set of singular points of P are the set poles of $\frac{a_i(x)}{a_0(x)}$ $(i = 1,\dots,n)$. We also say that $x = \infty$ is a singular point of P, if

$$P^{(\infty)} = \sum_{i=0}^{n} a_i(\frac{1}{x})(-x^2\partial)^i$$

has singular point at $x=0$. Suppose that $x=c(\neq\infty)$ is a singular point of P. Let us take the $\mathrm{wt}_c(P)$-homogeneous part of P,

$$\sum_{i-j=\mathrm{wt}_c(P)} a_{i,j}(x-c)^i\partial^j.$$

Then the characteristic polynomial of P at $x=c$ is defined by

$$\mathrm{C}_c(P)(t)=\sum_{i-j=\mathrm{wt}_c(P)} a_{i,j}t(t-1)\cdots(t-j+1).$$

If

$$\deg_{K[t]}\mathrm{C}_c(P)(t)=\mathrm{rank}\,P,$$

we say that $x=c$ is a regular singular point of P. If otherwise, we say that $x=c$ is an irregular singular point of P. For $x=\infty$, we can define regular and irregular singular points as well as the above if we replace $x-c$ to $\frac{1}{x}$.

Suppose that $x=c$ is an irregular singular point of P. For the simplicity of notations, we put $c=0$. There exists an algebraic extension $K((x^{\frac{1}{q}}))$ of $K((x))$ for a positive integer q and we can decompose M_P as the direct sum of regular singular $W_q((x))$-modules. Here $W_q((x))$ is the ring of ∂ with coefficients in $K((x^{\frac{1}{q}}))$.

More precisely, there exist $P_i\in W_q((x))$ and distinct polynomials w_i of $x^{-\frac{1}{q}}$ with no constant terms for $1\leq i\leq r$ such that we have the following.

1. Each P_i has $x=0$ as a regular singular point.
2. We have the decomposition $P=P_1(\vartheta-w_1)\cdots P_r(\vartheta-w_r)$ in $W_q((x))$.
3. We have the decomposition

   $$W_q((x))/W_q((x))P\simeq\bigoplus_{i=1}^{r}W_q((x))/W_q((x))P_i(\vartheta-w_i)$$

   as $W_q((x))$-modules.

This decomposition is unique in the following sense. If there is another polynomials v_i of $x^{-\frac{1}{q}}$ and $P'_i\in W_q((x))$ $(1\leq i\leq s)$ which satisfy the above conditions, then $s=r$ and there exist a permutation σ of $\{1,2,\ldots,r\}$ such that $w_i=v_{\sigma(i)}$ and

$$W_q((x))/W_q((x))P_i(\vartheta-w_i)\simeq W_q((x))/W_q((x))P'_{\sigma(i)}(\vartheta-v_{\sigma(i)})$$

for $1\leq i\leq r$.

Let us summarize these facts below.

Definition 1.1 (The local decomposition)**.** *For $P \in W(x)$ with an irregular singular point $x = c$, there exists the algebraic extension $K(((x-c)^{\frac{1}{q}}))$ of $K((x-c))$, distinct polynomials w_i of $(x-c)^{-\frac{1}{q}}$ with no constant terms, and $P_i \in W_q((x-c))$ for $1 \leq i \leq r$ such that we have the following.*

1. *Each P_i has $x = c$ as a regular singular point.*
2. *We can write P as the least left common multiple of*
$$\{P_1(\vartheta - w_1), \ldots, P_r(\vartheta - w_r)\}.$$
Namely there exist $R_i \in W_q((x-c))$ such that
$$P = R_i P_i(\vartheta_c - w_i) \text{ for } i = 1, \ldots, r.$$
Here $\vartheta_c = (x-c)\partial$.
3. *We can decompose*
$$W_q((x-c))/W_q((x-c))P \simeq \bigoplus_{i=1}^{r} W_q((x-c))/W_q((x-c))P_i(\vartheta_c - w_i)$$
as $W_q((x-c))$-modules.

We call the decomposition in 3 the local decomposition of P at $x = c$. Let us call $P_i \in W_L((x-c))$ local factors and w_i the exponential factors of P_i for $1 \leq i \leq r$.

The positive integer q is called the index of ramification. In particular when $q = 0$, we say that $x = c$ is an unramified irregular singular point. If otherwise, $x = c$ is called a ramified irregular singular point.

We define characteristic exponents of P at $x = c$ by characteristic exponents of each local factors.

Definition 1.2 (Characteristic exponents)**.** *Let us consider $P \in W(x)$ with an irregular singular point at $x = c$. Let us take local factors of P, $\{P_1, \ldots, P_r\} \subset W((x-c))$ at $x = c$. Define*
$$\mathrm{Exp}_c(P_i) = \{\lambda \in K \mid \mathrm{C}_c(P_i)(\lambda) = 0\}.$$
Then the set of characteristic exponents of P at $x = c$ is
$$\mathrm{Exp}_c(P) = \bigcup_{i=1}^{r} \mathrm{Exp}_c(P_i).$$

1.2 The local decomposition and the Newton polygon

Let us give a review of the Newton polygon of $P = \sum_{i=0}^{n} a_i(x)\partial^i$. We associate the point

$$(i, \mathrm{wt}_c(a_i(x)\partial^i)) \in \mathbb{N} \times \mathbb{Z}$$

for each i-th terms $a_i(x)\partial^i$ of P. Then the convex hull of the set

$$\bigcup_{i=0}^{n} \{(i-s, \mathrm{wt}_c(a_i(x)) + t) \mid s, t \in \mathbb{Z}_{\geq 0}\}$$

is called the Newton polygon of P at $x = c$. We denote it by $N_c(P)$.

Let us suppose that P has the local decomposition as in Definition 1.1 and we use the same notations. Let

$$a_1 = (i_1, j_1), \ldots, a_l = (i_l, j_l) \quad (0 \leq i_1 < \cdots < i_l)$$

be the set of vertices of $N_c(P)$. We denote λ_i slopes of edges connecting a_i and a_{i+1} for $i = 1, \ldots, l-1$. Then we can see that $q\lambda_i \in \mathbb{Z}$ for all $i = 1, \ldots, l-1$ and

$$\begin{aligned}
i_1 &= \begin{cases} \mathrm{rank}\, P_i & \text{if there exists the local factor } P_i \text{ with } w_i = 0, \\ 0 & \text{if otherwise,} \end{cases} \\
i_{k+1} - i_k &= \sum_{\{i \mid \deg_{K[x^{-\frac{1}{q}}]} w_i = q\lambda_k\}} \mathrm{rank}\, P_i \quad (k = 1, \ldots, l-1), \\
j_1 &= \mathrm{wt}_{\,c}(P), \\
j_{k+1} - j_k &= \lambda_k \sum_{\{i \mid \deg_{K[x^{-\frac{1}{q}}]} w_i = q\lambda_k\}} \mathrm{rank}\, P_i \quad (k = 1, \ldots, l-1).
\end{aligned}$$

For $x = \infty$, we can also define the Newton polygon $N_\infty(P)$. Let us denote vertices of $N_\infty(P)$ by

$$a_1 = (i_1, j_1), \ldots, a_l = (i_l, j_l) \quad (0 \leq i_1 < \cdots < i_l)$$

as above and suppose that P has local factors

$$\{P_1, \ldots, P_r\} \subset W_{q'}((x^{-1}))$$

with exponents $w_1, \ldots, w_r$ at infinity. Then we can obtain the following similar formulas which show the relationship between $N_\infty(P)$ and local factors.

$$\begin{aligned}
i_l &= n = \operatorname{rank} P,\\
j_l &= n - \deg_{K[x]} a_n(x),\\
i_{k+1} - i_k &= \sum_{\{i \mid \deg_{K[x^{\frac{1}{q'}}]} w_i = q'\lambda_k\}} \operatorname{rank} P_i \quad (k = 1, \ldots, l-1),\\
j_{k+1} - j_k &= \lambda_k \sum_{\{i \mid \deg_{K[x^{\frac{1}{q'}}]} w_i = q'\lambda_k\}} \operatorname{rank} P_i \quad (k = 1, \ldots, l-1).
\end{aligned}$$

Let $a\,(\in \{1, \ldots, l\})$ be the index such that $\lambda_a > 1$ and $\lambda_{a-1} \le 1$. Then we can see $\deg P = i_a - j_a$. Hence we have

$$\deg P = \deg_{K[x]} a_n(x) + \sum_{s=a}^{l-1} (\lambda_s - 1) \sum_{\{i \mid \deg_{K[x^{\frac{1}{q'}}]} w_i = q'\lambda_s\}} \operatorname{rank} P_i. \tag{1}$$

Also we can compute $\operatorname{wt}_\infty(P)$ as follows,

$$\begin{aligned}
\operatorname{wt}_\infty(P) = j_1 &= n - \deg_{K[x]} a_n(x) - \sum_{s=a}^{l-1} \lambda_s \sum_{\{i \mid \deg_{K[x^{\frac{1}{q'}}]} w_i = q'\lambda_s\}} \operatorname{rank} P_i\\
&= -\deg_{K[x]} a_n(x) - \sum_{s=a}^{l-1} (\lambda_s - 1) \sum_{\{i \mid \deg_{K[x^{\frac{1}{q'}}]} w_i = q'\lambda_s\}} \operatorname{rank} P_i.
\end{aligned} \tag{2}$$

1.3 Spectral types

We introduce the notion of spectral types around singular points. To do this, we define the spectral type of a matrix first.

Definition 1.3. *Let us take $A \in M(n, K)$, i.e., A is an $n \times n$ matrix with K-components. Let us take a partition of n,*

$$\mathbf{m} = (m_1, m_2, \ldots, m_N),$$

i.e., these $m_i (\neq 0)$ are positive integers satisfying

$$n = \sum_{i=1}^{N} m_i.$$

And we take a tuple of elements in K,

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in K^N.$$

Then we say that A has the spectrum

$$(\lambda, \mathbf{m}),$$

if A satisfies

$$\operatorname{rank} \prod_{\nu=1}^{j} (A - \lambda_\nu) = n - (m_1 + \cdots + m_j)$$

for all $j = 1, \ldots, N$.

In this case, we call $\mathbf{m}$ the spectral type of A.

Let us take $P \in W((x))$. We can regard $M_P = W((x))/W((x))P$ as the $K((x))$-vector space of

$$\dim M_P = \operatorname{rank} P.$$

For a basis $\{u_1, \ldots, u_n\}$ of M_P as $K((x))$-vector space, we can write the action of ∂ by the matrix as follows. For $u \in M_P$, there exists $a_{ij}(\vartheta) \in K((x))$ such that

$$\vartheta u_i = \sum_{j=1}^{n} a_{ij}(\vartheta) u_j.$$

Then we define $A(\vartheta) = {}^t(a_{ij}(\vartheta))_{\substack{1 \le i \le n \\ 1 \le j \le n}} \in M(n, K((x)))$. Moreover if P has $x = 0$ as a regular singular point, there exists a basis such that we can take $A(\vartheta)$ as the element in $M(n, K)$. Then we can choose the partition of n, written by $\mathbf{m} = (m_1, \ldots, m_N)$ and N-tuple of K, written by $\lambda = (\lambda_1, \ldots, \lambda_N)$. And $A(\vartheta)$ has the spectrum

$$(\lambda, \mathbf{m}).$$

This partition $\mathbf{m}$ is independent of choices of bases. The λ is also uniquely determined modulo $\mathbb{Z}^N$. Hence if we put $\bar{\lambda} = \lambda + \mathbb{Z}^N \in K^N/\mathbb{Z}^N$, the pair $(\bar{\lambda}, \mathbf{m})$ is unique for P. We call $(\bar{\lambda}, \mathbf{m})$ the spectrum of P and $\mathbf{m}$ the spectral type of P at $x = 0$.

For the other regular singular point $x = c$, we can define the spectral type at each points in the same way.

2 Algebraic transformations and local data

In this section, we introduce some transformations on $W[x]$, $W(x)$. We shall investigate changes of spectra caused by these transformations.

2.1 The addition and the Fourier-Laplace transform

For

$$g_\mu(x) = \mu x^{-1} + a_0 + a_1 x + a_2 x^2 + \cdots ,$$

let us consider the K-algebra automorphism

$$\begin{array}{rccc} j_\mu\colon & W((x-c)) & \longrightarrow & W((x-c)) \\ & x-c & \longmapsto & x-c \\ & \partial & \longmapsto & \partial - g_\mu(x-c) \end{array} .$$

Proposition 2.1. *Let j_λ be as above. We consider $P \in W((x-c))$ with a regular singular point $x = c$. Suppose that P has the spectrum $(\lambda, \mathbf{m}) \in K^N/\mathbb{Z}^N \times \mathbb{Z}^N$ at $x = c$. Then if $\bar{\mu}$ is the image of $(\mu, \ldots, \mu) \in K^N$ into $K^N/\mathbb{Z}^N$, the spectrum of $j_\mu(P)$ is $(\lambda + \bar{\mu}, \mathbf{m})$ and*

$$\mathrm{C}_c(j_\mu(P))(t) = \mathrm{C}_c(P)(t-\mu).$$

Proof. Put $n = \operatorname{rank} P$. Let $A(P) \in M(n, K)$ be a matrix such that

$$\vartheta_c u = A(P)u \quad (u \in W((x-c))/W((x-c))P)$$

for a basis of $W((x-c))/W((x-c))P$ as $K((x-c))$-vector space. Then we show that there exists a basis of $W((x-c))/W((x-c))j_\mu(P)$ such that

$$\vartheta_c v = (A(P) + \mu I_n)v \quad (v \in W((x-c))/W((x-c))j_\mu(P))$$

where I_n is the unit matrix of size n.

We define a new action of $W((x-c))$ on $W((x-c))/W((x-c))P$ as follows:

$$(x-c) \circ u = j_\mu^{-1}(x-c)u,$$
$$\partial \circ u = j_\mu^{-1}(\partial)u$$

for $u \in W((x-c))/W((x-c))P$. Then

$$j_\mu \colon W((x-c))/W((x-c))P \longrightarrow W((x-c))/W((x-c))j_\mu(P)$$

gives a $W((x-c))$-module isomorphism. Let us compute the matrix of $j_\mu^{-1}(\vartheta_c)$. Since there exists $h \in W((x-c))$ such that $\mathrm{wt}_c(h) \geq 1$ and $j_\mu^{-1}(\vartheta_c) = \vartheta_c + \mu + h$, the matrix of $j^{-1}(\vartheta_c) = \vartheta_c + \mu + h$ is

$$A(P) + \mu I_n + H(x),$$

where $H(x) = h(x)I_n$. We can assume that any pair of eigenvalues of $A(P) + \mu I_n$ do not differ by an integer (see Theorem 5.2.2 in [13] for example). Then we can find a new basis of $W((x-c))/W((x-c))P$ and the matrix of $j^{-1}(\vartheta_c)$ is

$$A(P) + \mu I.$$

Since $\mathrm{wt}_c(g_\mu(x-c)-\mu(x-c)^{-1})\geq 0$, we can see $\mathrm{wt}\,_c(j_\mu(P))=\mathrm{wt}\,_c(P)$ and

$$\mathrm{C}_c(j_\mu(P))(t)=\mathrm{C}_c(P)(t-\mu).$$

$\square$

Definition 2.2 (The addition)**.** *For $f(x)\in K(x)$, we define the following K-algebra automorphism of $W(x)$,*

$$\begin{array}{rccc} \mathrm{Ad}\,(e^{\int f(x)\,dx})\colon & W(x) & \longrightarrow & W(x) \\ & x & \longmapsto & x \\ & \partial & \longmapsto & \partial - f(x) \end{array}.$$

Especially, the following automorphim,

$$\begin{array}{rccc} \mathrm{Ad}\,((x-c)^{\lambda})\colon & W(x) & \longrightarrow & W(x) \\ & x & \longmapsto & x \\ & \partial & \longmapsto & \partial - \frac{\lambda}{x-c} \end{array}$$

for $c,\lambda\in K$ is called the addition at $x-c$ with the parameter λ.

Definition 2.3 (The Fourier-Laplace transform)**.** *The Fourier-Laplace transform is the following K-algebra automorphism of $W[x]$,*

$$\begin{array}{rccc} \mathcal{L}\colon & w[x] & \longrightarrow & W[x] \\ & x & \longmapsto & -\partial \\ & \partial & \longmapsto & x \end{array}.$$

We recall how spectra change by the Fourier-Laplace transform. The following propositions are special cases of results of J. Fang and C. Sabbah ([6],[12]).

Proposition 2.4. *Let $c\in K$ be an unramified irregular singular point of $P\in W[x]$. Suppose that $P\in W[x]$ has the local factor $Q\in W((x-c))$ with the exponential factor $w(x-c)$, written by*

$$w(x)=w_nx^{-n}+w_{n-1}x^{-n+1}+\ldots+w_1x^{-1}\ (w_n\neq 0)$$

where $n\geq 1$. We write the spectrum of Q by $(\lambda,\mathbf{m})\in K^N/\mathbb{Z}^N\times\mathbb{Z}^N$.

Then there exist elements $\alpha_1,\ldots,\alpha_{n+1}\in K$ and distinct polynomials $g_1(x),\ldots,g_{n+1}(x)\in K[x]$ of $\deg g_i(x)=n$ with no constant terms. For these α_i and $g_i(x)$, the Laplace transform $\mathcal{L}(P)$ satisfies the following properties.

1. *The Laplace transform $\mathcal{L}(P)$ has a ramified irregular singular point at $x=\infty$.*

2. *The set of local factors at $x=\infty$ of $\mathcal{L}(P)$ contains $R_1,\ldots,R_{n+1}\in W_{(n+1)}((x^{-1}))$ which satisfy the following.*

- *Each exponential factors of R_i for $i = 1, \dots, n+1$ is*

$$-cx + g_i(x^{\frac{1}{n+1}}).$$

- *Each spectra of R_i is*

$$(\lambda + \bar{\alpha}_i, \mathbf{m}),$$

where $\bar{\alpha}_i$ is the image of $(\alpha_i, \dots, \alpha_i) \in K^N$ into $K^N/\mathbb{Z}^N$.

- *Each characteristic polynomials of R_i is*

$$C_\infty(R_i)(t) = C_c(Q)(t - \alpha_i).$$

- *These $g_i(x)$ and α_i depend only on w.*

Proof. The translation $x \mapsto x - c$ corresponds to $\mathrm{Ad}\,(e^{-cx}) : \partial \mapsto \partial + c$ by the Laplace transform. Thus it is enough to consider the case $c = 0$.

Let us define

$$P_w(x, \partial) = P(x, \partial - x^{-1}w(x))) \in W((x)).$$

If we write $P_w = \sum_{i=0}^{N} a_i(x)(x^{n+1}\partial)^i$, then

$$P = \sum_{i=0}^{N} a_i(x)(x^{n+1}\partial + x^n w(x)))^i.$$

Let us write $x^n w = \tilde{w}(x) = \sum_{i=0}^{n-1} w_i x^i$. Then the Laplace transform

$$\mathcal{L}(P) = \sum_{i=0}^{N} a_i(-\partial)((-\partial)^{n+1}x + \tilde{w}(-\partial))^i \in W[x].$$

Here we notice that $a_i(-\partial)$ are elements in the ring of formal microlocal differential operators, $E[x] = \{\sum_{i \geq r} b_i(x)\partial^{-i} \mid b_i \in K((x)), r \in \mathbb{Z}\}$.

Let us define the homomorphism from $W((x^{-1}))$ to $W((x))$ as follows,

$$\begin{array}{cccc} r_\alpha \colon & W((x^{-1})) & \longrightarrow & W((x)) \\ & x & \longmapsto & x^\alpha \\ & \partial & \longmapsto & \frac{1}{\alpha}x^{1-\alpha}\partial \end{array}$$

for a negative integer α. Then we need to show that there exist polynomials $g_1, \dots, g_{n+1}$ and $\alpha_1, \dots, \alpha_{n+1} \in K$ such that

$$\mathrm{wt}_0(\mathrm{Ad}\,(e^{\int -g_i(x^{-1})dx})r_{-n-1}\mathcal{L}P) = \mathrm{wt}_0(P_w)$$

and

$$\mathrm{C}_0(\mathrm{Ad}\,(e^{\int -g_i(x^{-1})dx})r_{-n-1}\mathcal{L}P)(t) = \mathrm{C}_0(P_w)(t - \alpha_i).$$

This shows the following. Let us put $\bar{g}_i(x^{-1}) = \int g_i(x^{-1})dx$, i.e.,

$$\partial \bar{g}_i(x^{-1}) = g_i(x^{-1}).$$

Then there exist local factors $R_i \in W_{(n+1)}((x^{-1}))(i = 1, \dots, n+1)$ of $\mathcal{L}(P)$ with exponential factors $\bar{g}_i(x)$ such that

$$C_\infty(R_i)(t) = C_0(Q)(t - \alpha_i).$$

Then Theorem 1.1 of J. Fang [6] and Theorem 5.1 of C. Sabbah [12] tell us that these R_i have the same spectral types of the local factor of Q, namely, spectra of R_i are

$$(\lambda + \bar{\alpha}_i, \mathbf{m}).$$

To compute characteristic polynomials of $\mathcal{L}(P)$, we prepare the following lemma.

Lemma 2.5. *For a polynomial $f \in K[x]$ of degree $n-1$, there exist $n+1$ polynomials $g_i(x) = \sum_{j=1}^{n} g_{i,j} x^{j+1}$ $(i = 1, \dots, n+1)$ and $\alpha_1, \dots, \alpha_{n+1} \in K$ and we have following.*

For $i = 1, \dots, n+1$ there exist $S_i(x, \partial), T_i(x, \partial) \in K[x, x^{-1}][\partial]$ such that $wt_0(S_i) > 1$, $\mathrm{wt}_0(T_i) > n$ and we have

$$\mathrm{Ad}\,(e^{\int -g_i(x^{-1})dx}) r_{-n-1} \partial = \frac{g_{i,n}}{n+1} x + S_i, \tag{3}$$

$$\mathrm{Ad}\,(e^{\int -g_i(x^{-1})dx}) r_{-n-1} ((-\partial)^{n+1} x + f(-\partial)) = (\frac{g_{i,n}}{n+1})^n x^{n+1} \partial - \alpha_i x^n + T_i. \tag{4}$$

proof of Lemma 2.5. We write $f(x) = \sum_{i=0}^{n-1} f_i x^i$ and put

$$M(x, \partial) = (-\partial)^{n+1} x + f(-\partial),$$

then for a $g(x) = \sum_{i=1}^{n} g_i x^{i+1} \in K[x]$,

$$\begin{aligned} &\mathrm{Ad}\,(e^{\int -g(x^{-1})dx}) r_{-n-1} M(x, \partial) \\ &\qquad = (\frac{1}{n+1} x^{n+2} \partial + \tilde{g}(x))^{n+1} x^{-n-1} + f(\frac{1}{n+1} x^{n+2} \partial + \tilde{g}(x)). \end{aligned}$$

Here $\tilde{g}(x) = \frac{1}{n+1} x^{n+2} (g(x^{-1})) = \sum_{i=1}^{n} \tilde{g}_i x^i \in K[x]$. Since $\mathrm{wt}_0(x^{n+2}\partial) = n+1$ and $\mathrm{wt}_0(\tilde{g}) = 1$, if we put

$$N_i(x, \partial) = (\frac{1}{n+1} x^{n+2} \partial + \tilde{g}(x))^i - \tilde{g}(x)^i$$

for $i = 1, \dots, n-1$, then $wt_0(N_i) = n + i$. We also put

$$N_n(x, \partial) = (\frac{1}{n+1} x^{n+2} \partial + \tilde{g}(x))^{n+1} x^{-n-1} - x^{-n-1} (\tilde{g}(x))^{n+1}.$$

Then $\mathrm{wt}_0(N_n) = n$. By these, we can write

$$M(x, \partial) = N_n + \sum_{i=1}^{n-1} f_i N_i + f_0 + x^{-n-1}(\tilde{g}(x))^{n+1} + \sum_{i=1}^{n-1} f_i(\tilde{g}(x))^i.$$

Let us put $(\tilde{g}(x))^i = \sum_{j=i}^{ni} G_j^{(i)} x^j$ for $i = 1, \ldots, n+1$. Then we can see that $G_i^{(i)} = (\tilde{g}_1)^i$ and $G_{i+k}^{(i)}$ are polynomials of $\tilde{g}_1, \ldots, \tilde{g}_k$ for $k = 1, \ldots, n-1$. Moreover for each $k = 1, \ldots, n-1$, $G_{i+k}^{(i)}$ is linear on $\tilde{g}_k$ and the coefficient of $\tilde{g}_k$ is the powers of $\tilde{g}_1$.

Then if we choose $\tilde{g}_i\,(i = 1, \ldots, n)$ so that they satisfy the following equations:

$$\begin{array}{l} G_{n+1}^{(n+1)} + f_0 = 0 \\ G_{n+2}^{(n+1)} + f_1 G_1^{(1)} = 0 \\ \cdots \\ G_{n+1+j}^{(n+1)} + f_1 G_j^{(1)} + \cdots + f_j G_j^{(j)} = 0 \ (j \leq n-1) \end{array}, \tag{5}$$

then the wt_0 of

$$f_0 + x^{-n-1}(\tilde{g}(x))^{n+1} + \sum_{i=1}^{n-1} f_i(\tilde{g}(x))^i$$

becomes less than or equal to n. Hence the wt_0 of

$$\mathrm{Ad}\,(e^{\int -g(x^{-1})dx}) r_{-n-1} M(x, \partial)$$

is n.

Let us notice that the first equation $(\tilde{g}_n)^{n+1} + f_0 = 0$ has $n+1$ solutions on K. Then if we fix a solution $\tilde{g}_n$, remaining $\tilde{g}_{n-1}, \ldots, \tilde{g}_1$ are inductively determined by the other equations.

The homogeneous component of weight n of $\mathrm{Ad}\,(e^{\int -g(x^{-1})dx}) r_{-n-1} M(x, \partial)$ is the sum of the weight n homogeneous element of $N_{n+1}(x, \partial) x^{n+1}$ and that of $f_0 + \tilde{g}(x)^{n+1} x^{-n-1} + \sum_{j=1}^{n-1} \tilde{g}(x)^j$. Hence there exists $\alpha \in K$ and the weight n homogeneous component of $\mathrm{Ad}\,(e^{\int -g(x^{-1})dx}) r_{-n-1} M$ is

$$(\tilde{g}_n)^n x^{n+1} \partial - \alpha x^n.$$

This tells us that

$$\mathrm{Ad}\,(e^{\int -g(x^{-1})dx}) r_{-n-1} M(x, \partial) = (g_n)^n x^{n+1} \partial - \alpha x^n + M'(x, \partial)$$

where $\mathrm{wt}_0 M'(x, \partial) \geq n+1$.

The other equation can be obtained similarly.

□

Let us take polynomials g_k for $k = 1, \dots, n+1$ which are obtained by this Lemma if we put $f = \tilde{w}$. Then we have

$$\mathrm{Ad}\,(e^{\int -g_k(x^{-1})dx})r_{-n-1}\mathcal{L}(P) = \sum_{i=0}^{N} a_i(M_1^k(x,\partial))(M_2^k(x,\partial))^i$$

where M_1^k and M_2^k are (3) and (4) in Lemma 2.5 respectively. Then we see $\mathrm{wt}_0(\mathrm{Ade}(-g_k(x^{-1}))r_{-n-1}\mathcal{L}P) = \mathrm{wt}_0(P_f)$ and

$$\mathrm{C}_0(\mathrm{Ad}\,(e^{\int -g_k(x^{-1})dx})r_{-n-1}\mathcal{L}P)(t) = \mathrm{C}_0(P_w)(t - \alpha_k').$$

Here $\alpha_k' = \alpha_k(\frac{g_{k,n}}{n+1})^{-1}$.

□

Proposition 2.6. *Let $x = \infty$ be an unramified irregular singular point of $P \in W[x]$. Suppose that $P \in W[x]$ has the local factor $Q \in W((x^{-1}))$ with the exponential factor*

$$w = w_n x^n + w_{n-1}x^{n-1} + \dots + w_1 x \ (w_n \neq 0)$$

where $n \geq 2$. We write the spectrum of Q by $(\lambda, \mathbf{m}) \in K^N/\mathbb{Z}^N \times \mathbb{Z}^N$.

Then there exist elements $\alpha_1, \dots, \alpha_{n-1} \in K$ and distinct polynomials $g_1(x), \dots, g_{n-1}(x) \in K[x]$ of $\deg g_i(x) = n$ with no constant terms. For these α_i and $g_i(x)$, the Laplace transform $\mathcal{L}(P)$ satisfies the following properties.

1. *The Laplace transform $\mathcal{L}(P)$ has a ramified irregular singular point at $x = \infty$.*
2. *The set of local factors at $x = \infty$ of $\mathcal{L}(P)$ contains $R_1, \dots, R_{n-1} \in W_{(n-1)}((x^{-1}))$ which satisfy the following.*
   - *Each exponents of R_i for $i = 1, \dots, n-1$ is*
     $$g_i(x^{\frac{1}{n-1}}).$$
   - *Each spectra of R_i is*
     $$(\lambda + \bar{\alpha}_i, \mathbf{m}),$$
     where $\bar{\alpha}_i$ is the image of $(\alpha_i, \dots, \alpha_i) \in K^N$ into $K^N/\mathbb{Z}^N$.
   - *Each characteristic polynomials of R_i is*
     $$C_\infty(R_i)(t) = C_c(Q)(t - \alpha_i).$$
   - *These $g_i(x)$ and α_i depend only on w.*

This proposition can be shown by the same argument as in Proposition 2.4. Also we notice that inversion formulas of these propositions can be obtained as well.

2.2 The primitive component

Let us take $P \in W[x]$. Then there exist integers r, N, and polynomials $p_i(t)$ such that we can write

$$P = \sum_{i=r}^{N} (x-c)^i p_i(\vartheta_c) \ (p_r(t) \neq 0)$$

for $c \in K$. Here we notice that the first term $p_r(t)$ is the characteristic polynomial of P at $x = c$, i.e.,

$$p_r(t) = \mathrm{C}_c(P)(t).$$

Lemma 2.7. *Let us consider*

$$P = \sum_{i=r}^{N} (x-c)^i p_i(\vartheta_c) \in W[x] \ (p_r(x) \neq 0)$$

as above.

For a positive integer s, we have that $(x-c)^{-s}P$ is still in $W[x]$ if and only if $r - s \geq 0$ or the following equations are satisfied for $m = s - r$,

$$\begin{aligned} &p_r(0) = p_r(1) = \cdots = p_r(m-1) = 0, \\ &p_{r+1}(0) = p_{r+1}(1) = \cdots = p_{r+1}(m-2) = 0, \\ &\cdots \\ &p_{r+m-1}(0) = 0. \end{aligned} \tag{6}$$

Here we put $p_j(t) = 0$ for $j > N$.

Proof. We consider only the case $c = 0$. If equations (6) are satisfied, we have

$$\begin{aligned} x^{r+i} p_{r+i}(\vartheta) &= x^{r+i} \vartheta(\vartheta - 1) \cdots (\vartheta - m + i + 1) \tilde{p}_{r+i}(\vartheta) \\ &= x^{r+i} x^{m-i} \partial^{m-i} \tilde{p}_{r+i}(\vartheta) = x^{r+m} \partial^{m-i} \tilde{p}_{r+i}(\vartheta) \end{aligned}$$

for $i = 0, 1, \ldots, m-1$ where $\tilde{p}_{r+i} \in K[x]$. Thus we have $x^{-(r+m)}P \in W[x]$.

Conversely, let us suppose that $x^{-s}P \in W[x]$. We can write $x^{-s}P = \sum_{i=0}^{N} x^{i-m} p_{r+i}(\vartheta)$. Since weights of $x^{i-m}p_{r+i}(\vartheta)$ are $i - m$, they are linear combinations of $x^\alpha \partial^{\alpha+m-i}$ $(\alpha \geq 0)$ for $i = 0, \ldots, m$. Recalling that

$$x^\alpha \partial^{\alpha+m-i} = \vartheta(\vartheta - 1) \cdots (\vartheta - \alpha + 1) \partial^{m-i}$$

for $i = 0, 1, \ldots, m$, we can write

$$\begin{aligned} x^{i-m} p_{r+i}(\vartheta) &= \bar{p}_{r+i}(\vartheta) \partial^{m-i} \\ &= \partial^{m-i} \bar{p}_{r+i}(\vartheta - m + i) \\ &= x^{i-m} \vartheta(\vartheta - 1) \cdots (\vartheta - m + i + 1) \bar{p}_{r+i}(\vartheta - m + i) \end{aligned}$$

for $i = 0, 1, \ldots, m$. Here $\bar{p}_{r+1} \in K[x]$. Thus we have equations (6). □

Definition 2.8 (The primitive component)**.** *We say that* $P = \sum_{i=0}^{n} a_i(x)\partial^n \in W[x]$ *is primitive if*

1. $\gcd_{K[x]}\{a_i(x) \mid i = 0, \ldots, n\} = 1,$
2. *the highest term* $a_n(x)$ *is monic.*

For $P \in W(x)$*, there exist* $f(x) \in K(x)$ *and the primitive element* $\tilde{P} \in W[x]$*, and then we can decompose* P *as*

$$P = f(x)\tilde{P},$$

uniquely.

We denote this primitive element by $\mathrm{Prim}(P)$ *and call this the primitive component of* P*.*

Proposition 2.9 (Cf. Tsai [15])**.** *Let us consider* $P \in W[x]$ *which has singular points* $c_1, \ldots, c_p \in K$ *and no other singular points in* K*. At each* $x = c_i$ $(i = 1, \ldots, p)$*, we write*

$$P = \sum_{j=r_i}^{N_i} (x - c_i)^j p_j^{(i)}(\vartheta_{c_i})$$

by integers r_i*,* N_i*, and polynomials* $p_j^{(i)}(t)$ $(p_{r_i}^{(i)}(t) \neq 0)$*. Let us suppose that there exist positive integers* m_i *for* $i = 1, \ldots, p$ *such that*

$$\begin{aligned} &p_{r_i}^{(i)}(0) = p_{r_i}^{(i)}(1) = \cdots = p_{r_i}^{(i)}(m_i - 1) = 0, \\ &p_{r_i+1}^{(i)}(0) = p_{r_i+1}^{(i)}(1) = \cdots = p_{r_i+1}^{(i)}(m_i - 2) = 0, \\ &\cdots \\ &p_{r_i+m_i-1}^{(i)}(0) = 0. \end{aligned} \tag{7}$$

Here we put $p_j^{(i)}(t) = 0$ *for* $j > N_i$*. Moreover we assume that characteristic polynomials* $\mathrm{C}_{c_i}(P)(t) = p_{r_i}^{(i)}(t)$ *have no integer root* $\geq m_i$ *for* $i = 1, \ldots, p$*.*

Then if P *is irreducible in* $W(x)$*, i.e.,* P *generates the maximal left ideal of* $W(x)$*, then the primitive component* $\mathrm{Prim}\,(P)$ *of* P *generates the maximal ideal of* $W[x]$*.*

Proof. This is a direct consequence of Corollary 5.5 in [15]. □

Proposition 2.10 (Oshima [11])**.** *Let us take* $P = \sum_{i=r}^{N} (x - c)^i p_i(\vartheta_c) \in W[x]$ $(p_r(t) \neq 0)$ *as above. Also we take* $m_1, \ldots, m_s \in \mathbb{Z}_{>0}$ *and* $\lambda_1, \ldots, \lambda_s \in K$ *which satisfy*

$$\lambda_i - \lambda_j \notin \mathbb{Z} - \{0\} \ (i \neq j).$$

Let us suppose that the characteristic polynomial $\mathrm{C}_c(P)(t) = p_r(t)$ *is decomposed as follows,*

$$p_r(t) = C \prod_{i=1}^{s} \prod_{j=0}^{m_i-1} (t - (\lambda_i + j))$$

for a constant C*. Then the following are equivalent.*

1. *We have equations*

$$\begin{aligned} &p_r(\lambda_i) = p_r(\lambda_i + 1) = \cdots = p_r(\lambda_i + m_i - 1) = 0, \\ &p_{r+1}(\lambda_i) = p_{r+1}(\lambda_i + 1) = \cdots = p_{r+1}(\lambda_i + m_i - 2) = 0, \\ &\cdots \\ &p_{r+m_i-1}(\lambda_i) = 0. \end{aligned}$$

   for all $i = 1, \ldots, s$*.*

2. *There exists the local factor* P_{loc} *of* P *at* $x = c$ *such that*

$$\mathrm{C}_c(P_{loc})(t) = \mathrm{C}_c(P)(t)$$

   and the spectrum is

$$((\bar{\lambda}_1, \ldots, \bar{\lambda}_s), (m_1, \ldots, m_s))$$

   where $\bar{\lambda}_j$ *is the image of* λ_j *into* $K/\mathbb{Z}$*.*

Proof. This follows from Proposition 6.14 in [11]. □

Definition 2.11 (The spectral data)**.** *Let us take* $m_1, \ldots, m_s \in \mathbb{Z}_{>0}$ *and* $\lambda_1, \ldots, \lambda_s \in K$ *which satisfy*

$$\lambda_i - \lambda_j \notin \mathbb{Z} \ (i \neq j).$$

We say $P \in W(x)$ *has the spectral data*

$$\{(\lambda_1, \ldots, \lambda_s); (m_1, \ldots, m_s)\},$$

at $x = c$*, if* $P \in W(x)$ *has a regular singular point at* $x = c$ *and satisfies the following,*

1. $$\mathrm{C}_c(P)(t) = C \prod_{i=1}^{s} \prod_{j=0}^{m_i-1} (t - (\lambda_i + j))$$

   for a constant C*,*

2. *the spectrum is*

$$((\bar{\lambda}_1, \ldots, \bar{\lambda}_s), (m_1, \ldots, m_s))$$

   where $\bar{\lambda}_j$ *is the image of* λ_j *into* $K/\mathbb{Z}$*.*

Lemma 2.12. *Let $P \in W[x]$ be a primitive element, i.e.,* $\mathrm{Prim}(P) = P$. *We assume that $x = c \in K$ is an unramified singular point of P and P has the local factor P_{loc} with the spectral data*

$$\{(0, \lambda_1, \ldots, \lambda_l); (m_0, m_1, \ldots, m_l)\} \tag{8}$$

at $x = c$. Then $Q = \mathrm{Prim}(\mathrm{Ad}(x^{-\lambda_1})P)$ has the local factor with the spectral data

$$\{(-\lambda_1, 0, \lambda_2 - \lambda_1, \ldots, \lambda_l - \lambda_1); (m_0, m_1, \ldots, m_l)\}$$

at $x = c$.

In particular, if P has the local factor P_{loc} at $x = c$ with the exponential factor $w_{loc} = 0$ and the spectral data (8), then

$$\deg Q - \deg P = m_0 - m_1.$$

Proof. The first assertion follows from Proposition 2.1. We may assume $c = 0$. Let us write $P = \sum_{i=0}^{n} p_i(x)\partial^i$, $Q = \sum_{i=0}^{n} q_i(x)\partial^i$. Then there exists $r(x) \in K[x]$ satisfying $r(0) \neq 0$ such that $p_n(x) = x^M r(x)$ and $q_n(x) = x^N r(x)$ for some $N, M \in \mathbb{Z}_{>0}$. On the other hand, let us note that both of Newton polygons of P and Q have the same shapes. Moreover Lemma 2.7 tells us that $N_0(Q)$ can be obtained by moving $N_0(P)$ to the vertical direction $m_0 - m_1$. Thus $N - M = m_0 - m_1$. From the equation (1), there are $s, s' \in \mathbb{Z}$ and

$$\deg P = \deg p_n(x) + s,$$
$$\deg Q = \deg q_n(x) + s'.$$

By the definition of s and s', they are invariant by $\mathrm{Ad}(x^{-\lambda_1})$, i.e., $s = s'$. Hence

$$\deg Q - \deg P = \deg q_n(x) - \deg p_n(x) = N - M = m_0 - m_1.$$

□

2.3 The Fourier-Laplace transform of rank 1 irregular singular point at infinity

Proposition 2.13. *Let us take a primitive element $P \in W[x]$. If P has the local factor with the exponential factor* 0 *and the spectral data*

$$\{(0, \lambda_1, \ldots, \lambda_l); (m_0, m_1, \ldots, m_l)\}$$

at $x = c$, then $Q = \mathcal{L}(P)$ has the local factor $Q_{loc} \in W((x^{-1}))$ with the exponential factor $-cx$ and the spectral data

$$\{(\lambda_1 - 1, \ldots, \lambda_l - 1); (m_1, \ldots, m_l)\}$$

at $x = \infty$.

Proof. It is enough to consider the case $c = 0$. Let us write $P = \sum_{i=r}^{N} x^i p_i(\vartheta)$. Lemma 2.7 and Proposition 2.10 tell that $\mathrm{wt}_0(P) = r = -m_0$ and

$$\begin{aligned} x^{-m_0+i} p_{-m_0+i}(\vartheta) &= x^{-m_0+i} \vartheta \cdots (\vartheta - m_0 + i + 1) \bar{p}_{-m_0+i}(\vartheta) \\ &= x^{-m_0+i} x^{m_0-i} \partial^{m_0+i} \bar{p}_{-m_0+i}(\vartheta) \\ &= \partial^{m_0-i} \bar{p}_{-m_0+i}(\vartheta), \end{aligned}$$

for $i = 0, \dots m_0 - 1$. Here $\bar{p}_j(t)$ are polynomials. Since $\mathrm{wt}_0(P) = \mathrm{wt}_\infty(\mathcal{L}(P))$, we can see

$$\mathcal{L}(P) = \sum_{i=-m_0}^{N} x^{-i} \bar{p}_i(-\vartheta + 1).$$

Then the proposition follows from Proposition 2.10. □

Similarly we have the following.

Proposition 2.14. *Let us take $P \in W[x]$. Suppose that P has the local factor at infinity with the exponential factor cx and the spectral data*

$$\{(\lambda_1, \dots, \lambda_l); (m_1, \dots, m_l)\}$$

for $\lambda_i \in K \backslash \mathbb{Z}$ $(i = 1, \dots, l)$. Moreover we assume that

$$\mathrm{wt}_\infty(\mathrm{Ad}(e^{-cx})P) = -m_0 < 0.$$

Then $\mathcal{L}(P)$ has the local factor with the exponential factor 0 and the spectral data

$$\{(0, \lambda_1 + 1, \dots, \lambda_l + 1); (m_0, m_1, \dots, m_l)\}$$

at $x = c$.

Proof. This follows from the same argument as in Proposition 2.13. □

2.4 The Euler transform

From results shown in the previous sections, we shall compute explicit changes of spectra caused by the Euler transform.

Definition 2.15 (The Euler transform, see [11])**.** *The Euler transform of $P \in W(x)$ with the parameter λ is defined by*

$$E(\lambda)P = \mathcal{L} \circ \mathrm{Prim} \circ \mathrm{Ad}\,(x^\lambda) \circ \mathcal{L}^{-1} \circ \mathrm{Prim}(P) \in W[x].$$

Remark 2.16. *This definition is an analogue of the following classical description of Euler transform:*

$$I_c^\mu g(z) = \frac{1}{\Gamma(\mu)} \int_c^z g(x)(z-x)^{\mu-1}\, dx = \int_{-i\infty}^{i\infty} y^{-\mu} \int_c^\infty g(x) e^{-xy}\, dx\, e^{zy}\, dy.$$

Remark 2.17 (A comparison with the Katz middle convolution)**.** *Although we only deal with differential operators with polynomial coefficients, this can be seen as a special case of $\mathscr{D}$-module setting which is investigated by N. Katz for the Fuchsian case ([8]) and D. Arinkin for the irregular singular case ([1]). In the $\mathscr{D}$-module case, the middle convolution plays the same role as the Euler transform in this paper. Let us see the relationship between our Euler transform and the middle convolution by Katz. We follow Arinkin's paper [1] for the definition of the middle convolution.*

Let us take $P \in W(x)$ with singular points $c_1, \dots, c_p \in K$ and $c_0 = \infty$. As we see in Proposition 2.9, the result of Tsai (see [15]) tells us that if P is irreducible in $W(x)$ and satisfies the conditions in this proposition, then $\mathrm{Prim}\,(P)$ *generates the maximal ideal in $W[x]$. Thus we can see that*

$$W(x)/W(x)P \to W[x]/W[x]\mathrm{Prim}\,(P)$$

gives an analogue of the minimal extension (Deligne-Goresky-Macpherson extension) in the $\mathscr{D}$-module setting. We also notice that tensoring a 1-dimensional local system corresponds to $\mathrm{Ad}(x^\mu)$ for some $\mu \in K$ in our setting. Hence under the suitable assumptions, we can say that our Euler transform agrees with the middle convolution.

Theorem 2.18. *Let us consider $P \in W(x)$ which satisfies the following.*

1. *All singular points $c_0 = \infty, c_1, \dots, c_p \in K$ are unramified.*
2. *At these c_i $(i = 0, \dots, p)$, let us write local factors of P by $P_{i,1}, \dots, P_{i,l_i} \in W((x - c_i))$. Exponential factors of $P_{i,j}$ are written by $w_{i,j}$ which are polynomials of $\frac{1}{x-c_i}$ with no constant terms.*

For this $P \in W(x)$, we assume the following.

1. *For each $c_i \in K$ $(i = 1, \dots, p)$, the exponential $w_{i,1} = 0$ and the corresponding local factor $P_{i,1}$ has the spectral data*

$$\{(0, \lambda_1^i, \dots, \lambda_{k_i}^i); (m_0^i, m_1^i, \dots, m_{k_i}^i)\}.$$

For c_0, we assume $w_{0,1} = 0$ and if the exponential factor $\deg_{K[x]} w_{0,j} \leq 1$ then the corresponding local factor $P_{0,j}$ has the spectral data

$$\{\lambda_j; \mathbf{m}_j\}$$

where each components of λ_j is not an integer. Especially we write

$$\lambda_1 = (\lambda_1^0, \dots, \lambda_{k_0}^0), \quad \mathbf{m}_1 = (m_1^0, \dots, m_{k_0}^0)$$

for the local factor $P_{0,1}$ with the exponential factor $w_{0,1} = 0$.

2. *For $i = 1, \ldots, p$, we assume*

$$\lambda^i_j + \lambda^0_1 \notin \mathbb{Z} \quad (j = 1, \ldots, k_i).$$

Under these conditions, we have the following facts for $E(1-\lambda^0_1)P$. Let us put $\mu = 1 - \lambda^0_1$ for the simplicity.

(i) *The rank of $E(\mu)P$ is*

$$\operatorname{rank} E(\mu)P = \operatorname{rank} P + d$$

where

$$d = \deg \operatorname{Prim}(P) - \sum_{j=1}^{k_0} m^0_j - m^0_1.$$

(ii) *At c_i ($i = 0, \ldots, p$), let us write local factors of $E(\mu)P$ by $\tilde{P}_{i,1}, \ldots, \tilde{P}_{i,\tilde{l}_i}$ and corresponding exponential factors by $\tilde{w}_{i,1}, \ldots, \tilde{w}_{i,\tilde{l}_i}$. Then*

$$\tilde{l}_i = l_i \ (i = 0, \ldots, p),$$

and

$$\tilde{w}_{i,j} = w_{i,j} \ (i = 0, \ldots, p,\ j = 1, \ldots, l_i).$$

Moreover for c_i ($i = 1, \ldots, p$), each local factors $\tilde{P}_{i,j}$ ($j = 2, \ldots, l_i$) has the characteristic polynomial

$$\mathrm{C}_{c_i}(\tilde{P}_{i,j})(t) = \mathrm{C}_c(P_{i,j})(t + \mu_{i,j})$$

where

$$\mu_{i,j} = (\mathrm{wt}_{\,c_i}(w_{i,j}) + 1)(1 - \lambda^0_1)$$

and has the spectrum

$$(\lambda(\tilde{P}_{i,j}), \mathbf{m}(\tilde{P}_{i,j})) = (\lambda(P_{i,j}) - \mu_{i,j}, \mathbf{m}(P_{i,j}))$$

where $(\lambda(P_{i,j}), \mathbf{m}(P_{i,j})) \in K^{k_{i,j}}/\mathbb{Z}^{k_{i,j}} \times \mathbb{Z}^{k_{i,j}}$ ($k_{i,j} \in \mathbb{Z}_{>0}$) is the spectrum of $P_{i,j}$. In particular, the local factors $\tilde{P}_{i,1}$ for each $i = 1, \ldots, p$ has the spectral data

$$\{(0, \lambda^i_1 - \mu, \ldots, \lambda^i_{k_i} - \mu); (m^i_0 + d, m^i_1, \ldots, m^i_{k_i})\}.$$

On the other hand, for c_0 each local factors $\tilde{P}_{0,j}$ ($j = 2, \ldots, l_0$) has the characteristic polynomial

$$\mathrm{C}_{c_0}(\tilde{P}_{0,j})(t) = \mathrm{C}_c(P_{0,j})(t + \mu_{0,j})$$

where

$$\mu_{0,j} = (\mathrm{wt}_{\,c_0}(w_{0,j}) - 1)(1 - \lambda^0_1)$$

and has the spectrum

$$(\lambda(\tilde{P}_{0,j}), \mathbf{m}(\tilde{P}_{0,j})) = (\lambda(P_{0,j}) - \mu_{0,j}, \mathbf{m}(P_{0,j})).$$

In particular, the local factor $\tilde{P}_{0,1}$ *has the spectral data*

$$\{(1+\mu, \lambda_2^0 + \mu, \ldots, \lambda_{k_0}^0 + \mu); (m_1^0 + d, m_2^0, \ldots, m_{k_0}^0)\}.$$

Proof. We know that $\mathcal{L}^{-1}(\mathrm{Prim}\,(P))$ has the spectral data

$$\{(0, \lambda_1^0 - 1, \ldots, \lambda_{k_0}^0 - 1; (N_0, m_1^0, \ldots, m_{k_0}^0))\}$$

at $x = 0$, where $N_0 = \deg \mathrm{Prim}\,(P) - \sum_{j=1}^{k_0} m_j^0$ by Proposition 2.14. We note that $\mathcal{L}^{-1}(P)$ is a primitive element. Indeed if there exist $f(x)(\neq 0) \in K[x]$ and $R \in W[x]$ such that

$$\mathcal{L}^{-1}(\mathrm{Prim}\,(P)) = f(x)R,$$

then P can be divided

$$f(-\partial) = C(\partial - \alpha_1) \cdots (\partial - \alpha_k)$$

for some constants $C, \alpha_1, \ldots, \alpha_k \in K$. However this means that P has local factors with exponential factors $\alpha_j x$ ($\deg_{K[x]} \alpha_j x \leq 1$) and characteristic polynomials of these local factors have integer roots. This contradicts the assumption.

Hence $Q_\mu = \mathrm{Prim} \circ \mathrm{Ad}(x^\mu) \circ \mathcal{L}^{-1} \circ \mathrm{Prim}\,(P)$ has the spectral data

$$\{(\mu, 0, \ldots, \lambda_{k_0}^0 + \mu - 1); (N_0, m_1^0, \ldots, m_{k_0}^0)\}$$

and

$$\begin{aligned} \deg Q_\mu &= \deg \mathcal{L}^{-1}(P) + N_0 - m_1^0 \\ &= \mathrm{rank}\, P + N_0 - m_1^0 \end{aligned}$$

by Lemma 2.12. Thus

$$\mathrm{rank}\, E(\mu)P = \deg Q_\mu = \mathrm{rank}\, P + N_0 - m_1^0.$$

This shows (i). Also we see that $E(\mu)P$ has the local factor with the spectral data

$$\{(\mu + 1, \lambda_2^0 + \mu, \ldots, \lambda_{k_0}^0 + \mu); (N_0, m_2^0, \ldots, m_{k_0}^0)\}$$

at $x = 0$.

By Proposition 2.4, Proposition 2.6, and these inversion formula, we see that $E(\mu)P$ has local factors $\tilde{P}_{i,j}$ ($i = 0, \ldots, p$, $j = 2, \ldots, k_i$) as in the statement.

Similarly, for finite singular points c_i $(i = 1, \ldots, p)$ we can see that there exist integers N_0^i and $E(\mu)P$ has local factors with spectral data

$$\{(0, \lambda_1^i - \mu, \ldots, \lambda_{k_i}^i - \mu); (N_0^i, m_2^i, \ldots, m_{k_i}^i)\}.$$

By Proposition 2.14 we can see that

$$m_0^i = -\mathrm{wt}_\infty(\mathrm{Ad}\,(e^{cx})\mathcal{L}^{-1}(\mathrm{Prim}\,(P))), \qquad N_0^i = -\mathrm{wt}_\infty(\mathrm{Ad}\,(e^{cx})Q_\mu).$$

If we write $\mathcal{L}^{-1}(\mathrm{Prim}\,(P)) = \sum_{i=0}^N a_i(x)\partial^N$ and $Q_\mu = \sum_{i=0}^N b_i(x)\partial^i$, then by the equation (2) we have

$$\begin{aligned} m_0^i &= \deg_{K[x]} a_N(x) + t \\ N_0^i &= \deg_{K[x]} b_N(x) + t' \end{aligned}$$

for some integers t, t'. Recalling that Q_λ is obtained by applying $\mathrm{Ad}\,(x^\mu)$ to $\mathcal{L}^{-1}(\mathrm{Prim}\,(P))$ and the above t is invariant by $\mathrm{Ad}(x^\mu)$, we see that

$$t = t'.$$

Thus

$$\begin{aligned} N_0^i - m_0^i &= \deg_{K[x]} b_N(x) - \deg_{K[x]} a_N(x) = \deg Q_\mu - \deg \mathcal{L}^{-1}(\mathrm{Prim}\,P) \\ &= N_0 - m_1^0. \end{aligned}$$

□

3 The Euler transform and the Weyl group action of a Kac-Moody root system

In the previous section, we compute the Euler transform explicitly. At first glance, this computation is very complicated, hence we would like to understand this in more sophisticated way.

We shall investigate the relationship between the Euler transform and the action of the Weyl group of a Kac-Moody Lie algebra. Also we show the correspondence between differential operators and elements of the root lattice of the Kac-Moody Lie algebra. Moreover we define an analogue of the root system of this Kac-Moody root system and show that if a differential operator is irreducible, then corresponding element becomes a root of our generalized root system.

3.1 The working hypothesis

In the remaining of this paper, we keep the following assumptions. We consider the $P \in W(x)$ which satisfies following assumptions.

1. Singular points of P are $c_0 = \infty, c_1, \ldots, c_p \in K$. All these are unramified singular points.

2. Let us write the set of local factors of P at $x = c_i$ by $\{P_{i,1}, \ldots, P_{i,k_i}\}$. Then there exist positive integers $m_{i,j,s}$ and $\lambda_{i,j,s} \in K$ for $i = 0, \ldots, p$, $j = 1, \ldots, k_i$, $s = 1, \ldots, l_{i,j}$ such that local factors $P_{i,j}$ have spectral data

$$\{(\lambda_{i,j,1}, \ldots, \lambda_{i,j,l_{i,j}}); (m_{i,j,1}, \ldots, m_{i,j,l_{i,j}})\}$$

   respectively.

We write each exponential factors of $P_{i,j}$ by $w_{i,j}$ respectively.

Furthermore we shall discuss in a generic setting, that is, we regard above $\lambda_{i,j,s}$ as independent indeterminants which satisfy only one relation, so-called Fuchs relation ([3],[2]). Let us write $K(\lambda)$ the field generated by these $\lambda_{i,j,s}$ for $i = 0, \ldots, p$, $j = 1, \ldots, k_i$, $s = 1, \ldots, l_{i,j}$ and fix an algebraic closure Λ of $K(\lambda)$. We denote $W_\Lambda[x]$ and $W_\Lambda(x)$ rings of differential operators with coefficients in $\Lambda[x]$ and $\Lambda(x)$ respectively.

Let us define the subset

$$\mathcal{U}_\Lambda(P)$$

of $W_\Lambda(x)$ whose elements satisfy the following.

1. Singular points of $Q \in \mathcal{U}_\Lambda(P)$ are $c_0 = \infty, c_1, \ldots, c_p \in K$. All these are unramified singular points.

2. Let us write the set of local factors of Q at $x = c_i$ by $\{Q_{i,1}, \ldots, Q_{i,k_i}\}$. Then there exist positive integers $m'_{i,j,s}$ and $\lambda'_{i,j,s} \in \Lambda$ for $i = 0, \ldots, p$, $j = 1, \ldots, k_i$, $s = 1, \ldots, l_{i,j}$ such that local factors $Q_{i,j}$ have spectral data

$$\{(\lambda'_{i,j,1}, \ldots, \lambda'_{i,j,l_{i,j}}); (m'_{i,j,1}, \ldots, m'_{i,j,l_{i,j}})\}$$

   respectively.

   Here we allow that $m'_{i,j,s} = 0$ and $Q_{i,j,s} = 0$ for some (i,j,s). Let us write

$$\lambda(Q) = \prod_{i=0}^{p} \prod_{j=1}^{k_i} (\lambda'_{i,j,1}, \ldots, \lambda'_{i,j,l_{i,j}}),$$

$$\mathbf{m}(Q) = \prod_{i=0}^{p} \prod_{j=1}^{k_i} (m'_{i,j,1}, \ldots, m'_{i,j,l_{i,j}}).$$

3. Exponential factors of $Q_{i,j}$ are $w_{i,j}$ which are same exponential factors of local factors $P_{i,j}$ of P.

Here we note that $k_i, l_{i,j}$ $(i = 1,\dots,p,\ j = 1,\dots,k_i)$ used in the above are same one used for P. We define the product of indices

$$\mathcal{T}(P) = \prod_{i=0}^{p} \{(i,j) \mid j = 1,\dots,k_i\}.$$

3.2 The lattice transformations induced from the Euler transform

Definition 3.1 (The twisted Euler transform)**.** *Let us consider $Q \in \mathcal{U}_\lambda(P)$ whose spectral data are*

$$\{(\lambda(Q)_{i,j,s}); (m(Q)_{i,j,s})\} \quad (i = 0,\dots,p,\ j = 1,\dots,k_i,\ s = 1,\dots,l_{i,j})$$

at c_i. Then for $t = (t_0,\dots,t_p) \in \mathcal{T}(P)$, we define the twisted Euler transform $E(t)Q$ by

$$\begin{aligned} E(t)Q = &\prod_{i=0}^{p} \mathrm{Ad}\,(e^{w_{t_i}}) \prod_{i=1}^{p} \mathrm{Ad}((x-c_i)^{\lambda(Q)_{t_i,1}}) \\ &\circ E(1-\lambda(Q;t)) \prod_{i=1}^{p} \mathrm{Ad}((x-c_i)^{-\lambda(Q)_{t_i,1}}) \prod_{i=0}^{p} \mathrm{Ad}(e^{-w_{t_i}}) Q \end{aligned}$$

where

$$\lambda(Q;t) = \sum_{i=0}^{p} \lambda(Q)_{t_i,1}.$$

Theorem 3.2. *For $t \in \mathcal{T}(P)$, we have $E(t)P \in \mathcal{U}_\Lambda(P)$. If we write the spectral data of $Q_t = E(t)P$ by*

$$\{(\lambda(Q_t)_{i,j,s}); (m(Q_t)_{i,j,s})\} \quad (i = 0,\dots,p,\ j = 1,\dots,k_i,\ s = 1,\dots,l_{i,j})$$

at each c_i, then we have

$$\begin{aligned} m(Q_t)_{i,j,1} &= m_{i,j,1} + d(t) && \text{if } (i,j) = t_i, \\ m(Q_t)_{i,j,s} &= m_{i,j,s} && \text{otherwise}, \end{aligned}$$

where

$$\begin{aligned} d(t) = &\sum_{i=1}^{p} \sum_{j=1}^{k_i} (\mathrm{wt}_{c_i}(w_{i,j} - w_{t_i}) + 1) \sum_{s=1}^{l_{i,j}} m_{i,j,s} \\ &+ \sum_{j=1}^{k_0} (\mathrm{wt}_{c_0}(w_{0,j} - w_{t_0}) - 1) \sum_{s=1}^{l_{0,j}} m_{0,j,s} - \sum_{i=0}^{p} m_{t_i,1}. \end{aligned}$$

Also we have that for $i = 1, \dots, p$, $j = 1, \dots, k_i$, $s = 1, \dots, l_{i,j}$,

$$\lambda(Q_t)_{i,j,s} = \begin{cases} \lambda_{i,j,s} & \textit{if } t_i = (i,j) \textit{ and } s = 1, \\ \lambda_{i,j,s} - (\mathrm{wt}_{c_i}(w_{i,j} - w_{t_i}) + 1)(1 - \lambda(P;t)) & \textit{if otherwise,} \end{cases}$$

and for $i = 0$, $j = 1, \dots, k_0$, $s = 1, \dots, l_{0,j}$,

$$\lambda(Q_t)_{0,j,s} = \begin{cases} \lambda_{t_0,1} + 2(1 - \lambda(P;t)) & \textit{if } t_0 = (0,j) \textit{ and } s = 1, \\ \lambda_{0,j,s} - (\mathrm{wt}_{c_0}(w_{0,j} - w_{t_0}) - 1)(1 - \lambda(P;t)) & \textit{if otherwise.} \end{cases}$$

Proof. First we note that if we write $P = \sum_{i=0}^{n} a_n(x)\partial^i$, then we may assume $a_n(x)$ has zeros only at $x = c_1, \dots, c_p$.

For $t \in \mathcal{T}(P)$, Let us consider $\tilde{P}_t = \prod_{i=0}^{p} \mathrm{Ad}(e^{-w_{t_i}})P$ and write $P_t = \mathrm{Prim}\,(\tilde{P}_t) = \sum_{i=0}^{n} b_i(x)\partial^i$. Then

$$\deg_{\Lambda[x]} b_n(x) = \sum_{i=1}^{p}\sum_{j=1}^{k_i}(\mathrm{wt}_{c_i}(w_{i,j} - w_{t_i}) + 1)\sum_{s=1}^{l_{i,j}} m_{i,j,s}$$

from the Newton polygon at each $x = c_1, \dots, c_p$. Thus for

$$\tilde{P}'_t = \prod_{i=1}^{p} \mathrm{Ad}((x - c_i)^{-\lambda(P)_{t_i,1}}) \prod_{i=1}^{p} \mathrm{Ad}(e^{-w_{t_i}})P,$$

if we write $P'_t = \mathrm{Prim}\,(\tilde{P}'_t) = \sum_{i=0}^{n} c_n(x)\partial^i$, then

$$\deg_{\Lambda[x]} c_n(x) = \deg_{\Lambda[x]} b_n(x) - \sum_{i=1}^{p} m_{t_i,1}$$

by Lemma 2.12. Hence we have

$$\begin{aligned}
&\deg P'_t - \sum_{s=1}^{l_{t_0}} m_{t_0,s} - m_{t_0,1} \\
&\quad = \sum_{j=1}^{p}\sum_{j=1}^{k_i}(\mathrm{wt}_{c_i}(w_{i,j} - w_{t_i}) + 1)\sum_{s=1}^{l_{i,j}} m_{i,j,s} \\
&\qquad + \sum_{j=1}^{k_0}(\mathrm{wt}_{c_0}(w_{0,j} - w_{t_0}) - 1)\sum_{s=1}^{l_{0,j}} m_{0,j,s} - \sum_{i=0}^{p} m_{t_t,1}
\end{aligned}$$

by the equation (1).

Thus this follows from Theorem 2.18. □

Let us define the following $\mathbb{Z}$-lattice from P,

$$L(P) = \{\prod_{i=0}^{p}\prod_{j=1}^{k_i} \mathbf{a}_{i,j} \,|\mathbf{a}_{i,j} = (a_{i,j,1},\dots,a_{i,j,l_{i,j}}) \in \mathbb{Z}^{l_{i,j}},$$
$$\sum_{j=1}^{k_i}\sum_{s=1}^{l_{0,j}} a_{0,j,s} = \cdots = \sum_{j=1}^{k_i}\sum_{s=1}^{l_{p,j}} a_{p,j,s}\}.$$

We define the rank of $\mathbf{a} = \prod_{i=0}^{p}\prod_{j=1}^{k_i}(a_{i,j,1},\dots,a_{i,j,l_{i,j}}) \in L(P)$ by

$$\mathrm{rank}\,(\mathbf{a}) = \sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}} a_{i,j,s}$$

for any $i = 0,\dots,p$.

We can see that $\mathbf{m}(P) = \prod_{i=0}^{p}\prod_{j=1}^{k_i}(m_{i,j,1},\dots,m_{i,j,l_{i,j}}) \in L(P)$. As we show in Theorem 3.2, if we put $Q_t = E(t)P$, then we can also see that

$$\mathbf{m}(Q_t) = \prod_{i=0}^{p}\prod_{j=1}^{k_i}(m(Q_t)_{i,j,1},\dots,m(Q_t)_{i,j,l_{i,j}}) \in L(P).$$

Thus $E(t)$ define transformations of $L(P)$ for $t \in \mathcal{T}(P)$ as follows. For $t = (t_0,\dots,t_p) \in \mathcal{T}(P)$, we define $\mathbb{Z}$-endomorphism of $L(P)$ by

$$\begin{array}{cccc} \sigma(t)\colon & L(P) & \longrightarrow & L(P) \\ & \mathbf{a} = \prod_{i=0}^{p}\prod_{j=1}^{k_i}(a_{i,j,1},\dots,a_{i,j,l_{i,j}}) & \longmapsto & \prod_{i=0}^{p}\prod_{j=1}^{k_i}(\tilde{a}_{i,j,1},\dots,\tilde{a}_{i,j,l_{i,j}}) \end{array}$$

where

$$\begin{array}{ll} \tilde{a}_{i,j,1} = a_{i,j,1} + d(\mathbf{a};t) & \text{if } (i,j) = t_i, \\ \tilde{a}_{i,j,s} = a_{i,j,s} & \text{otherwise,} \end{array}$$

and

$$d(\mathbf{a};t) = \sum_{i=1}^{p}\sum_{j=1}^{k_i}(\mathrm{wt}_{c_i}(w_{i,j} - w_{t_i}) + 1)\sum_{s=1}^{l_{i,j}} a_{i,j,s}$$
$$+ \sum_{j=1}^{k_0}(\mathrm{wt}_{c_0}(w_{0,j} - w_{t_0}) - 1)\sum_{s=1}^{l_{0,j}} a_{0,j,s} - \sum_{i=0}^{p} a_{t_i,1}.$$

For $i_0 = 0,\dots,p$, $j_0 = 1,\dots,k_{i_0}$, $s_0 = 1,\dots,l_{i_0,j_0} - 1$, we also define permutations on $L(P)$,

$$\begin{array}{ccccl} \sigma(i_0,j_0,s_0)\colon & L(P) & \longrightarrow & L(P) & \\ & a_{i_0,j_0,s_0} & \longmapsto & a_{i_0,j_0,s_0+1}, & \\ & a_{i_0,j_0,s_0+1} & \longmapsto & a_{i_0,j_0,s_0}, & \\ & a_{i,j,s} & \longmapsto & a_{i,j,s} & \text{if } (i,j,s) \neq (i_0,j_0,s_0). \end{array}$$

Let us define the group $\tilde{W}(P)$ generated by these $\sigma(t), \sigma(i,j,s)$, i.e.,

$$\tilde{W}(P) = \\ \langle \sigma(t), \sigma(i,j,s) \mid t \in \mathcal{T}(P),\ i = 0, \ldots, p,\ j = 1, \ldots, k_i,\ s = 1, \ldots, l_{i,j} - 1 \rangle.$$

Thus we could define $\mathbb{Z}$-linear action of $\tilde{W}(P)$ on $L(P)$.

Similarly we define the space of local exponents $R(P)$ by

$$R(P) = \prod_{i=0}^{p} \prod_{j=1}^{k_i} \Lambda^{l_{i,j}}.$$

And we extend the translation which we see in Theorem 3.2 to this $R(P)$ as follows. For $t = (t_0, \ldots, t_p) \in \mathcal{T}(P)$, we define transformations $\sigma(t)$ of $R(P)$ by

$$\begin{array}{cccc} \sigma(t)\colon & R(P) & \longrightarrow & R(P) \\ & \prod_{i=0}^{p} \prod_{j=1}^{k_i} \nu_{i,j} & \longmapsto & \prod_{i=0}^{p} \prod_{j=1}^{k_i} \tilde{\nu}_{i,j} \end{array}$$

where

$$\tilde{\nu}_{i,j,s} = \begin{cases} \nu_{i,j,s} & \text{if } t_i = (i,j) \text{ and } s = 1, \\ \nu_{i,j,s} - (\mathrm{wt}_{c_i}(w_{i,j} - w_{t_i}) + 1)(1 - \nu(t)) & \text{if otherwise,} \end{cases}$$

and for $i = 0$,

$$\tilde{\mu}_{0,j,s} = \begin{cases} \nu_{t_0,1} + 2(1 - \nu(t)) & \text{if } t_0 = (0,j) \text{ and } s = 1, \\ \nu_{0,j,s} - (\mathrm{wt}_{c_0}(w_{0,j} - w_{t_0}) - 1)(1 - \nu(t)) & \text{if otherwise,} \end{cases}$$

where

$$\nu(t) = \sum_{i=0}^{p} \nu_{t_i,1}.$$

Also we define permutations $\sigma(i_0, j_0, s_0)$ on $R(P)$ as it is defined on $L(P)$.

3.3 The Euler transform and the Weyl group action on a root lattice

In the previous section, we consider the $\tilde{W}(P)$-module $L(P)$ from the translations caused by the Euler transform. We shall see that there exists a Kac-Moody root system with the Weyl group $W(P)$ and the root lattice $Q(P)$. Then $L(P)$ can be seen as a quotient $W(P)$-module of $Q(P)$.

We retain the notations of the previous section. We define the root system induced from the lattice $L(P)$ as follows. The root lattice $Q(P)$ is the $\mathbb{Z}$-lattice with the basis

$$\mathcal{C} = \{c_t \mid t \in \mathcal{T}(P)\} \\ \cap \{c(i,j,s) \mid i = 0, \ldots, p,\ j = 1, \ldots, k_i,\ s = 1, \ldots, l_{i,j} - 1\}.$$

Namely,

$$Q(P) = \sum_{c \in \mathcal{C}} \mathbb{Z}c.$$

We define the following symmetric bilinear form $\langle\ ,\ \rangle$ on $Q(P)$,

$$\begin{aligned}\langle c_t, c_{t'} \rangle = & -\sum_{i=1}^{p} (\mathrm{wt}_{c_i}(w_{t_i} - w_{t'_i}) + 1) - \mathrm{wt}_{c_0}(w_{t_0} - w_{t'_0}) + 1 \\ & + \#\{i \mid t_i = t'_i,\ i = 0, \ldots, p\},\end{aligned}$$

$$\langle c_t, c(i,j,s) \rangle = \begin{cases} -1 & \text{if } t_i = (i,j) \text{ and } s = 1 \\ 0 & \text{if otherwise} \end{cases},$$

$$\langle c(i,j,s), c(i',j',s') \rangle = \begin{cases} 2 & \text{if } (i,j,s) = (i',j',s') \\ -1 & \text{if } (i,j) = (i',j') \text{ and } |s - s'| = 1 \\ 0 & \text{if otherwise} \end{cases}.$$

For $c \in \mathcal{C}$, the reflections with respect to c is defined by

$$\sigma_c(\alpha) = \alpha - 2\frac{\langle c, \alpha \rangle}{\langle c, c \rangle} c$$

for $\alpha \in Q(P)$. Then the Weyl group $W(P)$ is the group generated by these reflections σ_c for all $c \in \mathcal{C}$.

Theorem 3.3. *Let us define the* $\mathbb{Z}$*-module homomorphism*

$$\Phi \colon Q(P) \longrightarrow L(P)$$

as follows. For

$$\alpha = \sum_{t \in \mathcal{T}(P)} \alpha_t c_t + \sum_{i=0}^{p} \sum_{j=1}^{k_i} \sum_{s=1}^{l_{i,j}-1} \alpha(i,j,s) c(i,j,s) \in Q(P),$$

the image $\Phi(\alpha) = \prod_{i=0}^{p} \prod_{j=1}^{k_i} (a_{i,j,1}, \ldots, a_{i,j,l_{i,j}})$ *is*

$$\begin{aligned} a_{i,j,1} &= \sum_{\{t \in \mathcal{T}(P) \mid t_i = (i,j)\}} \alpha_t - \alpha(i,j,1), \\ a_{i,j,s} &= \alpha(i,j,s-1) - \alpha(i,j,s) \quad \text{for } 2 \le s \le l_{i,j}. \end{aligned}$$

Here we put $\alpha(i,j,l_{i,j}) = 0$*. Then we have the following.*

1. *The map* Φ *is surjective.*

2. *This* Φ *is injective if and only if at most one* k_i *in* $\{k_0, \ldots, k_p\}$ *satisfies*

$$k_i > 1,$$

that is to say, we have

$$\#\{k_i \mid k_i > 1,\ i = 0, \ldots, p\} \le 1.$$

3. *The Weyl group action on $Q(P)$ corresponds to actions of $\tilde{W}(P)$ on $L(P)$. Namely, we have*

$$\Phi(\sigma_{c_t}\alpha) = \sigma(t)\Phi(\alpha),$$
$$\Phi(\sigma_{c(i,j,s)}\alpha) = \sigma(i,j,s)\Phi(\alpha),$$

for all $\alpha \in Q(P)$.

4. *For $\mathbf{a} \in L(P)$, if we define*

$$\operatorname{idx}\mathbf{a} = \langle \Phi^{-1}(\mathbf{a}), \Phi^{-1}(\mathbf{a})\rangle,$$

then it is well-defined.

5. *The Weyl group $W(P)$ acts on $R(P)$ as follows,*

$$\sigma_{c_t}\mu = \sigma(t)\mu,$$
$$\sigma_{c(i,j,s)}\mu = \sigma(i,j,s)\mu$$

for $\mu \in R(P)$.

Proof. We have

$$\sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}} a_{i,j} = \sum_{t\in\mathcal{T}(P)} \alpha_t \ (i = 0,\ldots,p)$$

for $\Phi(\alpha) = \prod_{i=1}^{p}\prod_{j=1}^{k_i}(a_{i,j,1},\ldots,a_{i,j,l_{i,j}})$ which are images of $\alpha \in Q(P)$. Hence Φ is well-defined.

We write $\mathcal{T}_i = \{(i,j) \mid j = 1,\ldots,k_i\}$ for $i = 0,\ldots,p$. Then $\mathcal{T}(P) = \prod_{i=0}^{p}\mathcal{T}_i$. Let us choose an element $\tau = (\tau_0,\ldots,\tau_p) \in \mathcal{T}(P)$. Then we see that images of

$$\begin{cases} c(i,j,s) & \text{for } i = 0,\ldots,p,\ j = 1,\ldots,k_i,\ s = 1,\ldots,l_{i,j}-1, \\ c_t & \text{for } t \in \{\tau\}\cup\bigcup_{i=0}^{p}((\mathcal{T}_i\backslash\{\tau_i\})\times\prod_{i\neq j}\{\tau_j\}) \end{cases}$$

generate L_P. Hence Φ is surjective.

Let us show 2. Ranks of free $\mathbb{Z}$-modules $Q(P)$ and $L(P)$ are

$$\operatorname{rank}_{\mathbb{Z}\text{-mod}}Q(P) = \prod_{i=0}^{p} k_i + \sum_{i=0}^{p}\sum_{j=1}^{k_i}(l_{i,j}-1),$$
$$\operatorname{rank}_{\mathbb{Z}\text{-mod}}L(P) = \sum_{i=0}^{p}\sum_{j=1}^{k_i} l_{i,j} - p.$$

Hence

$$\operatorname{rank}_{\mathbb{Z}\text{-mod}}Q(P) - \operatorname{rank}_{\mathbb{Z}\text{-mod}}L(P) = \prod_{i=0}^{p} k_i - \sum_{i=0}^{p} k_i + p.$$

Here we notice that

$$\prod_{i=0}^{p} k_i - \sum_{i=0}^{p} k_i + p \geq 0.$$

Indeed $g(k_0, \ldots, k_p) = \prod_{i=0}^{p} k_i - \sum_{i=0}^{p} k_i$ is the increasing function of each k_i for $i = 0, \ldots, p$, since

$$\frac{\partial}{\partial k_i} g(k_0, \ldots, k_p) = \prod_{j \neq i} k_j - 1 \geq 0.$$

Thus,

$$g(k_0, \ldots, k_p) \geq g(1, \ldots, 1) = -p.$$

If we assume that there exists at least two k_{i_1} and k_{i_2} satisfying $k_{i_1} \geq 2$ and $k_{i_2} \geq 2$, then

$$g(k_0, \ldots, k_p) \geq 4 - (2 + 2 + (p-1)) = -p + 1.$$

Hence

$$\mathrm{rank}_{\mathbb{Z}\text{-mod}} Q(P) - \mathrm{rank}_{\mathbb{Z}\text{-mod}} L(P) \geq 1.$$

This shows Φ is not injective. On the contrary, if all k_i are $k_i = 1$ except only one k_{i_0}, then

$$g(k_0, \ldots, k_p) = k_{i_0} - k_{i_0} - p.$$

Hence

$$\mathrm{rank}_{\mathbb{Z}\text{-mod}} Q(P) = \mathrm{rank}_{\mathbb{Z}\text{-mod}} L(P).$$

Then since we know Φ is surjective, hence Φ is injective.

Let us show 3. We have

$$\begin{aligned}
d(\Phi(\alpha);t) &= \sum_{i=1}^{p}\sum_{j=1}^{k_i}(\mathrm{wt}_{c_i}(w_{i,j}-w_{t_i})+1)\sum_{s=1}^{l_{i,j}} a_{i,j,s} \\
&\quad + \sum_{j=1}^{k_0}(\mathrm{wt}_{c_0}(w_{0,j}-w_{t_0})-1)\sum_{s=1}^{l_{0,j}} a_{0,j,s} - \sum_{i=0}^{p} a_{t_i,1} \\
&= \sum_{i=1}^{p}\sum_{r=0}^{\infty}(r+1)\sum_{\{t'\in\mathcal{T}(P)\mid \mathrm{wt}_{c_i}(w_{t'_i}-w_{t_i})=r\}}\alpha_{t'} \\
&\quad + \sum_{r=0}^{\infty}(r-1)\sum_{\{t''\in\mathcal{T}(P)\mid \mathrm{wt}_{c_0}(w_{t''_0}-w_{t_0})=r\}}\alpha_{t''} \\
&\quad - \sum_{i=0}^{p}\sum_{\{t'''\in\mathcal{T}(P)\mid t'''_i=t_i\}}\alpha_{t'''} + \sum_{i=0}^{p}\sum_{s\in\mathcal{T}_i}\alpha(i,s,1) \\
&= \sum_{t'\in\mathcal{T}(P)}\alpha_{t'}(\sum_{i=1}^{p}(\mathrm{wt}_{c_i}(w_{t'_i}-w_{t_i})+1)+\mathrm{wt}_{c_0}(w_{t'_0}-w_{t_0})-1 \\
&\quad - \#\{i \mid t'_i=t_i, i=0,\dots,p\}) \\
&\quad + \sum_{i=0}^{p}\sum_{s\in\mathcal{T}_i}\alpha(i,s,1) \\
&= -\langle c_t,\alpha\rangle.
\end{aligned}$$

Hence we have

$$\Phi(\sigma_{c_t}\alpha)=\sigma(t)\Phi(\alpha).$$

And the equations

$$\Phi(\sigma_{d(i,s,j)}\alpha)=\sigma(i,s,j)\Phi(\alpha)$$

are similarly obtained.

Let us show 4. If $\alpha\in\operatorname{Ker}\Phi$, then $d(\Phi(\alpha);t)=0$. Thus $\langle c_t,\alpha\rangle=0$ for all $t\in\mathcal{T}(P)$. And similarly we have $\langle c(i,j,s),\alpha\rangle=0$ for all $i=0,\dots,p$, $j=1,\dots,k_i$, $s=1,\dots,l_{i,j}-1$. Thus we have that if $\alpha\in\operatorname{Ker}\Phi$, then $\langle\beta,\alpha\rangle=0$ for all $\beta\in Q(P)$.

Finally we show 5. We need to see that the $W(P)$ action on $R(P)$ defined as above is well-defined. Namely, if for $c,c'\in\mathcal{C}$ reflections $\sigma_c,\sigma_{c'}$ satisfy Coxeter relations

$$\begin{aligned}
\sigma_c^2=\sigma_{c'}^2&=\mathrm{id}, \\
(\sigma_c\sigma_c)^{m(c,c')}&=\mathrm{id}
\end{aligned}$$

in $W(P)$ for some positive integer $m(c,c')$ (sometimes it is ∞), then we have

$$\sigma_c^2\mu = \sigma_{c'}^2\mu = \mu, \tag{9}$$

$$(\sigma_c\sigma_{c'})^{m(c,c')}\mu = \mu \tag{10}$$

for all $\mu \in R(P)$. The involutive relations (9) are directly follows from the definition. We check the relations (10).

Let us take $t, t' \in \mathcal{T}(P)$ $(t \neq t')$ and compute $(\sigma(t)\sigma(t))^m$ on $R(P)$. For $\nu = \prod_{i=0}^{p}\prod_{j=1}^{k_i}(\nu_{i,j,1},\ldots,\nu_{i,j,l_{i,j}}) \in R(P)$, we can see

$$(\sigma(t')\sigma(t))^m\nu = \nu^{(m)} = \prod_{i=0}^{p}\prod_{j=1}^{k_i}(\nu^{(m)}_{i,j,1},\ldots,\nu^{(m)}_{i,j,l_{i,j}})$$

becomes as follows. For $i = 1,\ldots,p$, we have the following.

- If $t_i = t'_i$,

$$\nu^{(m)}_{i,j,s} = \begin{cases} \nu_{i,j,1} & \text{for } t_i = t'_i = (i,j), \\ \nu_{i,j,s} - (\mathrm{wt}_{c_i}(w_{i,j} - w_{t'_i}) + 1)\displaystyle\sum_{u=1}^{m}\mu^{(u)}(t') & \\ \qquad - (\mathrm{wt}(w_{i,j} - w_{t_i}) + 1)\displaystyle\sum_{u=1}^{m}\mu^{(u)}(t) & \text{for the other } (i,j,s). \end{cases}$$

- If $t_i \neq t'_i$,

$$\nu^{(m)}_{i,j,s} = \begin{cases} \nu_{i,j,1} - (\mathrm{wt}_{c_i}(w_{t_i} - w_{t'_i}) + 1)\sum_{u=1}^{m}\mu^{(u)}(t') & \text{for } t_i = (i,j), \\ \nu_{i,j,1} - (\mathrm{wt}_{c_i}(w_{t'_i} - w_{t_i}) + 1)\sum_{u=1}^{m}\mu^{(u)}(t) & \text{for } t'_i = (i,j), \\ \nu_{i,j,s} - (\mathrm{wt}_{c_i}(w_{i,j} - w_{t'_i}) + 1)\displaystyle\sum_{u=1}^{m}\mu^{(u)}(t') & \\ \qquad - (\mathrm{wt}(w_{i,j} - w_{t_i}) + 1)\displaystyle\sum_{u=1}^{m}\mu^{(u)}(t) & \text{for the other } (i,j,s). \end{cases}$$

Also $\nu^{(m)}_{0,j,s}$ are as follows.

- If $t_0 = t'_0$,

$$\nu^{(m)}_{0,j,s} = \begin{cases} \nu_{0,j,1} + 2(\sum_{u=1}^{m}(\mu^{(u)}(t) + \mu^{(u)}(t'))) & \text{for } t_0 = t'_0 = (0,j), \\ \nu_{0,j,s} - (\mathrm{wt}_{c_0}(w_{0,j} - w_{t'_0}) - 1)\displaystyle\sum_{u=1}^{m}\mu^{(u)}(t') & \\ \qquad - (\mathrm{wt}(w_{0,j} - w_{t_0}) - 1)\displaystyle\sum_{u=1}^{m}\mu^{(u)}(t) & \text{for the other } (0,j,s). \end{cases}$$

- If $t_0 \neq t'_0$,

$$\nu^{(m)}_{0,j,s} = \begin{cases} \nu_{0,j,1} + 2\sum_{u=1}^{m} \mu^{(u)}(t) \\ \qquad - (\mathrm{wt}_{c_0}(w_{t_0} - w_{t'_0}) - 1)\sum_{u=1}^{m} \mu^{(u)}(t') & \text{for } t_0 = (0,j), \\ \nu_{0,j,1} + 2\sum_{u=1}^{m} \mu^{(u)}(t') \\ \qquad - (\mathrm{wt}_{c_0}(w_{t'_0} - w_{t_0}) - 1)\sum_{u=1}^{m} \mu^{(u)}(t) & \text{for } t'_0 = (0,j), \\ \nu_{0,j,s} - (\mathrm{wt}_{c_0}(w_{0,j} - w_{t'_0}) - 1)\sum_{u=1}^{m} \mu^{(u)}(t') \\ \qquad - (\mathrm{wt}(w_{0,j} - w_{t_0}) - 1)\sum_{u=1}^{m} \mu^{(u)}(t) & \text{for the other } (0,j,s). \end{cases}$$

Here $\mu^{(u)}(t)$ and $\mu^{(u)}(t')$ are defined as follows.

$$\begin{aligned} \mu^{(1)}(t) &= 1 - \sum_{i=0}^{p} \nu_{t_i,1}, \\ \mu^{(u)}(t) &= -\mu^{(u-1)}(t) + E\mu^{(u-1)}(t'), \\ \mu^{(1)}(t') &= 1 - \sum_{i=0}^{p} \nu_{t'_i,1} + E\mu^{(1)}(t), \\ \mu^{(u)}(t') &= -\mu^{(u-1)}(t') + E\mu^{(u)}(t), \end{aligned}$$

where

$$\begin{aligned} E = &\sum_{i=1}^{p} (\mathrm{wt}_{c_i}(w_{t_i} - w_{t'_i}) + 1) + \mathrm{wt}_{c_0}(w_{t_0} - w_{t'_0}) - 1 \\ &- \#\{i \mid t_i = t'_i,\ i = 0, \dots, p\}. \end{aligned}$$

Hence we have

$$\sum_{u=1}^{m} \mu^{(u)}(t) = \sum_{u=1}^{m} \mu^{(u)}(t') = 0 \text{ for } \begin{cases} m = 2 & \text{if } E = 0, \\ m = 3 & \text{if } E = 1, \\ m = 4 & \text{if } E = 2, \\ m = 6 & \text{if } E = 3, \end{cases}$$

and if $E \geq 4$, then $\sum_{u=1}^{m} \mu^{(u)}(t)$ and $\sum_{u=1}^{m} \mu^{(u)}(t')$ never become zero (see

Proposition 3.13 in [7]). This shows that

$$(\sigma(t)\sigma(t'))^m = \mathrm{id}|_{R(P)} \text{ for } \begin{cases} m=2 & \text{if } E=0, \\ m=3 & \text{if } E=1, \\ m=4 & \text{if } E=2, \\ m=6 & \text{if } E=3, \\ m=\infty & \text{if } E\geq 4. \end{cases}$$

Similarly the direct computation shows that

$$(\sigma(t)\sigma(i,j,s))^m = \mathrm{id}|_{R(P)} \text{ for } \begin{cases} m=3 & \text{if } t_i=(i,j) \text{ and } s=1, \\ m=2 & \text{if otherwise,} \end{cases}$$

and

$$(\sigma(i,j,s)\sigma(i',j',s'))^m = \mathrm{id}|_{R(P)} \text{ for } \begin{cases} m=3 & \text{if } (i,j)=(i',j') \text{ and } |s-s'|=1, \\ m=2 & \text{if otherwise.} \end{cases}$$

□

3.4 Examples : affine Weyl group symmetries of Heun equations.

Let us see some examples of Theorem 3.3. As examples, we consider the Heun differential equation and its confluent equations (see [14] for instance).

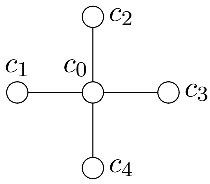
(1) The Heun differential operator.
The Heun differential operator is the differential operator of the form

$$\begin{aligned} P = x(x-1)(x-t)\partial^2 + \{c(x-1)(x-t) + dx(x-t) \\ + (a+b+1-c-d)z(z-1)\}\partial + (abx-\lambda). \end{aligned}$$

This has regular singular points at $x = 0, 1, t, \infty$ and at these singular points, it has the following spectral data,

$$\begin{aligned} &\{(0, 1-c; (1,1))\} \quad \text{at } x=0, \\ &\{(0, 1-d); (1,1)\} \quad \text{at } x=1, \\ &\{(0, c+d-a-b); (1,1)\} \quad \text{at } x=t, \\ &\{(a,b); (1,1)\} \quad \text{at } x=\infty. \end{aligned}$$

By Theorem 3.3, we can define the root lattice $Q(P)$ with the following extended Dynkin diagram.



Here this diagram is drawn by the following rule. If c_i and c_j in the basis of $Q(P)$ satisfy $\langle c_i, c_j \rangle = -m(i,j)$, then corresponding vertices c_i and c_j are connected by $m(i,j)$ edges. We can see

$$\langle c_1, c_0 \rangle = -1, \quad \langle c_1, c_2 \rangle = 0$$

from this diagram for example.

This diagram is that of the affine $D_4^{(1)}$ type root system. For

$$\mathbf{m}(P) = \prod_{i=0}^{4} (1,1),$$

we can associate the element in $Q(P)$,

$$\Phi^{-1}(\mathbf{m}(P)) = 2c_0 + \sum_{i=1}^{4} c_i.$$

This is a imaginary root of $Q(P)$.

We can see that $\delta(P) = \Phi^{-1}(\mathbf{m}(P))$ is invariant by the action of $W(P)$. Namely, twisted Euler transforms $E(t)$ and permutations $\sigma(i,j,s)$ preserves the spectral type of P. On the other hand, characteristic exponents are changed by $E(t)$ and permutations. As we see in Theorem 3.3, the Weyl group $W(P)$ acts on the space of characteristic exponents $R(P)$ as well. Thus we can conclude that characteristic exponents of the Heun differential operator has affine $D_4^{(1)}$ Weyl group symmetry generated by twisted Euler transform and permutations.

(2) The confluent Heun differential operator.

Let us consider the confluent operators of Heun operator. The confluent Heun differential operator is

$$P^c = x(x-1)\partial^2 + \{-tx(x-1) + c(x-1) + dx\}\partial + (-taz + \lambda).$$

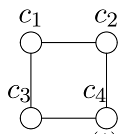
This has regular singular points at $x = 0, 1$ and irregular singular point at $x = \infty$. The spectral data are

$$\begin{aligned} &\{(0, 1-c); (1,1)\} \quad \text{at } x = 0, \\ &\{(0, 1-d); (1,1)\} \quad \text{at } x = 1, \end{aligned}$$

for regular singular points and

$$\{(a);(1)\} \quad \text{with the exponential factor } w_1 = 0,$$
$$\{(c+d-a);(1)\} \quad \text{with the exponential factor } w_2 = tx,$$

for the irregular singular point $x = \infty$. Then the corresponding root system has the following extended Dynkin diagram.



This corresponds to the affine $A_4^{(1)}$ root system. And we have

$$\delta(P^c) = \Phi^{-1}(\mathbf{m}(P^c)) = \sum_{i=1}^{4} c_i.$$

This $\delta(P^c)$ is imaginary root of $Q(P^c)$ and $W(P^c)$-invariant. Hence as well as the Heun differential operator, we can conclude that the characteristic exponents of confluent Heun differential operator has affine $A_4^{(1)}$ Weyl group symmetry generated by twisted Euler transforms and permutations.

(3) The biconfluent Heun differential equation.
Let us consider the biconfluent Heun differential equation,

$$P^{bc} = x\partial^2 + (-x^2 - tx + c)\partial + (-ax + \lambda).$$

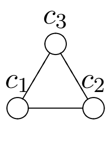
This has regular singular point at $x = 0$ with the spectral data,

$$\{(0, 1-c);(1,1)\},$$

and irregular singular point at $x = \infty$ with the spectral data,

$$\{(a);(1)\} \quad \text{with the exponential factor } w_1 = 0,$$
$$\{(c+1-a);(1)\} \quad \text{with } w_2 = x + t.$$

The corresponding diagram and the element in $Q(P^{bc})$ are as follows,



,

$$\delta(P^{bc}) = \Phi^{-1}(\mathbf{m}(P^{bc})) = \sum_{i=1}^{3} c_i.$$

Hence this is the affine $A_3^{(1)}$ root system and $\delta(P^{bc})$ is a imaginary root of this root system. As well as the above examples, we can see that P^{bc} has the affine $A_3^{(1)}$ Weyl group symmetry generated by twisted Euler transforms and permutations.

(4) The triconfluent Heun differential equation.
The triconfluent Heun differential equation is

$$P^{tc} = \partial^2 + (-x^2 - t)\partial + (-ax + \lambda).$$

As well as the above examples, we can see that P^{tc} has affine $A_2^{(1)}$ Weyl group symmetry generated by twisted Euler transforms.

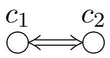
Indeed the spectral data are

$$\begin{aligned} &\{(a);(1)\} \quad \text{with the exponential factor } w_1 = 0,\\ &\{(2-a);(1)\} \quad \text{with } w_2 = x^2 + t \end{aligned}$$

at the irregular singular point $x = \infty$. And we can see that

$$\delta(P^{tc}) = \Phi^{-1}(\mathbf{m}(P^{tc})) = c_1 + c_2$$

is a imaginary root of the root system with the extended Dynkin diagram,



.

(5) The doubly confluent Heun differential operator.
The doubly confluent Heun differential operator is

$$P^{dc}x^2 = \partial^2 + (-x^2 + cx + t)\partial + (-ax + \lambda).$$

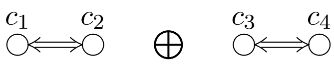
This has the $A_1^{(1)} \oplus A_1^{(1)}$ Weyl group symmetry. The spectral data are

$$\begin{aligned} &\{(0);(1)\} \quad \text{with the exponential factor } W_1^0 = 0,\\ &\{(2-c);(1)\} \quad \text{with } w_1^{(1)} = \frac{-t}{x}, \end{aligned}$$

at $x = 0$ and

$$\begin{aligned} &\{(a);(1)\} \quad \text{with the exponential factor } w_1^\infty = 0,\\ &\{(c-a);(1)\} \quad \text{with } w_2^\infty = x \end{aligned}$$

at $x = \infty$. Then the corresponding diagram is



,

and

$$\delta(P^{dc}) = \Phi^{-1}(\mathbf{m}(P^{dc})) = \sum_{i=1}^{4} c_i + a(c_1 + c_2 - c_3 - c_4) \ (a \in \mathbb{Z}).$$

Here we notice that $(c_1 + c_2 - c_3 - c_4) \in \operatorname{Ker}\Phi$. We can see that $\delta(P^{dc})$ is $W(P^{dc})$-invariant. Hence we can conclude that P^{dc} has Weyl group $W(P^{dc})$ symmetry.

Let us give comments about the relationship with Painlevé equations. As is known, if we put an apparent singular point to each these operators and consider the isomonodromic deformation, then we can obtain Painlevé equations, namely, P_{VI} from the Heun operator, P_V from the confluent Heun operator, P_{VI} from the biconfluent Heun operator, P_{III} from the doubly confluent Heun operator, and P_{II} from the triconfluent Heun operator respectively.

It is known that these Painlevé equations have following affine Weyl group symmetries generated by birational transformations.

P_{VI}	P_V	P_{IV}	P_{II}	P_{III}
$D_4^{(1)}$	$A_3^{(1)}$	$A_2^{(1)}$	$A_1^{(1)}$	$(A_1 \oplus A_1)^{(1)}$

Our Weyl groups recover these Painlevé symmetries.

3.5 The irreducibility and the Φ-root system

We shall define an analogue of the root system in $L(P)$, called Φ-roots and show that if P is irreducible, then $\mathbf{m}(P)$ becomes a Φ-root.

Proposition 3.4. *Let us suppose that P is irreducible in $W_\Lambda(x)$. If* $\operatorname{rank} P > 1$*, then $E(t)P$ are irreducible and* $\operatorname{rank} E(t)P \geq 1$ *for $t \in \mathcal{T}(P)$. Moreover if we put*

$$Q_{u-1} = E(t_{u-1}) \circ E(t_{u-2}) \circ \ldots \circ E(t_1)P$$

and assume that $\operatorname{rank} Q_r > 1$ *for $r = 1, \ldots, u-1$, then $E(t_u)Q_{u-1}$ is irreducible and* $\operatorname{rank} E(t_u)Q_{u-1} \geq 1$ *for $t_1, \ldots, t_u \in \mathcal{T}(P)$.*

Proof. We denote

$$\bar{P} = \prod_{i=1}^{p} \operatorname{Ad}((x - c_i)^{-\lambda_{t_i,1}}) \prod_{i=0}^{p} \operatorname{Ad}(-e^{w_{t_i}})P.$$

Proposition 2.9 implies that $\operatorname{Prim}(\bar{P})$ generates the maximal ideal of $W_\Lambda[x]$. Thus $\mathcal{L}^{-1}\operatorname{Prim}(\bar{P})$ also generates the maximal ideal of $W_\Lambda[x]$. If

$$\mathcal{L}^{-1}\operatorname{Prim}(\bar{P}) \notin \Lambda[x],$$

then $\mathcal{L}^{-1}\operatorname{Prim}(\bar{P})$ generates the maximal ideal in $W_\Lambda(x)$ as well.

Suppose that $\mathcal{L}^{-1}\operatorname{Prim}(\bar{P}) = f(x) \in \Lambda[x]$. Then $\operatorname{Prim}(\bar{P}) = f(-\partial)$. Since P is irreducible in $W_\Lambda(x)$, we have $\deg_{\Lambda[x]} f(x) = 1$. This contradicts $\operatorname{rank} P > 1$. Hence $\mathcal{L}^{-1}\operatorname{Prim}(\bar{P})$ is irreducible in $W_\Lambda(x)$.

Moreover since $\operatorname{Ad}(x^{1-\lambda(P;t)})$ does not change the irreducibility, we have $\operatorname{Ad}(x^{1-\lambda(P;t)})\mathcal{L}\operatorname{Prim}(\bar{P})$ is irreducible in $W_\Lambda(x)$. Then Proposition 2.9 tells that $\mathrm{Prim}\,\mathrm{Ad}(x^{1-\lambda(P;t)})\mathcal{L}^{-1}\mathrm{Prim}\,(\bar{P})$ generates the maximal ideal in $W_\Lambda[x]$. Thus if $E(1-\lambda(P;t))\bar{P} \notin \Lambda[x]$, then $E(1-\lambda(P;t))\bar{P}$ is irreducible in $W_\Lambda(x)$.

Suppose that $E(1-\lambda(P;t))\bar{P} = g(x) \in \Lambda[x]$. Then

$$\mathrm{Prim}\,\mathrm{Ad}(x^{1-\lambda(P;t)})\mathcal{L}^{-1}\mathrm{Prim}\,(\bar{P}) = g(\partial).$$

Hence $g(x) = ax + b$ for some $a, b \in \Lambda$ from the irreducibility. Thus there exists $f(x) \in \Lambda[x]$ such that

$$\mathrm{Ad}(x^{1-\lambda(P;t)})\mathcal{L}^{-1}\mathrm{Prim}\,(\bar{P}) = f(x)(ax\partial + bx - 1 + \lambda(P;t)).$$

Hence

$$\mathrm{Prim}\,\bar{P} = f(-\partial)(-ax\partial - b\partial - 1 + \lambda(P;t)).$$

This is the contradiction since $\mathrm{rank}\,P > 1$ and P is irreducible in $W_\Lambda(x)$.

The second assertion follows from the same argument. □

We shall define Φ-root system of $L(P)$ which is an analogue of the root system of $Q(P)$. For this purpose, we recall roots of $Q(P)$ first. The set of real roots is

$$\Delta_{\mathrm{re}} = \bigcup_{c\in\mathcal{C}} W(P)c,$$

i.e., the union of $W(P)$-orbit of $c \in \mathcal{C}$. To define imaginary roots, let us consider the set

$$F = \{\alpha \in Q(P)^+ = \sum_{c\in\mathcal{C}} \mathbb{Z}_{\geq 0}c \mid {\scriptstyle \langle\alpha,c\rangle\leq 0 \text{ for all } c\in\mathcal{C}, \atop \mathrm{supp}\,\alpha \text{ is connected.}}\}\backslash\{\mathbf{0}\}.$$

Here we say $\mathrm{supp}(\alpha)$ is connected if $\alpha = \sum_{c\in\mathcal{C}} \alpha_c c$ satisfies the following. If $I = \{c \in \mathcal{C} \mid \alpha_c \neq 0\}$ is decomposed by a disjoint union $I = I_1 \amalg I_2$ such that we have $\langle c_1, c_2\rangle = 0$ for all $c_1 \in I_1$ and $c_2 \in I_2$, then $I_1 = \emptyset$ or $I_2 = \emptyset$.

We define the set of imaginary roots by

$$\Delta_{\mathrm{im}} = W(P)F \cup -(W(P)F).$$

And we define the set of roots by

$$\Delta = \Delta_{\mathrm{re}} \cup \Delta_{\mathrm{im}}.$$

Let us define Φ-roots as an analogue of Δ. We consider the following subset of $L(P)$,

$$\Delta^\Phi_{\mathrm{re}} = \bigcup_{t\in\mathcal{T}(P)} \tilde{W}(P)\Phi(c_t),$$

i.e., the union of $\tilde{W}(P)$-orbit of $\Phi(c_t)$. We call this the set of Φ-real roots. We also consider the subset

$$F^\Phi = \{\mathbf{a} \in L(P) \cap \prod_{i=0}^{p}\prod_{j=1}^{k_i} \mathbb{Z}^{l_{i,j}}_{\geq 0}\backslash\{\mathbf{0}\} \mid {\scriptstyle a_{i,j,1}\geq a_{i,j,2}\geq\cdots\geq a_{i,j,l_{i,j}},\, d(\mathbf{a};t)\geq 0 \atop \text{for all } i=0,\ldots,p,\, j=1,\ldots,k_i,\, t\in\mathcal{T}(P)}\}.$$

Then let us define the set of Φ-imaginary roots by

$$\Delta^{\Phi}_{\mathrm{im}} = \tilde{W}(P)F^{\Phi} \cup -(\tilde{W}(P)F^{\Phi}).$$

We call

$$\Delta^{\Phi} = \Delta^{\Phi}_{\mathrm{re}} \cup \Delta^{\Phi}_{\mathrm{im}}$$

the set of Φ-roots and

$$\Delta^{\Phi+} = \Delta^{\Phi} \cap \prod_{i=0}^{p}\prod_{j=1}^{k_i} \mathbb{Z}_{\geq 0}^{l_{i,j}}$$

the set of Φ-positive roots. Elements in Δ^{Φ} and $\Delta^{\Phi+}$ are called Φ-roots and Φ-positive roots respectively.

The following proposition assures that Δ^{Φ} can be seen as a natural generalization of the root system Δ.

Proposition 3.5. *If Φ is injective, then*

$$\Phi^{-1}(\Delta^{\Phi}) \subset \Delta.$$

Proof. It is clear that $\Phi^{-1}(\Delta^{\Phi}_{\mathrm{re}}) \subset \Delta_{\mathrm{re}}$. Hence we need to check $\Phi^{-1}(F^{\Phi}) \subset F$. To show this, it suffices to see that $\mathrm{supp}(\Phi^{-1}(\alpha))$ are connected for all $\alpha \in F^{\Phi}$. Let us suppose the contrary, i.e., there exists $\alpha = \sum_{c\in\mathcal{C}} \alpha_c c \in \Phi^{-1}(F^{\Phi})$ such that $\mathrm{supp}(\alpha)$ is not connected. Since Φ is injective, there exists $i_0 \in \{0,\ldots,p\}$ and we have

$$\mathcal{T}(P) = \mathcal{T}_{i_0}$$

and all k_i are $k_i = 1$ except k_{i_0} by Theorem 3.3. Hence for any $t \in \mathcal{T}(P)$ and $i \in \{0,\ldots,p\}\backslash\{i_0\}$, we have

$$\langle c_t, c(i,1,1)\rangle = -1.$$

This shows that $\alpha_{c(i,1,1)} = 0$ for all $i \in \{0,\ldots,p\}\backslash\{i_0\}$. And since

$$\alpha_{c(i,1,1)} \geq \alpha_{(i,1,2)} \geq \cdots,$$

we have $\alpha_{c(i,1,s)} = 0$ for $i \in \{0,\ldots,p\}\backslash\{i_0\}$ and $s = 1,\ldots,l_{i,1}$.

Next we show that for any $t, t' \in \mathcal{T}(\alpha) = \{t \in \mathcal{T}(P) \mid \alpha_{c_t} \neq 0\}$, we have

$$\mathrm{wt}_{c_{i_0}}(w_{t'_{i_0}} - w_{t_{i_0}}) = 1.$$

Suppose that we can show this. Then we have

$$\langle c_t, c_{t'}\rangle = 0$$

for $t, t' \in \mathcal{T}(\alpha)$ $(t \neq t')$. And for $t \in \mathcal{T}(\alpha)$, we have

$$\langle \alpha, c_t \rangle > 0,$$

since $\alpha_{c_t} \geq \alpha_{c(t_{i_0},1)}$. This is the contradiction.

Let us show the above claim. If there exist $t, t' \in \mathcal{T}(\alpha)$ such that

$$\mathrm{wt}_{c_{i_0}}(w_{t'_{i_0}} - w_{t_{i_0}}) \geq 2.$$

Then

$$\langle c_{t'}, c_t \rangle \neq 0.$$

Let us take $t'' \in \mathcal{T}(\alpha)$ satisfying

$$\mathrm{wt}_{c_{i_0}}(w_{t''_{i_0}} - w_{t_{i_0}}) = 1.$$

Then $\langle c_{t''}, c_t \rangle = 0$. However we have

$$\mathrm{wt}_{c_{i_0}}(w_{t''_{i_0}} - w_{t'_{i_0}}) \geq 2,$$

thus $\langle c_{t''}, c_{t'} \rangle \neq 0$. This contradicts the assumption that $\mathrm{supp}(\alpha)$ is not connected.

□

The following theorem shows that the irreducible condition for the differential operator P relate to the root condition of $\mathbf{m}(P) \in L(P)$.

Theorem 3.6. *If P is irreducible in $W_\Lambda(x)$, then we have the following.*

1. *We have that $\mathbf{m}(P) \in L(P)$ is the element in $\Delta^{\Phi+}$.*
2. *If $\mathrm{idx}\,\mathbf{m}(P) > 0$, then $\mathrm{idx}\,\mathbf{m}(P) = 2$.*
3. *We have*

$$\mathbf{m}(P) \in \begin{cases} \Delta^{\Phi}_{re} & \text{if } \mathrm{idx}\,\mathbf{m}(P) = 2, \\ \Delta^{\Phi}_{im} & \text{if } \mathrm{idx}\,\mathbf{m}(P) \leq 0. \end{cases}$$

Proof. Proposition 3.4 tells that if

$$\tilde{W}(P)\mathbf{m}(P) \not\subset L^+(P) = L(P) \cup \prod_{i=0}^{p} \prod_{j=1}^{k_i} \mathbb{Z}_{\geq 0}^{l_{i,j}},$$

then there exist $t^{(1)}, \ldots, t^{(r)} \in \mathcal{T}(P)$ such that

$$\mathrm{rank}\, E(t^{(r)}) \circ \cdots \circ E(t^{(1)})P = 1.$$

Thus for $Q_r = E(t^{(r)}) \circ \cdots \circ E(t^{(1)})P$, there exist $w \in \tilde{W}(P)$ and $t \in \mathcal{T}(P)$ such that

$$w\mathbf{m}(Q_r) = \Phi(c_t).$$

Hence we have

$$\mathbf{m}(P) \in \Delta^{\Phi}_{\mathrm{re}}$$

and

$$\operatorname{idx} \mathbf{m}(P) = 2.$$

Next we assume

$$\tilde{W}(P)\mathbf{m}(P) \subset L^{+}(P).$$

First we show that $\operatorname{idx} \mathbf{m}(P) \leq 0$. To do this, we suppose the contrary, i.e., $\operatorname{idx} \mathbf{m}(P) > 0$. Let us take an element $\mathbf{b} \in \tilde{W}(P)\mathbf{m}(P)$ which has the least rank in $\tilde{W}(P)\mathbf{m}(P)$ and

$$b_{i,j,1} \geq b_{i,j,2} \geq \dots b_{i,j,l_{i,j}} \text{ for all } i = 0, \dots, p,\ j = 1, \dots, k_i. \tag{11}$$

Since $\operatorname{idx} \mathbf{b} > 0$, we can show that there exist $t \in \mathcal{T}(P)$ such that

$$\operatorname{rank} \sigma(t)\mathbf{b} < \operatorname{rank} \mathbf{b}.$$

Indeed, since $\langle \Phi^{-1}(\mathbf{b}), \Phi^{-1}(\mathbf{b}) \rangle > 0$, there exist $c \in \mathcal{C}$ such that

$$\langle \Phi^{-1}(\mathbf{b}), c \rangle > 0.$$

The condition (11) implies $c \in \{c_t \mid t \in \mathcal{T}(P)\}$. This shows the above claim. However this contradicts the choice of $\mathbf{b}$. Hence $\operatorname{idx} \mathbf{m}(P) \leq 0$.

Next we show $\mathbf{m}(P) \in \Delta^{\Phi}_{\mathrm{im}}$. Let us take one of the least rank element $\mathbf{b} \in \tilde{W}(P)\mathbf{m}(P)$ which satisfies the condition (11) as above.

Since $\langle \Phi^{-1}(\mathbf{b}), \Phi^{-1}(\mathbf{b}) \rangle \leq 0$, we have $\langle \Phi^{-1}(\mathbf{b}), c \rangle \leq 0$ for all $c \in \mathcal{C}$. This shows that

$$\mathbf{b} \in F^{\Phi}.$$

Hence we have

$$\mathbf{m}(P) \in \Delta^{\Phi}_{\mathrm{im}}.$$

□

In the theory of the middle convolution, the Katz algorithm is one of the most important results. This shows that if an irreducible Fuchsian differential operator or a local system is rigid, i.e., uniquely determined by local structures around their singular points, then this operator or local system can be reduced to rank 1 element by finite iteration of the middle convolutions and the additions. This rigidity condition is estimated by the certain number, so-called the index of rigidity. Namely, one can show that a Fuchsian differential operator or local system are rigid if and only if their index of rigidity is 2.

A generalization of this theorem for non-Fuchsian differential operators is obtained by D. Arinkin and D. Yamakawa independently (see [1] and [17]). We can show an analogue of their results as a immediate consequence of Theorem 3.6.

Corollary 3.7 (Cf. Arinkin and Yamakawa [1],[17]). *Suppose that P is irreducible in $W_\Lambda(x)$. There exist $t^{(1)}, \dots, t^{(r)} \in \mathcal{T}(P)$ such that*

$$\operatorname{rank} E(t^{(r)}) \circ \cdots \circ E(t^{(1)})P = 1,$$

if and only if

$$\operatorname{idx} \mathbf{m}(P) = 2.$$

Proof. For any $t \in \mathcal{T}(P)$, we have

$$\operatorname{rank} \Phi(c_t) = 1.$$

Hence this follows from Theorem 3.6 and the definition of $\Delta_{\mathrm{re}}^{\Phi}$ immediately. □

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